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Automatic Control III

Lecture 9 – Optimal control

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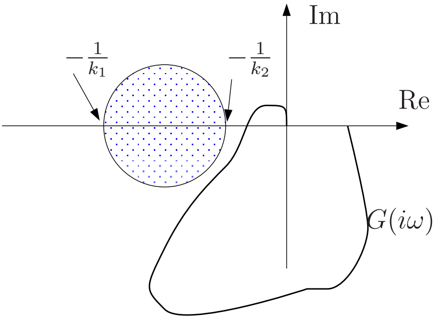
Contents – lecture 9

Summary of lecture 8 (I/II)

Close the loop around a linear system $G(s)$ using a static nonlinearity $f(y)$, where

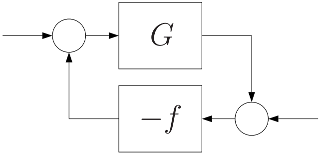
$$f(0) = 0, \qquad k_1 \leq \frac{f(y)}{y} \leq k_2,$$



The closed-loop system is stable if the Nyquist curve for $G(i\omega)$ does not encircle or enters the circle.

Summary of lecture 8 (II/II)

- Describing function: Self-oscillation in the following structure:



- f is represented using an amplitude-dependent gain $Y_f(C)$, where C is the amplitude.
- Condition for self-oscillation: $Y_f(C)G(i\omega) = -1$.
- Graphical representation: The intersection between the Nyquist curve $G(i\omega)$ and $-1/Y_f(C)$.
- The stability of the oscillation is not guaranteed.
- The method is approximative.

Optimal control - a classical application

Goddard's rocket problem: How should the thrust of a vertically ascending rocket be optimized to reach as high altitudes as possible (taking into account atmospheric drag and the gravitational field)?

Robert Goddard (on March 16, 1926), holds the launching frame of his most notable invention — the **first liquid-fueled rocket**.

Optimal control - a classical application

Goddard's diary entry the day after (March 17, 1926) the successful launch:

> *"The first flight with a rocket using liquid propellants was made yesterday at Aunt Effie's farm in Auburn... Even though the release was pulled, the rocket did not rise at first, but the flame came out, and there was a steady roar. After a number of seconds it rose, slowly until it cleared the frame, and then at express train speed, curving over to the left, and striking the ice and snow, still going at a rapid rate."*

To read more about Goddard and his endeavours, see
`http://en.wikipedia.org/wiki/Robert_H._Goddard`
R. Goddard "Rockets - A Method of Reaching Extreme Altitudes/Liquid-Propellant Rocket Developments", 1946

Goddards rocket problem (I/II)

1. Newton's force law gives us

$$\dot{v} = \frac{1}{m}(u - D(v, h)) - g,$$

where m denotes the mass of the rocket, v denotes the velocity, u denotes the engine thrust, h denotes altitude and g denotes gravity. Furthermore, $D(v, h)$ denotes the air drag.

2. The rocket ascends vertically: $\dot{h} = v$

3. The mass of the rocket is reduced as more fuel is consumed. Assume that the fuel consumption is proportional to the thrust

$$\dot{m} = -\gamma u$$

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Goddards rocket problem (II/II)

4. The thrust is limited

$$0 \leq u \leq u_{\mathsf{max}}.$$

5. We start from the ground. Fully fueled, the rocket mass is m_0 and empty the rocket mass is m_1,

$$v(0) = 0, \quad h(0) = 0, \quad m(0) = m_0, \quad m(t_f) \geq m_1,$$

where t_f is the final time.

6. Resulting **optimization** problem:

$$\max h(t_f)$$

subject to the above restrictions 1 – 5.

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The maximum principle – special case (I/III)

Study the following special case:

$$
\begin{aligned}
&\min \phi(x(t_f))\\
&\dot{x}(t) = f(x(t), u(t)),\\
&u(t) \in U, \quad 0 \le t \le t_f,\\
&x(0) = x_0.
\end{aligned}
$$

Step 1. Assume that we have a control signal $u^*(t)$ with a corresponding state $x^*(t)$ that fulfils the constraints.

Let us now test if $u^*(t)$ and $x^*(t)$ are also optimal (i.e. minimizes $\phi(x(t_f))$) by investigating what happens if we perturb the control signal somewhat.

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The maximum principle – special case (II/III)

We can derive the following **variational equation**

$$\dot{\eta}(t) = f_x(x^*(t), u^*(t))\eta(t)$$

Step 2 continued. The change in the cost function is given by (first order approximation)

$$\epsilon\phi_x(x^*(t_f))\eta(t_f)$$

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The maximum principle – special case (III/III)

Step 3 continued. The cost function change can also be written

$$\lambda^T(t_1)\eta(t_1) = \lambda^T(t_1)(f(x^*(t_1),\bar{u}) - f(x^*(t_1),u^*(t_1))),$$

where λ is given by

$$\begin{aligned}\dot{\lambda}(t) &= -f_x(x^*(t),u^*(t))^T\lambda(t),\\ \lambda(t_f) &= \phi_x(x^*(t_f))^T.\end{aligned}$$

This means that optimality requires

$$\lambda^T(t_1)\left(f(x^*(t_1),\bar{u}) - f(x^*(t_1),u^*(t_1))\right) \geq 0.$$

Since $\bar{u}$ and t_1 were arbitrary in our derivation this must hold for all choices of $\bar{u} \in U$ and for all $0 \leq t_1 \leq t_f$. This concludes our derivation of the maximum principle.

The maximum principle – special case (III/III)

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The maximum principle

Theorem: Assume that the optimization problem

$$\begin{aligned}
&\min \phi(x(t_f)) \\
&\dot{x}(t) = f(x(t), u(t)), \\
&u(t) \in U, \quad 0 \leq t \leq t_f, \\
&x(0) = x_0.
\end{aligned}$$

has a solution $u^*(t), x^*(t)$. Then it must hold that

$$\min_{u \in U} \lambda^T(t) f(x^*(t), u) = \lambda^T(t) f(x^*(t), u^*(t)), \qquad 0 \leq t \leq t_f,$$

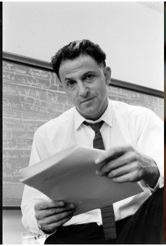
where $\lambda(t)$ fulfils

$$\dot{\lambda}(t) = -f_x(x^*(t), u^*(t))^T \lambda(t), \qquad \lambda(t_f) = \phi_x(x^*(t_f))^T.$$

Short history of optimal control

Roots in calculus of variations (Bernoulli, Euler, Lagrange, Weierstrass, ...)

- Optimal control emerged in the 1950s during the space race
- Dynamic programming (Richard Bellman in the US)
- Maximum principle (Lev Pontryagin in the former Soviet Union)
- Linear quadratic control (Rudolph Kalman)


Motion control – industrial robots

The are many many industrial application, let me just show you one.

Generation of suitable reference trajectories is a very common application of optimal control.

Computing optimal trajectories (position, velocity, ...) for the tool. This reference trajectory is then handed to the robot control system that controls the various parts of the robot such that the trajectory is followed as well as possible.

http://www.youtube.com/watch?v=SOESSCXGhFo

Motion control – bipedal robots

- An optimization algorithm transforms descriptions of each maneuver into dynamically-feasible reference motions.
- Then the motions are tracked using a model predictive controller.
- Using this approach, a performance success rate is about 80%.

https://www.youtube.com/watch?v=_sBBaNYex3E

A few concepts to summarize lecture 9

Optimal control: Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved.

Maximum principle: The maximum principle is used in optimal control to find the best possible control for taking a dynamical system from one state to another. A necessary condition for an optimum.

Variational equation: A variational equation describes how perturbations evolve along a trajectory.