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Automatic Control III

Lecture 5 – H_2 and H_∞ loop shaping

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Contents – lecture 5

1. Summary of lecture 4
2. Specifications – closed loop system
3. An extended system
4. $\mathcal{H}_2$ control
5. $\mathcal{H}_\infty$ control
6. Design example

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Summary of lecture 4 (I/II)

The key difficulty in controlling multivariable systems is that there are **cross couplings** between the input and the output signals.

The **relative gain array (RGA)** is a way of measuring the amount of cross couplings in a system

$$\mathsf{RGA}(G) \triangleq G. * (G^{\dagger})^T.$$

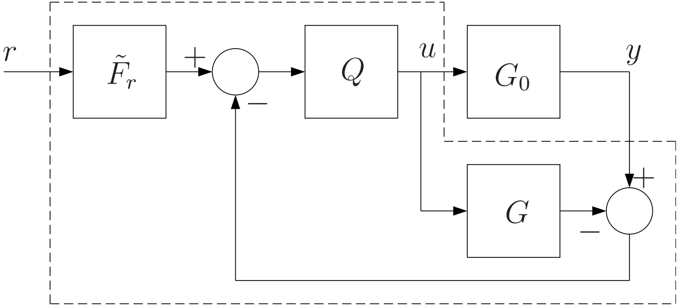
Decentralized control means that we let every input be determined by feedback from one single output.

The **pairing problem** is to select which input-output pairs that should be used for the feedback.

Decoupled control makes use of a change of variables such that suitable pairings of measurements and control signals becomes easier to see.

Summary of lecture 4 (II/II)

If there were no model errors and no disturbances we can make y follow r perfectly by choosing $Q = G^{-1}$ in $y = GQr$.



The idea in **Internal Model Control (IMC)** is to choose $Q \approx G^{-1}$ and feedback only using the new information $y - Gu$.

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Linear quadratic synthesis – pros and cons

$$
\begin{aligned}
z &= Gu + w; \quad \operatorname{cov}(w) = R_1 \delta(t) \\
y &= z + n; \quad e = z - r; \quad \operatorname{cov}(n) = R_2 \delta(t) \\
u &= F_r r - F_y y; \quad \min_u \int E\left\{ e^T(t) Q_1 e(t) + u^T(t) Q_2 u(t) \right\} \, dt
\end{aligned}
$$

- (+) All reasonable (stabilizable and detectable, $Q_1 \geq 0$, $Q_2 > 0$, $R_1 \geq 0$, $R_2 > 0$) choices of Q_1, Q_2, R_1, R_2 result in a closed-loop system with poles strictly in the left half plane.
- (+) Handles the trade-offs between the sizes of the different components in the states x and u well.
- (+) It is easy to adjust Q_1, Q_2 such that we obtain nice transients in the time domain.
- (+) Some possibilities of handling robustness.
- (-) Hard to see how Q_1, Q_2, R_1, R_2 affect S, T, G_{wu}, etc.

Problem formulation

For the system

$$y = G(p)u + w,$$

find the feedback control law

$$u = -F_y(p)y,$$

such that the closed-loop system has good properties in terms of the sensitivity functions S and T, and the transfer function from disturbance to input G_{wu}.

Direct synthesis of S, T and G_{wu}

- Classical methods (lead-lag) – scalar systems. Intuitive design using Bode diagrams.
- Finding a compromize using optimization: $\mathcal{H}_2$–design
- Fulfilling bounds on S, T and G_{wu}: $\mathcal{H}_\infty$–design

Today: $\mathcal{H}_2$–design and $\mathcal{H}_\infty$–design.

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Today: $\mathcal{H}_2$–design and $\mathcal{H}_\infty$–design.

Recall – the $\mathcal{H}_2$-norm

Definition: The $\mathcal{H}_2$-norm of the system $y = G(p)u$ is given by

$$\|G\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(i\omega)|_2^2 \, d\omega$$

where $|G(i\omega)|_2^2 = \text{tr}(G^*(i\omega)G(i\omega))$.

For a square matrix $A = [a_{ij}]$ the trace is

$$\text{tr}(A) = \sum_{k=1}^{n} a_{kk} = \sum_{k=1}^{n} \lambda_k(A).$$

Interpretations due to Parseval's theorem:

- Deterministic: $\|G\|_2 = \|g(t)\|_2$, i.e. the 2-norm of the impulse response $g(t)$.
- Stochastic: If u is a stochastic process with $\Phi_u(\omega) = I$, then $\|G\|_2^2 = \frac{1}{2\pi} \int |\Phi_y(\omega)|_2^2 \, d\omega = \text{tr}(R_y) = \|y\|_2^2$.

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$$\|G\|_\infty = \sup_u \frac{\|y\|_2}{\|u\|_2} = \sup_\omega \bar{\sigma}(G(i\omega))$$

For scalar systems $\|G\|_\infty = \sup_\omega |G(i\omega)|$.

The input u that maximizes the size of the outputs is a sinusoid with the frequency ω for which $|G(i\omega)|$ attains its maximum.

Interpretation: Highest peak/ "worst case".

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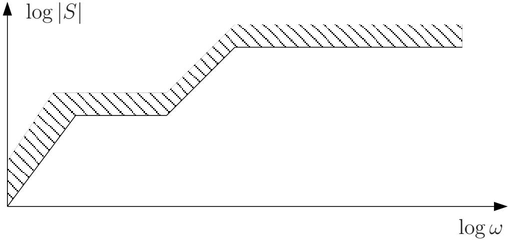
Example of requirements: S

We typically want the sensitivity S to be small for low frequencies where we are likely to have process disturbances (e.g. not to be sensitive to the slope of a hill).

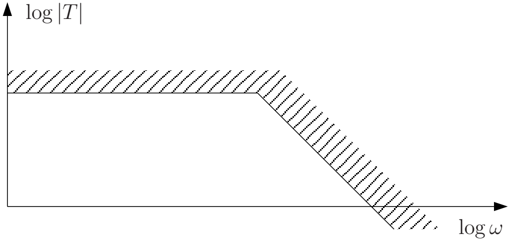


The desired shape is encoded using a weighting function $W_S(i\omega)$

$$|S(i\omega)| \leq |W_S^{-1}(i\omega)|, \ \forall \omega \quad \Rightarrow \quad ||W_S S||_\infty \leq 1.$$

Example of requirements: T

We typically want the complementary sensitivity T to be small for high frequencies where we are likely to have measurement noise.



The desired shape is encoded using a weighting function $W_T(i\omega)$

$$|T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|}\ \forall\omega \quad \Rightarrow \quad ||W_T T||_\infty \leq 1;$$

$$G_0 = (I + \Delta_G)G.$$

Example of requirements: G_{wu}

- Recall that $|G_{wu}|$ (or $|G_{ru}|$) is the gain from a disturbance (or reference) to the control signal.
- This gain is important in order to make sure that the control signals does not become too big.
- Analogously to what we did before, we obtain

$$|G_{wu}(i\omega)| \leq |W_u^{-1}| \; \forall \omega \Rightarrow ||W_u G_{wu}||_\infty \leq 1$$

- It is important to note that the requirements on S, T and G_{wu} can "collide", implying that a compromize must be made.

Design spec. in the frequency domain

Weightings of S, T and G_{wu}:

$$\begin{aligned} &W_S(i\omega)S(i\omega) \\ &W_T(i\omega)T(i\omega) \\ &W_u(i\omega)G_{wu}(i\omega) \end{aligned}$$

Design criterion $\mathcal{H}_2$-design: Choose the controller such that

$$V = \|W_S S\|_2^2 + \|W_T T\|_2^2 + \|W_u G_{wu}\|_2^2$$

is minimized.

Design criterion $\mathcal{H}_\infty$-design: Choose the controller such that

$$\|W_S S\|_\infty \leq 1, \qquad \|W_T T\|_\infty \leq 1, \qquad \|W_u G_{wu}\|_\infty \leq 1.$$

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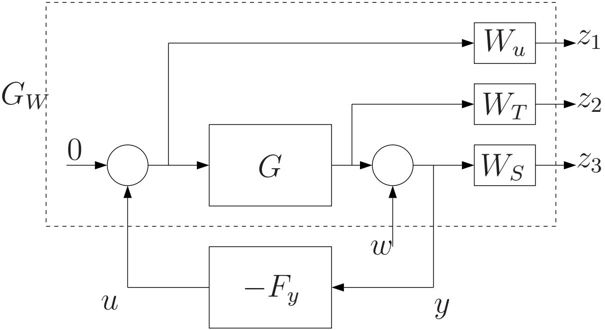
Useful formulae

$$
\begin{aligned}
S &= (I + GF_y)^{-1}, \\
T &= I - S = (I + GF_y)^{-1} GF_y = G(I + F_y G)^{-1} F_y, \\
G_{wu} &= -(I + F_y G)^{-1} F_y = -F_y (I + GF_y)^{-1} = -F_y S.
\end{aligned}
$$

An extended "imaginary" system G_W (I/II)



When we close the system using $u = -F_y y$ we have:

$$z_1 = W_u G_{wu} w, \quad z_2 = -W_T T w, \quad z_3 = W_S S w$$

An extended "imaginary" system G_W (II/II)

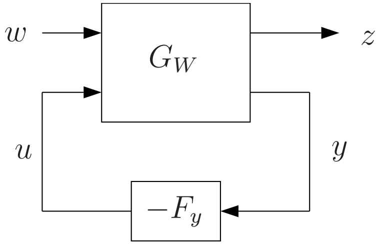
Introduce the performance variables

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} W_u G_{wu} \\ -W_T T \\ W_S S \end{bmatrix} w \triangleq G_{ec} w$$

as (artificial) outputs of an extended closed loop system G_{ec} with w as input.

This way z_1, z_2 and z_3 represent the requirements for the closed-loop system.

Compact representation



State space realization of G_W (i.e. state-space model of the open-loop system $(u, w) \to (z, y)$):

$$\begin{aligned} \dot{x} &= Ax + Bu + Nw \\ z &= Mx + Du \\ y &= Cx + w \end{aligned}$$

A technical assumption: Assume $D^T \begin{bmatrix} M & D \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$; always possible via change of variables when $\det(D^T D) \neq 0$

Closed loop system: $z = G_{ec} w$ where $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

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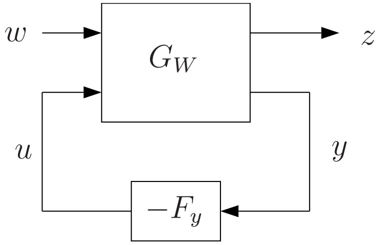

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Closed loop system: $z = G_{ec} w \qquad$ where $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

Structural property

The system

$$
\begin{aligned}
\dot{x} &= Ax + Bu + Nw \\
z &= Mx + Du \\
y &= Cx + w
\end{aligned}
$$

is on innovation form since $v_1 = v_2 = w$, which implies that "the system is its own observer" and the Kalman filter ($K = N$) is simply

$$\dot{\hat{x}} = A\hat{x} + Bu + N(y - C\hat{x}),$$

if $A - NC$ has all its eigenvalues strictly in the LHP.

VertiGo: ETH & Disney Research

VertiGo Combines Car and Helicopter

- Maximize the ratio between thrust output and vehicle weight.
- Ground to wall transition: the rear propellor thrusts against the wall while the front propellor thrusts upward
- This particular control problem was "somewhat of a step in the dark"

https://www.youtube.com/watch?v=tmm6etLlleA

Recall the $\mathcal{H}_2$ norm and the design criterion

Definition: The $\mathcal{H}_2$-norm of the system $y = G(p)u$ is given by

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|_2^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{tr}\left(G^*(i\omega)G(i\omega)\right) d\omega.$$

Design criterion $\mathcal{H}_2$-design: Choose the controller such that

$$V = \|W_S S\|_2^2 + \|W_T T\|_2^2 + \|W_u G_{wu}\|_2^2$$

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Optimal $\mathcal{H}_2$ control – the problem

Minimizing the criterion

$$V(F_y) = \|G_{ec}\|_2^2 = \|W_S S\|_2^2 + \|W_T T\|_2^2 + \|W_u G_{wu}\|_2^2$$

is equivalent to minimizing

$$\|z\|_2^2 = \|Mx\|_2^2 + \|u\|_2^2.$$

This is the **LQG problem!** (with $z' = Mx$, $Q_1 = I$ and $Q_2 = I$)

Optimal $\mathcal{H}_2$ control – the solution

Solution provided by Theorem 9.1 (page 270). The optimal $\mathcal{H}_2$ controller is given by (recall that the system is given on innovations form $\Rightarrow K = N$ in the Kalman filter)

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + N(y - C\hat{x}),\\ u &= -L\hat{x},\end{aligned}$$

with

$$\begin{aligned}L &= B^T S,\\ 0 &= A^T S + SA + M^T M - SBB^T S.\end{aligned}$$

Hence,

$$F_y(s) = L(sI - A + BB^T S + NC)^{-1} N$$

(If $A - NC$ not stable, compute and use the Kalman filter.)

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Recall the $\mathcal{H}_\infty$ norm and the design objective

Definition: The $\mathcal{H}_\infty$-norm of the system $y = G(p)u$ is given by

$$\|G\|_\infty = \sup_u \frac{\|y\|_2}{\|u\|_2} = \sup_\omega \bar{\sigma}(G(i\omega))$$

Design objective $\mathcal{H}_\infty$-design: Find the controller that minimize

$$\|G_{ec}\|_\infty = \max_\omega \bar{\sigma}(G_{ec}(i\omega)).$$

This is a hard (non-convex) problem, instead we search for controllers that satisfy

$$\|G_{ec}\|_\infty < \gamma$$

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Optimal $\mathcal{H}_\infty$ control – the solution

Assume

$$A^T S + SA + M^T M + S(\gamma^{-2} N N^T - BB^T)S = 0$$

has a positive semidefinite solution $S = S_\gamma$, and that $A - BB^T S_\gamma$ is stable.

Consider the controller

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + N(y - C\hat{x}), \\ u &= -L_\infty \hat{x}, \end{aligned}$$

with $L_\infty = B^T S_\gamma$. Then

$$F_y(s) = L_\infty (sI - A + BB^T S_\gamma + NC)^{-1} N$$

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$\mathcal{H}_\infty$ control – design steps

1. G is given.
2. Choose weights W_u, W_S, W_T.
3. Form the extended system.
4. Choose a constant γ.
5. Solve the Riccati equation and compute L_∞.
   5.1 If no solution exists, increase γ and go to step 4.
   5.2 If a solution exists, accept it, or decrease γ and go to step 4.
6. Check the properties of the closed loop system. If not acceptable, go to step 2.

$\mathcal{H}_2$, $\mathcal{H}_\infty$ synthesis – pros and cons

- (+) Directly handles the specifications on S, T and G_{wu}
- (+) Let us know when certain specifications are impossible to achieve (via γ).
- (+) Easy to handle several different specifications (in the frequency domain)
- (-) Can be hard to control the behaviour in the time domain in detail.
- (-) Often results in complex controllers (number of states in the controller = number of states in G, W_u, W_S, W_T).

Linear multivariable controller synthesis

Summary:

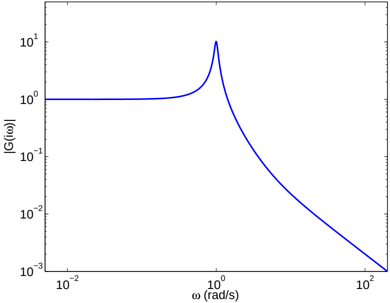
1. Perform an RGA analysis
2. Employ simple PID controllers if the RGA indicates that it is possible.
3. Otherwise make use of LQ, MPC or $\mathcal{H}_2/\mathcal{H}_\infty$-synthesis.

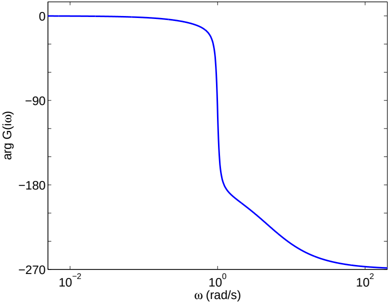
Control joke

You might be a control engineer if:
Your license plate reads: hNFiNTY, LMS ALG, ADAPTiv, LQGAUSS, OPTiMAL, or PiDRULZ.

Design example (I/V)

Control the system $G(s) = \frac{-0.2s+1}{s^2+0.1s+1}$. Bode plots





Difficulties: Highly resonant system, non-minimum phase zero in +5.

Design example (II/V)

Specifications:

1. $|G_{wu}(i\omega)| < 4, \forall\omega \;\; \Leftrightarrow \;\; \|G_{wu}\|_\infty < 4.$
2. $|T(i\omega)| < 1.25, \forall\omega \;\; \Leftrightarrow \;\; \|T\|_\infty < 1.25.$
3. Integral action $\Leftrightarrow S(0) = 0.$
4. Bandwidth $\omega_G \approx 2$ rad/s

Rule of thumb: $\omega_B < z/2$ for RHP zero $z \Rightarrow \omega_B < 2.5$ rad/s here!

Try with the following weighting functions

$$W_u = \frac{1}{4} = 0.25, \quad W_T = \frac{1}{1.25} = 0.8, \quad W_S(s) = (0.5s)^{-1} = \frac{2}{s}$$

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Design example (III/V)

Four control strategies were employed:

1. **PID control/lead-lag compensation:**

$$F(s) = K \frac{\tau_D s + 1}{\beta \tau_d s + 1} \frac{\tau_I s + 1}{\tau_I s}$$

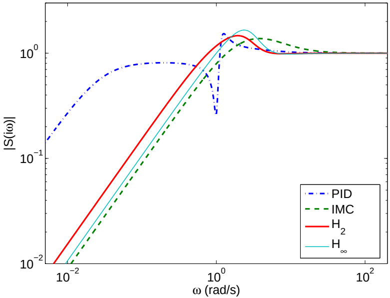
2. **IMC**, with

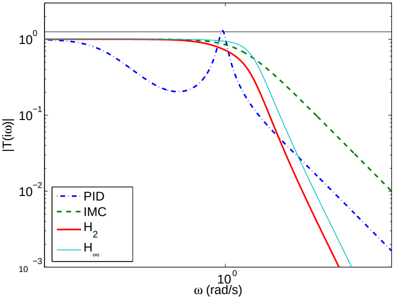
$$Q(s) = \frac{1}{0.5s + 1} \left(\frac{0.2s + 1}{s^2 + 0.1s + 1} \right)^{-1} \Rightarrow$$
$$T(s) = \frac{-0.2s + 1}{(0.2s + 1)(0.5s + 1)}$$

3. $\mathcal{H}_2$ **control**
4. $\mathcal{H}_\infty$ **control** with $\gamma = 2.9$.

Design example (IV/V)

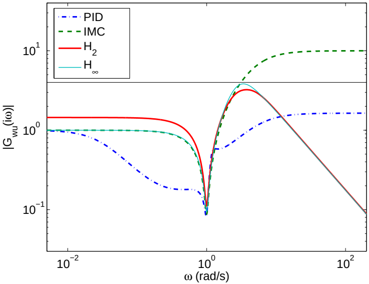
Results: the sensitivity functions





Design example (V/V)

Results: G_{wu} and comparisons with the specifications.



	$\|T\|_\infty$	$\|G_{wu}\|_\infty$	ω_B
PID	1.29	1.64	0.030
IMC	1.00	10.0	2.00
$\mathcal{H}_2$	1.00	3.23	1.28
$\mathcal{H}_\infty$	1.00	3.83	2.45

Outline – entire course

1. **Lecture 1:** Introduction and linear multivariable systems
2. **Lecture 2–5:** Multivariable linear control theory
   a) systems theory, closed loop system
   b) Basic limitations
   c) Controller structures and control design
   d) $\mathcal{H}_2$ and $\mathcal{H}_\infty$ loop shaping
1. **Lecture 6–8:** Nonlinear control theory
   a) Linearization and phase portraits
   b) Lyapunov theory and the circle criterion
   c) Describing functions
2. **Lecture 9:** Optimal control
3. **Lecture 10:** Summary and repetition

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A few concepts to summarize lecture 5

Extended system: Allows us to systematically incorporate design specifications in the frequency domain by extending the system G. This is done by introducing three performance variables z_1, z_2 and z_3 which are viewed as artificial outputs of the system.

$\mathcal{H}_\infty$-norm: The $\mathcal{H}_\infty$-norm of the system $y = G(p)u$ is given by $\|G\|_\infty = \sup_u \frac{\|y\|_2}{\|u\|_2} = \sup_\omega \bar{\sigma}(G(i\omega))$. Interpretation: highest peak/ “worst case”.

$\mathcal{H}_2$-norm: The $\mathcal{H}_2$-norm of the system $y = G(p)u$ is given by $\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|_2^2 d\omega$. Interpretation: total “energy”.