Automatic Control III, Homework 2

Jonathan Sundell, Philip Ahl, Felix Elmgren, Carl Tysk
Uppsala University
Ångströmslaboratoriet, Uppsala

October 16, 2019

Problem 1, H_∞ control and basic limitations

a)

Expanding the state space model to incorporate the frequency weightings

$$z_1 = K_u u \tag{1}$$

$$z_2 = K_T G u \frac{s+\beta_T}{s+\alpha_T} \tag{2}$$

$$z_3 = \frac{K_s}{s+\alpha_s}(w + Gu) \tag{3}$$

$$z_2 = K_T C x\left(\frac{s+\alpha_T}{s+\alpha_T} + \frac{s+\beta_T - s + \alpha_T}{s+\alpha_T}\right) \tag{4}$$

$$= K_T C x\left(1 + \frac{\beta_T + \alpha_T}{s+\alpha_T}\right) \tag{5}$$

$$(z_2 - K_T C x)s = -\alpha_T(z_2 - K_T C x) + K_T C x(\beta_T + \alpha_T) \tag{6}$$

Here we set $x_1' = z_2 - K_T C x$ as one of the extended states, hence we get

$$\dot{x}_1' = -\alpha_T x_1' + K_T C x(\beta_T + \alpha_T) \tag{7}$$

$$z_3 = \frac{K_s}{s+\alpha_S}(w + Cx) \Leftrightarrow s z_3 = -\alpha_S z_3 + K_s(w + Cx) \tag{8}$$

Here we set $x_2' = z_3$ as an extended state and get

$$\dot{x}_2' = -\alpha_S x_2' + K_s(w + Cx) \tag{9}$$

When we extend the model on state-space form we now get

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ K_T C(\beta_T + \alpha_T) & K_T C(\beta_T + \alpha_T) & -\alpha_T & 0 \\ K_s C & K_s C & 0 & -\alpha_S \end{bmatrix} x + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} X \\ 0 \\ K_s \end{bmatrix} w \tag{10}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ K_T C & K_T C & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} K_u \\ 0 \\ 0 \end{bmatrix} u \tag{11}$$

$$y = C'x + w \tag{12}$$

The state vector is defined as $x = \begin{bmatrix} X \\ x_1' \\ x_2' \end{bmatrix}$, where X are the states of the unextended system, A_{11}, A_{12}, A_{12} and A_{22} are the upper left, upper right, lower right and lower left quadrant of the unextended A matrix. X are a row vector of zeros of length of the original state vector. C' is now with dimension matching the length of the new state vector. z is the new introduced variables.

b)

The frequency weightings are set on the following form:

$$W_S(s) = \frac{K_S}{s + \alpha_S}, \quad W_T(s) = K_T \frac{s + \beta_T}{s + \alpha_T}, \quad W_u(s) = K_u. \tag{13}$$

One specification for the closed loop system is that for the input, $|u| < 4$ should hold, meaning that the transfer function from the noise w to the input u should be limited:

$$|G_{wu}| < 4 \tag{14}$$

The design criteria for the H_∞ controller for G_{wu} is:

$$||W_u G_{wu}||_\infty < 1 \tag{15}$$

$$\rightarrow ||K_u G_{wu}||_\infty < 1 \tag{16}$$

$$\rightarrow |G_{wu}| < \frac{1}{K_u} = 4 \tag{17}$$

This will lead to the following design criteria for the weighting matrix W_u:

$$\rightarrow K_u = \frac{1}{4} \tag{18}$$

Another specification for the closed loop system is that the controller $F_y(s)$ should have integral action. $F_y(s)$ has an integral action if $S(0) = 0$. We know that

$$||S(s)||_\infty \leq ||W_s(s)^{-1}||_\infty \tag{19}$$

and $||W_s(0)^{-1}||_\infty = \frac{\alpha_s}{K_s}$ so in order to get the integral action we choose $\alpha_s = 0$.

The bandwidth for small frequencies is given by the condition that $S(iw) \leq 1$. With the relation given by Equation 19 and the specification states that the frequency should be 2rad/s we understand that $|\frac{K_s}{iw}| \geq 1$. We rewrite the imaginary expression as $|\frac{-K_s iw}{w^2}| = \frac{K_s}{w} \geq 1$ hence to fulfill the specification that the bandwidth should be 2rad/s we choose $K_s = 2$.

The relation between T and W_T is that

$$\parallel W_T T \parallel_\infty < 1 \tag{20}$$

T should never be amplified by more than 50%, the largest T is obtained for low frequencies. Therefore we choose $\omega = 0$ and get the following equation

$$\frac{\alpha_T}{K_T \beta_T} < \frac{3}{2}. \tag{21}$$

The second condition on T, which attenuates high frequencies, is that it should be less than 0.01 for $\omega > 314 rad/s$. Thus,

$$\parallel W_T^{-1} \parallel^2 > (1/100)^2 \tag{22}$$

Therefore, the parameters should be chosen so that

$$\frac{(-\omega^2 - \alpha_T \beta_T)^2 - \omega^2(\alpha_T - \beta_T)^2}{K_T^2(-\omega^2 - \beta_T^2)^2} < 10^{-4}. \tag{23}$$

Thus, there are two equations for three unknowns. Consequently, one parameter was put to a certain value and the others were tested back and forth until both conditions were filled. K_T was put to 100 since when analyzing equation Equation 23 it is clear that the dominating terms are the w^4 which gives that the ratio tends towards $\frac{1}{K_T^2}$. The ratio should, according to the specifications, be lower than 10^{-4} for high frequencies ($100\pi rad/s$). α_T and β_T were chosen so that Equation 21 was fullfilled. As a result, β_T was put to 0.01 and α_T was put to 1.

Problem 2, Equilibria and stability

a)

$$\dot{x}(t) = r * x(t), \quad r > 0 \tag{24}$$

The order is 1 since the derivate is of first order and the system is linear since

$$f(x,t) = rx(t)\alpha f(x,t) = r(\alpha * x) = f(\alpha x, t) \tag{25}$$

where α is a constant. Thus, the equality holds under scaling. Furthermore, the system should hold the property of superposition

$$f(x1 + x2, t) = r(x1 + x2) = rx1 + rx2 = f(x1, t) + f(x2, t) \tag{26}$$

b)

$$f(x,t) = 0 \rightarrow x(t) = 0 \tag{27}$$

since r is a non zero constant. To determine the stability we use Taylorexpansion around the stationary point and get r, since r>0 we have instability.

c)

The differential equation is

$$\dot{x}(t) = r * x(t) \rightarrow \tag{28}$$

which gives that the analytical solution is

$$x(t) = Ce^{rt}x(0) = x_0 \rightarrow c = x_0 \tag{29}$$

$$x = x_0 e^{rt} \tag{30}$$

Thus, the limit when t tends to infinity is infinity.

d)

To analyze whether the system is linear, the property of superposition will be checked.

$$f(x1 + x2, t) = r(x1 + x2) - r * \frac{(x1 + x2)^2}{K} = \tag{31}$$

$$r(x1+x2) - r\frac{x_1^2+x_2^2}{K} - 2*r*\frac{(x1*x2)}{K} \tag{32}$$

$$f(x1,t)+f(x2,t) = r(x1+x2) - r*\frac{(x_1^2+x_2^2)}{K} \tag{33}$$

Thus, the system is *not* linear. The highest order of derivative is 1 hence the order of the system is 1.

e)

$$\dot{x}(t) = r*x(t)(1-\frac{x(t)}{K}) \tag{34}$$

$$f(x,t) = r*x(t)(1-\frac{x(t)}{K}) \tag{35}$$

$$f(x,t) = 0 \rightarrow \tag{36}$$

the equilibria are in $x_i = 0$, $x_{ii} = K$. To check the stability the function will be Taylor expanded around the equilibria.

$$f(x+h) = f'(x)*h + \mathcal{O}(h^2) \rightarrow \tag{37}$$

$$f'(x) = r - \frac{2r}{K}x \tag{38}$$

$$f'(0) = r, \quad > 0 \tag{39}$$

$$f'(K) = -r, \quad < 0 \tag{40}$$

which gives that $x_i = 0$ is unstable and $x_{ii} = K$ is stable.

f)

$$x(t) = \frac{Kx_0*e^{rt}}{K+x_0*(e^{rt}-1)} \tag{41}$$

Put Equation 42 into the differential equation gives

$$\dot{x}(t) = r*x(t) - r\frac{x^2(t)}{K} = \tag{42}$$

$$r\frac{Kx_0 * e^{rt}}{K + x_0 * (e^{rt} - 1)} - r\frac{K^2x_0^2 * e^{2rt}}{K(K + x_0(e^{rt} - 1)^2} \tag{43}$$

Finding the least common deminator gives

$$r\frac{K^2x_0 * e^{rt}(K + x_0(e^{rt} - 1) - rK^2x_0^2e^{2rt}}{K(K + x_0(e^{rt} - 1))^2} = \tag{44}$$

$$rK(\frac{x_0 * e^{rt}(K + x_0(e^{rt} - 1) - Kx_0^2e^{2rt}))}{(K + x_0(e^{rt} - 1))^2} \tag{45}$$

Taking the derivate, using the quotient rule, gives

$$\dot{x} = \frac{Krx_0e^{rt}(K + x_0(e^rt - 1)) - Kx_0e^{rt}rx_0e^{rt}}{(K + x_0(e^rt - 1))^2} = \tag{46}$$

$$rK\frac{(x_0 * e^{rt}(K + x_0(e^{rt} - 1) - Kx_0^2e^{2rt}}{(K + x_0(e^rt - 1))^2} \tag{47}$$

Comparing Equation 45 and Equation 47 they are identical. Thus the handed solution solves the differential equation.

$$x(0) = \frac{KC * e^0}{K + C * (e^0 - 1)} = \tag{48}$$

$$x(0) = KC/K = C = x_0 \tag{49}$$

which proofs the initial condition. C is the constant determined by the initial condition. When t tends to infinity the terms that dominate are e^{rt}. Therefore the ratio goes toward the value of K, which we stated in e) was an equilibria. When K goes to ∞ we see that $x(t) = x_0e^{rt}$ which is the same as our answer in c) Equation 30.

g)

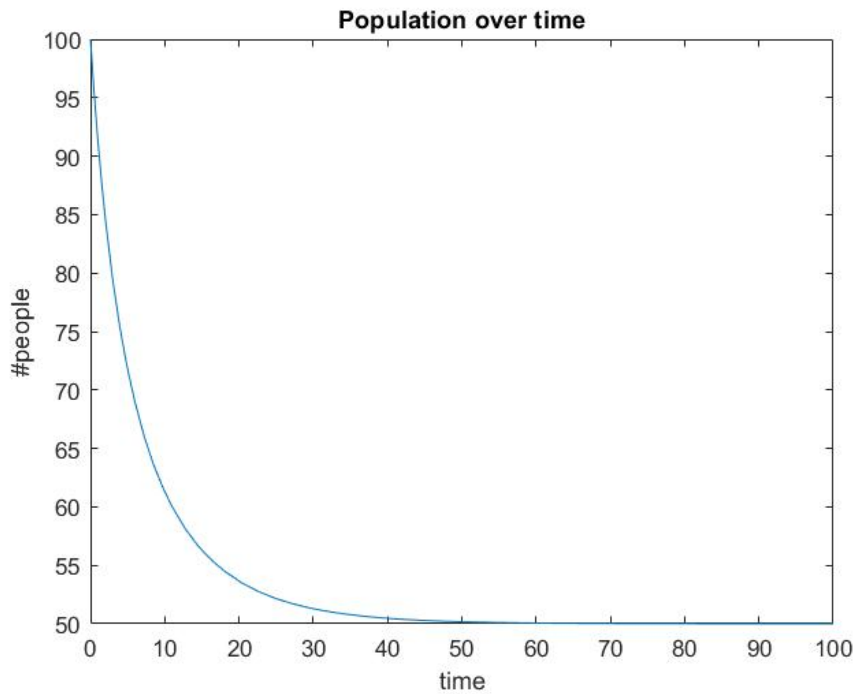

Figure 1: *Simulated population over time. Here K is smaller than initial guess* x_0.

Figure 2 visualizes that the solution converges exponentially from the starting guess x_0 towards the value of K. Here $x_0 = 100$ and $K = 50$. r determines how rapidly the solution converges towards K, here $r = 1$.

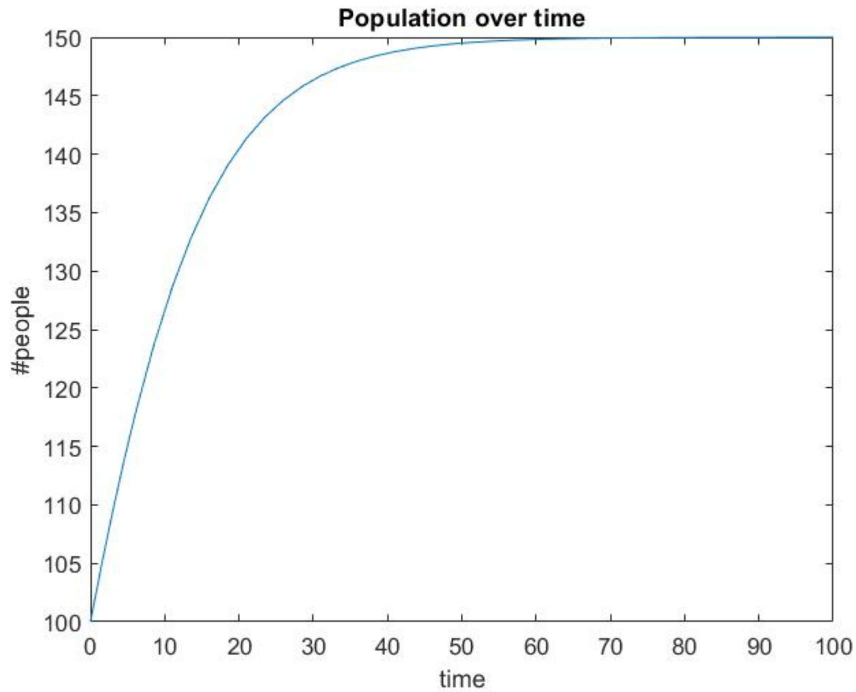

Figure 2: *Simulated population over time. Here K is larger than initial guess x_0.*

Figure 2 visualizes that the solution converges exponentially from the starting guess x_0 towards the value of K. Here $x_0 = 100$ and $K = 150$.

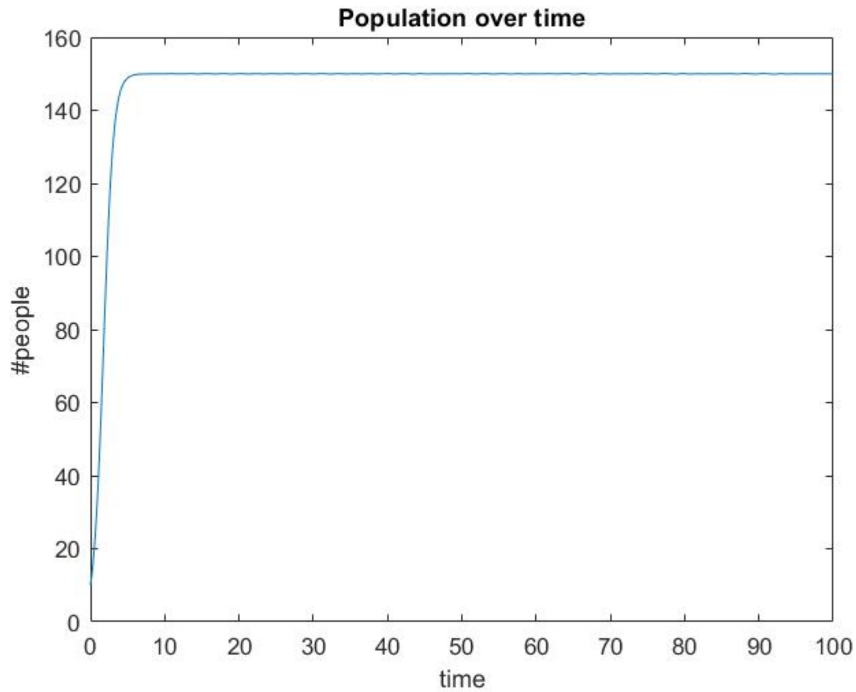

Figure 3: *Simulated population over time. Here K is larger than initial guess* x_0.

Figure 3 illustrates the result when r is increased. The function converges much more rapidly towards the value of K.

Problem 3, Lyapunov stability

a)

The function $f(e)$ is enclosed by two linear functions defined by $k1$ and $k2$. By choosing k1 very small, close to zero, the circle spanned by k1 and k2 covers most of the left half plane. By testing different values of k2, which will determine how close the circle comes to the imaginary axis and the value of K a stable system was constructed.

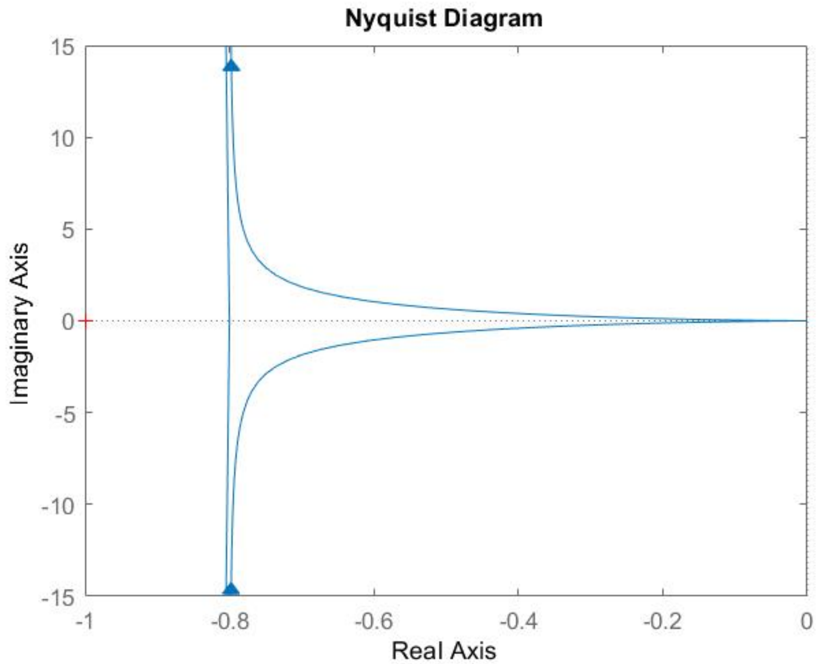

Figure 4: *Nyquist plot of G(s) and the circle spanned by k1 and k2.*

Figure 4 illustrates the Nyquist plot of $G(s)$ and the circle spanned by k1, 10^-6, and k2, 1.25. The circle covers almost the entire left plane. According to the Nyquist criterion for stability the system is stable. Different values of K were tested and the condition was found to be

$$K > 1/k_2 \tag{50}$$

which in the case $k = 1.25$ becomes

$$K < 0.8. \tag{51}$$

However $k2$ could be chosen only with the condition that it is larger than 1. Thus the condition becomes

$$0 < K < 1. \tag{52}$$

b)

When writing the system on state space form, the states are first chosen as $x_1 = y$ and $x_2 = \dot{y}$. The state space form becomes:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ K \end{bmatrix} u(t) \tag{53}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x \tag{54}$$

c)

The Lyapunov function is given on the form:

$$V(x) = \frac{1}{2}{x_2}^2 + Kg(x_1) \tag{55}$$

To chose $g(x_1)$ theorem 12.4 in the book is used:

$$V(x_0) = 0 \tag{56}$$

$$V(x) > 0 \quad \forall \quad x \neq x_0 \tag{57}$$

$$\dot{V}(x) = V_x(x)\dot{x} < 0 \tag{58}$$

In eq.39 x_0 is the equilibrium and is found by putting the states $\dot{x}$ to zero:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ K \end{bmatrix} u(t) = 0 \tag{59}$$

$$\rightarrow x_2 = 0 \tag{60}$$

$$\rightarrow Ku(t) = -(x_1) = 0 \rightarrow x_1 = 0 \tag{61}$$

$$x_0 = [0 \quad 0]^T \tag{62}$$

Putting result eq.45 in eq.38 gives us a condition for $g(x_1)$:

$$V(x) = \frac{1}{2}0^2 + Kg(0) \tag{63}$$

$$\rightarrow g(0) = 0 \tag{64}$$

By analyzing theorem 12.4 further we look at eq.40 to get another condition for $g(x_1)$:

$$V(x) = \frac{1}{2}{x_2}^2 + Kg(x_1) > 0 \tag{65}$$

and since:

$$\frac{1}{2}{x_2}^2 > 0 \quad \forall \quad x \neq x_0 \tag{66}$$

we get that:

$$g(x_1) > 0 \quad \forall \quad x \neq x_0 \tag{67}$$

When analyzing the last criteria in theorem 12.4 (eq.41) the following is found:

$$V_x(x) = [Kg_x(x_1) \quad x_2] \tag{68}$$

$$\dot{V}(x) = [Kg_x(x_1) \quad x_2][x_2 \quad -x_2 - Kfx_1]^T = Kx_2 g_x(x_1) - {x_2}^2 - Kf(x_1)x_2 < 0 \tag{69}$$

$$\rightarrow Kx_2(g_x(x_1) - f(x_1)) - {x_2}^2 < 0 \tag{70}$$

To be able to decide $g(x_1)$ we need to look at 3 different cases because of the non linearity in $f(x_1)$.
Case 1, $x_1 > 1$:

$$\rightarrow Kx_2(g_x(x_1) - 1) - {x_2}^2 < 0 \tag{71}$$

$$\rightarrow g_x(x_1) = 1 \tag{72}$$

$$\rightarrow \int g_x(x_1)dx_1 = \int 1dx_1 \tag{73}$$

$$\rightarrow g(x_1) = x_1 + a \tag{74}$$

Case 2, $x_1 < -1$:

$$\rightarrow Kx_2(g_x(x_1) + 1) - {x_2}^2 < 0 \tag{75}$$

$$g_x(x_1) = -1 \tag{76}$$

$$\rightarrow \int g_x(x_1)dx_1 = \int -1dx_1 \tag{77}$$

$$\rightarrow g(x_1) = -x_1 + b \tag{78}$$

Case 3, $-1 < x_1 < 1$:

$$\rightarrow Kx_2(g_x(x_1) - x_1) - {x_2}^2 < 0 \tag{79}$$

$$g_x(x_1) = x_1 \tag{80}$$

$$\rightarrow \int g_x(x_1)dx_1 = \int x_1 dx_1 \tag{81}$$

$$\rightarrow g(x_1) = \frac{1}{2}{x_1}^2 + c \tag{82}$$

To decide the constants a, b, c we use the fact that $g(x_1)$ should be continuous and the result in eq.47 stating $g(0) = 0$. This leads to $c = 0$ and $a = b = -\frac{1}{2}$:

$$g(x_1) = \begin{cases} x_1 + a, & x_1 > 1 \\ \frac{1}{2}{x_1}^2 + c, & -1 < x_1 < 1 \\ -x_1 + b, & x_1 < -1 \end{cases} \tag{83}$$

$\dot{V}(x) = 0$ means that there is no movement in any direction for $V(x)$. When looking at equation $\dot{V}(x) = 0$ we see that $x_2 = 0$ solves the equation regardless of x_1, but what we need to consider is that when $x_1 \neq 0$ we get $\dot{x}_2 = C$ where C is a constant $\neq 0$. This means we actually have movement in the x_2 direction. Hence $\dot{V}(x) = 0$ only holds for $x_1 = x_2 = 0$, which is the equilibrium state.

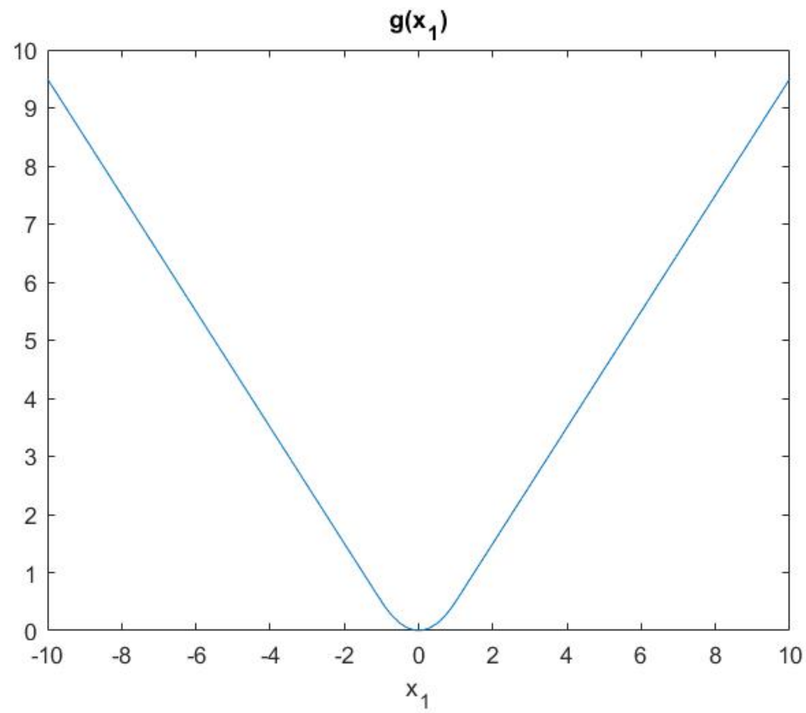

Figure 5: *Plot of $g(x_1)$*.

In Figure 5 the final design of $g(x_1)$ can be seen. Since our Lyapunov function $V(x)$ now fulfill theorem 12.4 regarding the stability, a conclusion can be drawn that our closed loop system is stable.

Matlab Code

```
%%Homework2
clear all
close all
if true
alpha=1;
beta=2/3;
%% Exercise 1

for i=1:length(alpha)

r(i)=(314^2+alpha(i)^2)/((314^2+alpha(i)*beta(i))^2+(alpha(i)-beta(i))^2*314^2);
```

```

if(r(i)<10^-4)
correct(i)=r(i)
alpha_correct(i)=alpha(i);
beta_correct(i)=beta(i);
end

r(i)=1;
end



end


%% Exercise 2
if false
clear all
close all
t_period=[0 100];
% x=linspace(0,100);
x0_1=10;
x0_2=0;
[t,x]=ode45('population',t_period,x0_1);
plot(t,x)
xlabel('time')
ylabel('#people')
title('Population over time')

end



%% Exercise 3
if false
k2=1.25;
k1=10^-6;
r=(1/k1-1/k2)/2;
w=linspace(0,2*pi,10000);
unit_circle=(cos(w)+1i*sin(w))*r-(1/k2+r);
K=0.8;
G=tf([K],[1 1 0]);
G_inv=1/G;
nyquist(G)
hold on
plot(unit_circle)
```

```
end
if false
    g=linspace(-10,10,100);

    for i=1:length(g)
        if(g(i)<-1)
            G(i)=-g(i)-0.5;

        elseif(g(i)>1)
            G(i)=g(i)-0.5;

        else
            G(i)=0.5*g(i)^2;
        end
    end

plot(g,G)
title('g(x_1)')
xlabel('x_1')


end
```