Automatic Control III

Lecture 6 – Linearization and phase portraits

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Summary of lecture 5 (I/III)

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis:

- Make $W_u G_{wu}, W_S S, W_T T$ small.
- $\mathcal{H}_2$: Minimize $\int \left(|W_u G_{wu}|_2^2 + |W_S S|_2^2 + |W_T T|_2^2 \right) \mathrm{d}\omega$.
- $\mathcal{H}_\infty$: Set an upper bound for $|W_u G_{wu}|, |W_S S|, |W_T T| \; \forall \; \omega$.
- Results in algebraic Riccati equations.

Summary of lecture 5 (II/III)

$\mathcal{H}_2$, $\mathcal{H}_\infty$ synthesis – pros and cons:

(+) Directly handles the specifications on S, T and G_{wu}

(+) Let us know when certain specifications are impossible to achieve (via γ).

(+) Easy to handle several different specifications (in the frequency domain)

(-) Can be hard to control the behaviour in the time domain in detail.

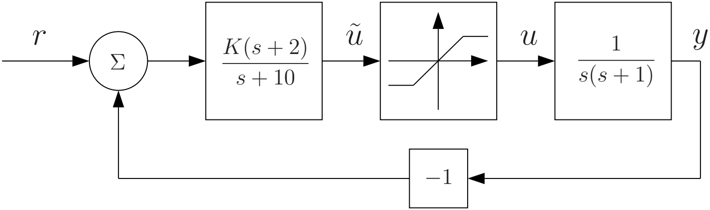
(-) Often results in complex controllers (number of states in the controller = number of states in G, W_u, W_S, W_T).

Summary of lecture 5 (III/III)

Linear multivariable controller synthesis summary:

1. Perform an RGA analysis
2. Use simple SISO controllers of PID type if the RGA analysis indicates that it might be possible.
3. Otherwise make use of LQ, MPC or $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis.

DC motor – saturated control signal (I/V)



- DC motor controlled using a lead controller.
- We want to control the motor angle.
- The saturation

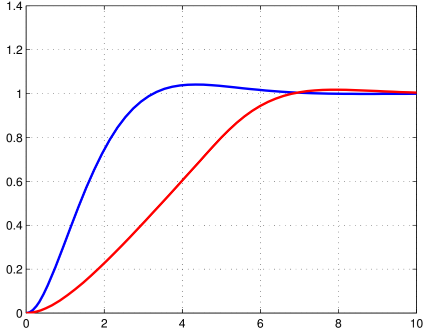
$$u = \mathsf{sat}(\widetilde{u}) = \begin{cases} \widetilde{u} & |\widetilde{u}| \leq 1, \\ 1 & \widetilde{u} > 1, \\ -1 & \widetilde{u} < -1, \end{cases}$$

renders the system nonlinear.

DC motor – saturated control signal (II/V)

Step responses for two different amplitudes of the reference signal r.

Blue: Amplitude 1
Red: Amplitude 5 (scaled with $1/5$)



Conclusion: The step response is **amplitude dependent**. If the system would have been linear the two step responses would have coincided.

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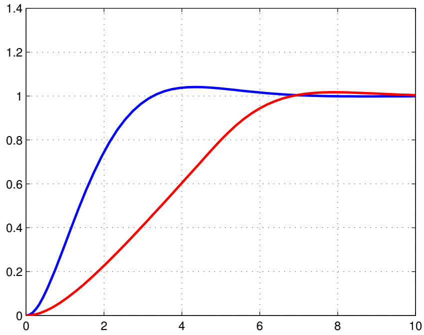
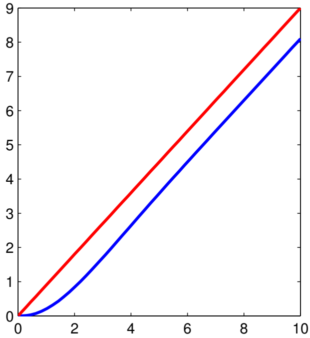
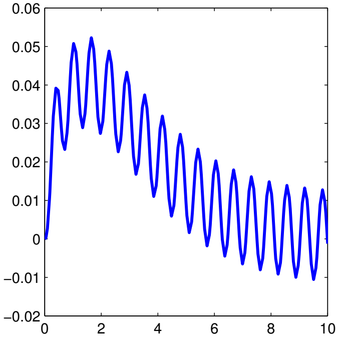


DC motor – saturated control signal (III/V)

Red: Reference signal r. Blue: Output signal y.

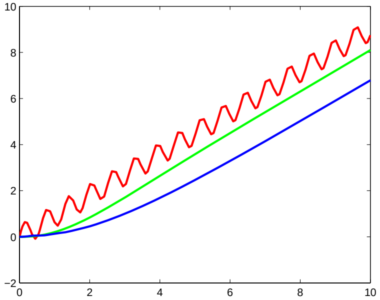




Both the ramp (left) and the sine responses (right) are roughly the same as for a linear system.

DC motor – saturated control signal (IV/V)

Red: r. Blue: y. Green: y when r is a ramp (same as on the previous slide).



Something happens here: There is no sine present in the response and the ramp error has increased...

This violates the superposition principle and the frequency fidelity!

DC motor – saturated control signal (IV/V)

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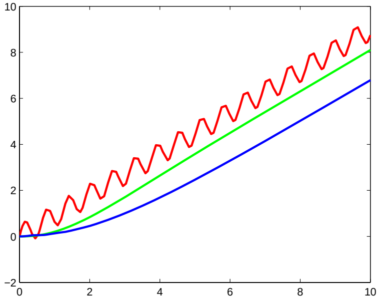

Something happens here: There is no sine present in the response and the ramp error has increased...
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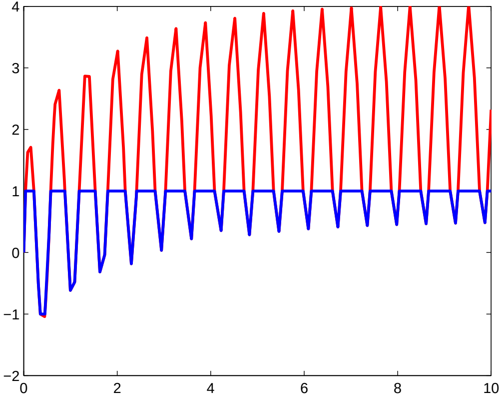
DC motor – saturated control signal (V/V)

Red: before the saturation ($\tilde{u}$). Blue: after the saturation (u).



Linearization of a nonlinear system

We can approximate a nonlinear system

$$\dot{x} = f(x,u), \qquad y = h(x,u),$$

by linearizing the system around an equilibrium (stationary) point (x_0, u_0). Intuitively this amounts to approximating the right-hand side of the system equation by a flat hyperplane (straight line in the scalar case).

Let $\Delta x(t) = x(t) - x_0, \quad \Delta u(t) = u(t) - u_0, \quad \Delta y(t) = y(t) - y_0.$

A Taylor expansion (only keeping the linear terms) results in

$$\begin{aligned} \frac{d}{dt}\Delta x &= \overbrace{\frac{\partial f(x_0,u_0)}{\partial x}}^{A} \Delta x + \overbrace{\frac{\partial f(x_0,u_0)}{\partial u}}^{B} \Delta u, \\ \Delta y &= \underbrace{\frac{\partial h(x_0,u_0)}{\partial x}}_{C} \Delta x + \underbrace{\frac{\partial h(x_0,u_0)}{\partial u}}_{D} \Delta u. \end{aligned}$$

Phase portraits for linear systems

- **Sign:** Is the solution moving towards the origin or away from the origin (along the eigenvector)?
- **Relative size:** "fast" and "slow" eigenvectors, which is dominating the solution behaviour for $t \approx 0$ and $t \gg 0$?
- **Complex/real:** Complex conjugated eigenvalues result in circles and spirals.

Cases to consider:

1. Two distinct real-valued eigenvalues imply two eigenvectors.
2. Multiple eigenvalues.
3. Complex eigenvalues.

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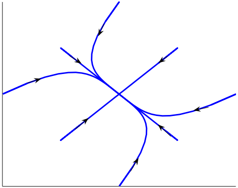
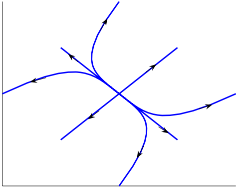
Two distinct eigenvalues with the same sign

The solution is $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$.

Stable node: For eigenvalues $\lambda_1 < \lambda_2 < 0$. The first term dominates for small t, the second term dominates for large t.



Unstable node: For eigenvalues $0 < \lambda_1 < \lambda_2$. Also here, the first term dominates for small t, the second term dominates for large t.



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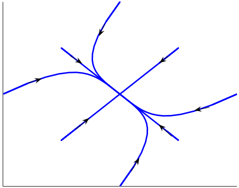

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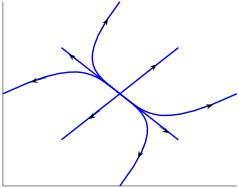

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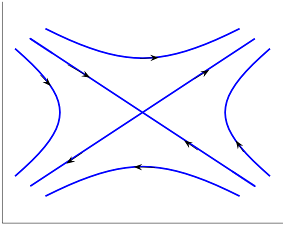
Two distinct eigenvalues with opposite sign

For eigenvalues $\lambda_1 < 0 < \lambda_2$ (with corresponding eigenvectors v_1, v_2) the solution is

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2.$$

Trajectories close to v_1 will approach the origin. v_1 is called the **stable eigenvector**.

Trajectories close to v_2 will move away from the origin. v_2 is called the **unstable eigenvector**.



Saddle point

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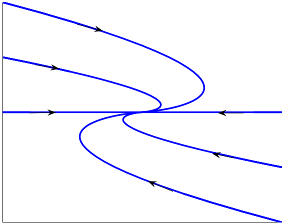
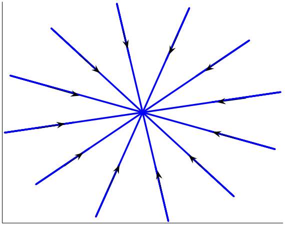
A multiple eigenvalue

For multiple eigenvalues $\lambda_1 = \lambda_2$.



Stable node (unstable: change direction).



Stable star node (unstable: change direction).

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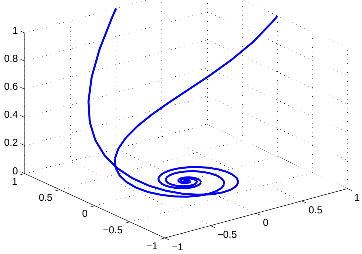
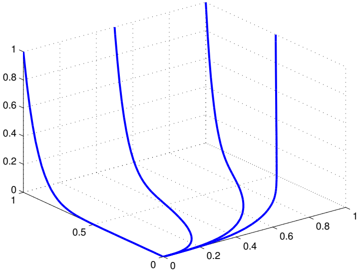
Two examples in 3D

Example of a generalization to 3D.





Left: Focus + one real eigenvalue.

Right: Three real eigenvalues.

A few concepts to summarize lecture 6

Equilibrium points: An equilibrium point is a point x_0, u_0 where the system is at rest, i.e. $f(x_0, u_0) = 0$. Also referred to as stationary points.

Linearization: Find a Taylor expansion of the nonlinear system around an equilibrium point and only keep the linear parts. This means that we are approximating the system using a flat hyperplane.

Phase plane: A two-dimensional state space that is simple to visualize graphically.

Phase portraits: A plot where one state variable is plotted against another state variable.

Limit cycle: A limit cycle is a closed trajectory in phase space having the property that at least one other trajectory spirals into it either as time approaches $\pm$ infinity.