Uppsala University
Department of Information Technology
Division of Systems and Control
September 3, 2019

Automatic control III

Homework assignment 2
2019

Deadline (for this assignment):
Monday October 7, 23.59

All homework assignments are compulsory and form an important part of the examination.

The assignments are to be solved in groups of up to 4 students (sign up on studentportalen). All group members should understand and be able to explain the entire solution of each problem. There will be an oral examination (October 22), based on all three assignments. Each group member will be given approximately 5 minutes to present the solution of one (by the examiner) arbitrarily chosen problem from one of the three assignments.

Instructions on how to hand in the assignments can be found on the course homepage. The solution should be handed in as one pdf. It should include a clearly presented and well motivated answer to all questions, with satisfactory equation typesetting, equation numbering, complete sentences, etc. All figures should have correct labels, a relevant figure text and be referenced in the text. If you use MATLAB (or some other language) for plotting or computations, you must *include your code* with your solution. However, a full report is not required, so no introduction, abstract, etc is needed, just solutions to the problems.

Before handing in your solutions, make sure (for each exercise) that

(I) You have answered all questions

(II) Your answers are reasonable

Solutions not fulfilling (I) and (II) will be rejected.

Problem I $\mathcal{H}_\infty$ control and basic limitations

The system

$$\begin{cases} \dot{x} = Ax + Bu, \\ \bar{z} = Cx, \\ y = Cx + w, \end{cases} \qquad \Leftrightarrow \qquad G(s) = C(sI - A)^{-1}B, \tag{1}$$

where w is a measurement disturbance $\big(|w| \leq 1\big)$, is to be controlled by an $\mathcal{H}_\infty$ controller, designed using the frequency weightings

$$W_S(s) = \frac{K_S}{s + \alpha_S}, \quad W_T(s) = K_T \frac{s + \beta_T}{s + \alpha_T}, \quad W_u(s) = K_u.$$

(a) Give a state space model representing the extended open loop system (i.e. incorporating the frequency weightings), expressed using the matrices in (1) and the parameters of the frequency weightings.

(b) We want the closed loop system to fulfill the following specifications:

(i) The controller $F_y(s)$ should have integral action.

(ii) The bandwidth of the closed loop system should be approximately 2 rad/s.

(iii) The effect of w on $\bar{z} = Cx$ should never be amplified more than 50 %, and it should be attenuated by at least a factor 100 for frequencies $\omega \geq 314$ rad/s ($= 50$ Hz).

(iv) For the input, $|u| < 4$ should hold.

For the $\mathcal{H}_\infty$ controller,

$$\| W_S S \|_\infty < 1, \quad \| W_T T \|_\infty < 1, \quad \| W_u G_{wu} \|_\infty < 1.$$

should hold. Based on this, suggest appropriate values for the parameters $K_S, \alpha_S, K_T, \beta_T, \alpha_T, K_u$ in the frequency weightings, so that the specifications i–iv are fulfilled. (Hint: First try to express the specifications i–iv in terms of conditions on S, T, G_{wu}.)

Problem II Equilibria and stability

In 1798, Reverend Thomas Robert Malthus postulated[1] the following dynamical system to model growth of a human population:

$$\dot{x}(t) = rx(t) \tag{2}$$

where $r > 0, x(t) \in \mathbb{R}$ and $t \in \mathbb{R}_+$.

(a) Is the system (2) linear? What is the order of the system?

(b) Compute all equilibria of the system and determine their stability.

(c) Compute the solution $x(t)$ of system (2) starting at $x(0) = x_0 > 0$. What is its limit as $t \to \infty$?

Subsequent research in population dynamics postulated[2] the following system to model population growth in the presence of limited resources:

$$\dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{K}\right) \tag{3}$$

where $r > 0, K > 0, x(t) \in \mathbb{R}$ and $t \in \mathbb{R}_+$.

(d) Is the system (3) linear? What is the order of the system?

(e) Compute all equilibria of the system and determine their stability using linearization.

(f) Show (analytically) that

$$x(t) = \frac{Kx_0e^{rt}}{K + x_0(e^{rt} - 1)} \tag{4}$$

is the solution of system (3) starting at initial condition $x(0) = x_0 > 0$. Compute the limit of the solution as $t \to \infty$ and relate to your anwer to **(e)**. Comment on the relation of this solution and the one computed in **(c)** as $K \to \infty$.

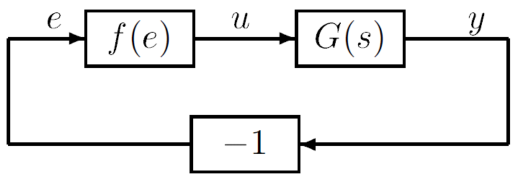
(g) Simulate the system (3) for some different values on r, x_0 and K (make sure to try both $x_0 > K$ and $x_0 < K$), without using the closed-form solution (4). If using MATLAB, a useful command could be **ode45**. Give an interpretation of what the parameters represent.

[1]T.R. Malthus, “An Essay on the Principle of Population” *Printed for J. Johnson, in St. Paul’s Church-Yard, London, UK*, 1798.

[2]Hutchinson, G. Evelyn. “Circular causal systems in ecology.” *Annals of the New York Academy of Sciences* 50.4 (1948): 221-246.

Problem III Lyapunov stability

Consider the system in the block diagram below, where a DC motor is controlled using a saturated amplifier.



The individual blocks are given by

$$G(s) = \frac{K}{s(s+1)} \ , \quad (K > 0)$$
$$f(e) = \begin{cases} -1 & e < -1 \\ e & -1 < e < 1 \\ 1 & e > 1 \end{cases}$$

(a) Use the circle criterion to find a condition on K which ensures that the closed loop system is stable.

(b) Write the system on state space form. (Hint: Use $x_1 = y$ and $x_2 = \dot{y}$ as state variables.)

(c) For the state space model in (b), analyse the stability of the closed loop system using Lyapunov theory. Use a Lyapunov function of the form

$$V(x) = \frac{1}{2}x_2^2 + Kg(x_1).$$

(Hint: Start by selecting an appropriate function $g(x_1)$, equation (12.4) in the book may be of use.)