Automatic Control III

Lecture 2 – The closed-loop system

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Outline – Lecture 2

1. Summary of lecture 1
2. Poles and zeros
3. A general block diagram
4. Stability for the closed loop system
5. Sensitivity
6. Robustness
7. Specification

Summary of lecture 1 (I/III)

Formalized how we can measure the **size** of signals and systems using various norms.

Signal size

$$||z||_2^2 = \int_{-\infty}^{\infty} |z(t)|^2 dt = \int_{-\infty}^{\infty} z^T(t)z(t)dt,$$
$$||z||_\infty = \sup_t |z(t)|.$$

The gain of system $\mathcal{S}$, where $y = \mathcal{S}(u)$:

$$\|\mathcal{S}\| = \sup_u \frac{\|y\|_2}{\|u\|_2} = \sup_u \frac{\|\mathcal{S}(u)\|_2}{\|u\|_2}$$

The gain of a linear stable SISO system G was shown to be

$$||G|| = \sup_\omega |G(i\omega)|.$$

Summary of lecture 1 (II/III)

For a multivariable system with the transfer matrix $G(s)$, we showed:

Gain $= ||G||_\infty = \sup_\omega \bar{\sigma}(G(i\omega))$, where

$$\bar{\sigma}(G(i\omega)) = \text{the largest singular value of } G(i\omega)$$

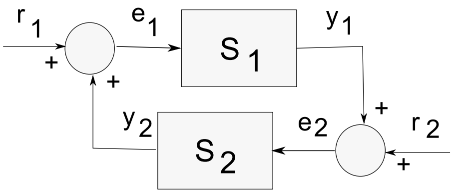
We also showed that

$$||y||_2 \leq ||G||_\infty ||u||_2$$

Plotting the singular values of $G(i\omega)$ as a function of ω corresponds to the plot of the amplitude curve for SISO systems.

Summary of lecture 1 (III/III)

Small gain theorem



Assume that the two systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are stable. Then the closed-loop system is stable if

$$\|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\| < 1.$$

Poles (of multivariable $G(s)$)

The **poles** are given by the eigenvalues (counted with multiplicity) of the system matrix A of a minimal (controllable and observable) state-space realization of the system.

What if $G(s)$ is given?

The pole polynomial is the **least common denominator** of all minors of $G(s)$. The system poles are then given by the roots of the pole polynomial.

Note that this also provides a way of finding the order of a minimal realization! The order of the system = the number of poles (counted with multiplicity)

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Zeros (of multivariable $G(s)$)

Definition (quadratic $G(s)$): The zero polynomial is given by the numerator of $\det(G(s))$ (after it has been normalized to have the pole polynomial as its denominator).

Definition (general $G(s)$): The zero polynomial is the **greatest common divisor** of the numerators of the maximal minors (after they have been normalized to have the pole polynomial as its denominator).

Maximal minor (*maximal underdeterminant*): A determinant of maximal size.

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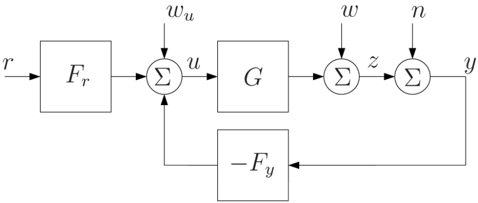
The closed-loop system: important signals

1. u, control signal
2. z, the quantity we want to control
3. r, reference signal (i.e., what we want z to be)
4. y, output signal
5. disturbances
    - w_u, input disturbance
    - w, output disturbance
    - n, measurement disturbance

"The canonical block diagram"



Open-loop system:

$$z(t) = Gu(t) + w(t),$$
$$y(t) = z(t) + n(t).$$

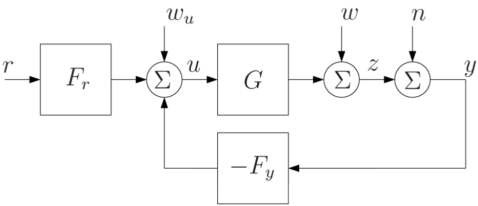
Feedback (linear system):

$$u(t) = F_r r(t) - F_y y(t) + w_u(t).$$

The closed-loop system: important transfer functions

Transfer functions:

$$\begin{aligned} G_c &= (I + GF_y)^{-1}GF_r \\ S &= (I + GF_y)^{-1} \\ S_u &= (I + F_yG)^{-1} \\ T &= (I + GF_y)^{-1}GF_y \end{aligned}$$



Relations among signals:

$$\begin{aligned} z &= G_c r + Sw - Tn + GS_u w_u \\ u &= \underbrace{S_u F_r}_{G_{ru}} r \underbrace{- S_u F_y}_{G_{wu}} (w + n) + S_u w_u \end{aligned}$$

Stability of the closed-loop system

Consider the closed-loop system as a system with two input signals (w_u, w) and two output signals (u, y). For now $r = n = 0$.

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} GS_u & S \\ S_u & -S_uF_y \end{bmatrix} \begin{bmatrix} w_u \\ w \end{bmatrix} \tag{1}$$

If G and F_y are represented by controllable and observable state-space descriptions, it can be shown that the closed-loop system in (1) is also controllable and observable. Checking all 4 transfer functions (the transfer matrix blocks)

$$GS_u, S, S_u, S_uF_y$$

we will give all the poles for the systems (and we can then e.g. establish the stability).

Internal stability

From the example we saw that we have to be careful with pole-zero cancellations.

Definition (internal stability): The closed-loop system is said to be *internally stable* if the following four transfer functions are stable (after possible cancellations), as well as F_r,

$$
\begin{aligned}
w_u(t) \to u(t) &: S_u = (I + F_y G)^{-1}, \\
w_u(t) \to y(t) &: SG = (I + G F_y)^{-1} G, \\
w(t) \to u(t) &: -S_u F_y = -(I + F_y G)^{-1} F_y, \\
w(t) \to y(t) &: S = (I + G F_y)^{-1}.
\end{aligned}
$$

Sensitivity (I/II)

The true system is given by $G_0(s)$ while the model is $G(s)$. Let

$$G_0 = (I + \Delta_G)G.$$

Neglecting disturbances, we have

$$z_0 = (I + G_0 F_y)^{-1} G_0 F_r r$$

for the true system G_0 and

$$z = (I + G F_y)^{-1} G F_r r$$

for the model G.

Let us study the relationship between z_0 and z by introducing Δ_z and S_0 according to

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The sensitivity function S_0 (for the true system) describes how the relative model error Δ_G is transformed into a relative output signal error Δ_z according to

$$\Delta_z = S_0 \Delta_G.$$

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Robustness

What model errors Δ_G can be allowed without endangering the stability of the closed loop system?

The **small gain theorem** can be used to prove that the closed loop system remains stable if

$$\|\Delta_G T\|_\infty < 1,$$

which in turn is fulfilled if

$$|T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|}, \quad \forall \omega$$

Formalizing the control problem (spec.)

Formulation of the control problem in words

> *"Choose a controller such that the controlled variable resembles the reference signal as close as possible, despite disturbances, measurement errors and model errors, while at the same time not making use of too large control signals."*

We now have a machinery to formalize these words. Recall the following relationships:

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\begin{aligned}
e &= (I - G_c)r - Sw + Tn, \; e = r - z, \\
u &= G_{ru}r + G_{wu}(w + n), \\
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Formalizing the control problem (spec.)

1. $|I - G_c|$ small $\Rightarrow$ The controlled variable follows the reference signal.
2. S small $\Rightarrow$ Small influence of model errors and process disturbances.
3. T small $\Rightarrow$ Small influences of measurement errors and to make sure that model errors do not jeopardize the stability.
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Important to note that these requirements are in conflict. For example:

$$S + T = I,$$
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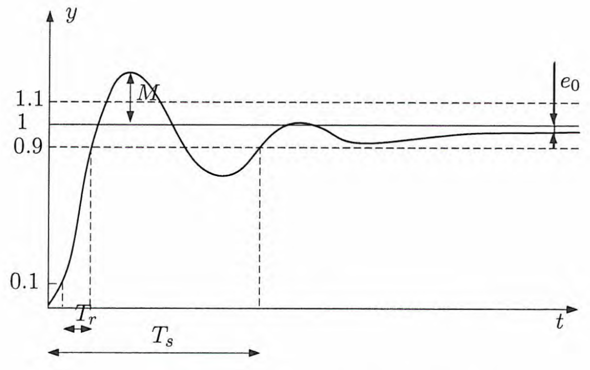
Design spec. in the time domain (I/III)

Design criterion 1: Choose a controller so that M, e_0, T_r and T_s for the step response of G_c are smaller than some given values.



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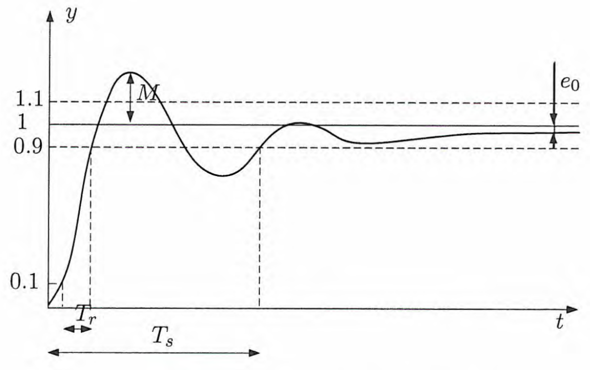

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Design spec. in the time domain (II/III)

Recall the *error coefficients* from the basic course. The first error coefficient, the *static error coefficient* is given by

$$e_0 = \lim_{t \to \infty} e(t) = I - G_c(0).$$

Assume $F_y = F_r$. This means that $I - G_c = I - T = S$, which in turn implies that

$$e_0 = S(0).$$

Usual design requirement: $e_0 = S(0) = 0$.

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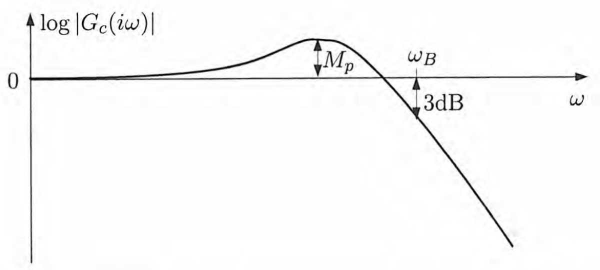
Design spec. in the time domain (III/III)

Design criterion 2: Choose the controller so that G_c and S are equal to given transfer functions. *(Note: Internal Model Control, (IMC) in Chapter 8)*
Design criterion 3: Choose the controller so that the poles of G_c and S are placed within given areas. *(Note: Recall pole placement in the basic course)*
Design criterion 4: Choose the controller such that

$$\|e\|_{Q_1}^2 + \|u\|_{Q_2}^2$$

is minimized. *(Note: linear quadratic controllers (LQG))*

Design spec. in the frequency dom. (I/II)



Requirements:

$$|S(i\omega)| \leq |W_S^{-1}(i\omega)|, \qquad |T(i\omega)| \leq \frac{1}{|\Delta_G(i\omega)|}, \quad \forall \omega.$$

In the multivariable case

$$\|W_S S\|_\infty \leq 1, \qquad \|W_T T\|_\infty \leq 1.$$

Design spec. in the frequency dom. (II/II)

Design criterion 5: Choose the controller such that

$$\|W_S S\|_\infty \leq 1, \qquad \|W_T T\|_\infty \leq 1, \qquad \|W_{ru} G_{ru}\|_\infty \leq 1.$$

Note: This leads to $\mathcal{H}_\infty$-controllers (Chapter 10).

Design criterion 6: Choose the controller such that

$$V = \|W_S S\|_2^2 + \|W_T T\|_2^2 + \|W_{ru} G_{ru}\|_2^2$$

is minimized. *Note: This leads to $\mathcal{H}_2$-controllers (Chapter 10).*

Design spec. in the frequency dom. (II/II)

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A few concepts to summarize lecture 2

Minor: The determinant of a quadratic matrix of A obtained by crossing out rows and columns in A.

Pole of a multivariable system: The pole polynomial is the least common denominator of all minors of $G(s)$. The system poles are then given by the zeros of the pole polynomial.

Zero of a multivariable system: The zero polynomial is the greatest common divisor of the numerators of the maximal minors (after they have been normalized to have the pole polynomial as its denominator).

Internal stability: The concept of internal stability allows us to assess the stability of the closed-loop system, without "missing any unseen modes".

Robustness: What model errors Δ_G can be allowed without endangering the stability of the closed-loop system?