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Exercises
Automatic Control III

2019

Foreword

This exercise manual is designed for the course "Automatic Control III", given by the Division of Systems and Control. The numbering of the chapters follows the book by Torkel Glad and Lennart Ljung: "Control Theory. Multivariable and nonlinear methods." (Taylor and Francis, 2000), and its Swedish translation from Studentlitteratur, 2003.

Most problems in this manual has been collected from the older exercise manual, originally compiled by Mikael Johansson and Torsten Söderström (at KTH and UU, respectively) in 2006. In this manual, only problems relevant to the current course have been kept. Some solutions have been clarified and a few exercises have been added.

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Contents

1 Introduction

1.1 Small gain & Nyquist Criterion (LiU)

Show how the small gain theorem follows from the Nyquist criterion for linear systems!

1.2 Gain of ideal relay (LiU)

What is the gain of an ideal relay? An ideal relay is given by the function

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

1.3 Small gain theorem for linear system with saturation (LiU)

Consider the linear system

$$G(s) = \frac{2}{s^2 + 2s + 2}$$

The control signal first passes through a saturating valve:

$$\tilde{u} = \begin{cases} 1 & \text{if } u > 2 \\ \frac{1}{2}u & \text{if } |u| < 2 \\ -1 & \text{if } u < -2 \end{cases}$$

Hereby the control signal becomes

$$y(t) = G(p)\tilde{u}(t)$$

We use P-control, i.e. $u(t) = -Ky(t)$. For which values of K is the closed loop system guaranteed to be stable according to the small gain theorem?

1.4 Signal norms (LiU)

Calculate the norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$ of the continuous time signals

a) $y(t) = \begin{cases} a\sin(t), & t > 0 \\ 0, & t \leq 0 \end{cases}$

b) $y(t) = \begin{cases} \frac{1}{t}, & t > 1 \\ 0, & t \leq 1 \end{cases}$

c) $y(t) = \begin{cases} e^{-t}(1 - e^{-t}), & t > 0 \\ 0, & t \leq 0 \end{cases}$.

1.5 Gain for second order linear system (LiU)

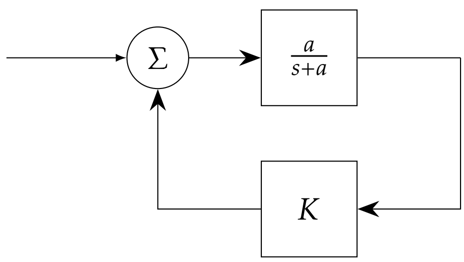
Consider the linear system

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}.$$

Determine the gain $\|G\|$ for the system for all values $\omega_0 > 0$ and $\zeta > 0$.

1.6 Small gain theorem vs poles (LiU)

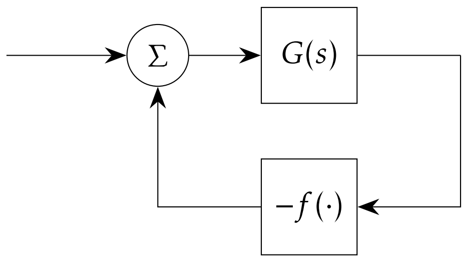
Examine the stability for the below system using both the small gain theorem and by computing the poles of the closed loop system.



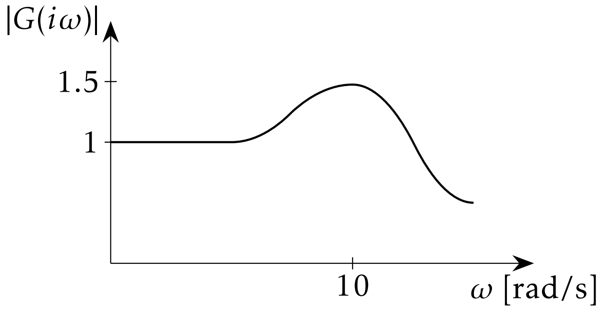
Explain the possible differences!

1.7 Small gain theorem with nonlinear static feedback (LiU)

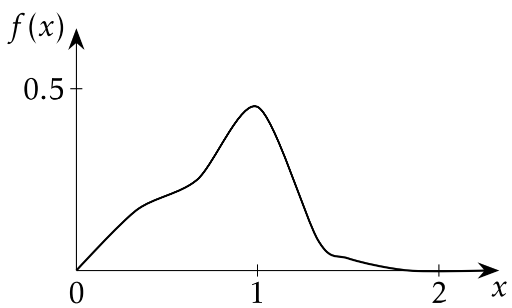
Consider two subsystems coupled with feedback as in the figure below.



where $G(s)$ is a stable linear system with Bode plot



and $f(\cdot)$ is an amplifier with the following input-output characteristics



Is the closed-loop system stable?

3 Properties of linear systems

3.1 Poles and zeros of MIMO systems I

Determine the poles and zeros for the following system with three inputs and two outputs:

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

3.2 Poles and zeros of MIMO systems II (LiU)

Determine the poles and the zeros of the system

$$G(s) = \frac{1}{(s+1)(s+3)} \begin{pmatrix} 1 & 0 \\ -1 & 2(s+1)^2 \end{pmatrix}$$

3.3 Poles and zeros of MIMO systems III (LiU)

Determine the poles (with multiplicity) to the system

$$G(s) = \frac{1}{(s+1)^2} \begin{bmatrix} 1-s & \frac{1}{3}-s \\ 2-s & 1-s \end{bmatrix}.$$

What is the order for a minimal realization?

3.4 Poles and zeros of MIMO systems IV (KTH)

a) The system G_1 has the transfer function

$$G_1(s) = \begin{pmatrix} 2 & \frac{s-2}{s+3} \\ \frac{s+6}{s+1} & \frac{s+1}{s+3} \end{pmatrix}$$

Calculate the poles and zeros of G_1 with multiplicity.

b) The system G_2 is

$$G_2(s) = \begin{pmatrix} -(s+1) & \frac{s+2}{s+6} \\ \frac{3}{s+1} & \frac{s^2-1}{4s^2+4s+1} \end{pmatrix}$$

Assume that a constant input signal u shall be used to obtain

$$y(t) = y_{\text{ref}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

in stationarity. Explain why this is impossible.

3.5 Poles, zeros and singular values for parallel tanks (LiU)

A multivariable system consists of two coupled parallel tanks, and has the dynamics

$$\begin{aligned} \frac{d}{dt}x &= \begin{pmatrix} -1-\alpha & \alpha \\ \alpha & -1-\alpha \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x \end{aligned}$$

The coefficient α depends on the size of an opening between the two tanks.

(a) What is the transfer function of the system?

(b) Verify that one of the poles is always located in $s = -1$, no matter what value α has. Where is the other pole located when α is varied from 0 and upwards?

(c) Verify that the system has no zeros, no matter what value α takes.

(d) Determine the singular values of the system.

3.6 Poles, zeros, singular values and IMC for a double tank (UU)

Consider a double tank with inflows to both the upper (u_1) and the lower (u_2) tank. Assuming both tank levels can be measured, a local model around a working point can be written as

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -a & 0 \\ a & -a \end{pmatrix} x + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} u \\ y &= x \end{aligned}$$

where a and b are positive constants.

(a) Determine the transfer function of the system, and its poles and zeros.

(b) Determine the singular values of the system.

(c) Design a regulator based on Internal model control, using λ-tuning. Determine the sensitivity function of the closed loop system.

(d) What are the singular values of the sensitivity function?

6 The closed-loop system

6.1 Internal stability I

(LiU)

For a given system G and a given controller F we have defined the four transfer functions required to be stable, for the system to be stable, as

$$H_{11} = (I + FG)^{-1}, \;\; H_{12} = -(I + FG)^{-1} F$$
$$H_{21} = (I + GF)^{-1} G, \;\; H_{22} = (I + GF)^{-1}$$

Show that

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} I & F \\ -G & I \end{bmatrix}^{-1}$$

6.2 Internal stability II

(LiU)

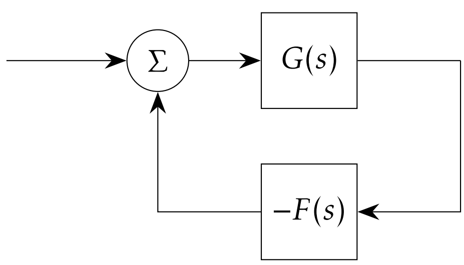
A system

$$G(s) = \frac{s-1}{s+1}$$

is controlled in feedback using the controller

$$F(s) = \frac{s+2}{s-1}$$

according to



Compute G_c, T and S. Are they stable? Is the system internally stable?

6.3 Stability and model errors I (LiU)

For the discrete-time model

$$y(t) = G(q)u(t)$$

there is a feedback

$$u(t) = F(q)(r(t) - y(t))$$

so that the closed-loop system becomes

$$y(t) = \frac{1-\alpha}{q-\alpha} r(t), \qquad 0 < \alpha < 1$$

Show that this control system can tolerate up to 100% relative changes in $G(q)$, that is $|\Delta_G| < 1$, without jeopardizing stability of the closed-loop system.

6.4 Stability and model errors II (KTH)

Assume that we have designed a controller F for the scalar system G such that the loop-gain $L = GF$ and the corresponding closed-loop system are stable. Due to model errors the true loop-gain is equal to L_p. Define the relative model error as

$$L_p = (1 + \Delta_L)L.$$

We assume that Δ_L is stable. Derive, using the Nyquist criterion, constraints on Δ_L that guarantee stability for the true system.

6.5 Stability and model errors III (KTH)

Consider a process with the nominal model

$$G(s) = \frac{1}{s+1}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Suppose that the gain for the second input is uncertain. The real system is therefore given by

$$G_0(s) = \frac{1}{s+1}\begin{bmatrix} 2 & \alpha \\ 0 & 2\alpha \end{bmatrix}$$

where α represents the uncertain gain.

(a) Use the nominal model and design a decoupling regulator according to

$$F(s) = \frac{k}{s} G^{-1}(s).$$

Choose k so that the closed-loop system is as fast the original open-loop system.

(b) Represent the uncertainty as a relative output model error. Use a pertinent robustness criterion to determine for which values of α that stability of the closed-loop system can be guaranteed.

(c) Represent the uncertainty as a relative input model error. Use a pertinent robustness criterion to determine for which values of α that stability of the closed-loop system can be guaranteed. Will the answer differ from what was obtained in part (b)?

(d) Determine the poles of the real closed-loop system. For which values of α is the system stable?

6.6 From state space form to sensitivity function (UU)

Consider a system where the open loop part is given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + w \end{aligned}$$

and the feedback is

$$\begin{aligned} \dot{z} &= Fz + Gy \\ u &= -Hz - Jy \end{aligned}$$

a) Represent the closed loop system in state space form.

b) Give the sensitivity function $S(s)$ expressed in the matrices A, B, C, F, G, H, J.

c) In the simple case when the open loop system has a transfer function $G(s) = 1/(s+1)$, and is controlled by a proportional regulator, $u = -Ky$, determine the sensitivity function $S(s)$ by direct calculations, and verify that the expression derived in part b) leads to the same result.

7 Basic limitations in control design

7.1 Limitations and stability (LiU)

Given a system

$$G(s) = \frac{s-3}{s+1}.$$

we want the complementary sensitivity function

$$T(s) = \frac{5}{s+5}.$$

(a) Compute a feedback $F_r = F_y = F$ which gives this T. Will it work?

(b) Suggest an alternative T, that has the bandwidth 5 rad/s as before, but gives a stable closed loop system with $F_r = F_y = F$.

(c) What is the corresponding sensitivity function?

(d) Could a solution with F_r and F_y different be a good alternative? More specific, could one choose F_r and F_y so that none of them has a pole in s=3 and that the closed loop system transfer function becomes

$$G_c = \frac{GF_r}{1+GF_y} = \frac{5}{s+5}$$

7.2 Conflicting requirements? I (LiU)

A multivariable system is required to dampen all system disturbances (w) by at least a factor 10 under 0.1 rad/sek. It should also dampen measurement noise (n) by at least a factor of 10 over 2 rad/sek. Constant disturbances should be dampened by at least a factor 100 in stationarity.

(a) Formulate requirements on S and T (their singular values) which assure fulfillment of the specifications.

(b) Translate the specifications to requirements on the loop gain GF.

(c) Formulate the requirements using $\|\cdot\|_\infty$ and the weight functions W_S and W_T according to the text book.

(d) What crossover frequency and phase margin could be expected according to (b)? Which lower limit does this impose on $\|T\|_\infty$?

(e) How is this lower limit on $\|T\|_\infty$ related to the requirements according to (c)?

7.3 Conflicting requirements? II (LiU)

On a certain closed loop control system we specify the following requirements:

i. Noise on the output signal with frequency under 2 rad/s should be damped by a factor 1000.

ii. The system should remain stable despite a model uncertainty

$$|\Delta G| \leq 100|G|$$

for frequencies over 20 rad/s where G is the nominal open loop system frequency function and ΔG is the absolute error in the same.

Can this be achieved using a linear time-invariant controller?

7.4 Conflicting requirements? III (UU)

The engineer Civerth was given the task to design a SISO control system with the specifications:

- The stationary control error, when the reference signal is a ramp, should be zero (that is, the error coefficient $e_1 = 0$).
- The step response for the closed loop system should not have an overshoot.

Did he succeed?

Investigate by treating the subproblems a) - c) below.

a) Rewrite the requirement $e_1 = 0$ as a condition on the sensitivity function $S(s)$.

b) Assume that the condition in part a) holds. Show that, for a step in the reference signal

$$\int_0^\infty e(t)dt = \lim_{s \to 0} E(s) = 0$$

c) What does the relation in b) tell about the possible overshoot in the step response?

7.5 Conflicting requirements? IV

As a control engineer, you are asked to design a (linear, time-invariant) controller, which at 5 rad/s has the following properties:

- It dampens disturbances on the output 10 times, i.e., the absolute value of the transfer function from disturbances on the output to the controlled variable is not bigger than 0.1 at this frequency.
- Noise in the measurements may at most affect the controlled variable with a tenth of the amplitude of the noise. I.e., the absolute value of the transfer function from measurement noise to the controlled variable is at most 0.1 at this frequency.

How can you easily tell you have been assigned an impossible task?

7.6 Time scales

The continuous-time description of an unstable system has a pole in +0.005.

a) On what time-scale will the closed loop system have to work (at least), with a well-designed controller? Are we talking about hours, minutes, seconds, milliseconds, ...?

b) If we are given the information that the system also contains a time-delay of 10 second, what can you say about the possibilities to control the system?

8 Controller structures and control design

8.1 RGA (LiU)

Given the multivariable system

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{0.1s+1} \begin{pmatrix} \frac{0.6}{s+1} & -0.4 \\ 0.3 & 0.6 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Assume that we want to control the system using a decentralized controller, and want to use the *relative gain array* (RGA) to decide which control signal should be used to control each output signal. Further assume that we aim for a crossover frequency of $\omega_c = 10$ rad/s. Determine how the signals should be paired.

8.2 RGA and decentralized control (LiU)

Consider the multivariable system

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{10s+1} & \frac{-2}{2s+1} \\ \frac{1}{10s+1} & \frac{s-1}{2s+1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

(a) Using RGA-analysis determine which control signal should be used to control each output signal.

(b) Assume we want to control the system using a decentralized controller, i.e.

$$F^{\text{diag}}(s) = \begin{pmatrix} F_{11}(s) & 0 \\ 0 & F_{22}(s) \end{pmatrix}.$$

Further assume that we do not want the stationary error in one channel to affect the stationary error in the other channel. Give the structure of a controller $F(s)$, expressed in $F^{\text{diag}}(s)$ above, that achieves this.

8.3 IMC for a first order system (LiU)

Design a controller using the IMC method for a stable first order process

$$G(s) = \frac{K}{\tau s + 1}, \qquad \tau > 0.$$

Which kind of controller is obtained? Determine the sensitivity function $S(s)$ and the complementary sensitivity function $T(s)$. Sketch the Bode plot of $S(i\omega)$. What would Bode's integral theorem tell in this case?

8.4 IMC for a MIMO system (LiU)

The multivariable system

$$G(s) = \frac{1}{s/20+1} \begin{pmatrix} \frac{9}{s+1} & 2 \\ 6 & 4 \end{pmatrix}.$$

is given.

(a) Determine the poles and the zeros of the system.

(b) Design a controller based on IMC for the system.

10 Loop shaping

10.1 Simple $\mathcal{H}_2$ control I (LiU)

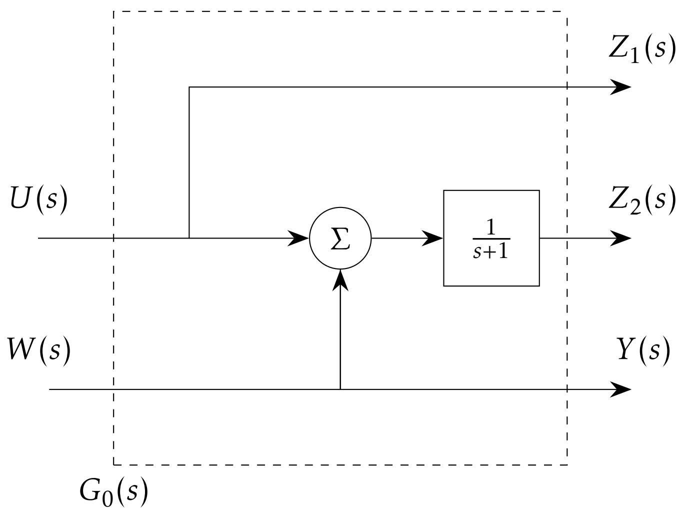
$$y = \frac{1}{p+1}u.$$

We would like to design a closed-loop system with S, T and G_{wu}, such that

$$\int |S(i\omega)/\omega|^2 + |0.5T(i\omega)|^2 + |5G_{wu}(i\omega)|^2 \, dw$$

is minimized. Determine a controller that achieves this!

10.2 Simple $\mathcal{H}_2$ control II (KTH)



Consider the extended system $G_0(s)$ given by the block diagram above.

(a) Show that

$$\begin{aligned}\frac{dx(t)}{dt} &= -x(t) + u(t) + w(t)\\ z(t) &= \begin{bmatrix}0\\1\end{bmatrix}x(t) + \begin{bmatrix}1\\0\end{bmatrix}u(t)\\ y(t) &= w(t)\end{aligned}$$

is a state-space description of $G_0(s)$.

(b) An observer-based controller is given by

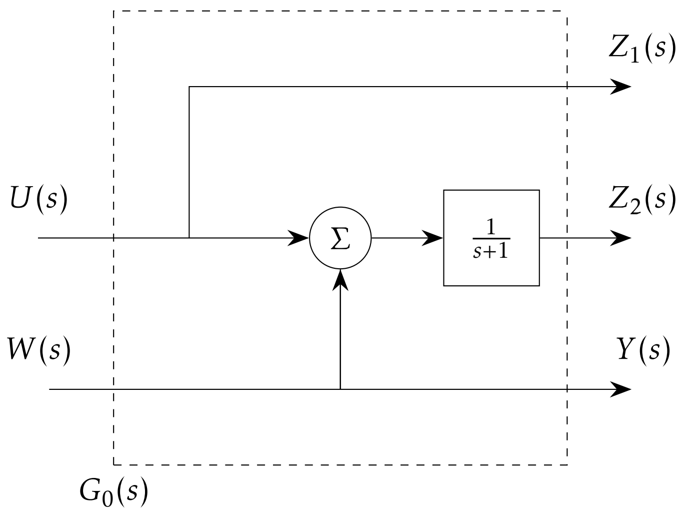
$$\begin{aligned}\frac{d\hat{x}(t)}{dt} &= -\hat{x}(t) + u(t) + y(t)\\ u(t) &= -L\hat{x}(t)\end{aligned}$$

Determine L such that $\|G_{ec}\|_2$ is minimized, where $G_{ec}(s)$ is the closed-loop transfer function from $W(s)$ to $Z(s)$. Compute the controller transfer function from $Y(s)$ to $U(s)$ for this optimal L.

(c) What is the value of $\|G_{ec}\|_2$ for the optimal controller determined in (b)? *Hint:* $\|G_{ec}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\Big(G_{ec}(i\omega)G_{ec}^T(-i\omega)\Big)d\omega$

(d) Let $U(s) = -KY(s)$, i.e. consider proportional control. Determine the value of K that minimizes $\|G_{ec}\|_2$, where $G_{ec}(s)$ is the closed-loop transfer function from $W(s)$ to $Z(s)$ under proportional control. What is the minimal value of $\|G_{ec}\|_2$?

10.3 Simple $\mathcal{H}_\infty$ control (KTH)



Consider the extended system $G_0(s)$ given by the block diagram above.

(a) Show that

$$\frac{dx(t)}{dt} = -x(t) + u(t) + w(t)$$
$$z(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = w(t)$$

is a state-space description of $G_0(s)$.

(b) An observer-based controller is given by

$$\frac{d\hat{x}(t)}{dt} = -\hat{x}(t) + u(t) + y(t)$$
$$u(t) = -L\hat{x}(t)$$

Determine L such that $\|G_{ec}\|_\infty \leq 1$ and such that the closed-loop system is internally stable, where $G_{ec}(s)$ is the closed-loop transfer function from $W(s)$ to $Z(s)$. (Observere that the claim that **(a)** F_y satisfies all conditions (10.19) on page 318 i the text book can be replaced by **(a)** F_y satisfies (10.18).)

(c) Determine the same observer-based controller as in (b). Determine the value of L that minimizes $\|G_{ec}\|_\infty$ and that ensures that the closed-loop system is internally stable. Compute the controller transfer function from $Y(s)$ to $U(s)$ for this optimal L.

(d) Let $U(s) = -KY(s)$, i.e. assume proportional control. Determine the value of K which minimizes $\|G_{ec}\|_\infty$, where $G_{ec}(s)$ is the closed-loop system transfer function from $W(s)$ to $Z(s)$ under proportional control. What is the smallest value of $\|G_{ec}\|_\infty$?

10.4 Weight functions and observers (LiU)

Consider the SISO system with state-space representation

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

We would like to use loop shaping with the weights

$$W_S = \frac{1}{s}, \quad W_T = 1, \quad W_u = 1$$

(a) State the equations which determine the optimal controller in the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ sense, respectively.

(b) Write out the observer equations for the extended state-vector and show that the optimal controller can be written as

$$u(t) = \frac{\alpha}{1 + L(pI - A)^{-1}B} \int_0^t y(\tau)\, d\tau$$

for some L, where $\alpha = 1$ for the $\mathcal{H}_2$ controller and $\alpha > 1$ for the $\mathcal{H}_\infty$ controller. State the equation that determines L.

(c) Show that the controller implements integral action unless $G(s)$ itself contains an integrator.

12 Stability of nonlinear systems

12.1 Lyapunov stability I

The nonlinear differential equation

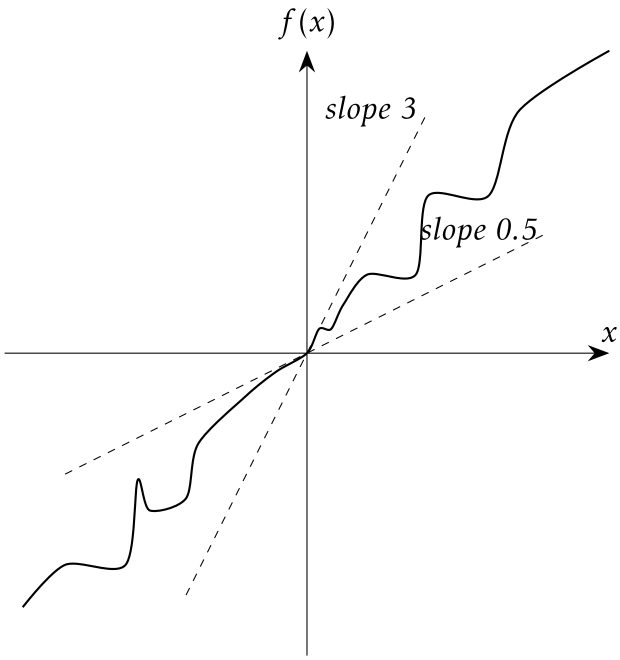
$$\ddot{y} + 0.2(1 + \dot{y}^2)\dot{y} + y = 0$$

is given. Introduce the state variables $x_1 = y$ and $x_2 = \dot{y}$. Derive the stability conditions for the zero solution. Try the Lyapunov function

$$V = \frac{1}{2}(x_1^2 + x_2^2).$$

12.2 Circle criterion I

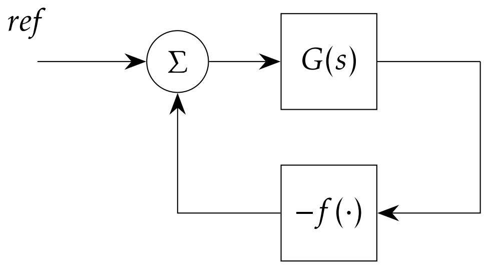
A nonlinear function lies between the lines in the figure.



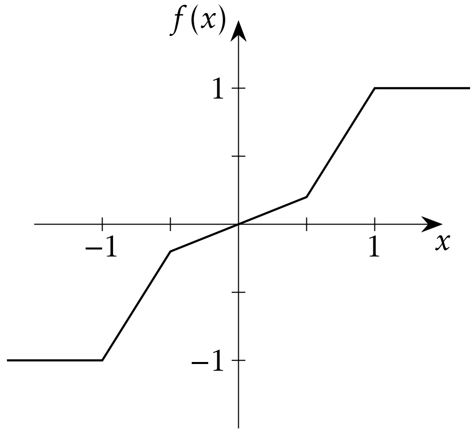
Which circle in the complex plane does this nonlinearity correspond to, when the circle stability criterion is applied?

12.3 Circle criterion II

A nonlinear system is given by the following block diagram.



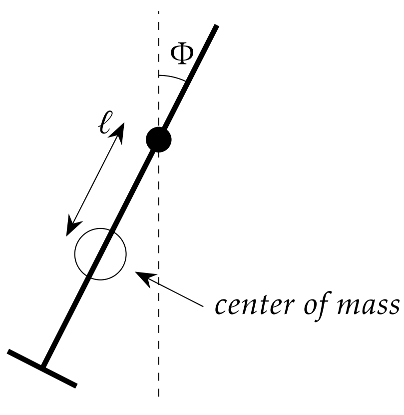
where $G(s)$ is a linear system, and the static nonlinearity is given in the figure below.



Which constraints must be imposed on $G(s)$, when the circle criterion is applied to guarantee the stability of the closed-loop system?

12.4 Lyapunov stability II

Consider the swing in the figure below.



The movements of the swing are described by the equation

$$J\frac{d^2\Phi}{dt^2} + mg\ell\sin\Phi = 0$$

where m is the mass and J the moment of inertia. When a person is repeatedly bending and stretching his legs, the swing can be controlled. The distance to the point of gravity is the control variable, and we regard for simplicity J as a constant.

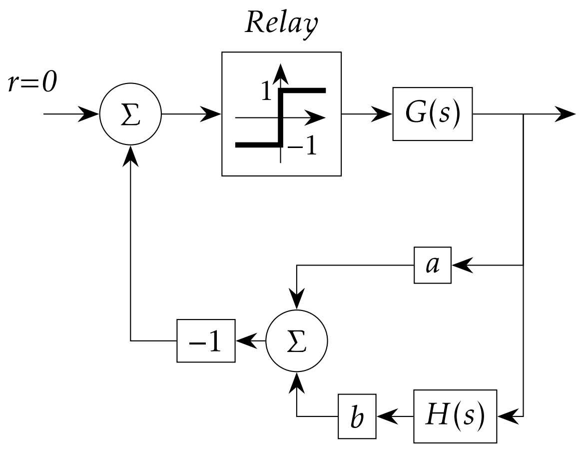
Show that the input

$$\ell = \ell_0 + \varepsilon\Phi\dot{\Phi}, \qquad \varepsilon > 0$$

will force the swing to its rest position $\Phi = 0$.

12.5 Lyapunov stability and circle criterion for controller design

Consider the feedback system in the figure below.



where

$$H(s) = s, \quad G(s) = \frac{1}{(s+1)(s+2)}$$

(a) Write the open loop system in state space form, with $x_1 = y$ and $x_2 = \dot{y}$ as state variables.

(b) Use Lyapunov theory to design the feedback parameters a and b such that the closed loop system is guaranteed to be asymptotically stable. Try Lyapunov functions of the form

$$V(x) = \frac{1}{2}\left(\alpha x_1^2 + \beta x_2^2\right)$$

(c) Use the circle criterion to design the feedback parameters a and b such that the closed loop system is guaranteed to be asymptotically stable.

13 Phase plane analysis

13.1 Equilibrium points and phase portrait

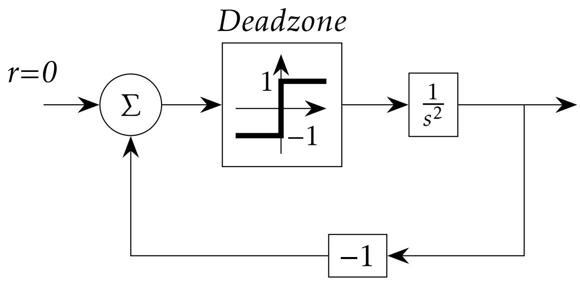
The differential equation

$$\ddot{x} - (0.1 - \frac{10}{3}\dot{x}^2)\dot{x} + x + x^2 = 0$$

is given. Determine the equilibrium points and their characters. Further, sketch the phase portrait.

13.2 Phase portrait for a closed loop system with relay

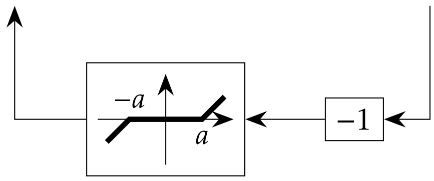
The following system is given



(a) When the input to the relay is zero, the output is +1 or −1, depending the previous value of the input. The relay does not change its output until the input has switched sign.

Determine the phase portrait.

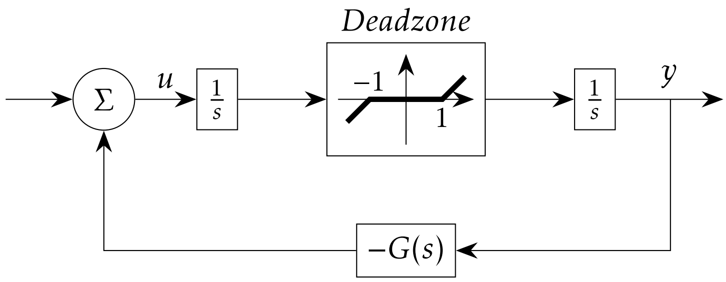
(b) Assume that the feedback is modified according to the figure below.



Determine the phase portrait in this case.

13.3 Analysis of car behavior (LiU)

Linus is on his way home after the examination period. On the highway outside Linköping the car is hit by a wind gust, which moves it away sidewize. The problem is, with help of phase plane analysis, to determine how the movement of the car will develop. (Will the car return to its desired position or not?) If the car has a constant speed, its dynamics can be described as in the figure below.



Here, u is the torque the driver apply to the driving wheel. The deadzone comes from a gear in the control equipment. Further, y is the error in the sidewise position of the car, and $G(s)$ is the transfer function from the driver's perception of the position error to the force with which he effects the driving wheel.

Consider two specific cases.

1. $G(s) = 1$ (the examination period finished with a party)
2. $G(s) = 1 + s$ (the examination period finished without a party)

13.4 Analysis of fish system

In a simple ecosystem, there are two kind of fishes. One sorts is eating algae, while the second is a predator and is feeding on the first type of fish. Let the amount of fishes of the two kinds be denoted by x_1 and x_2, respectively.

The interaction between the species is modelled as

$$\dot{x}_1 = 2x_1 - \frac{x_1 x_2}{1 + \frac{1}{6}x_1} - 0.2x_1^2 \tag{1}$$

$$\dot{x}_2 = -3x_2 + \frac{x_1 x_2}{1 + \frac{1}{6}x_1} \tag{2}$$

(a) Determine the singular points to this model.

(b) Determine the characters of the singular points, and sketch the phase portraits in the close neighboorhood.

(c) Merge the phase portraits in a way that is reasonable, without making explicit calculations. It is sufficient to consider the case $x_1 > x_2 > 0$.

The interpretation of the given model is as follows.

If the fishes feeding on algae have unlimited amount of food and no enemies, they would grow in number exponentially according to

$$\dot{x}_1 = 2x_1$$

As the amount of algae is limited, there is saturation effect of the growth modelled as

$$\dot{x}_1 = 2x_1 - 0.2x_1^2.$$

Further, if the enemies (predators) x_2 are present, the first type of fishes will be consumed at the speed

$$\frac{x_1 x_2}{1 + \frac{1}{6}x_1}$$

The interpretation of this term is that for x_1 large, every predatory will eat to be satisfied, and this hunger corresponds to a consumption of 6 x_1 fishes per time unit. When x_1 is smaller, the consumption by the x_2 fishes will be reduced.

The second equation means that if the supply of food is unlimited ($x_1 = \infty$), the x_2-fishes are reproducing at the rate

$$\dot{x}_2 = 3x_2.$$

Should there be no food at all for the predator ($x_1 = 0$) the x_2-fishes will decline in number according to

$$\dot{x}_2 = -3x_2$$

13.5 Phase portrait and Lyapunov stability for control design

Consider the system

$$\dot{x} = \begin{pmatrix} -x_1^3 + u \\ x_1 \end{pmatrix}$$

(a) Sketch the phase portrait for the case $u = 0$.

(b) Use the Lyapunov function

$$V(x) = x_1^2 + x_2^2$$

and derive a nonlinear feedback

$$u = f(x_1, x_2)$$

which makes the origin globally asymptotically stable. Sketch the phase portrait for the closed loop system, close to the origin.

13.6 Equilibrium points (UU)

For each of the following systems, find all equilibrium points, and determine the type of each isolated equilibrium.

(a)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \frac{x_1^3}{6} - x_2\end{aligned}$$

(b)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)\end{aligned}$$

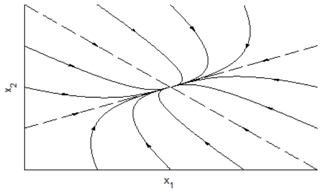
(c)

$$\begin{aligned}\dot{x}_1 &= (1 - x_1)x_1 - \frac{2x_1 x_2}{1 + x_1} \\ \dot{x}_2 &= \left(2 - \frac{x_2}{1 + x_1}\right)x_2\end{aligned}$$

13.7 Phase plane and time evolution

Pair each phase plane plot (left) with its corresponding state evolution over time (right)!

a



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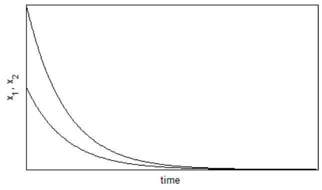

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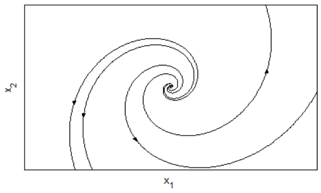

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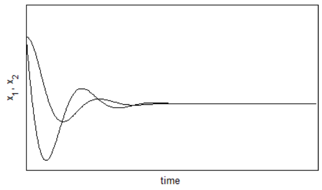

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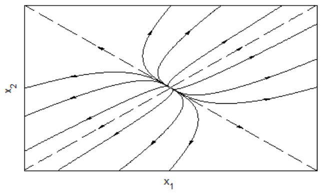

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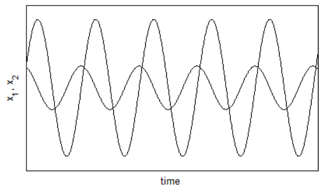

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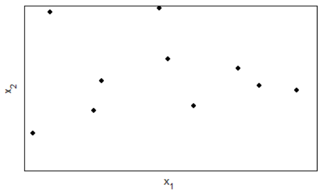

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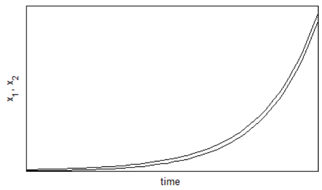

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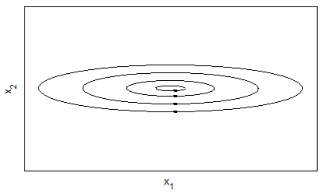

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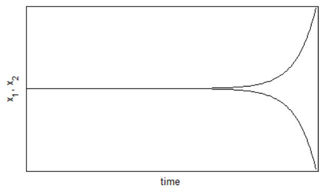

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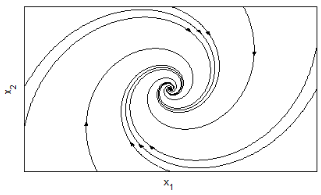

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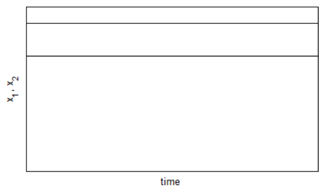

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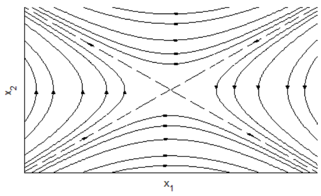

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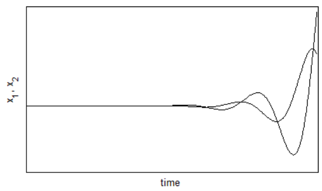
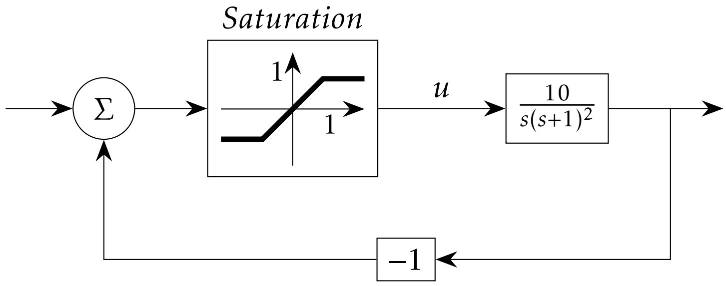


14 Oscillations and describing functions

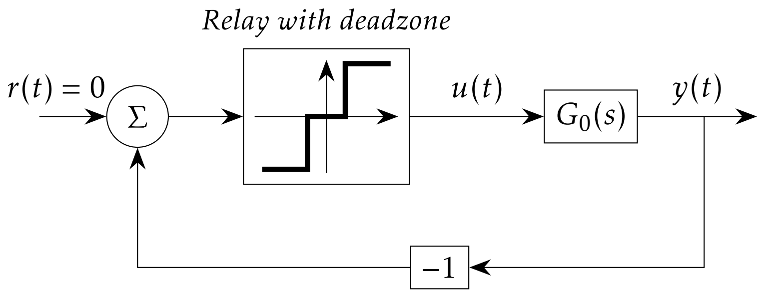
14.1 Limit cycle for a closed loop with saturation



Examine the stability of the system. If a limit cycle exists, determine its frequency and amplitude.

14.2 Limit cycle for a closed loop with deadzone relay

A temperature control system contains a relay with a deadzone:



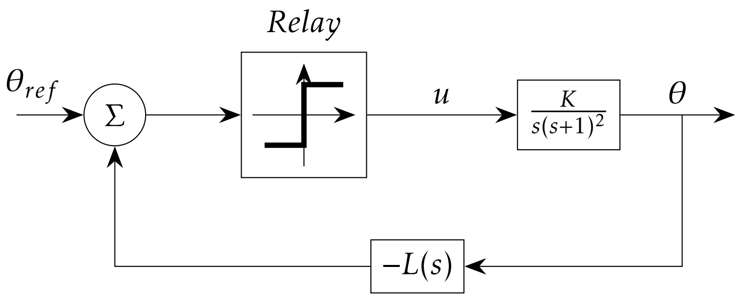
where $G_0(s) = \frac{1}{s(1+s)^2}$, $\pm D$ is the width of the deadzone, and $\pm H$ is its amplitude level. The parameters D and H have values such that a limit cycle barely exists (if H is decreased, or D is increased somewhat, there is no limit cycle in the system). Determine D, H and the frequency of the limit cycle.

The describing function for a relay with deadzone reads

$$\mathrm{Re}\{Y_f(C)\} = \frac{4H}{\pi \cdot C}\sqrt{1 - D^2/C^2}; \quad C \geq D$$
$$\mathrm{Im}\{Y_f(C)\} \equiv 0$$

14.3 Limit cycles for controller design I

Consider a servo system with a relay, as given in the figure below.



The gain K is positive.

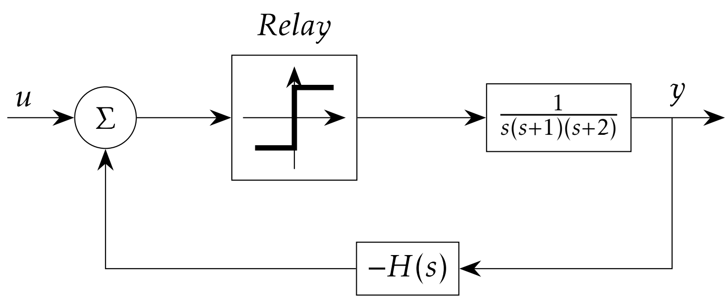
(a) Use the feedback $L(s) = 1$. Show that a limit cycle exists for all values of K.

(b) In order to reduce the wear of the system, it is desired that the amplitude in θ should be less than 0.1. For which values of K does this happen?

(c) It is desired to use a higher gain K than the value computed in part (b). Give a feedback controller $L(s)$ with $L(0) = 1$, which makes this possible. (No detailed calculations are requested. Give the transfer function $L(s)$ and motivate why it will solve the problem.)

14.4 Limit cycles for controller design II

Consider the nonlinear control system



(a) With the proportional regulator $H(s) = 1$, there is a limit cycle. Determine its amplitude and frequency.

(b) In order to eliminate the limit cycle, use the PD controller $H(s) = 1 + Ks$. Determine how K is to be chosen in order to eliminate the limit cycle.

18 Optimal control

18.1 The maximum principle

We want to control the system

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2^3 + (1 + x_1)u \end{cases}$$

with initial state $x_1(0) = 1, x_2(0) = 1$, such that

$$\int_0^1 e^{x_1^2} + x_2^2 + u^2 \, dt$$

is minimized. Find the optimal u!

(For this exercise, it is sufficient to express u using a system of differential equations; you do not have to present a closed-form expression of u.)

18.2 Optimal bang-bang control

We want to control the system

$$\dot{x} = \begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

from the initial state $x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $x(t_f) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in as short time as possible ($\min_{t_f}$), under the constraint

$$|u(t)| \leq 3.$$

Show that the optimal control law is

$$u(t) = \begin{cases} -3 & 0 \leq t \leq t_1 \\ +3 & t_1 \leq t \leq t_f \end{cases}$$

or

$$u(t) = \begin{cases} +3 & 0 \leq t \leq t_1 \\ -3 & t_1 \leq t \leq t_f \end{cases}$$

for some value of t_1.

18.3 The maximum principle and bang-bang control

Consider the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \quad |u| \leq 1 \end{cases}$$

with the criterion

$$\min \int_0^{t_f} \frac{1}{2} x_1^2 dt, \qquad x(0) = x_0, \qquad x(t_f) = 0$$

The final time t_f is fixed.

Write down the equations corresponding to the maximum principle. Show that they give a bang-bang control if a certain condition on λ id fulfilled. There is one obvious initial state $x(0)$ not giving a bang-bang control. Which one?