Automatic control III - Homework assignment 3

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November 10, 2017

Problem I

a)

We introduce

$$f_1(x_1, x_2) = x_2 \tag{1}$$
$$f_2(x_1, x_2) = \mu(1 - x_1^2)x_2 - x_1 \tag{2}$$

To determine the stationary point we set

$$f_1(x_1, x_2) = 0 \tag{3}$$
$$f_2(x_1, x_2) = 0 \tag{4}$$

which gives:

$$x_2 = 0 \tag{5}$$
$$x_1 = 0 \tag{6}$$

The next step is the linearization around the stationary points:

$$\frac{\partial f_1}{x_1} = 0 \tag{7}$$
$$\frac{\partial f_1}{x_2} = 1 \tag{8}$$
$$\frac{\partial f_2}{x_1} = -2\mu x_2 x_1 - 1 = -1 \tag{9}$$
$$\frac{\partial f_2}{x_2} = \mu(1 - x_1^2) = \mu \tag{10}$$

The eigenvalues are determined from

$$0 = det(\lambda I - \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}) = \lambda^2 - \mu\lambda + 1 \tag{11}$$

where

$$\lambda = \frac{\mu}{2} \pm \sqrt{\frac{\mu^2}{4} - 1} \tag{12}$$

From this we can characterize the stationary point in (0,0) in three different ways. For $0 < \mu < 2$ we have an unstable focus since λ_{12} is complex and has the real part > 0. For $\mu > 2$ we have an unstable node since $0 < \lambda_1 < \lambda_2$ and both are real. When $\mu = 2$ we have $\lambda_1 = \lambda_2 = 1$ which gives either an unstable one-point node or an unstable star node. To figure out which of them it is we look at the eigenvectors.

$$\begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} x_2 \\ -x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{14}$$

This gives an unstable one-point node since there are only one linearly independent eigenvector.

b)

Phase portraits for $\mu = 0.1, 1$ and 4 is presented in Figure 1, 2 and 3. In each case one can see that we have closed trajectories which means that we have a limit cycle in all three cases.

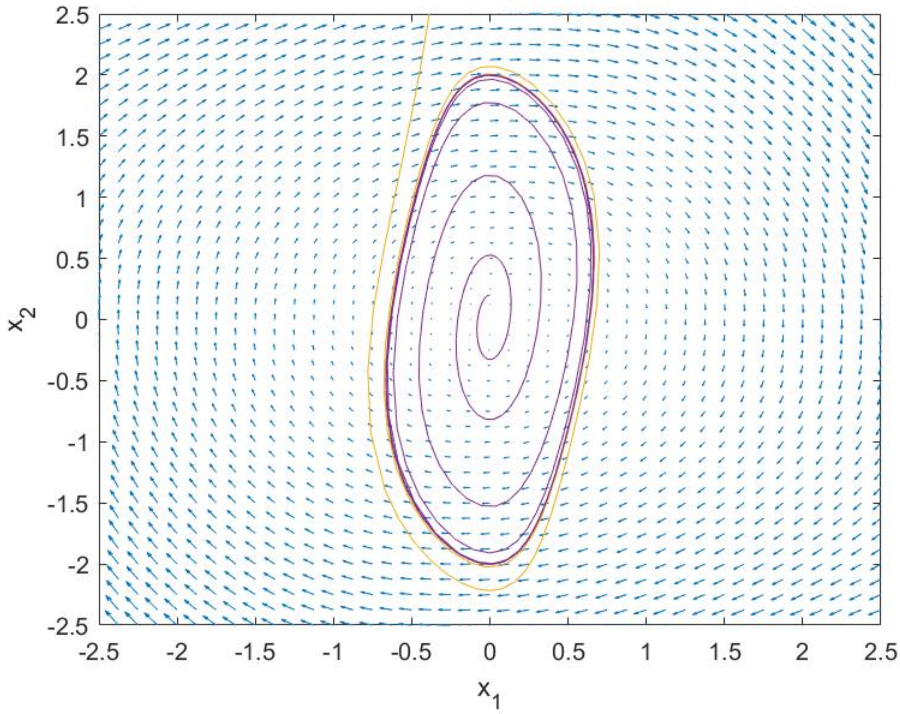

Figure 1: Phase portrait when $\mu = 0.1$ for three trajectories with different start values.

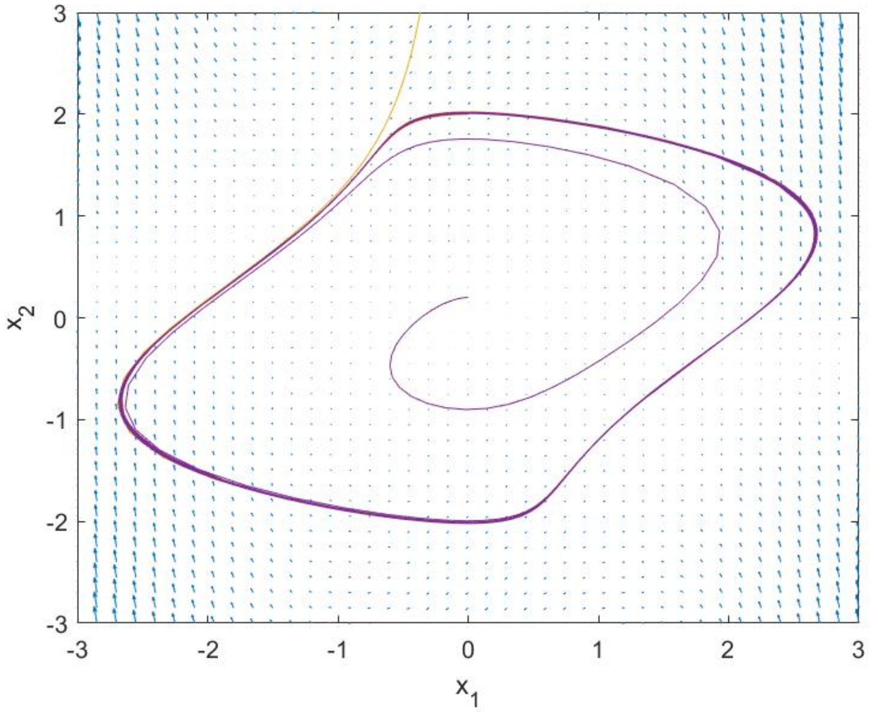

Figure 2: Phase portrait when $\mu = 1$ for three trajectories with different start values.

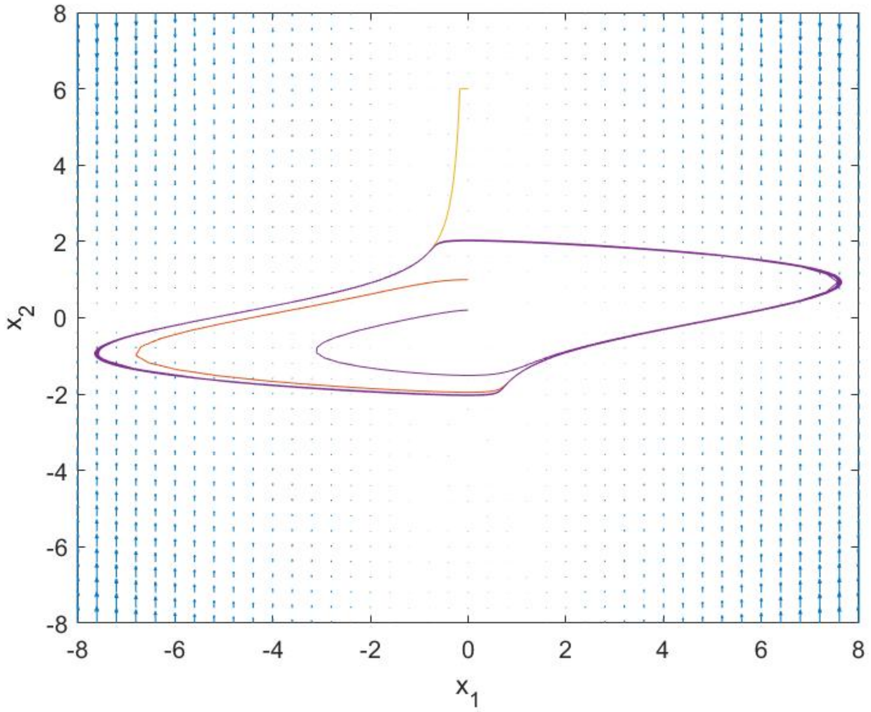

Figure 3: Phase portrait when $\mu = 4$ for three trajectories with different start values.

c)

We want to show that the system

$$\frac{d^2y}{dt^2} - \mu\frac{dy}{dt} + y = \frac{\mu}{3}\frac{du}{dt} \tag{15}$$

is equivalent with:

$$\frac{d^2y}{dt^2} - \mu(1-y^2)\frac{dy}{dt} + y = 0 \tag{16}$$

With use of the chain rule and rewriting the equation we get

$$\frac{du}{dt} = \frac{du}{dy}\frac{dy}{dt} = -3y^2\frac{dy}{dt} \Rightarrow \tag{17}$$

$$\frac{d^2y}{dt^2} - \mu\frac{dy}{dt} + y = -\mu y^2\frac{dy}{dt} \Rightarrow \tag{18}$$

$$\frac{d^2y}{dt^2} - \mu(1-y^2)\frac{dy}{dt} + y = 0 \tag{19}$$

which is equivalent with (14).

d)

G(s) is determined by laplace transform (13) and rewriting:

$$s^2Y - s\mu Y + Y = s\frac{\mu}{3}U \Rightarrow \tag{20}$$

$$(s^2 - s\mu + 1)Y = s\frac{\mu}{3}U \Rightarrow \tag{21}$$

$$\frac{Y}{U} = G(s) = \frac{\mu}{3}\frac{s}{s^2 - s\mu + 1} \tag{22}$$

Since the input to $f(\cdot)$ is $-y$ and the output is $-y^3$, $f(\cdot)$ is a cubic function.

e)

Since the minimum and maximum slope in the cubic funtion $f(x) = x^3$ is 0 and ∞ we have $k_1 = 0$ and $k_2 = \infty$. This corresponds to a circle that intersect the real axis in $-\frac{1}{k_1} = -\infty$ and $-\frac{1}{k_2} = 0$. In other words the circle fills the whole LHP.

The circle criterion is not fulfilled since the Nyquist curve enters the LHP. (When $\omega = 1$ $G(i\omega) = \frac{\mu}{3}\frac{i}{-\mu i} = -\frac{1}{3}$).

Note: I am not sure if the circle criteria can be used here since the poles in G(s) is in the RHP. From 12.7 in Reglerteori, Torkel Glad, Lennart Ljung: Anta att överföringsfunktionen G inte har poler i höger halvplan...

f)

The describing function is $Y_f(C) = \frac{b(C)+ia(C)}{C}$ where a(C) = 0 and

$$\begin{aligned} b(C) &= \frac{1}{\pi}\int_0^{2\pi} f(C\sin\alpha) sin\alpha \, d\alpha = && (23) \\ &= \frac{1}{\pi}\int_0^{2\pi} C^3 \sin^4\alpha \, d\alpha = \frac{1}{\pi}\frac{3\pi C^3}{4} = \frac{3C^3}{4} \Rightarrow && (24) \\ Y_f(C) &= \frac{3C^2}{4} && (25) \end{aligned}$$

We want $Y_f(C)G(i\omega) = -1$ to hold. Since $Y_f(C)$ is real this can only hold if $G(i\omega)$ is real. This happens when $\omega = 1$ which gives

$$G(i\omega) = G(i) = -\frac{1}{3}. \tag{26}$$

Now we can calculate C:

$$Y_f(C)G(i) = -\frac{C^2}{4} = -1 \Rightarrow C = \pm 2 \tag{27}$$

We have a stable limit cycle with amplitude 2 and frequency 1 $\forall \mu$. Simulation of the system for $\mu = 0.1, 1$ and 4 is presented in Figure 4, 5 and 6. For all three simulations the amplitude is 2. For simulation when $\mu = 0.1$ and 1 the frequency $\omega = 1$. In the last simulation $\omega \approx 0.6$.

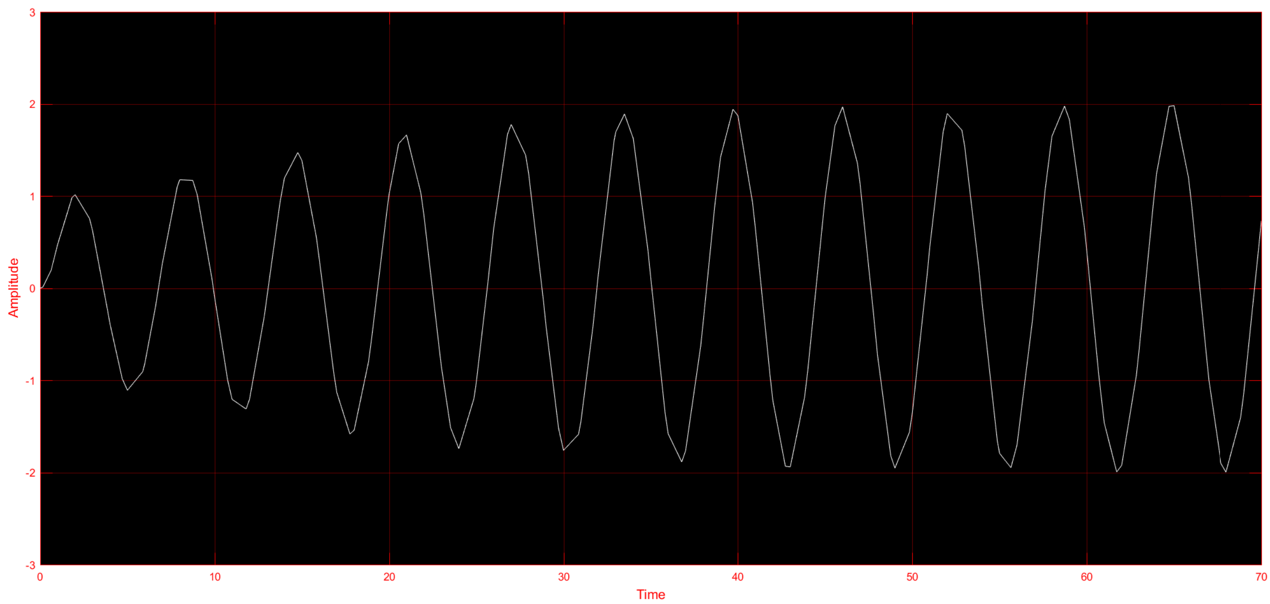

Figure 4: Simulation of the system when $\mu = 0.1$ using Simulink.

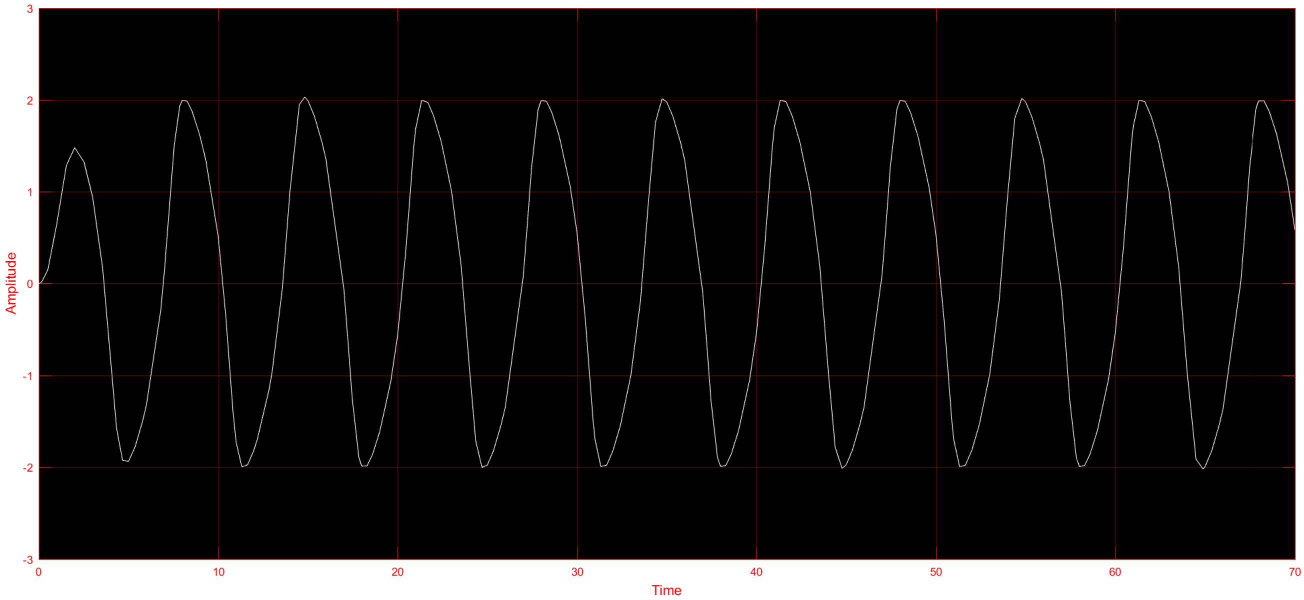

Figure 5: Simulation of the system when $\mu = 1$ using Simulink.

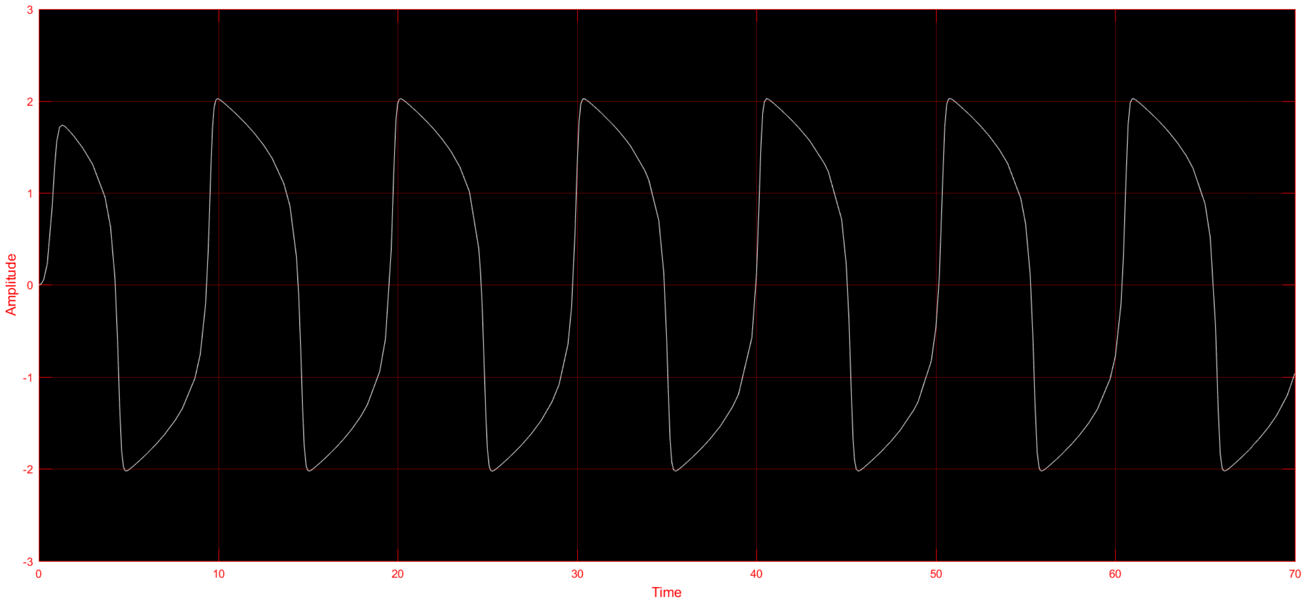

Figure 6: Simulation of the system when $\mu = 4$ using Simulink.

Problem II

a)

We want to solve the problem

$$Y_f(C)\tilde{G}(i\omega) = -1 \tag{28}$$

which is the same as

$$\tilde{G}(i\omega) = -\frac{1}{Y_f(C)} \tag{29}$$

where $\tilde{G}(i\omega)$ and Y_f are given by

$$\tilde{G}(i\omega) = KG(i\omega) = \frac{K}{i\omega(i\omega+1)} \tag{30}$$

$$Re(Y_f(C)) = \frac{1}{\pi}[\frac{\pi}{2} + arcsin(1-\frac{0.04}{C}) + 2(1-\frac{0.04}{C})\sqrt{\frac{0.02}{C}(1-\frac{0.02}{C})}] \tag{31}$$

$$Im(Y_f(C)) = -\frac{0.08}{\pi C}(1-\frac{0.02}{C}) \tag{32}$$

This is very hard (maybe impossible) to solve algebraic so I will solve this graphically. By plotting $\tilde{G}(i\omega)$ as a function of ω i.e. the Nyquist curve and $-\frac{1}{Y_f(C)}$ as a function of C in the same graph I can determine for which K's the curves intersect one an other. In other words, I can determine how large values of the gain K can be used so a limit cycle is to be avoided. After several attempts for different K, I concluded that the largest gain that can be used is ~ 3.4. Plots when $K = 3.7$ and $K = 3.4$ is shown in Figure 7, 8. In Figure 9 it can easily be seen that the curves does not intersect when $K = 3.4$ hence the limit cycle is avoided.

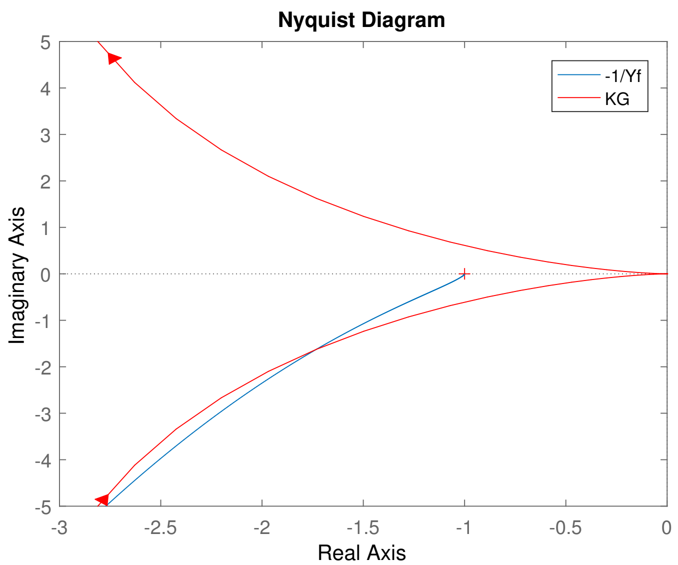

Figure 7: Nyquist curve and describing function when K=3.7.

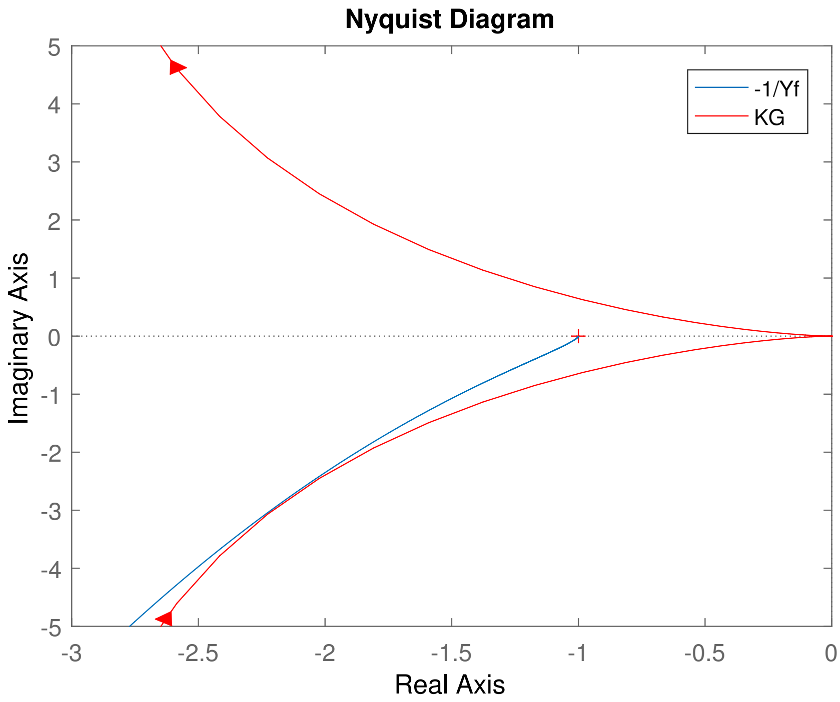

Figure 8: Nyquist curve and describing function when K=3.4.

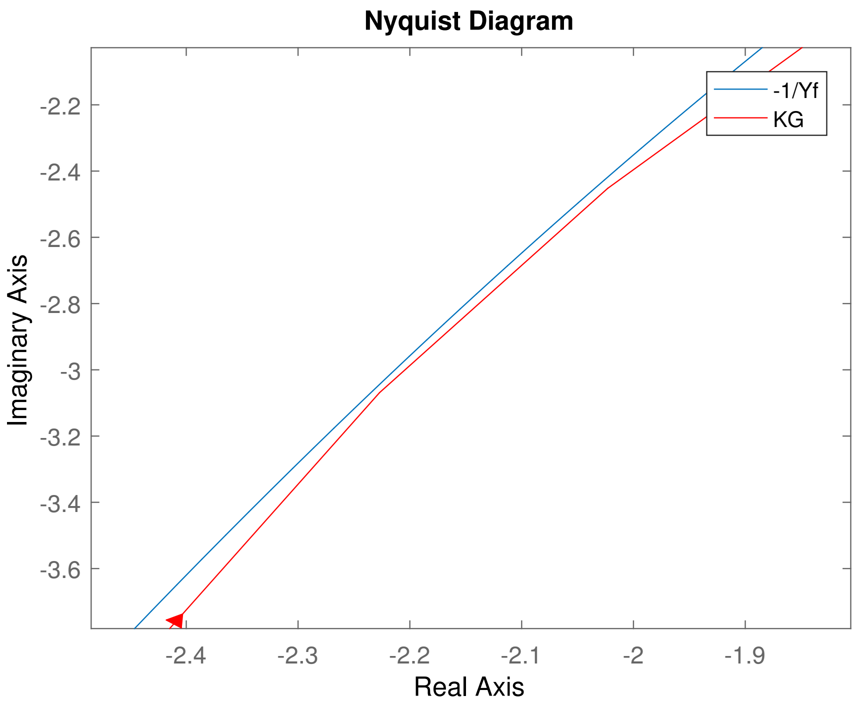

Figure 9: Nyquist curve and describing function when K=3.4 zoomed in.

Simulation of the system when $K = 3.4$ and $K = 10$ is presented in Figure 10 and 11.

The simulation in Figure 10 corresponds to when the limit cycle is avoided and therefore the oscillation decays. In Figure 11 the amplitude of the oscillation decays as well. This implies that the limit cycle is stable.

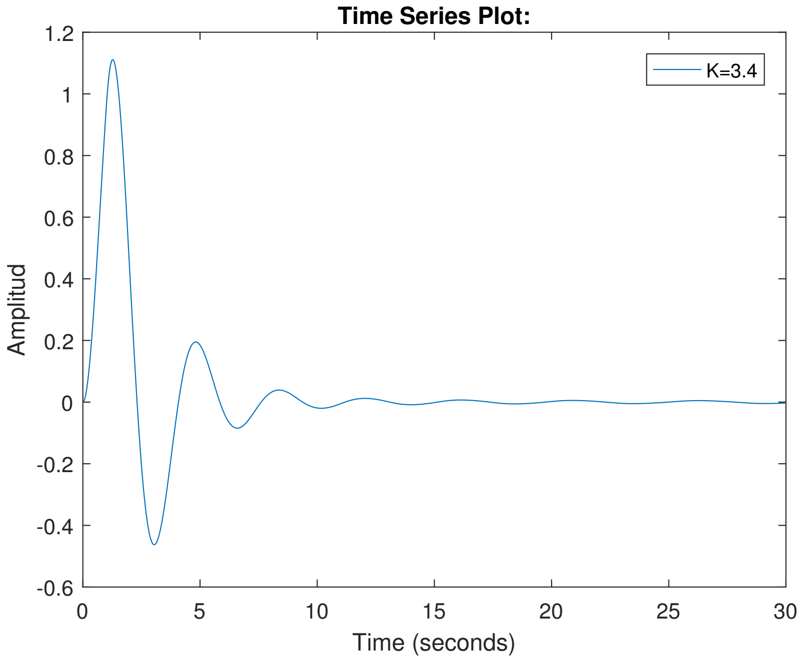

Figure 10: Simulation of the system when $K = 3.4$.

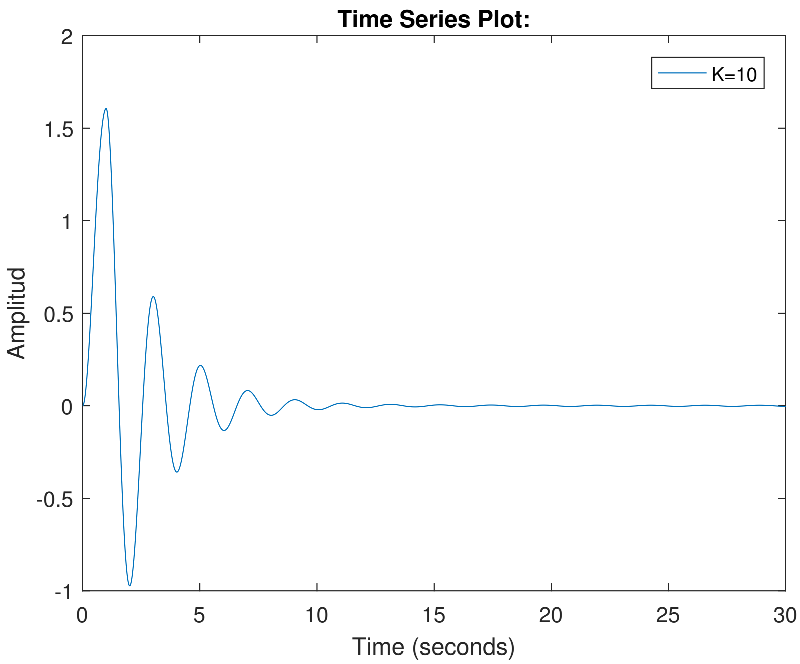

Figure 11: Simulation of the system when $K = 10$.

b)

I solved the problem using IMC and used n=2 to get a proper system Q.

$$Q = \frac{1}{(\lambda+1)^n} G(s)^{-1} = \frac{1}{(\lambda+1)^n} G(s)^{-1} = \frac{s^2+s}{\lambda s^2 + 2\lambda s + 1} \tag{33}$$

To achieve the requirements I chose $\lambda = 0.01$. In Figure 12 The step response of the system is presented. The rise time and the overshoot is within it's limits, there are no oscillations and the error goes to zero $\Rightarrow$ integral action. In Figure 13 the Nyquist curve and the describing function is presented. The curves does not intersect which means that a limit cycle does not exist.

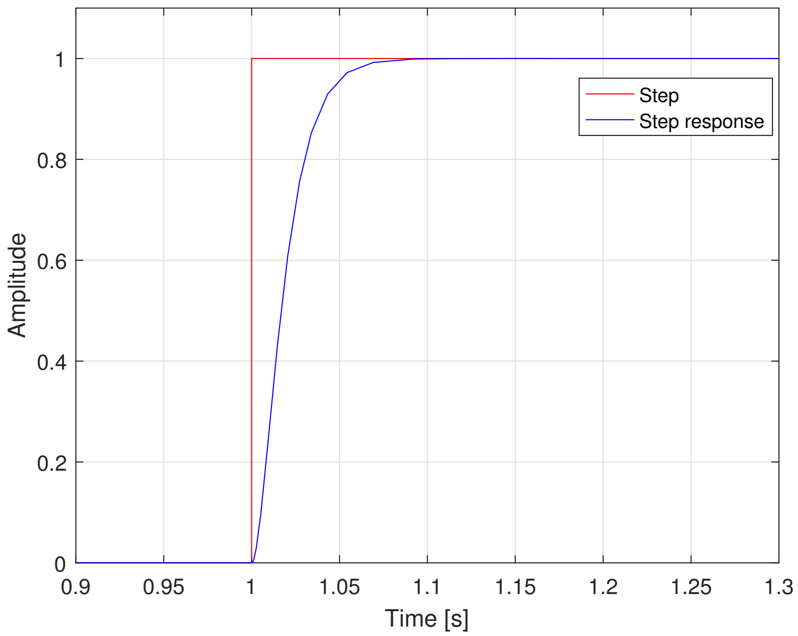

Figure 12: Step and step response of the system when $\lambda = 0.01$.

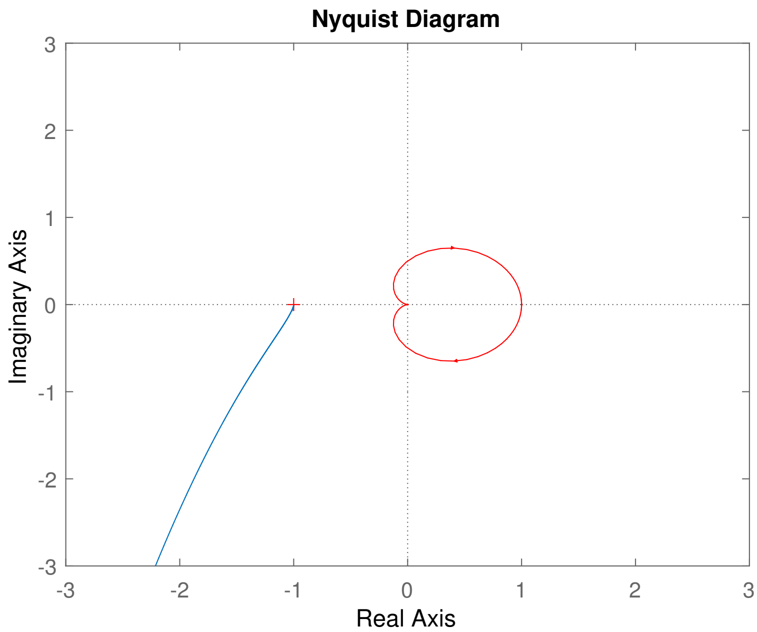

Figure 13: Nyquist curve and describing function when $\lambda = 0.01$.

Problem III

We want to minimize the criterion

$$\int_0^T (x(t) + u^2(t))\,dt \tag{34}$$

for the system

$$\dot{x}(t) = x(t) + u(t) + 1 \tag{35}$$

with $x(0) = 0$. We can use Theorem 18.2 in Glad/Ljung and identify the components.

$$\dot{x}(t) = f(x(t), u(t)) = x(t) + u(t) + 1 \tag{36}$$

$$L(x(t), u(t)) = x(t) + u^2(t) \tag{37}$$

$$t_f = T \tag{38}$$

$$\phi(x(t_f)) = 0 \tag{39}$$

$$H(x, y, \lambda) = L(x, u) + \lambda^T f(x, u) \tag{40}$$

$$min\, H = min\, x(t) + u^2(t) + \lambda(x(t) + u(t) + 1) \tag{41}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -1 - \lambda \tag{42}$$

$$min\, H : 0 = \frac{\partial H}{\partial u} = 2u + \lambda \Rightarrow u = -\frac{\lambda}{2} \tag{43}$$

What we have now is that the u that minimizes H is given by $u = -\frac{\lambda}{2}$. (It is a minimum point since $\frac{\partial^2 H}{\partial u^2} = 2 > 0$). We need to solve the differential equation for λ.

$$\dot{\lambda} = -1 - \lambda \tag{44}$$

$$\lambda = Ce^{-t} - 1 \tag{45}$$

With

$$\lambda(t_f) = \lambda(T) = \frac{\partial \phi}{\partial x} = 0 \tag{46}$$

we get

$$0 = Ce^{-T} - 1 \Rightarrow C = e^{T} \tag{47}$$

and

$$\lambda = e^{T-t} - 1. \tag{48}$$

Finally we found that the feedback control u(t) that minimizes the criterion is

$$u(t) = -\frac{e^{T-t} - 1}{2} = \frac{1 - e^{T-t}}{2} \tag{49}$$

The feedback control u and x as a function of time is presented in Figure 14 with T=3. One can relate the graph to the criterion that should be minimized (34) and see that it makes sense. The curves goes to different directions and the slope for x is much more steeper than the slope for u. This is expected since the integral in equation 34 consist of u^2 and x and we want to minimize is. One can also notice that $u(0)^2 \approx x(T)$.

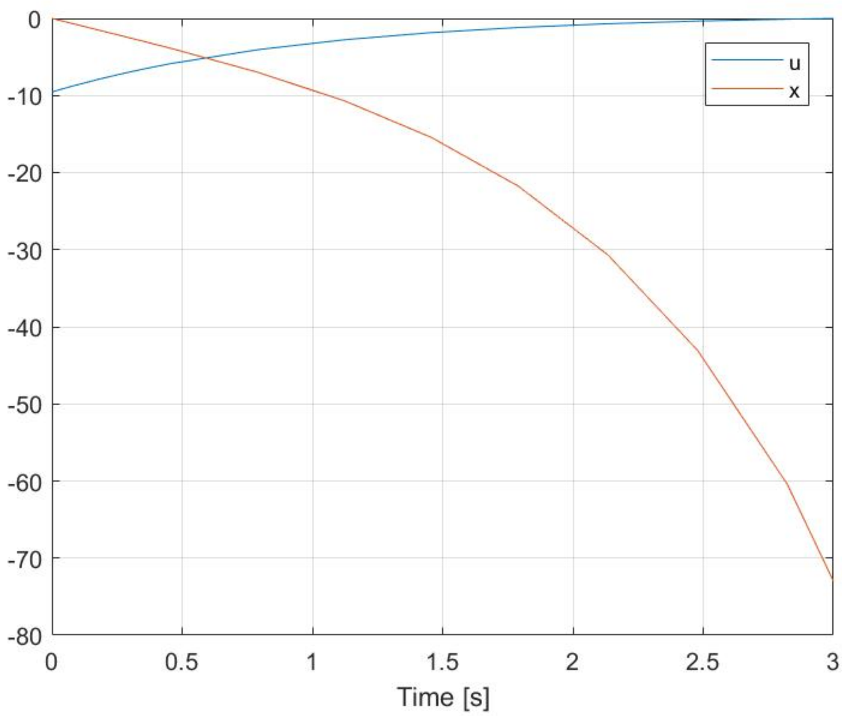

Figure 14: Simulation of the system when T=3.

Problem IV

Poles and zeros of MIMO systems are used to characterize and describe a linear systems behavior.

For linear SISO and MIMO systems, a closes loop system is internally stable if $S_u, G_{w_u u}, G_{wu}, S$ and F_r are all stable.

How good a controller can be is limited with it's performance limitations, for example we need to make trade offs and compromises of how to handle T, S and $G_w u$ because the requirements of these cannot be fulfilled for all frequencies.

RGA is used on linear MIMO systems to measure the level of cross coupling or interaction from input to output signals.

The concept IMC is used on MIMO-systems when describing and calculating a feedback controller based on the "new information" $y - Gu$ through a transfer function Q.

H_2 design is implemented on linear MIMO-systems when we want to minimize the cost function with respect to F_y.

H_∞ design is implemented on linear MIMO-systems when we want to minimize the H_∞-norm i.e. the biggest singular value of $G_{ec}(i\omega)$.

An equilibrium point is a point where the system is in rest.

A phase plane is a two dimensional state space where the characteristics of the equilibrium points easily can be visualized.

A function V that fulfills $V_X(x(t))f(x(t)) \leq 0$ around an equilibrium point is called a Lyapunov function and is used for both linear and nonlinear systems

Describing function is an approximate method for examining existence of periodic solutions for systems involving a static nonlinearity in the feedback loop.

The circle criterion is used on nonlinear time-varying systems for examination of the stability i.e. generalizing the Nyquist criterion.

Optimal control is kind of an expansion of LQ-control and is more general where you use mathematical optimization methods to get to the best solution.