# Automatic Control III

*Lecture 7 – Lyapunov theory*

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# Summary of lecture 6 (I/II)

We can approximate a nonlinear system

$$\dot{x} = f(x, u), \qquad y = h(x, u),$$

by linearizing the system around an equilibrium (stationary) point $(x_0, u_0)$. Intuitively this amounts to approximating the system by a flat hyperplane (straight line in the scalar case).

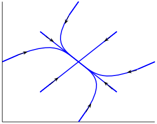
Let $\Delta x(t) = x(t) - x_0, \quad \Delta u(t) = u(t) - u_0, \quad \Delta y(t) = y(t) - y_0.$

A Taylor expansion (only keeping the linear terms) results in

$$\begin{aligned}
\frac{d}{dt}\Delta x &= \overbrace{\frac{\partial f(x_0, u_0)}{\partial x}}^{A} \Delta x + \overbrace{\frac{\partial f(x_0, u_0)}{\partial u}}^{B} \Delta u, \\
\Delta y &= \underbrace{\frac{\partial h(x_0, u_0)}{\partial x}}_{C} \Delta x + \underbrace{\frac{\partial h(x_0, u_0)}{\partial u}}_{D} \Delta u.
\end{aligned}$$

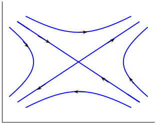
# Summary of lecture 6 (II/II)

- Node



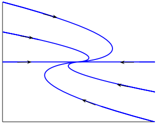
(Two real e.v. same sign)

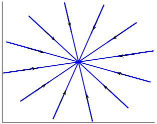
- Saddle



(Two real e.v. different sign)

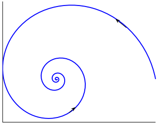
- One eigenvector node and star node

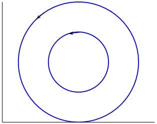




(two equal eigenvalues)

- Focus and center





(Complex eigenvalues)

# Phase portraits for nonlinear systems

Close to an equilibrium point the dynamics are determined by the linearized dynamics.

- If the linearized system has a node, a focus, or a saddle point, the same applies for the nonlinear system.
- If the linearized system has a **center**, the nonlinear system has a center or a focus.
- If the linearized system has a **star node**, additional conditions have to be checked for the nonlinear system.

# Phase portraits far away from equilibrium

Second-order nonlinear system

$$\dot{x}_1 = f_1(x_1, x_2),$$
$$\dot{x}_2 = f_2(x_1, x_2).$$

Equation for the trajectories ("eliminate time")

$$\frac{dx_2}{dx_1} = \frac{dx_2}{dt}\frac{dt}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}.$$

- The trajectories have slope 0 when $f_2(x_1, x_2) = 0$.
- The trajectories have slope $\infty$ when $f_1(x_1, x_2) = 0$.
- Study the limits

$$\lim_{x_1 \to \pm\infty} \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}, \qquad \lim_{x_2 \to \pm\infty} \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)},$$

to find out what the phase portrait looks like far from the origin.

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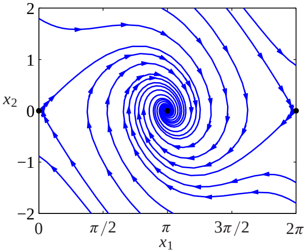
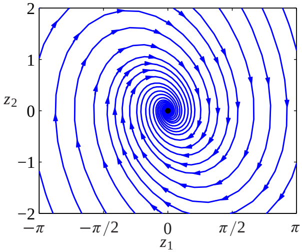
# Phase portrait of an inverted pendulum

Open loop (autonomous) dynamics

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = \sin x_1 - \gamma x_2.$$




Left: Nonlinear phase portrait. Right: Phase portrait of the linearized dynamics.

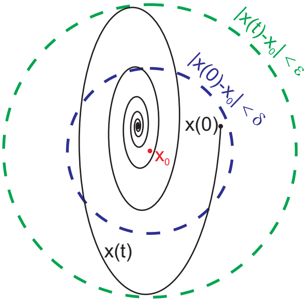
# Control joke

**You might be a control engineer if**

You have ever used the term 'going nonlinear' to describe human behavior

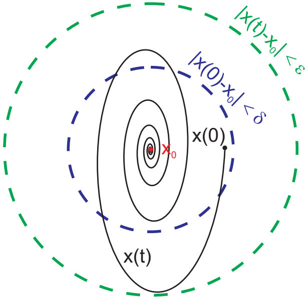
# Stability of an equilibrium

Given the requirement that the solution must remain within a distance of $\epsilon$ from the equilibrium point $x_0$ for all future $t$, we must be able to find a ball of radius $\delta$ centered in $x_0$, such that the requirement is fulfilled for all starting points $x(0)$ in this ball.



# Asymptotic stability of an equilibrium

If the system is started sufficiently close to $x_0$, it will eventually end up in $x_0$ as $t \to \infty$.




Globally asymptotically stable if asymptotic stability holds for the equilibrium in the whole state space.

# Lyapunov functions and stability

$$V_x(x) = [\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}]$$

**Theorem:** If a Lyapunov function $V$ satisfying

$$V_x(x(t))f(t) < 0, x \neq x_0, \qquad V(x) \to \infty \quad \text{as} \quad |x| \to \infty$$

can be found, then the equilibrium point $x_0$ is globally asymptotically stable.

The tricky part is to **find** the Lyapunov function!

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# Lyapunov functions for linear systems

**Lemma:** 1. If $A$ has all its eigenvalues in the LHP, then for every matrix $Q = Q^T > 0$ (or $Q \geq 0$), there is a matrix $P = P^T > 0$ (or $P \geq 0$) that satisfies the Lyapunov equation,

$$A^T P + PA = -Q.$$

2. If there are matrices $P = P^T \geq 0$ and $Q = Q^T \geq 0$ satisfying the Lyapunov equation and $(A, Q)$ is detectable, then $A$ has all its eigenvalues strictly in the LHP.

Not very useful in itself, since we already new this... However, it opens up for more interesting results.

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# Lyapunov functions for linearized systems

Consider the nonlinear system

$$\dot{x} = Ax + g(x),$$

where $g(x)$ is sufficiently small close to the origin. Let $A$ have eigenvalues in the LHP and let $P$ and $Q$ be positive definite matrices fulfilling

$$A^T P + PA = -Q.$$

Then $V(x) = x^T Px$ is a Lyapunov function also for the nonlinear system in an area around the origin.

# A few concepts to summarize lecture 7

**Globally asymptotically stable:** The equilibrium point $x_0$ is globally asymptotically stable (GAS) if it is stable and $x(t) \to x_0, t \to \infty$ for every starting point $x(0)$.

**Lyapunov function:** A Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ is a generalized distance from $x$ to an equilibrium point $x_0$, satisfying $V(x_0) = 0, V(x) > 0, x \neq x_0$ and $V_x(x) f(x) \neq 0$.

**Lyapunov equation:** The Lyapunov equation $A^T P + PA = -Q$ can be used to find Lyapunov functions for linear (and almost linear) systems.