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Automatic Control III

Lecture 10 – Summary of the course

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Summary of lecture 9 (I/II)

Characteristics of an optimal control problem:

1. Infinite dimensional optimization problem (chose $u(t)$ for $0 \leq t \leq t_f$).
2. The cost function is related to the control signal via the system dynamics.
3. Constraints on the control signal.
4. Constraints on the states at the start and end.
5. The final time t_f is unknown and part of the problem.

Summary of lecture 9 (II/II)

Theorem [the maximum principle]: Assume that the optimization problem

$$\begin{aligned}
&\min \phi(x(t_f)) \\
&\dot{x}(t) = f(x(t), u(t)), \\
&u(t) \in U, \quad 0 \le t \le t_f, \\
&x(0) = x_0.
\end{aligned}$$

has a solution $u^*(t), x^*(t)$. Then it must hold that

$$\min_{u \in U} \lambda^T(t) f(x^*(t), u) = \lambda^T(t) f(x^*(t), u^*(t)), \qquad 0 \le t \le t_f,$$

where $\lambda(t)$ fulfils

$$\dot{\lambda}(t) = -f_x(x^*(t), u^*(t))^T \lambda(t), \qquad \lambda(t_f) = \phi_x(x^*(t_f))^T.$$

Measuring the size of signals and systems

We formalized how to measure the **size** of signals and systems using various norms.

Signal size

$$||z||_2^2 = \int_{-\infty}^{\infty} |z(t)|^2 dt = \int_{-\infty}^{\infty} z^T(t) z(t) dt,$$
$$||z||_\infty = \sup |z(t)|.$$

The gain of system $\mathcal{S}$, where $y = \mathcal{S}(u)$:

$$\|\mathcal{S}\| = \sup_u \frac{\|y\|_2}{\|u\|_2} = \sup_u \frac{\|\mathcal{S}(u)\|_2}{\|u\|_2}$$

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Gain of linear systems

The gain of a linear stable SISO system G was shown to be

$$||G|| = \sup_{\omega} |G(i\omega)|.$$

For a multivariable system with transfer matrix $G(s)$ we showed: Gain $= ||G||_\infty = \sup_\omega \bar{\sigma}(G(i\omega))$, where

$$\bar{\sigma}(G(i\omega)) = \text{the largest singular value of } G(i\omega)$$

Plotting the singular values of $G(i\omega)$ as a function of ω corresponds to the plot of the amplitude curve for SISO systems.

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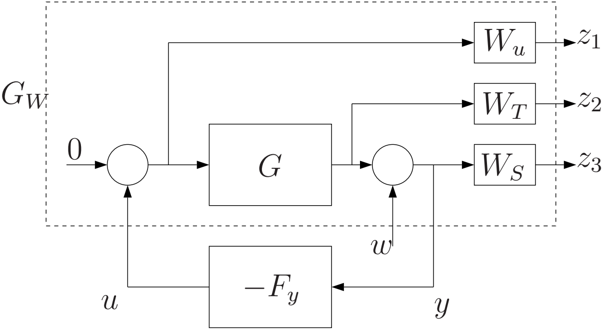
An extended "imaginary" system G_W



When we close the system using $u = -F_y y$ we have:

$$z_1 = W_u G_{wu} w, \quad z_2 = -W_T T w, \quad z_3 = W_S S w$$

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$\mathcal{H}_2$ control

Definition: The $\mathcal{H}_2$-norm of the system $y = G(p)u$ is given by

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|_2^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\left(G^*(i\omega)G(i\omega)\right) d\omega.$$

Design criterion $\mathcal{H}_2$-design: Choose the controller such that

$$V = \|W_S S\|_2^2 + \|W_T T\|_2^2 + \|W_u G_{wu}\|_2^2$$

is minimized.

We proved that an equivalent problem is given by minimizing

$$\|z\|_2^2 = \|Mx\|_2^2 + \|u\|_2^2,$$

which is an **LQG problem!**

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$\mathcal{H}_\infty$ control

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$$\|G\|_\infty = \sup_u \frac{\|y\|_2}{\|u\|_2} = \sup_\omega \bar{\sigma}(G(i\omega))$$

Design objective $\mathcal{H}_\infty$-design: Find the controller that minimize

$$\|G_{ec}\|_\infty = \max_\omega \bar{\sigma}(G_{ec}(i\omega)).$$

This is a hard (non-convex) problem, instead we search for controllers that satisfies

$$\|G_{ec}\|_\infty < \gamma,$$

by solving a sequence of Riccati equations.

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Basic limitations and conflicts

- All signal amplitudes have limits.
- $S(s) + T(s) = 1.$
- For stable systems we derived **Bode's integral theorem**,

$$\int_0^\infty \log |S(i\omega)| d\omega = 0.$$

- Compromise between S and T (Bode's relation)
- Poles in the RHP (unstable poles)
- Zeros in the RHP (Non-minimum phase zeros)
- Bounded inputs
- Time delays (steal phase)
- Model uncertainties (robustness)

Lyapunov functions

A Lyapunov function $V(x)$ "measures the distance to the goal":

- Let $V(x)$ denote a (generalized) distance from x to an equilibrium point x_0.
- The distance must remain positive until the system has arrived in the equilibrium point x_0,

$$V(x) > 0, \qquad x \neq x_0, \qquad V(x_0) = 0.$$

- The distance must decrease until the final destination is reached,

$$\frac{d}{dt}V(x(t)) = V_x(x(t))\dot{x}(t) = V_x(x(t))f(x(t)) < 0, \, x(t) \neq x_0.$$

- If the system "diverge", this must be clearly visible

$$V(x) \to \infty, \;\; |x| \to \infty.$$

Lyapunov stability

Theorem: If a Lyapunov function V satisfying

$$V_x(x(t))f(t) < 0, x \neq x_0, \qquad V(x) \to \infty \text{ as } |x| \to \infty$$

can be found, then the equilibrium point x_0 is globally asymptotically stable.

The tricky part is to **find** the Lyapunov function!

Summary for the exam (in one slide)

- Signal sizes and gains, singular values
- Small gain theorem and the circle criterion
- Computing poles and zeros for transfer matrices
- Block scheme calculations for MIMO systems
- Stability – internal stability
- Basic limitations and conflicts – general understanding
- The pairing problem and RGA
- Decentralized and decoupled control
- IMC, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers
- Computing and using linearizations
- Understanding and using phase portraits
- Lyapunov stability
- Describing functions

Summary for life

What should you remember from automatic control?

- The **principles**: Feedback (and feed forward)
- Stability and instability: That they **exist**
- The **possibilities**: Use automatic control where it has never been used before.

The TSTF-principle: **Try Simple Things First**

Summary for life

What should you remember from automatic control?

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Automatic control is used almost everywhere

We have seen many examples where automatic control has been successfully used.

To summarize it is fair to say that automatic control is used almost everywhere, but it is often hidden.

Automatic control is sometimes refereed to as the **"hidden science"**.

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Control engineering joke

You might be a control engineer if

You wonder why the term "control freak" is considered an insult.

Tack!

"Automatic control is the art of getting things to behave as you want."

"Tack för mig och lycka till med allt ni tar er för framöver!!"