



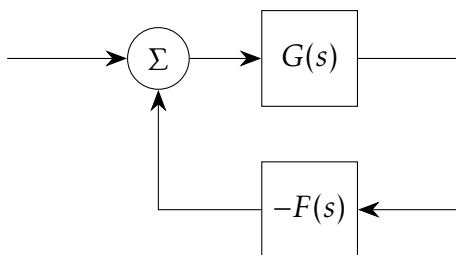
UPPSALA
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Solutions
Automatic Control III
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1 Introduction

1.1 Small gain & Nyquist Criterion

Suppose $G(s)$ and $F(s)$ are stable transfer functions in the closed-loop system:



The closed-loop system is then stable if $|G(i\omega)| \cdot |F(i\omega)| < 1, \quad \forall \omega$.

(A more restrictive version of sufficient condition is: $\sup_{\omega} |G(i\omega)| \cdot \sup_{\omega} |F(i\omega)| < 1 \Rightarrow \|G\| \cdot \|F\| < 1$ since $|G(i\omega)| \cdot |F(i\omega)| \leq \|G\| \cdot \|F\|$)

Proof: The closed-loop system is given by $G_c(s) = \frac{G(s)}{1+G(s)F(s)}$ and is, according to the Nyquist criterion, stable if the Nyquist curve for $G(i\omega)F(i\omega)$ is *not* encircling -1 . (Using that G as well as F are stable.)

That is: If $|G(i\omega)F(i\omega)| \leq |G(i\omega)| \cdot |F(i\omega)| < 1$, we can see that the Nyquist criterion is fulfilled, and we are done.

(Note that asymptotic stability \Rightarrow input-output stability. Input-output stability is used in the small gain theorem.)

1.2 Gain of ideal relay

We have $y(t) = f(u(t))$ where $f(\cdot)$ describes the ideal relay.

The gain is

$$\|f\| = \sup_u \frac{\|y\|_2}{\|u\|_2}$$

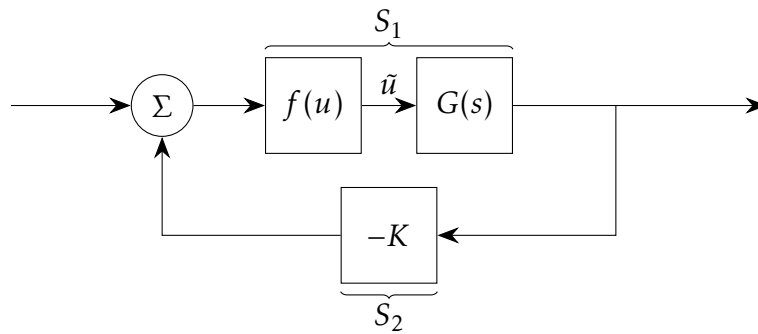
We have that $|f(u)| \equiv 1, \quad \forall u(t) \neq 0$, which implies

$$\|y\|_2^2 = \int_{-\infty}^{\infty} y^2(t) dt = \lim_{T \rightarrow \infty} \int_{-T}^T [f(u(t))]^2 dt = \lim_{T \rightarrow \infty} 2T = \infty$$

for all choices of inputs $u(t) \neq 0$ such that $0 < \|u\|_2 < \infty$. (For example, take $u(t) = \frac{1}{t}$)

Hence: An ideal relay has infinite gain.

1.3 Small gain theorem for linear system with saturation



According to the small gain theorem the above system is stable if $\|S_1\| \cdot \|S_2\| < 1$.

$$\text{We have: } \begin{cases} \|S_1\| \leq \|f(u)\| \cdot \|G\| \\ \|S_2\| = K \end{cases}$$

where

$$\|G\| = \sup_{\omega} |G(i\omega)| = \sup_{\omega} \frac{2}{\sqrt{(2 - \omega^2)^2 + 4\omega^2}} = 1 \quad (\text{for } \omega = 0)$$

$$\begin{aligned} \|f(u)\|^2 &= \frac{\|\tilde{u}\|_2^2}{\|u\|_2^2} = \frac{\int_{-\infty}^{\infty} (f(u(t)))^2 dt}{\|u\|_2^2} \leq \left[|f(u(t))| \leq \frac{1}{2} |u(t)| \right] \\ &\leq \frac{\frac{1}{4} \|u\|_2^2}{\|u\|_2^2} = \frac{1}{4} \Rightarrow \|f(u)\| \leq \frac{1}{2} \end{aligned}$$

$$\|S_1\| \cdot \|S_2\| \leq \frac{1}{2} \cdot K < 1$$

i.e., we have to choose $K < 2$ to guarantee (input-output) stability.

1.4 Signal norms

- a) $\|y\|_\infty = |a|, \quad \|y\|_2 = \infty$
b) $\|y\|_\infty = 1, \quad \|y\|_2 = 1$
c) $\|y\|_\infty = \frac{1}{4}, \quad \|y\|_2 = \frac{1}{\sqrt{12}}$

1.5 Gain for second order linear system

$$\|G\| = 1, \quad \zeta > \frac{1}{\sqrt{2}}$$
$$\|G\| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}, \quad 0 < \zeta < \frac{1}{\sqrt{2}}$$

1.6 Small gain & Nyquist Criterion

- I. $a > 0 \Rightarrow$ The linear system is stable. Small gain theorem \Rightarrow stable if $|K| < 1$.
Linear theory \Rightarrow Stable if $K < 1$.
- II. $a < 0 \Rightarrow$ The linear system is unstable \Rightarrow we can not use the small gain theorem.
Linear theory \Rightarrow stable if:
 $a(1 - K) > 0$ (the pole comes from $s + a(1 - K) = 0$) $\Rightarrow K > 1$.

1.7 Small gain theorem with nonlinear static feedback

The gain of f is 0.5. The gain of G is 1.5. As

$$0.5 \cdot 1.5 < 1$$

the closed-loop system is stable according to the small-gain theorem.

3 Properties of linear systems

3.1 Poles and zeros of MIMO systems I

The system has the following minors:

$$(1 \times 1): \quad \frac{2}{s+1}, \quad \frac{3}{s+2}, \quad \frac{3}{s+2}, \quad \frac{1}{s+1}, \quad \frac{1}{s+1}, \quad \frac{1}{s+1}$$
$$(2 \times 2): \quad \frac{-s+1}{(s+1)^2(s+2)}, \quad \frac{-s+1}{(s+1)^2(s+2)}, \quad \frac{0}{(s+1)(s+2)}.$$

Least common denominator is $(s+1)^2(s+2)$ and the poles become $\{-1, -1, -2\}$.

To determine the zeros, normalize the maximal minors s.t. the pole polynomial is in the denominator.

$$(2 \times 2): \quad \frac{-s+1}{(s+1)^2(s+2)}, \quad \frac{-s+1}{(s+1)^2(s+2)}, \quad \frac{0}{(s+1)^2(s+2)}.$$

The largest common denominator to the numerators is $(-s+1)$. Hence, there is a multivariable zero in $\{+1\}$.

3.2 Poles and zeros of MIMO systems II

The transfer function has the determinant

$$\det G(s) = \frac{2}{(s+3)^2}$$

and the minors (the matrix elements, in this case) of lower order

$$\frac{1}{(s+1)(s+3)}, \quad \frac{-1}{(s+1)(s+3)}, \quad \frac{2(s+1)}{(s+3)}$$

The pole polynomial, that is the least common denominator for all the minors, will hence become

$$p(s) = (s+1)(s+3)^2$$

which means that the system has its poles in $-1, -3$ and -3 .

The minor of maximal order is

$$\frac{2}{(s+3)^2}$$

and after normalization with the pole polynomial, it becomes

$$\frac{2(s+1)}{(s+1)(s+3)^2}$$

The zero polynomial is therefore

$$n(s) = (s + 1)$$

which gives that the system has a zero in $s = -1$.

3.3 Poles and zeros of MIMO systems III

Minors

$$\underbrace{\frac{1-s}{(s+1)^2}}_{\text{mult. 2}}, \quad \frac{2-s}{(s+1)^2}, \quad \frac{1/3-s}{(s+1)^2}, \quad \frac{1/3}{(s+1)^3}$$

We have 3 poles in $s = -1$ ($\text{LCD} = (s+1)^3$). Hence a minimal realization must be of order 3.

3.4 Poles and zeros of MIMO systems IV

- a) The poles and eigenvalues of a square system is given by the determinant.

$$\begin{aligned} \det(G(s)) &= 2 \frac{s+1}{s+3} - \frac{(s-2)(s+6)}{(s+3)(s+1)} \\ &= \frac{2(s+1)^2 - (s-2)(s+6)}{(s+1)(s+3)} \\ &= \frac{s^2 + 14}{(s+1)(s+3)} \end{aligned}$$

The zeros are the roots of the numerator polynomial. The poles are the roots of the denominator polynomial.

Answer: The zeros are $z = \pm i\sqrt{14}$ and the poles are $p = -1, -3$

- b) G_{22} is strictly stable because the denominator has strictly positive coefficients and is second order. The other elements are stable. Because the poles of the system is the poles of the elements (disregarding multiplicity), G is strictly stable. Assuming constant input signal u_0 (step input), the output in stationarity will be

$$y_\infty = \lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} sG(s)u_0 \frac{1}{s}$$

according to the final value theorem. The limit exists because $sG(s)u_0 \frac{1}{s}$ is strictly stable.

$$\lim_{s \rightarrow 0} G(s)u_0 = G(0)u_0 = \begin{pmatrix} -1 & \frac{1}{3} \\ 3 & -1 \end{pmatrix} u_0$$

It is seen that $\text{span}(G(0)) = \text{span}\left(\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right)$. Because $y_{ref} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \notin \text{span}(G(0))$, there is no u_0 such that y_{ref} can be attained.

Answer: Because the static gain of the system does not span the desired output direction, it is impossible to achieve y_{ref} with a constant input signal.

3.5 Poles, zeros and singular values for parallel tanks

- (a) The transfer function becomes

$$\begin{aligned} G(s) &= (sI - A)^{-1} \\ &= \frac{1}{s^2 + 2(1 + \alpha)s + 1 + 2\alpha} \begin{pmatrix} s + 1 + \alpha & \alpha \\ \alpha & s + 1 + \alpha \end{pmatrix} \end{aligned}$$

- (b) The poles of the system are the eigenvalues of the matrix A , that is the solutions to

$$0 = \det(sI - A) = (s + 1 + \alpha)^2 - \alpha^2 = (s + 1)(s + 1 + 2\alpha)$$

One eigenvalue is hence always in $s = -1$, while the other is in $s = -1 - 2\alpha$.

- (c) The zeros of the system can be found as the poles of $G^{-1}(s)$ (as the system has an equal number of inputs and outputs). Noting that

$$G^{-1}(s) = sI - A = \begin{pmatrix} s + 1 + \alpha & -\alpha \\ -\alpha & s + 1 + \alpha \end{pmatrix}$$

which has no poles, we note that the system has no zeros.

- (d) The singular values are the square roots of the eigenvalues of $G^T(i\omega)G(-i\omega)$. Using the results of parts (a) and (b), we get

$$\begin{aligned} G^T(i\omega)G(-i\omega) &= \frac{1}{|(i\omega + 1)(i\omega + 1 + 2\alpha)|^2} \times \\ &\quad \times \begin{pmatrix} i\omega + 1 + \alpha & \alpha \\ \alpha & i\omega + 1 + \alpha \end{pmatrix} \begin{pmatrix} -i\omega + 1 + \alpha & \alpha \\ \alpha & -i\omega + 1 + \alpha \end{pmatrix} \\ &= \frac{1}{(\omega^2 + 1)(\omega^2 + (1 + 2\alpha)^2)} \times \\ &\quad \times \begin{pmatrix} (1 + \alpha)^2 + \omega^2 + \alpha^2 & 2\alpha(1 + \alpha) \\ 2\alpha(1 + \alpha) & (1 + \alpha)^2 + \omega^2 + \alpha^2 \end{pmatrix} \end{aligned}$$

We therefore find that the singular values satisfy

$$\begin{aligned} \sigma^2(i\omega) &= \frac{(1 + \alpha)^2 + \omega^2 + \alpha^2 \pm 2\alpha(1 + \alpha)}{(\omega^2 + 1)(\omega^2 + (1 + 2\alpha)^2)} \\ &= \begin{cases} \frac{(1 + 2\alpha)^2 + \omega^2}{(\omega^2 + 1)(\omega^2 + (1 + 2\alpha)^2)} = \frac{1}{\omega^2 + 1} \\ \frac{(1 + \omega^2)}{(\omega^2 + 1)(\omega^2 + (1 + 2\alpha)^2)} = \frac{1}{\omega^2 + (1 + 2\alpha)^2} \end{cases} \end{aligned}$$

Hence,

$$\begin{aligned}\sigma_1(\omega) &= \sqrt{\frac{1}{\omega^2 + 1}} \\ \sigma_2(\omega) &= \sqrt{\frac{1}{\omega^2 + (1 + 2\alpha)^2}}\end{aligned}$$

3.6 Poles, zeros, singular values and IMC for a double tank

(a) The transfer function is

$$G(s) = b \begin{pmatrix} s+a & 0 \\ -a & s+a \end{pmatrix}^{-1} = \frac{b}{(s+a)^2} \begin{pmatrix} s+a & 0 \\ a & s+a \end{pmatrix}$$

The pole polynomial is easily found to be $(s+a)^2$. The system has a double pole in $s = -a$, and no zero.

(b) To determine the singular values of the system, we first determine the eigenvalues of the matrix

$$\begin{aligned}G(i\omega)G^*(i\omega) &= \frac{b^2}{[(i\omega+a)(-i\omega+a)]^2} \begin{pmatrix} i\omega+a & 0 \\ a & i\omega+a \end{pmatrix} \begin{pmatrix} -i\omega+a & a \\ 0 & -i\omega+a \end{pmatrix} \\ &= \frac{b^2}{(\omega^2+a^2)^2} \begin{pmatrix} a^2+\omega^2 & a^2+ai\omega \\ a^2-ai\omega & 2a^2+\omega^2 \end{pmatrix} = \frac{b^2}{(\omega^2+a^2)^2} Q(\omega)\end{aligned}$$

The characteristic polynomial of the matrix $Q(\omega)$ becomes

$$\det[\lambda - Q(\omega)] = \lambda^2 + \lambda(-3a^2 - 2\omega^2) + (a^4 + 2a^2\omega^2 + \omega^4)$$

The eigenvalues are thus

$$\begin{aligned}\lambda &= \frac{3a^2 + 2\omega^2}{2} \pm \left[\left(\frac{3a^2 + 2\omega^2}{2} \right)^2 - (a^4 + 2a^2\omega^2 + \omega^4) \right]^{1/2} \\ &= \frac{3a^2 + 2\omega^2}{2} \pm \frac{1}{2} [5a^4 + 4a^2\omega^2]^{1/2}\end{aligned}$$

The singular values of the system are the square roots of the eigenvalues to $G(i\omega)G^*(i\omega)$, which gives the two singular values

$$\sigma(\omega) = \frac{b}{\sqrt{2}(a^2 + \omega^2)} \sqrt{3a^2 + 2\omega^2 \pm \sqrt{4a^2\omega^2 + 5a^4}}$$

(c) Set the Q factor to

$$Q(s) = \frac{1}{\lambda s + 1} G^{-1}(s) = \frac{1}{\lambda s + 1} \begin{pmatrix} s+a & 0 \\ -a & s+a \end{pmatrix} \frac{1}{b}$$

The regulator becomes

$$\begin{aligned} F_y(s) &= [I - Q(s)G(s)]^{-1}Q(s) \\ &= \frac{1}{\lambda s}G^{-1}(s) = \frac{1}{\lambda s} \begin{pmatrix} s+a & 0 \\ -a & s+a \end{pmatrix} \frac{1}{b} \end{aligned}$$

The sensitivity function will be

$$S(s) = I - G(s)Q(s) = \frac{\lambda s}{\lambda s + 1}I$$

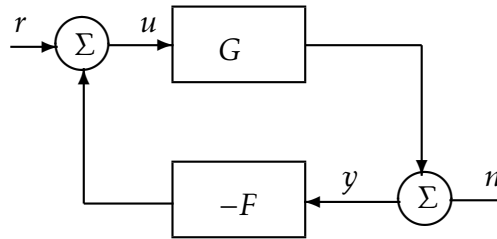
(d) $S(s)$ has two singular values, which are identical:

$$\sigma(\omega) = \left| \frac{\lambda i \omega}{\lambda i \omega + 1} \right| = \frac{\lambda \omega}{\sqrt{\lambda^2 \omega^2 + 1}}$$

6 The closed-loop system

6.1 Internal stability I

Alt 1. Consider the block diagram



which gives

$$y = (I + GF)^{-1}(n + Gr) = G_{ny}n + G_{ry}r$$

and

$$u = (I + FG)^{-1}(r - Fn) = G_{ru}r + G_{nu}n$$

which gives the I/O model

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} G_{ru} & G_{nu} \\ G_{ry} & G_{ny} \end{bmatrix} \begin{bmatrix} r \\ n \end{bmatrix}.$$

from the block diagram we see that

$$r = u + Fy$$

and

$$n = y - Gu$$

In matrix form this becomes

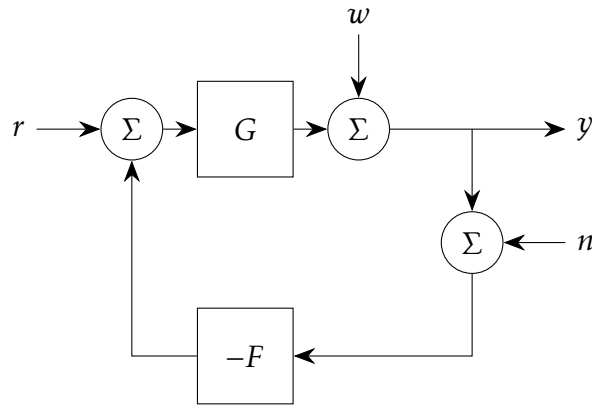
$$\begin{bmatrix} r \\ n \end{bmatrix} \begin{bmatrix} I & F \\ -G & I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$

Hereby we have shown

$$\begin{bmatrix} G_{ru} & G_{nu} \\ G_{ry} & G_{ny} \end{bmatrix}^{-1} = \begin{bmatrix} I & F \\ -G & I \end{bmatrix}$$

Alt. 2 Show that the product of the two matrices is equal to the identity matrix.

6.2 Internal stability II



With transfer functions

$$G = \frac{s-1}{s+2}, \quad F = \frac{s+2}{s-1}$$

we get

$$\begin{aligned} Y &= G(R - F(Y + N)) + W \Rightarrow (1 + GF)Y = GR - GFN + W \\ \Rightarrow Y &= (1 + GF)^{-1}GR - (1 + GF)^{-1}GFN + (1 + GF)^{-1}W \end{aligned}$$

Now

$$G_c = G_{ry} = (1 + GF)^{-1}G = \frac{s-1}{2s+3}$$

$$S = G_{wy} = (1 + GF)^{-1} = \frac{s+1}{2s+3}$$

$$T = 1 - S = \frac{s+2}{2s+3}$$

which all are stable.

Internal stability?

We check the following four transfer functions

$$H_{11} = (1 + FG)^{-1} = \frac{s+1}{2s+3}$$

$$H_{12} = (1 + FG)^{-1}F = \frac{(s+2)(s+1)}{(s-1)(2s+3)}$$

$$H_{21} = (1 + GF)^{-1}G = \frac{s-1}{2s+3}$$

$$H_{22} = (1 + GF)^{-1} = \frac{s+1}{2s+3}$$

The system is not internally stable since H_{12} is not stable.

6.3 Stability and model errors I

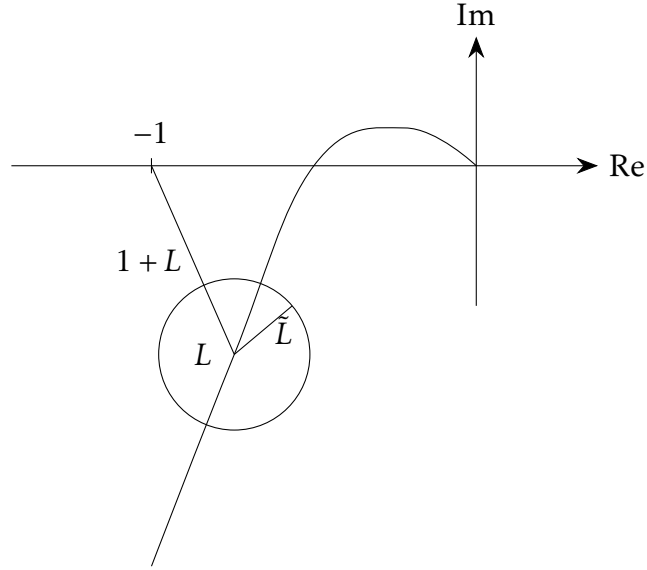
$$T = \frac{1-\alpha}{q-\alpha}, \quad \|\Delta_G T\|_\infty < 1 \Rightarrow \text{stab}$$

$$\|T\|_\infty = \sup_{\omega} \left| \frac{1-\alpha}{e^{i\omega}-\alpha} \right| = \left| \frac{1-\alpha}{1-\alpha} \right| = 1$$

$$\|\Delta_G T\|_\infty \leq \underbrace{\|\Delta_G\|_\infty}_{<1} \cdot \underbrace{\|T\|_\infty}_{=1} < 1$$

6.4 Stability and model errors II

The true loop gain is given by $L_p = L + \tilde{L}$, $\tilde{L} = \Delta_L L$. A typical Nyquist diagram is given in the figure below.



The worst case is when L_p is the point on the circle closest to -1. We get the condition

$$\begin{aligned} |1+L| - |\tilde{L}| &> 0 \quad \forall \omega \\ \Leftrightarrow |\Delta_L| &< \frac{|1+L|}{|L|} = \frac{1}{|T|} \quad \forall \omega \end{aligned}$$

which is the robustness criterion for the scalar case!

6.5 Stability and model errors III

(a) The feedback becomes

$$F(s) = \frac{k(s+1)}{s} \begin{bmatrix} 1/2 & -1/4 \\ 0 & 1/2 \end{bmatrix}$$

This leads to the loop gain $GF = \frac{k}{s}I$ and the closed-loop system

$$G_c = (I + GF)^{-1}GF = \frac{k}{s+k}I.$$

The open-loop system has two poles in -1 and G_c two poles in $-k$. Choose $k = 1$ to let the systems be equally fast.

(b) The true system is in this case

$$G_0 = (I + \Delta_o)G$$

where Δ_o is the relative output model error. The corresponding robustness criterion is $\|\Delta_o T\|_\infty < 1$ where

$$T = (I + GF)^{-1}GF = \frac{1}{s+1}I.$$

In this case

$$\Delta_o = (G_0 - G)G^{-1} = (\alpha - 1) \begin{bmatrix} 0 & 1/2 \\ 0 & 1 \end{bmatrix}.$$

Set $H = \Delta_o T$, and determine

$$\|H\|_\infty = \max_{\omega} \bar{\sigma}(H(i\omega)) = \max_{\omega} \sqrt{\lambda_{\max}(H^*(i\omega)H(i\omega))}$$

where $\lambda_{\max}(H^*(i\omega)H(i\omega))$ is the largest eigenvalue of $H^*(i\omega)H(i\omega) = H^T(-i\omega)H(i\omega)$. This can be determined as follows:

$$\begin{aligned} \lambda_{\max}(H^T(-i\omega)H(i\omega)) &= \frac{(\alpha - 1)^2}{1 + \omega^2} \lambda_{\max}\left(\begin{bmatrix} 0 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1/2 \\ 0 & 1 \end{bmatrix}\right) \\ &= \frac{(\alpha - 1)^2}{1 + \omega^2} \frac{5}{4}. \end{aligned}$$

Hence, $\|H\|_\infty = |\alpha - 1| \frac{\sqrt{5}}{2}$, and the robustness criterion implies that the closed-loop system is guaranteed to be stable for all α fulfilling

$$\begin{aligned} 1 - \frac{2}{\sqrt{5}} &< \alpha < 1 + \frac{2}{\sqrt{5}} \\ 0.11 &< \alpha < 1.89 \end{aligned}$$

(c) This time we write the true system as

$$G_0 = G(I + \Delta_I)$$

where Δ_I is the relative input uncertainty. The robustness criterion is in this case $\|\Delta_I T_I\|_\infty < 1$ where $T_I = (I + FG)^{-1}FG$. Note the difference between the definitions of T and T_I ! In this problem it happens though to hold that $T_I = T$ as $FG = GF = \frac{k}{s}I$. The input uncertainty is given by

$$\Delta_I = G^{-1}(G_0 - G) = (\alpha - 1) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This in turn implies that

$$\Delta_I T_I = \frac{\alpha - 1}{s + 1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

which gives $\|\Delta_I T_I\|_\infty = |\alpha - 1|$. The robustness criterion says that the closed-loop system is guaranteed to be stable for all α fulfilling

$$0 < \alpha < 2.$$

Note the difference compared to part (b). For multivariable systems, the way to represent the modelling error can have a significant importance.

(d) The closed-loop system is given by

$$(I + G_0 F)^{-1} G_0 F = \frac{1}{(s+1)(s+\alpha)} \begin{bmatrix} s+\alpha & \frac{\alpha-1}{2}s \\ 0 & \alpha(s+1) \end{bmatrix}.$$

This system has poles in -1 and $-\alpha$, and the system is in fact stable for all $\alpha > 0$! Note that the robustness criteria deal with the worst case (worst type of modelling error) and do not take into account that we in this case have a special structure of the modelling error.

6.6 From state space form to sensitivity function

a) Use $(x^T \ z^T)^T$ as state vector. One gets

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} x \\ z \end{pmatrix} &= \begin{pmatrix} A - BJC & -BH \\ GC & F \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} -BJ \\ G \end{pmatrix} w \\ y &= \begin{pmatrix} C & 0 \end{pmatrix} x + w \end{aligned}$$

b) Straightforward calculations give

$$S(s) = I + \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} sI - A + BJC & BH \\ -GC & sI - F \end{pmatrix}^{-1} \begin{pmatrix} -BJ \\ G \end{pmatrix}$$

c) With $G(s) = 1/(s+1)$ and $F_y(s) = K$, we have

$$S(s) = (1 + G(s)F_y(s))^{-1} = \frac{1}{1 + K/(s+1)} = \frac{s+1}{s+1+K}$$

Applying the formula derived in part b) with

$$A = -1, B = 1, C = 1, F = 0, G = 0, H = 0, J = K$$

gives

$$S(s) = 1 + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s+1+K & 0 \\ 0 & s \end{pmatrix}^{-1} \begin{pmatrix} -K \\ 0 \end{pmatrix} = 1 + \frac{-K}{s+1+K} = \frac{s+1}{s+1+K}$$

7 Basic limitations in control design

7.1 Limitations and stability

(a) Using a controller with one degree of freedom, the complementary sensitivity function become

$$T(s) = \frac{F(s)G(s)}{1 + F(s)G(s)}$$

To compute the controller $F(s)$ which gives the desired $T(s)$ we express $F(s)$ as a function of $T(s)$ and $G(s)$ in the following way. The above expression gives us

$$T(s) + T(s)F(s)G(s) = F(s)G(s)$$

and

$$T(s) = G(s)(1 - T(s))F(s)$$

which yields

$$F(s) = G^{-1}(s) \frac{T(s)}{1 - T(s)}$$

With the provided $T(s)$ and $G(s)$ we get

$$F(s) = \frac{5(s+1)}{s(s-3)}$$

This results in the zero in $s = 3$ to be canceled when we look at the control system from reference signal to the output signal. However, if we look at the system from reference signal to control signal we have the relation

$$U(s) = \frac{5(s+1)}{(s-3)(s+5)} R(s)$$

We see that this system has a pole in $s = 3$ and hence it is unstable. Alternatively, we can see that the system is not internally stable.

- (b) The bandwidth 5 rad/s can be achieved if we keep the zero and in addition place a pole in $s = -3$, i.e.

$$T(s) = \frac{5(3-s)}{(s+3)(s+5)}$$

The relation

$$F(s) = G^{-1}(s) \frac{T(s)}{1 - T(s)}$$

gives

$$F(s) = \frac{-5(s+1)}{s(s+13)}$$

In this case there is no cancellation of poles and zeros, and all transfer functions for the closed loop system are stable. If there are no cancellations between unstable poles and zeros in the control system, this indicates that the system is internally stable. A more thorough check of internal stability can be done as:

$$\begin{aligned} \begin{pmatrix} 1 & -G(s) \\ F(s) & 1 \end{pmatrix}^{-1} &= \frac{1}{1 + F(s)G(s)} \begin{pmatrix} 1 & G(s) \\ -F(s) & 1 \end{pmatrix} \\ &= \frac{1}{1 + \frac{-5(s-3)(s+1)}{(s+1)s(s+13)}} \begin{pmatrix} 1 & \frac{s-3}{s+1} \\ \frac{5(s+1)}{s(s+13)} & 1 \end{pmatrix} \\ &= \frac{s(s+13)}{s^2 + 8s + 15} \begin{pmatrix} 1 & \frac{s-3}{s+1} \\ \frac{5(s+1)}{s(s+13)} & 1 \end{pmatrix} \end{aligned}$$

All four elements are stable and hence the closed loop system is internally stable.

Alternative solution

From the calculations in (a), it is clear that to avoid an unstable $F(s)$, the desired $T(s)$ must contain the non-minimum phase zero from $G(s)$. (Otherwise, the non-minimum phase zero in $G(s)$ will appear as an unstable pole in $F(s)$).

In order not to affect the bandwidth of the system (which is defined only by the amplitude curve in the bode plot), a pole with the same amplitude (but different sign) as the zero can be added to the transfer function;

$$T_{new}(s) = \frac{5}{s+5} \cdot \frac{s-3}{s+3}.$$

With this,

$$|T_{new}(i\omega)| = \left| \frac{5}{i\omega+5} \cdot \frac{i\omega-3}{i\omega+3} \right| = \left| \frac{5}{i\omega+5} \right| \cdot \left| \frac{i\omega-3}{i\omega+3} \right| = \left| \frac{i\omega-3}{i\omega+3} \right| = \frac{\sqrt{\omega^2+9}}{\sqrt{\omega^2+9}} = 1 = |T_{old}(i\omega)|.$$

Hence the amplitude curve and the bandwidth will be identical. See the compendium for calculations of $F(s)$ and stability check.

According to a rule of thumb, the bandwidth for a non-minimum phase system cannot realistically be greater than half the value of the non-minimum phase zero, i.e., $3/2 = 1.5$ rad/s. However, with this controller, the bandwidth is clearly greater than 1.5 rad/s. Has the rule of thumb been circumvented?

The designed closed loop has, in fact, many drawbacks. Firstly, the phase for the closed loop is $< -90^\circ$ for $\omega > 2$ rad/s, which is bad for reference tracking. Secondly, the amplitude of the sensitivity function is greater than 1 for $\omega \geq 1.3$ rad/s, that is, high-frequent noise will be amplified.

(c) The sensitivity function is

$$S(s) = 1 - T(s) = \frac{s(s+13)}{s^2 + 8s + 15}$$

In particular it satisfies $S(0) = 0$, which means that constant disturbances are fully damped, and no stationary error is obtained. This is due to the integrator in the regulator.

(d) Since $G_c = \frac{GF_r}{1+GF_y}$ and G_c has no zero in $s = 3$, the factor $(s-3)$ in $G(s)$ must, in this case, be cancelled. This directly gives stability issues.

Put in another way, demand that F_r and F_y are finite for $s = 3$. For $s = 3$ evaluate:

$$G_c(3) = \frac{G(3)F_r(3)}{1 + G(3)F_y(3)} = 0$$

which contradicts $G_c(s) = \frac{5}{s+5}$. Hence there exists no such controller.

7.2 Conflicting requirements? I

- (a) The requirements on $|S(i\omega)| = \bar{\sigma}(S(i\omega))$ and $|T(i\omega)| = \bar{\sigma}(T(i\omega))$ can be formulated as

$$\begin{aligned} |S(i\omega)| &\leq \frac{1}{10}, \quad \omega \leq 0.1, & |T(i\omega)| &\leq \frac{1}{10}, \quad \omega \geq 2 \\ |S(0)| &\leq \frac{1}{100} \end{aligned}$$

- (b) The corresponding requirements on the loop gain GF_y become

$$\begin{aligned} |G(0)F_y(0)| &> 100 \\ |G(i\omega)F_y(i\omega)| &> 10, \quad \omega \leq 0.1 \\ |G(i\omega)F_y(i\omega)| &< \frac{1}{10}, \quad \omega \geq 2 \end{aligned}$$

- (c) The requirements in (a) can be formulated using weight functions W_S and W_T such that

$$\begin{aligned} |S(i\omega)| &\leq |W_S^{-1}(i\omega)|, \quad \forall \omega \\ |T(i\omega)| &\leq |W_T^{-1}(i\omega)|, \quad \forall \omega \end{aligned}$$

If W_S^{-1} and W_T^{-1} are chosen as first order functions according to

$$W_S^{-1}(s) = a_1 \left(1 + \frac{s}{b_1} \right), \quad W_T^{-1}(s) = \frac{a_2}{s} \left(1 + \frac{s}{b_2} \right)$$

one get, for example

$$W_S^{-1}(s) = \frac{1}{100}(1 + 100s), \quad W_T^{-1}(s) = \frac{0.14}{s} \left(1 + \frac{s}{2} \right)$$

- (d) The minimal slope of the loop gain in the interval $[0.1, 2]$ is approximately given by straight line that is tangent to the forbidden areas in (b).

$$\text{Slope in the Bode diagram: } \frac{\log 0.1 - \log 10}{\log 2 - \log 0.1} \approx \frac{-1 - 1}{0.3 - (-1)} \approx -1.53$$

This yield

$$\frac{\log 1 - \log 10}{\log \omega_c - \log 0.1} = -1.53 \Rightarrow \log \omega_c = -\frac{0.53}{1.53} = 0.346 \Rightarrow \omega_c = 0.45 \text{ rad/s}$$

From Bodes relation we get

$$\arg GF_y \approx -1.53 \cdot \frac{\pi}{2} = -138^\circ$$

This gives a phase margin $\varphi_m \approx 40^\circ$.

Lower bound on $\|T\|_\infty$?

$$G(i\omega_c)F_y(i\omega_c) = 1 \cdot e^{-i \cdot 138^\circ} = -0.743 + 0.669i$$

$$|T(i\omega_c)| = \left| \frac{G(i\omega_c)F_y(i\omega_c)}{1 + G(i\omega_c)F_y(i\omega_c)} \right| \approx 1.4$$

$$\begin{aligned} \|T\|_\infty &= \sup_{\omega} |T(i\omega)| \Rightarrow \|T\|_\infty \geq |T(i\omega)|, \quad \forall \omega \\ &\Rightarrow \|T\|_\infty \geq 1.4 = |T(i\omega_c)| \end{aligned}$$

(e)

$$|T(i\omega_c)| = 1.4$$

$$|W_T^{-1}(i\omega_c)| = \frac{0.14}{0.45} \sqrt{1 + \frac{0.45^2}{2^2}} = 0.32$$

Thus it is not possible to find a solution *with these weight functions*.

7.3 Conflicting requirements? II

The first requirement says

$$|S(i\omega)| < 10^{-3} \quad \omega \leq 2$$

where

$$S(s) = \frac{1}{1 + F(s)G(s)}$$

When $|F(i\omega)G(i\omega)|$ is “large” we approximately have that

$$|S(i\omega)| \approx \frac{1}{|F(i\omega)G(i\omega)|}$$

This gives us the requirement

$$|F(i\omega)G(i\omega)| > 10^3 \quad \omega \leq 2$$

Further it is required that the control system remains stable despite a model uncertainty

$$|\Delta G(i\omega)| \leq 100|G(i\omega)| \quad \omega \geq 20$$

where $\Delta G(s)$ is the absolute model error in the model $G(s)$. This gives a relative model error as

$$\left| \frac{\Delta G(i\omega)}{G(i\omega)} \right| \leq 100$$

To maintain stability, it is required

$$|T(i\omega)| < 10^{-2} \quad \omega \geq 20$$

where

$$T(s) = \frac{F(s)G(s)}{1 + F(s)G(s)}$$

When $|F(i\omega)G(i\omega)|$ is “small” we approximately have that

$$|T(i\omega)| \approx |F(i\omega)G(i\omega)|$$

this gives us the requirement

$$|F(i\omega)G(i\omega)| < 10^{-2} \quad \omega \geq 20$$

To fulfill both requirements we thus need the loop gain to drop from 10^3 to 10^{-2} during the interval $\omega = 2$ to $\omega = 20$, i.e. 100 dB over one decade (slope -5). According to Bode’s relation we then have $\arg G(i\omega) \approx -5 \cdot 90^\circ$ within this interval. This means that the closed loop system becomes unstable. Thus we can not fulfill the requirements.

7.4 Conflicting requirements? III

a) As $e(t) = S(p)r(t) + \dots$, we have for $r(t)$ a ramp

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} sS(s) \frac{1}{s^2}$$

and it must hold

$$S(0) = 0, \quad \frac{dS(s)}{ds} \Big|_{s=0} = 0$$

b) Use Laplace transform (the reference signal is a step function in this case)

$$\int_0^\infty e(t) dt = \lim_{s \rightarrow 0} \int_0^\infty e^{-st} e(t) dt = \lim_{s \rightarrow 0} E(s) = \lim_{s \rightarrow 0} S(s) \frac{1}{s} = \lim_{s \rightarrow 0} \frac{dS(s)}{ds} = 0$$

c) As

$$\int_0^\infty e(t) dt = 0$$

the control error $e(t)$ must take both positive and negative values. Hence there must be an overshoot in the step response. (Civerth, who is a well trained engineer, realized that there is no way to satisfy the two design objectives simultaneously.)

7.5 Conflicting requirements? IV

The sensitivity function $S(i\omega)$ and the complementary sensitivity function $T(i\omega)$ always fulfills the relation $S(i\omega) + T(i\omega) = 1$ for all ω . This gives $1 = |S(i\omega) + T(i\omega)| \leq |S(i\omega)| + |T(i\omega)|$. However, you would need $|S(i5)| \leq 0.1$ and $|T(i5)| \leq 0.1$, which cannot satisfy $S + T = 1$.

7.6 Time scales

- a) Using the rule of thumb $\omega_0 > 2p_1$, where w_0 is the lower bandwidth and p_1 is the unstable pole, a bandwidth of at least 0.01 rad/s is necessary. That corresponds to a timescale of $1/\omega_0 = 100 \text{ s} \approx 1 \text{ min}$.
- b) For system with time-delays, the bandwidth ω_B is approximately limited by $\omega_B < 1/T$, where T is the time delay. Hence cannot the bandwidth exceed 0.1 rad/s due to the time delay, which is compatible with the (approximate) lower limit of 0.01 rad/s.

8 Controller structures and control design

8.1 RGA

$$\text{RGA}(G(s)) = \begin{pmatrix} \frac{0.6}{s+1} & -0.4 \\ 0.3 & 0.6 \end{pmatrix} \cdot \frac{s+1}{0.12(s+4)} \begin{pmatrix} 0.6 & -0.3 \\ 0.4 & \frac{0.6}{s+1} \end{pmatrix} = \begin{pmatrix} \frac{3}{s+4} & \frac{s+1}{s+4} \\ \frac{s+1}{s+4} & \frac{3}{s+4} \end{pmatrix}$$

Observe that we use *element-by-element* multiplication above. For the DC-gain we get

$$\text{RGA}(G(0)) = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$$

Since all the elements in $\text{RGA}(G(0))$ are positive all combinations are possible. At the crossover frequency we get

$$\text{RGA}(G(10i)) = \begin{pmatrix} \frac{12-30i}{116} & \frac{104+30i}{116} \\ \frac{104+30i}{116} & \frac{12-30i}{116} \end{pmatrix}.$$

We get the diagonal elements closest to 1 if we pair u_1 with y_2 , and u_2 with y_1 .

8.2 RGA and decentralized control

(a)

$$\text{RGA}(G(s)) = \begin{pmatrix} \frac{s-1}{s+1} & \frac{2}{s+1} \\ \frac{2}{s+1} & \frac{s-1}{s+1} \end{pmatrix}$$

gives

$$\text{RGA}(G(0)) = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}.$$

Since one wants to avoid to pair elements with negative $\text{RGA}(0)$ the choice must be $u_1 \leftrightarrow y_2$ and $u_2 \leftrightarrow y_1$.

(b)

$$G(0) = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$$

choose $W_1 = G^{-1}(0)$ and $W_2 = I$, a controller achieving decoupling is given by

$$\begin{aligned} F(s) &= W_1 F^{\text{diag}}(s) W_2 = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} F_{11}(s) & 0 \\ 0 & F_{22}(s) \end{pmatrix} \\ &= \begin{pmatrix} -F_{11}(s) & 2F_{22}(s) \\ -F_{11}(s) & F_{22}(s) \end{pmatrix}. \end{aligned}$$

8.3 IMC for a first order system

The ideal choice of $Q(s)$ is $Q(s) = G^{-1}(s)$, the causal choice is

$$Q(s) = \frac{\tau s + 1}{K(\lambda s + 1)}.$$

This leads to

$$F(s) = \frac{Q(s)}{1 - Q(s)G(s)} = \frac{\tau}{K\lambda} \left(1 + \frac{1}{\tau s} \right).$$

which is a PI controller. The constant gain is $K_{PI} = \frac{\tau}{K\lambda}$ and the integration time is $T_I = \tau$. The sensitivity functions become

$$S(s) = 1 - G(s)Q(s) = \frac{\lambda s}{\lambda s + 1}.$$

$$T(s) = G(s)Q(s) = \frac{1}{\lambda s + 1}.$$

The above calculations imply

$$|S(i\omega)| \leq 1 \quad \forall \omega$$

which may seem to violate Bode's integral theorem. However, the loop gain does not decrease faster than $1/\omega$, and the theorem therefore does not apply.

8.4 IMC for a MIMO system

(a) The transfer function is

$$G(s) = \frac{1}{s/20 + 1} \begin{pmatrix} \frac{9}{s+1} & 2 \\ 6 & 4 \end{pmatrix}.$$

The poles are given as the largest common divisor to the minors

$$\begin{aligned} g_{11}(s) &= \frac{9}{(s/20+1)(s+1)} & g_{12}(s) &= \frac{2}{s/20+1} \\ g_{21}(s) &= \frac{6}{s/20+1} & g_{22}(s) &= \frac{4}{s/20+1} \end{aligned}$$

and

$$\det G(s) = \frac{1}{(s/20 + 1)^2} \left(\frac{36}{s+1} - 12 \right) = \frac{24(1 - s/2)}{(s/20 + 1)^2(s+1)}.$$

which implies that there are two poles in -20 and one in -1 . The zeros are given by $\det G(s)$, normalized such that the denominator is equal to the pole polynomial. Therefore, there is a zero in $s = 2$. As there is a zero in the right half plane, special care must be exercised when making an IMC design.

(b)

$$\begin{aligned} G^{-1}(s) &= (s/20 + 1) \begin{pmatrix} \frac{9}{s+1} & 2 \\ 6 & 4 \end{pmatrix}^{-1} \\ &= \frac{(s/20 + 1)(s+1)}{24(-s/2 + 1)} \begin{pmatrix} 4 & -2 \\ -6 & \frac{9}{s+1} \end{pmatrix} \end{aligned}$$

Choose to reflect the unstable zero of $G(s)$ and add the factor $(\lambda s + 1)$.

$$Q(s) = \frac{(s/20 + 1)(s+1)}{24(\lambda s + 1)(s/2 + 1)} \begin{pmatrix} 4 & -2 \\ -6 & \frac{9}{s+1} \end{pmatrix}$$

We now get

$$Q(s)G(s) = \frac{2-s}{(\lambda s + 1)(2+s)} I$$

and the controller becomes

$$F_y(s) = (I - Q(s)G(s))^{-1} Q(s) = \frac{(s/20 + 1)(s+1)}{12s(\lambda s + 2\lambda + 2)} \begin{pmatrix} 4 & -2 \\ -6 & \frac{9}{s+1} \end{pmatrix}$$

10 Loop shaping

10.1 Simple \mathcal{H}_2 control I

The criterion to be minimized is the \mathcal{H}_2 -norm of G_{ec} .

The system on state-space form

$$\begin{aligned} \dot{x}_1 &= -x_1 + u \\ y &= x_1 \end{aligned}$$

Weight functions

$$W_u(s) = 5, \quad W_T(s) = 0.5, \quad W_S(s) = \frac{1}{s}$$

Form the extended system G_0 :

$$\begin{aligned}z_1 &= W_u u = 5u \\z_2 &= W_T Gu = 0.5x_1 \\z_3 &= W_S(Gu + w) = x_2\end{aligned}$$

where x_2 is introduced as a new state via

$$x_2 = \frac{1}{p}(Gu + w) \quad \Leftrightarrow \quad \dot{x}_2 = x_1 + w$$

This gives

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w \\z &= \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} u \\y &= (1 \quad 0)x + w\end{aligned}$$

Is this on innovation form? Examine the eigenvalues to $A - NC$!

$$A - NC = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 \quad 0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

which has eigenvalues in $\lambda = -1$ and $\lambda = 0$. The system is thus marginally in innovation form. Check M and D .

$$D^T(M \quad D) = (5 \quad 0 \quad 0) \begin{pmatrix} 0 & 0 & 5 \\ 0.5 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (0 \quad 0 \quad 25) \neq (0 \quad 0 \quad 1)$$

Thus, form a new input \tilde{u} as

$$\tilde{u} = (D^T D)^{1/2} u + (D^T D)^{-1/2} D^T M x = 5u \quad \Leftrightarrow \quad u = \frac{1}{5} \tilde{u}$$

This is a pure scaling of the input, and we get a new B -matrix

$$\tilde{B} = \frac{1}{5} B$$

Solve the Riccati equation: $A^T S + SA + M^T M - S \tilde{B} \tilde{B}^T S = 0$

Postulate

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$$

which gives

$$\begin{pmatrix} -s_1 + s_2 & -s_2 + s_3 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -s_1 + s_2 & 0 \\ -s_2 + s_3 & 0 \end{pmatrix} + \begin{pmatrix} 0.25 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{25} \begin{pmatrix} s_1^2 & s_1 s_2 \\ s_1 s_2 & s_2^2 \end{pmatrix} = 0$$

Hence

$$\begin{cases} -2s_1 + 2s_2 + 0.25 - \frac{1}{25}s_1^2 = 0 \\ -s_2 + s_3 - \frac{1}{25}s_1 s_2 = 0 \\ 1 - \frac{1}{25}s_2^2 = 0 \end{cases}$$

which has the positive definite solution $s_1 = -2.5 + 2.5\sqrt{141} = 4.686$ and $s_2 = 5$. The state feedback for the scaled system is thus given by

$$\tilde{L} = \tilde{B}^T S = \left(\frac{1}{5}s_1 \quad \frac{1}{5}s_2 \right) = (0.937 \quad 1)$$

For the original system, we get

$$L = \frac{1}{5}\tilde{L} = (0.187 \quad 0.2)$$

The controller is hence given by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + N(y - C\hat{x}) \\ u &= -L\hat{x} \end{aligned}$$

The transfer function of the controller becomes

$$F_y(s) = \frac{\ell_2(s+1)}{s(s+1+\ell_1)} = \frac{0.2(s+1)}{s(s+1.187)}$$

10.2 Simple \mathcal{H}_2 control II

(a) Laplace transformation of the state space equations yields

$$\begin{aligned} sX(s) &= -X(s) + U(s) + W(s) \\ Z(s) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} X(s) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U(s) \\ Y(s) &= W(s) \end{aligned}$$

from which it follows that $X(s) = \frac{1}{s+1}(U(s) + W(s))$ and that $Z_1(s) = U(s)$, $Z_1(s) = X(s) = \frac{1}{s+1}(U(s) + W(s))$, and $Y(s) = W(s)$.

- (b) We first verify that the state-space description is on the form (10.7) in the text book, and that the condition (10.8) is satisfied. We then observe that the problem is an \mathcal{H}_2 -optimal control problem according to Ch. 10.3. Thus, solve the Riccati equation

$$-S - S + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} - S^2 = 0$$

This equation has the positive solution $S = \sqrt{2} - 1$. This gives $L = S = \sqrt{2} - 1$. The transfer function of the controller is obtained by taking the Laplace transform of the state-space equations describing the controller dynamics:

$$s\hat{X}(s) = -\hat{X}(s) + U(s) + Y(s); \quad U(s) = -(\sqrt{2} - 1)\hat{X}(s)$$

which gives $U(s) = -\frac{\sqrt{2}-1}{s+\sqrt{2}}Y(s)$.

- (c) It holds that

$$\begin{aligned} Z(s) &= \begin{bmatrix} 1 & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} U(s) \\ W(s) \end{bmatrix} \\ Y(s) &= W(s) \\ U(s) &= -\frac{\sqrt{2}-1}{s+\sqrt{2}}Y(s) = -\frac{\sqrt{2}-1}{s+\sqrt{2}}W(s) \end{aligned}$$

This implies that

$$Z(s) = \underbrace{\frac{1}{s+\sqrt{2}} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}}_{G_{ec}(s)} W(s)$$

Furthermore, it holds that

$$\begin{aligned} \|G_{ec}\|_2^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \left(G_{ec}(i\omega) G_{ec}^T(-i\omega) \right) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{ec}^T(-i\omega) G_{ec}(i\omega) d\omega \\ &= \frac{2-\sqrt{2}}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\left(\frac{\omega}{\sqrt{2}}\right)^2 + 1} d\omega \\ &= \frac{2-\sqrt{2}}{2\pi} \left[\sqrt{2} \arctan \frac{\omega}{\sqrt{2}} \right]_{-\infty}^{\infty} = \sqrt{2} - 1 \end{aligned}$$

Hence, $\|G_{ec}\|_2 = \sqrt{\sqrt{2} - 1}$.

- (d) The closed-loop transfer function can be computed from

$$\begin{aligned} Z(s) &= \begin{bmatrix} 1 & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} U(s) \\ W(s) \end{bmatrix} \\ Y(s) &= W(s) \\ U(s) &= -KY(s) = -KW(s) \end{aligned}$$

This gives

$$Z(s) = \underbrace{\begin{bmatrix} -K \\ \frac{1-K}{s+1} \end{bmatrix}}_{G_{ec}(s)} W(s)$$

Furthermore, it holds that

$$\begin{aligned} \|G_{ec}\|_2^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \left(G_{ec}(i\omega) G_{ec}^T(-i\omega) \right) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{ec}^T(-i\omega) G_{ec}(i\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[K^2 + \frac{(1-K)^2}{\omega^2 + 1} \right] d\omega \end{aligned}$$

This integral is finite only if $K = 0$. Thus, it has to be this value of K that minimizes $\|G_{ec}\|_2^2$. Furthermore, for $K = 0$, the integral evaluates to

$$\frac{1}{2\pi} [\arctan \omega]_{-\infty}^{\infty} = \frac{1}{2}$$

Hence, it holds that $\|G_{ec}\|_2 = \frac{1}{\sqrt{2}}$.

10.3 Simple \mathcal{H}_∞ control

(KTH)

(a) Taking the Laplace transform of the state-space equations gives

$$\begin{aligned} sX(s) &= -X(s) + U(s) + W(s) \\ Z(s) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} X(s) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U(s) \\ Y(s) &= W(s) \end{aligned}$$

from which it follows that $X(s) = \frac{1}{s+1}(U(s) + W(s))$ and that $Z_1(s) = U(s)$, $Z_2(s) = X(s) = \frac{1}{s+1}(U(s) + W(s))$, and $Y(s) = W(s)$.

(b) We first verify that the state-space description is on the form (10.7) in the text book, and that the condition (10.8) is satisfied. We then note that the problem is an \mathcal{H}_∞ -optimal control problem according to Ch. 10.4. We should hence solve the Riccati equation

$$-S - S + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} + S(\gamma^{-2} - 1)S = 0$$

for $\gamma = 1$. This equation has the solution $S = 1/2$ which is positive (and thus positive semidefinite). This gives $L = S = 1/2$. We observe that $A - BL = -1 - 1 \times L = -3/2$ is a stable matrix, and hence $L = 1/2$ is a valid solution.

- (c) Now, study the same problem as in (b) with 1 replaced by γ , i.e. determine L such that $\|G_{ec}\|_\infty \leq \gamma$ and the closed-loop system is internally stable. Then find the smallest value of γ for which there is a solution. To this end, solve the Riccati equation

$$-S - S + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} + S(\gamma^{-2} - 1)S = 0$$

for arbitrary γ . If $\gamma \neq 1$ the solution is given by

$$S = \frac{1}{\gamma^{-2} - 1} \pm \frac{\sqrt{2 - \gamma^{-2}}}{|\gamma^{-2} - 1|}$$

A real solution exists if and only if $2 - \gamma^{-2} \geq 0$ which is equivalent with the condition that $\gamma \geq 1/\sqrt{2}$. For $\gamma = 1/\sqrt{2}$ we find $S = 1$ which is positive (and hence positive semi-definite). This gives $L = S = 1$. For this value of L , $A - BL = -1 - 1 \times L = -2$ which is a stable matrix, hence $L = 1$ is a solution that minimizes $\|G_{ec}\|_\infty$ and renders the closed-loop system internally stable. The transfer function of the controller is obtained by taking the Laplace transform of the state-space equations for the controller dynamics:

$$s\hat{X}(s) = -\hat{X}(s) + U(s) + Y(s); \quad U(s) = -\hat{X}(s)$$

which gives $U(s) = -\frac{1}{s+2}Y(s)$.

- (d) The closed-loop transfer function can be computed from

$$\begin{aligned} Z(s) &= \begin{bmatrix} 1 & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} U(s) \\ W(s) \end{bmatrix} \\ Y(s) &= W(s) \\ U(s) &= -KY(s) = -KW(s) \end{aligned}$$

We find

$$Z(s) = \underbrace{\begin{bmatrix} -K \\ \frac{1-K}{s+1} \end{bmatrix}}_{G_{ec}(s)} W(s)$$

Furthermore, it holds that

$$\bar{\sigma}^2(G_{ec}(i\omega)) = G_{ce}^T(-i\omega)G_{ec}(i\omega) = K^2 + \frac{(1-K)^2}{\omega^2 + 1}$$

It is obvious that

$$\|G_{ec}\|_\infty^2 = \sup_{\omega} \bar{\sigma}^2(G_{ec}(i\omega)) = K^2 + (1-K)^2 =: f(K)$$

Finally, we have $f'(K) = 4K - 2 = 0$ for $K = 1/2$ and $f''(K) = 4 > 0$ so $f(K)$ is minimized by $K = 1/2$. Since $f(1/2) = 1/2$ it follows that

$$\min_K \|G_{ec}\|_\infty = \frac{1}{\sqrt{2}}.$$

10.4 Weight functions and observers

(a) The frequency weights $W_S = \frac{1}{s}$ and $W_T = W_u = 1$ give

$$\begin{aligned} z_1 &= W_u u = u \\ z_2 &= W_T G u = Cx \\ z_3 &= W_S(Gu + w) \quad \Leftrightarrow \quad \dot{z}_3 = Cx + w = y \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{z}_3 \end{pmatrix} &= \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ z_3 \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w \\ y &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ z_3 \end{pmatrix} + w \end{aligned}$$

Controllers for the \mathcal{H}_2 - and \mathcal{H}_∞ -criteria can be determined using the equations in the text book.

(b) The observer is given by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu \quad (*) \\ \dot{\hat{z}}_3 &= C\hat{x} + (y - C\hat{x}) = y \quad \Leftrightarrow \quad \hat{z}_3 = \int y d\tau \end{aligned}$$

The state-feedback is given by

$$u = -\begin{pmatrix} L & -\alpha \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{z}_3 \end{pmatrix} = -L\hat{x} + \alpha \hat{z}_3 = -L\hat{x} + \alpha \int y d\tau$$

which, together with (*), gives the desired form for the controller. The controller thus has integral action.

(c) If the system itself includes an integrator, then $\det(pI - A) = p \cdot \xi(p)$ which implies that

$$u = \frac{\alpha}{1 + \frac{1}{p\xi(p)}L(pI - A)^a B} \cdot \frac{1}{p} y = \frac{\alpha \xi(p)}{p\xi(p) + L(pI - A)^a B} y$$

i.e. the integral term of the controller is cancelled.

12 Stability of nonlinear systems

12.1 Lyapunov stability I

Using the state variables $x_1 = y$ and $x_2 = \dot{y}$ gives

$$\dot{x} = \begin{pmatrix} x_2 \\ -0.2(1 + x_2^2)x_2 - x_1 \end{pmatrix} = f(x)$$

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) \Rightarrow$$

$$\dot{V} = V_x f(x) = x_1 \dot{x}_1 + x_2 \dot{x}_2 = -0.2x_2^2(1 + x_2^2)$$

$$\dot{V} < 0 \text{ except the case } x_2 = 0$$

$x_2 \equiv 0 \Rightarrow x_1 = \text{constant} = 0$. Hence, the zero solution is asymptotically stable.

12.2 Circle criterion I

$$\begin{cases} k_1 = 0.5 \\ k_2 = 3 \end{cases} \Rightarrow$$

This leads to a circle which passes through the points $-1/3$ and -2 .

12.3 Circle criterion II

The nonlinearity is limited by straight lines with slopes $k_1 = 0$ and $k_2 = 1$. Hence, the non-admissible area, according to the circle criterion, is the half plane: $\text{Re } s < -1$. Therefore $G(i\omega)$ must lie outside the area $\text{Re } s < -1$, or $\text{Re}(G(i\omega)) \geq -1, \forall \omega$.

12.4 Lyapunov stability II

The state variables $x_1 = \Phi, x_2 = \dot{\Phi}$ gives the model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{mg}{J} \sin x_1 l \end{aligned}$$

The control law

$$l = l_0 + \varepsilon \Phi \dot{\Phi}$$

gives

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{mg}{J} \sin x_1 (l_0 + \varepsilon \Phi \dot{\Phi}) \end{aligned}$$

As a tentative Lyapunov function, try

$$V(x) = \frac{1}{2} J x_2^2 + mgl_0(1 - \cos x_1)$$

which corresponds to the total energy in the system. This leads to

$$\dot{V} = J x_2 \dot{x}_2 + mgl_0 \sin x_1 \dot{x}_1 = -\varepsilon mg x_2^2 x_1 \sin x_1 \leq 0 \quad (-\pi/2 < x_1 < \pi/2)$$

$\dot{V} = 0$ only when $x_1 \equiv 0$ or $x_2 \equiv 0$. $x_1 \equiv 0 \Rightarrow x_2 = 0$ and $x_2 \equiv 0 \Rightarrow x_1 = 0$.

12.5 Lyapunov stability and circle criterion for controller design

(a) The state space form becomes

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 3x_2 + u\end{aligned}$$

(b) With the feedback

$$u = -\text{sign}(ax_1 + bx_2)$$

the closed loop system is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 3x_2 - \text{sign}(ax_1 + bx_2)\end{aligned}$$

The Lyapunov function

$$V(x) = \frac{\alpha}{2}x_1^2 + \frac{\beta}{2}x_2^2$$

gives

$$\dot{V} = (\alpha - 2\beta)x_1x_2 - 3\beta x_2^2 - \beta x_2 \text{sign}(ax_1 + bx_2)$$

Choose now $\alpha = 2\beta > 0$, for example $\alpha = 2, \beta = 1$, and next choose $a = 0, b = 1$, which gives

$$\dot{V} = -3x_2^2 - |x_2| \leq 0$$

Hence $x = 0$ is at least a stable solution. Further, $\dot{V} \equiv 0$ implies

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_1 \equiv 0 \Rightarrow x_1(t) = c$$

The second state equation implies

$$0 = -2c + u(t)$$

The chosen input implies, however, $u(t) = -\text{sign}(x_2(t))$ which is a contradiction. Hence there is no solution satisfying $\dot{V} = 0$, and $x = 0$ is an asymptotically stable solution.

(c) The relay gives the bounds $k_1 = 0, k_2 = \infty$. The circle in the circle criterion becomes the full left half plan. The linear part must hence lie fully in the right half plan.

The linear part has transfer function

$$G_0(s) = G(s)[a + bH(s)] = \frac{a + bs}{(s+1)(s+2)}$$

The requirement $\operatorname{Re} G_0(i\omega) \geq 0$ is equivalent to

$$\begin{aligned}
 \operatorname{Re} \frac{a + bi\omega}{(i\omega + 1)(i\omega + 2)} &\geq 0 \\
 &\iff \\
 \operatorname{Re} (a + bi\omega)(-i\omega + 1)(-i\omega + 2) &\geq 0 \\
 &\iff \\
 \operatorname{Re} [(a(2 - \omega^2) + 3b\omega^2) + i((2 - \omega^2)b\omega - 3a\omega)] &\geq 0 \\
 &\iff \\
 a(2 - \omega^2) + 3b\omega^2 &\geq 0 \quad \forall \omega \\
 &\iff \\
 2a + (3b - a)\omega^2 &\geq 0 \quad \forall \omega \\
 &\iff \\
 b &\geq a/3 \geq 0
 \end{aligned}$$

13 Phase plane analysis

13.1 Equilibrium points and phase portrait

$$\ddot{x} - (0.1 - 10\dot{x}^2/3)\dot{x} + x + x^2 = 0$$

Introduce state variables as $x_1 = x$ and $x_2 = \dot{x}$. The state-space model becomes

$$\begin{aligned}
 \dot{x}_1 &= x_2 = f_1(x_1, x_2) \\
 \dot{x}_2 &= -x_1(1 + x_1) + x_2(0.1 - 10x_2^2/3) = f_2(x_1, x_2)
 \end{aligned}$$

Now determine the singular points and determine the character.

1. Singular points

$$f(x^\circ) = 0 \Rightarrow x_2^\circ = 0 \text{ samt } x_1^\circ(1 + x_1^\circ) = 0.$$

$$\text{SP I } \begin{cases} x_1^\circ = 0 \\ x_2^\circ = 0 \end{cases}, \quad \text{SP II } \begin{cases} x_1^\circ = -1 \\ x_2^\circ = 0 \end{cases}$$

2. Linearize around the singular points

Use Taylor's formula:

$$f(x) = f(x^\circ) + \left. \frac{df}{dx}(x) \right|_{x=x^\circ} (x - x^\circ) + o|x - x^\circ| = \left. \frac{df}{dx}(x) \right|_{x=x^\circ} (x - x^\circ) + o|x - x^\circ|$$

as $f(x^\circ) = 0$. The matrix $\frac{df}{dx}(x)$ is the Jacobian of the function f : Its ij -element is $\frac{\partial f_i}{\partial x_j}(x)$

$$\begin{aligned}\frac{\partial f_1}{\partial x_1} &= 0 & \frac{\partial f_1}{\partial x_2} &= 1 \\ \frac{\partial f_2}{\partial x_1} &= -1 - 2x_1 & \frac{\partial f_2}{\partial x_2} &= 0.1 - 10x_2^2\end{aligned}$$

Make the change of variables $z = x - x^\circ$ for the different singular points

3. SP I

Linear approximation $\dot{z} = Az$, with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0.1 \end{pmatrix}$$

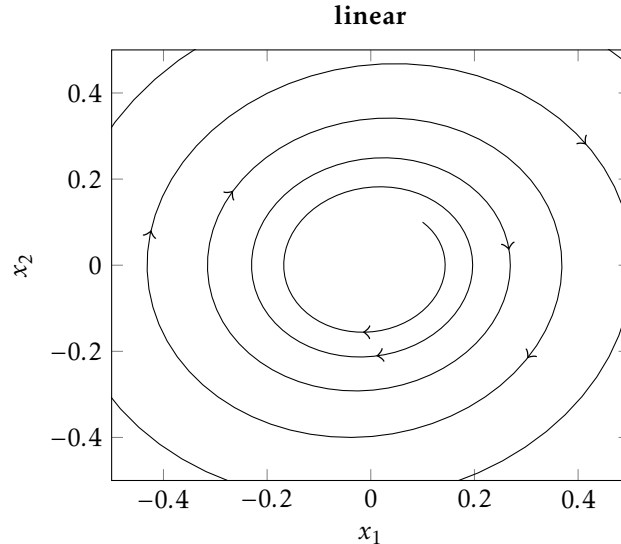
The eigenvalues of the matrix are determined from

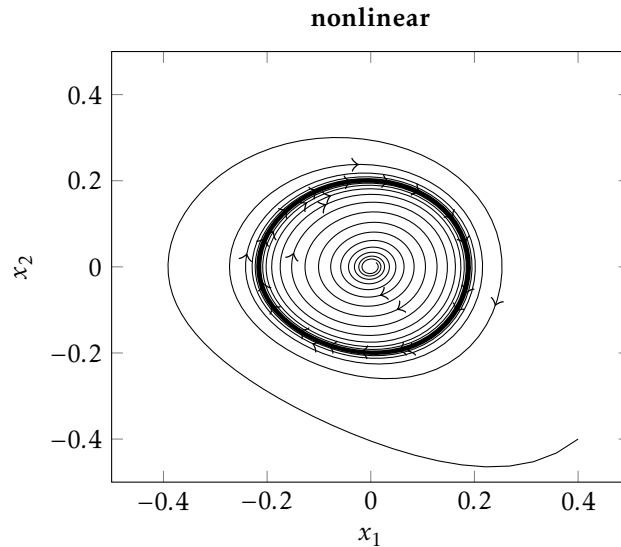
$$0 = \det(\lambda I - A) = \lambda(\lambda - 0.1) + 1,$$

leading to

$$\lambda = 0.05 \pm \sqrt{0.05^2 - 1}$$

The linear approximation thus has an unstable focus in $(0,0)$. The nonlinear differential equation has then the same type of singularity.





4. SP II

The linear approximation $\dot{z} = Bz$, with

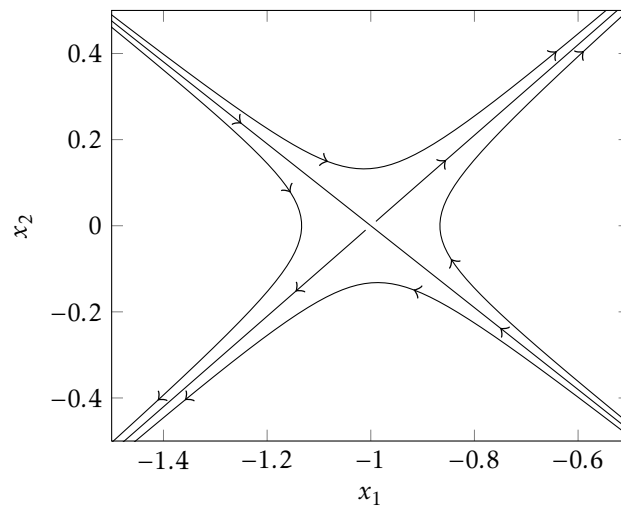
$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0.1 \end{pmatrix}$$

Eigenvalues of B ,

$$0 = \det(\lambda I - B) = \lambda(\lambda - 0.1) - 1$$

$$\lambda = 0.05 \pm \sqrt{0.05^2 + 1}, \quad \lambda_1 \approx -0.95, \quad \lambda_2 \approx 1.05$$

The linearized equation has a saddle point in $(-1, 0)$. This also applies for the nonlinear differential equation. The stable eigenvector is $(1, -0.95)^T$, and the unstable one is $(1, 1.05)^T$.

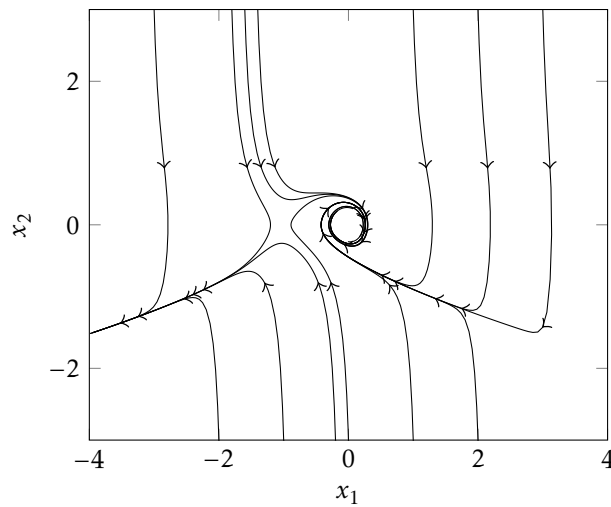


5. Far from the singular points

What do the trajectories look like far from the origin? Check the derivative

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{-x_1(1+x_1) + x_2(0.1 - 10x_2^2/3)}{x_2}$$

When x_1 is bounded and $x_2 \rightarrow \pm\infty$, it holds that $\dot{x}_2/\dot{x}_1 \rightarrow -\infty$. Hence the trajectories become parallel to the x_2 axis when $|x_2|$ grows (and also when $x_2 \rightarrow 0$).



13.2 Phase portrait for a closed loop system with relay

(a) Introduce $x_1 = y, x_2 = \dot{y}$, which leads to

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\operatorname{sgn} x_1\end{aligned}$$

There are no singular points. The phase portrait will be sketched in two parts, depending on $\operatorname{sgn} x_1$. When $x_1 > 0$ it holds

$$\frac{dx_2}{dx_1} = -\frac{1}{x_2}$$

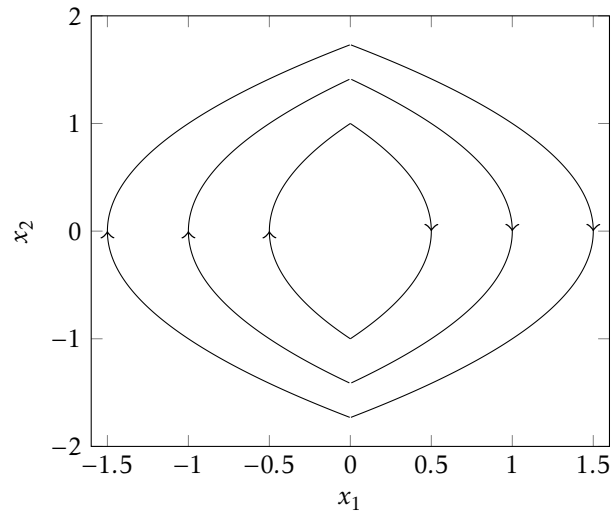
This differential equation has the solution

$$\frac{1}{2}x_2^2 + x_1 = \text{konst}$$

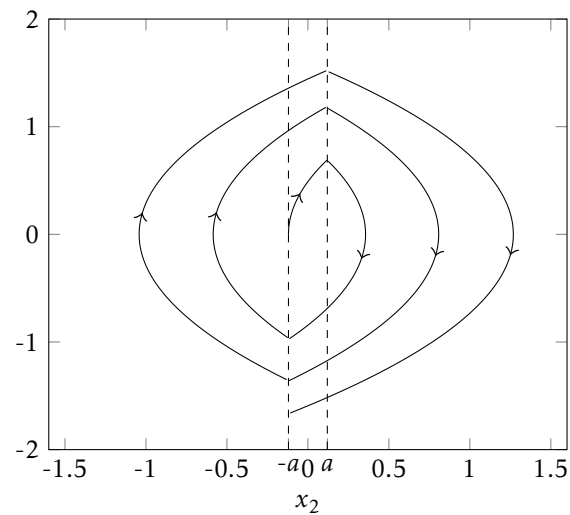
so x_1 as a function of x_2 is described as a parabola. For $x_1 < 0$ we get in a similar way

$$\frac{1}{2}x_2^2 - x_1 = \text{konst}$$

The phase portrait:



- (b) For $x_1 > a$ it holds as above, $\frac{1}{2}x_2^2 + x_1 = \text{const}$, and when $x_1 < -a$ we have instead $\frac{1}{2}x_2^2 - x_1 = \text{const}$. For the case $|x_1| \leq a$ the relay will keep its former value, and therefore the parabola will 'continue' passed the deadzone.



13.3 Analysis of car behavior

- a) $G(s) = 1$

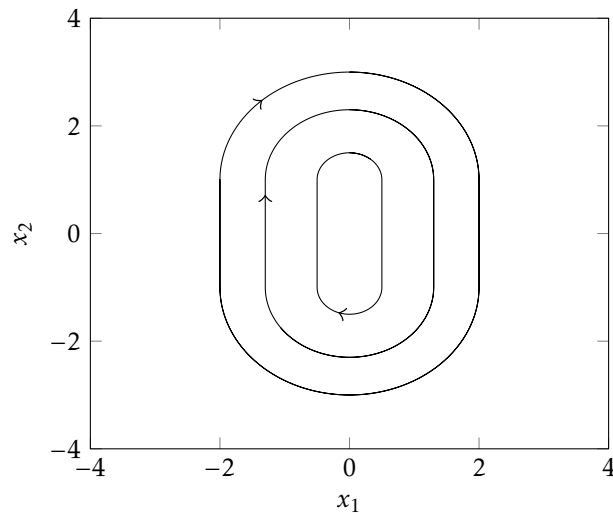
Let $x_1 = y$ and let x_2 denote the input to the nonlinearity (the deadzone). The state-space model becomes

$$\begin{aligned}\dot{x}_1 &= f(x_2) \\ \dot{x}_2 &= -x_1\end{aligned}$$

where

$$f(x) = \begin{cases} x+1, & x < -1 \\ 0, & -1 \leq x \leq 1 \\ x-1, & x > 1 \end{cases}$$

The singular points are centers and appear in $x = (0, -1)^T$ (for $x_2 \leq -1$ equation) and $x = (0, 1)^T$ (for $x_2 \geq 1$ equation). For $-1 < x_2 < 1$ we get $x_1 = \text{const}$. This leads to the phase portrait



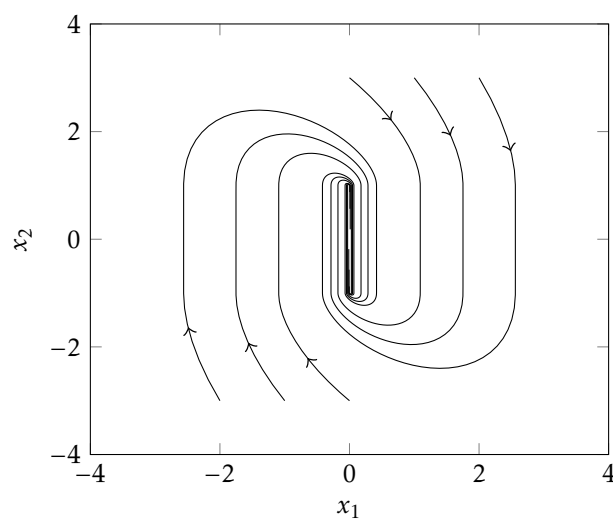
The car does not return to the desired position with this “P-controller”.

b) $G(s) = 1 + s$

In this case the state-space representation will be

$$\begin{aligned} \dot{x}_1 &= f(x_2) \\ \dot{x}_2 &= -x_1 - f(x_2) \end{aligned}$$

The difference as compared to the previous case is that the singular points become stable focuses, instead of centers. The phase portrait becomes



The car returns to its desired position, but Linus will have to repeatedly apply some force to the steering wheel.

13.4 Analysis of fish system

$$\dot{x}_2 = 0 \Rightarrow 0 = -3x_2(1 + \frac{1}{6}x_1) + x_1x_2 = \frac{1}{2}(x_1 - 6)x_2$$

that is $x_1 = 6$ or $x_2 = 0$.

We have two cases:

- $x_2 = 0$ and $\dot{x}_1 = 0 \Rightarrow 0 = 2x_1 - 0.2x_1^2 = 0.2(10 - x_1)x_1$ that is $x_1 = 0$ or $x_1 = 10$.
- $x_1 = 6$ and $\dot{x}_1 = 0 \Rightarrow$

$$0 = 2 \cdot 6(1 + \frac{1}{6} \cdot 6) - 6 \cdot x_2 - 0.2 \cdot 6^2(1 + \frac{1}{6} \cdot 6) = 24 - 6x_2 - 14.4$$

The stationary points are thus

$$\text{SP I: } \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}, \quad \text{SP II: } \begin{cases} x_1 = 10 \\ x_2 = 0 \end{cases}, \quad \text{SP III: } \begin{cases} x_1 = 6 \\ x_2 = 1.6 \end{cases}$$

The Jacobian for the system is

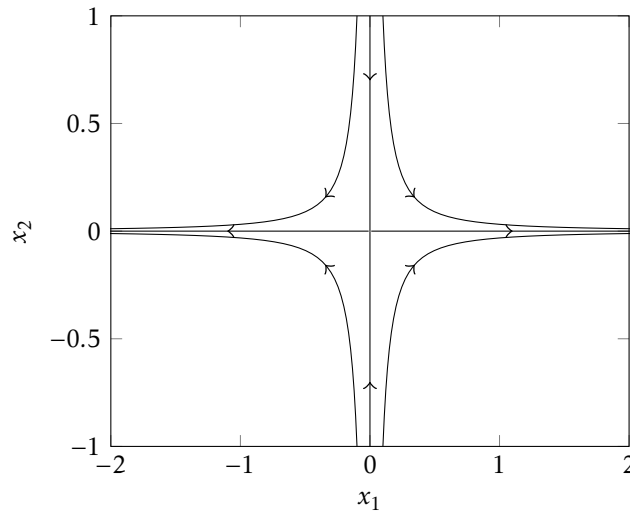
$$H(x) = \frac{df}{dx}(x) = \begin{pmatrix} 2 - 0.4x_1 - x_2/(1 + x_1/6)^2 & -x_1/(1 + x_1/6) \\ x_2/(1 + x_1/6)^2 & -3 + x_1/(1 + x_1/6) \end{pmatrix}$$

SP I:

$x_1 = x_2 = 0$ gives

$$H_1 = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

The origin is a saddle point with trajectories according to

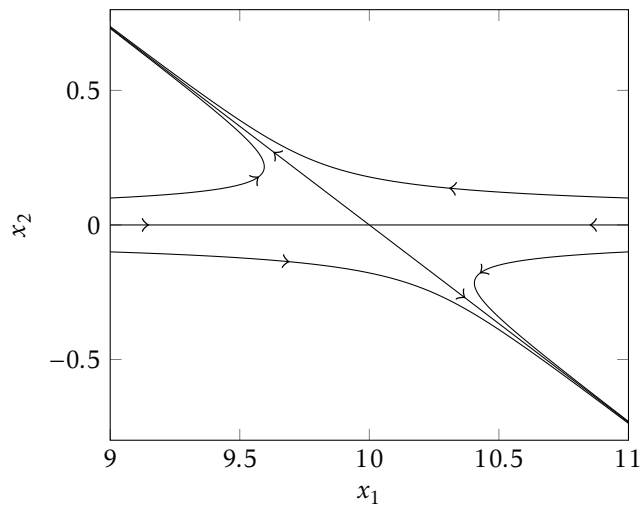


SP II:

When $x_1 = 10$ and $x_2 = 0$ the Jacobian becomes

$$H_2 = \begin{pmatrix} -2 & -3.75 \\ 0 & 0.75 \end{pmatrix}$$

which is also a saddle point. The unstable eigenvector is $(3.75, -2.75)^T$, while the stable one is $(1, 0)^T$. The phase portrait looks like

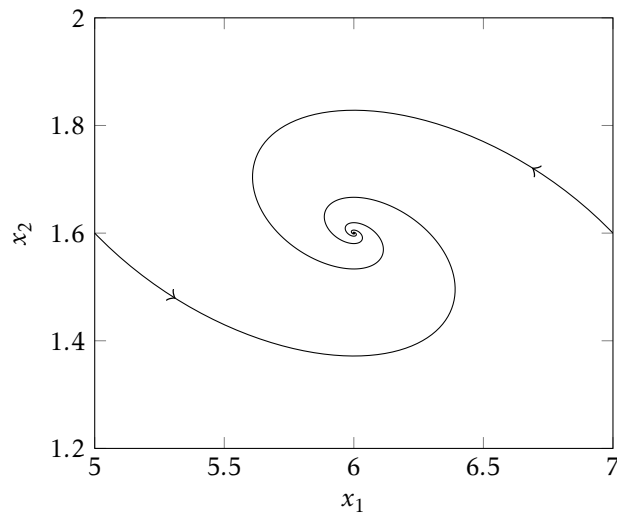


SP III:

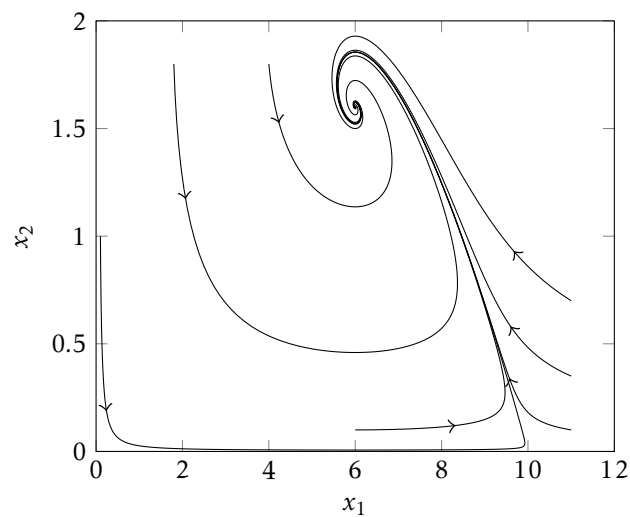
$x_1 = 6$, $x_2 = 1.6$ gives

$$H_3 = \begin{pmatrix} -0.8 & -3 \\ 0.4 & 0 \end{pmatrix}$$

the eigenvalues are $-0.4 \pm 1.02i$. This means that we have a stable focus, with the corresponding phase portrait



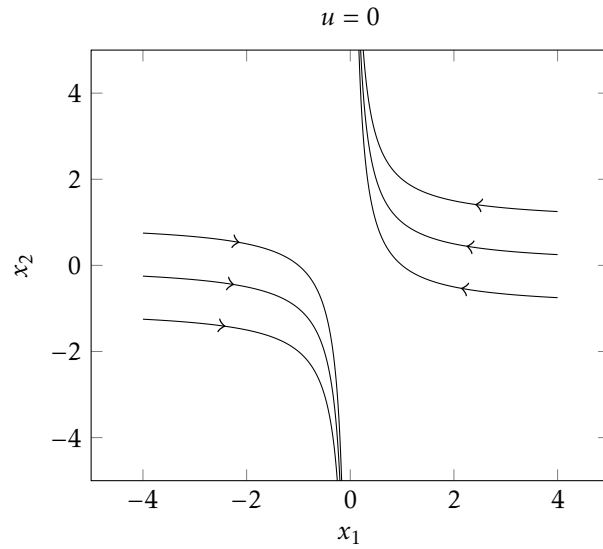
Merging the phase portraits is straightforward:



13.5 Phase portrait and Lyapunov stability for control design

- (a) The singular points are given by $x_1 = 0$, that is the full x_2 -axis, when $u = 0$. The trajectories are given by

$$\frac{dx_2}{dx_1} = -\frac{1}{x_1^2} \quad \Leftrightarrow \quad x_2 = \frac{1}{x_1} + C.$$



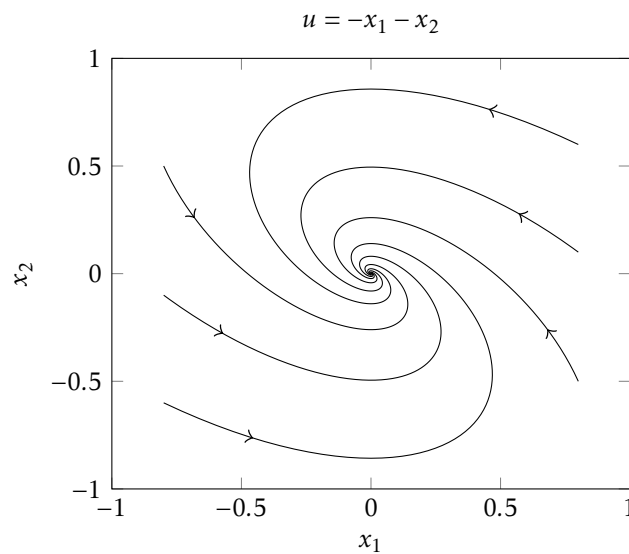
(b)

$$\dot{V} = V_x \dot{x} = -2x_1^4 + 2x_1 u + 2x_1 x_2 \quad \Rightarrow \quad \text{Choose } u = -x_1 - x_2$$

This gives $\dot{V} = -2x_1^4 - 2x_1^2 < 0$ and a stationary point $x_1 = x_2 = 0$. The corresponding linearization is

$$\dot{x} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} x, \quad \text{with eigenvalues } \lambda = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}.$$

Therefore, the phase portrait becomes a stable focus.



13.6 Equilibrium points

(a) There are three equilibrium points

$$x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x = \begin{pmatrix} \sqrt{6} \\ 0 \end{pmatrix}, \quad x = \begin{pmatrix} -\sqrt{6} \\ 0 \end{pmatrix}$$

The Jacobian is

$$\frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ -1 + x_1^2/2 & -1 \end{pmatrix}$$

For the first point, we have

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

with eigenvalues in $s = -0.5 \pm i\sqrt{0.75}$. The equilibrium is a stable focus.

For the second point, we have

$$A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$$

with eigenvalues in $s = 1$ and $s = -2$. The equilibrium is a saddle point.

For the third point, we have

$$A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$$

with eigenvalues in $s = 1$ and $s = -2$. The equilibrium is a saddle point.

(b) There is a unique stationary point, $x = (0 \ 0)^T$. The Jacobian becomes

$$A = \frac{\partial f}{\partial x}|_{x=0} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

As it has eigenvalues in $s = 0.5 \pm i\sqrt{0.75}$, it is an unstable focus.

(c) From the second equation we find that either $x_2 = 0$ or $x_2 = 2(1 + x_1)$. Inserting this into the first equation, we find that there are in total four equilibrium points, as described below. Further, the Jacobian is in the general case

$$\frac{\partial f}{\partial x} = \begin{pmatrix} 1 - 2x_1 - \frac{2x_2}{(1+x_1)^2} & -2\frac{x_1}{(1+x_1)} \\ \frac{x_2^2}{(1+x_1)^2} & 2 - \frac{2x_2}{(1+x_1)} \end{pmatrix}$$

The first stationary point $x = (0 \ 0)^T$ gives

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

It has eigenvalues in $s = 1$ and $s = 2$, and is an unstable focus.

The second stationary point $x = (1 \ 0)^T$ gives

$$A = \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}$$

It has eigenvalues in $s = -1$ and $s = 2$, and is a saddle point.

The third stationary point $x = (0 \ 2)^T$ gives

$$A = \begin{pmatrix} -3 & 0 \\ 4 & -2 \end{pmatrix}$$

It has eigenvalues in $s = -3$ and $s = -2$, and is a stable node.

The fourth stationary point $x = (-3 \ -4)^T$ gives

$$A = \begin{pmatrix} 9 & -3 \\ 4 & -2 \end{pmatrix}$$

It has eigenvalues in $s = 7.772$ and $s = -0.772$, and is a saddle point.

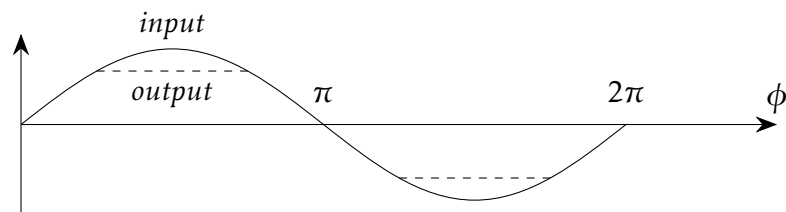
13.7 Phase plane and time evolution

a – 1, b – 7, c – 5, d – 6, e – 3, f – 2, g – 4

14 Oscillations and describing functions

14.1 Limit cycle for a closed loop with saturation

1. We first derive the describing function. Use the signal $e(t) = C \sin \Phi$, with $\Phi = \omega t$, before the nonlinearity. If $C \leq 1$, then the output of the nonlinearity becomes $u(t) = e(t)$. If $C > 1$, the output $u(t)$ behaves as in the dashed line in the figure below.



Here, Φ_1 is given by $C \sin \Phi_1 = 1$, that is $\Phi_1 = \arcsin(1/C)$.

2. Determine the coefficients a_1 and b_1 as

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} u(\Phi) \cos \Phi \, d\Phi, \quad b_1 = \frac{1}{\pi} \int_0^{2\pi} u(\Phi) \sin \Phi \, d\Phi$$

As $u(\Phi)$ is an odd function, and $\cos \Phi$ is even, we get $a_1 = 0$. Due to symmetry, we can write b_1 as (assuming $C > 1$)

$$\begin{aligned} b_1 &= \frac{4}{\pi} \int_0^{\pi/2} u(\Phi) \sin \Phi \, d\Phi \\ &= \frac{4}{\pi} \left(\int_0^{\Phi_1} C \sin^2 \Phi \, d\Phi + \int_{\Phi_1}^{\pi/2} \sin \Phi \, d\Phi \right) \\ &= \frac{4C}{\pi} \left(\frac{\Phi_1}{2} - \frac{\sin 2\Phi_1}{4} + \frac{\cos \Phi_1}{C} \right) \end{aligned}$$

As $\sin 2\Phi_1 = 2 \sin \Phi_1 \cos \Phi_1$, $\sin \Phi_1 = 1/C$, and $\cos \Phi_1 = \sqrt{C^2 - 1}/C$, we get

$$b_1 = \frac{2C}{\pi} \left(\arcsin \frac{1}{C} + \frac{\sqrt{C^2 - 1}}{C^2} \right)$$

3. The describing function is now found to be $Y_f(C) = (b_1 + ia_1)/C$,

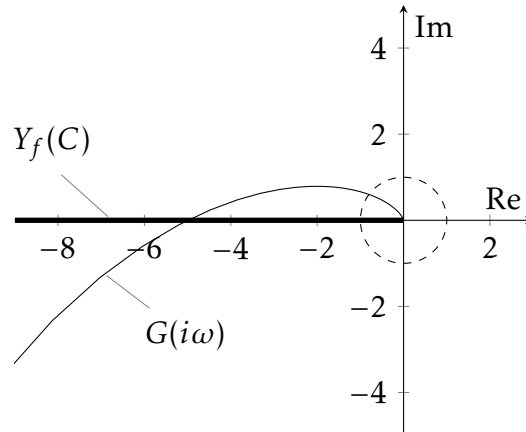
$$Y_f(C) = \frac{2}{\pi} \left(\arcsin \frac{1}{C} + \frac{\sqrt{C^2 - 1}}{C^2} \right)$$

(applies for $C > 1$, $Y_f(C) = 1$ if $C \leq 1$)

Next, draw the Nyquist curve for $G(s) = 10/(s(1+s)^2)$, and the curve $-1/Y_f(C)$ in the same diagram.

$$\begin{aligned} \arg G(i\omega) &= -\pi/2 - 2 \arctan \omega \\ |G(i\omega)| &= 10/(\omega(1+\omega^2)) \end{aligned}$$

Nyquist curve

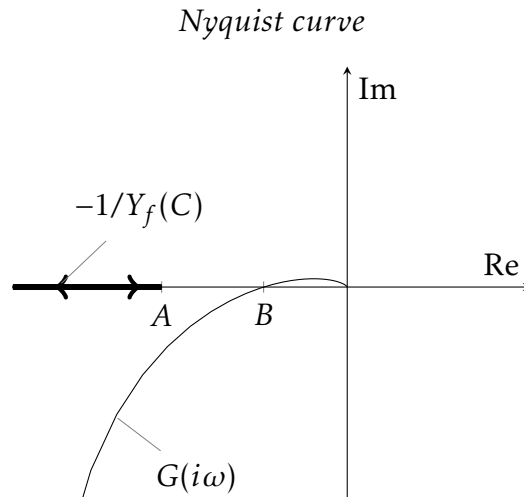


The curves are crossing when $\arg G(i\omega) = -\pi \Rightarrow \omega = 1$. Now use $|G(i1)| = 5$, and

$$-1/Y_f(C) = G(i1) = -5$$

leading to $C \approx 6.3$. There is hence a periodic solution (limit cycle) with approximative amplitude 6.3, and angular frequency 1 rad/s. Due to the modified Nyquist criterion, the system is 'unstable' if the Nyquist curve encircles $-1/Y_f(C)$. This happens when $C < 6.3$, which means that the amplitude is going to increase. On the other hand, if $C > 6.3$, the system is stable and the amplitude of the oscillation will decrease. The conclusion is hence that the oscillation is 'stable', that is, a limit cycle exists.

14.2 Limit cycle for a closed loop with deadzone relay



The describing function 'turns' in the point A, which occurs for $C = \sqrt{2}D$ and gives $A = -\frac{\pi D}{2H}$ (as $\frac{4H}{\pi C} \sqrt{1 - D^2/C^2}$ is maximized for $C = \sqrt{2}D$. This gives the value $\frac{2H}{\pi D}$). A possible crossing with the Nyquist curve occurs when the Nyquist curve crosses the negative real axis. This leads to $\arg G(i\omega) = -\pi$ for $\omega = 1$, and $|G(i1)| = 1/2$. The point B corresponds hence to $B = -1/2$. That a limit cycle barely exists, means that $B \approx A$ and the amplitude = 2.5 gives $\sqrt{2}D = 2.5$. Hence we can conclude that $D = 5 \cdot \sqrt{2}/4$ and $H = \pi \cdot 5 \cdot \sqrt{2}/4$. The frequency of the limit cycle is $\omega = 1$.

14.3 Limit cycles for controller design I

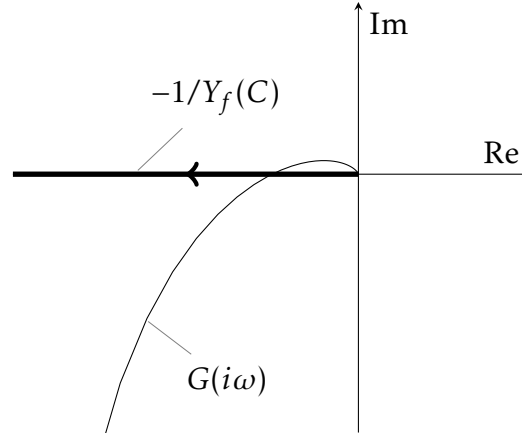
(a) The describing function for the relay is given by

$$Y_f(C) = 4/(\pi C), \Rightarrow -1/Y_f(C) = -\pi C/4$$

The curve $-1/Y_f(C)$ passes the whole negative real axis. For the Nyquist curve we have

$$\begin{aligned} G(i\omega) &= \frac{K}{i\omega(i\omega + 1)^2} = \frac{K(1 - i\omega)^2 \cdot (-i\omega)}{\omega^2(1 + \omega^2)^2} \\ &= \frac{K(1 - \omega^2 - 2i\omega)(-i\omega)}{\omega^2(1 + \omega^2)^2} = \frac{-2K\omega - iK(1 - \omega^2)}{\omega(1 + \omega^2)^2} \end{aligned}$$

Nyquist curve



There will always be a limit cycle, as the Nyquist curve crosses $-1/Y_f(C)$ (negative real axis) for all values of K .

- (b) The two curves cross when $\arg G(i\omega) = -\pi$, leading to $\omega = 1$. As $|G(i1)| = K/2$, the amplitude C , is given by the relation

$$-\frac{K}{2} = -\frac{\pi C}{4}$$

The requirement $C < 0.1$ now implies that $K < \pi/20$.

- (c) A lead compensation (or a PD compensator), designed so that $G(i\omega)$ advances the phase for $\omega > 1$, would increase the amplitude margin. This means that a larger value of K can be used.

14.4 Limit cycles for controller design II

The describing function for an ideal relay is given by

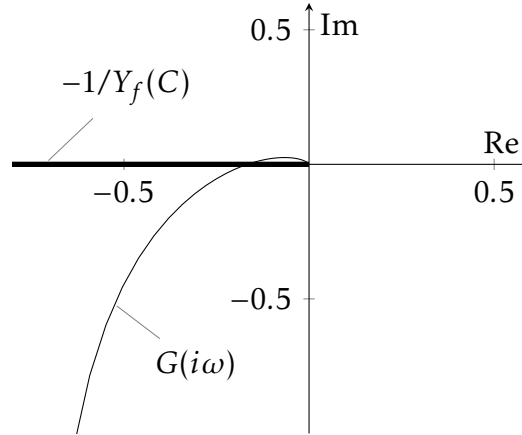
$$Y_f(C) = \frac{4}{\pi \cdot C} \Rightarrow -\frac{1}{Y_f(C)} = -\frac{\pi}{4} \cdot C$$

(a) $H(s) = 1$

Draw the Nyquist curve for $G(s)H(s) = G(s)$

$$\begin{aligned} G(i\omega) &= \frac{1}{i\omega(i\omega + 1)(i\omega + 2)} = \frac{-i(1 - i\omega)(2 - i\omega)}{\omega(\omega^2 + 1)(\omega^2 + 4)} \\ &= -\frac{3}{(\omega^2 + 1)(\omega^2 + 4)} - i\frac{2 - \omega^2}{\omega(\omega^2 + 1)(\omega^2 + 4)} \end{aligned}$$

Nyquist curve



If the Nyquist curve encloses the point $-1/Y_f(C)$ the amplitude of the output would increase, and vice versa. In this case we will have a stable oscillation (limit cycle). The frequency and the amplitude are given by the joint point of the two curves. This point is characterized by $\text{Im } G(i\omega) = 0$, that is $\omega = \sqrt{2}$. As $\text{Re } G(i\sqrt{2}) = -1/6$, we get

$$-1/6 = -\frac{\pi C}{4} \Rightarrow C = \frac{2}{3\pi}$$

Hence, the limit cycle has amplitude $2/(3\pi)$ and frequency $\omega = \sqrt{2}$.

(b) $H(s) = 1 + Ks$

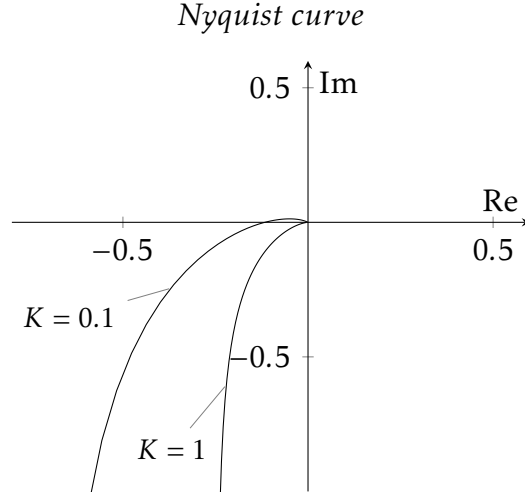
Consider now $G(i\omega)H(i\omega)$

$$\begin{aligned} G(i\omega) \cdot H(i\omega) &= \frac{-i(1 - i\omega)(2 - i\omega)(1 + Ki\omega)}{\omega(\omega^2 + 1)(\omega^2 + 4)} \\ &= \frac{-3 + 2K - K\omega^2}{(\omega^2 + 1)(\omega^2 + 4)} + i\frac{-2 + \omega^2 - 3K\omega^2}{\omega(\omega^2 + 1)(\omega^2 + 4)} \end{aligned}$$

According to part (a), limit cycles are avoided if the imaginary part $< 0 \forall \omega$.

$$\begin{aligned} -2 + \omega^2 - 3K\omega^2 &< 0 \Rightarrow \\ K &> \frac{\omega^2 - 2}{3\omega^2} \end{aligned}$$

As $(\omega^2 - 2)/(3\omega^2) < 1/3 \forall \omega$ we can take $K > 1/3$. See the figure below for illustration.



18 Optimal control

18.1 The maximum principle

We note that Theorem 18.2 is suitable for this problem. We start by writing it on the standard form

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2^3 + (1 + x_1)u \end{cases} \quad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\min_{u \in \mathcal{U}} \int_0^1 (e^{x_1^2} + x_2^2 + u^2) dt \Rightarrow \begin{cases} L = e^{x_1^2} + x_2^2 + u^2 \\ \phi = 0 \end{cases}$$

where $\mathcal{U} = \{u(t) : |u(t)| < \infty\}$

By Theorem 18.2 (with $\mu_0 = 1$, normal case)

$$\min_{u \in \mathcal{U}} H = e^{x_1^2} + x_2^2 + u^2 + \lambda_1 x_2 + \lambda_2 (-x_1 - x_2^3 + (1 + x_1)u)$$

$$\begin{cases} \dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = -2x_1 e^{x_1^2} - \lambda_2(-1 + u) \\ \dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -2x_2 - \lambda_1 + 3x_2^2 \lambda_2 \end{cases}$$

$$\lambda(1) = \frac{\partial \phi}{\partial x} = 0$$

$$\min_{u \in \mathcal{U}} H : \frac{\partial H}{\partial u} = 0 \Rightarrow 2u + \lambda_2(1 + x_1) = 0 \Rightarrow u = -\frac{1}{2}\lambda_2(1 + x_1)$$

$$\frac{\partial^2 H}{\partial u^2} = 2 > 0 \text{ i.e., a minimum}$$

By plugging $u = -\frac{\lambda_2}{2}(1 + x_1)$ into the \dot{x} - and $\dot{\lambda}$ equations, we obtain the system of differential equations

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2^3 - \frac{\lambda_2}{2}(1 + x_1)^2 \\ \dot{\lambda}_1 = -2x_1 e^{x_1^2} + \lambda_2(1 + \frac{\lambda_2}{2}(1 + x_1)) \\ \dot{\lambda}_2 = -2x_2 - \lambda_1 + 3\lambda_2 x_2^2 \\ \text{At } t_f = 1 \text{ it must hold that } \lambda_1(1) = 0 = \lambda_2(1) \end{cases}$$

whose solution, together with $u = -\frac{\lambda_2}{2}(1 + x_1)$, gives the optimal $u(t)$.

18.2 Optimal bang-bang control

Use Theorem 18.5 and 18.6 (you have to carefully check that the conditions of the theorems are satisfied).

18.3 The maximum principle and bang-bang control

$$L(x, u) = \frac{1}{2}x_1^2, \quad \phi(t_f, x(t_f)) = 0, \quad \psi(t_f, x(t_f)) = x(t_f)$$

The maximum principle:

$$\min_{|u| \leq 1} H(x, u, \lambda, n_0) = n_0 \frac{1}{2}x_1^2 + \lambda_1 x_2 + \lambda_2 u$$

where

$$\begin{aligned} \dot{\lambda} &= -H_x^T = \begin{pmatrix} -n_0 x_1 \\ -\lambda_1 \end{pmatrix} \\ \lambda(t_f) &= \psi_x^T \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \end{aligned}$$

Minimum is obtained at

$$u = \begin{cases} +1, & \lambda_2 < 0 \\ -1, & \lambda_2 > 0 \\ ?, & \lambda_2 = 0 \end{cases}$$

This is bang-bang control unless $\lambda_2 = 0$ on any interval.

The initial state $x_0 = 0$ gives the control signal $u \equiv 0$, which is not a bang-bang controller.