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Automatic Control III

Lecture 8 – The circle criterion and describing functions

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Summary of lecture 7 (I/III)

- Defined stability of equilibrium (stationary) points; stable, asymptotically stable and globally asymptotically stable.
- Investigated stability of an equilibrium of a nonlinear system by studying how the distance to the origin changes over time.
- The above idea lead us into Lyapunov theory.

Summary of lecture 7 (II/III)

A Lyapunov function $V(x)$ "measures the distance to the goal":

- Let $V(x)$ denote a (generalized) distance from x to an equilibrium point x_0.
- The distance must remain positive until the system has arrived in the equilibrium point x_0,

$$V(x) > 0, \qquad x \neq x_0, \qquad V(x_0) = 0.$$

- The distance must decrease until the final destination is reached,

$$\frac{d}{dt}V(x(t)) = V_x(x(t))\dot{x}(t) = V_x(x(t))f(x(t)) < 0,\ x(t) \neq x_0.$$

- If the system "diverges", this must be clearly visible

$$V(x) \to \infty, \ \ |x| \to \infty.$$

Summary of lecture 7 (III/III)

Theorem: If a Lyapunov function V satisfying

$$V_x(x(t))f(t) < 0, x \neq x_0, \qquad V(x) \to \infty \quad \text{as} \quad |x| \to \infty$$

can be found, then the equilibrium point x_0 is globally asymptotically stable.

The tricky part is to **find** the Lyapunov function!

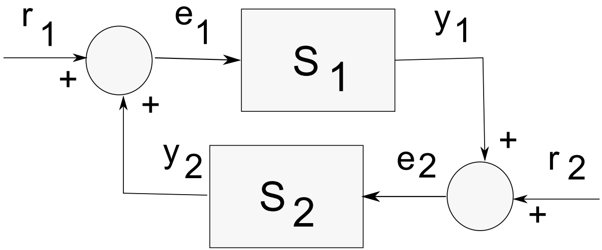
We also showed that finding a Lyapunov function for a linear system amounts to solving the **Lyapunov equation**,

$$A^T P + PA = -Q.$$

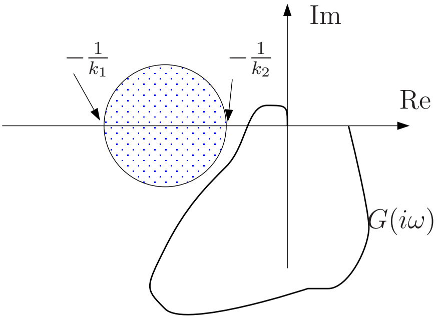
Stability – the small gain theorem

Two stable systems $\mathcal{S}_1$ and $\mathcal{S}_2$ which are connected according to the figure below result in a closed loop system that is stable if

$$\|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\| < 1.$$

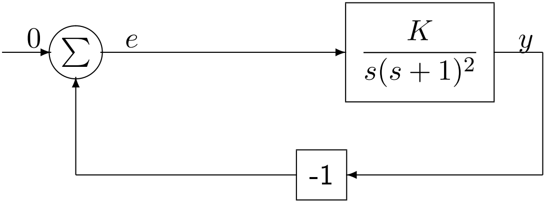


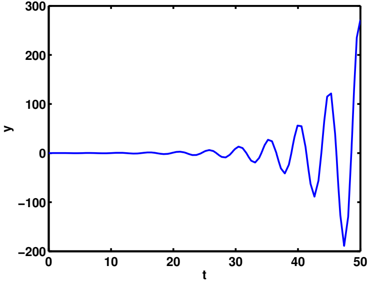
Circle criterion



Theorem: [Circle criterion] Assume that $G(s)$ has no poles in the RHP and that $f(0) = 0, k_1 \leq f(y)/y \leq k_2$ for $y \neq 0$. Then the closed-loop system is input-output stable if the Nyquist curve $G(i\omega)$ does not enter, nor encircle the circle which intersects the negative real axis (perpendicularly) in $-1/k_1$ and $-1/k_2$.

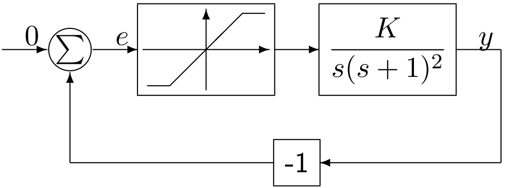
A simple feedback system



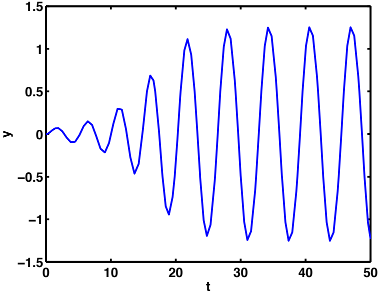


Unstable for K=4!

A simple feedback system – with saturation



The same system, but now with a saturation (a static nonlinearity) in the loop.



Note the **stability of the periodic solution** (limit cycle)!

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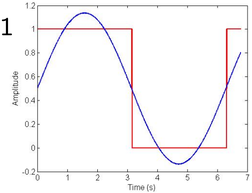
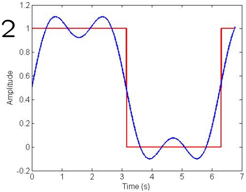
Recall – Fourier series

A Fourier series decomposes periodic signals into the sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines (or complex exponentials).

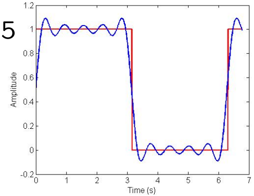
$N = 1$



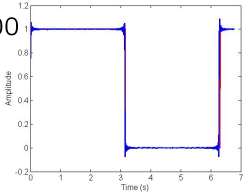
$N = 2$



$N = 5$



$N = 100$



Passing a sine through a static nonlinearity

$u = C \sin \omega t$ $\qquad$ $w = f(C \sin \omega t)$

$$u \longrightarrow \boxed{f} \xrightarrow{\;w\;}$$

Fourier series expansion of w:

$$w = \frac{1}{2}\widetilde{A}_0(C) + \sum_{n=1}^{\infty}(\widetilde{A}_n(C)\cos(n\omega t) + \widetilde{B}_n(C)\sin(n\omega t))$$

$$= A_0(C) + \sum_{n=1}^{\infty} A_n(C)\sin(n\omega t + \phi_n(C))$$

Define the **describing function** as

$$Y_f(C) = \frac{A_1(C)e^{i\phi_1(C)}}{C},$$

where $|Y_f(C)|$ is the gain and $\arg Y_f(C)$ is the phase shift.

Passing a sine through a static nonlinearity

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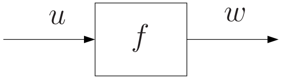

Fourier series expansion of w:

$$
\begin{aligned}
w &= \frac{1}{2}\widetilde{A}_0(C) + \sum_{n=1}^{\infty} \left(\widetilde{A}_n(C)\cos(n\omega t) + \widetilde{B}_n(C)\sin(n\omega t)\right) \\
&= A_0(C) + \sum_{n=1}^{\infty} A_n(C)\sin(n\omega t + \phi_n(C))
\end{aligned}
$$

Define the **describing function** as

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Y_f(C) = \frac{A_1(C)e^{i\phi_1(C)}}{C},
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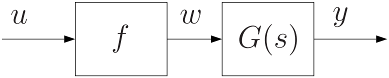
Sine through a static nonlinearity and $G(s)$

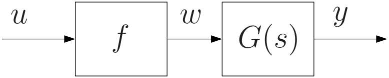


$$u = C \sin \omega t$$

$$w = A_0(C) + \sum_{n=1}^{\infty} A_n(C) \sin(n\omega t + \phi_n(C))$$

$$y = \underset{\text{NL sys.}}{A_0(C)}\,\underset{\text{Lin. sys.}}{|G(0)|} + \sum_{n=1}^{\infty} \underset{\text{NL sys.}}{A_n(C)}\,\underset{\text{Lin. sys.}}{|G(in\omega)|} \sin(n\omega t + \underset{\text{NL. sys.}}{\phi_n(C)} + \underset{\text{Lin. sys.}}{\psi(n\omega)})$$

Sine through a static nonlinearity and $G(s)$



Assume:

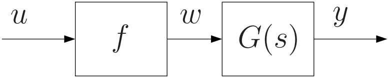
- $A_0 = 0$ (valid for example if f is an odd function).
- $|G(ki\omega)| << |G(i\omega)|, \quad |k| > 1$, i.e. G "steep LP filter".

Then we have

$$y \approx A_1(C)|G(i\omega)|\sin(\omega t + \phi_1(C) + \psi(\omega))$$

where $\psi(\omega) = \arg G(i\omega)$.

Sine through a static nonlinearity and $G(s)$



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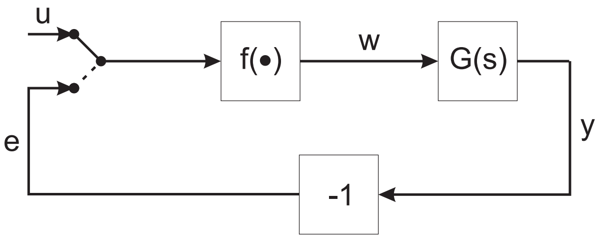
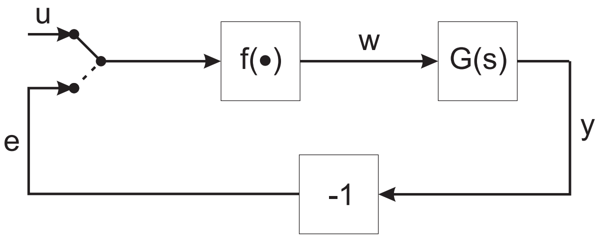
Follow the sine around the loop (I/III)



Only keep the fundamental frequency:

$$
\begin{aligned}
u &= C \sin \omega t \\
w &= A_1(C) \sin(\omega t + \phi_1(C)) \\
y &= A_1(C)|G(i\omega)| \sin(\omega t + \phi_1(C) + \psi(\omega)) \\
e &= -y
\end{aligned}
$$

Follow the sine around the loop (II/III)

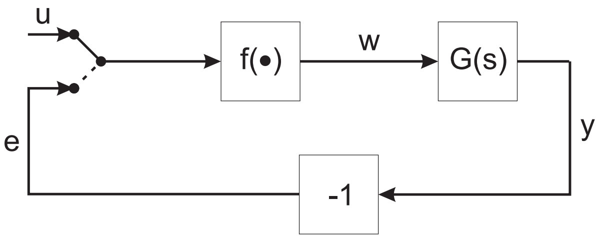


Conditions for oscillation: $e = u$, i.e.

$$e = A_1(C)|G(i\omega)| \sin(\omega t + \phi_1(C) + \psi(\omega) + \pi) = C \sin(\omega t) = u$$

The same amplitude: $A_1(C)|G(i\omega)| = C$
The phase is the same, save for 2π: $\phi_1(C) + \psi(\omega) = \pi + \nu 2\pi$.

Follow the sine around the loop (III/III)



or, more compactly (phase and amplitude in *one* equation)

$$Y_f(C)G(i\omega) = -1$$

since $G(i\omega) = |G(i\omega)|e^{i\psi(\omega)}$.

Describing function – interpretation

The describing function is given by

$$Y_f(C) = \frac{A_1(C)e^{i\phi_1(C)}}{C}$$

- **Interpretation:** The "transfer function" for the nonlinearity for a stationary sine (the fundamental frequency). An "amplitude-dependent gain".
- The gain is given by $|Y_f(C)|$ and the phase shift is given by $\arg Y_f(C)$.

A few concepts to summarize lecture 8

Circle criterion: The circle criterion generalizes the Nyquist criterion to static nonlinearities.

Describing function: An approximate method for examining existence of periodic solutions for systems involving a static nonlinearity in the feedback loop.