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Automatic Control III

Lecture 3 – Basic limitations and conflicts

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Summary of lecture 2 (I/II)

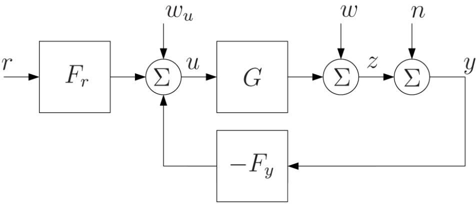
Find the poles and the zeros directly from $G(s)$ for a MIMO system:

- The pole polynomial is the **least common denominator** of all minors of $G(s)$. The system poles are then given by the zeros of the pole polynomial.
- The zero polynomial is the **greatest common divisor** of the numerators of the maximal minors (after they have been normalized to have the pole polynomial as their denominator)

Summary of lecture 2 (II/II)

Important transfer functions:

$$G_c = (I + GF_y)^{-1} GF_r$$
$$S = (I + GF_y)^{-1}$$
$$S_u = (I + F_y G)^{-1}$$
$$T = (I + GF_y)^{-1} GF_y$$



Relations among signals:

$$z = G_c r + Sw - Tn + GS_u w_u$$
$$u = \underbrace{S_u F_r}_{G_{ru}} r \underbrace{- S_u F_y}_{G_{wu}} (w + n) + S_u w_u$$

Stability of S, S_u, G_{wu}, $G_{w_u y}$ (and F_r) guarantees **internal stability** of the closed-loop system.

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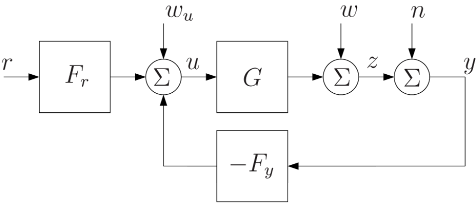

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Stability of S, S_u, G_{wu}, $G_{w_u y}$ (and F_r) guarantees **internal stability** of the closed-loop system.

Compromise S and T – bounding loop gain

Ideally, the sensitivity functions S and T should both be small, but

$$S + T = I.$$

Compromise:

- Let S be small for **low** frequencies (dampen process disturbances, insensitivity to modeling errors)
- Let T be small for **high** frequencies (reduce the effect of measurement errors, stability)

Note that both S and T are uniquely determined by the loop gain GF_y.

Compromise S and T – bounding loop gain

For a small $\epsilon > 0$ we have the following approximate relationships

$$|S| < \epsilon \Leftrightarrow |GF_y| > \frac{1}{\epsilon},$$
$$|T| < \epsilon \Leftrightarrow |GF_y| < \epsilon.$$

This is another way of seeing that S and T can not be made "small" at the same frequencies.

Relevant question: How fast can we transition from "a small S" to "a small T"?

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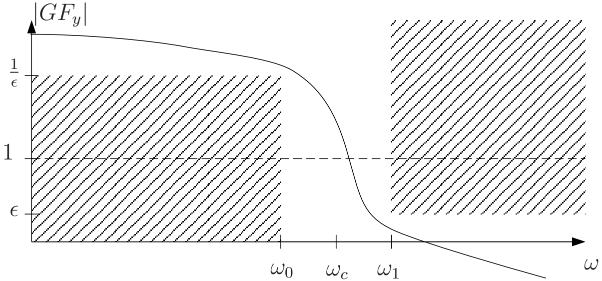
Relevant question: How fast can we transition from "a small S" to "a small T"?

How fast can $|GF_y|$ change?



How small can $\omega_1 - \omega_0$ become?

There is a **relationship between the amplitude and the phase** of transfer functions (e.g. GF_y) that prevents us from making an arbitrarily fast transition...

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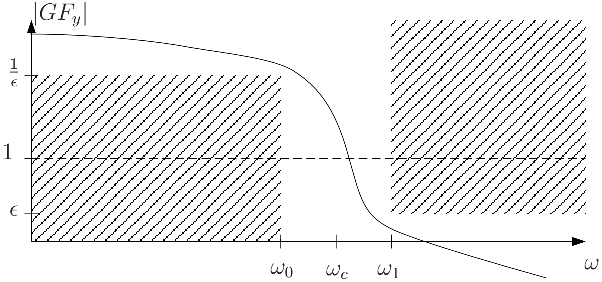

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There is a **relationship between the amplitude and the phase** of transfer functions (e.g. GF_y) that prevents us from making an arbitrarily fast transition...

Bode's relation – coupling of amplitude and phase

Theorem: Let $f(x) = \log|G(ie^x)|$. We have

$$\arg G(i\omega) \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d}{dx} f(x) \cdot \psi(x - \log\omega) dx$$

where the weight function ψ is given by

$$\psi(x) = \log \frac{e^x + 1}{|e^x - 1|}.$$

(The inequality is replaced by an equality if $G(s)$ do not have any zeros in the RHP.)

Interpretation: Bode's relationship provides an upper bound on the phase, which depends on the derivative of the amplitude curve.

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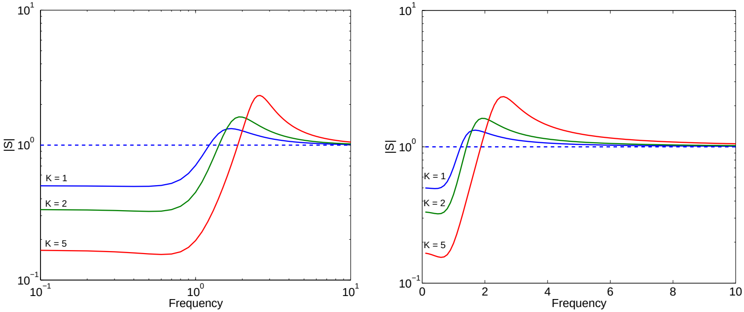
How small can S become? – an example

$$G(s) = \frac{1}{s^2 + s + 1}, \qquad F_y(s) = K.$$



The frequency scale is logarithmic to the left and linear to the right.

This means even if we neglect the fact that $S + T = 1$, we cannot make S arbitrarily small everywhere!

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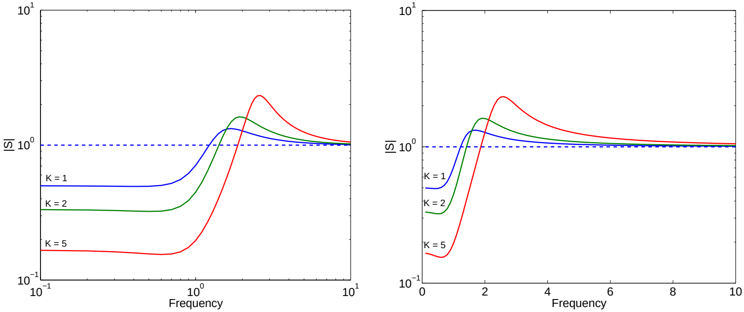

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How small can S become? – Bode's integral theorem

Assume that the loop gain GF_y has M poles in the RHP: $p_i; i = 1, \dots, M$ and that $|GF_y|$ decays at least as $|s|^{-2}$ when $|s| \to \infty$. Then, (scalar case)

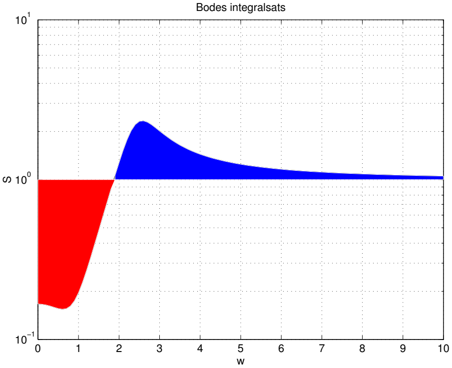
$$\int_0^\infty \log |S(i\omega)| d\omega = \pi \sum_{i=1}^{M} \mathrm{Re}(p_i).$$

Multivariable square:

$$\int_0^\infty \log |\det S(i\omega)| \, d\omega = \pi \sum_{i=1}^{M} \mathrm{Re}(p_i).$$

where $|\det S| = \sigma_1 \cdots \sigma_m$.

Invariance property of S



A sensitivity $|S(i\omega)| < 1$ for certain frequencies (red region) has to be payed back with $|S(i\omega)| > 1$ for other frequencies (blue region). If GF_y is unstable the situation gets even worse.

The waterbed effect

Stein, G. **Respect the unstable**. *IEEE Control Systems Magazine*, 23(4): 12–25, 2003.

Bode's integral theorem – consequences

1. **Stable systems:** The sensitivity cannot be < 1 for all frequencies, $\int_0^\infty \log |S(i\omega)| \mathrm{d}\omega = 0$.
2. $\int_0^\infty \log |S(i\omega)| \mathrm{d}\omega$ is **invariant** to the choice of the controller.
3. The assumption on decay rate ($|s|^{-2}$ for large s) is fulfilled if both $G(s)$ and $F_y(s)$ are strictly proper (physically reasonable).
4. For unstable loop gains the situations gets worse. The regions where $|S(i\omega)| > 1$ dominate. The faster the unstable poles, the worse the situation becomes.

Controlling unstable systems

- Requires very reliable controllers! If it breaks...

Gripen JAS39 prototype accident on 2 February 1989 due to rate limit of the control surfaces (`https://www.youtube.com/watch?v=k6yVU_yYtEc`)

- A limited control signal implies that stabilization is normally only possible in part of the state space.
- An unstable real pole p_1 sets a lower bound for the bandwidth:

$$\omega_B > 2p_1 \text{ (roughly)}$$

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Balancing a stick (I/III)

Consider the task of balancing a stick of length l and mass m on your finger.

Input (the acceleration of the finger): $u = \ddot{x}$.
Output (stick angle): $y = \varphi$.

Position of the center of mass:

$$\left(x + \frac{l}{2}\sin(\varphi),\ \ \frac{l}{2}\cos(\varphi) \right).$$

Newton tells us (in the x direction):

$$\begin{aligned} F\sin(\varphi) &= m\ddot{x} + m\frac{d^2}{dt^2}\left(\frac{l}{2}\sin(\varphi)\right) \\ &= mu + m\frac{l}{2}\left(\ddot{\varphi}\cos(\varphi) - \dot{\varphi}^2\sin(\varphi)\right). \end{aligned}$$

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Balancing a stick (II/III)

Newton tells us (in the y direction):

$$
\begin{aligned}
F\cos(\varphi) - mg &= m\frac{d^2}{dt^2}\left(\frac{l}{2}\cos(\varphi)\right) \\
&= m\frac{l}{2}\left(-\ddot{\varphi}\sin(\varphi) - \dot{\varphi}^2\cos(\varphi)\right).
\end{aligned}
$$

Combining these two equations will now result in

$$
\frac{l}{2}\ddot{\varphi} - g\sin(\varphi) = -u\cos(\varphi)
$$

Assume that φ is small (i.e. $\sin(\varphi) \approx \varphi, \cos(\varphi) \approx 1$) results in

$$
G(s) = \frac{-2/l}{s^2 - \frac{2g}{l}}
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Balancing a stick (III/III)

We have now shown that the "stick-system" is unstable with poles in

$$\pm\sqrt{\frac{2g}{l}}$$

Assuming that we have a $\tau = 0.1$ s delay in our eye-hand coordination (0.1 s $\Leftrightarrow 10$ Hz)

$$\omega_{\mathsf{B}} < 1/\tau; \quad \omega_{\mathsf{B}} = 2\pi 10 \approx 2\pi\sqrt{\frac{2g}{l}} \quad \Rightarrow l \approx 20\text{cm}$$

$l = 1$ m $\Rightarrow \omega_{\mathsf{B}} \approx 4.4$ Hz.

If you do not want to balance the stick yourself you can of course have someone (or rather something) else balance it for you:

`https://www.youtube.com/watch?v=XxFZ-VStApo`

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Yet another example of an unstable system

The transfer function for an **ordinary bike** (from steering angle to tilt angle):

$$\text{const} \cdot V \frac{s + V/a}{s^2 - g/h}$$

V: speed, h: height of the center of masses (CoM), a: distance CoM to the rear wheel, $g = 9.82\,\text{m/s}^2$.

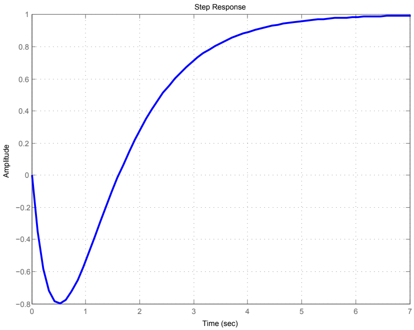
- Similar to an inverted pendulum.
- Pole in the RHP
  - Depends on the height of the bike: A low bike is harder to balance than a tall bike (the unstable pole is moved further into the RHP)
- Zero in the LHP
  - Depends on the speed and the location of the CoM.

Zeros in the RHP – example

Step response for

$$\frac{-4s+2}{(s+1)(s+2)}$$

"The step response initially heads in the wrong direction"



Zeros in the RHP – intuition

- "The system gain for fast changes has the opposite sign compared to changes with slow changes."
- "A controller that controls slow changes well makes use of the wrong sign for fast changes and could thereby destabilize the system."
- "Avoid fast changes by giving the system a low bandwidth."
- **Conclusion:** A zero z_0 in the RHP appears to impose an **upper** limit on the bandwidth.

More rigorously we can make use of Theorem 7.4 to obtain the following rule of thumb:

$$\omega_B \leq \frac{z_0}{2}$$

Zeros and feedback

Consider

$$G_c = \frac{GF_r}{1 + GF_y}, \quad T = \frac{GF_y}{1 + GF_y}$$

If $G(z_0) = 0$, then we also have $G_c(z_0) = 0$ and $T(z_0) = 0$.

- The zeros of $G(s)$ will also become the zeros of $G_c(s)$ and $T(s)$, i.e. feedback cannot move the zeros.
- Zeros can sometimes be cancelled.
- But, a zero in the RHP cannot be cancelled, since that would require an unstable pole in F_y and/or F_r.
- Hence, it is impossible to get around the limitations imposed by zeros in the RHP.

Both a pole and a zero in the RHP – example

Transfer function for a bicycle with rear wheel steering:

$$\text{const} \cdot V \frac{-s + V/a}{s^2 - g/h}$$

If the pole is to the right of the zero the systems is extremely hard to control.

"Our rules of thumb for the crossover frequency collide."

Dynamics and control of bicycles:

Åström, K. J., Klein, R. E. and Lennartsson, A. Bicycle dynamics and control, *IEEE Control systems magazine*, 25(4):26-47, 2005.

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Poles and zeros in the RHP – summary

1. Pole in the RHP
   - Implies a **lower** limit on the bandwidth (fast control)
   - High operational reliability requirements (recall Gripen accident)
   - Harder when the pole is moved to the **right**
2. Zero in the RHP
   - Implies an **upper** limit on the bandwidth (slow control)
   - Harder when the zero is moved to the **left**
3. Both a zero and a pole in the RPH
   - Zero to the right of the pole: Can be ok.
   - Zero to the left of the pole: extremely hard to control.

Stein, G. **Respect the unstable**. *IEEE Control Systems Magazine*, 23(4): 12–25, 2003.

Bounds on the control signal – example

On 24 November 2004, the passenger ferry Casino Express was grounded while entering the port of Umeå due to high winds.

Crash investigation:

- The wind power on the upper parts was at least 600 kN (20 m/s wind speed).
- No combination of control signals (propellers, rudders) could have compensated this.
- Not even tug assistance (up to 260 kN) was enough.

Control signal – compensate for disturbances

Control signal: u, disturbance signal: d

$$y = G(s)u + G_d(s)d,$$

for some G, G_d (scalar for simplicity)

- Assume that $|u(t)| \leq u_0$ and $|d(t)| \leq d_0$.
- Then it must hold that

$$u_0 \geq \frac{|G_d(i\omega)|}{|G(i\omega)|} d_0, \quad \forall \omega$$

  if d is to be perfectly eliminated.
- If this is not fulfilled, there is no controller (linear or nonlinear) that can provide perfect disturbance attenuation.

A few concepts to summarize Lecture 3

Bode's relation: reveals a fundamental coupling between the phase and the amplitude. More specifically, it provides an upper bound on the phase, which depends on the derivative of the amplitude curve.

Bode's integral: Provides a fundamental limitation in terms of what can be achieved by control. Views control design as a way of redistributing disturbance attenuation over different frequencies.

Waterbed effect: If disturbance attenuation is improved in one frequency range, it will be worse in another.

Poles in RHP: Impose a **lower** limit on the bandwidth and it impose very high operational reliability requirements. Harder when the pole is moved to the **right**.

Zeros in RHP: Impose an **upper** limit on the bandwidth. Harder when the zero is moved to the **left**.