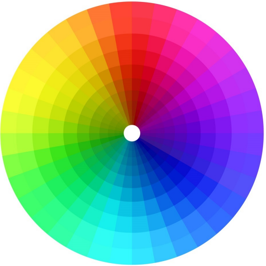
Filter bank methods



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The spectral estimation problem

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Problem

Estimate $\phi(\omega),\ \omega \in [-\pi,\ \pi]$ from a finite number of samples $\{y(t)\}_{t=1}^{N}$.

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- Non-parametric: Assume $\phi(\omega)$ is approx. constant in intervals $[\omega - \beta\pi, \omega + \beta\pi]$ for some $\beta \ll 1$.

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- The non-parametric typically used when we do not have enough information to setup a parametric model.

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Non-parametric approach

Alternative formulation

From a finite dataset, estimate how the power is distributed over narrow frequency bins $[\omega - \beta\pi, \omega + \beta\pi]$.

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- Identifiability: Want $1/\beta \leq N \iff N\beta \geq 1$.
- Variance/resolution: β sets resolution and number of parameters.

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The filter bank (FB) approach

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The filter bank approach

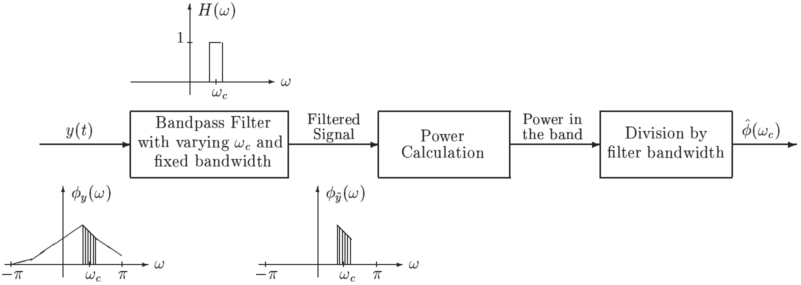
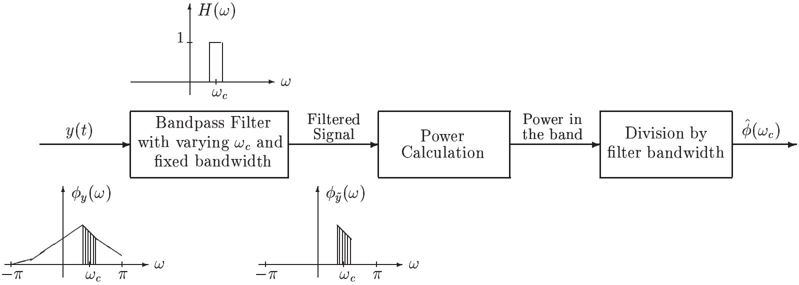


Figure 5.1. The filter bank approach to PSD estimation.

- Design filter that extract part of the spectrum around ω_c.

The filter bank approach



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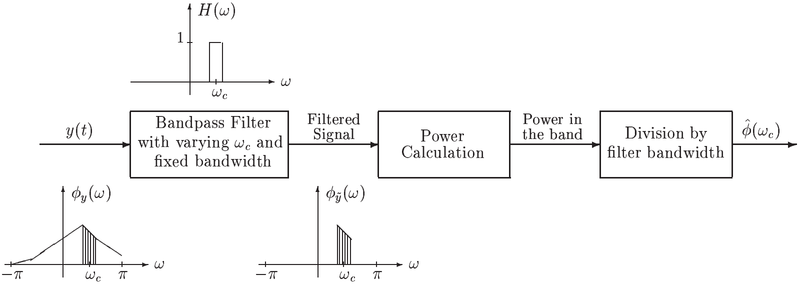

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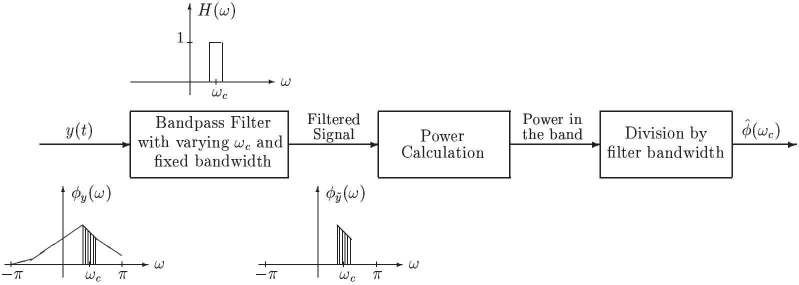

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- Design filter that extract part of the spectrum around ω_c.
- Compute filtered signal $y_F(t) = H(q)y(t)$.
- Estimate power of $y_F(t)$ and divide with bandwidth.
- Sweep over ω using filters with different ω_c.

Power estimation

Let $y_F(t) = H(q)y(t)$, and N_F be the number of filtered samples.

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Then a good approximation of $\phi(\omega_c)$ is

$$\hat{\phi}_{\mathrm{FB}}(\omega_c) = \frac{1}{N_F\beta}\sum |y_F(t)|^2.$$

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Discrete time filters

- Linear filter can be written as $y_F(t) = H(q)y(t)$.

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- If $m = N$ we only compute one filtered value $y_F(N)$.
- Long filter in time $\implies$ Few filtered values $\implies$ high variance in estimate

$$\hat{\phi}_{\mathrm{FB}}(\omega) = \frac{1}{N_F \beta} \sum |y_F(t)|^2.$$

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Conflicting requirements

Assumptions for good FB-based PSD estimate

1. $\phi(\omega)$ is (nearly) constant over the filter passband;
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Again, bias/variance trade-off!

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The periodogram as an FB-method

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The periodogram

$$\hat{\phi}_p(\tilde{\omega}) = \frac{1}{N}\left|\sum_{t=1}^{N} y(t)e^{-i\tilde{\omega}t}\right|^2 = \frac{1}{N}\left|\sum_{k=0}^{N-1} y(N-k)e^{i\tilde{\omega}k}\right|^2$$

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- $\beta = 1/N$ and $N_F = 1$ (number of filtered samples).

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Connection to the Periodogram (2)

Conclusion

The Periodogram $\hat{\phi}_p(\omega)$ can be exactly obtained by the filter bank approach with the derived $H(\omega)$ filters

- The bandwidth is $\beta = 1/N$.
- Due to long filter, power calculation is made using **a single sample** of filtered signal $y_F(N) \implies$ high variance

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Baseband processing

- One passband filter $H_{\tilde{\omega}}(\omega)$ for each $\tilde{\omega}$?

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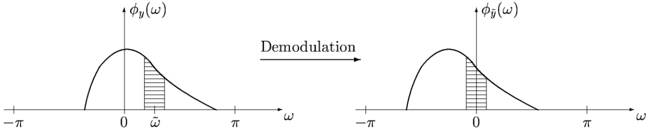

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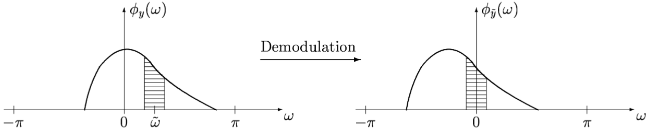
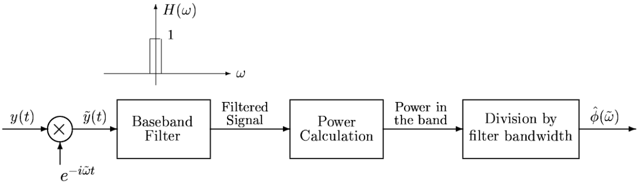


- Can then apply a baseband filter $H(\omega)$ on $\tilde{y}(t)$ for each $\tilde{\omega}$.



Refined filter bank approach

Refined filter bank (RFB) approach

- Goal: Pass the baseband $[-\beta\pi, \beta\pi]$ as undistorted as possible, and attenuate all other frequencies.

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- Idea: Choose h_k so that power in baseband is large relative to total power.

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Slepian baseband filters

This leads to the following optimization problem

$$\underset{h}{\text{maximize}}\ h^*\Gamma h \quad \text{subject to } h^*h = 1$$

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Slepian baseband filters

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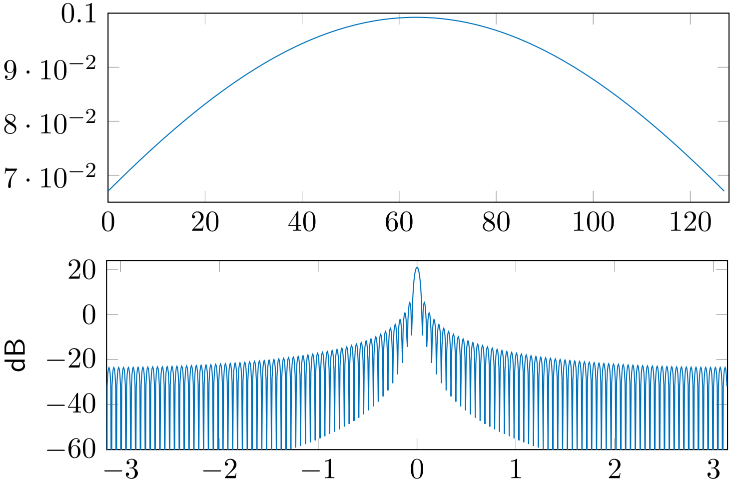
- If $K = N\beta > 1$ (assume K is an integer):
  The K eigenvectors of Γ corresponding to the largest eigenvalues give a set of orthogonal impulse responses that approximately cover $[-\beta\pi, \beta\pi]$.
- Technical detail: With $h^*h = 1$, the gain of the filter is approximately $\frac{1}{\beta}$ in the baseband (and not 1). Hence, the estimated power of the filtered signal should not be multiplied by $1/\beta$ to get estimate of ϕ in this case, since the factor is already built into the filter.

Example

$$N = 256, \quad \beta = 1/N, \quad K = N\beta = 1.$$



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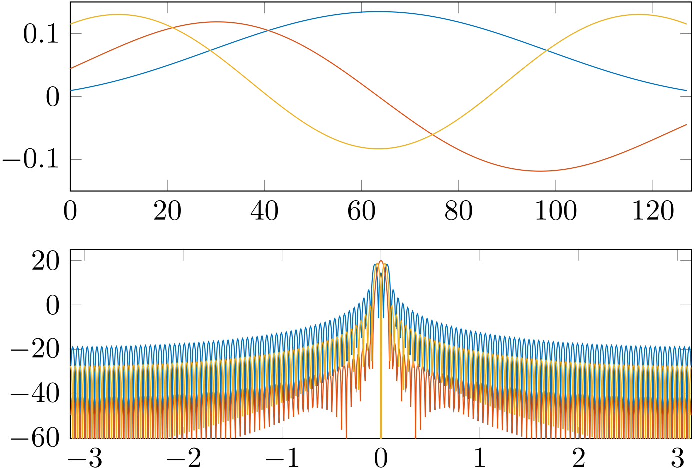
Example

$$N = 256, \quad \beta = 3/N, \quad K = N\beta = 3.$$



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- Time-windowed periodogram.
- $h_k \to 1/\sqrt{N}$. That is, for large N approximately equal to periodogram.

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- Variance will almost reduce by a factor K.

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$$E\left\{ |\tilde{y}_F(t)|^2 \right\} = h^* R h,$$

$$R = E\left\{ \begin{bmatrix} y(t) \\ \vdots \\ y(t-m) \end{bmatrix} \begin{bmatrix} y^*(t) & \cdots & y^*(t-m) \end{bmatrix} \right\}.$$

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- Idea: Minimize the total power while passing the frequency $\tilde{\omega}$ undistorted.

The Capon method (2)

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- Needed: Order m, estimate $\hat{R}$, and bandwidth β.

Variants of Capon

In practice, we use the sample covariance matrix $\hat{R}$. The two expressions for β derived in the book give:

$$\text{CM1: } \beta = \frac{1}{m+1}, \quad \hat{\phi}(\omega) = \frac{m+1}{a^*(\omega)\hat{R}^{-1}a(\omega)}$$

$$\text{CM2: } \beta = h^*h, \quad \hat{\phi}(\omega) = \frac{a^*(\omega)\hat{R}^{-1}a(\omega)}{a^*(\omega)\hat{R}^{-2}a(\omega)}$$

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- The first is from the time-bandwidth product
- The second is derived using a rectangle of the same energy
  - The bandwidth is then both data and frequency dependent!
  - Often a more exact expression for β

Capon properties

$$\hat{\phi}(\omega) = \frac{1}{\beta a^*(\omega)\hat{R}^{-1}a(\omega)}$$

Choice of model order m:

- $m < N/2$ so that $\hat{R}$ is invertible.
- Smaller m leads to more accurate estimates of $\hat{R}$ (less variance), but worse resolution.

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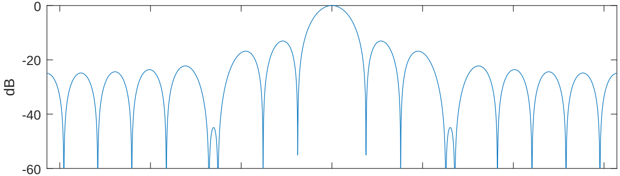
Data-dependent:

- Can adjust side/main-lobes to the data.
- Can outperform the Periodogram-based methods for fine details (e.g. closely spaced peaks).



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Summary

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- Connection to the Periodogram
- Bias/variance trade-off (filter length/power calc.)
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Later:

- Examples in Lab 4!
- Only repetition and guest lecture left.

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Guest lecture

Professor Petre Stoica (`http://user.it.uu.se/~ps/ps.html`), who wrote the course book together with Randy Moses, will give the planned guest lecture on October 14, 11.15-12, in ITC 1146 (it's in your schedules as lecture 9). One of the top researchers in all of engineering, world wide! The topic will be "Array processing" taking up applications of spectral analysis such as RADAR.

See Chapter 6 in the book or `https://en.wikipedia.org/wiki/Array_processing` for more details