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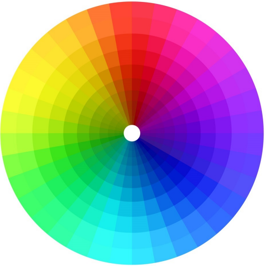
Subspace methods for line spectra



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2019-09-19

Summary from last lecture

- Signal model for line spectra
- Estimating the spectrum $\Longrightarrow$ frequency estimation
- Nonlinear least squares
- Models of the covariance matrix and ARMA-type model
- Singular value decomposition (SVD)

Today

- A recap from L5
- Subspace methods for frequency estimation
  - High order Yule-Walker
  - MUSIC
  - Min-Norm
  - ESPRIT
- Makes use of the covariance structure

Line spectra

Signal model:

$$\boxed{y(t) = \sum_{k=1}^{n} \alpha_k e^{i(\omega_k t + \varphi_k)} + e(t)}$$

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Theoretical spectrum

$$\boxed{\phi(\omega) = 2\pi \sum_{p=1}^{n} \alpha_p^2 \delta(\omega - \omega_p) + \sigma^2}$$

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Problem

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- Hard (impossible) to use in general.
- Today we will instead use subspace methods.

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Singular value decomposition (SVD)

Definition

Factor a matrix $A \in \mathbb{C}^{m \times n}$ as

$$A = U \Sigma V^*$$

where $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are unitary matrices, and $\Sigma \in \mathbb{R}^{m \times n}$ is diagonal with non-negative diagonal values σ_i (singular values) ordered in decreasing order

(See A.4 in the book for the details)

- Low-rank approximations
- Useful to find the best fit subspaces of A
- Projection onto subspace – perpendicular distance
- LS fit – vertical distance

SVD (2)

$A \in \mathbb{C}^{m\times n}$ is of rank $r < \min(m,n)$ then

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} = U_1 \Sigma_1 V_1^*$$

where $\Sigma_1 \in \mathbb{R}^{r\times r}$, then

- U_1 is an orthonormal basis in $\mathcal{R}(A)$
- U_2 is an orthonormal basis in $\mathcal{N}(A^*)$
- V_1 is an orthonormal basis in $\mathcal{R}(A^*)$
- V_2 is an orthonormal basis in $\mathcal{N}(A)$

Moreover, $\mathcal{R}(A)$ and $\mathcal{N}(A^*)$ are orthogonal to each other and they span $\mathbb{C}^m$
($\mathcal{N}(A^*)$ is the orthogonal complement to $\mathcal{R}(A)$ and vice versa)

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- Multiply both sides with some filter $\tilde{A}(z)$ to get

$$B(z)y(t) = B(z)e(t),$$

where $B(z) = A(z)\tilde{A}(z)$ has order $L > n$ ("high order").

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- Idea: Estimate $B(z)$ and take the n roots $\{r_k\}_{k=1}^{n}$ closest to the unit circle to get estimates $\hat{\omega}_k = \arg(r_k)$.

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- From before we know that

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- Important property: $\text{rank}(\Omega) = n$.
- When $L > n$ this rank-condition gives extra information about the structure of Ω.

HOYW-equations with estimated ACS

- Form $\widehat{\Omega}$ and $\hat{\rho}$ by replacing $r(k)$ with $\hat{r}(k)$ in Ω and ρ,

$$\widehat{\Omega} b \approx -\hat{\rho}. \tag{1}$$

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- $\widehat{\Omega}_n$ should be closer to the true Ω since we exploited
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- Find $\hat{b}$ by solving (LS)
$$\widehat{\Omega}_n \hat{b} \approx -\hat{\rho} \Longrightarrow \hat{b} = -\widehat{\Omega}_n^{\dagger} \hat{\rho}.$$

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HOYW in practice

- Use SVD on $\widehat{\Omega}$,

$$\widehat{\Omega} = U\Sigma V^* = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix},$$

where Σ_1 contains the n largest singular values.

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- Note: SVD can also give an idea about the size of n.

MUltiple SIgnal Classification (MUSIC)

The Covariance Matrix Equation

Let

$$a(\omega) = \begin{bmatrix} 1 & e^{-i\omega} & \cdots & e^{-i(m-1)\omega} \end{bmatrix}^\top \text{ for some } m > n$$
$$A = \begin{bmatrix} a(\omega_1) & \cdots & a(\omega_n) \end{bmatrix} \in \mathbb{C}^{m \times n}$$

Let $\tilde{y}(t) = \begin{bmatrix} y(t) & \cdots & y(t-m+1) \end{bmatrix}^\top$, then

$$\tilde{y}(t) = A\tilde{x}(t) + \tilde{e}(t)$$
$$\tilde{x}(t) = \begin{bmatrix} x_1(t) & \cdots & x_n(t) \end{bmatrix}^\top, \quad \tilde{e}(t) = \begin{bmatrix} e(t) & \cdots & e(t-m+1) \end{bmatrix}^\top$$

Covariance matrix: $R = E\{\tilde{y}(t)\tilde{y}^*(t)\} = APA^* + \sigma^2 I$, where

$$P = E\{\tilde{x}(t)\tilde{x}^*(t)\} = \begin{bmatrix} \alpha_1^2 & & 0 \\ & \ddots & \\ 0 & & \alpha_n^2 \end{bmatrix}$$

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Some rank conditions

- $\text{rank}(A) = n$ (if $m \geq n$ and $\omega_k \neq \omega_p$ for $k \neq p$).

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- The eigenvalues λ_k of $R = APA^* + \sigma^2 I$ are thus given by
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- The eigenvalues λ_k of $R = APA^* + \sigma^2 I$ are thus given by
$$\lambda_k = \tilde{\lambda}_k + \sigma^2 > 0, \quad k = 1, \ldots, m,$$
- and thus
$$\text{rank}(R) = m.$$

Eigendecomposition of R

$$R = APA^* + \sigma^2 I \quad (m > n)$$

- Eigenvalues: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > \lambda_{n+1} = \cdots = \lambda_m = \sigma^2$

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- Eigenvectors: $s_1, \ldots, s_n,\ g_1, \ldots, g_{m-n}$ orthonormal

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$$0 = A^*G = \begin{bmatrix} a^*(\omega_1)G \\ \vdots \\ a^*(\omega_n)G \end{bmatrix} \implies a^*(\omega)G = 0 \text{ if } \omega = \omega_k.$$

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- (Any $a(\tilde{\omega})$ that satisfy the equation belongs to $\mathcal{N}(G^*)$, which is a subspace of dimension n. At most n vectors on the form $a(\omega)$ can belong to this subspace, which by the above must be $\{a(\omega_k)\}_{k=1}^n$.)

MUSIC

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closest to the unit circle

Min-Norm method

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- Which vector to use? Are strong empirical evidence that the vector

$$g = \begin{bmatrix} 1 \\ \tilde{g} \end{bmatrix} \in \mathcal{R}(G)$$

with minimum Euclidean norm have good properties (accuracy, and reduced risk of spurious ω-estimates).

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Estimation of Signal Parameters by Rotational Invariance Techniques (ESPRIT)

esprit (noun) – vivacious cleverness or wit

Theory of ESPRIT

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- If estimates of α_k are needed, use LS with the estimated frequencies.

Performance

- All methods can achieve super resolution (beyond the Periodogram)
- Finding the frequencies through subspace methods is tractable as opposed to NLS (typically)
- ESPRIT is typically slightly better than the others
- We still have to choose the order n

Method	Computational Burden	Accuracy / Resolution	Risk for False Freq Estimates
Periodogram	small	medium-high	medium
Nonlinear LS	very high	very high	very high
Yule-Walker	medium	high	medium
Pisarenko	small	low	none
MUSIC	high	high	medium
Min-Norm	medium	high	small
ESPRIT	medium	very high	none

MATLAB

Useful MATLAB functions:

- `hoyw()`
- `music()`
- `minnorm()`
- `esprit()`
- `freqaphi()`
- `lsa()`

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Next lecture:

- How to choose the model order?
- Information criteria
- Heuristic approaches
- Cross validation