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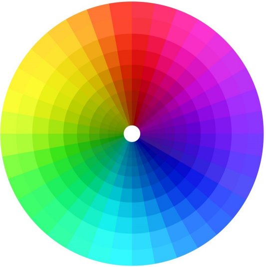
Nonparametric methods: The Periodogram and the Correlogram



Per Mattsson

Systems and Control
Department of Information Technology
Uppsala University

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Summary

The Spectral Estimation Problem

From a finite dataset $\{y(1), y(2), \ldots, y(N)\}$ of a stationary signal, determine an estimate $\hat{\phi}(\omega)$ of the PSD $\phi(\omega)$ for $\omega \in [-\pi,\ \pi]$

- Power spectral density (PSD)

$$\phi(\omega) = \lim_{N\to\infty} E\left\{ \frac{1}{N} \left| \sum_{t=1}^{N} y(t)e^{-i\omega t} \right|^2 \right\}$$

- Autocovariance sequence (ACS)

$$r(k) = E\{y(t)y^*(t-k)\}$$

- DTFT of the ACS gives PSD

$$\phi(\omega) = \sum_{k=-\infty}^{\infty} r(k)e^{-i\omega k}$$

Today

- Estimation theory.
- The Periodogram and Correlogram methods
- Properties and equivalences
- Computing the periodogram in practice.
- Useful Matlab functions.

Time allocation during the course

- Lectures: Present key concepts and methods, Q&A
- Exercise sessions: You/we solve theory problems (prove/show/verify something)
- Labs/homeworks: Simulate things in MATLAB and try the methods
- All over: Discuss properties, interpretations, and intuition

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Beyond the lectures

- You will need to read more in the book to get the full picture
- The computer labs will let you compare theory and practice in a systematical way
- The homework will require you to code, try different approaches, make engineering decisions, and present your results

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per.mattsson@it.uu.se

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- For any experiment (collected data $\mathbf{y}$) you get one realization $\hat{\boldsymbol{\theta}}$. If you redo the experiment, you will probably get a different $\hat{\boldsymbol{\theta}}$.

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- With this view, we can talk about the expected value and variance of $\hat{\boldsymbol{\theta}}$.

Bias, variance, consistency

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- **Consistency:** The way we estimate (our estimator) is consistent if $\hat{\boldsymbol{\theta}} \to \boldsymbol{\theta}$ as the number of data points used for estimation goes to infinity.

Parametric vs. non-parametric

Nonparametric (Classical) Methods

No model imposing constraints, or rather, a very general model (i.e. many “parameters”)

Parametric (Modern) Methods

Model the data and estimate the model parameters.

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Model the data and estimate the model parameters.

Trade-offs: (Robustness, and Resolution vs. Variance)

- Parametric methods: better estimates **if** model is good, otherwise the results can be heavily biased.
- Non-parametric methods: works for any dataset, but have relatively high variance, even when $N \to \infty$.
- Resolution – detecting/separating spectral components.

Periodogram and correlogram

Non-parametric PSD estimation

Both

$$\phi(\omega) = \lim_{N\to\infty} E\left\{ \frac{1}{N} \left| \sum_{t=1}^{N} y(t) e^{-i\omega t} \right|^2 \right\}$$

and

$$\phi(\omega) = \sum_{k=-\infty}^{\infty} r(k) e^{-i\omega k} \quad \text{where} \quad r(k) = E\{y(t)y^*(t-k)\}$$

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- Idea used by Schuster (~ 1900) to determine "hidden periodicities".

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  where $\hat{r}(k)$ is an estimate of $r(k) = E\{y(t)y^*(t-k)\}$.
- Finite sums $\Longrightarrow$ can be computed in practice.

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Estimation of ACS

Standard unbiased estimate:

$$\hat{r}(k) = \frac{1}{N-k} \sum_{t=k+1}^{N} y(t) y^*(t-k), \quad 0 \leq k \leq N-1.$$

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For both estimates: $\hat{r}(k) = \hat{r}^*(-k)$ for $k < 0$.

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Equivalence

If the biased ACS estimator $\hat{r}(k)$ is used in $\hat{\phi}_c(\omega)$, then

$$\hat{\phi}_p(\omega) = \frac{1}{N}\left|\sum_{t=1}^{N} y(t)e^{-i\omega t}\right|^2 = \sum_{k=-(N-1)}^{N-1} \hat{r}(k)e^{-i\omega k} = \hat{\phi}_c(\omega)$$

$$\boxed{\hat{\phi}_p(\omega) = \hat{\phi}_c(\omega)}$$

- Can analyze the estimates simultaneously!

Analysis

Analysis of the periodogram

- Since $\phi_p(\omega) = \phi_c(\omega)$ we can use both formulations in the analysis.
- Bias-variance tradeoff:

$$\mathsf{MSE} = \mathsf{var}(\hat{\phi}_p(\omega)) + |\mathsf{bias}(\hat{\phi}_p(\omega))|^2.$$

- Wanted: Small bias, small variance, consistent.

The bias of the periodogram

- With the biased ACS estimate we get

$$E\{\hat{\phi}_c(\omega)\} =$$

per.mattsson@it.uu.se

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$$=$$

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$$
\begin{aligned}
E\{\hat{\phi}_c(\omega)\} &= \sum_{k=-(N-1)}^{N-1} E\{\hat{r}(k)\}e^{-i\omega k} = \text{[Board]} \\
&= \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) r(k)e^{-i\omega k} \\
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$$= \sum_{k=-\infty}^{\infty} w_B(k) r(k) e^{-i\omega k},$$

where the Bartlett (triangular) window is given by

$$w_B(k) = \begin{cases} 1 - \frac{|k|}{N}, & |k| \le N-1 \\ 0, & |k| \ge N \end{cases}.$$

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- Large bias for small N since $w_B(k) \neq 1$.

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- Ideally $W_B(\omega)$ should approximate a Dirac impulse $\delta(\omega)$.

The Bartlett window

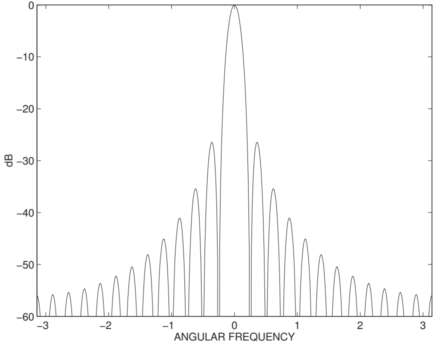

Figure 2.1. $W_B(\omega)/W_B(0)$, for $N = 25$.

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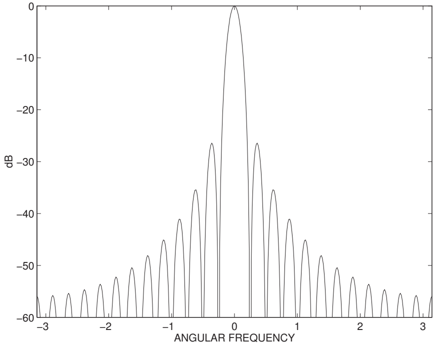

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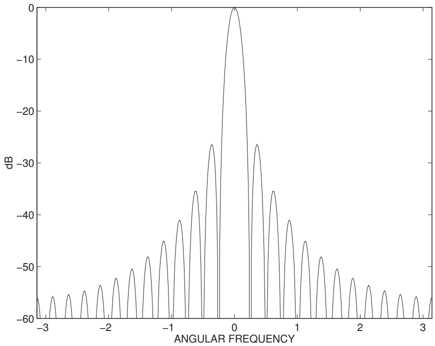

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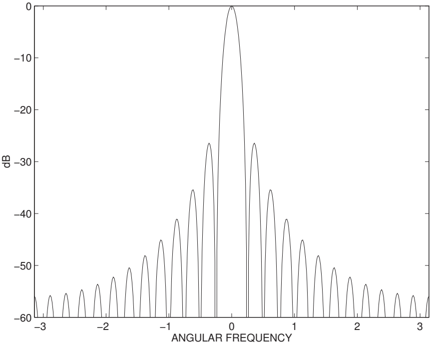

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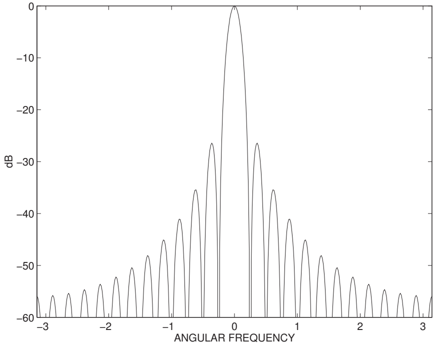

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- Sidelobes $\Longrightarrow$ Leakage.

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- That is, the periodogram is asymptotically unbiased.

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Summary of the periodogram bias properties

- For small N, severe bias.
- However,

$$\lim_{N\to\infty} W_B(\omega) = 2\pi\delta(\omega),$$

so,

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- That is, the periodogram is asymptotically unbiased.
- If bias was the only problem with periodogram, we could solve it by increasing the number of data points used. However....

The asymptotic variance/covariance

- As $N \to \infty$,

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$$=$$

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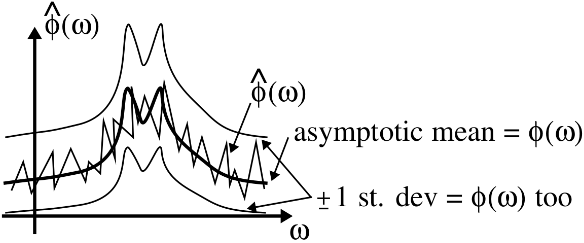

Summary of periodogram properties

$$\hat{\phi}_p = \frac{1}{N}\left|\sum_{t=1}^{N} y(t)e^{-i\omega t}\right|^2 = \sum_{k=-(N-1)}^{N-1} \hat{r}(k)e^{-i\omega k} = \phi_c(\omega)$$

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- Thus, $\hat{\phi}(\omega)$ has bad performance.

Intuitive explanation:

- $\hat{r}(k) - r(k)$ may be large for large $|k|$.
- Even if this is not case, the sum of many small errors can become large.

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Computing the periodogram

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- Can be computed using Fast Fourier Transform (FFT) (computations in the order of $N \log_2 N$ vs N^2 for direct approach). Often assumes $N = 2^m$.

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$$\{\underbrace{y(1), \ldots, y(N), 0, \ldots 0}_{L}\}$$

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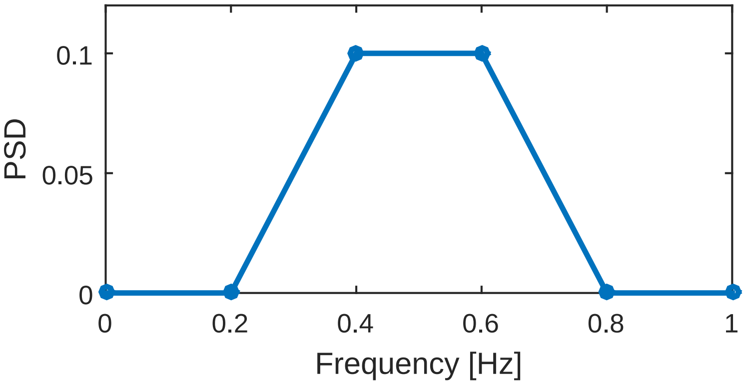
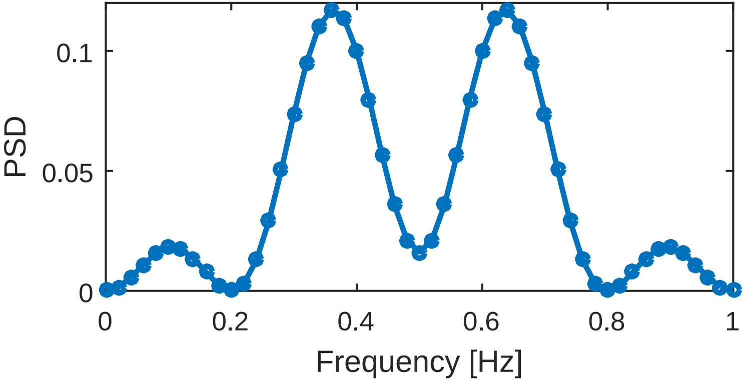
- There are FFT algorithms that eliminate operations on zeros.

Example: Zeropadding



6 samples (no zeropadding)

Example: Zeropadding



60 samples ($L = 10N$ zeropadding)

Summary

- The periodogram and correlogram.
- Smearing and leakage.
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- Using the FFT for computing the periodogram.

Try the following:

- Simulate a sinusoid/complex exponential in noise and plot the Periodogram (compute the periodogram manually and with FFT!)
- Add another periodic component, study the effect of smearing.
- Exercise 1.5, 1.7, 1.10, 2.3, 2.6

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MATLAB

Useful functions:

- `randn(N,1)` – Generates Gaussian white noise.
- `fft(y,L)` – Performs FFT padded with zeros.
- `fftshift(phi)` – Shifts frequencies from $[0, 2\pi]$ to $[-\pi, \pi]$.
- `periodogramse(y,v,L)` – (Windowed) periodogram.
- `filter(B,A,e)` – Generate $y(t) = \frac{B(q)}{A(q)}e(t)$. `A` and `B` are vectors with the coefficients of the polynomials $A(q), B(q)$.

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If you have also applied `fftshift`, the normalized frequencies are

```
» omega = (0:L-1)*2*pi/L - pi;
```