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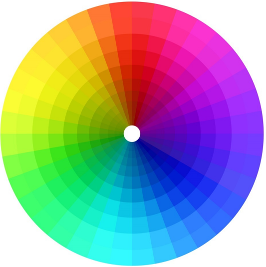
Line spectra, Covariance structure, and NLS



Per Mattsson

Systems and Control
Department of Information Technology
Uppsala University

2019-09-17

Summary from last lecture

- Parametric methods $\Longrightarrow$ fewer parameters to estimate (bias/variance)
- AR and ARMA models for rational (continuous) spectra
- AR – only peaks
- ARMA – peaks and valleys
- 4 approaches:
  - YW (AR)
  - LSAR (AR)
  - MYW (ARMA, two-stage)
  - LSARMA (ARMA, two-stage)
- ARMA is nonlinear, methods have user parameters

Today: Estimating line spectra.

per.mattsson@it.uu.se

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Sinusoidal model

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- Model with n components and noise,

$$x(t) = \sum_{k=1}^{n} \alpha_k e^{i(\omega_k t + \varphi_k)} = \sum_{k=1}^{n} x_k(t),$$
$$y(t) = x(t) + e(t).$$

Sinusoidal model, assumptions

$$y(t) = \underbrace{\sum_{k=1}^{n} \underbrace{\alpha_k e^{i(\omega_k t + \varphi_k))}}_{x_k(t)}}_{x(t)} + e(t)$$

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  - Some methods are poor for colored noise!

The spectrum

- PSD: $\phi(\omega) = \sum_{k=-\infty}^{\infty} r(k) e^{-i\omega k}, \quad r(k) = E\{y(t)y^*(t-k)\}$.
- With the sinusoidal model

$$E\{y(t)y^*(t-k)\} = \text{[Board]} =$$

per.mattsson@it.uu.se

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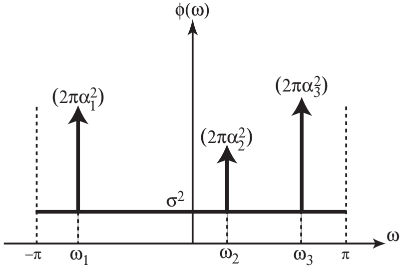
$$\phi(\omega) = [\text{Board}] = 2\pi \sum_{p=1}^{n} \alpha_p^2 \delta(\omega - \omega_p) + \sigma^2.$$

The theoretical spectrum

$$\boxed{\phi(\omega) = 2\pi \sum_{p=1}^{n} \alpha_p^2 \delta(\omega - \omega_p) + \sigma^2}$$



Need to estimate $\{\omega_k\}$, $\{\alpha_k\}$ and σ^2.

Parameter estimation

Parameter estimation

- If ω_k is given, then α_k (and φ_k) can be estimated from

$$y(t) = \sum_{k=1}^{n} \beta_k e^{i\omega_k t} + e(t),$$

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- Using least squares, we get

$$\min_{\beta} \|y - B\beta\|_2^2,$$

$$y = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix}, \quad B = \begin{bmatrix} e^{i\omega_1} & \cdots & e^{i\omega_n} \\ \vdots & & \vdots \\ e^{iN\omega_1} & \cdots & e^{iN\omega_n} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

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- However, in general ω_k are not given!

Nonlinear least squares

- What if we optimize over ω_k too? That is, minimize

$$f(\omega, \alpha, \varphi) = \sum_{k=1}^{N} \left| y(t) - \sum_{k=1}^{n} \alpha_k e^{i(\omega_k t + \varphi_k)} \right|^2 = \|y - B\beta\|_2^2.$$

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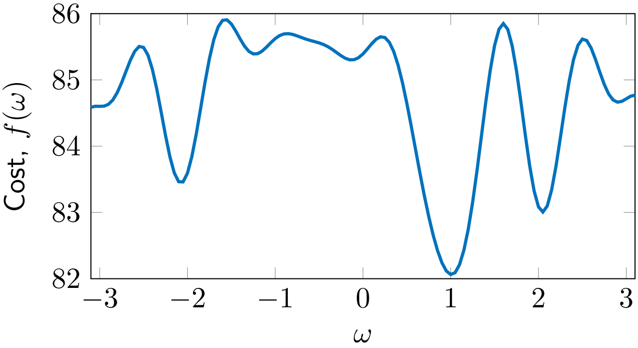
Nonlinear least squares, properties

$$\boxed{\begin{aligned}\hat{\omega} &= \underset{\omega}{\operatorname{argmin}} \|y - B(B^*B)^{-1}B^*y\|_2^2 \\ \hat{\beta} &= (B^*B)^{-1}B^*y\Big|_{\omega=\hat{\omega}}\end{aligned}}$$

- Gives a consistent estimate.
- For $N \gg 1$, $\operatorname{var}(\hat{\omega}_k) = \frac{6\sigma^2}{N^3\alpha_k^2}$.
- Best (minimum variance) unbiased estimator of ω_k.
- Problem: No (general) good method available to carry out the minimization.

Nonlinear least squares, example

The cost function when $y(t) = e^{it} + e(t)$, and $N = 100$:



No general way to find the global minimum!

Periodogram vs NLS

- For $n = 1$ (one complex sinusoid) it can be shown that the NLS estimate is given by

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- For tightly spaced peaks, super resolution are needed!

Models of sinusoidal signals in noise

ARMA-model

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- Use Modified Yule-Walker?

ARMA-model

$$y(t) = \sum_{k=1}^{n} \underbrace{\alpha_k e^{i(\omega_k t + \varphi_k)}}_{x_k(t)} + e(t).$$

- Annihilating filter: $(1 - e^{i\omega_k} z^{-1}) x_k(t) = 0.$
- Let
$$A(z) = \prod_{k=1}^{n} (1 - e^{i\omega_k} z^{-1}),$$
then
$$A(z)y(t) = A(z)e(t)$$
- Can estimate ω_k by estimating $A(z)$
- Note: Can't divide away $A(z)$ due to zeros on the unit circle!
- Use Modified Yule-Walker?
- Can be improved by instead estimating $B(z) = A(z)\tilde{A}(z)$ in
$$B(z)y(t) = B(z)e(t).$$

Covariance model (1)

Notation:

$$a(\omega) \triangleq [1,\ e^{-i\omega}, \ldots, e^{-i(m-1)\omega}]^{\mathsf{T}} \qquad (m \times 1)$$

$$A_m = [a(\omega_1), \ldots, a(\omega_n)] \qquad (m \times n)$$

For some $m \geq n$

$$\boxed{\operatorname{rank}(A_m) = n \quad \text{if} \quad \omega_k \neq \omega_p \text{ for } k \neq p}$$

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For some $m \geq n$

$$\boxed{\operatorname{rank}(A_m) = n \quad \text{if} \quad \omega_k \neq \omega_p \text{ for } k \neq p}$$

We can write

$$\tilde{y}_m(t) \triangleq \begin{bmatrix} y(t) \\ y(t-1) \\ \vdots \\ y(t-m+1) \end{bmatrix} = A_m \tilde{x}_m(t) + \tilde{e}_m(t)$$

where

$$\tilde{x}_m(t) = [x_1(t), \dots, x_n(t)]^\mathsf{T}, \quad \tilde{e}_m(t) = [e(t), \dots, e(t-m+1)]^\mathsf{T}.$$

Covariance model (2)

This gives the covariance matrix model

$$\boxed{R \triangleq E\{\tilde{y}_m(t)\tilde{y}_m^*(t)\} = A_m P A_m^* + \sigma^2 I, \qquad P = \begin{bmatrix} \alpha_1^2 & & 0 \\ & \ddots & \\ 0 & & \alpha_n^2 \end{bmatrix}}$$

The eigenstructure of R contains complete frequency information.

per.mattsson@it.uu.se

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The eigenstructure of R contains complete frequency information.

Furthermore, using non-overlapping parts of $\{y(t)\}$ (uncorrelated)

$$\boxed{\Gamma \triangleq E\{\tilde{y}_M(t-L-1)\tilde{y}_{L+1}^*(t)\} = A_M P_{L+1} A_{L+1}^*}$$

where

$$P_K = \begin{bmatrix} \alpha_1^2 e^{-i\omega_1 K} & & 0 \\ & \ddots & \\ 0 & & \alpha_n^2 e^{-i\omega_n K} \end{bmatrix}$$

Null space of Γ ($L, M \geq n$) contains complete frequency info.

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Singular value decomposition (SVD)

Definition

Factor a matrix $A \in \mathbb{C}^{m \times n}$ as

$$A = U \Sigma V^*$$

where $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are unitary matrices, and $\Sigma \in \mathbb{R}^{m \times n}$ is diagonal with non-negative diagonal values σ_i (singular values) ordered in decreasing order

(See A.4 in the book for more details)

- Low-rank approximations
- Useful to find the best fit subspaces of A
- Projection onto subspace – perpendicular distance
- LS fit – vertical distance

SVD (2)

$A \in \mathbb{C}^{m\times n}$ is of rank $r < \min(m,n)$ then

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} = U_1 \Sigma_1 V_1^*$$

where $\Sigma_1 \in \mathbb{R}^{r\times r}$, then

- U_1 is an orthonormal basis in $\mathcal{R}(A)$
- U_2 is an orthonormal basis in $\mathcal{N}(A^*)$
- V_1 is an orthonormal basis in $\mathcal{R}(A^*)$
- V_2 is an orthonormal basis in $\mathcal{N}(A)$

Moreover, $\mathcal{R}(A)$ and $\mathcal{N}(A^*)$ are orthogonal to each other and they span $\mathbb{C}^m$
($\mathcal{N}(A^*)$ is the orthogonal complement to $\mathcal{R}(A)$ and vice versa)

Summary

- Signal model for line spectra
- Estimating the spectrum $\implies$ frequency estimation
- Models of the covariance matrix and ARMA-type model
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Next lecture:

- Methods for estimating line spectra (super-resolution): HOYW, MUSIC, Min-norm, ESPRIT
- All exploit eigen-decomposition of the covariance matrix and corresponding subspaces

Interesting complements: 4.9.5 Minimization methods, Appendix B Cramér-Rao bound