## MORE ON LINES IN EUCLIDEAN RAMSEY THEORY

#### DAVID CONLON AND YU-HAN WU

ABSTRACT. Let  $\ell_m$  be a sequence of m points on a line with consecutive points at distance one. Answering a question raised by Fox and the first author and independently by Arman and Tsaturian, we show that there is a natural number m and a red/blue-colouring of  $\mathbb{E}^n$  for every n that contains no red copy of  $\ell_3$  and no blue copy of  $\ell_m$ .

#### 1. Introduction

Let  $\mathbb{E}^n$  denote n-dimensional Euclidean space, that is,  $\mathbb{R}^n$  equipped with the Euclidean metric. Given two sets  $X_1, X_2 \subset \mathbb{E}^n$ , we write  $\mathbb{E}^n \to (X_1, X_2)$  if every red/blue-coloring of  $\mathbb{E}^n$  contains either a red copy of  $X_1$  or a blue copy of  $X_2$ , where a copy for us will always mean an isometric copy. Conversely,  $\mathbb{E}^n \to (X_1, X_2)$  means that there is some red/blue-coloring of  $\mathbb{E}^n$  which contains neither a red copy of  $X_1$  nor a blue copy of  $X_2$ .

The study of which sets  $X_1, X_2 \subset \mathbb{E}^n$  satisfy  $\mathbb{E}^n \to (X_1, X_2)$  is a particular case of the Euclidean Ramsey problem, which has a long history going back to a series of seminal papers [6, 7, 8] of Erdős, Graham, Montgomery, Rothschild, Spencer and Straus in the 1970s. Despite the vintage of the problem, surprisingly little progress has been made since these foundational papers (though see [9, 12] for some important positive results). For instance, it is an open problem, going back to the papers of Erdős et al. [7], as to whether, for every n, there is m such that  $\mathbb{E}^n \to (X, X)$  for every  $X \subset \mathbb{E}^n$  with |X| = m.

Write  $\ell_m$  for the set consisting of m points on a line with consecutive points at distance one. Perhaps because it is a little more accessible than the general problem, the question of determining which n and X satisfy the relation  $\mathbb{E}^n \to (\ell_2, X)$  has received considerable attention. For instance, it is known [11, 14] that  $\mathbb{E}^2 \to (\ell_2, X)$  for every four-point set  $X \subset \mathbb{E}^2$  and that  $\mathbb{E}^2 \to (\ell_2, \ell_5)$ . On the other hand [5], there is a set X of 8 points in the plane, namely, a regular heptagon with its center, such that  $\mathbb{E}^2 \to (\ell_2, X)$ .

In higher dimensions, by combining results of Szlam [13] and Frankl and Wilson [10], it was observed by Fox and the first author [4] that  $\mathbb{E}^n \to (\ell_2, \ell_m)$  provided  $m \leq 2^{cn}$  for some positive constant c (see also [1, 2] for some better bounds in low dimensions). Our concern here will be with a question raised independently by Fox and the first author [4] and also by Arman and Tsaturian [2], namely, as to whether an analogous result holds with  $\ell_2$  replaced by  $\ell_3$ . That is, for every natural number m, is there a natural number n such that  $\mathbb{E}^n \to (\ell_3, \ell_m)$ ? We answer this question in the negative.

**Theorem 1.1.** There exists a natural number m such that  $\mathbb{E}^n \to (\ell_3, \ell_m)$  for all n.

Before our work, the best result that was known in this direction was a 50-year-old result of Erdős et al. [6], who showed that  $\mathbb{E}^n \to (\ell_6, \ell_6)$  for all n. Their proof uses a spherical colouring, where all points at the same distance from the origin receive the same colour. We will also use a spherical colouring, though, unlike the colouring in [6], which is entirely explicit, our colouring will be partly random.

### 2. Preliminaries

In this short section, we note two key lemmas that will be needed in our proof. The first says that certain real-valued quadratic polynomials are reasonably well-distributed modulo a prime q.

**Lemma 2.1.** Let  $p(x) = x^2 + \alpha x + \beta$ , where  $\alpha$  and  $\beta$  are real numbers, and let q be a prime number. Then, for  $m = q^3$ , the set  $\{p(i)\}_{i=1}^m$  overlaps with at least q/6 of the intervals [j, j+1) with  $0 \le j \le q-1$  when considered mod q.

*Proof.* By a standard argument using the pigeonhole principle, there exists some  $k \leq q^2$  such that  $|k\alpha| \leq 1/q \mod q$ . We split into two cases, depending on whether k is a multiple of q or not.

Suppose first that  $k \not\equiv 0 \mod q$  and consider the set of values  $\{p(ki)\}_{i=1}^q$ . Note first that  $\{i^2\}_{i=1}^q$  is a set of (q+1)/2 distinct integers mod q, so, since k is not a multiple of q, the same is also true of the set  $\{k^2i^2\}_{i=1}^q$ . Hence, letting  $p_1(x) = x^2 + \beta$ , we see that the set  $\{p_1(ki)\}_{i=1}^q$  overlaps with at least q/2 of the intervals [j, j+1) with  $0 \le j \le q-1$  when considered mod q. But  $|ki\alpha| \le 1$  mod q for all  $1 \le i \le q$ , so that  $|p(ki) - p_1(ki)| \le 1$  for all  $1 \le i \le q$ . Therefore, since exactly three different intervals are within distance one of any particular interval, the set  $\{p(ki)\}_{i=1}^q$  overlaps with at least q/6 of the intervals [j, j+1) mod q.

Suppose now that k = sq for some  $s \le q$ . Then  $sq\alpha = rq + \epsilon$  for some  $|\epsilon| \le 1/q$ , which implies that  $\alpha = \frac{r}{s} + \epsilon'$ , where  $|\epsilon'| \le 1/q^2$ . Without loss of generality, we may assume that r and s have no common factors. Consider now the polynomial  $p_2(x) = x^2 + \frac{r}{s}x$  and the set  $\{p_2(si)\}_{i=1}^q$ . Since  $p_2(si) = s^2i^2 + ri$ , it is easy to check that  $p_2(si) \equiv p_2(sj) \mod q$  if and only if  $s^2(i+j) + r \equiv 0 \mod q$ . Since r and s are coprime, this implies that the set  $\{p_2(si)\}_{i=1}^q$  takes at least q/2 values mod q. Hence, letting  $p_3(x) = x^2 + \frac{r}{s}x + \beta$ , we see that the set  $\{p_3(si)\}_{i=1}^q$  overlaps with at least q/2 of the intervals [j, j+1) with  $0 \le j \le q-1$  when considered mod q. But, since  $|\alpha - r/s| \le 1/q^2$ , we have that  $|p(si) - p_3(si)| = |\alpha - \frac{r}{s}|si \le 1$ , so that, as above, the set  $\{p(si)\}_{i=1}^q$  overlaps with at least q/6 of the intervals  $[j, j+1) \mod q$ .

Given M real polynomials  $p_1, \ldots, p_M$  in N variables, a vector  $\sigma \in \{-1, 0, 1\}^M$  is called a sign pattern of  $p_1, \ldots, p_M$  if there exists some  $x \in \mathbb{R}^N$  such that the sign of  $p_i(x)$  is  $\sigma_i$  for all  $1 \le i \le M$ . The second result we need is the Oleinik–Petrovsky–Thom–Milnor theorem (see, for example, [3]), which, for N fixed, gives a polynomial bound for the number of sign patterns.

**Lemma 2.2.** For  $M \ge N \ge 2$ , the number of sign patterns of M real polynomials in N variables, each of degree at most D, is at most  $\left(\frac{50DM}{N}\right)^N$ .

# 3. Proof of Theorem 1.1

Suppose that  $a_1, a_2, a_3 \in \mathbb{R}^n$  form a copy of  $\ell_3$  with  $|a_1 - a_2| = |a_2 - a_3| = 1$ . If the points are at distances  $x_1, x_2$  and  $x_3$ , respectively, from the origin o and the angle  $a_1 a_2 o$  is  $\theta$ , then we have

$$x_1^2 = x_2^2 + 1 - 2x_2 \cos \theta$$

and

$$x_3^2 = x_2^2 + 1 + 2x_2 \cos \theta.$$

Adding the two gives

$$x_1^2 + x_3^2 = 2x_2^2 + 2.$$

Similarly, if  $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$  form a copy of  $\ell_m$  with  $|a_i - a_{i+1}| = 1$  for all  $i = 1, 2, \ldots, m-1$ , then, again writing  $x_i$  for the distance of  $a_i$  from the origin, we have

$$x_{i-1}^2 + x_{i+1}^2 = 2x_i^2 + 2$$

for all i = 2, ..., m - 1. Given these observations, our aim will be to colour  $\mathbb{R}_{\geq 0}$  so that there is no red solution to  $y_1 + y_3 = 2y_2 + 2$  and no blue solution to the system  $y_{i-1} + y_{i+1} = 2y_i + 2$  with

 $i=2,\ldots,m-1$ . Assuming that we have such a colouring  $\chi$ , we can simply colour a point  $a\in\mathbb{R}^n$  by  $\chi(|a|^2)$  and it is easy to check that there is no red copy of  $\ell_3$  and no blue copy of  $\ell_m$ .

We have therefore moved our problem to one of finding a natural number m and a colouring  $\chi$  of  $\mathbb{R}_{\geq 0}$  with no red solution to  $y_1+y_3=2y_2+2$  and no blue solution to the system  $y_{i-1}+y_{i+1}=2y_i+2$  with  $i=2,\ldots,m-1$ . Let q be a prime number. We will take  $m=q^3$  and define  $\chi$  by choosing an appropriate colouring  $\chi'$  of  $\mathbb{Z}_q$  and then setting  $\chi(y)=\chi'(\lfloor y\rfloor \mod q)$  for all  $y\in\mathbb{R}_{\geq 0}$ . Our aim now is to show that there is a suitable choice for  $\chi'$ . For this, we consider a random red/blue-colouring  $\chi'$  of  $\mathbb{Z}_q$  and show that, for q sufficiently large, the probability that  $\chi$  contains either of the banned configurations is small.

Concretely, suppose that  $\mathbb{Z}_q$  is coloured randomly in red and blue with each element of  $\mathbb{Z}_q$  coloured red with probability  $p = q^{-3/4}$  and blue with probability 1 - p. With this choice, the expected number of solutions in red to any of the equations  $y_1 + y_3 = 2y_2 + c$  with  $c \in \{1, 2, 3\}$  is at most

$$3p^3q^2 + 9p^2q < 12q^{-1/4} < \frac{1}{2},$$

where we used that there are at most 3q solutions to any of our 3 equations with two of the variables  $\{y_1, y_2, y_3\}$  being equal and that q is sufficiently large. Note that if there are indeed no red solutions to these three equations over  $\mathbb{Z}_q$ , then there is no red solution to  $y_1 + y_3 = 2y_2 + 2$  in the colouring  $\chi$  of  $\mathbb{R}$ . Indeed, if  $y_i = n_i + \epsilon_i$  with  $0 \le \epsilon_i < 1$ , then  $n_i$  is coloured red in  $\chi'$  and

$$n_1 + n_3 = 2n_2 + 2 + 2\epsilon_2 - \epsilon_1 - \epsilon_3.$$

But  $|2\epsilon_2 - \epsilon_1 - \epsilon_3| < 2$ , so we must have

$$n_1 + n_3 = 2n_2 + c$$

for  $c \in \{1, 2, 3\}$ . However, we know that there are no red solutions to any of these equations in the colouring  $\chi'$ , so there is no red solution to  $y_1 + y_3 = 2y_2 + 2$  in the colouring  $\chi$ .

For the blue configurations, we first observe that if the  $y_i$  satisfy the equations  $y_{i-1}+y_{i+1}=2y_i+2$  with  $i=2,\ldots,m-1$  with  $y_1=a$  and  $y_2=a+d$ , then  $y_i=a+(i-1)d+(i^2-3i+2)$ . In particular, by Lemma 2.1, at least q/6 elements of the sequence  $y_1,\ldots,y_m$  lie in different intervals [j,j+1) with  $0 \le j \le q-1$  when considered mod q.

Our aim now is to apply Lemma 2.2 to count the number of different ways in which a set of solutions  $(y_1, y_2, \ldots, y_m)$  to our system of equations can overlap the collection of intervals [j, j+1) mod q. Without loss of generality, we may assume that  $0 \le a, d < q$ . Since, under this assumption, any set of solutions over  $\mathbb R$  to our system of equations is contained in the interval  $[0, 2m^2)$ , it will suffice to count the number of feasible overlaps with the intervals [j, j+1) with  $0 \le j \le 2m^2 - 1$ . Since we need to check at most two linear inequalities in the two variables a and d to check whether each of the m points are placed in each of the  $2m^2$  intervals, we can apply Lemma 2.2 with N=2, D=1 and  $M=2\cdot m\cdot 2m^2=4m^3$  to conclude that the points  $y_1,\ldots,y_m$  overlap the intervals [j,j+1) with  $0 \le j \le 2m^2-1$  in at most  $(100m^3)^2=10^4m^6$  different ways. But now, since at least q/6 of the  $y_i$  must always be in distinct intervals, a union bound implies that the probability we have a blue solution to our system of equations is at most

$$10^4 m^6 (1 - q^{-3/4})^{q/6} < \frac{1}{2}$$

for m sufficiently large. Combined with our earlier estimate for the probability of a red solution to  $y_1 + y_3 = 2y_2 + 2$ , we see that for m sufficiently large ( $m = 10^{50}$  will suffice) there exists a colouring with no red  $\ell_3$  and no blue  $\ell_m$ , as required.

### 4. Concluding remarks

We say that a set  $X \subset \mathbb{E}^d$  is Ramsey if for every natural number r there exists n such that every r-colouring of  $\mathbb{E}^n$  contains a monochromatic copy of X. In [4], it was shown that a set X is

Ramsey if and only if for every natural number m and every fixed  $K \subset \mathbb{E}^m$  there exists n such that  $\mathbb{E}^n \to (X, K)$ . We suspect that there may be an even simpler characterisation.

Conjecture 4.1. A set X is Ramsey if and only if for every natural number m there exists n such that  $\mathbb{E}^n \to (X, \ell_m)$ .

Of course, by the result mentioned above, we already know that if X is Ramsey, then  $\mathbb{E}^n \to (X, \ell_m)$  for n sufficiently large. It therefore remains to show that if X is not Ramsey, then there exists m such that  $\mathbb{E}^n \nrightarrow (X, \ell_m)$  for all n. To prove this in full generality might be difficult. However, an important result of Erdős et al. [6] says that if X is Ramsey, then it must be spherical, in the sense that it must be contained in the surface of a sphere of some dimension. Thus, a first step towards Conjecture 4.1 might be to prove the following.

**Conjecture 4.2.** For every non-spherical set X, there exists a natural number m such that  $\mathbb{E}^n \to (X, \ell_m)$  for all n.

The simplest example of a non-spherical set is the line  $\ell_3$ , so our main result may be seen as a verification of Conjecture 4.2 in this particular case. The next case of interest seems to be when X consists of three points  $a_1, a_2, a_3$  on a line, but now with  $|a_1 - a_2| = 1$  and  $|a_2 - a_3| = \alpha$  for some irrational  $\alpha$ .

#### References

- [1] A. Arman and S. Tsaturian, A result in asymmetric Euclidean Ramsey theory, *Discrete Math.* **341** (2018), 1502–1508.
- [2] A. Arman and S. Tsaturian, Equally spaced collinear points in Euclidean Ramsey theory, preprint available at arXiv:1705.04640 [math.CO].
- [3] S. Basu, R. Pollack and M.-F. Roy, **Algorithms in Real Algebraic Geometry**, 2nd Edition, Algorithms and Computation in Mathematics, 10, Springer-Verlag, Berlin, 2006.
- [4] D. Conlon and J. Fox, Lines in Euclidean Ramsey theory, Discrete Comput. Geom. 61 (2019), 218–225.
- [5] G. Csizmadia and G. Tóth, Note on a Ramsey-type problem in geometry, J. Combin. Theory Ser. A 65 (1994), 302–306.
- [6] P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer and E. G. Straus, Euclidean Ramsey theorems I, J. Combin. Theory Ser. A 14 (1973), 341–363.
- [7] P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer and E. G. Straus, Euclidean Ramsey theorems II, in Infinite and finite sets (Colloq., Keszthely, 1973), Vol. I, 529–557, Colloq. Math. Soc. János Bolyai, Vol. 10, North-Holland, Amsterdam, 1975.
- [8] P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer and E. G. Straus, Euclidean Ramsey theorems III, in Infinite and finite sets (Colloq., Keszthely, 1973), Vol. I, 559–583, Colloq. Math. Soc. János Bolyai, Vol. 10, North-Holland, Amsterdam, 1975.
- [9] P. Frankl and V. Rödl, A partition property of simplices in Euclidean space, J. Amer. Math. Soc. 3 (1990), 1–7.
  [10] P. Frankl and R. M. Wilson, Intersection theorems with geometric consequences, Combinatorica 1 (1981), 357–368.
- [11] R. Juhász, Ramsey type theorems in the plane, J. Combin. Theory Ser. A 27 (1979), 152–160.
- [12] I. Kříž, Permutation groups in Euclidean Ramsey Theory, Proc. Amer. Math. Soc. 112 (1991), 899–907.
- [13] A. D. Szlam, Monochromatic translates of configurations in the plane, J. Combin. Theory Ser. A 93 (2001), 173–176.
- [14] S. Tsaturian, A Euclidean Ramsey result in the plane, Electron. J. Combin. 24 (2017), Paper No. 4.35, 9 pp.

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CA 91125, USA *Email address*: dconlon@caltech.edu

ÉCOLE NORMALE SUPÉRIEURE - PSL, PARIS, FRANCE Email address: yu-han.wu@ens.psl.eu