Aidan Fischer 8558 HW2 3/5/2024

I pledge my honor that I have abided by the Stevens Honor System.

2a) $\sigma(x) = \frac{1}{1+e^{-x}}$

$$\sigma'(x) = \frac{d}{dx}\left(\frac{1}{1+e^{-x}}\right)$$

$$= \frac{-(-e^{-x})}{(1+e^{-x})^2} = \frac{e^{-x}}{(1+e^{-x})^2}$$

$$= \frac{1+e^{-x}}{(1+e^{-x})^2} - \frac{1}{(1+e^{-x})^2}$$

$$= \frac{1}{1+e^{-x}} - \frac{1}{(1+e^{-x})^2}$$

$$\boxed{= \sigma(x) - \sigma^2(x)}$$

2b) $\hat{y}_o = p(o|c) = \frac{\exp(u_o^T v_c)}{\sum_{w=1}^{W} \exp(u_w^T v_c)}$

$$J_{CE}(o, v_c, U) = CE(y, \hat{y}) = -\sum_i y_i \log(\hat{y}_i)$$

$$= -\sum_i y_i \log\left(\frac{\exp(u_i^T v_c)}{\sum_{w=1}^{W} \exp(u_w^T v_c)}\right)$$

$$= -\sum_i y_i \left(\log(\exp(u_i^T v_c)) - \log\left(\sum_{w=1}^{W} \exp(u_w^T v_c)\right)\right)$$

$$= -\sum_i y_i \left(u_i^T v_c - \log\left(\sum_{w=1}^{W} \exp(u_w^T v_c)\right)\right)$$

$$\frac{d}{dv_c}(J_{CE}) = \frac{d}{dv_c} \sum_i y_i u_i^T v_c - y_i \log\left(\sum_{w=1}^{W} \exp(u_w^T v_c)\right)$$

$$= -\sum_i \left(y_i u_i^T - y_i \frac{d}{dv_c}\left(\log\left(\sum_{w=1}^{W} \exp(u_w^T v_c)\right)\right)\right)$$

$$= -\sum_i y_i \left(u_i^T - \frac{d}{dv_c} \log\left(\sum_{w=1}^{W} \exp(u_w^T v_c)\right) \right)$$

$$= -\sum_i y_i \left(u_i^T - \frac{\frac{d}{dv_c}\sum_{w=1}^{W} \exp(u_w^T v_c)}{\sum_{w=1}^{W} \exp(u_w^T v_c)} \right)$$

$$= -\sum_i y_i \left(u_i^T - \frac{\sum_{w=1}^{W} \frac{d}{dv_c} \exp(u_w^T v_c)}{\sum_{w=1}^{W} \exp(u_w^T v_c)} \right)$$

$$\cancel{= -\sum_i y_i \left(u_i^T - \frac{\sum_{w=1} u_w \exp(u_w^T v_c)}{} \right.}$$

$$= -\sum_i y_i \left(u_i^T - \frac{\sum_{w=1}^{W} u_w^T \exp(u_w^T v_c)}{\sum_{w=1}^{W} \exp(u_w^T v_c)} \right)$$

$$= -\sum_i y_i \left(\frac{u_i^T \sum_{w=1}^{W} (\exp(u_w^T v_c)) - \sum_{w=1}^{W} (u_w^T \exp(u_w^T v_c))}{\sum_{w=1}^{W} \exp(u_w^T v_c)} \right)$$

$$= -\sum_i y_i \left(\frac{\sum_{w=1}^{W} (u_i^T - u_w^T) \exp(u_w^T v_c)}{\sum_{w=1}^{W} \exp(u_w^T v_c)} \right)$$

$$= -\sum_i \frac{\sum_{w=1}^{W} y_i (u_i^T - u_w^T) \exp(u_w^T v_c)}{\sum_{w=1}^{W} \exp(u_w^T v_c)}$$

$$\boxed{= \sum_i \frac{\sum_{w=1}^{W} y_i (u_w^T - u_i^T) \exp(u_w^T v_c)}{\sum_{w=1}^{W} \exp(u_w^T v_c)}}$$

2c) $J_{CE} = -\sum_i y_i \log\left(\frac{\exp(u_o^T v_c)}{\sum_{w=1}^{W} \exp(u_w^T v_c)}\right)$ (assuming $u_w = u_o$ for some w)

$$\frac{\partial J_{CE}}{\partial v_o} = -\sum_i y_i \frac{\partial}{\partial u_o}\left(\log(\exp(u_o^T v_c)) - \log\left(\sum_{w=1}^{W}\exp(u_w^T v_c)\right)\right)$$

$$= -\sum_i y_i \left(\frac{\partial}{\partial u_o}(u_o^T v_c) - \frac{\partial}{\partial v_o}\log\left(\sum_{w=1}^{W}\exp(u_w^T v_c)\right)\right)$$

$$= -\sum_i y_i\left(v_c - \frac{\frac{\partial}{\partial v_o}\sum_{w=1}^{W}\exp(u_w^T v_c)}{\sum_{w=1}^{W}\exp(u_w^T v_c)}\right)$$ $\frac{\partial}{\partial v_o}\exp(u_w^T v_c)\ \forall\, u_w \neq u_o$ because $u_w \neq u_o$ is constant w.r.t. u_o

$$= -\sum_i y_i\left(v_c - \frac{\frac{\partial}{\partial u_o}\exp(u_o^T v_c)}{\sum_{w=1}^{W}\exp(u_w^T v_c)}\right) = -\sum_i y_i\left(v_c - \frac{v_c\exp(u_o^T v_c)}{\sum_{w=1}^{W}\exp(u_w^T v_c)}\right)$$

$$= -\sum_i y_i\left(\frac{v_c\sum_{w=1}^{W}(\exp(u_w^T v_c)) - v_c\exp(u_o^T v_c)}{\sum_{w=1}^{W}\exp(u_w^T v_c)}\right) = \boxed{-\sum_i y_i\left(\frac{\sum_{w=1, w\neq o}^{W} v_c\exp(u_w^T v_c)}{\sum_{w=1}^{W}\exp(u_w^T v_c)}\right)}\ \frac{\partial J_{CE}}{\partial u_o}$$

$$\frac{\partial J_{CE}}{\partial u_{w_i}}\ (u_{w_i} \neq u_o)\ (i \neq o) = -\sum_i y_i\left(\frac{\partial}{\partial u_{w_i}}(u_o^T v_c) - \frac{\partial}{\partial u_{w_i}}\log\left(\sum_{w=1}^{W}\exp(u_w^T v_c)\right)\right)$$

$$= -\sum_i y_i\left(0 - \frac{\frac{\partial}{\partial u_{w_i}}\sum_{w=1}^{W}\exp(u_w^T v_c)}{\sum_{w=1}^{W}\exp(u_w^T v_c)}\right) = \sum_i y_i\left(\frac{\frac{\partial}{\partial u_{w_i}}\exp(u_{w_i}^T v_c)}{\sum_{w=1}^{W}\exp(u_w^T v_c)}\right)$$

$$= \boxed{\sum_i \frac{y_i v_c \exp(u_{w_i}^T v_c)}{\sum_{w=1}^{W}\exp(u_w^T v_c)}}\quad \frac{\partial J_{CE}}{\partial u_w},\ w \neq o$$

2d) $\frac{\partial J_{neg\text{-}sample}}{\partial v_c} = \cancel{-\frac{\partial}{\partial v_c}\log\left(\frac{1}{1+\exp(-}\right.} -\frac{\partial}{\partial v_c}\log(\sigma(u_o^T v_c)) - \sum_{k=1}^{K}\frac{\partial}{\partial v_c}\log(\sigma(-u_k^T v_c))$

$$= -\frac{\frac{\partial}{\partial v_c}\sigma(u_o^T v_c)}{\sigma(u_o^T v_c)} - \sum_{k=1}^{K}\frac{\frac{\partial}{\partial v_c}\sigma(-u_k^T v_c)}{\sigma(-u_k^T v_c)} \qquad \left(\frac{\partial}{\partial x}\sigma(x) = \sigma(x) - \sigma^2(x)\right)$$

$$\Rightarrow \left(\frac{\partial}{\partial v_c}(f(v_c)) = \frac{\partial}{\partial v_c}f(v_c)(\sigma(f(v_c)) - \right.$$

$$= -\frac{u_o^T(\sigma(u_o^T v_c) - \sigma^2(u_o^T v_c))}{\sigma(u_o^T v_c)} - \sum_{k=1}^{K}\frac{-u_k^T(\sigma(-u_k^T v_c) - \sigma^2(-u_k^T v_c))}{\sigma(-u_k^T v_c)}$$

$$= -u_o^T(1 - \sigma(u_o^T v_c)) - \sum_{k=1}^{K}(-u_k^T(1 - \sigma(-u_k^T v_c)))$$

$$\boxed{= u_o^T(\sigma(u_o^T v_c) - 1) + \sum_{k=1}^{K} u_k^T(1 - \sigma(-u_k^T v_c))} \quad \frac{\partial J_{neg\text{-}sample}}{\partial v_c}$$

$$\frac{\partial J_{neg\text{-}sample}}{\partial u_o} = -\frac{\partial}{\partial u_o}\log(\sigma(u_o^T v_c)) - 0 \qquad \left(u_o \neq o_k \text{ since } o \notin [1, \ldots, k]\right.$$

$$\text{so } \frac{\partial}{\partial u_o}f(u_k) = 0$$

$$= \frac{-\frac{\partial}{\partial u_o}\sigma(u_o^T v_c)}{\sigma(u_o^T v_c)} = \frac{-v_c(\sigma(u_o^T v_c) - \sigma^2(u_o^T v_c))}{\sigma(u_o^T v_c)} = \boxed{v_c(\sigma(u_o^T v_c) - 1)} \quad \frac{\partial J_{neg\text{-}sample}}{\partial u_o}$$

$$\frac{\partial J_{neg\text{-}sample}}{\partial u_k} = 0 - \cancel{\sum_{i=1}^{K}\frac{\partial}{\partial u_k}\frac{\log\sigma(-u_k^T v_c)}{}} \sum_{i=1}^{K}\frac{\partial}{\partial u_k}\log(\sigma(-u_i^T v_c)) \quad (= 0 \text{ when } i \neq k)$$

$$= -\frac{\partial}{\partial u_k}\log(\sigma(-u_k^T v_c)) = -\frac{\frac{\partial}{\partial u_k}\sigma(-u_k^T v_c)}{\sigma(-u_k^T v_c)}$$

$$= \frac{v_c(\sigma(-u_k^T v_c) - \sigma^2(-u_k^T v_c))}{\sigma(-u_k^T v_c)} = \boxed{v_c(1 - \sigma(-u_k^T v_c))} \quad \frac{\partial J_{neg\text{-}sample}}{\partial u_k}$$

$(u_k \neq v_c \quad [k \neq 0])$

2a) $\frac{\partial}{\partial u_k} \sum_{-m \leq j \leq m, j \neq 0} F(u_{c+j}, v_c)$

$= \frac{\partial F(u_k, v_c)}{\partial u_k}$

$\frac{\partial}{\partial v_c} \sum_{-m \leq j \leq m, j \neq 0} F(u_{c+j}, v_c)$

$= \sum_{-m \leq j \leq m, j \neq 0} \frac{\partial F(u_{c+j}, v_c)}{\partial v_c}$