数值分析 第四章

2. $x=\frac{1}{2}\cos x$

令 $f(x)=x-\frac{1}{2}\cos x$, $f'(x)=1+\frac{1}{2}\sin x\in[\frac{1}{2},\frac{3}{2}]$

$f(0)=-\frac{1}{2}<0$, $f(\frac{\pi}{2})=\frac{\pi}{2}>0$, 因此 $f(x)=0$ 有唯一实根

$x_0\in(0,\frac{\pi}{2})$

令 $\varphi(x)=\frac{1}{2}\cos x$, $\varphi'(x)=-\frac{1}{2}\sin x$, 对 $\forall x\in(-\infty,+\infty)$, 有

$|\varphi'(x)|\le\frac{1}{2}<1$, $\varphi(x)\in[-\frac{1}{2},\frac{1}{2}]\subseteq(-\infty,+\infty)$.

因此，对一切 $x_0\in(-\infty,+\infty)$, $x_{k+1}=\frac{1}{2}\cos x_k$ 收敛。

5. (1) $\varphi(x)=4+\frac{2}{3}\cos x$, $\varphi'(x)=-\frac{2}{3}\sin x$。$\varphi(x)$在$R$上连续

$\forall x_0\in R$, $\varphi(x)\in[4-\frac{2}{3},4+\frac{2}{3}]\subseteq R$, $|\varphi'(x)|\le\frac{2}{3}<1$

因此，对任意初值 x_0, $x_{k+1}=4+\frac{2}{3}\cos x_k$ 收敛。

(2) 要证线性收敛性，只要证明 $\varphi'(x^*)\neq 0$, x^* 满足 $x^*=\varphi(x^*)$

当 $\varphi'(x^*)=0$, $x=k\pi$, $k\in Z$,

又 $\varphi(x^*)=x^*$, 有 $x^*=4+\frac{2}{3}\cos x^*$

代入，有 $k\pi=4+(-1)^k$

$k=\frac{4+(-1)^k}{\pi}\in Z$, 无解。因此 $\varphi'(x^*)\neq 0$. 证毕。

(3) 初值 $x_0 = 0.4$

i	0	1	2	3	4
x_i	0.4	4.6140	3.9345	3.5321	3.3835
$\varphi(x_i)$	4.6140	3.9345	3.5321	3.3835	3.35268

i	5	6
x_i	3.35268	3.3481
$\varphi(x_i)$	3.3481	3.3475

$x^* = 3.3475$

19. $f(x) = (x^3-a)^2$, $f'(x) = 6x^2(x^3-a)$

令 $\varphi(x) = x - \frac{f(x)}{f'(x)} = \frac{5}{6}x + \frac{a}{6x^2}$, $\varphi'(x) = \frac{5}{6} - \frac{a}{3x^3}$

令 $\varphi(x^*) = 0$; 有 $x^* = \sqrt[3]{a}$, $\varphi'(x^*) = \frac{1}{2} \neq 0$

因此 $x_{k+1} = \frac{5}{6}x_k + \frac{a}{6x_k^2}$ 线性收敛

20. $f(x) = \frac{1}{x} - a$. $f'(x) = -\frac{1}{x^2}$

令 $\varphi(x) = x - \frac{f(x)}{f'(x)} = 2x - ax^2$ 为递推公式

21. $f(x) = \frac{1}{x^2} - a$. $\cancel{f'(x) = -\frac{1}{2x^3}}$ $f'(x) = -\frac{2}{x^3}$

令 $\varphi(x) = x - \frac{f(x)}{f'(x)} = \frac{3}{2}x - \frac{a}{2}x^2$ 为递推公式

7. (1) $\varphi(x)=1+\frac{1}{x^2}$, $\varphi'(x)=-\frac{2}{x^3}$. $\varphi(x)$在$[1.3,1.6]$上连续

在$[1.3,1.6]$上, $|\varphi'(x)|\le\left|-\frac{2}{1.3^3}\right|<1$, 又$\varphi(1.3)=1+\frac{1}{1.3^2}<1.6$

$\varphi(1.6)=1+\frac{1}{1.6^2}>1.3$。又$x_0\in[1.3,1.6]$.

因此 $x_{k+1}=1+\frac{1}{x_k^2}$收敛。

(2) $\varphi(x)=(1+x^2)^{\frac{1}{3}}$, $\varphi'(x)=\frac{2x}{3}(1+x^2)^{-\frac{2}{3}}$. $\varphi(x)$在$[1.3,1.6]$上连续

$|\varphi'(x)|\le\frac{2\times1.6}{2}(1+1.3^2)^{-\frac{2}{3}}<1$. $\varphi(1.3)>1.3$, $\varphi(1.6)<1.6$

$x_0\in[1.3,1.6]$

因此, $x_{k+1}=(1+x_k^2)^{\frac{1}{3}}$收敛

13. $\Phi(x)=1+2cx$

$c=0$时, $\Phi(x)=x$显然在R上满足$\Phi(x_k)=x_k$, 但$\Phi'(x)=1$,

不具有收敛性

$c\neq0$时, $\Phi(x)=x+c(x^2-3)$, $\Phi(x^*)=x^*$, 有$x^*=\pm\sqrt{3}$

$\Phi'(x^*)=1+2cx^*$. 当$\Phi(x)$局部收敛时满足

$|1+2cx^*|<1$对一x^*成立

$x^*=\sqrt{3}$时, ~~$0<c<\frac{\sqrt{3}}{3}$~~ $-\frac{\sqrt{3}}{3}<c<0$; $x^*=-\sqrt{3}$时, $0<c<\frac{\sqrt{3}}{3}$

因此, $0<|c|<\frac{\sqrt{3}}{3}$时 $x_{k+1}=\Phi(x_k)$局部收敛