数值分析

1. $f(x)=\sin x$. $f(0)=0$, $f'(0)=1$. $f^{(2)}(0)=0$, $f^{(3)}(0)=-1$
$f^{(4)}(0)=0$, $f^{(5)}(0)=1$, $f^{(6)}(x)=-\sin x$

$$P(x)=f(0)+xf'(0)+\frac{x^2}{2!}f^{(2)}(0)+\frac{x^3}{3!}f^{(3)}(0)+\frac{x^4}{4!}f^{(4)}(0)$$
$$+\frac{x^5}{5!}f^{(5)}(0)$$
$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}$$

误差 $f(x)-P_5(x)=\frac{x^6}{6!}f^{(6)}(\xi)\le\frac{1^6}{6!}$; $|x|\le 1$, 误差 $\frac{1}{120}$

2. $f(x)=e^{-\frac{x^2}{2}}$, $f'(x)=-xe^{-\frac{x^2}{2}}$, $f''(x)=(x^2-1)e^{-\frac{x^2}{2}}$.
$f^{(3)}(x)=xe^{-\frac{x^2}{2}}$, $f^{(4)}(x)=(1-x^2)e^{-\frac{x^2}{2}}$

$$P_3(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f^{(3)}(0)$$
$$\cancel{=1+}=1-\frac{x^2}{2}$$

~~误差f~~

5. 由定理2, $P_n(x)$ 对某 $n+1$ 个节点 $(x_0,y_0)\cdots(x_n,y_n)$ 唯一确定

设 $(x_0,y_0)\cdots(x_n,y_n)$ 在 $f(x)$ 上, 由此 $n+1$ 个节点确定 $P_n(x)$, 误差 $f(x)-P_n(x)$ 有至少 $n+1$ 个零点
$x=x_i$, $i=0,1,2,\cdots,n$

又 $f(x)-P_n(x)$ 为不多于 n 次的多项式, 由代数基本定理, $f(x)-P_n(x)$ 至多有 n 个零点, 除非 $f_n(x)=P_n(x)$

综上, $f(x)=P_n(x)$

~~6.(1) $l_0(x)=\frac{x-x_1}{x_0-x_1}=-\frac{x+1}{2}$, $l_1(x)=\frac{x-x_0}{x_1-x_0}=\frac{x-1}{2}$~~

~~$P_1(x)=l_0(x)y_0+l_1(x)y_1=$~~

6.(1), $x_0=-1$, $y_0=-1$; $x_1=1$, $y_1=1$

$l_0(x)=\frac{x-x_1}{x_0-x_1}=-\frac{x-1}{2}$; $l_1(x)=\frac{x-x_0}{x_1-x_0}=\frac{x+1}{2}$

$P_1(x)=l_0(x)y_0+l_1(x)y_1=\frac{x-1}{2}+\frac{x+1}{2}=x$

i	0	1	2
(2) x_i	-1	0	1
y_i	-1	0	1

$l_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\frac{x(x-1)}{2}$, $l_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}=\cancel{\frac{x^2-1}{1-x^2}}$

$l_2(x)=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}=\frac{x(x+1)}{2}$

$P_2(x)=\frac{x(x-1)}{2}\cdot(-1)+l_1(x)\cdot 0+\frac{x(x+1)}{2}\cdot 1$

$=x$

(3) i	0	1	2	3
x_i	-1	0	1	2
y_i	-1	0	1	8

$l_0(x)=-\frac{x(x-1)(x-2)}{6}$, $l_1(x)=\frac{(x+1)(x-1)(x-2)}{2}$, $l_2(x)=-\frac{x(x+1)(x-2)}{2}$

$l_3(x)=\frac{x(x+1)(x-1)}{6}$

$P_3(x)=\sum_{i=0}^{3}l_i(x)y_i=x^3$

7. i 0 1 2

x_i 0 1 2

y_i 1 2 3

$l_0(x)=\frac{(x-1)(x-2)}{2}$, ~~$l_1(x)=-(x-2)x$~~

$l_1(x)=-x(x-2)$, $l_2(x)=\frac{x(x-1)}{2}$

$P_2(x)=\sum_{i=0}^{2} l_i(x)y_i = x+1$

9. 基函数与y_i关系

取$f(x)=1$, $\sum_{i=0}^{n} l_i(x)y_i = \sum_{i=0}^{n} l_i(x)=1$ $(n\geq 1)$

$n=3$ 即为结果: $l_0(x)+l_1(x)+l_2(x)-1=0$

~~12. (1) $f(x)=4x^3-3x+2$, $f^{(4)}(x)=0$~~

~~$f(x)-P_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{k=0}(x-x_k)=0$~~

~~(2) $f(x)=x^4-2x^3$, $f^{(4)}(x)=24$~~

$f(x)-P_n(x)=\frac{f^{(4)}}{(4+1)!}$

12. (1) $f(x)=4x^3-3x+2$; $f^{(3)}(x)=24$, $f^{(4)}(x)=\cancel{24}\ 0$

$$f(x)-P_n(x)=\frac{f^{(4)}(\xi)}{(3+1)!}\prod_{k=0}^{n}(x-x_k)=0$$

~~(2) $f(x)=f(x^4)-2x^3$~~

(2) $f(x)=x^4-2x^3$, $f^{(4)}(x)=24$

$$f(x)-P_4(x)=\frac{f^{(4)}(\xi)}{(3+1)!}\prod_{k=0}^{n}(x-x_k)=(x+1)(x-1)(x-3)(x-4)$$

14. $l_0(x)=\dfrac{x-x_1}{x_0-x_1}=-x+1$, $l_1(x)=\dfrac{x-x_0}{x_1-x_0}=x$

$P_1(x)=(-x+1)\cdot 1+x\cdot\dfrac{1}{e}=\dfrac{x}{e}-x+1$

$f^{(2)}(x)=e^{-x}$,

$$|f(x)-P_1(x)|=\left|\frac{f^{(2)}(\xi)}{2!}\prod_{k=0}^{1}(x-x_k)\right|=\left|\frac{e^{-x}}{2}x(x-1)\right|\leq\frac{1}{8}$$

15. $$|f(x)-P_1(x)|=\left|\frac{f''(\xi)}{2!}\prod_{k=0}^{1}(x-x_k)\right|=\left|\frac{(x-x_0)(x-x_1)}{2}f''(\xi)\right|$$

$$\leq\left|\frac{\frac{(x_0-x_1)^2}{4}}{2}f''(\xi)\right|\leq\frac{(x_0-x_1)^2}{8}\max_{x_0\leq x\leq x_1}f''(x)$$

~~16~~.

16. ~~f(1,2)=~~ $f(1)=7$, $f(2)=21$, $f(3)=59$, $f(4)=133$, $f(5)=255$

$f(1,2)=14$. $f(2,3)=38$, $f(3,4)=74$. $f(4,5)=122$

$f(1,2,3)=12$. $f(2,3,4)=18$, $f(3,4,5)=24$

$f(1,2,3,4)=2$, $f(2,3,4,5)=2$.

$f(1,2,3,4,5)=0$

20. (1) $f(x_0, x_1, \cdots, x_n)=\sum_{k=0}^{n}\frac{f(x_k)}{\prod_{\substack{j=0\\j\neq k}}^{n}(x_k-x_j)}$ (书P25)

$F(x_0, x_1, \cdots, x_n)=\sum_{k=0}^{n}\frac{cf(x_k)}{\prod_{\substack{j=0\\j\neq k}}^{n}(x_k-x_j)}=cf(x_0, x_1, \cdots, x_n)$

(2) $F(x_0, x_1, \cdots, x_n)=\sum_{k=0}^{n}\frac{f(x_k)+g(x_k)}{\prod_{\substack{j=0\\j\neq k}}^{n}(x_k-x_j)}=\sum_{k=0}^{n}\frac{f(x_k)}{\prod_{\substack{j=0\\j\neq k}}^{n}(x_k-x_j)}+\sum_{k=0}^{n}\frac{g(x_k)}{\prod_{\substack{j=0\\j\neq k}}^{n}(x_k-x_j)}$

$=f(x_0, x_1, \cdots, x_n)+g(x_0, x_1, \cdots, x_n)$

20.(1) $n=0$时，$F(x_0)=cf(x_0)$

设$n=k-1$时，$F(x_0,\cdots x_{k-1})=cf(x_0,\cdots,x_{k-1})$

则 $F(x_0,\cdots x_k)=\dfrac{F(x_0,\cdots,x_{k-1})-F(x_1,\cdots,x_k)}{x_0-x_k}$

$=c\dfrac{f(x_0,\cdots,x_{k-1})-f(x_1,\cdots,x_k)}{x_0-x_k}$

$=cf(x_0,\cdots,x_k)$ 成立

$\therefore \forall n\geqslant 0, F(x_0,\cdots,x_n)=cf(x_0,\cdots x_n)$

(2) $n=0$时，$F(x_0)=f(x_0)+g(x_0)$

设$n=k-1$时，$F(x_0,\cdots,x_{k-1})=f(x_0,\cdots x_{k-1})+g(x_0,\cdots,x_{k-1})$

$F(x_0,\cdots,x_k)=\dfrac{F(x_0,\cdots x_{k-1})-F(x_1,\cdots x_k)}{x_0-x_k}$

$=\dfrac{f(x_0,\cdots x_{k-1})-f(x_1,\cdots,x_k)+g(x_0,\cdots,x_{k-1})-g(x_1,\cdots x_k)}{x_0-x_k}$

$=f(x_0,\cdots,x_k)+g(x_0,\cdots,x_k)$

23. ~~设 $p(x)=ax^2+bx+c$~~

设 $P(x)=ax(x-1)^2$

$P'(x)=a(x-1)^2+2ax(x-1)=3ax^2-4ax+a=a(3x-1)(x-1)$

$P(0)=0,\ P(1)=0,\ P'(1)=0.\ P(2)=2a=1$

$a=\frac{1}{2},\ P(x)=\frac{1}{2}x(x-1)^2$

24. $P(x)=ax(x-1)(x-2)^2$

$P(3)=6a=3,\ a=\frac{1}{2}$

$P(x)=\frac{1}{2}\cancel{x}(x-1)(x-2)^2$

33. $f(x)=x^3+x^2,\ g(x)=2x^3+bx^2+cx-1$

$f'(1)=5,\ g'(1)=6+2b+c$

$f(1)=2,\ g(1)=1+b+c$

$$\begin{cases} f(1)=g(1) \\ f'(1)=g'(1) \end{cases} \Rightarrow \begin{cases} b=-2 \\ c=3 \end{cases}$$

35. $A=\begin{pmatrix} 2 & 4 \\ 3 & -5 \\ 1 & 2 \\ 2 & 1 \end{pmatrix},\ A^T=\begin{pmatrix} 2 & 3 & 1 & 2 \\ 4 & -5 & 2 & 1 \end{pmatrix},\ A^TA=\begin{pmatrix} 18 & -3 \\ -3 & 46 \end{pmatrix}$

$b=\begin{pmatrix} 11 \\ 3 \\ 6 \\ 7 \end{pmatrix}\ A^Tb=\begin{pmatrix} 51 \\ 48 \end{pmatrix}$ $\quad A^TA\begin{pmatrix} \overset{x}{\cancel{x_1}} \\ \underset{y}{\cancel{x_2}} \end{pmatrix}=A^Tb$, 解得 $\begin{cases} \overset{x}{\cancel{x_1}}=3.04029 \\ \underset{y}{\cancel{x_2}}=1.24176 \end{cases}$

36.

P_2

$P_2($

36. 设 $p_1(x) = a_0 + a_1 x$

$$\begin{pmatrix} 9 & 0 \\ 0 & \cancel{4.25}\ 3.75 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 18.1183 \\ 8.4437 \end{pmatrix}$$

$a_0 = 2.01314,\ a_1 = 2.2516$

$p_1(x) = 2.01314 + 2.2516$

$p_2(x) = a_0 + a_1 x + a_2 x^2$

$$\begin{pmatrix} 9 & 0 & 3.75 \\ 0 & 3.75 & 0 \\ 3.75 & 0 & 2.7656 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 18.1183 \\ 8.4437 \\ 7.5870 \end{pmatrix}$$

$a_0 = 2.0001 \quad a_1 = 2.2516, \quad a_2 = 0.0313$

$p_2(x) = 2.0001 + 2.2516\cancel{x} + 0.0313x^2$