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BORDEAUX

ScalFMM Periodic Model

Author:

Berenger BRAMAS

Olivier COULAUD

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Working note

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1 Abstract

The current report describes a new algorithm in order to perform an periodic Multipole Method (FMM). This algorithm uses the usual FMM operators ($M2M$, $M2L$, $L2L$, $P2P$) why makes it kernels independent in a way explained in the later sections. The computational cost of the method is small and is using the same error bound as the FMM.

Keywords: FMM, octree, periodic.

2 Introduction

The FMM is widely use to execute physical simulations with pairs interactions. Direct approaches compute all interactions and have a complexity of $O(n^2)$. The FMM reduce this complexity to a $O(n)$. But many physical problems need to compute the periodic conditions. Several solutions relies on analytical methods.

Type of interactions. In the current report we speak of potential at particle x_i given by:

$$V(x_i) = \sum_{j=0, i \neq j}^N \frac{q_j}{\|x_i - x_j\|}$$

and the force on atom x_i writes

$$f(x_i) = \sum_{j=0, i \neq j}^N q_j \frac{x_i - x_j}{\|x_i - x_j\|^3}.$$

Finally, the total energy of the system is

$$U = \frac{1}{2} \sum_{i=0}^N q_i V(x_i).$$

Such formulas are supported by ScalFMM.

3 State of the art

In APPLICATION OF FAST MULTIPOLE METHOD TO MICROMAGNETIC SIMULATION OF PERIODIC SYSTEMS, an approach is proposed but is not using the power of FMM and the usual shape of its operators.

4 Usual Operators

In this section we remind the common FMM operators in an algorithm view.

4.1 Octree

The FMM operators rely on the spacial decomposition of the problem which is done using an octree. The system is included in a cube where the root of the tree at level 0 contains all the data. Then we divide this root-cell by 2 in each dimension which creates 2^{DIM} sub-cells. We repeat the previous step until a level L . We have a maximum of $2^{DIM \cdot l}$ cells at level l . Some implementation subdivides a cell until a condition is respected, like for example having less than N particles in each leaves. Such approaches leads to a tree where all leaves may not be at the same level, but our method can even be applied them.

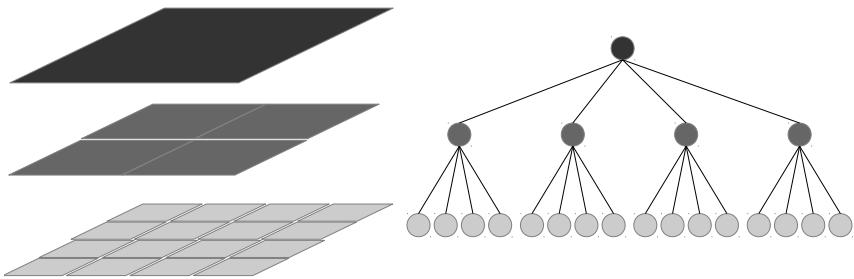


Figure 1: 2D Octree

4.2 Top Down Operators

These operators are performing computation between a cell and its children (the cells it contains one lower level in the tree) called $M2M$ and $L2L$, and inside leaves between cells and particles called $P2M$ and $L2P$. Some implementation defines optimizations which make them having operators that skips level during $M2M$ or $L2L$. But in most FMM it exists a $M2M$ (and a $L2L$) which is able to compute the interaction between a cells and its 2^{DIM} children. The children are positioned at equal distance from their common parent r and can be positioned using spherical coordinates: (r, θ, ϕ) , with $(\theta, \phi) \in \{\pi/4, 3\pi/4\} \times \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$ in 3D and (r, ϕ) , with $(\phi) \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$ in 2D. Depending on the level where

the operator is applied, r is equal to half of a cube diagonal: $r = \text{Diagonal}/2 = \sqrt{3 * \text{Edge}^2}/2 = \sqrt{3 * (\text{BoxWidth}/2^l)^2}/2$.

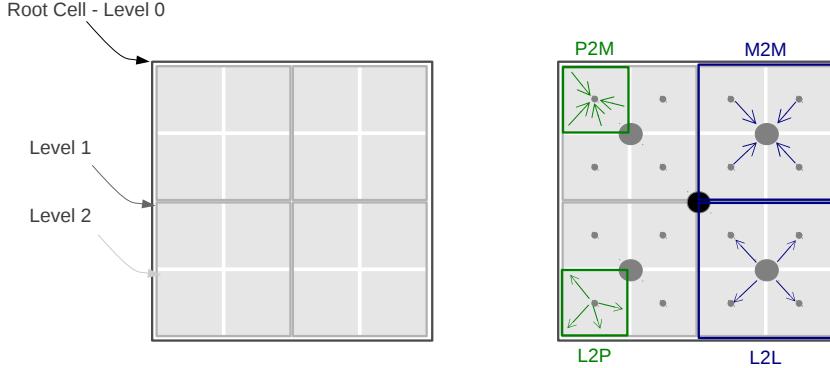


Figure 2: Top-Down Operators

4.3 Interactions operators

The operators that operate on one level and do not rely on the parent or sub-cells are the $P2P$ and the $M2L$. The $P2P$ operates between neighbors leaves. If the tree is full (each cell exists), each leaf has a number of neighbors ng with $2^{DIM} - 1 \leq ng \leq 3^{DIM} - 1$. The minimum is for leaves that are positioned in the corners. If the octree is not full then the minimum can be 0 since leaf may not have any neighbors.

The $M2L$ operates between a cell and what is called its interaction list. The interaction list is composed of cells from the same level and which are not neighbors but are closed. The closed properties is true for the cells which have their parents as neighbors respectively. So each cell has a unique interactions list. Depending on the position of a cell relatively to its parent, the interactions list is taken cells from $[-3; +2]$ or $[-2; +3]$ in each dimension. In $3D$ and with a full octree, each cell has a maximum of $6^3 - 3^3 = 189$ interactions. The 6^3 is coming from the cube of size 5 which is encapsulate all the potential interactions and the 3^3 represents the cell itself and its neighbors. Any FMM implementation has the capacity to compute the $M2L$ between two cell of the same level that are separated by at least one and a maximum of two in each direction.

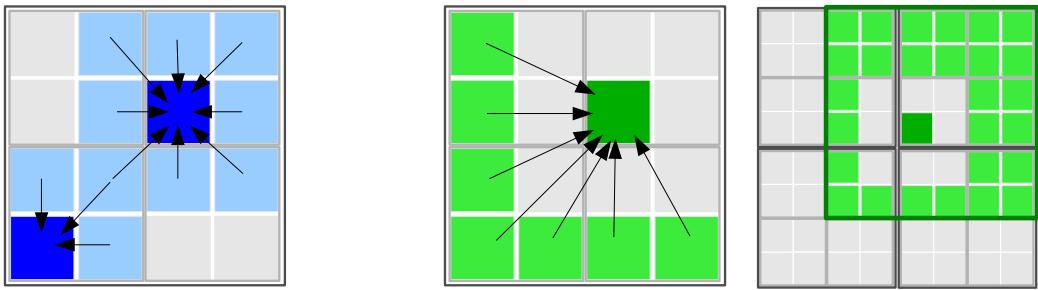


Figure 3: Interactions Operators, P2P (Blue), M2L (Green)

4.4 Global view of the algorithm

The FMM algorithm can be split in three major parts, an upward pass, a transfer pass and a downward pass. The upward pass is composed by the $P2M$ and the $M2M$, it centralized the information contained by each cell from leaf level to the top of the tree. The transfer pass, composed by the $M2L$, is translating the values at each level using the interactions lists. Then the downward pass, using $L2L$ and $L2P$, merge the results of the transfer pass and impact the leaves by starting from the root. In the non periodic FMM, there is no computation above level 2 since no $M2L$ happens at level 1 because the interactions lists are empty.

In addition of these steps called the far field, we compute the near field which is the $P2P$ operators.

5 Virtual Repetition

When using periodicity, a cell or a leaf can have its interactions or neighbors lists extended. In fact, when working with periodicity the simulation box should be repeated. A cell on the border or separated by one cell from the border should have interactions from the other side of the box. In the next paragraph we supposed when speaking about interactions or neighbors lists that they are a periodic lists.

5.1 Periodicity during normal pass

During the upward pass the periodicity does not change the operators. In fact, they are operate between cells and sub-cells.

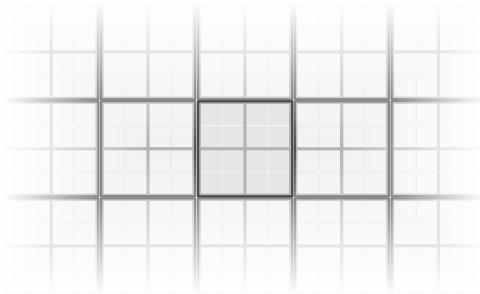


Figure 4: Repetitions of the simulation box create new neighbors

Whereas, the transfer pass ($M2L$ and $P2P$) should not work with periodic lists. The normal FMM stops at level 2 because interactions lists are empty at level 1. By doing this, it makes a repetition of original box $rep(2) = 1/2$. If we use the upward pass, transfer pass and downward pass at level 1 then $rep(1) = 1$. In fact, at level 1 the periodic interactions lists are not empty and the $M2L$ at level 1 is covering the neighbors of our root. It is important to have in mind that without any special operation the box is already repeated once it we go at level 1 and use periodic lists.

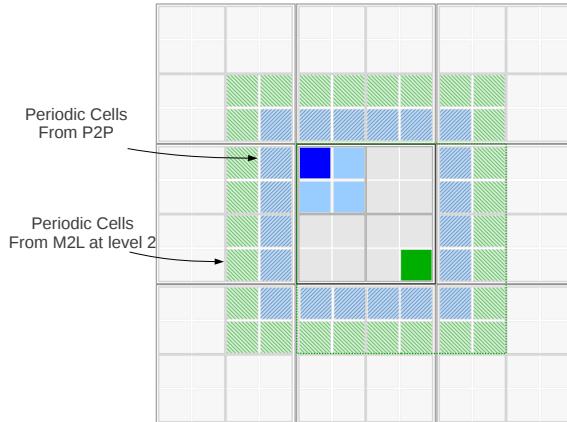


Figure 5: Extension of the box with a periodicity until level 2

5.2 Periodicity at root level

If the $M2M$ goes until the root at level 0, the root cell is containing all the information of the entire tree. Using this cell we can perform a $M2L$ from -3 to

3 by using it same cell inside the interactions lists. This $M2L$ will not compute the interaction with the neighbors cells, but this has been already done. We then have a repetition of 2 cells in each direction so it like having a grid of $(2 * 2) + 1$ and working on the cell in the middle. Finally we can compute the downward pass from level 0 to the leaves.

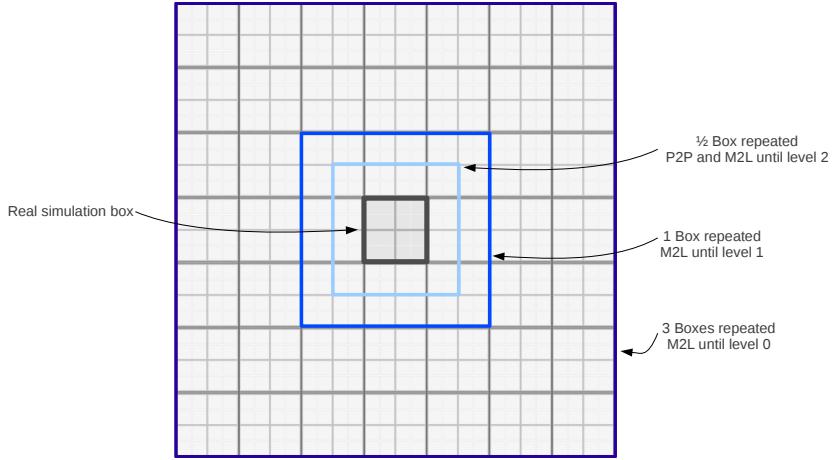


Figure 6: Extension of the box with a periodicity until level 0

5.3 Periodicity above the root

In the previous paragraph we stopped the operation at level 0. But the key idea of the periodic algorithm is to continue with the usual operators above the level 0 of the tree.

If we compute the $M2L$ between $[-3; +3]$ at level 0 it is then impossible to continue with the usual operators. In fact, the $M2L$ is covering what sub-cells do not cover and should let too far cells to upper $M2L$. The $M2L$ at level 0 has to be limited to $[-3; +2]$ or $[-2; +3]$ in each direction.

This choice is made by choosing where does our current root is situated relatively to its parent. Any choice can be made, but we consider that the root cell is at position $(1, 1, 1)$ relatively to its parent. So when performing the $M2L$ at level 0 the repetition of the box is not odd and so it is not equilibrate. It is similar as having a grid of width 6 and working on the cell of coordinate $(4, 4, 4)$. We can continue to perform similar operations, $M2M$ followed by $M2L$ and considering that our working cell is in $(1, 1, 1)$ relatively to its parent.

Then, at the last level above the root, the final $M2L$ should be perform by considering that the working cell is in position $(0, 0, 0)$ relatively to its parent. This put our working cell in the middle of the grid.

It remains that the repetition is not symmetric. In fact, the grid has a width of $(6 * 2^d)$ which is odd. There are $(6 * 2^d)/2$ repetition in one side and $((6 * 2^d)/2) - 1$ in the other side for all the dimensions. The have a perfect symmetric periodicity we need to add a border.

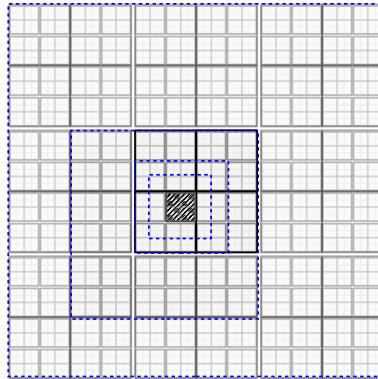


Figure 7: Extension of the box with a periodicity until level -1 without border

We can group the root cell to perform a $M2M$ from level 0 to level -1 . The results contains the information of a box which is nothing more than our original box repeated 2 times in each direction. Again, we can do a $M2L$ at level -1 and we choose that our cell is at position $(0, 0, 0)$ relatively to its parent. These operations can be repeated until we have enough repetition.

Every time we perform a $M2M$ d levels above 0 we create a new cell which contain 2^d times our original box. Then when we perform a $M2M$ at a level we repeat the cell from -2 to 3 in any direction. So at d levels above 0 performing a $M2L$ leads to create a repetition of $6 * 2^d$ original box in all the directions. So we create a grid of size $(6 * 2^d)^3$ and we are working on the center cell at position $((6 * 2^d)/2, (6 * 2^d)/2, (6 * 2^d)/2)$.

5.4 Adding a border

THIS IS NOT USE ANYMORE!

After having proceed d levels above 0, a border of the simulation box is missing.

To take into consideration this border, we need to use the root cell and create all the configuration in order to have create cells one levels above d which contains all the information. Then we can apply a final $M2L$ between those cells and our working cell.

In $2D$ there are three kinds of border cells: the horizontal border, the vertical border and the angle. In $3D$ there are seven kinds of border cells: the top, the side, the back and the four different links between them. More generally, there are DIM different sides and 2^{DIM-1} different connexion between sides.

Computing the cell which represent a side is done in $d - 1$ $M2M$. Each of this $M2M$ is computing the relation between one or two sub-cells and their parent.

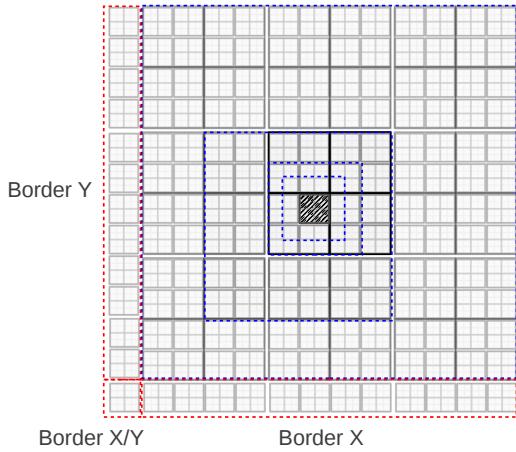


Figure 8: Extension of the box with a periodicity until level -1 with border

5.5 Limiting the periodicity direction

If the simulation needs that the periodicity is limited in some direction, for example $(+X, + - Y, No Z)$, the only changes is in the construction to the periodic lists including those made above the root. Then, during the addition of the border only the periodic directions have to be applied.

6 Theoretical repetition

The periodic-algorithm does not allow to choose a specific grid size. The only parameter D is the number of level above the root of the box where the algorithm is applied. As it has been illustrated the final grid size has a width of $\dim = 2^{\text{Abs}(D)+2}$

D	Grid Dim
1	3
0	4
1	8
5	32
20	1 048 576

6.1 Cost above the root

We determine the cost of the periodicity in term of FMM operators. The following cost is based on a full octree.

There are $2^{DIM} M2M$, $1 L2L$, and $6^{DIM} - 3^{DIM} M2L$ per level above the root.

For example, in $3D$, the cost for $D = 5$ is $5 * (2^3 + 1^2 + 2^2 + 3^2) = 110 M2M$ and $5 * 189 = 945 M2L$. For example, in $3D$, the cost for $D = 20$ is $20 * (2^3 + 1^2 + 2^2 + 3^2) = 440 M2M$ and $20 * 189 = 3780 M2L$.

7 Implementation details

7.1 The algorithm

The algorithm above the root can be spited in 4 steps:

- Upward pass, for level 0 to $-d$ we compute the $2^{DIM} M2M$ (8 per level in $3D$).
- Transfer pass, for level 0 to $-d$ we compute the $6^{DIM} - 3^{DIM} M2L$ (189 per level in $3D$). For level 0 to $-d + 1$ we put the cell at position $(1, 1, 1)$ relatively to its parent. For the last $M2L$ at level $-d$ we put the cell at position $(0, 0, 0)$ relatively to its parent.
- Border pass, at level $-d - 1$ we compute the border and apply it using $M2L$.

- Downward pass, from level $-d - 1$ to 0 we perform one $L2L$. There is only one sub-cell per $L2L$ and the position has to be the same as the one used during the Transfer pass.

7.2 Low level details

In our library, all the main part of the FMM are dissociated. To ensure that our kernel is able to compute all the periodic operation, we pass to our kernel initialization method an offset for the box width and for the tree height. Our kernel compute the requested operation without any difference between the real FMM and the operation above the root for the periodic FMM. In fact, let the original FMM box width be w and a tree height be h . Then, if is required to apply periodicity until level d above the root. The kernel should be able to proceed usual FMM operator in a tree of height of size $h + d + 5$ with a initial box width of $w * 2^{d+3}$.

8 Results

8.1 Ewald summation comparisons

The energy computed by molecular dynamics codes for a finite system is given by

$$U = \frac{1}{4\pi\epsilon_0} \sum_{i=0}^N \sum_{j < i} \frac{q_i q_j}{\|x_i - x_j\|}$$

and the force on atom x_i is

$$f(x_i) = \frac{q_i}{4\pi\epsilon_0} \sum_{j=0, i \neq j}^N q_j \frac{x_i - x_j}{\|x_i - x_j\|^3}$$

Now consider a periodic system, the energy computed by molecular dynamics codes atypically by Ewald method is

$$U = \frac{1}{8\pi\epsilon_0} \sum_{|n|=0}^{\infty} \sum_{i,j}^N \sum_{j < i} \frac{q_i q_j}{\|x_i - x_j + nL\|}$$

where L is the box size and ' specifies that for $n = 0$ the omit i equal j in the sum. The force on atom x_i is

$$f(x_i) = \frac{q_i}{8\pi\epsilon_0} \sum_{|n|=0}^{\infty} \sum_{i,j}^N q_j \frac{x_i - x_j}{\|x_i - x_j + nL\|^3}$$

In the Ewald method, to original box is replicated up to the infinity, and then the method is equivalent to consider a surrounding conductor ($\epsilon = \infty$).

The method described in this document and implemented in ScalFMM consider a finite set of the original box surrounded by a vacuum with dielectric constant $\epsilon = 1$. The results by these two approaches are different [?, ?] and for a very large sphere surrounding all boxes we have the following equation

$$E_{Ewald} = E_{FMM} - \frac{2\pi}{3V} \left| \sum_{i=1}^N q_i x_i \right|^2. \quad (1)$$

By taking the gradient of this equation, the relationship between FMM forces and Ewald forces is

$$F_{Ewald}(x_k) = -\nabla_k E_{Ewald} = F_{FMM}(x_k) + \frac{4\pi}{3V} q_k \sum_{i=1}^N q_i x_i. \quad (2)$$

If we denote by $\mathbf{D} = \sum_{i=1}^N q_i x_i$ the dipole moment of the system then the periodic energy is

$$E_{Ewald} = E_{FMM} - \frac{2\pi}{3V} \mathbf{D} \cdot \mathbf{D} = \sum_{i=1}^N q_i V_i^{Ewald} \quad (3)$$

the potential

$$V^{Ewald}(r) = V^{FMM}(r) - \frac{4\pi}{3V} r \cdot \mathbf{D} \quad (4)$$

and the forces are

$$F_{Ewald}(x_k) = -\nabla_k E_{Ewald} = F_{FMM}(x_k) + \frac{4\pi}{3V} q_k \mathbf{D}. \quad (5)$$

8.2 Units

Consider the following dimensional units

type	Expression
time	t_0
length	l_0
mass	m_0
charge	q_0
energy	$E_0 = m_0(l_0/t_0)^2$

In adimensional variables the energy writes

$$U = \frac{q_0^2}{4\pi\epsilon_0 l_0} \sum_{i=0}^N \sum_{j < i} \frac{q_i q_j}{\|x_i - x_j\|} = C_{energy} \sum_{i=0}^N \sum_{j < i} \frac{q_i q_j}{\|x_i - x_j\|}$$

and the forces write

$$f(x_i) = -\frac{q_0^2}{4\pi\epsilon_0 l_0^2} q_i \sum_{j=0, i \neq j}^N q_j \frac{x_i - x_j}{\|x_i - x_j\|^3} = C_{force} q_i \sum_{j=0, i \neq j}^N q_j \frac{x_i - x_j}{\|x_i - x_j\|^3}$$

The Energy conversion factor is $\gamma_0 = \frac{q_0^2}{4\pi\epsilon_0 l_0}/E_0$.

8.2.1 DL_Poly comparisons

DL_POLY_2 uses the following internal molecular units

type	Expression	Numerical value	
time	t_0	$10^{-12} s$	picosecond
length	l_0	$10^{-10} m$	\AA Angstrom
mass	m_0	$1.6605402 \cdot 10^{27} kg$	atomic mass unit
charge	q_0	$1.60217733 \cdot 10^{19} Coulombs$	unit of proton charge
energy	$E_0 = m_0(l_0/t_0)^2$	$1.6605402 \cdot 10^{23} Joules$	$10 J mol^{-1}$

The Energy conversion factor is $\gamma_0 = \frac{q_0^2}{4\pi\epsilon_0 l_0}/E_0 = 138935.4835$. The energy unit is in Joules and if you want $kcal mol^{-1}$ unit the the factor becomes $\gamma_0/418.400$.

To compare with the molecular dynamics code, the forces and the energy is multiplied by C_{force} (C_{energy} respectively).

	Constant	Value	Unit
Energy	C_{energy}	$138935.4835/418.4$	$kcal mol^{-1}$
Force	C_{force}	-138935.4835	$10 J mol^{-1} \text{\AA}^{-1}$

The force unit is the internal DL_Poly unit.

The first test consists in a small crystal $4 \times 4 \times 4$ of NaCl. It is composed of 128 atoms. The second test is a larger crystal $10 \times 10 \times 10$ of NaCl and have 2000 atoms. The positions and the forces are stored in the REVCON file and the energy in the STATIS file. The Ewald's parameters are chosen such that the Ewald sum precision is 10^{-14} . In the test1 (reps. test2) the cut off is 8 \AA (reps. 12 \AA) and the reciprocal lattice vector in 18 (resp. 25) in each direction.

Crystal	Cube size \AA	lattice \AA	Nb atoms	Energy $kcal mol^{-1}$	Energy/atom $kcal mol^{-1}$
test1	16	$4x4x4$	128	$-2.346403 10^4$	-183.312727
test2	40	$10x10x10$	2000	$-3.666255 10^5$	183.312729

8.2.2 Stamp comparisons

Stamp uses the International unit system as internal molecular units

type	Expression	Numerical value	
time	t_0	s	second
length	l_0	m	\AA meter
mass	m_0	kg	atomic mass unit
charge	q_0	C	Coulombs
energy	$E_0 = m_0(l_0/t_0)^2$	$\frac{kgm^2}{s^2} = J$	Joules

Constant	Numerical value
ϵ_0	$8.854187 10^{-12}$
$\frac{1}{4\pi\epsilon_0}$	$8.98755179 10^9$

To compare with the molecular dynamics code, the forces and the energy is multiplied by C_{force} (C_{energy} respectively).

	Constant	Value	Unit
Energy	C_{energy}	$8.98755179 10^9$	J
Force	C_{force}	$-C_{energy}$	Jm^{-1}

T

9 Conclusion

A new approach has been proposed to compute a periodic FMM. This approach can be used with any kernel and without the need of developing new operators. Moreover, it is possible to compute an important repetition size for a small cost.

Appendix 1

DL_poly CONTROL file

```
timestep 1.e-15
steps 1
temperature 0.0
ensemble nve
delr 0.5
cutoff 8.0
ewald precision 1.e-8
no vdw
traj 1,1,2
print 1
stats 1
job time 3600
close time 10
finish
```

Appendix 2

Binary output file

We introduce the following structure for atoms coming from MD simulation. Here the type of the atom is not considered and you can find it with the index of the atom.

```
struct MDParticle {  
    FPoint position;  
    FReal forces[3];  
    FReal physicalValue;  
    FReal potential;  
    int index;  
};
```

We save the direct evaluation of the forces in a binary file. To read the file you should proceed as follow

```
std::ifstream filein(filename,  
                    std::ifstream::in | std::ios::binary);  
  
    filein.seekg (std::ios::beg);  
1 filein.read( (char*)(&nbParticles), sizeof(int));  
2 filein.read( (char*)(&boxsize[0]), sizeof(double)*3);  
3 filein.read((char*)&PER[0],sizeof(int)*4);  
4 filein.read( (char*)(&denergy), sizeof(denergy));  
//  
particles = new MDParticle[nbParticles*2];  
fileout.read((char*)&particles[0], sizeof(MDParticle)*nbParticles);
```

Line 1 reads the number of particles, line 2 the boxe size in the three dimension of space. Line 3 describes if we are periodic or not. If PER[0] is 0 the system is not periodic and the three others values are null. If PER[0] is 0 then we consider a periodic system ans for the FMM algorithm the periodic deep is given by PER[1]xPER[2]xPER[3].