Stability of Image-Reconstruction Algorithms

Pol del Aguila Pla[1,2], Sebastian Neumayer[2], and Michael Unser[2,1]

2023 IEEE International Conference on Acoustics, Speech, and Signal Processing

[1]: CIBM Center for Biomedical Imaging, Switzerland. [2]: EPFL's Biomedical Imaging Group, Lausanne, Switzerland.

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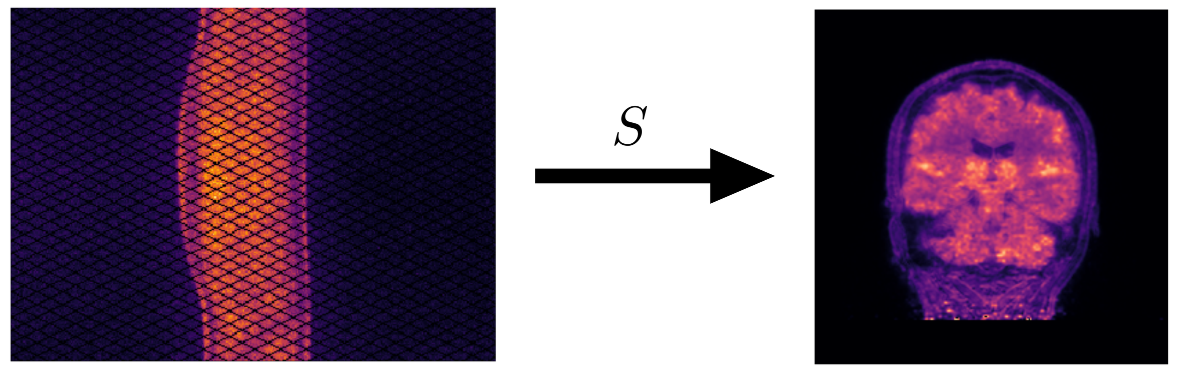
Motivation

- Image reconstruction

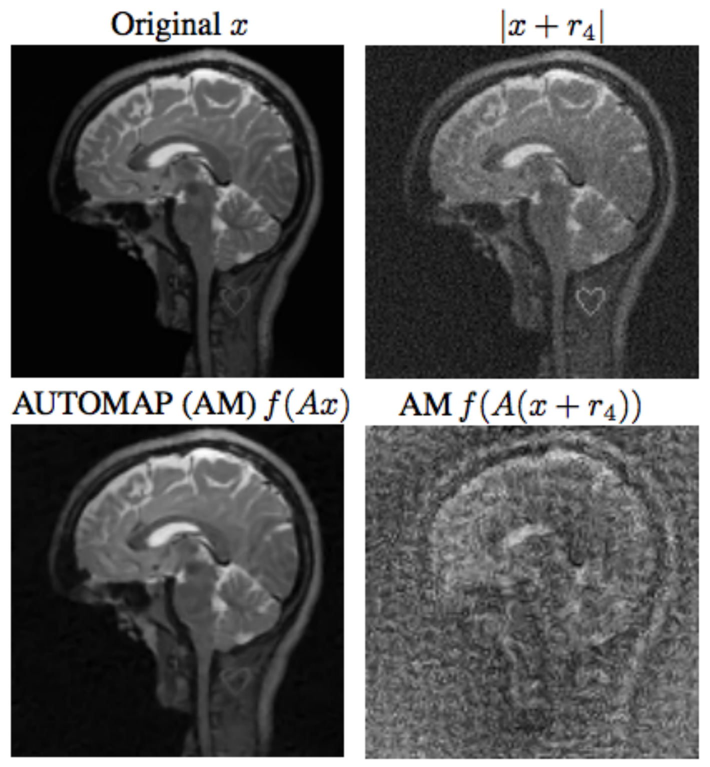


Measurement $\mathbf{y}$ Reconstruction $S(\mathbf{y}) = s_{\mathbf{y}}$

Example borrowed from NiftyPET (https://github.com/NiftyPET)

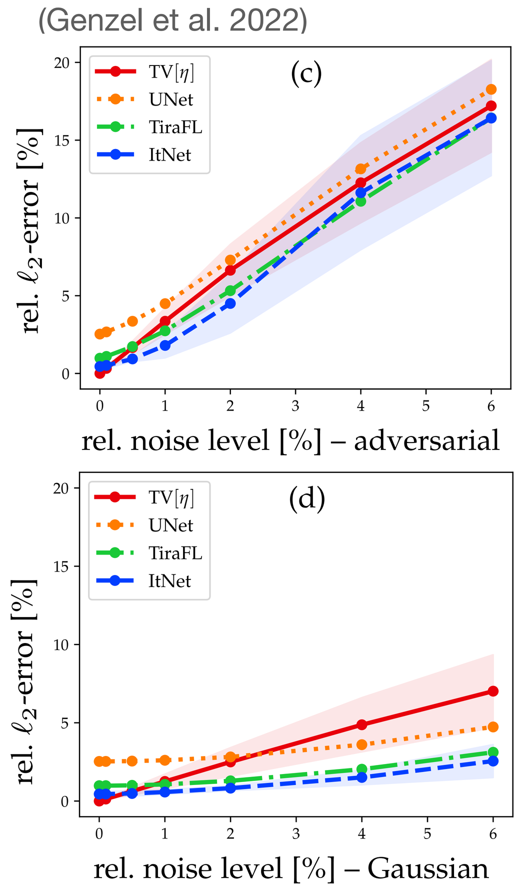
- Reliability critical to applications
  - State-of-the-art approaches are not stable

Learning-based image rec. (Antun et al. 2020)



AUTOMAP (Zhu et al. 2018)

Variational image rec. (Genzel et al. 2022)



Figures borrowed from (Antun et al. 2020) and (Genzel et al. 2022).

- Going back to fundamentals

Which methods guarantee

$$\|s_{\mathbf{y}_1} - s_{\mathbf{y}_2}\|_{\ell_p} \leq K\,\|\mathbf{y}_1 - \mathbf{y}_2\|_2^{\beta} \quad ?$$

Discussion: How should K be?

Stability through the history of image rec.

- A reference variational problem

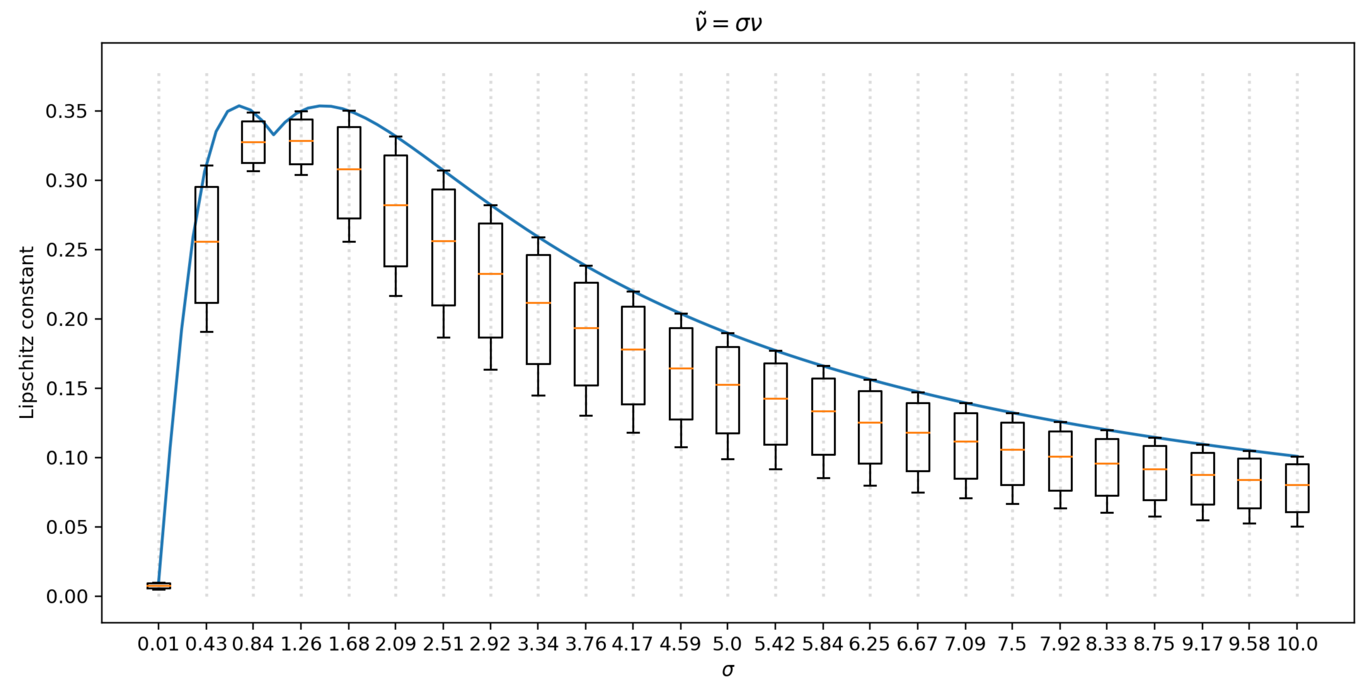
$$\min_{f\in\mathcal{X}} \left\{ E\left(\mathbf{y}, \boldsymbol{\nu}(f)\right) + \psi\left(\|f\|_{\mathcal{X}}\right) \right\}$$

measurement operator $\boldsymbol{\nu}(f) = (\langle \nu_1, f\rangle_{\mathcal{X}'\times\mathcal{X}}, \dots, \langle \nu_M, f\rangle_{\mathcal{X}'\times\mathcal{X}})$.

- Tikhonov regularization

$E(\mathbf{y}_1,\mathbf{y}_2) = \frac{1}{2}\|\mathbf{y}_1 - \mathbf{y}_2\|_2^2$, $\mathcal{X} = \ell_2$, and $\psi(x) = x$

$$\Rightarrow f_{\mathbf{y}} = \sum_{m=1}^{M} a_{\mathbf{y},m}\varphi_m \quad \text{with } \varphi_m = \mathcal{R}(\nu_m) \ \Rightarrow \mathbf{a}_{\mathbf{y}} = (\mathbf{H} + 2\lambda\mathbf{Id})^{-1}\mathbf{y}$$



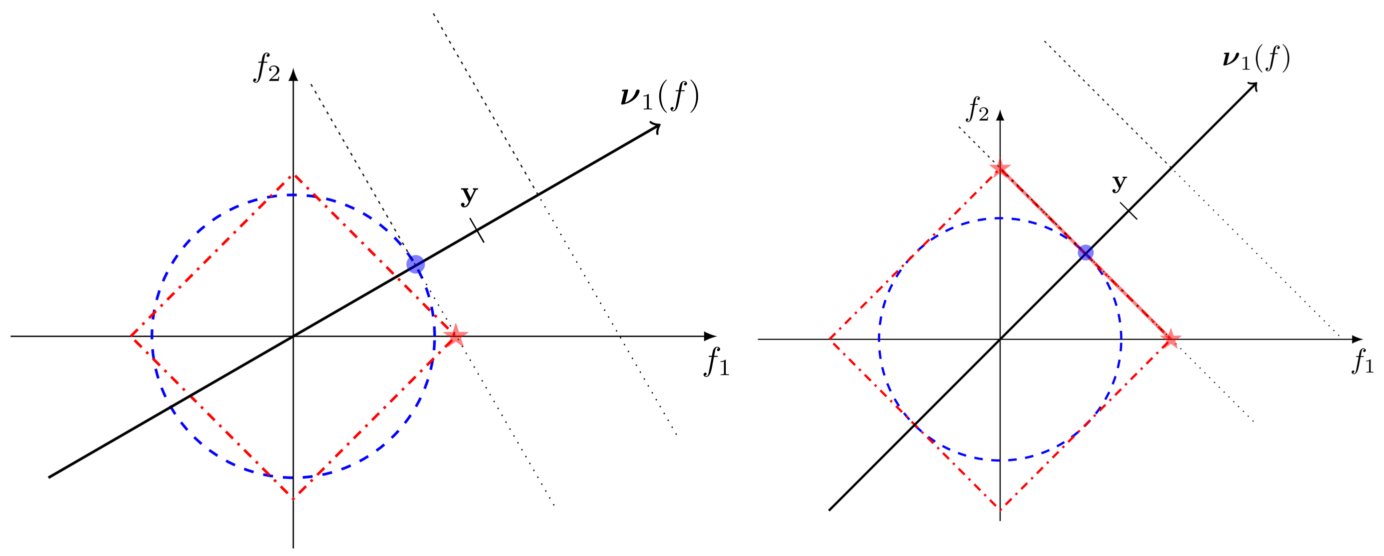
Lipschitz-stable

$$\|f_{\mathbf{y}_1} - f_{\mathbf{y}_2}\|_{\mathcal{H}}^2 \leq \max_{m\in\{1,2,\dots,M\}} \left\{ \frac{\sigma_m}{(\sigma_m + 2\lambda)^2} \right\} \|\mathbf{y}_1 - \mathbf{y}_2\|_2^2$$

where $\{\sigma_m\}$ are the eigenvalues of $\mathbf{H}$ with $\mathbf{H}_{m,n} = \langle \varphi_m, \varphi_n\rangle_{\ell_2}$.

Discussion: The role of the operator norm

- Sparsity-based regularization



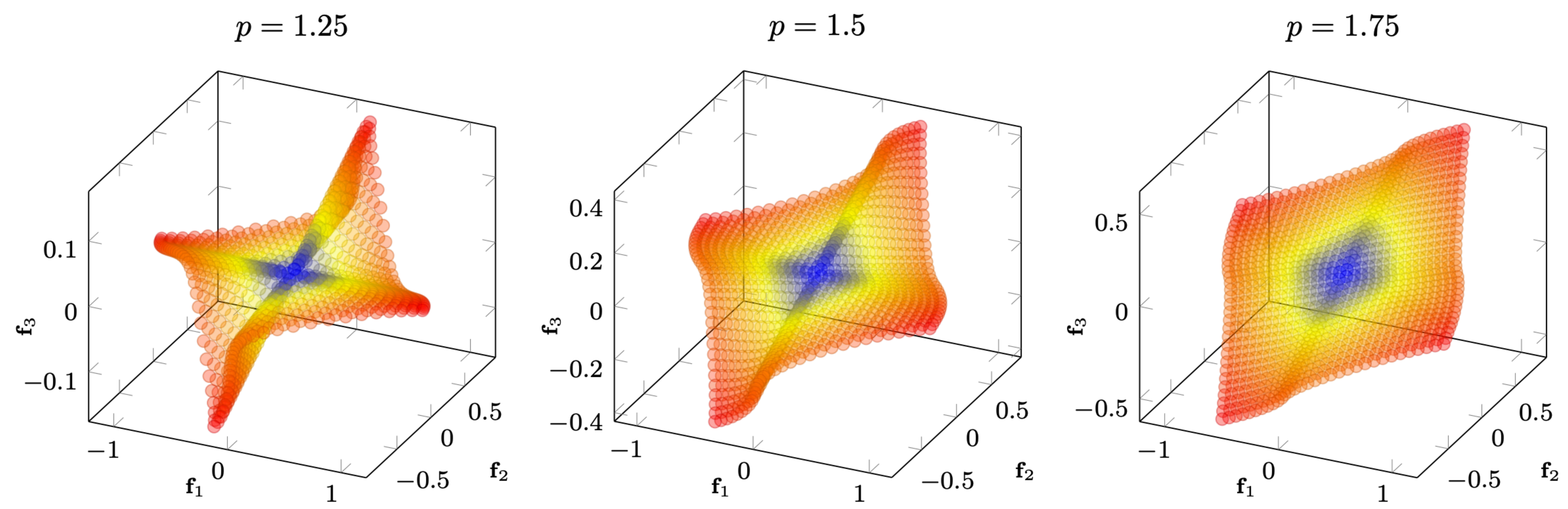
Lipschitz-stable when there is a unique solution

K hard to compute or bound

Our study

- ℓ_p regularization for $p \in (1,\infty)$

$\mathcal{X} = \ell_p$ or $\mathcal{X} = L_p(\Omega)$ and $\psi(x) = x^p$



Sets of possible solutions to an ℓ_p regularized inverse problem for different values of p.

Under technical conditions on E and ψ,

$$\Rightarrow f_{\mathbf{y}} = \mathcal{J}_{\mathcal{X}}\left(\sum_{m=1}^{M} a_{\mathbf{y},m}\nu_m\right) \quad \text{(Unser 2021)}$$

with $\mathcal{J}_{\mathcal{X}}(\nu) = \dfrac{|\nu|^{p-2}}{\|\nu\|_{L_p}^{p-2}}\nu$.

Results

- For $p \in (1,2)$

$$\|f_{\mathbf{y}_1} - f_{\mathbf{y}_2}\|_{L_p} \leq \frac{(2\,r_p(Y))^{2-p}K_p}{\lambda p(p-1)}\|\mathbf{y}_1 - \mathbf{y}_2\|_2$$

Locally Lipschitz-stable

K depends on the locality

K depends on the operator norm

Discussion: The role of the operator norm

- For $p \in (2,\infty)$

$$\|f_{\mathbf{y}_1} - f_{\mathbf{y}_2}\|_{L_p} \leq \left(\frac{2^{p-2}K_p}{\lambda p}\right)^{\frac{1}{p-1}} \|\mathbf{y}_1 - \mathbf{y}_2\|_2^{\frac{1}{p-1}}$$

Hölder stable

K depends on the operator norm