CIBM
Center for Biomedical Imaging

STABILITY OF IMAGE RECONSTRUCTION ALGORITHMS

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CIBM Breakfast and Science, March 29, 2022

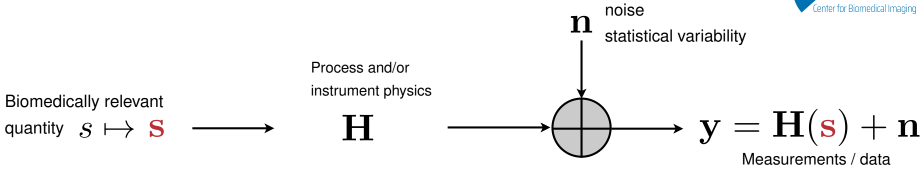
CHUV · UNIL | Université de Lausanne · EPFL · UNIVERSITÉ DE GENÈVE · HUG Hôpitaux Universitaires Genève

C I B M . C H

Joint work with Dr. S. Neumayer and Prof. M. Unser

IMAGING AS AN INVERSE PROBLEM

BIOMEDICAL IMAGING AS AN INVERSE PROBLEM



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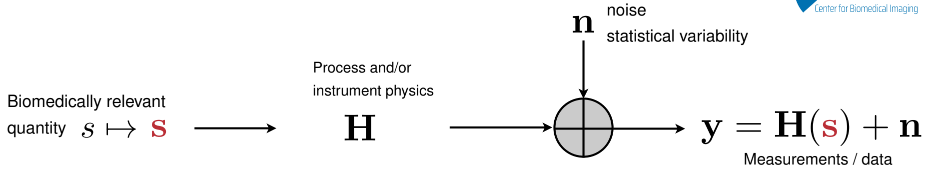


- $\mathbf{n}$: (approximately) additive statistical variability

BIOMEDICAL IMAGING AS AN INVERSE PROBLEM

Biomedically relevant quantity $s \mapsto \mathbf{s}$ $\longrightarrow$ Process and/or instrument physics $\mathbf{H}$ $\longrightarrow \oplus \longrightarrow$ $\mathbf{y} = \mathbf{H}(\mathbf{s}) + \mathbf{n}$ Measurements / data

$\mathbf{n}$ noise statistical variability

- $\mathbf{n}$: (approximately) additive statistical variability
- Ill-posed inverse problem: Many $\mathbf{s}$ could lead to the same $\mathbf{y}$

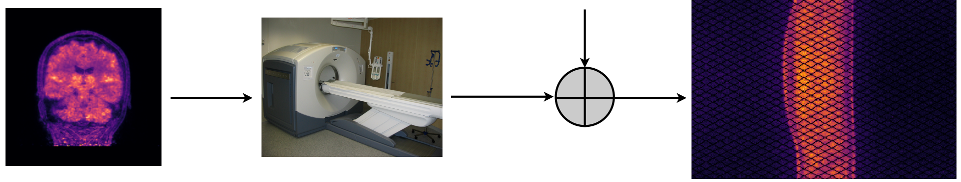
BIOMEDICAL IMAGING AS AN INVERSE PROBLEM

Example: PET

$s(\cdots)$ (2D/3D/+time) $\mapsto \mathbf{s}$

noise $\mathbf{n}$

$\mathbf{y} = \mathbf{H}(\mathbf{s}) + \mathbf{n}$



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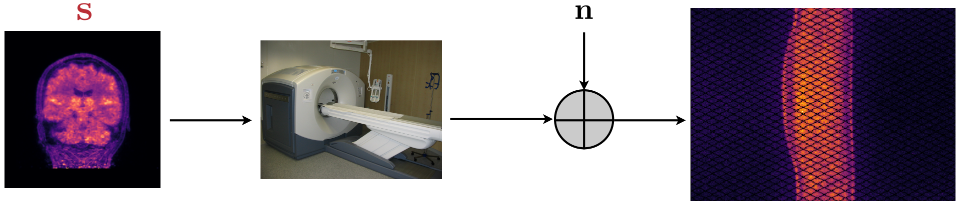

FORWARD IMAGING MODEL

Unknown molecular/anatomical map: $s(\boldsymbol{r}), \boldsymbol{r} = (x, y, z, t) \in \mathbb{R}^d$

defined over a continuum in space-time

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Imaging operator $\mathrm{H} : s \mapsto \mathbf{y} = (y_1, \cdots, y_M) = \mathrm{H}\{s\}$

from continuum to discrete (finite dimensional)

$$\mathrm{H} : L_2(\mathbb{R}^d) \to \mathbb{R}^M$$

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$$\Rightarrow \quad [\mathbf{y}]_m = y_m = \langle \eta_m, s \rangle = \int_{\mathbb{R}^d} \eta_m(\boldsymbol{r}) s(\boldsymbol{r}) \mathrm{d}\boldsymbol{r}$$

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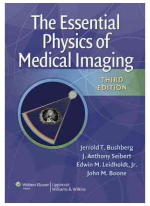

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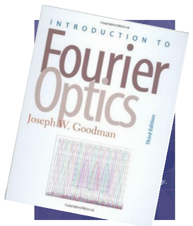

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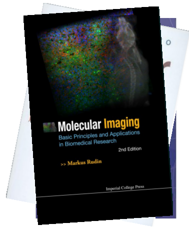

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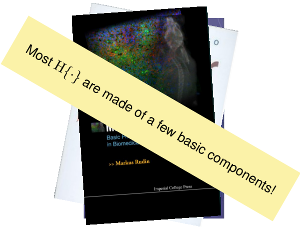

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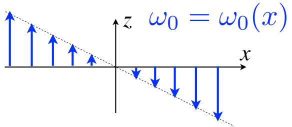

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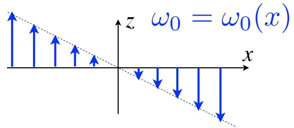

- Linear forward model for MRI $\quad \boldsymbol{r} = (x, y, z)$

$$\hat{s}(\boldsymbol{\omega}_m) = \int_{\mathbb{R}^3} s(\boldsymbol{r}) \mathrm{e}^{-\mathrm{j}\langle \boldsymbol{\omega}_m, \boldsymbol{r} \rangle} \mathrm{d}\boldsymbol{r}$$

(sampling of Fourier transform)

Equivalent analysis functions: $\eta_m(\boldsymbol{r}) = \mathrm{e}^{\mathrm{j}\langle \boldsymbol{\omega}_m, \boldsymbol{r} \rangle}$ (complex sinusoids)

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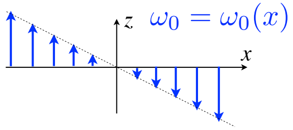

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- Extended forward model with coil sensitivity

$$\hat{s}_w(\boldsymbol{\omega}_m) = \int_{\mathbb{R}^3} w(\boldsymbol{r}) s(\boldsymbol{r}) \mathrm{e}^{-\mathrm{j}\langle \boldsymbol{\omega}_m, \boldsymbol{r} \rangle} \mathrm{d}\boldsymbol{r}$$

$$\eta_m(\boldsymbol{r}) = w(\boldsymbol{r}) \mathrm{e}^{\mathrm{j}\langle \boldsymbol{\omega}_m, \boldsymbol{r} \rangle}$$

FORWARD IMAGING MODEL

In summary

$$[\mathrm{H}s]_m = y_m = \langle \eta_m, s \rangle = \int_{\mathbb{R}^d} \eta_m(\boldsymbol{r}) s(\boldsymbol{r}) \mathrm{d}\boldsymbol{r}$$

Modality	Radiation	Analysis function η_m

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2D or 3D tomography	coherent x-ray	$\delta(t_m - \langle \boldsymbol{r}, \boldsymbol{\theta}_m \rangle)$ $\boldsymbol{\theta}_m$: mth sampled direction
3D deconvolution microscopy	fluorescence	$h(\boldsymbol{r}_m - \boldsymbol{r})$ h: PSF of microscope
Structured illumination microscopy (SIM)	fluorescence	$h(\boldsymbol{r}_m - \boldsymbol{r}) w_m(\boldsymbol{r})$ w_m: mth illumination pattern
Positron emission tomography (PET)	gamma rays	$\delta(t_m - \langle \boldsymbol{r}, \boldsymbol{\theta}_m \rangle)$ $(\boldsymbol{\theta}_m, t_m)$: mth LoR
Magnetic resonance imaging (MRI)	radio frequency	$\mathrm{e}^{\mathrm{j}\langle \boldsymbol{\omega}_m, \boldsymbol{r} \rangle}$
Cardiac MRI (parallel, non-uniform)	radio frequency	$w_m(\boldsymbol{r}) \mathrm{e}^{\mathrm{j}\langle \boldsymbol{\omega}_m, \boldsymbol{r} \rangle}$ w_m: sensitivity of the mth coil

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How do we represent $s \in L_2(\mathbb{R}^d)$ as $\mathrm{s} \in \mathbb{R}^K$?

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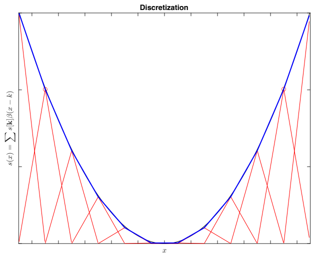
Represent the continuous signal on a finite combination of synthesis functions $\beta_{\boldsymbol{k}}(\boldsymbol{r})$

$s(\boldsymbol{r}) = \sum_{\boldsymbol{k}\in\Omega} s[\boldsymbol{k}]\beta_{\boldsymbol{k}}(\boldsymbol{r})$ with the signal vector (of coefficients): $\boldsymbol{s} = \left[s[\boldsymbol{k}]\right]_{\boldsymbol{k}\in\Omega}$ of dimension $K = |\Omega|$.



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- Applying the forward imaging model to such a signal

$$
\begin{aligned}
y_m &= \int_{\mathbb{R}^d} s(\boldsymbol{r})\eta_m(\boldsymbol{r})\mathrm{d}\boldsymbol{r} + n[m] = \langle s, \eta_m\rangle + n[m], \quad (m = 1, \dots, M) \\
&= \sum_{\boldsymbol{k}\in\Omega} \langle \beta_{\boldsymbol{k}}, \eta_m\rangle s[\boldsymbol{k}] + n[m]
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η_m: analysis function, $\beta_{\boldsymbol{k}}$: synthesis function
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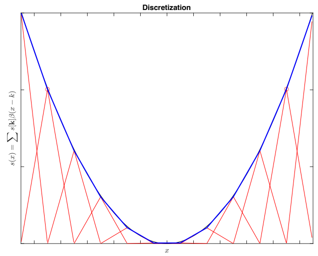

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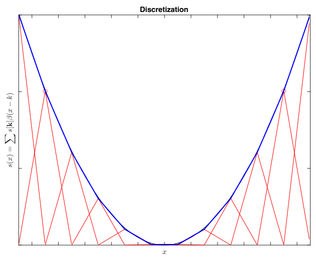

$\Rightarrow \boxed{\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n}}$ $\quad (M \times K)$ system matrix : $\quad [\mathbf{H}]_{m,\boldsymbol{k}} = \langle \eta_m, \beta_{\boldsymbol{k}}\rangle = \int_{\mathbb{R}^d} \eta_m(\boldsymbol{r})\beta_{\boldsymbol{k}}(\boldsymbol{r})\mathrm{d}\boldsymbol{r}$

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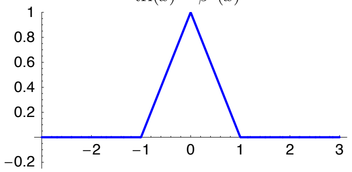

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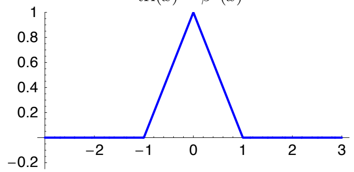
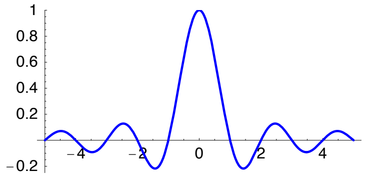
- Bandlimited representation

  $\beta(x) = \mathrm{sinc}(x)$

$\mathrm{tri}(x) = \beta^1(x)$





A CONDENSED HISTORY OF IMAGE RECONSTRUCTION

See my 40 min tutorial at: go.epfl.ch/ImRecTutorial

HISTORY OF IMAGE RECONSTRUCTION

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HISTORY OF IMAGE RECONSTRUCTION

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Classical image reconstruction (1st gen., 20th century)

Sparsity-based image reconstruction (2nd gen., 21st century)

The learning revolution (3rd gen.)

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HISTORY OF IMAGE RECONSTRUCTION

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HISTORY OF IMAGE RECONSTRUCTION

Least-squares: $\min_{\tilde{\mathbf{s}}\in\mathbb{R}_+^M}\{\|\mathbf{y}-\mathbf{H}\tilde{\mathbf{s}}\|^2\}$

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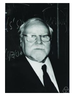

Tikhonov

Wiener

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HISTORY OF IMAGE RECONSTRUCTION

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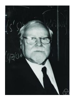
HISTORY OF IMAGE RECONSTRUCTION



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HISTORY OF IMAGE RECONSTRUCTION

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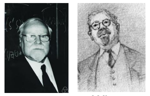
Center for Biomedical Imaging

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HISTORY OF IMAGE RECONSTRUCTION

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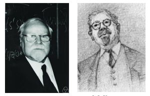

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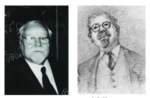

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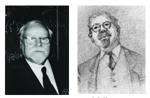

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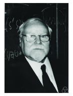

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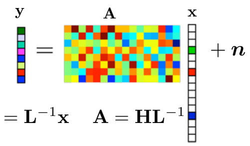

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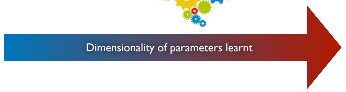
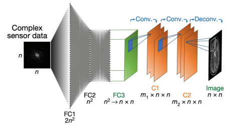


- λ
- Cal. $\mathbf{H}$
- $\mathbf{L}$
- $\mathcal{R}$
- "Task" learning



STABILITY OF IMAGE RECONSTRUCTION ALGORITHMS

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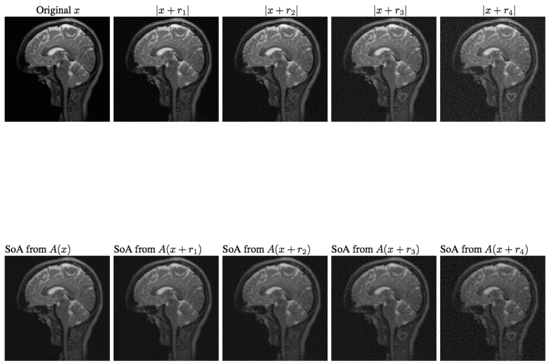
Learning-based reconstruction

Tiny perturbations of increasing strength

Ground-Truth

State-of-the-Art (SoA) Compressed Sensing



V. Antun, F. Renna, C. Poon, B. Adcock, and A.C. Hansen, "On instabilities of deep learning in image reconstruction and the potential costs of AI," Proceedings of the National Academy of Sciences 48 (117), pp. 30088–30095, 2020

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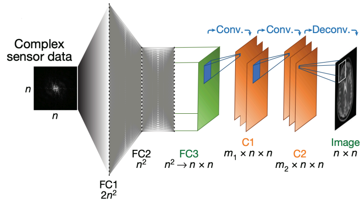

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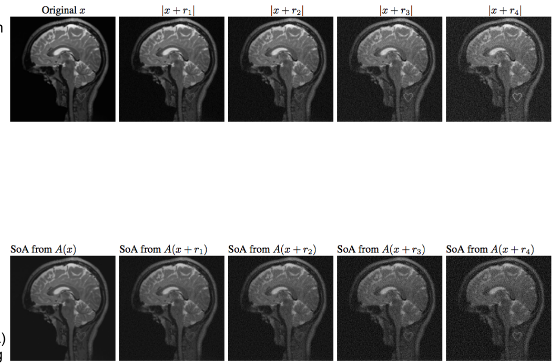

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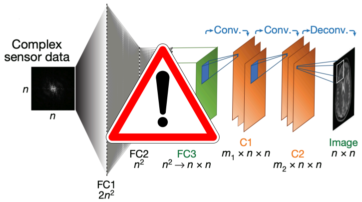

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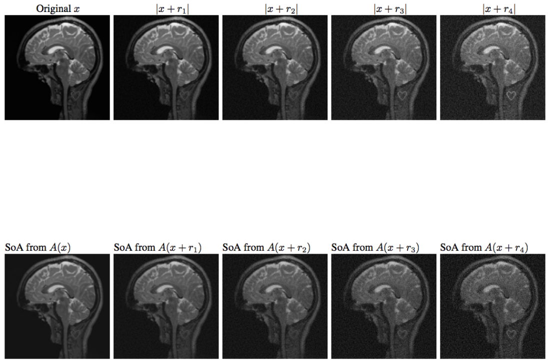

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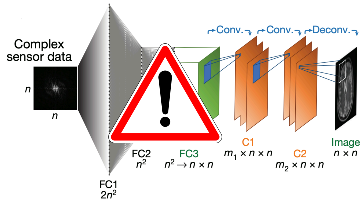

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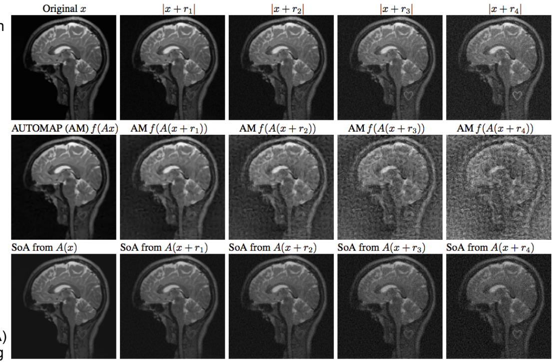

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Sparsity-based reconstruction

Stability of the reconstruction

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Continuity of the solution map

e.g., Lipschitz continuity:

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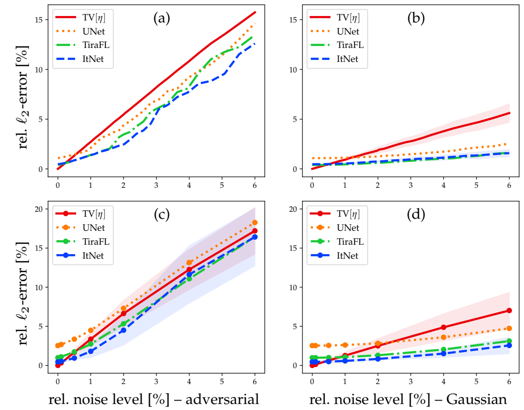

M. Genzel, J. Macdonald, and M. Marz "Solving inverse problems with deep neural networks - Robustness included," IEEE Transactions on Pattern Analysis and Machine Learning - Early Access (2022)

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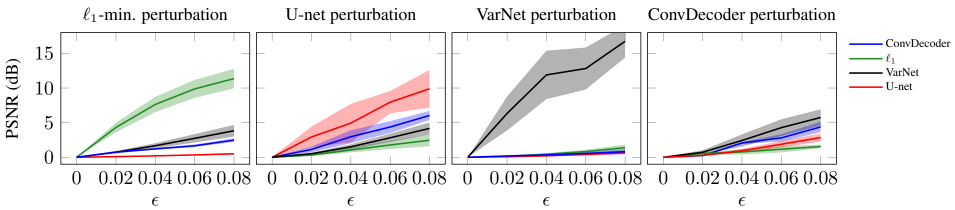

Figure 3: **Both trained and un-trained reconstruction methods are vulnerable to small adversarial perturbations.** Performance loss as a function of the perturbation strength, $\epsilon = \frac{\|\text{perturbation}\|_2}{\|k-\text{space}\|_2}$, for all methods. In each plot, the perturbations are obtained by attacking one method (specified in the plot title), and are applied to all methods. The results are averaged over 10 randomly-chosen proton density knee images from the fastMRI validation set. Shaded areas denote 95% confidence intervals.

M. Z. Darestani, A. S. Chaudhari, and R. Heckel, "Measuring robustness in deep learning based compressive sensing," Proceedings of the 38th International Conference on Machine Learning, PMLR 139:2433–2444, 2021

DISCRETIZATION: REVISITED

Exact discretization ⇔ Representer theorems for optimization in abstract spaces

- Very general setting

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M. Unser "A unifying representer theorem for inverse problems and machine learning," Foundations of Computational Mathematics (2021), in press.

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Classical image reconstruction: Tikhonov regularization

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$$\begin{aligned}
\|f_{\mathbf{y}_1} - f_{\mathbf{y}_2}\|_{\mathcal{H}}^2 &= (\mathbf{a}_{\mathbf{y}_1} - \mathbf{a}_{\mathbf{y}_2})^{\mathrm{T}}\mathbf{H}\,(\mathbf{a}_{\mathbf{y}_1} - \mathbf{a}_{\mathbf{y}_2}) \\
&= (\mathbf{y}_1-\mathbf{y}_2)^{\mathrm{T}}(\mathbf{H}+2\lambda\mathbf{Id})^{-\mathrm{T}}\mathbf{H}(\mathbf{H}+2\lambda\mathbf{Id})^{-1}(\mathbf{y}_1-\mathbf{y}_2) \\
&= (\mathbf{y}_1-\mathbf{y}_2)^{\mathrm{T}}\mathbf{P}\frac{\mathbf{\Lambda}}{(\mathbf{\Lambda}+2\lambda\mathbf{Id})^2}\mathbf{P}^{\mathrm{T}}(\mathbf{y}_1-\mathbf{y}_2) \\
&\le \max_{m\in\{1,2,\dots,M\}}\left\{\frac{\sigma_m}{(\sigma_m+2\lambda)^2}\right\}\|\mathbf{y}_1-\mathbf{y}_2\|_2^2,
\end{aligned}$$

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STABILITY OF IMAGE RECONSTRUCTION ALGORITHMS

Classical image reconstruction: Tikhonov regularization

- $\mathcal{X}$: Hilbert space $\Rightarrow \mathcal{J}_{\mathcal{X}}$ is linear $\implies f_{\mathbf{y}} = \sum_{m=1}^{M} a_{\mathbf{y},m}\varphi_m$ where $\varphi_m = \mathcal{J}_{\mathcal{X}}\{\nu_m\}$
  - For the simplified case: $E(\mathbf{y},\tilde{\mathbf{y}}) = \frac{1}{2}\|\mathbf{y}-\tilde{\mathbf{y}}\|^2$ and $\psi(x) = \lambda x^2$ $\Rightarrow \mathbf{a}_{\mathbf{y}} = (\mathbf{H}+2\lambda\mathbf{Id})^{-1}\mathbf{y}$

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\end{aligned}$$

Lipschitz continuity

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Classical image reconstruction: Tikhonov regularization

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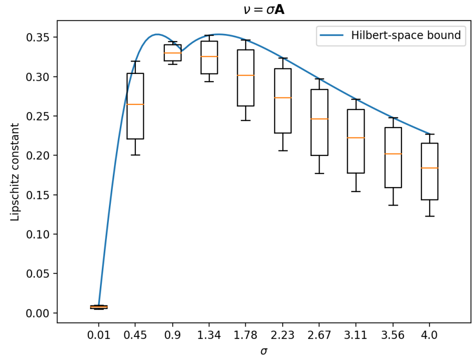
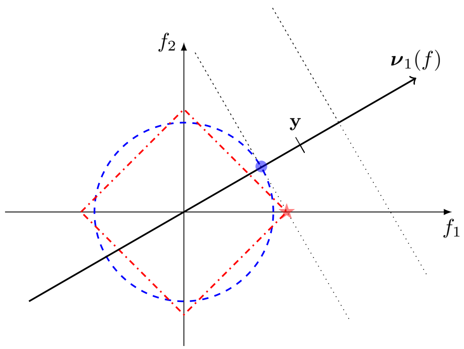
Classical image reconstruction: Tikhonov regularization



STABILITY OF IMAGE RECONSTRUCTION ALGORITHMS

Sparsity-based reconstruction



$\min_f \|f\|_p$ subject to $\|\nu_1(f) - y\|_2 \le \sigma$ for $p \in \{1, 2\}$.

E. J. Candès, J. K. Romberg, and T. Tao "Stable signal recovery from incomplete and inaccurate measurements," Communications on Pure and Applied Mathematics 59 (8), pp. 1207–1223, 2006

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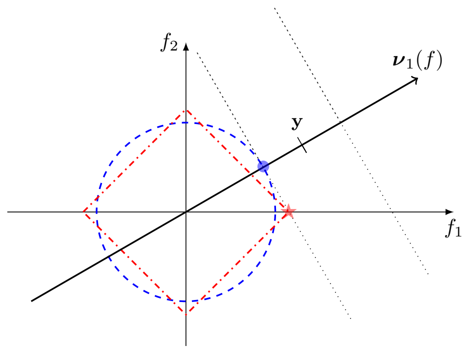

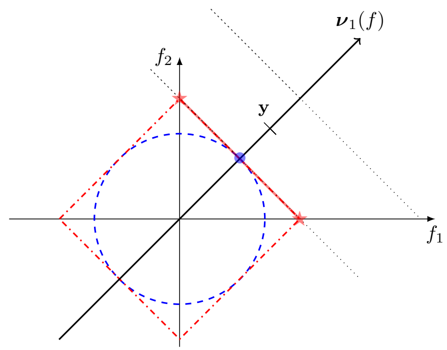

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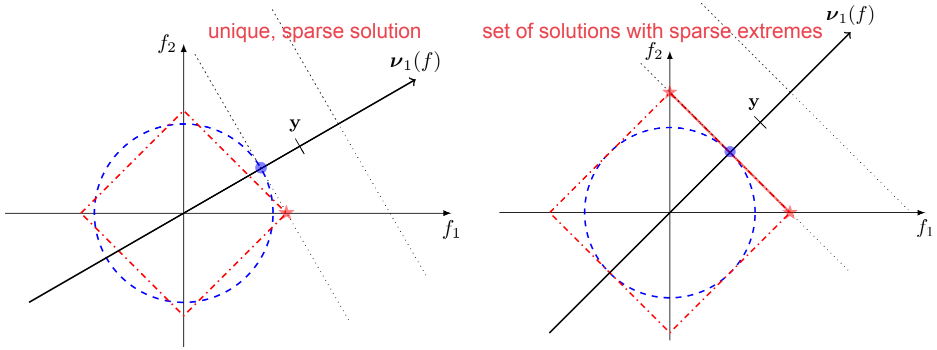

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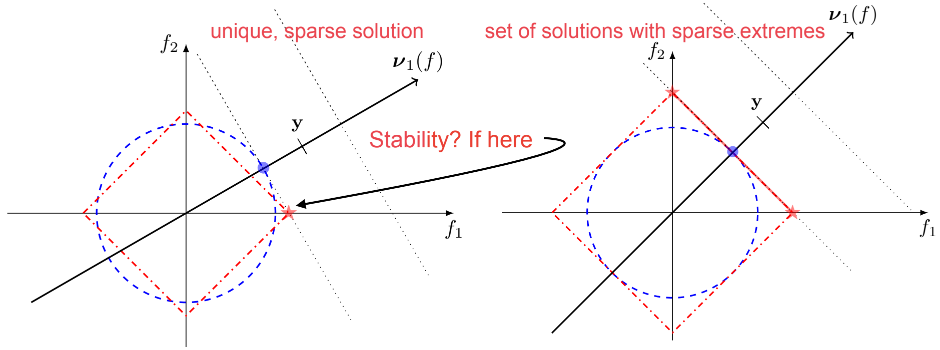

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STABILITY OF IMAGE RECONSTRUCTION ALGORITHMS

L_p-regularization

$\mathcal{R}(\mathbf{s}) = \|\mathbf{L}\mathbf{s}\|_p^p \Leftrightarrow \mathcal{X}' = L_p(\Omega)$ and $\psi(x) = \lambda x^p$

$p = 1$

(if unique)

local Lipschitz

$p \in (1, 2)$

$p = 2$

Lipschitz

Hilbert result

Hölder

$p \in (2, \infty)$

∞

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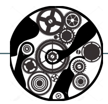

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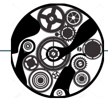

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