

Cálculo Diferencial e Integral en Varias Variables

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Ejercicio 1

Consigna

Hallar la solución general de las siguientes ecuaciones diferenciales de variables separables:

1. $y' = y^2 - 1$
2. $(1 + y^2)yy' + (1 + y^2) = 0$
3. $xe^{2y}y' - (1 + e^{2y}) = 0$

Resolución

Parte 1

- $y' = y^2 - 1$

$$\begin{aligned}
y' &= y^2 - 1 \\
&\iff \\
\frac{y'}{y^2 - 1} &= 1 \\
&\iff \\
\int \frac{y'}{y^2 - 1} dx &= \int 1 dx \\
&\iff (\text{u=y(x), du=y'(x)dx}) \\
\int \frac{1}{u^2 - 1} du &= x + k_1 \\
&\iff (\text{fracciones simples}) \\
\int \frac{1}{2} \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du &= x + k_1 \\
&\iff \\
\frac{1}{2} \left(\int \frac{1}{u-1} du - \int \frac{1}{u+1} du \right) &= x + k_1 \\
&\iff \\
\frac{1}{2} (\ln|u-1| + k_2 - \ln|u+1| + k_3) &= x + k_1 \\
&\iff (C_1 = \frac{1}{2}(k_2 + k_3)) \\
\frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + C_1 &= x + k_1 \\
&\iff (C_2 = 2k_1 - C_1) \\
\ln \left| \frac{u-1}{u+1} \right| &= 2x + C_2 \\
&\iff \\
\frac{u-1}{u+1} &= \pm e^{2x+C_2} \\
&\iff (K = \pm e^{C_2}) \\
\frac{u-1}{u+1} &= Ke^{2x} \\
&\iff \\
u-1 &= Ke^{2x}(u+1) \\
&\iff \\
u-1 &= Ke^{2x}u + Ke^{2x} \\
&\iff \\
u - Ke^{2x}u &= Ke^{2x} + 1 \\
&\iff \\
u(1 - Ke^{2x}) &= Ke^{2x} + 1 \\
&\iff (\text{suponemos que } 1 - Ke^{2x} \neq 0) \\
u &= \frac{1 + Ke^{2x}}{1 - Ke^{2x}} \\
&\iff (\text{deshago cambio de variable}) \\
y(x) &= \frac{1 + Ke^{2x}}{1 - Ke^{2x}}
\end{aligned}$$

Parte 2

$$\bullet \quad (1 + y^2)yy' + (1 + y^2) = 0$$

$$(1 + y^2)yy' + (1 + y^2) = 0$$

$$\iff (\forall x \in \mathbb{R}: (1+y^2) > 0)$$

$$y'y = \frac{-(1 + y^2)}{(1 + y^2)}$$

$$\iff$$

$$y'y = -1$$

$$\iff$$

$$\int y'y dx = \int -1 dx$$

$$\iff (u=y(x); du=y'(x)dx)$$

$$\int u du = -x + k_1$$

$$\iff$$

$$\frac{u^2}{2} = -x + k_1$$

$$\iff (C = k_1 - k_2)$$

$$u = \sqrt{2}\sqrt{-x + C}$$

$$\iff (\text{deshago cambio de variable})$$

$$y(x) = \sqrt{2}\sqrt{-x + C}$$

Parte 3

$$\bullet \quad xe^{2y}y' - (1 + e^{2y}) = 0$$

$$\begin{aligned}
xe^{2y}y' - (1 + e^{2y}) &= 0 \\
\Leftrightarrow \\
y' &= \frac{1 + e^{2y}}{e^{2y}} \cdot \frac{1}{x} \\
\Leftrightarrow \\
\frac{e^{2y}y'}{1 + e^{2y}} &= \frac{1}{x} \\
\Leftrightarrow \\
\int \frac{e^{2y}y'}{1 + e^{2y}} dx &= \int \frac{1}{x} dx \\
\Leftrightarrow (u=y(x);du=y'(x)dx) \\
\int \frac{e^{2u}}{1 + e^{2u}} du &= \ln|x| + k_1 \\
\Leftrightarrow (\text{preparando otro cambio de variable}) \\
\frac{1}{2} \int \frac{2e^{2u}}{1 + e^{2u}} du &= \ln|x| + k_1 \\
\Leftrightarrow (v=1+e^{2u};dv=2e^{2u}du) \\
\frac{1}{2} \int \frac{dv}{v} &= \ln|x| + k_1 \\
\Leftrightarrow \\
\frac{1}{2} \ln|v| + k_2 &= \ln|x| + k_1 \\
\Leftrightarrow (\text{deshago cambios de variable}) \\
\frac{1}{2} \ln|1 + e^{2y}| + k_2 &= \ln|x| + k_1 \\
\Leftrightarrow (C=2(k_1-k_2)) \\
\ln|1 + e^{2y}| &= 2\ln|x| + C \\
\Leftrightarrow (n \cdot \ln(x) = \ln(x^n) \text{ y } (1+e^{2y}) > 0) \\
1 + e^{2y} &= e^{\ln|x^2| + C} \\
\Leftrightarrow (x^2 > 0 \text{ y } K = e^C) \\
1 + e^{2y} &= Kx^2 \\
\Leftrightarrow \\
e^{2y} &= Kx^2 - 1 \\
\Leftrightarrow \\
2y &= \ln(Kx^2 - 1) \\
\Leftrightarrow \\
y &= \frac{1}{2} \ln(Kx^2 - 1)
\end{aligned}$$