Cálculo Diferencial e Integral en Varias Variables

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Ejercicio 1

Consigna

Hallar la solución general de las siguientes ecuaciones diferenciales de variables separables:

1. $y' = y^2 - 1$
2. $(1 + y^2)yy' + (1 + y^2) = 0$
3. $xe^{2y}y' - (1 + e^{2y}) = 0$

Resolución

Parte 1

- $y' = y^2 - 1$

$$y' = y^2 - 1$$
$$\iff$$
$$\frac{y'}{y^2-1} = 1$$
$$\iff$$
$$\int \frac{y'}{y^2-1}dx = \int 1dx$$
$\iff$ (u=y(x),du=y'(x)dx)
$$\int \frac{1}{u^2-1}du = x + k_1$$
$\iff$ (fracciones simples)
$$\int \frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right)du = x + k_1$$
$$\iff$$
$$\frac{1}{2}\left(\int \frac{1}{u-1}du - \int \frac{1}{u+1}du\right) = x + k_1$$
$$\iff$$
$$\frac{1}{2}(ln|u-1| + k_2 - ln|u+1| + k_3) = x + k_1$$
$\iff$ ($C_1 = \frac{1}{2}(k_2+k_3)$)
$$\frac{1}{2}ln\left|\frac{u-1}{u+1}\right| + C_1 = x + k_1$$
$\iff$ ($C_2 = 2k_1 - C_1$)
$$ln\left|\frac{u-1}{u+1}\right| = 2x + C_2$$
$$\iff$$
$$\frac{u-1}{u+1} = \pm e^{2x+C_2}$$
$\iff$ ($K = \pm e^{C}_{2}$)
$$\frac{u-1}{u+1} = Ke^{2x}$$
$$\iff$$
$$u - 1 = Ke^{2x}(u+1)$$
$$\iff$$
$$u - 1 = Ke^{2x}u + Ke^{2x}$$
$$\iff$$
$$u - Ke^{2x}u = Ke^{2x} + 1$$
$$\iff$$
$$u(1-Ke^{2x}) = Ke^{2x} + 1$$
$\iff$ (suponemos que $1 - Ke^{2x} \neq 0$)
$$u = \frac{1+Ke^{2x}}{1-Ke^{2x}}$$
$\iff$ (deshago cambio de variable)
$$y(x) = \frac{1+Ke^{2x}}{1-Ke^{2x}}$$

Parte 2

- $(1+y^2)yy' + (1+y^2) = 0$

$$
\begin{aligned}
&(1+y^2)yy' + (1+y^2) = 0 \\
&\quad \iff_{(\forall x \in \mathbb{R}: (1+y^2)>0)} \\
&y'y = \frac{-(1+y^2)}{(1+y^2)} \\
&\quad \iff \\
&y'y = -1 \\
&\quad \iff \\
&\int y'y dx = \int -1 dx \\
&\quad \iff_{(u=y(x); du=y'(x)dx)} \\
&\int u du = -x + k_1 \\
&\quad \iff \\
&\frac{u^2}{2} = -x + k_1 \\
&\quad \iff_{(C=k_1-k_2)} \\
&u = \sqrt{2}\sqrt{-x+C} \\
&\quad \iff_{(\text{deshago cambio de variable})} \\
&y(x) = \sqrt{2}\sqrt{-x+C}
\end{aligned}
$$

Parte 3

- $xe^{2y}y' - (1+e^{2y}) = 0$

$xe^{2y}y' - (1 + e^{2y}) = 0$

$\iff$

$y' = \dfrac{1 + e^{2y}}{e^{2y}} \cdot \dfrac{1}{x}$

$\iff$

$\dfrac{e^{2y}y'}{1 + e^{2y}} = \dfrac{1}{x}$

$\iff$

$\displaystyle\int \frac{e^{2y}y'}{1 + e^{2y}}dx = \int \frac{1}{x}dx$

$\iff$ $(u{=}y(x); du{=}y'(x)dx)$

$\displaystyle\int \frac{e^{2u}}{1 + e^{2u}}du = ln|x| + k_1$

$\iff$ (preparando otro cambio de variable)

$\displaystyle\frac{1}{2}\int \frac{2e^{2u}}{1 + e^{2u}}du = ln|x| + k_1$

$\iff$ $(v{=}1{+}e^{2u}; dv{=}2e^{2u}du)$

$\displaystyle\frac{1}{2}\int \frac{dv}{v} = ln|x| + k_1$

$\iff$

$\dfrac{1}{2}ln|v| + k_2 = ln|x| + k_1$

$\iff$ (deshago cambios de variable)

$\dfrac{1}{2}ln|1 + e^{2y}| + k_2 = ln|x| + k_1$

$\iff$ $(C{=}2(k_1{-}k_2))$

$ln|1 + e^{2y}| = 2ln|x| + C$

$\iff$ $(n{\cdot}ln(x){=}ln(x^n)$ y $(1{+}e^{2y}){>}0)$

$1 + e^{2y} = e^{ln|x^2|+C}$

$\iff$ $(x^2{>}0$ y $K{=}e^C)$

$1 + e^{2y} = Kx^2$

$\iff$

$e^{2y} = Kx^2 - 1$

$\iff$

$2y = ln(Kx^2 - 1)$

$\iff$

$y = \dfrac{1}{2}ln(Kx^2 - 1)$