

Lógica

Mauro Polenta Mora

Ejercicio 1

Consigna

Sean φ, ψ, σ proposiciones cualesquiera de *PROP*. Construya derivaciones que demuestren que las siguientes proposiciones son teoremas del cálculo proposicional:

- (a) $\varphi \rightarrow \varphi$
- (b) $\perp \rightarrow \varphi$
- (c) $\neg(\varphi \wedge \neg\varphi)$
- (d) $\varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi)$
- (e) $(\varphi \rightarrow \psi) \leftrightarrow \neg(\varphi \wedge \neg\psi)$
- (f) $(\varphi \wedge \psi) \leftrightarrow \neg(\varphi \rightarrow \neg\psi)$
- (g) $(\varphi \rightarrow (\psi \rightarrow \sigma)) \leftrightarrow (\psi \rightarrow (\varphi \rightarrow \sigma))$
- (h) $(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi$
- (i) $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow (\varphi \rightarrow \sigma))$
- (j) $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \sigma)) \rightarrow (\varphi \rightarrow (\psi \rightarrow \sigma))$
- (k) $((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$

Resolución

Parte (a)

Queremos probar $\vdash \varphi \rightarrow \varphi$, veamos la prueba:

Parte (b)

Queremos probar $\vdash \perp \rightarrow \varphi$, veamos la prueba:

$$\frac{[\alpha]^1}{\alpha \rightarrow \alpha} I \rightarrow^{(1)}$$

Figure 1: Figura 1

$$\frac{\frac{[\perp]^1}{\varphi} E \perp}{\perp \rightarrow \varphi} I \rightarrow^{(1)}$$

Figure 2: Figura 2

Parte (c)

Queremos probar $\vdash \neg(\varphi \wedge \neg\varphi)$, veamos la prueba:

$$\frac{\frac{[\alpha \wedge \neg\alpha]^1}{\neg\alpha} E \wedge_2 \quad \frac{[\alpha \wedge \neg\alpha]^1}{\alpha} E \wedge_1}{\frac{\perp}{\neg(\alpha \wedge \neg\alpha)} I \neg^{(1)}} E \neg$$

Figure 3: Figura 3

Parte (d)

Queremos probar $\vdash \varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi)$, veamos la prueba:

Parte (e)

Queremos probar $\vdash (\varphi \rightarrow \psi) \leftrightarrow \neg(\varphi \wedge \neg\psi)$, veamos la prueba:

Parte (f)

Queremos probar $\vdash (\varphi \wedge \psi) \leftrightarrow \neg(\varphi \rightarrow \neg\psi)$, veamos la prueba:

Parte (g)

Queremos probar $\vdash (\varphi \rightarrow (\psi \rightarrow \sigma)) \leftrightarrow (\psi \rightarrow (\varphi \rightarrow \sigma))$, veamos la prueba:

$$\begin{array}{c}
\frac{[\varphi]^1 \quad [\psi]^2}{\varphi \wedge \psi} I\wedge \\
\frac{\varphi \wedge \psi}{\psi \rightarrow \varphi \wedge \psi} I\rightarrow^{(2)} \\
\frac{\psi \rightarrow \varphi \wedge \psi}{\varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi)} I\rightarrow^{(1)}
\end{array}$$

Figure 4: Figura 4

$$\begin{array}{c}
\frac{[\neg(\varphi \wedge \neg\psi)]^1 \quad \frac{[\varphi]^2 \quad [\neg\psi]^3}{\varphi \wedge \neg\psi} I\wedge}{\frac{\perp}{\psi} RAA^{(3)} \quad \frac{\varphi \wedge \neg\psi}{\varphi \rightarrow \psi} I\rightarrow^{(2)}} E\neg \\
\frac{\varphi \rightarrow \psi}{(\varphi \rightarrow \psi) \leftrightarrow \neg(\varphi \wedge \neg\psi)} I\leftrightarrow^{(1)} \\
\frac{[\varphi \wedge \neg\psi]^4}{\neg\psi} E\wedge_2 \quad \frac{[\varphi \rightarrow \psi]^1 \quad \frac{[\varphi \wedge \neg\psi]^4}{\varphi} E\wedge_1}{\psi} E\rightarrow \\
\frac{\perp}{\neg(\varphi \wedge \neg\psi)} I\neg^{(4)} \quad \frac{\psi}{\neg(\varphi \wedge \neg\psi)} E\neg \\
\frac{\neg(\varphi \wedge \neg\psi)}{(\varphi \rightarrow \psi) \leftrightarrow \neg(\varphi \wedge \neg\psi)} I\leftrightarrow^{(1)}
\end{array}$$

Figure 5: Figura 5

$$\begin{array}{c}
\frac{[\neg(\varphi \wedge \psi)]^3 \quad \frac{[\varphi]^4 \quad [\psi]^5}{\varphi \wedge \psi} I\wedge}{\frac{\perp}{\neg\psi} I\neg^{(5)} \quad \frac{\varphi \wedge \psi}{\varphi \rightarrow \neg\psi} I\rightarrow^{(4)}} E\neg \\
\frac{[\neg(\varphi \rightarrow \neg\psi)]^1 \quad \frac{\perp}{(\varphi \wedge \psi)} RAA^{(3)} \quad \frac{\varphi \rightarrow \neg\psi}{\neg\psi} I\rightarrow^{(4)}}{(\varphi \wedge \psi) \leftrightarrow \neg(\varphi \rightarrow \neg\psi)} I\leftrightarrow \\
\frac{[\varphi \rightarrow \neg\psi]^2 \quad \frac{[\varphi \wedge \psi]^1}{\varphi} E\wedge_1}{\neg\psi} E\rightarrow \quad \frac{[\varphi \wedge \psi]^1}{\psi} E\wedge_2 \\
\frac{\perp}{\neg(\varphi \rightarrow \neg\psi)} I\neg^{(2)} \quad \frac{\psi}{\neg(\varphi \rightarrow \neg\psi)} E\neg \\
\frac{\neg(\varphi \rightarrow \neg\psi)}{(\varphi \wedge \psi) \leftrightarrow \neg(\varphi \rightarrow \neg\psi)} I\leftrightarrow
\end{array}$$

Figure 6: Figura 6

$$\begin{array}{c}
\frac{[\varphi \rightarrow (\psi \rightarrow \sigma)]^1 \quad [\varphi]^3}{\psi \rightarrow \sigma} E\rightarrow \quad \frac{[\psi]^2}{\psi \rightarrow \sigma} E\rightarrow \\
\frac{\sigma}{\varphi \rightarrow \sigma} I\rightarrow^{(3)} \quad \frac{\psi \rightarrow \sigma}{\varphi \rightarrow (\psi \rightarrow \sigma)} I\rightarrow^{(2)} \\
\frac{\varphi \rightarrow (\psi \rightarrow \sigma)}{(\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \sigma))} I\rightarrow^{(1)} \\
\frac{[\psi \rightarrow (\varphi \rightarrow \sigma)]^1 \quad [\psi]^5}{\varphi \rightarrow \sigma} E\rightarrow \quad \frac{[\varphi]^4}{\varphi \rightarrow \sigma} E\rightarrow \\
\frac{\sigma}{\psi \rightarrow \sigma} I\rightarrow^{(5)} \quad \frac{\psi \rightarrow \sigma}{\varphi \rightarrow (\psi \rightarrow \sigma)} I\rightarrow^{(4)} \\
\frac{\varphi \rightarrow (\psi \rightarrow \sigma)}{(\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \sigma))} I\leftrightarrow^{(1)}
\end{array}$$

Figure 7: Figura 7

Parte (h)

Queremos probar $\vdash (\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi$, veamos la prueba:

$$\begin{array}{c}
 \frac{[(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \neg\psi)]^1}{\varphi \rightarrow \neg\psi} E\wedge_2 \quad \frac{[(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \neg\psi)]^1}{\varphi \rightarrow \psi} E\wedge_1 \\
 \frac{\varphi \rightarrow \neg\psi \quad [\varphi]^2}{\neg\psi} E\rightarrow \quad \frac{\varphi \rightarrow \psi \quad [\varphi]^2}{\psi} E\rightarrow \\
 \frac{\neg\psi \quad \psi}{\perp} E\neg \\
 \frac{\perp}{\neg\varphi} I\neg^{(2)} \\
 \frac{\neg\varphi}{(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi} I\rightarrow^{(1)}
 \end{array}$$

Figure 8: Figura 8

Parte (i)

Queremos probar $\vdash (\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow (\varphi \rightarrow \sigma))$, veamos la prueba:

$$\begin{array}{c}
 \frac{[\varphi \rightarrow (\psi \rightarrow \sigma)]^2 \quad [\varphi]^3}{\psi \rightarrow \sigma} E\rightarrow \quad \frac{[\varphi \rightarrow \psi]^1 \quad [\varphi]^3}{\psi} E\rightarrow \\
 \frac{\psi \rightarrow \sigma \quad \psi}{\sigma} E\rightarrow \\
 \frac{\sigma}{\varphi \rightarrow \sigma} I\rightarrow^{(3)} \\
 \frac{\varphi \rightarrow \sigma}{(\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow (\varphi \rightarrow \sigma)} I\rightarrow^{(2)} \\
 \frac{(\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow (\varphi \rightarrow \sigma)}{(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow (\varphi \rightarrow \sigma))} I\rightarrow^{(1)}
 \end{array}$$

Figure 9: Figura 9

Parte (j)

Queremos probar $\vdash ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \sigma)) \rightarrow (\varphi \rightarrow (\psi \rightarrow \sigma))$, veamos la prueba:

$$\begin{array}{c}
 \frac{[(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \sigma)]^1 \quad \frac{[\psi]^3}{\varphi \rightarrow \psi} I\rightarrow^{(4)}}{\varphi \rightarrow \sigma} E\rightarrow \quad \frac{[\varphi]^2}{\sigma} E\rightarrow \\
 \frac{\varphi \rightarrow \sigma \quad \sigma}{\psi \rightarrow \sigma} I\rightarrow^{(3)} \\
 \frac{\psi \rightarrow \sigma}{\varphi \rightarrow (\psi \rightarrow \sigma)} I\rightarrow^{(2)} \\
 \frac{\varphi \rightarrow (\psi \rightarrow \sigma)}{((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \sigma)) \rightarrow (\varphi \rightarrow (\psi \rightarrow \sigma))} I\rightarrow^{(1)}
 \end{array}$$

Figure 10: Figura 10

Parte (k)

Queremos probar $\vdash ((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$, veamos la prueba:

$$\begin{array}{c}
\frac{[\neg\varphi]^2 \quad \frac{[(\varphi \rightarrow \psi) \rightarrow \varphi]^1 \quad \frac{\frac{\frac{[\neg\varphi]^2 \quad [\varphi]^3}{E\neg} \quad \frac{\perp}{E\perp}}{\psi} \quad \frac{\perp}{\varphi \rightarrow \psi} \quad I\rightarrow^3}{E\rightarrow}}{\varphi} \quad E\neg}{\frac{\perp}{\varphi} \quad RAA^{(2)}} \quad I\rightarrow^{(1)} \\
\frac{(\varphi \rightarrow \psi) \rightarrow \varphi \rightarrow \varphi}{\perp}
\end{array}$$

Figure 11: Figura 11