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Real-Time Inverse Kinematics of the Human Arm

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Abstract

A simple inverse kinematics procedure is proposed for a seven degree of freedom model of the human arm. Two schemes are used to provide an additional constraint leading to closed-form analytical equations with an upper bound of two or four solutions. Multiple solutions can be evaluated on the basis of their proximity from the rest angles or the previous configuration of the arm. Empirical results demonstrate that the procedure is well suited for real-time applications.

1 Introduction

Real-time inverse kinematics is a key component of a human modeling system. For example, in interactive figure manipulation it is convenient for the user to specify the desired location of the hand instead of explicitly manipulating the arm joints. Another important application of inverse kinematics occurs in the playback of motion capture data where joint angle trajectories must be inferred from the movement of sensors positioned on the body.

Mechanisms with more than six degrees of freedom, such as the human arm, are said to be redundant because they have more flexibility than required to achieve a given end-effector position and orientation. In robotics applications redundancy is often exploited to satisfy additional objectives such as torque optimization, and singularity and obstacle avoidance. It is generally impossible to solve for these additional criteria analytically so a numerical procedure must be used. For example, Klein (1984) used a generalized pseudoinverse technique to avoid joint limits and obstacles. Suh and Hollerbach (1987) also used the pseudoinverse to minimize torque utilization. In the computer graphics community, Maciejewski (1990) used a generalization of the pseudoinverse to generate smooth trajectories for articulated figures. Zhao and Badler (1994) implemented a flexible inverse kinematics technique that allows the user to specify multiple positioning and aiming goals for high degree of freedom figures. In their scheme, the inverse kinematics task is cast into a constrained nonlinear programming problem that is solved using a modified quasi-Newton algorithm. Koga, Kondo, Kuffner, and Latombe (1994) developed a two-phase inverse kinematics procedure for generating "natural" looking arm postures. They use a sensorimotor transformation model, proposed by Soechting and Flanders (1989a, b) to obtain an initial guess for an arm posture that matches physiological observations of human subjects. Because the solution is not exact, the joint angles are then refined by an optimization procedure until a precise solution is achieved.

The chief advantages of a numerical method are flexibility and generality: a

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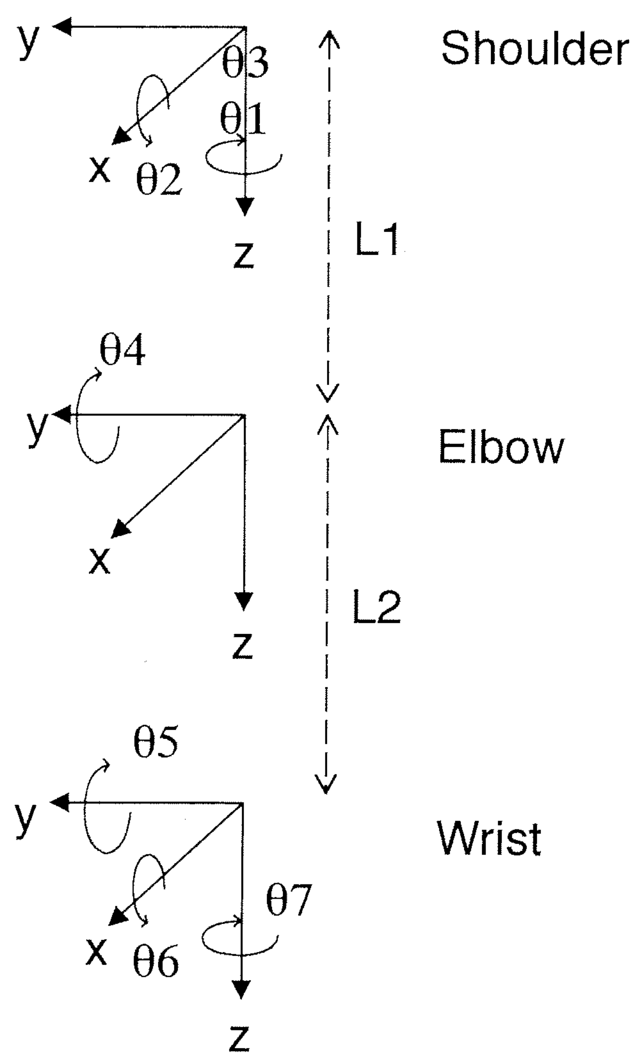

Figure 1. *A simplified model of the human arm.*

single procedure can be used for a variety of manipulators, optimization criteria and constraints can be incorporated, and solutions for redundant manipulators can be obtained. However, numerical algorithms are computationally expensive and often fall short of real-time requirements. Moreover, most numerical methods furnish only a single answer even though multiple solutions exist. Finally, stability and convergence problems often occur near a singularity of the Jacobian. For these reasons, an analytical solution is often preferable.

Because the human arm is redundant there are an infinite number of solutions for a given wrist position. We propose two simple schemes for reducing the degrees of freedom to yield a system of equations with closed-form solutions. The two methods are combined to construct a simple but effective inverse kinematics procedure for the human arm.

2 A Simple Model of the Human Arm

The human arm, particularly the shoulder-clavicle complex, is a complicated mechanical structure that is challenging to model accurately. Badler, Phillips, and Webber (1993) discuss a sophisticated model for the human arm and the implementation of its inverse kinematics for interactive tasks. However, in our application real-time performance is the chief concern so we are willing to use a simpler model that permits fast closed-form solutions to the inverse kinematics equations. The human arm can be crudely modeled as a seven degree of freedom mechanism consisting of a spherical joint for the shoulder, a revolute joint for the elbow, and a spherical joint for the wrist. This admittedly simple model neglects scapula movement and forearm pronation but is visually adequate for many applications. Placing a fixed coordinate system $\{0\}$ at the shoulder and moving coordinate systems $\{i\}$ $(i = 1..7)$ at each joint, define $\mathbf{A}_i$ as the 4×4 homogeneous coordinate transformation from frame $\{i - 1\}$ to frame $\{i\}$ as a function of joint variable θ_i. In the model shown in Figure 1, the values for $\mathbf{A}_i$ are

$$\mathbf{A}_1 = \mathbf{R}_z(\theta_i) = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_2 = \mathbf{R}_x(\theta_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c2 & -s2 & 0 \\ 0 & s2 & c2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_3 = \mathbf{R}_z(\theta_3)\mathbf{T}(0, 0, L_1) = \begin{bmatrix} c3 & -s3 & 0 & 0 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & L1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_4 = \mathbf{R}_y(\theta_4)\mathbf{T}(0, 0, L2) = \begin{bmatrix} c4 & 0 & s4 & s4L2 \\ 0 & 1 & 0 & 0 \\ -s4 & 0 & c4 & c4L2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_5 = \mathbf{R}_y(\theta_5) = \begin{bmatrix} c5 & 0 & s5 & 0 \\ 0 & 1 & 0 & 0 \\ -s5 & 0 & c5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_6 = \mathbf{R}_x(\theta_6) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c6 & -s6 & 0 \\ 0 & s6 & c6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_7 = \mathbf{R}_z(\theta_7) = \begin{bmatrix} c7 & -s7 & 0 & 0 \\ s7 & c7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where ci and si are used to denote cos (θ_i) and sin (θ_i) and $L1$ and $L2$ are constants representing the lengths of the upper and lower arm.

Given $\mathbf{A}_{\text{wrist}}$, the desired position and orientation of the wrist frame relative to the shoulder frame, the inverse kinematics problem is to find a set of angles $\theta_1, \ldots, \theta_7$ that satisfies the following equation:

$$\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4\mathbf{A}_5\mathbf{A}_6\mathbf{A}_7 = \mathbf{A}_{\text{wrist}} \tag{1}$$

If the components of $\mathbf{A}_{\text{wrist}}$ are denoted as

$$\mathbf{A}_{\text{wrist}} = \begin{bmatrix} g11 & g12 & g13 & g14 \\ g21 & g22 & g23 & g24 \\ g31 & g32 & g33 & g34 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

the vector $\mathbf{p} = [g14, g24, g34]^T$ is the position of the wrist measured in the fixed coordinate system. Given $\mathbf{p}$, the elbow angle θ_4 is uniquely determined by the distance of the wrist from the shoulder according to the formula

$$\theta_4 = \pi \pm \arccos\left(\frac{L1^2 + L2^2 - \|\mathbf{p}\|^2}{2L1L2}\right). \tag{3}$$

Although there are two solutions for θ_4 only one answer is physically realizable because of joint limits. Equation (1) is underconstrained as $\mathbf{A}_{\text{wrist}}$ specifies six, rather than seven, independent quantities so there are an infinite number of values for θ_1, θ_2, θ_3, θ_5, θ_6, and θ_7 that satisfy (1). To obtain a finite set of solutions an additional constraint must be provided. We propose two schemes that lead to simple closed-form analytic solutions.

3 Scheme 1: Specifying the Elbow Position

Korein (1985) first noted that a natural parameterization for the extra degree of freedom of the arm can be based on the observation that if the position of the wrist is fixed the elbow is still free to swivel about an axis from the wrist to the shoulder. Consider the diagram shown in figure two, where $\mathbf{s}$, $\mathbf{e}$, and $\mathbf{w}$ define the positions of the shoulder, elbow, and wrist, respectively. As the swivel angle ϕ varies, the elbow traces the arc of a circle lying on a plane whose normal is parallel to the wrist-to-shoulder axis. To mathematically describe the circle and ϕ we first define the normal vector of the plane by the unit vector in the direction of the wrist to the shoulder

$$\hat{\mathbf{n}} = \frac{\mathbf{w} - \mathbf{s}}{\|\mathbf{w} - \mathbf{s}\|}. \tag{4}$$

Additionally, we need two unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ that form a local coordinate system for the plane containing the circle. Setting $\hat{\mathbf{u}}$ to be the projection of the $-\mathbf{z}$ axis onto the plane gives

$$\hat{\mathbf{u}} = \frac{-\mathbf{z} + (\mathbf{z} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}}{\|-\mathbf{z} + (\mathbf{z} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}\|} \tag{5}$$

and $\hat{\mathbf{v}}$ is obtained by taking the crossproduct $\hat{\mathbf{v}} = \hat{\mathbf{n}} \times \hat{\mathbf{u}}$. The center of the circle $\mathbf{c}$, and its radius r can be derived by simple trigonometry

$$\begin{aligned} \mathbf{c} &= \mathbf{s} + \cos(\alpha)L1\hat{\mathbf{n}} \\ r &= L1 \sin(\alpha) \\ \cos(\alpha) &= \frac{L2^2 - L1^2 - \|\mathbf{w} - \mathbf{s}\|^2}{-2L1\|\mathbf{w} - \mathbf{s}\|} \\ \sin(\alpha) &= \frac{L2 \sin(\psi)}{\|\mathbf{w} - \mathbf{s}\|} \\ \psi &= \pi - \theta_4 \end{aligned} \tag{6}$$

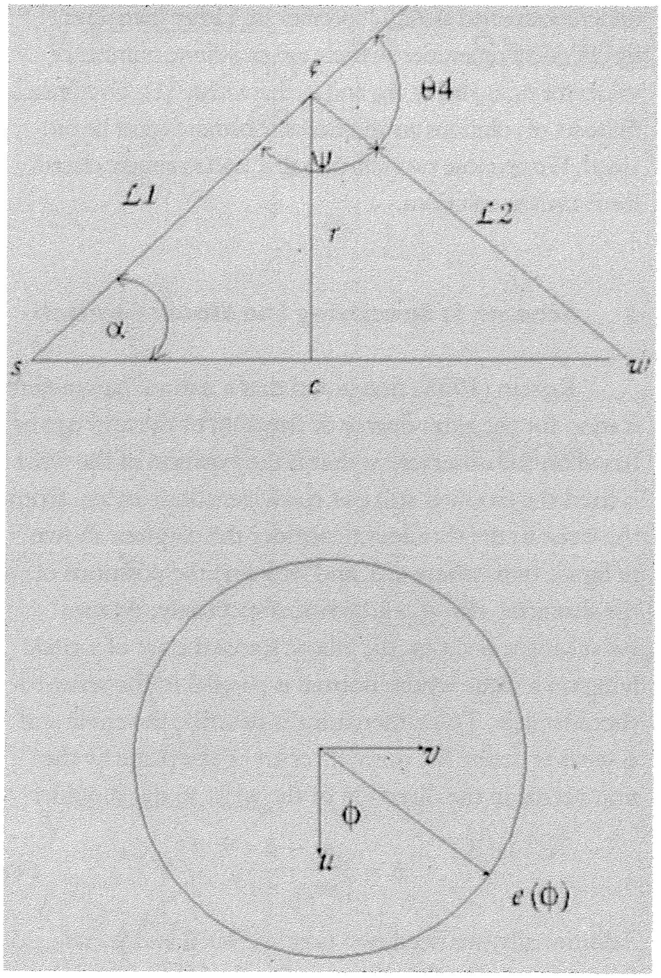

Figure 2. *The elbow position is parameterized by the swivel angle.*

and the elbow position can be parameterized as a function of ϕ (Fig. 2) about the $\hat{\mathbf{u}}$ axis

$$\mathbf{e} = r[\cos(\phi)\hat{\mathbf{u}} + \sin(\phi)\hat{\mathbf{v}}] + \mathbf{c}. \tag{7}$$

Defining $\hat{\mathbf{u}}$ as the projection of the $-\mathbf{z}$ axis allows ϕ to be interpreted as a variable that controls the elbow height. It is easy to verify that the elbow is at its lowest point when $\phi = 0$ and that increasing the magnitude ϕ of will elevate the elbow.

For the time being assume that a suitable value of ϕ is known. Once the elbow position $\mathbf{e}$ has been calculated, the inverse kinematics problem can be solved analytically. First note that the elbow position is only a function of the first three joints

$$\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}ex\\ey\\ez\\1\end{bmatrix}.$$

Expanding the matrix equation above yields the following unique solutions for θ_1 and θ_2

$$\begin{aligned}\theta_1 &= \operatorname{arctan2}(ex, -ey)\\ \theta_2 &= \operatorname{arctan2}\left(s2, \frac{ez}{L1}\right)\\ s2 &= \begin{cases}\dfrac{-ey}{c1L1} & c1 \neq 0\\ \dfrac{ex}{s1L1} & \text{otherwise}\end{cases}\end{aligned} \tag{8}$$

where arctan2 is the two argument arctangent function.

To solve θ_3, we make use of the fact that the position of the wrist depends only upon the first four joints. For known values of θ_1, θ_2, and θ_4 the value of θ_3 can be obtained by solving the equation

$$\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}g14\\g24\\g34\\1\end{bmatrix}. \tag{9}$$

Premultiplying both sides by $(\mathbf{A}_1\mathbf{A}_2)^{-1}$ and using the first two components of the vector equation leads to

$$\theta_3 = \operatorname{arctan2}(-s1c2g14 + c1c2g24 + s2g34, c1g14 + s1g24). \tag{10}$$

3.1 Solving for the Wrist Angles

Once the values of θ_1, θ_2, θ_3, and θ_4 have been calculated the wrist angles can be isolated by rearranging (1) as

$$\mathbf{A}_5\mathbf{A}_6\mathbf{A}_7 = (\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4)^{-1}\mathbf{A}_{\text{wrist}} \tag{11}$$

Denoting the elements of the right hand matrix by rij ($i = 1..3$, $j = 1..3$) and expanding the rotational components of both sides give the following nine equations:

$$\begin{bmatrix} c5c7 + s5s6s7 & -c5s7 + s5s6c7 & s5c6 \\ c6s7 & c6c7 & -s6 \\ -s5c7 + c5s6s7 & s5s7 + c5s6c7 & c5c6 \end{bmatrix} = \begin{bmatrix} r11 & r12 & r13 \\ r21 & r22 & r23 \\ r31 & r32 & r33 \end{bmatrix}. \quad (12)$$

Two possible values for θ_6 are determined by

$$\theta_6 = \begin{cases} \arcsin(-r23) \\ -[\pi + \arcsin(-r23)] \end{cases} \quad (13)$$

and the values of θ_5 and θ_7 are obtained by taking the appropriate arctangents

$$c6 \neq 0 \begin{cases} \theta_5 = \arctan 2\left(\frac{r13}{c6}, \frac{r33}{c6}\right) \\ \theta_7 = \arctan 2\left(\frac{r21}{c6}, \frac{r22}{c6}\right) \end{cases}$$

$$c6 = 0 \begin{cases} \theta_5 = \arctan 2(-r31, r11) \\ \theta_7 = 0. \end{cases} \quad (14)$$

Since each set of shoulder and elbow angles leads to two sets of solutions for the wrist joints there are a total of two possible solutions to the inverse kinematics equation for a specified value of ϕ.

3.2 Special Cases

There are two special cases that occur in the scheme described above. If the shoulder–wrist axis is parallel to the **z** axis the $\hat{\mathbf{u}}$ vector is undefined. When this occurs the possible elbow positions lie on a circle parallel to the **x–y** plane and the swivel angle has lost its interpretation as an elbow elevation parameter. The elbow equation also degenerates when the arm is completely outstretched. In this case, there is only one possible position for the elbow irrespective of the value of ϕ.

4 Scheme 2: Fixing a Shoulder or Wrist Joint

An even simpler scheme for resolving the extra degree of redundancy is to forfeit one degree of freedom by fixing one of the wrist or shoulder joints to a suitable value and solving for the remaining joint variables. In this scheme, Eq. (1) becomes an inverse kinematics problem for a six degree of freedom manipulator with a spherical joint at either the shoulder or the wrist. It is a well known result in robotics that any six degree of freedom manipulator with a spherical joint has a closed-form solution (Craig, 1989). The derivation of the solutions for each of the six cases are given in the following sections.

4.1 Solving for a Fixed Shoulder Angle

4.1.1 Case 1: Joint 1 Is Fixed. If θ_1 is known we can premultiply both sides of (9) by $\mathbf{A}_1^{-1}$ to give

$$\begin{aligned} s4L2c3 &= c1g14 + s1g24 \\ s4L2s3c2 + (-c4L2 - L1)s2 &= -s1g14 + c1g24 \\ (c4L2 + L1)c2 + s4L2s3s2 &= g34 \end{aligned} \quad (15)$$

which yields two solutions for θ_3

$$\theta_3 = \pm\arccos\left(\frac{c1g14 + s1g24}{s4L2}\right). \quad (16)$$

For a given value of θ_3, θ_2 is computed from

$$\begin{aligned} \theta_2 &= \arctan 2(ad - bc, ac + bd) \\ a &= s4L2s3 \\ b &= c4L2 + L1 \\ c &= -s1g14 + c1g24 \\ d &= g34. \end{aligned} \quad (17)$$

4.1.2 Case 2: Joint 2 Is Fixed. If θ_2 is fixed, the last equation of (15) is used to solve for two values of θ_3

$$\theta_3 = \begin{cases} \arcsin(\eta) \\ -[\pi - \arcsin(\eta)] \end{cases}$$

$$\eta = \frac{g34 - (c4L2 + L1)c2}{s4L2s2} \quad (18)$$

and θ_1 is given by

$$\begin{aligned}\theta_1 &= \arctan2(ad - bc, ac + bd)\\ a &= g24\\ b &= g14\\ c &= s4L2s3c2 + (-c4L2 - L1)s2\\ d &= s4L2c3.\end{aligned} \tag{19}$$

4.1.3 Case 3: Joint 3 Is Fixed. Premultiplying Eq. (9) by $(\mathbf{A}_1\mathbf{A}_2)^{-1}$ gives

$$\begin{aligned} s4L2c3 &= c1g14 + s1g24\\ s4L2c3 &= (-s1g14 + c1g24)c2 + s2g34\\ c4L2 + L1 &= c2g34 + (s1g14 - c1g14)s2\end{aligned} \tag{20}$$

which leads to the following solutions for θ_1 and θ_2

$$\begin{aligned}\theta_1 &= \arctan2(g24, g14) \pm \arctan2\\ &\quad (\sqrt{g14^2 + g24^2 - (s4c3L2)^2}, s4c3L2)\\ \theta_2 &= \arctan2(ad - bc, ac + bd)\\ a &= g34\\ b &= (-s1g14 + c1g24)\\ c &= c4L2 + L1\\ d &= s4L2s3.\end{aligned} \tag{21}$$

Finally, once the values of θ_1, θ_2, θ_3, and θ_4 have been calculated the wrist angles can be determined as previously discussed. In each of the first three cases, fixing a joint gives two sets of solutions for the other two shoulder angles. For each set of shoulder angles there are up to two possible solutions for the wrist joints. Therefore, fixing one of the shoulder joints leads to an upper bound of four distinct solutions to Eq. (1).

4.2 Solving for a Constant Wrist Angle

Let us now consider the case where one of the wrist angles is fixed. Since the position of the shoulder joint expressed in coordinate system six is a function of only the elbow and wrist joints, it is possible to obtain three equations that involve just θ_4, θ_5, θ_6, and θ_7:

$$\begin{aligned}\mathbf{A}_7(\mathbf{A}_{\text{wrist}})^{-1}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} &= (\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4\mathbf{A}_5\mathbf{A}_6)^{-1}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\\ \begin{bmatrix}\alpha c7 - \beta s7\\ \alpha s7 + \beta c7\\ \gamma\\ 1\end{bmatrix} &= (\mathbf{A}_5\mathbf{A}_6)^{-1}\begin{bmatrix}s4L1\\0\\-c4L1 - L2\\1\end{bmatrix}\end{aligned} \tag{22}$$

where

$$\begin{aligned}\alpha &= -(g11g14 + g21g24 + g31g34)\\ \beta &= -(g12g14 + g22g24 + g33g34)\\ \gamma &= -(g13g14 + g23g24 + g33g34).\end{aligned}$$

4.2.1 Case 4: Joint Angle 7 Is Constant. Moving $\mathbf{A}_5$ and $\mathbf{A}_6$ to the left hand side of (22) and expanding gives

$$\begin{aligned}&(\alpha c7 - \beta s7)c5 + (\alpha s7 + \beta c7)s5s6 + \gamma c6s5 = s4L1\\ &\qquad (\alpha s7 + \beta c7)c6 - \gamma s6 = 0\\ &-(\alpha c7 - \beta s7)s5 + (\alpha s7 + \beta c7)c5c6\\ &\qquad + \gamma c5c6 = -c4L1 - L2.\end{aligned} \tag{23}$$

If θ_7 is constant the only unknowns in (4) are θ_5 and θ_6. The second equation gives two values for θ_6

$$\theta_6 = \arctan2(-\gamma, \alpha s7 + \beta c7) \pm \frac{\pi}{2}. \tag{24}$$

Substituting a value for θ_6 into the first and last equations of (23) gives two values for θ_5

$$\begin{aligned}\theta_5 &= \arctan2(ad - bc, ac + bd)\\ ac5 - bs5 &= c\\ as5 + bc5 &= d\\ a &= (\alpha s7 + \beta c7)s6 + \gamma c6\\ b &= (\alpha c7 - \beta s7)\\ c &= -c4L1 - L2\\ d &= s4L1.\end{aligned} \tag{25}$$

4.2.2 Case 5: Joint Angle 5 Is Constant. If θ_5 is constant it is convenient to write (22) as

$$A_6A_7(A_{wrist})^{-1}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = (A_1A_2A_3A_4A_5)^{-1}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \tag{26}$$

which expands into the scalar equations

$$\begin{aligned} \alpha c7 - \beta s7 &= c5s4L1 + s5(c4L1 + L2) \\ \alpha c6s7 + \beta c6c7 - \gamma s6 &= 0 \\ \alpha s6s7 + \beta s6c7 + \gamma c6 &= s5s4L1 + c5(-L2 - c4L1). \end{aligned} \tag{27}$$

The first equation of (27) gives two solutions for θ_7

$$\begin{aligned} \theta_7 &= \arctan2(-\beta, \alpha) \pm \arctan2(\sqrt{\alpha^2 + \beta^2 - c^2}, c^2) \\ c &= c5s4L1 + s5(c4L1 + L2). \end{aligned} \tag{28}$$

For a given value of θ_7 the corresponding value of θ_6 is

$$\begin{aligned} \theta_6 &= \arctan2(ad, \gamma d) \\ a &= \alpha s7 + \beta c7 \\ d &= s5s4L1 + c5(-L2 - c4L1). \end{aligned} \tag{29}$$

4.2.3 Case 6: Joint Angle 6 Is Constant. If θ_6 is constant, the only unknowns in (27) are θ_5 and θ_7. The second equation gives two values for θ_7

$$\begin{aligned} \theta_7 &= \arctan2(\beta c6, \alpha c6) \\ &\pm \arctan2(\sqrt{(\alpha c6)^2 + (\beta c6)^2 - (\gamma s6)^2}, \gamma s6) \end{aligned} \tag{30}$$

and θ_5 is determined by

$$\begin{aligned} \theta_5 &= \arctan2(ad - bc, ac + bd) \\ a &= s4L1 \\ b &= -c4L1 - L2 \\ c &= \alpha c7 - \beta s7 \\ d &= \alpha s6s7 + \beta s6c7 + \gamma c6. \end{aligned} \tag{31}$$

4.2.4 Solving for the Shoulder Angles. Finally, once the values of the wrist joints have been determined the values of $\theta_1, \theta_2, \theta_3$ can be extracted from the rotational components of $A_{wrist}(A_4A_5A_6A_7)^{-1}$. Rearranging (1) as

$$A_1A_2A_3 = A_{wrist}(A_4A_5A_6A_7)^{-1} \tag{32}$$

isolates all the known quantities on the right-hand side. Denoting the rotational component of the right hand side by rij yields the equations

$$\begin{bmatrix} c1c3 - s1c2s3 & -c1s3 - s1c2c3 & s1s2 \\ s1c3 + c1c2s3 & -s1s3 + c1c2c3 & -c1s2 \\ s2s3 & s2c3 & c2 \end{bmatrix} = \begin{bmatrix} r11 & r12 & r13 \\ r21 & r22 & r23 \\ r31 & r32 & r33 \end{bmatrix}.$$

Casual inspection reveals two sets of values for $\theta_1, \theta_2, \theta_3$

$$\begin{aligned} \theta_2 &= \pm\arccos(r33) \\ \theta_1 &= \begin{cases} s2 \neq 0 & \arctan2\left(\dfrac{r13}{s2}, \dfrac{-r23}{s2}\right) \\ s2 = 0 & \arctan2(r21, r11) \end{cases} \\ \theta_3 &= \begin{cases} s2 \neq 0 & \arctan2\left(\dfrac{r31}{s2}, \dfrac{-r32}{s2}\right) \\ s2 = 0 & 0 \end{cases} \end{aligned} \tag{33}$$

As with the case of fixing a shoulder joint, fixing a wrist angle produces an upper bound of four distinct solutions to Eq. (1).

5 Joint Limits and Incomplete Solutions

Because of joint limits and holes in the workspace, it is not always possible to find a solution. There are several compromise strategies that can be used under these circumstances. If a single joint exceeds its limit by a small margin, the swivel angle can be incremented or decremented and the inverse kinematics equations recomputed. If the new angle is closer to the joint limit the process is repeated iteratively until a solution is found or no further improvement is possible. Alternatively, the offending joint can be fixed to its limit and scheme two can be used to compute for the other joint angles in the hope that they will remain feasible. Additionally, if the orientation of the hand is not critical it only becomes necessary to find a solution for the shoulder and elbow joint variables.

6 A Real-Time Inverse Kinematics Tracking Procedure

The inverse kinematics equations derived in the previous sections can be used to construct a real-time tracking procedure for following an end-effector trajectory. Initially, a solution is obtained by using scheme one and selecting a suitable value for the swivel angle ϕ. Ideally the value selected for ϕ should produce a "realistic" looking arm posture for the starting hand configuration. In our applications, the user interactively adjusts ϕ until the solution looks visually acceptable. If user intervention is not possible a heuristic must be used instead. A naive but often satisfactory rule of thumb is to select a small positive value for ϕ; this has the effect of keeping the elbow down and spread slightly away from the torso. A more sophisticated scheme would be to use a biomechanical model to guess a "natural" looking placement for the elbow position. For example, Soechting and Flanders (1989a, b) conducted a set of experiments where subjects pointed in the dark to remembered targets using a stylus. Based on their observations, the authors developed a set of equations that predict the forearm and upper arm postures as a function of the hand position. Another possibility is to use data recorded from human subjects to build a table of swivel angles for a variety of hand positions. The table entries could then be used by the inverse kinematics procedure to interpolate a suitable value for ϕ.

After an initial solution has been determined, the tracking procedure switches to scheme two. When the goal position changes the inverse kinematics equations are then solved six times, once for each individual wrist and shoulder joint fixed at its previous value. This produces a maximum number of 24 possible solutions. To keep the arm motion smooth the procedure chooses the set of angles that is closest to the previous solution as measured by $\Sigma_{i=1}^{7} (\theta_i - \theta_i^0)^2$ where θ_i^0 denotes the previous value of the ith joint variable. Finally if a solution cannot be found the procedure solves for position only, leaving the wrist angles unchanged.

Although the inverse kinematics equations are evaluated six times with each iteration, our experimental results indicate that the performance is more than adequate for computer animation requirements. For example, on a 200-MHz SGI workstation, the inverse kinematics equations can be computed at rates exceeding 20,000 Hz. Smooth arm motions are achieved through the combined strategy of using the previous joint angles as the fixed values for the next iteration and by selecting the solution that is closest to the previous joint configuration. On occasion, an abrupt change in joint angles can occur. This usually happens when the arm is at the boundary of a joint limit and a change in end-effector position requires a sudden flip in the orientation of the arm. In this case, the tracking procedure interpolates the joint angles to produce a smooth motion.

7 Conclusions and Limitations

We have presented an analytical approach for computing the inverse kinematics of a simple model of the human arm. Using two different schemes we reduced the number of degrees of freedom of the arm by one to obtain closed-form equations that solve the inverse kinematics problem. These equations form the basis of a procedure for tracking the motion of an end-effector in real-time. The method is efficient, accurate, robust, and produces smooth-looking motions, but it suffers from some imperfections. In particular, the method does not have a well-founded theory for generating and evaluating realistic looking postures. Additionally, fixing a joint reduces the reachable workspace and also produces motions where only six of the seven joint variables change between successive time steps. Usually these limitations are not noticeable, but an ideal scheme would utilize an analytical solution that exploits the full redundancy of the manipulator.

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