Linear Algebra

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Chapter 1

Introduction

1.1 Notation

A vector is an ordered list of numbers:

$$\begin{pmatrix} -1.1 \\ 0.0 \\ 3.7 \end{pmatrix}$$

Vectors are denoted here using boldface: $\boldsymbol{a}$, $\boldsymbol{b}$, etc. The i-th element of an n-vector is a_i, where i is the index.

Two vectors $\boldsymbol{a}, \boldsymbol{b}$ are equal if $a_i = b_i$ for all i. A **stacked vector** with vectors $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ is:

$$\boldsymbol{a} = \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix}$$

A **unit vector** $\boldsymbol{e}_i$ has 1 in the i-th position and 0 elsewhere:

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

A vector is **sparse** if most entries are 0.

1.2 Addition and Scalar Multiplication

Vector addition is entry-wise:

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix}$$

1.2.1 Properties of Vector Addition

- Commutative: $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$
- Associative: $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$
- Identity: $\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$
- Inverse: $\boldsymbol{a} - \boldsymbol{a} = \boldsymbol{0}$

1.2.2 Scalar-Vector Multiplication

For scalar α and vector $\boldsymbol{a}$:

$$(-2)\begin{pmatrix}2\\4\\7\end{pmatrix} = \begin{pmatrix}-4\\-8\\-14\end{pmatrix}$$

- Associative: $\alpha(\beta\boldsymbol{a}) = (\alpha\beta)\boldsymbol{a}$
- Distributive: $(\alpha+\beta)\boldsymbol{a} = \alpha\boldsymbol{a} + \beta\boldsymbol{a}$
- Distributive: $\alpha(\boldsymbol{a}+\boldsymbol{b}) = \alpha\boldsymbol{a} + \alpha\boldsymbol{b}$

1.2.3 Linear Combinations & Inner Product

A **linear combination** of vectors $\boldsymbol{a}_1, ..., \boldsymbol{a}_m$ with scalars $\beta_1, ..., \beta_m$ is:

$$\beta_1\boldsymbol{a}_1 + \cdots + \beta_m\boldsymbol{a}_m$$

The **dot product** (inner product) of vectors $\boldsymbol{a}, \boldsymbol{b}$ is:

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

Example:

$$\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}2\\9\\-4\end{pmatrix} = (1)(2) + (-1)(9) + (0)(-4) = -7$$

Inner Product Properties

For vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and scalar γ:

- Commutative: $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$
- Scalar association: $(\gamma\boldsymbol{a})\cdot\boldsymbol{b} = \gamma(\boldsymbol{a}\cdot\boldsymbol{b})$
- Distributive: $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{c}$
- Relation to norm: $\boldsymbol{a}\cdot\boldsymbol{a} = \|\boldsymbol{a}\|^2$