Probability Theory Notes

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Chapter 1

Probability

1.1 Set theory basics

A set is a collection of objects. If S is a set, $x \in S$ means that the element x is in the set S:

- $\{1, 3, 5, 7, \dots\}$ is the set of all odd numbers.
- $\mathbb{R}$ is the set of all real numbers.
- $[3, 7]$ is the set of all real numbers between 3 and 7.

The **empty set** is the smallest set, containing no elements, denoted by $\emptyset$. If A and B are sets and $A \subseteq B$, it means that every element of A is in B, expressed by:

$$A \subseteq B \quad \text{if and only if} \quad \forall x \in A, x \in B. \tag{1.1}$$

Two sets are equal if they have the same elements. The **union** of sets A and B, denoted by $A \cup B$, represents the set of all elements that are in A or B, expressed by:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}. \tag{1.2}$$

Alternatively, using logical symbols, it can be written as:

$$A \cup B = \{x \mid x \in A \vee x \in B\}. \tag{1.3}$$

The **intersection** of the sets A and B, denoted by $A \cap B$, represents the set of elements that they have in common, expressed by:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}. \tag{1.4}$$

The **complement** of A, denoted by A^c, represents the elements that are not in the set A, expressed by:

$$A^c = \{x \mid x \notin A\}. \tag{1.5}$$

We say that A and B are **disjoint** if they have no elements in common (their intersection is empty), denoted by $A \cap B = \emptyset$. Two sets A and B are equal if both $A \subseteq B$ and $B \subseteq A$. The sets $A_1, A_2, \ldots, A_n$ are *mutually disjoint* if:

$$A_i \cap A_j = \emptyset \quad \text{for all} \quad i \neq j, \quad i, j = 1, 2, \ldots \tag{1.6}$$

In this case, we can express the union as:

$$A_1 \cup A_2 \cup \cdots \cup A_n = \sum_{j=1}^{n} A_j \quad \text{and for infinite sets,} \quad \bigcup_{j=1}^{\infty} A_j. \tag{1.7}$$

Similarly, the intersection for n sets is written as:

$$\bigcap_{i=1}^{n} A_i. \tag{1.8}$$

The idea holds even when dealing with infinite sets.
The **difference** between sets A and B is defined as:

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}. \tag{1.9}$$

The **symmetric difference** $A \Delta B$ is:

$$A \Delta B = (A - B) \cup (B - A). \tag{1.10}$$

De Morgan's laws give us a duality between unions and intersections:

- $(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$
- $(A_1 \cap A_2 \cap \cdots \cap A_n)^c = A_1^c \cup A_2^c \cup \cdots \cup A_n^c$

The first law says that not being in at least one of the A_n's is the same as not being in every single one of them.

The second law says that not being in all of the A_n's is the same as being outside at least one of the A_n's.

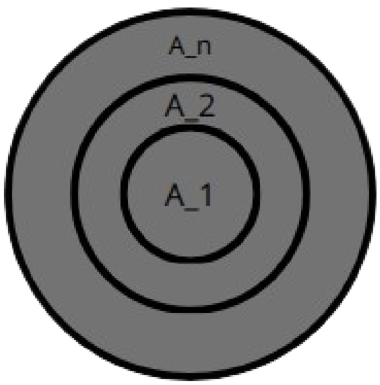
A **partition** is a collection of subsets $A_1, A_2, \ldots, A_n$ of S such that the union of those sets is equal to S and $A_i \cap A_j = \emptyset$ for $i \neq j$.

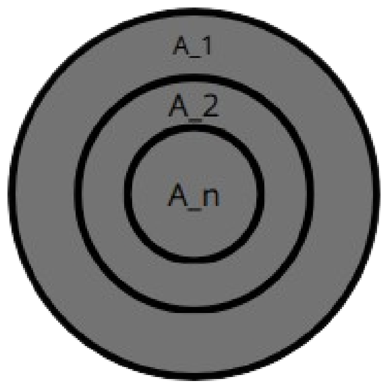
1.1.1 Sequences

The sequence $\{A_n\}, \quad n = 1, 2, \ldots$ is a **monotone sequence** of sets if:

1. $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ (A_n is increasing). We denote that by $A_n^{\uparrow}$.
2. $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ (A_n is decreasing). We denote that by $A_n^{\downarrow}$.

We can represent the sequences below:





The *limit* of the sequences is defined below:

1. If it's increasing, then $\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$
2. If it's decreasing, then $\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$

Generally:

$$\underline{A} = \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} A_j \tag{1.11}$$

and:

$$\bar{A} = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j \tag{1.12}$$

If these limits are equal, the sequence has a limit.