BNPlib for density estimation:

A nonparametric C++ library (part 3)

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https://github.com/poliprojects/BNPlib

Model

DP and DPM models

Having observed the iid sample $\{y_i\}_i$, $i = 1, \ldots, n$:

- Dirichlet process model (discrete):

$$y_i | G \overset{\text{iid}}{\sim} G$$
$$G \sim DP(MG_0)$$

- Dirichlet process mixture (**DPM**) model (continuous):

$$y_i | G \overset{\text{iid}}{\sim} f_G(\cdot) = \int_\Theta f(\cdot|\boldsymbol{\vartheta})\, G(\mathrm{d}\boldsymbol{\vartheta})$$
$$G \sim DP(MG_0)$$

Equivalent formulations (1)

- (DPM) is equivalent to:

$$
\begin{aligned}
y_i|\vartheta_i &\overset{\text{ind}}{\sim} f(\cdot|\vartheta_i), \quad i = 1, \dots, n \\
\vartheta_i|G &\overset{\text{iid}}{\sim} G, \quad i = 1, \dots, n \\
G &\sim DP(MG_0)
\end{aligned}
$$

- State: ϑ_i latent variables (discrete)

Equivalent formulations (2)

- (DPM) is also equivalent to:

$$
\begin{aligned}
y_i|\boldsymbol{\phi}, c_i &\overset{\text{ind}}{\sim} f(\cdot|\phi_{c_i}), \quad i = 1, \dots, n \\
c_i|\mathbf{p} &\overset{\text{iid}}{\sim} \sum_{j=1}^{K} p_j \delta_j(\cdot), \quad i = 1, \dots, n \\
\phi_c &\overset{\text{iid}}{\sim} G_0, \quad c = 1, \dots, k \\
\mathbf{p} &\sim \text{Dir}(M/K, \dots, M/K) \\
K &\to +\infty
\end{aligned}
$$

- State: c_i **allocations** to clusters
- State: ϕ_{c_i} **unique values** for each cluster
- Only the finitely many ϕ_c used are kept track of

Case study

- (DPM) with Normal Normal-InverseGamma (**NNIG**) prior:

$$
\begin{aligned}
y_i|\vartheta_i &\overset{\text{ind}}{\sim} f(\cdot|\vartheta_i), \quad i = 1, \ldots, n \\
\vartheta_i|G &\overset{\text{iid}}{\sim} G, \quad i = 1, \ldots, n \\
G &\sim DP(MG_0)
\end{aligned}
$$

$$
\begin{aligned}
f(y|\boldsymbol{\vartheta}) &= N(y|\mu, \sigma^2), \\
G_0(\boldsymbol{\vartheta}|\mu_0, \lambda_0, \alpha_0, \beta_0) &= N\left(\mu|\mu_0, \frac{\sigma^2}{\lambda_0}\right) \times \text{Inv-Gamma}(\sigma^2|\alpha_0, \beta_0)
\end{aligned}
$$

- State: $\boldsymbol{\vartheta} = (\mu, \sigma)$,

Algorithms

General structure

```
void step(){
    sample_allocations();
    sample_unique_values();
}

void run(){
    initialize();
    unsigned int iter = 0;
    while(iter < maxiter){
        step();
        if(iter >= burnin){
            save_iteration(iter);
        }
        iter++;
    }
}
```

aaa

aaa

Applications

Cluster estimation

```
unsigned int cluster_estimate();
```

$$\hat{k} = \arg\min_k \left\| D^{(k)} - \bar{D} \right\|_F^2 = \arg\min_k \sum_{i,j} \left(D_{ij}^{(k)} - \bar{D}_{ij} \right)^2$$

Density estimation

```
void eval_density(const std::vector<double> grid);
```

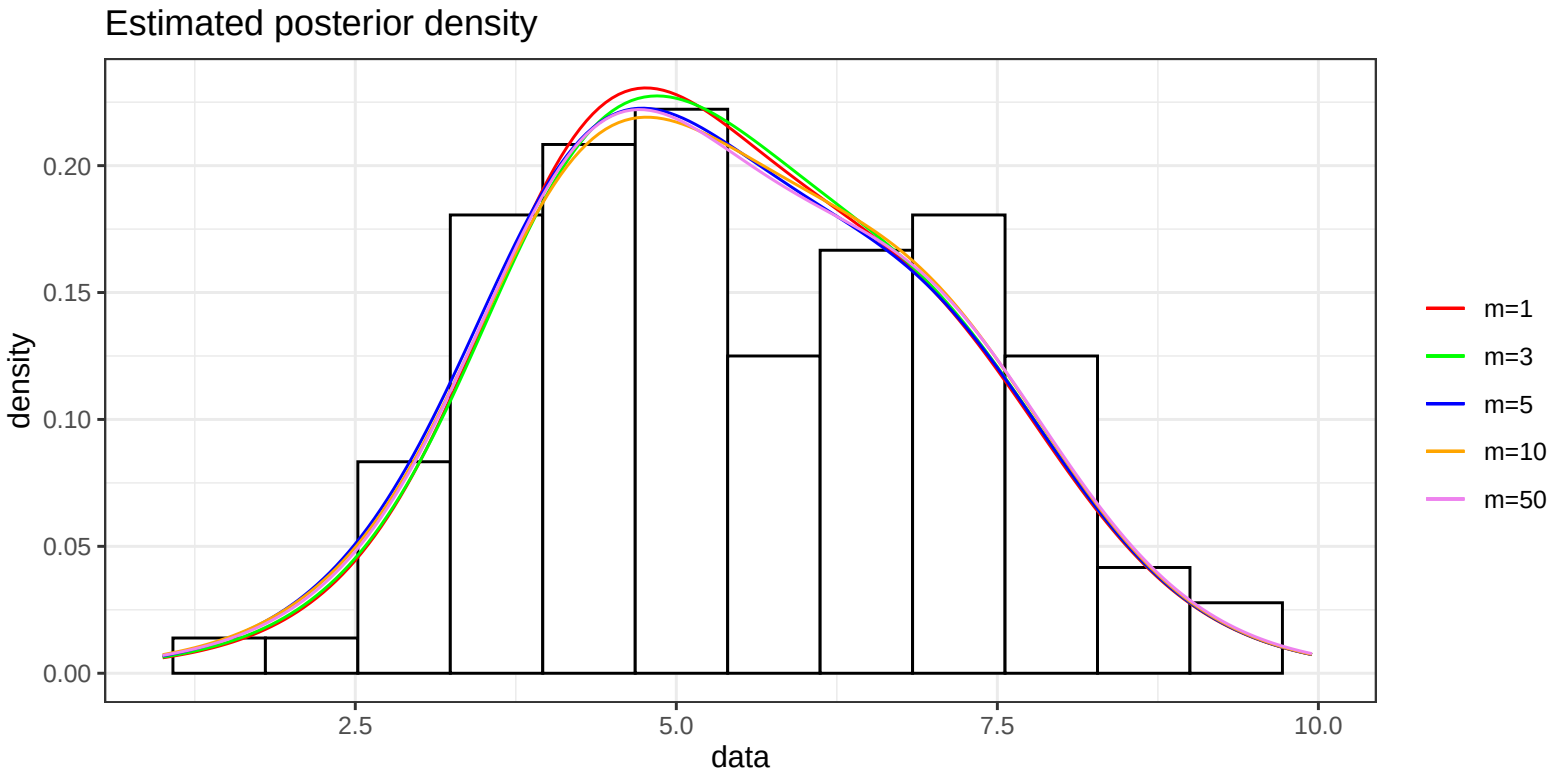
$$\hat{f}^{(k)}(x) = \sum_j \frac{n_j^{(k)}}{M+n} f\left(x|\phi_j^{(k)}\right) + \frac{M}{M+n} m(x)$$

$$\hat{m}(x) = \frac{1}{m} \sum_{h=0}^{m-1} f\left(x|\phi_h\right)$$

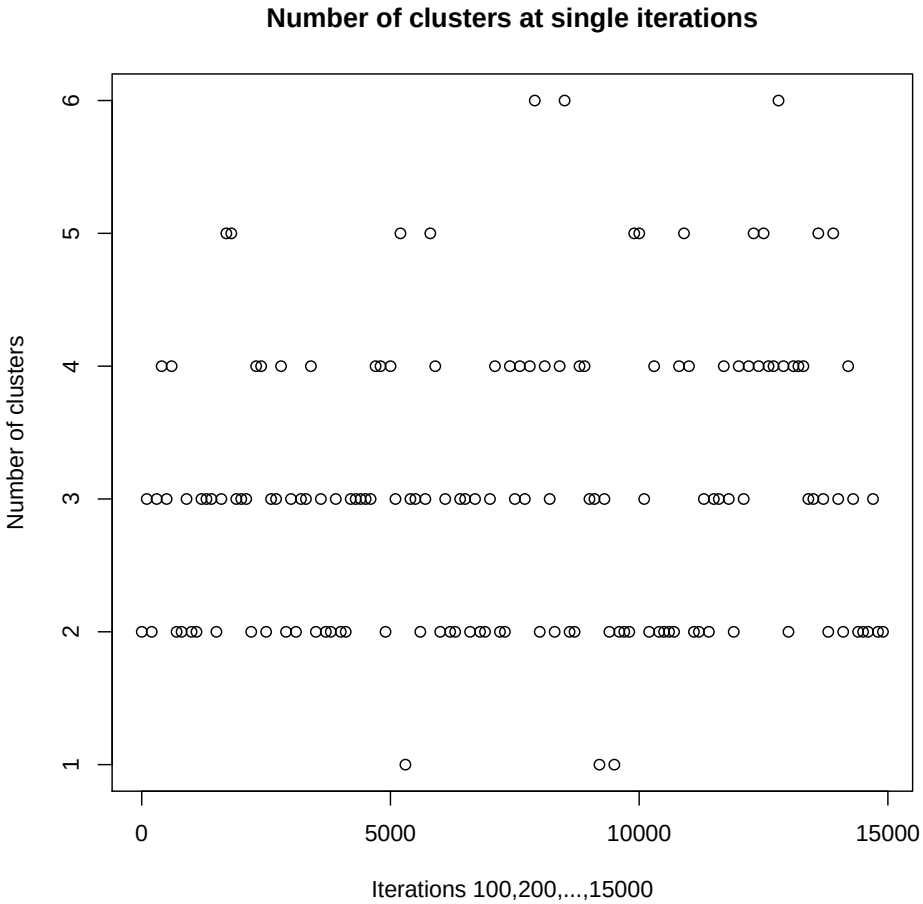
$$\Longrightarrow \hat{f}(x) = \frac{1}{K} \sum_k \hat{f}^{(k)}(x)$$

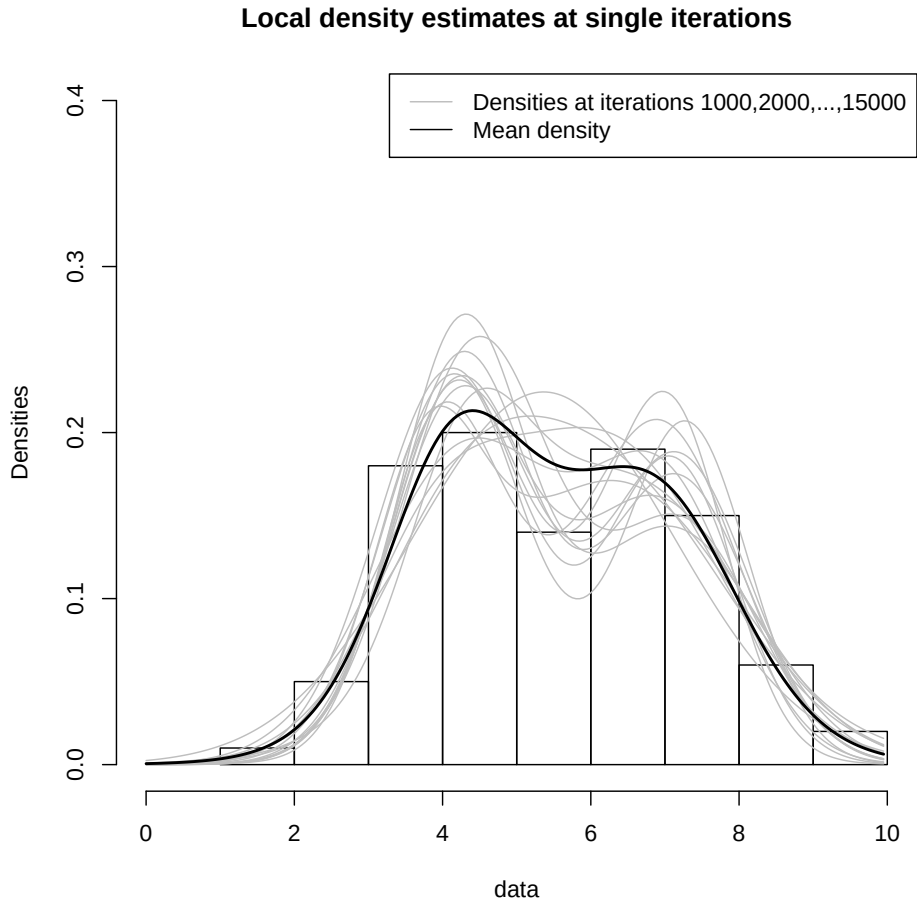
Results

Auxiliary parameters

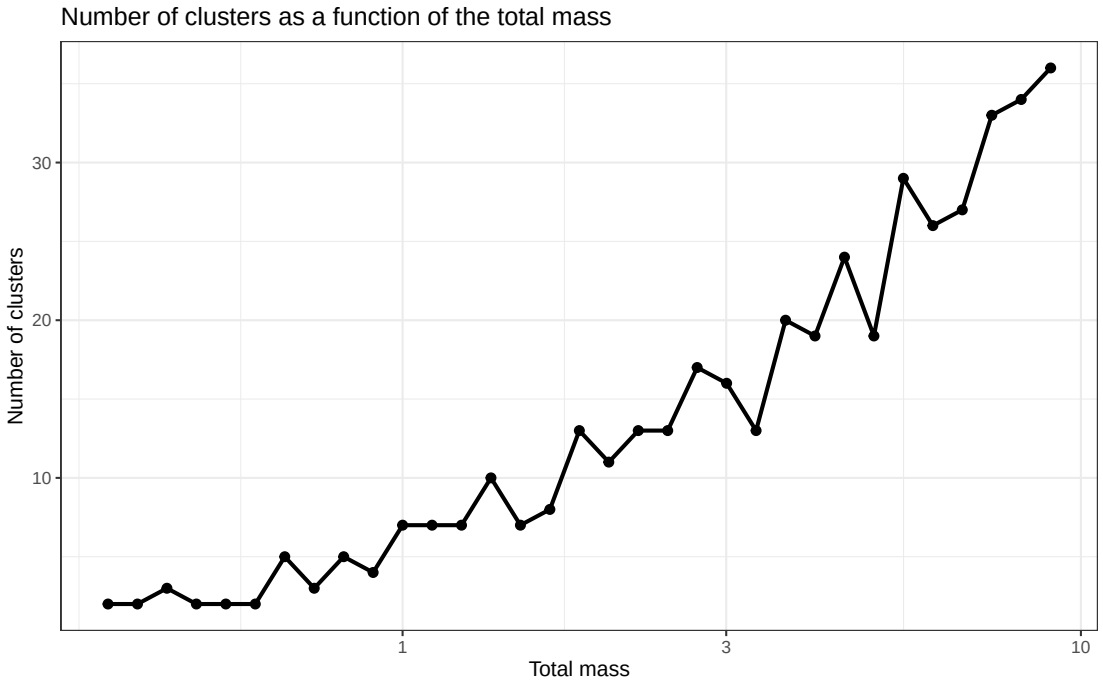


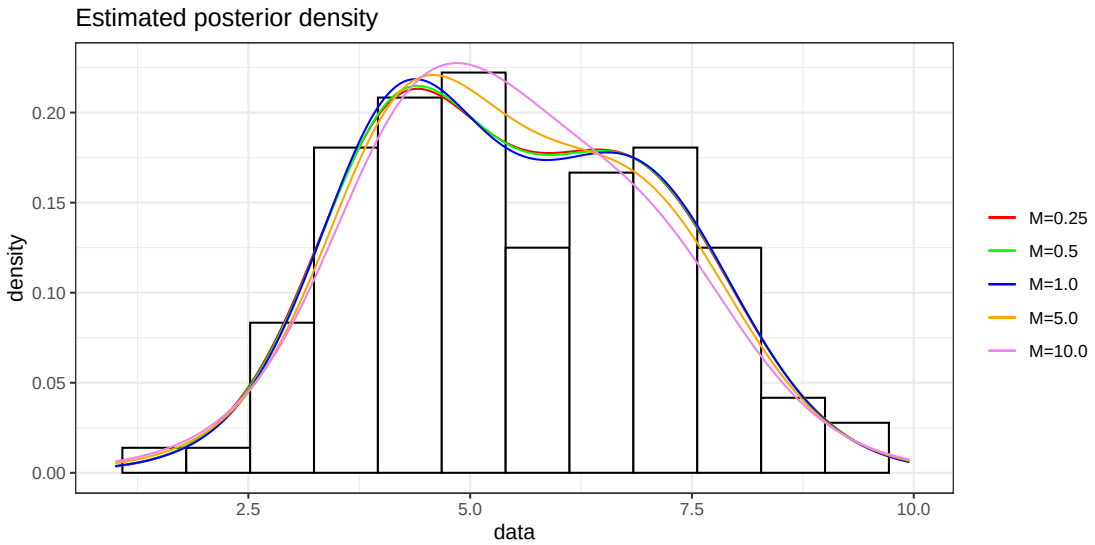
Oscillations



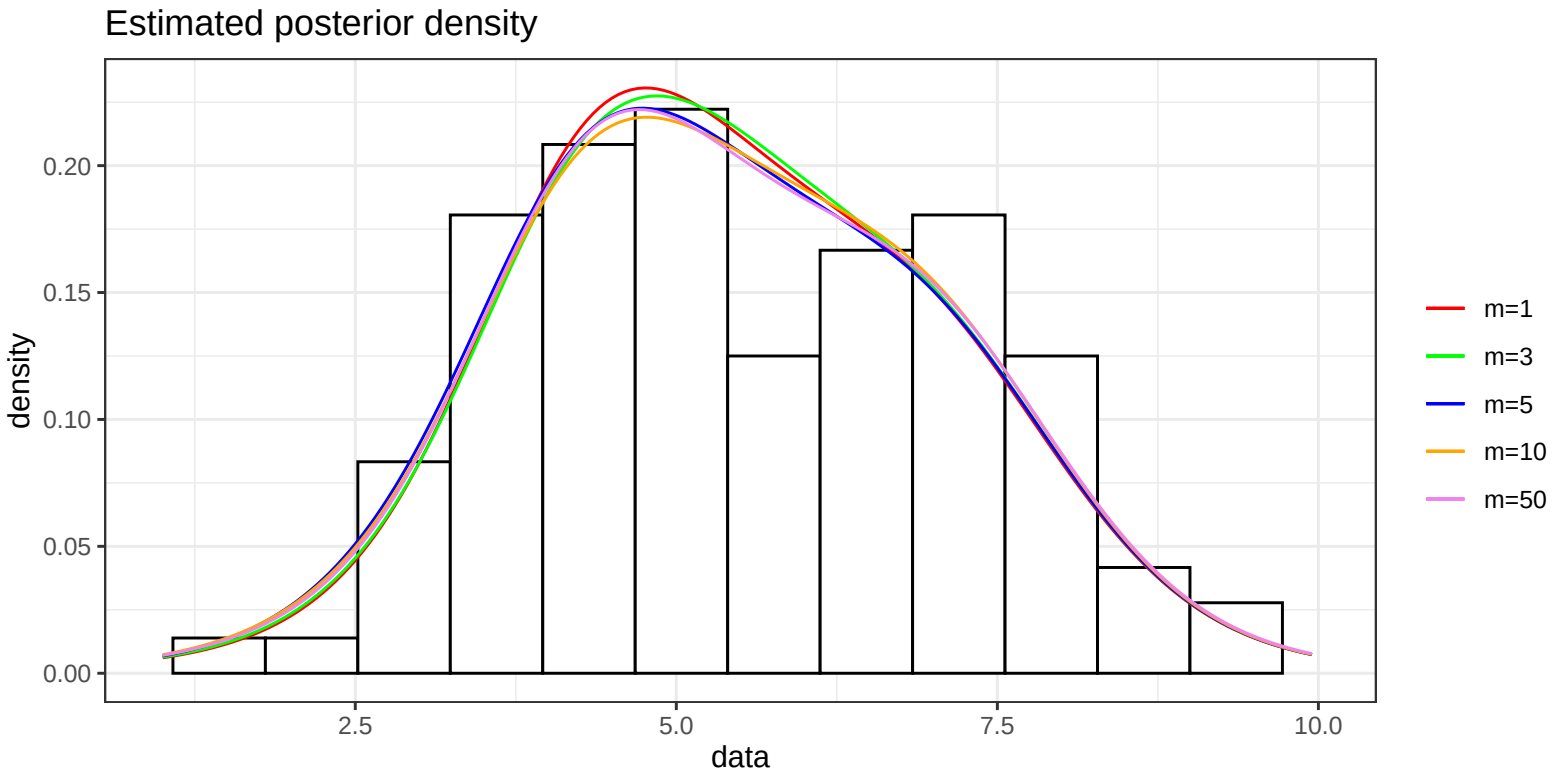


Total mass

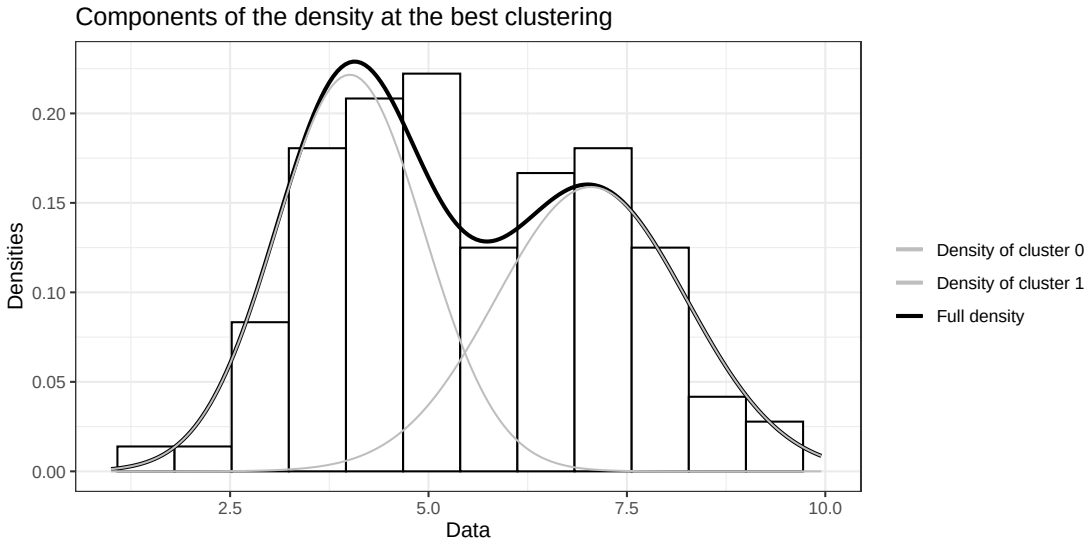

Number of clusters as a function of the total mass
Number of clusters
30
20
10
1
3
10
Total mass

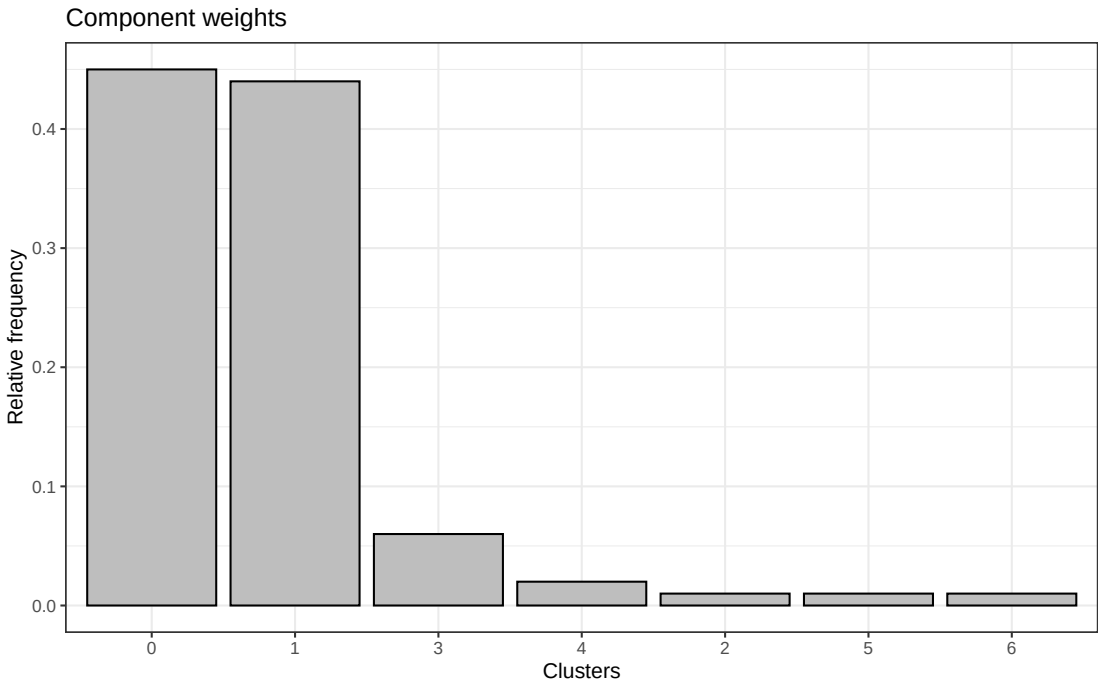

Estimated posterior density
density
0.20
0.15
0.10
0.05
0.00
2.5
5.0
7.5
10.0
data
M=0.25
M=0.5
M=1.0
M=5.0
M=10.0

Auxiliary parameters

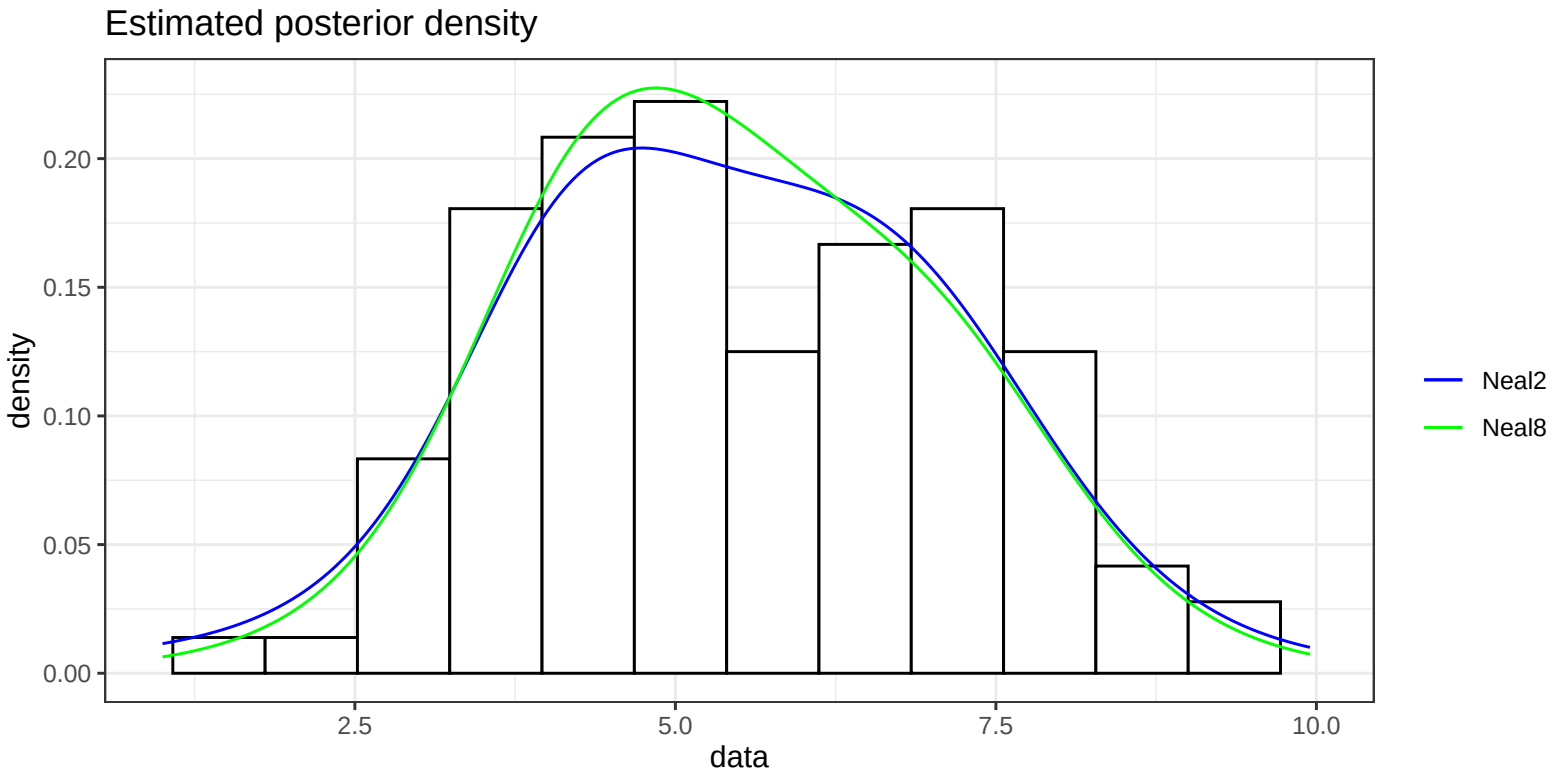


Density components





Neal2 vs Neal8



Bibliography

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- Ishwaran, James (2001), *Gibbs Sampling Methods for Stick-Breaking Priors*
- Murphy (2007), *Conjugate Bayesian analysis of the Gaussian distribution*
- Protocol Buffers: `https://developers.google.com/protocol-buffers/docs/cpptutorial`
- Stan: `http://mc-stan.org/math`
- Eigen: `https://eigen.tuxfamily.org/dox`
- GitHub codes of Mario Beraha and Riccardo Corradin for similar projects
- Course material for Bayesian Statistics: `https://beep.metid.polimi.it/web/2019-20-bayesian-statistics-alessandra-guglielmi-/`