The paper is structured as follows. Section 2 discusses existence and local stability of fixed points. In Section 3, local bifurcations analysis of fixed points are studied. Section 4 demonstrates the existence of a snap-back repeller in the sense of Marotto. Section 5 illustrates some numerical simulations for the complex dynamics of the map. In Section 6, we present the encryption method based on Marotto's map and spatiotemporal chaos and compound chaos. Finally, we conclude in Section 7.

2. Existence and local stability of fixed points

System (1.1) has at most two fixed points:

1. For all parameters values there exists one fixed point, namely $fix_1 = (0, 0)$,
2. For $a + b \neq 1$, there exists an interior fixed point $fix_2 = \left(\frac{a+b-1}{(a+b)^2}, \frac{a+b-1}{(a+b)^2}\right)$.

Lemma 1 ([58]). *Let $F(\lambda) = \lambda^2 + P\lambda + Q$. Suppose that $F(1) > 0$, and $F(\lambda) = 0$ has two roots λ_1 and λ_2. Then*

1. *$F(-1) > 0$ and $Q < 1$ if and only if $|\lambda_1| < 1$ and $|\lambda_2| < 1$;*
2. *$F(-1) < 0$ if and only if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$);*
3. *$F(-1) > 0$ and $Q > 1$ if and only if $|\lambda_1| > 1$ and $|\lambda_2| > 1$;*
4. *$F(-1) = 0$ and $P \neq 0, 2$ if and only if $\lambda_1 = -1$ and $|\lambda_2| \neq 1$;*
5. *$P^2 - 4Q < 0$ and $Q = 1$ if and only if λ_1 and λ_2 are complex and $|\lambda_{1,2}| = 1$.*

The Jacobian matrix calculated at (x^*, y^*) reads

$$J(x^*, y^*) = \begin{pmatrix} a - 2a^2x^* - 2aby^* & -2abx^* + b - 2b^2y^* \\ 1 & 0 \end{pmatrix}.$$

3. Local bifurcations analysis

In this section, a detailed bifurcation analysis is being performed at the fixed points of system (1.1) the two blue crosses at (0, 0) and (0.1875, 0.1875).

3.1. *Bifurcation of $fix_1(0, 0)$*

The Jacobian matrix at $fix_1(0, 0)$ reads

$$J(fix_1) = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}.$$

First we discuss the occurrence of a transcritical bifurcation at $fix_1(0, 0)$ in the following lemma.

Lemma 2. *If $a = 1 - b$, and $b \neq -1, 1$, then system (1.1) admits a transcritical bifurcation at $fix_1(0, 0)$.*

Proof. Let $a = 1 - b$, the two eigenvalues associated to the Jacobian matrix evaluated at $fix_1(0, 0)$ become $\lambda_1 = 1$ and $\lambda_2 = -b$. Let $\mu = a - 1 + b$ be a new and a dependent variable, the system (1.1) is transformed into the following form

$$\begin{pmatrix} x \\ y \\ \mu \end{pmatrix} \to \begin{pmatrix} -b+1 & b & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ \mu \end{pmatrix} + \begin{pmatrix} \mu x - (\mu - b + 1)^2 x^2 - 2b(\mu - b + 1)xy - b^2 y^2 \\ 0 \\ 0 \end{pmatrix}. \tag{3.1}$$

Constructing an invertible matrix

$$T = \begin{pmatrix} 1 & 0 & -b \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

transcritical bifurcation at $fix_1(0,0)$. □

Now we discuss the possibility of the occurrence of flip bifurcation of $fix(0,0)$. The Jacobian matrix $J(0,0)$ of system (1.1) has two eigenvalues $\lambda_{1,2} = \frac{a \pm \sqrt{a^2+4b}}{2}$. If $a = b-1$, then we have $\lambda_1 = -1$, $\lambda_2 = b$ with $|\lambda_2| \neq 1$ provided that $b \neq -1, 1$.

Lemma 3. *If $a = b-1$, and $b \neq -1, 1, \frac{3\pm\sqrt{5}}{2}$, then system (1.1) admits a flip bifurcation at $fix_1(0,0)$. In addition, the stable periodic-2 orbit bifurcates from this fixed point.*

Proof. Let $a = b-1$, the two eigenvalues associated to the Jacobian matrix evaluated at $fix_1(0,0)$ become $\lambda_1 = -1$ and $\lambda_2 = b$. Let $\mu = a - b + 1$ be a new and a dependent variable, the system (1.1) is transformed into the following form

$$\begin{pmatrix} x \\ y \\ \mu \end{pmatrix} \to \begin{pmatrix} b-1 & b & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ \mu \end{pmatrix} + \begin{pmatrix} \mu x - (\mu + b - 1)x^2 - 2b(\mu + b - 1)xy - b^2y^2 \\ 0 \\ 0 \end{pmatrix}. \tag{3.6}$$

Constructing an invertible matrix

$$T = \begin{pmatrix} -1 & 0 & b \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

Thus, system (1.1) undergoes a subcritical flip bifurcation at $fix_1(0,0)$. □

3.2. *Bifurcation of fix_2*

The Jacobian matrix at $fix_2 = \left(\frac{a+b-1}{(a+b)^2}, \frac{a+b-1}{(a+b)^2}\right)$, reads

$$J(fix_1) = \begin{pmatrix} a - 2a(ax^* + by^*) & b - 2b(ax^* + by^*) \\ 1 & 0 \end{pmatrix},$$

with $(x^*, y^*) = \left(\frac{a+b-1}{(a+b)^2}, \frac{a+b-1}{(a+b)^2}\right)$. The characteristic equation

$$F(\lambda) = \lambda^2 + B\lambda + A = 0, \tag{3.11}$$

where $B = \frac{a(a+b-2)}{a+b}$ and $A = \frac{b(a+b-2)}{a+b}$, has two eigenvalues $\lambda_{1,2} = \frac{-B \pm \sqrt{b^2 - 4A}}{2}$.

In this section, the occurrence of both flip and Neimark–Sacker bifurcations in system (1.1) is investigated at the interior fixed point fix_2 where a is taken as the bifurcation parameter. First of all, the occurrence of flip bifurcation of (1.1) is discussed.

Let

where $X_n = (x_n \;\; y_n)^T$. The eigenvalues corresponding to the fixed point fix_2 are given by

$$\lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - 4A}}{2},$$

where

$$B = \frac{b(a+b-2)}{a+b},$$

$$A = \frac{a(a+b-2)}{a+b}.$$

Let $\lambda_{1,2}$ be a pair of complex eigenvalues with $|\lambda_{1,2}| > 1$, that is

$$\begin{cases} B^2 - 4A < 0, \\ A > 1. \end{cases}$$

Let

$$S_1(x, y) = a^2(1 + 2a(ax + by))^2 + 4b(1 - 2(ax + by)),$$

if $y > 0$, then for

$$a^2(1 + 2a(ax + by))^2 + 4b(1 - 2(ax + by)) < 0,$$

we have

$$-O^* < x < O^*,$$

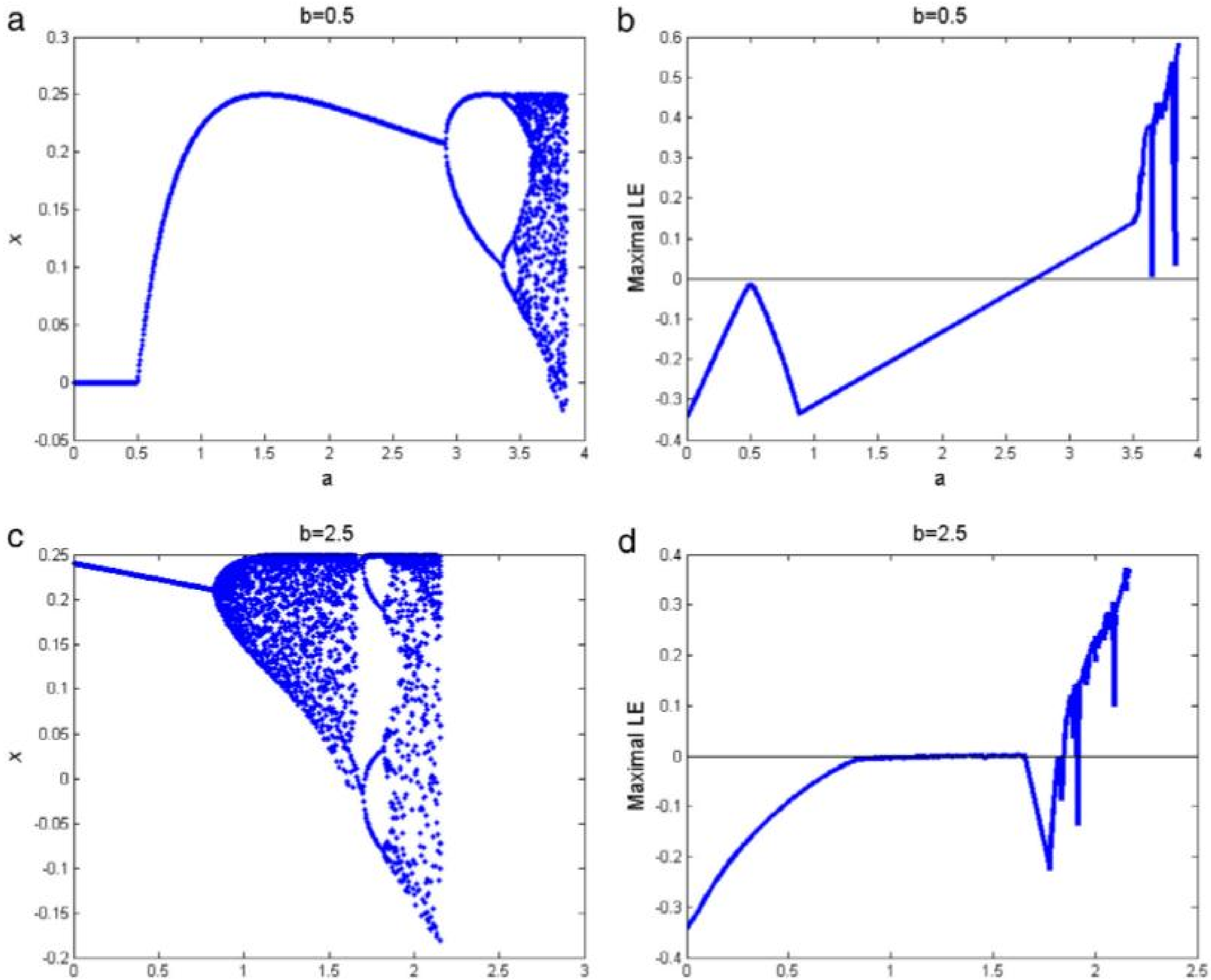
where

$$O^* = \sqrt{\frac{8ba^4(2by - 1) - 2a^6(2b^2y^2 + 1) + (4ab - 2a^3by)^2}{4a^8}}.$$

5. Numerical simulations

All the previous analytical findings are substantiated with the help of numerical simulations performed via Matlab. In all numerical simulations we choose a as the bifurcation parameter and the initial condition is taken as $(x_0, y_0) = (0.1, 0.1)$.

1. Fix $b = 0.5$, then a transcritical bifurcation of the fixed point $fix_1(0, 0)$ occurs at $a = 0.5$ as can be seen in Fig. 2(a) which agrees with Lemma 3. The corresponding maximal Lyapunov exponent is shown in Fig. 2(b). Now fix $b = 2.5$, then a flip bifurcation of the same fixed point occurs at $a = 1.5$ which agrees with Lemma 4 as depicted in Fig. 2(c) and its corresponding maximal Lyapunov exponent is shown in Fig. 2(d). If we choose $b = 2$, then the interior fixed point will be $fix_2 = (0.204, 0.204)$ and a flip bifurcation will occur at it when $a = 1.5$ as illustrated in Fig. 3(a). The corresponding maximal Lyapunov exponent is shown in Fig. 3(b). Next, if we take $b = 3$, then according to the analysis of the Neimark–Sacker bifurcation, $a_c = 0$ and according to these parameters we obtain $fix_2 = (0.2222, 0.2222)$, a Neimark–Sacker bifurcation occurs at it and this can be clearly seen in Fig. 3(c). The corresponding maximal Lyapunov exponent is shown in Fig. 3(d).
2. Choose $a = 0.67$ and $b = 2.5$. In this case we have $fix_2 = (0.2159, 0.2159)$ which is stable as depicted in Fig. 4(a) according to Proposition 2. Now we vary a and fix $b = 0.5$. The fixed point fix_2 loses its stability at $a = 1.78$ as seen in Fig. 4(b), on account of the norm of complex eigenvalues of its corresponding Jacobian matrix equal to 1. Consequently,


a
b=0.5
x
a
b
b=0.5
Maximal LE
a
c
b=2.5
x
d
b=2.5
Maximal LE

a closed invariant cycle appears at $a = 1.2$ as shown in Fig. 4(c). The breakdown of this cycle is depicted in Fig. 4(d) when $a = 1.35$. Strange attractors are illustrated in Fig. 4(e), (f), (g), (h), for different a and b.

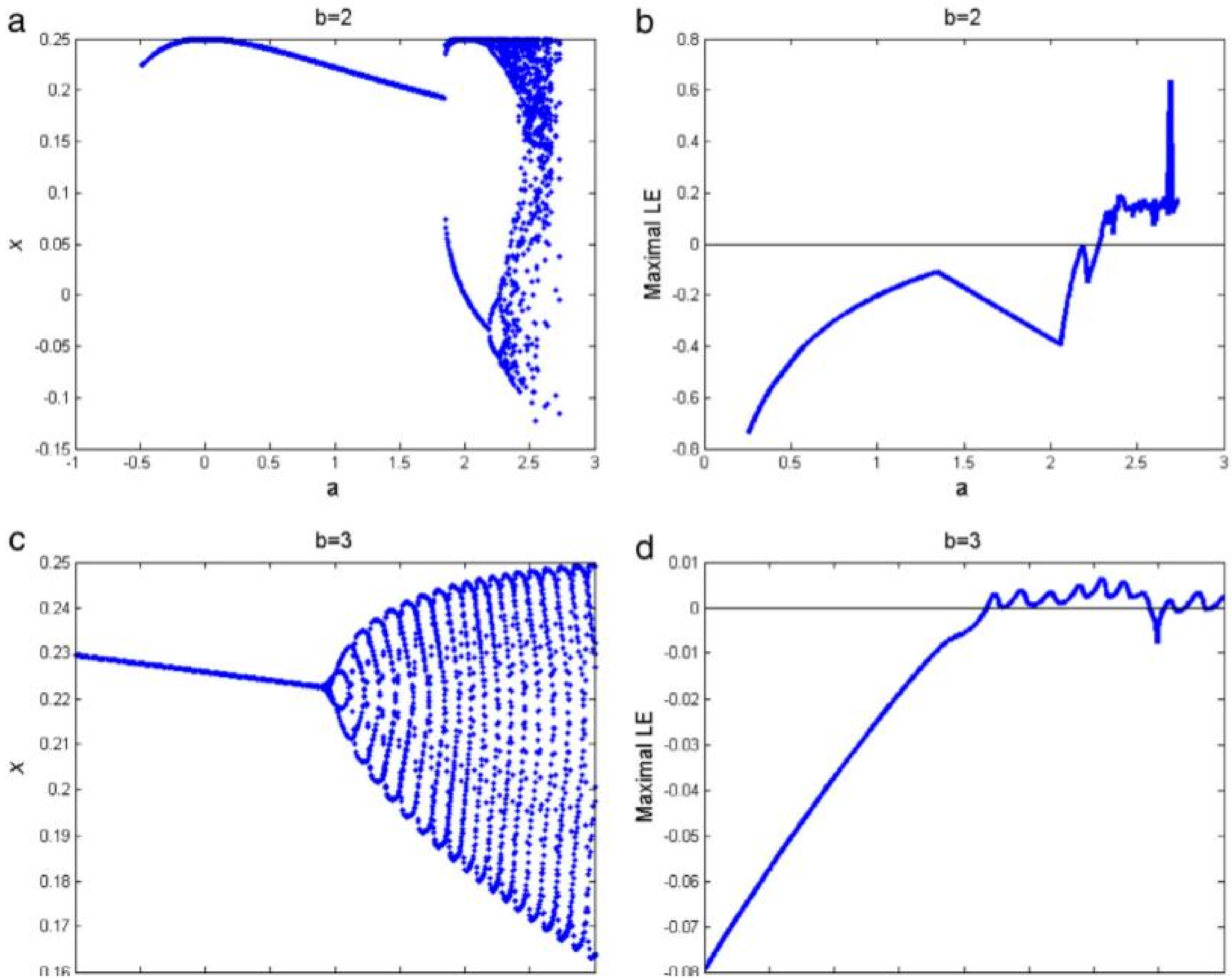
3. For the snap-back repeller we consider the following numerical example:
   Take $a = 0.4$ and $b = 3.5$, then map (1.1) has a fixed point $z^* = (x^*, y^*) = (0.7436, 0.7436)$. The eigenvalues of the Jacobian matrix associated to z^* are $\lambda_{1,2} = -0.8527 \pm 0.7295i$ with $|\lambda_{1,2}| > 1$. In this case and by Theorem 3 we have $D_1 = \{x : x > -4.89935$ and $y > 0.09869\}$ and $D_2 = \{-0.0708 < x < 0.0708\}$. Now we can find a point $(x_1, y_1) = (0.23011, 0.1568)$ which satisfies $F^2(x_1, y_1) = (x^*, y^*)$ and $|DF^2(x_1, y_1)| \neq 0$ therefore (x^*, y^*) is a snap-back repeller.

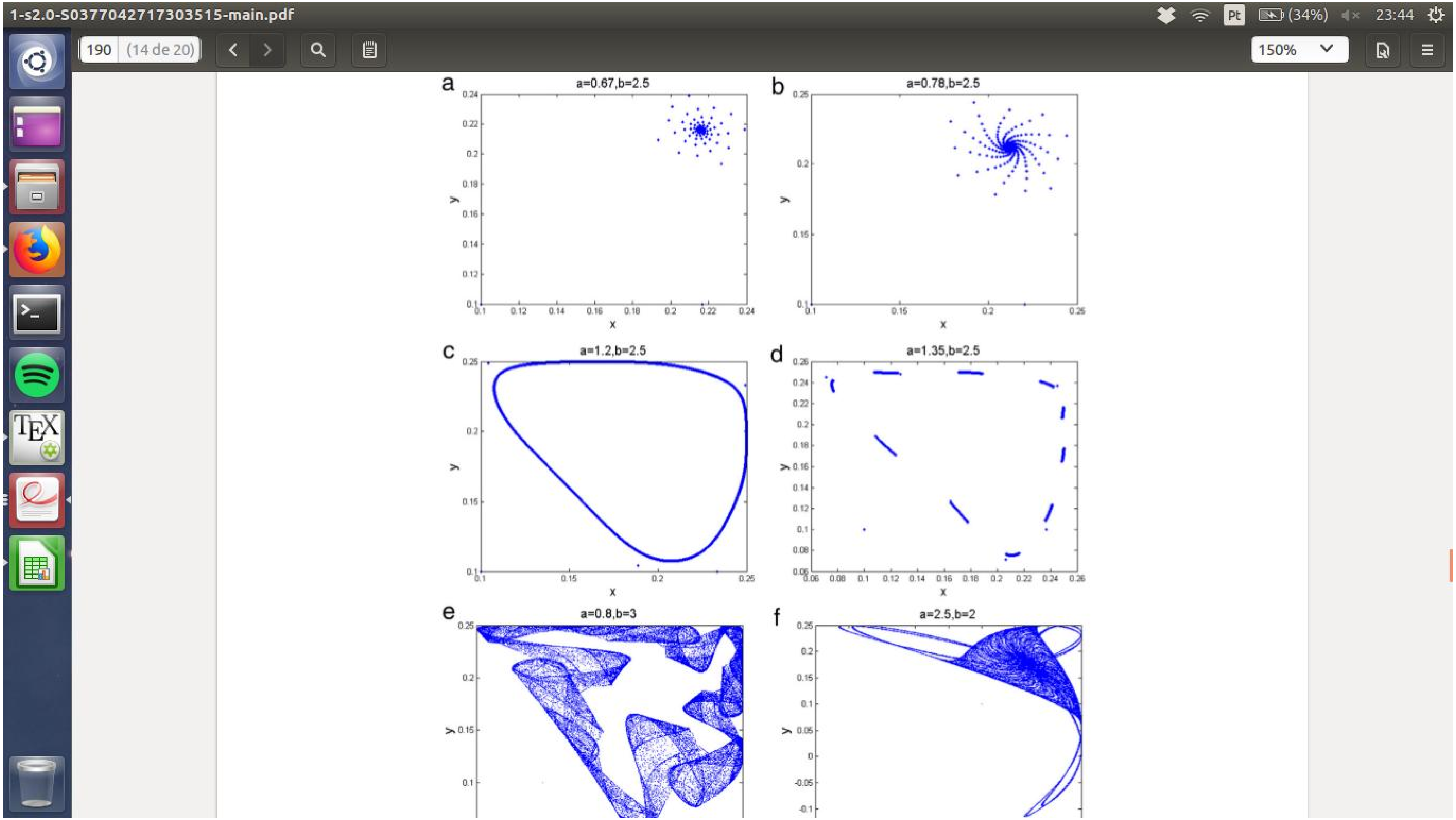
6. Encryption process

In this part of the paper, we ply the Marotto's map for image encryption. A little modification to the encryption scheme presented in [57] will be made which is summed up in replacing the Hénon map with the Marotto's map (1.1) with $a = 0.8$ and $b = 3$. Now the map (1.1) has the following form

$$\begin{aligned} x_{n+1} &= (0.8x_n + 3y_n)(1 - 0.8x_n - 3y_n), \\ y_{n+1} &= x_n. \end{aligned} \tag{6.1}$$

The encryption process is made up of two main steps. In the first step, the Marotto's map is used to implement the positions of pixel permutation. In the second step, a combination of a spatiotemporal chaos and compound chaos is used as value shuffling. The spatiotemporal chaotic map adopted here in the one-way coupled map lattice (OCML) given in the


a
b=2
x
a
b
b=2
Maximal LE
a
c
b=3
x
d
b=3
Maximal LE


1-s2.0-S0377042717303515-main.pdf
(34%)
23:44
190
(14 de 20)
150%
a
a=0.67,b=2.5
b
a=0.78,b=2.5
c
a=1.2,b=2.5
d
a=1.35,b=2.5
e
a=0.8,b=3
f
a=2.5,b=2
x
y

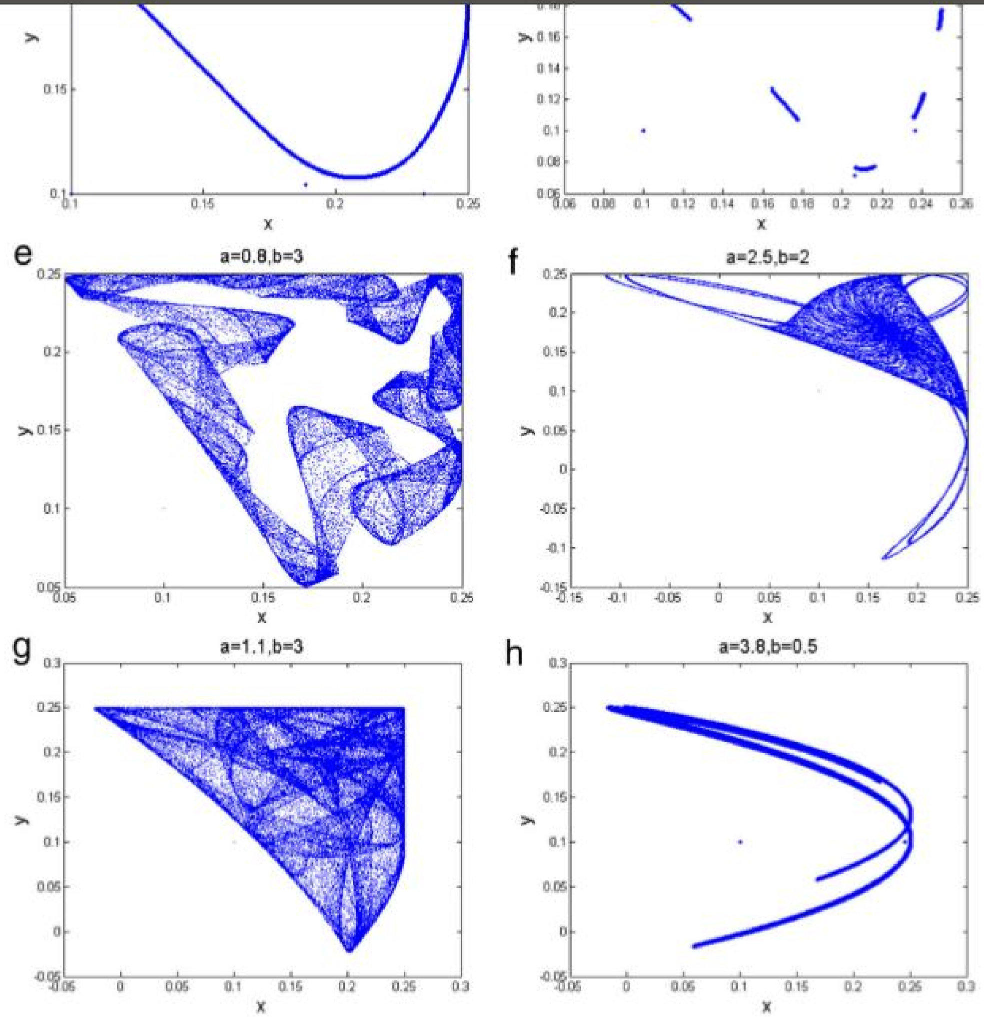


Fig. 4. Phase portraits of map (1.1) for different a and b.