Optimization approach to Tverberg type theorems

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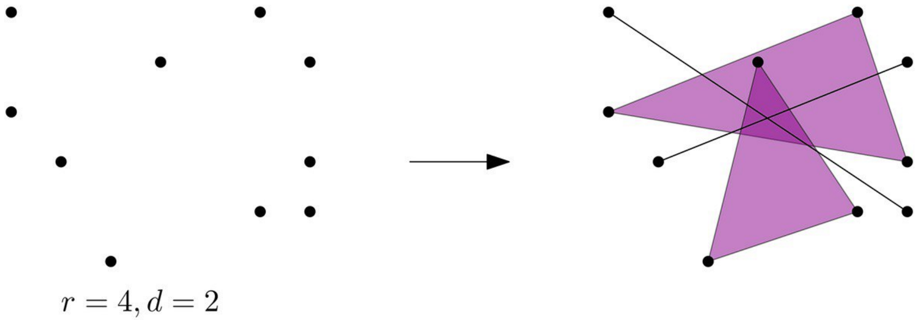
Tverberg Theorem

Tverberg Theorem (1966)

Given $(r-1)(d+1)+1$ points in $\mathbb{R}^d$, there is a partition of them into r parts whose convex hulls intersect.



Tverberg Theorem: Roudneff approach

Sketch. For an r-partition $\mathcal{P}$ of a set of $(r-1)(d+1)+1$ points, consider the function $H_{\mathcal{P}}: \mathbb{R}^d \to \mathbb{R}$ defined by

$$H_{\mathcal{P}}(x) = \sum_{Y \in \mathcal{P}} \operatorname{dist}^2(x, \operatorname{conv} Y),$$

where $\operatorname{dist}(A, B)$ is the distance between sets $A, B \subset \mathbb{R}^d$.

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Since $H_{\mathcal{P}}$ is convex, it attains its minimum.
Choose a partition $\mathcal{P}_0$ for which this minimum is the smallest possible; let it be m.
1. If $m = 0$, then we are done.

Tverberg Theorem: Roudneff approach

2. Suppose that m is positive and attained at $x = x_{\mathcal{P}_0}$.
Then, analyzing the arrangement of the point $x_{\mathcal{P}_0}$ and the convex hulls of $Y \in \mathcal{P}_0$, we find a partition $\mathcal{P}_0'$ such that

$$H_{\mathcal{P}_0'}(x_{\mathcal{P}_0}) < H_{\mathcal{P}_0}(x_{\mathcal{P}_0}) = m,$$

a contradiction.

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a contradiction.
Moreover, the partition $\mathcal{P}_0'$ is obtained from $\mathcal{P}_0$ by moving one element between two sets.

Optimization approach

Outline.

1. Choose a function $H_{\mathcal{P}} : \mathbb{R}^d \to \mathbb{R}$

2. Minimize it over $\mathbb{R}^d$

3. Find an optimal partition $\mathcal{P}_0$

4. Suppose that $\mathcal{P}_0$ does not satisfy the statement of the theorem

5. Find a new partition $\mathcal{P}_0'$ such that $\min H_{\mathcal{P}_0'}$ is less than $\min H_{\mathcal{P}_0}$

Tverberg matching

For points $x, y \in \mathbb{R}^d$, denote by $D(xy)$ the closed ball with diameter xy. A perfect matching $\mathcal{M}$ for an even set of points in $\mathbb{R}^d$ is called a **Tverberg matching** if

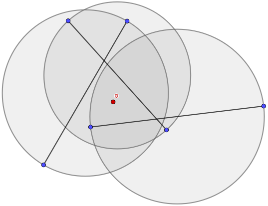
$$\bigcap_{xy \in \mathcal{M}} D(xy) \neq \varnothing.$$

Replacing **closed** balls by **open** balls in the definition of a Tverberg matching, we define an **open Tverberg matching**.

Tverberg type theorems

Theorem (Pirahmad, Polyanskii, Vasilevskii, 2021+).

For any even set of distinct points in $\mathbb{R}^d$, there exists an open Tverberg matching.



Tverberg type theorems

Theorem (Pirahmad, Polyanskii, Vasilevskii, 2021+).

For n red points and n blue points in $\mathbb{R}^d$, there exists a red-blue Tverberg matching (every edge of this matching connects a red vertex with a blue one).

Tverberg type theorems

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For n red points and n blue points in $\mathbb{R}^d$, there exists a red-blue Tverberg matching (every edge of this matching connects a red vertex with a blue one).

Moreover, this matching maximizes the function $Q(\mathcal{M})$ defined by

$$Q(\mathcal{M}) = \sum_{rb \in \mathcal{M}} (r - b)^2.$$

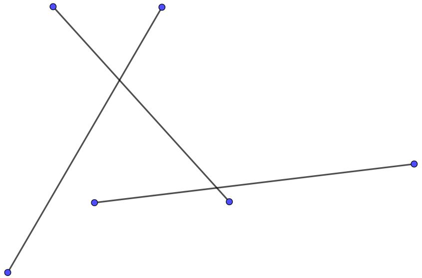
Generalized optimization approach

Outline.

1. Choose a function $H_{\mathcal{P}} : \mathbb{R}^d \to \mathbb{R}$
2. Minimize it over $\mathbb{R}^d$
3. Find an optimal partition $\mathcal{P}_0$ with respect to some function Q
4. Suppose that $\mathcal{P}_0$ does not satisfy the statement of the theorem
5. Find a new partition $\mathcal{P}_0'$ such that $Q(\mathcal{P}_0')$ is more extreme than $Q(\mathcal{P}_0)$

Maximum-sum matching

A partition $\mathcal{M}$ of an even point set into pairs is called **maximum-sum matching** if it maximizes the total Euclidean distance of the matched pairs $\sum_{ab \in \mathcal{M}} \|a - b\| \to max$.

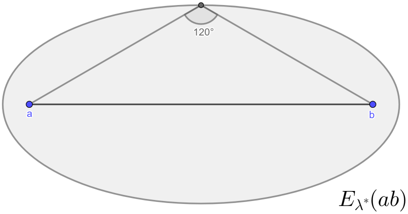


Ellipse

For points $a, b \in \mathbb{R}^2$ and a coefficient λ, denote by $E_\lambda(ab)$ the ellipse with foci a and b

$$E_\lambda(ab) := \{x \in \mathbb{R}^2 : \|a - x\| + \|b - x\| \leq \lambda \|a - b\|\}.$$

Put $\lambda^* := \frac{2}{\sqrt{3}} \approx 1.1547$.



Fingerhut Conjecture

Fingerhut Conjecture (1995)

For any maximum-sum matching $\mathcal{M}$ of an even point set in the plane, the intersection of the corresponding ellipses with coefficient λ^* is not empty, that is,

$$\bigcap_{ab \in \mathcal{M}} E_{\lambda^*}(ab) \neq \varnothing.$$

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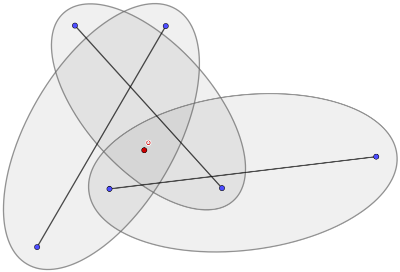
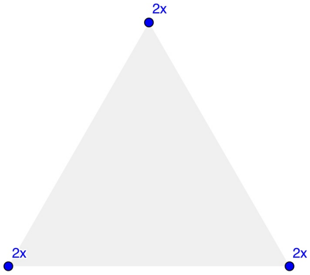


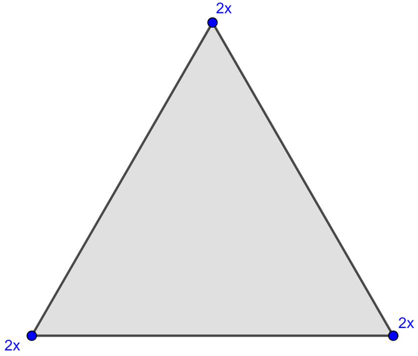
Fingerhut Conjecture: Lower bound

Consider an equilateral triangle, where at each vertex two points are located.



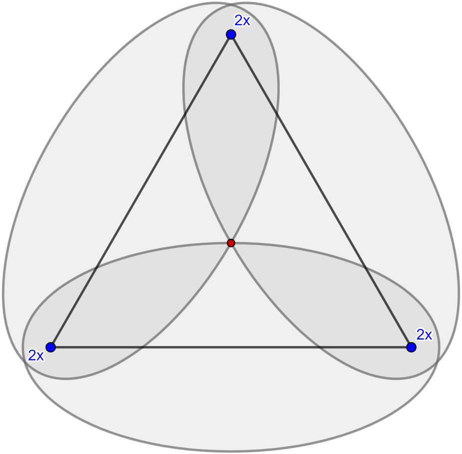
Fingerhut Conjecture: Lower bound

The maximum-sum matching of the six points is made of 3 edges.

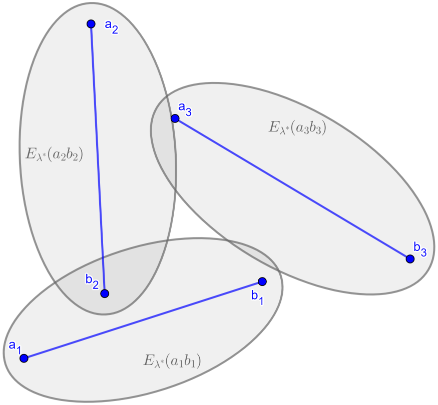


Fingerhut Conjecture: Lower bound

The corresponding 3 ellipses have exactly one point in common.



Fingerhut Conjecture: Idea of the proof



1. By Helly's theorem it is sufficient to prove conjecture for $n = 2, 3$

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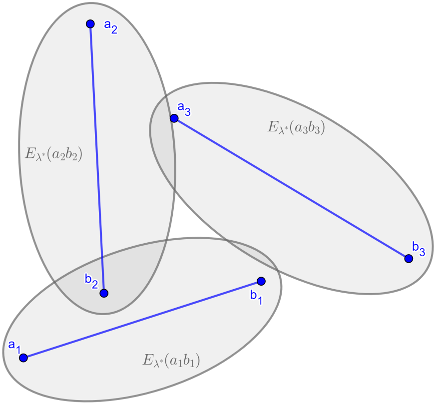

2. Suppose to the contrary that for an even set of points $S \subset \mathbb{R}^2$ and a maximum-sum matching $\mathcal{M}$ of S the corresponding ellipses do not intersect

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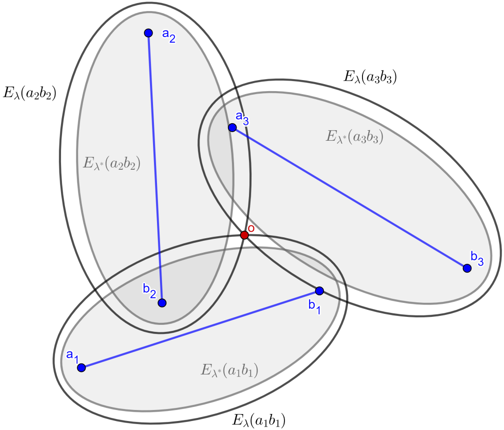

$$g_{ab}(x) := \frac{\|a-x\| + \|b-x\|}{\|a-b\|} \qquad H(x) := \max_{ab\in\mathcal{M}} g_{ab}(x) \to \min_{x\in\mathbb{R}^2}$$

$$o := \operatorname{argmin} H(x)$$

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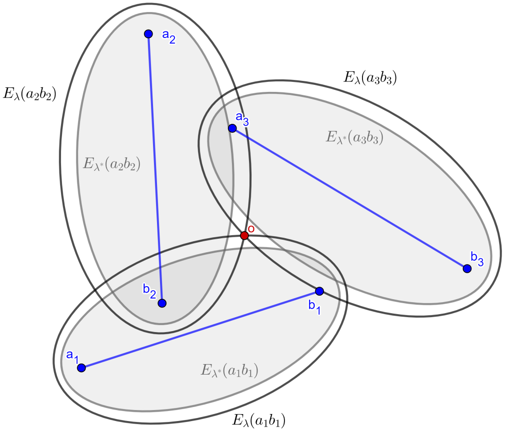

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$$o := \operatorname{argmin} H(x) \qquad \lambda := H(o) > \lambda^*$$

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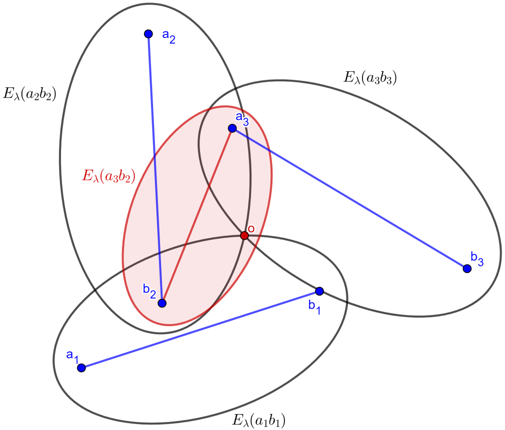
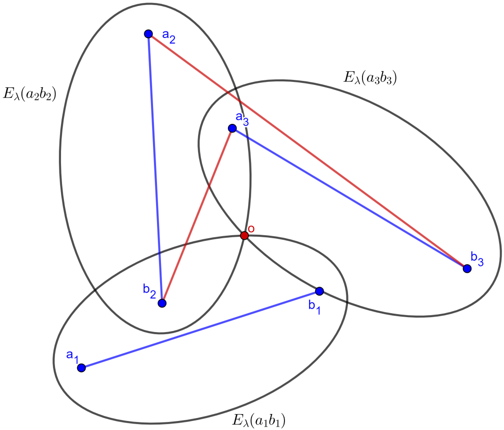


$$E_b = \mathcal{M} \qquad\qquad E_r = \{vu : \|v\| + \|u\| < \lambda\|v - u\|\}$$

Fingerhut Conjecture: Alternating cycle



$$\|a_2 - b_2\| = \frac{1}{\lambda}(\|a_2\| + \|b_2\|) \qquad \|a_2 - b_3\| > \frac{1}{\lambda}(\|a_2\| + \|b_3\|)$$

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Tverberg matching in $\mathbb{R}^2$

Theorem (Bereg, Chacón-Rivera, Flores-Peñaloza, Huemer, Pérez-Lantero, Seara, 2019)

For any even set S of distinct points in $\mathbb{R}^2$, any maximum-sum matching of S is a Tverberg matching.

Theorem

For any even set S of distinct points in $\mathbb{R}^2$, any maximum-sum matching of S is an open Tverberg matching.