MAXIMUM SPANNING TREE IS A TVERBERG GRAPH

ABSTRACT. For a finite set S of points in $\mathbb{R}^d$ consider a maximum-weight spanning tree T of S. Abu-Affash, Carmi and Maman [1] proved that for $d = 2$ the intersection of the discs whose diameters are edges of the tree T is not empty. We show how to obtain the same result for balls in higher dimensions with the help of optimization approach.

Let S be a set of points in the plane and let $K = (S, E)$ be a complete graph over S. A *maximum-weight spanning tree* T of S is a spanning tree of K with maximum edge weight, where the weight of an edge is the Euclidean distance between its endpoints.

For a pair of points (p, q) in $\mathbb{R}^d$, let $B(pq)$ be the ball having the segment pq as its diameter.

Theorem 1. *For any finite set $S \subset \mathbb{R}^d$ and a maximum-weight spanning tree T of S*

$$\bigcap_{pq \in E(T)} B(pq) \neq \varnothing. \tag{1}$$

Proof. Let $S \subset \mathbb{R}^d$ be a finite set and let T be a maximum-weight spanning tree of S. Consider the function $H : \mathbb{R}^d \to \mathbb{R}$ defined by

$$H(x) = \max_{p \in S} \|p - x\|.$$

This function attains its global minimum at a unique point, that is the center of the smallest enclosing sphere of S. Without loss of generality, we may assume that this point coincides with the origin o and $H(o) = 1$. Our goal is to show that for every $pq \in E(T)$ the origin o belongs to $B(pq)$.

Consider a subset

$$S_0 = \{p \in S : \|p\| = 1\}.$$

Note that every point $p \in S \setminus S_0$ lies strictly inside the unit sphere, that is $\|p\| < 1$.

We claim that $o \in \operatorname{conv} S_0$. Suppose to the contrary that $o \notin \operatorname{conv} S_0$. By the Separation Theorem, there exists a non-zero vector x such that $\langle x, y \rangle > 0$ for any $y \in \operatorname{conv} S_0$. Hence, for sufficiently small $\varepsilon > 0$, the point εx is closer than o to all points of S_0. Then, for some positive ε, we have $H(\varepsilon x) < H(o)$. Contradiction.

Thus, there are non-negative coefficients $\lambda_p \geq 0$ for $p \in S_0$ such that

$$\sum_{p \in S_0} \lambda_p p = 0 \text{ and } \sum_{p \in S_0} \lambda_p = 1.$$

Consider the graph G on the vertex set $V(G) := \{p \in S_0 : \lambda_p > 0\}$ with the edge set $E(G)$ defined as

$$E(G) := \{pq \in V(G) \times V(G) : \langle p, q \rangle \leq 0\}.$$

The graph G is connected. Indeed, for any $U \subset V(G)$

$$\left\langle \sum_{p \in U} \lambda_p p, \sum_{q \in V(G) \setminus U} \lambda_q q \right\rangle = \left\langle \sum_{p \in U} \lambda_p p, -\sum_{p \in U} \lambda_p p \right\rangle \leq 0.$$

Therefore, there is at least one edge between U and $V(G) \setminus U$.

Next we show that

$$o \in \bigcap_{pq \in E(T)} B(pq). \tag{2}$$

Suppose to the contrary that there is an edge $pq \in E(T)$ such that $o \notin B(pq)$. Then $\langle p, q\rangle > 0$. Since

$$\sum_{x \in V(G)} \lambda_x \langle x, p\rangle = \left\langle \sum_{x \in V(G)} \lambda_x x, p \right\rangle = 0,$$

there exists a vertex $x \in V(G)$ such that $\langle x, p\rangle \leq 0$. Analogously, there is a vertex $y \in V(G)$ such that $\langle y, q\rangle \leq 0$.

Consider a path $x = x_1, \cdots, x_k = y$ that connects x with y in G. To finish the proof we show that pq is the shortest edge in cycle $p, x_1, \cdots, x_k, q, p$. Indeed,

$$\|p - x_1\|^2 = \|p\|^2 + \|x_1\|^2 - 2\langle p, x_1\rangle > \|p\|^2 + 1 - 2\langle p, q\rangle \geq \|p\|^2 + \|q\|^2 - 2\langle p, q\rangle = \|p - q\|^2.$$

The same argument works for $\|x_k - q\|$ and $\|x_{i+1} - x_i\|$.

It is easy to see that the shortest edge of a cycle cannot be an edge of any maximum-weight spanning tree. Contradiction with the maximality of T. □

References

[1] A. K. Abu-Affash, P. Carmi, and M. Maman, *Piercing diametral disks induced by edges of maximum spanning tree*, arXiv, 2022.