Maximum Spanning tree is a Tverberg graph

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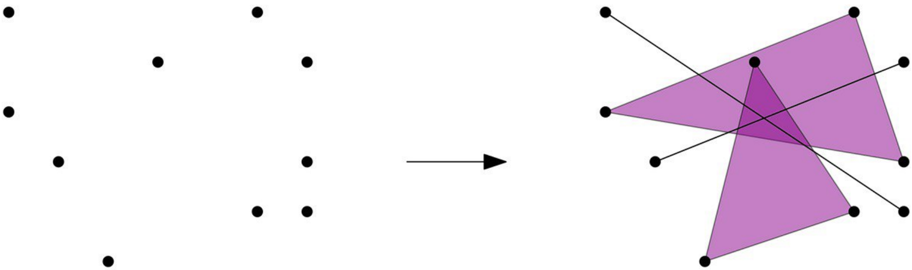
Tverberg Theorem

Tverberg Theorem (1966)

Given $(r-1)(d+1)+1$ points in $\mathbb{R}^d$, there is a partition of them into r parts whose convex hulls intersect.

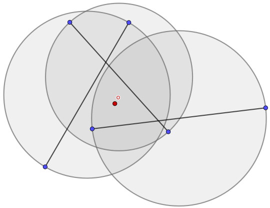


$r = 4, d = 2$

Tverberg graph

For points $x, y \in \mathbb{R}^d$, denote by $D(xy)$ the closed ball with diameter xy. A finite graph $\mathcal{G}$ whose vertices are points in $\mathbb{R}^d$ is called a **Tverberg graph** if

$$\bigcap_{xy \in \mathcal{E}(\mathcal{G})} D(xy) \neq \varnothing.$$



Examples of Tverberg graphs

A partition $\mathcal{M}$ of an even point set into pairs is called **maximum-sum matching** if it maximizes the total Euclidean distance of the matched pairs $\sum_{ab \in \mathcal{M}} \|a - b\| \to max$.

Theorem (Bereg, Chacón-Rivera, Flores-Peñaloza, Huemer, Pérez-Lantero, Seara, 2019)

For any even set S of distinct points in $\mathbb{R}^2$, any maximum-sum matching of S is a Tverberg graph.

Theorem (Pirahmad, Polyanskii, Vasilevskii, 2021+).

For any even set of distinct points in $\mathbb{R}^d$, there exists a Tverberg graph.

Maximum-weight spanning tree

Let S be a set of points in the plane and let $G = (S, E)$ be the complete graph over S. A **maximum-weight spanning tree** T of S is a spanning tree of G with maximum edge weight, where the weight of an edge is the Euclidean distance between its endpoints.

Piercing Diametral Disks Induced by Edges of Maximum Spanning Tree

Theorem (Karim Abu-Abbash, Carmi, Maman, 2022)

For any finite set $S \subset \mathbb{R}^2$ a maximum-weight spanning tree T of S is a Tverberg graph, that is

$$\bigcap_{xy \in E(T)} \mathcal{D}(xy) \neq \varnothing.$$

Moreover, let c be the center of the smallest enclosing circle of the points of S. Then, c lies in all of the given disks.

Results

Theorem

For any finite set $S \subset \mathbb{R}^{\mathbf{d}}$ a maximum-weight spanning tree T of S is a Tverberg graph, that is

$$\bigcap_{xy \in E(T)} \mathcal{D}(xy) \neq \varnothing.$$

Moreover, let c be the center of the smallest enclosing sphere of S. Then, c lies in all of the given balls.

Outline of the proof

1. Find a set of points that belong to the smallest enclosing sphere of S
2. Build an "obtuse" graph on this set and prove that it is connected
3. Choose any edge xy of T. Suppose that $c \notin D(xy)$
4. Use "obtuse" graph to find a path between x and y with all edges longer than xy
5. Contradiction with maximality of T