Helly type problems for d-intervals

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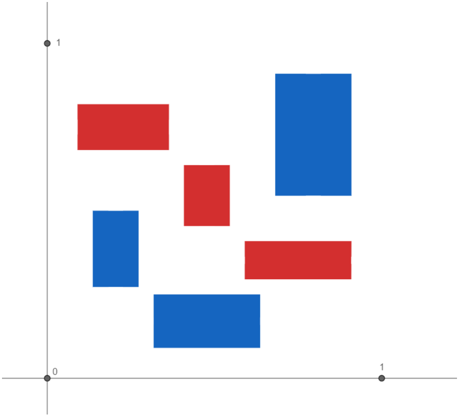
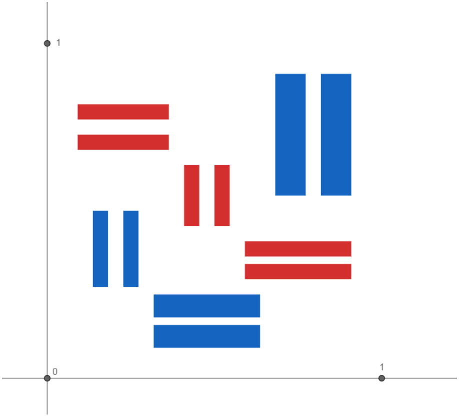
Illustrative example



Consider a family of red and blue axis-parallel rectangles. Assume that for every colorful pair of rectangles there is either horizontal or vertical line that intersects both of them. Then, there are two horizontal and one vertical line that together intersect all red or all blue rectangles.

Illustrative example



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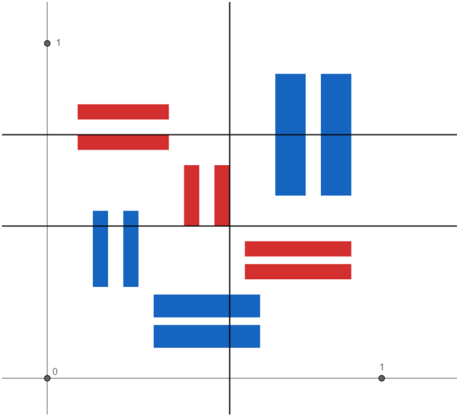

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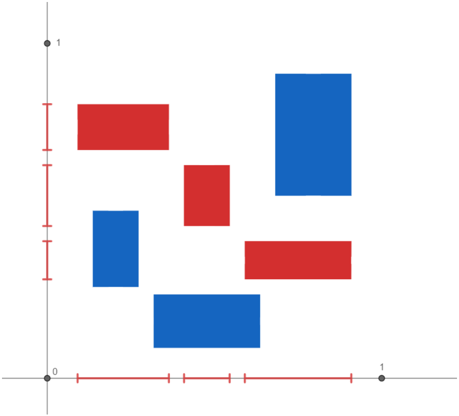
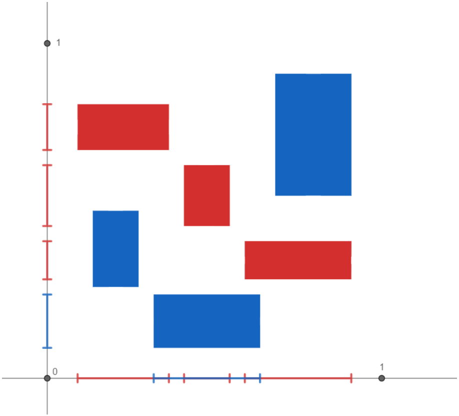


Project each rectangle on both axes.

Every axis-parallel rectangle is determined by a pair of segments.

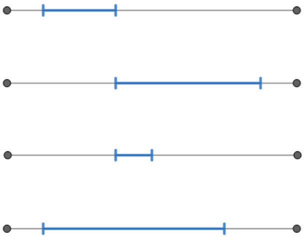
Illustrative example



Property of two rectangles "to be intersected by axis-parallel line" is equivalent to the property of the corresponding segment pairs "to have non-empty intersection".

d-intervals

Let $I_1, I_2, \ldots, I_d$ be disjoint parallel segments in the plane. We say a set $H \subset \bigcup_{i=1}^{d} I_i$ is a d-**interval** if its intersection with each I_i is a closed interval.



Piercing colorful d-intervals

Transversal of a set $\mathcal{S}$ is a set of points that intersect every member of $\mathcal{S}$. If a transversal consists of n_i points from I_i for each $i \in \{1, \ldots, d\}$, we say it is a $n_1 \times n_2 \times \cdots \times n_d$ transversal.

Theorem

Let $\mathcal{F}_i$, $i \in [d]$, be d sets of d-intervals with $d \geq 3$. If any d representatives $H_1 \in \mathcal{F}_1, \ldots, H_d \in \mathcal{F}_d$ have a nonempty intersection, then there exists index $i \in [d]$ such that $\mathcal{F}_i$ has a $(d-1) \times 1 \times \cdots \times 1$ transversal.

Method

Colorful polytopal KKMS Theorem (Frick, Zerbib, 2017)

Let P be a k-dimensional polytope with $0 \in P$. Suppose for every nonempty, proper face σ of P we are given $k+1$ points $y_\sigma^{(1)}, \dots, y_\sigma^{(k+1)} \in C_\sigma$ and $k+1$ closed sets $A_\sigma^{(1)}, \dots, A_\sigma^{(k+1)} \subset P$. If $\sigma \subset \bigcup_{\tau \subset \sigma} A_\tau^{(j)}$ for every face σ of P and every $j \in [k+1]$, then there exist faces $\sigma_1, \dots, \sigma_{k+1}$ of P such that $0 \in \mathrm{conv}\{y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}\}$ and $\bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)} \neq \emptyset$.