Control Theory.
Homework Three.
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Task 1
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Task 2
My variant:
$(11.6+2.7)x''-2.7*0.57cos(\theta)\theta''+2.7*0.57sin(\theta)(\theta')^2 = 14.3x''-1.539cos(\theta)\theta''+1.539sin(\theta)(\theta')^2 = F$
and $-cos(\theta)x''+0.57\theta''-9.81sin(\theta)=0$
A According to the task, $u=F$, $q=[x\ \theta]^T$. From calculus we know that derivative of a vector is just derivatives of its components, hence, $q'=[x'\ \theta']^T$ and $q''=[x''\ \theta'']^T$.

$$\textit{Manipulator form :} M(q)q''+n(q,q')=Bu$$

$$\textit{My variant :}\begin{bmatrix} 14.3 & -1.539cos(\theta) \\ -cos(\theta) & 0.57 \end{bmatrix}\begin{bmatrix} x'' \\ \theta'' \end{bmatrix}+\begin{bmatrix} 1.539sin(\theta)(\theta')^2 \\ -9.81sin(\theta) \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}F$$

$$M(q)=\begin{bmatrix} 14.3 & -1.539cos(\theta) \\ -cos(\theta) & 0.57 \end{bmatrix}$$

$$n(q,q')=\begin{bmatrix} 1.539sin(\theta)(\theta')^2 \\ -9.81sin(\theta) \end{bmatrix}, B=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

B
$$M(q)^{-1}=\frac{1}{8.151-1.539sin(\theta)cos(\theta)}\begin{bmatrix} 0.57 & 1.539cos(\theta) \\ cos(\theta) & 14.3 \end{bmatrix}$$

$$\textit{Target form: } z'=f(z)+g(z)u, wherez=[x,\ y,\ x',\ y']^T$$

Hence: $$\begin{bmatrix} x'' \\ \theta'' \end{bmatrix}=\frac{1}{8.151-1.539sin(\theta)cos(\theta)}\begin{bmatrix} 0.57 & 1.539cos(\theta) \\ cos(\theta) & 14.3 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}F+\begin{bmatrix} -1.539sin(\theta)(\theta'') \\ 9.81sin(\theta) \\ \theta'' \end{bmatrix}\right)$$

As $z=\begin{bmatrix} x \\ \theta \\ x' \\ \theta' \end{bmatrix}$, the answer is: $$\begin{bmatrix} x' \\ \theta' \\ x'' \\ \theta'' \end{bmatrix}=\begin{bmatrix} x' \\ \theta' \\ \frac{9.81cos(\theta)cos(\theta)-0.87723sin(\theta)(\theta)^2}{8.151-1.539sin(\theta)cos(\theta)} \\ \frac{-2.368521sin(\theta)cos(\theta)(\theta)^2+140.283sin(\theta)}{8.151-1.539sin(\theta)cos(\theta)} \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ \frac{0.57}{8.151-1.539sin(\theta)cos(\theta)} \\ \frac{1.539cos(\theta)}{8.151-1.539sin(\theta)cos(\theta)} \end{bmatrix}F$$

C
Lets derive in letters all previous steps:
Step A: $$\begin{bmatrix} (m+M) & -mlcos(\theta) \\ -cos(\theta) & l \end{bmatrix}\begin{bmatrix} x'' \\ \theta'' \end{bmatrix}+\begin{bmatrix} mlsin(\theta)(\theta')^2 \\ -gsin(\theta) \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}F$$

Step B: $$\bar{K}=\begin{bmatrix} x' \\ \theta' \\ x'' \\ \theta'' \end{bmatrix}=\begin{bmatrix} x' \\ \theta' \\ \frac{F-mlsin(\theta)(\theta')^2+mgsin(\theta)cos(\theta)}{(m+M)-mcos^2(\theta)} \\ \frac{Fcos(\theta)-mlsin(\theta)cos(\theta)(\theta')^2+(m+M)gsin(\theta)}{(m+M)l-mlcos^2(\theta)} \end{bmatrix}$$

Lets add the results of the previous step and get B in the same form :

$$\begin{bmatrix} x' \\ \theta' \\ x'' \\ \theta'' \end{bmatrix} = \begin{bmatrix} x' \\ \theta' \\ \frac{9.81cos(\theta)cos(\theta)-0.87723sin(\theta)(\theta)^2+0.57F}{8.151-1.539sin(\theta)cos(\theta)} \\ \frac{-2.368521sin(\theta)cos(\theta)(\theta)^2+140.283sin(\theta)+1.539cos(\theta)F}{8.151-1.539sin(\theta)cos(\theta)} \end{bmatrix}$$

$$\textit{Target form: } \delta z' = A\delta z + B\delta u$$

For this step we will need 2 matrixes A,B.To find them we should remember that $\bar{z} = [x, \theta, x', \theta']^T$, $u = F$, $\delta z = z' - \bar{z}$,$\delta u = u' - \bar{u}$, and, according to the task, $\bar{z} = [0, 0, 0, 0]^T$:

$$A = \frac{\partial \bar{K}}{\partial \bar{z}}|_{z=\bar{z},u=\bar{u}} = \begin{bmatrix} \frac{\partial k_1}{\partial z_1} & \cdots & \frac{\partial k_1}{\partial z_4} \\ & \ddots & \\ & & \frac{\partial k_4}{\partial z_4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{mg}{M} & 0 & 0 \\ 0 & \frac{(m+M)g}{Ml} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{26.487}{11.6} & 0 & 0 \\ 0 & \frac{140.283}{6.612} & 0 & 0 \end{bmatrix}$$

$$B = \frac{\partial \bar{r}}{\partial \bar{u}}|_{z=\bar{z},u=\bar{u}} = \begin{bmatrix} \frac{\partial k_1}{\partial u} \\ \vdots \\ \frac{\partial k_4}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ \frac{1}{Ml} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{11.6} \\ \frac{1}{6.612} \end{bmatrix}$$

D $eig(A) = det\left(\begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & 2.283 & -\lambda & 0 \\ 0 & 21.216 & 0 & -\lambda \end{bmatrix}\right) = -\lambda\left(\begin{bmatrix} -\lambda & 0 & 1 \\ 2.283 & -\lambda & 0 \\ 21.216 & 0 & -\lambda \end{bmatrix}\right) =$
$= -\lambda^2(-\lambda^2 - 21.216)$
Eigenvalues of matrix A are $0, 0, \approx \pm 4.6061$ and because on eigenvalues is positive $\Rightarrow$ the system is unstable.

E
In Matlab RANK(CTRB(A,B)), where A and B are from step C.
Manually we have controllability matrix : $M_c = [B, AB, A^2B...A^{n-1}B]$, where n is number of rows in A, and the system is controlable if this matrix is full rank.

In my case $M_c = [B, AB, A^2B, A^3B]$, $AB = \begin{bmatrix} 0.0862 \\ 0.1512 \\ 0 \\ 0 \end{bmatrix}$, $A^2B = \begin{bmatrix} 0 \\ 0 \\ 0.3453 \\ 3.2088 \end{bmatrix}$, $A^3B = \begin{bmatrix} 0.3453 \\ 3.2088 \\ 0 \\ 0 \end{bmatrix}$.

Hence, $M_c = \begin{bmatrix} 0 & 0.0862 & 0 & 0.3453 \\ 0 & 0.1512 & 0 & 3.2088 \\ 0.0862 & 0 & 0.3453 & 0 \\ 0.1512 & 0 & 3.2088 & 0 \end{bmatrix}$
$rank(M) = 4 \Rightarrow$ the system is controllable as M_c is full rank.

F
Code solution, plots and comments on them for F and G are via the link.
I have chosen 4 inputs:

- $y0 = [0, 0, 0, 0]$ - s stable point from the beginning,
- $y0 = [0, 0, 0, 0]$ - small initial deviation with 0 derivatives, hence, with 0 initial velocity,

- $y0 = [0, 0, 0, 0]$ - big initial deviation with 0 derivatives, hence, with 0 initial velocity,
- $y0 = [0, 0, 0, 0]$ - big initial deviation with non-0 derivatives, hence, with some initial velocity.

G

Code solution, plots and comments on them for F and G are via the link.