MSCF Financial Markets Prep Project - Chooser Option

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1 Introduction

Assume that there is a (risk-free) money market account with annual interest rate r (continuous compounding) and let S be a stock that pays no dividends. A European Chooser Option V on S has a maturity date T, a strike price K, and a choice date τ with $0 < \tau < T$.
At time τ the holder gets to choose whether the option will be a European call C or a European put P on S. (The put and call both have maturity T and strike price K.)
Make some interesting observations/discoveries about this type of option. Using put-call parity to compute $C_\tau - P_\tau$ should prove helpful. You can discover some important things without making any assumptions about how the stock price evolves (i.e. in a model-independent context). You can discover some more explicit things in the context of the Black-Scholes framework.

2 Math Derivation

We will derive a formula for the European chooser option with above mentioned assumptions and parameters.
In the formula, denote as follows the meaning of each variable:
$M_{t,t',k}$ = the price at time t of a European **chooser** option with maturity date t' and strike price k.
$C_{t,t',k}$ = the price at time t of a European **call** option with maturity date t' and strike price k.
$P_{t,t',k}$ = the price at time t of a European **put** option with maturity date t' and strike price k.

S_t = the price of the underlying stock at time t.

$$
\begin{aligned}
M_{0,T,K} &= e^{-r\tau}E(max(C_{\tau,T,K},\ P_{\tau,T,K})) && (1)\\
&= e^{-r\tau}E(max(S_\tau + P_{\tau,T,K} - Ke^{-r(T-\tau)}, P_{\tau,T,K})) && \text{by Put-Call parity at time } \tau \quad (2)\\
&= e^{-\tau t}E(P_{\tau,T,K} + max(S_\tau - Ke^{-r(T-\tau)}, 0)) && (3)\\
&= e^{-\tau t}E(P_{\tau,T,K}) + e^{-\tau t}E(max(S_\tau - Ke^{-r(T-\tau)}, 0)) && \text{by linearity of expectation} \quad (4)\\
&= e^{-\tau t}E(P_{\tau,T,K}) + e^{-\tau t}E(C_{\tau,\tau,Ke^{-r(T-\tau)}}) && (5)\\
&= P_{0,T,K} + C_{0,\tau,Ke^{-r(T-\tau)}} && \text{by risk-neutral evaluation} \quad (6)
\end{aligned}
$$

Notice that, from (4) to (5) it is because $max(S_\tau - Ke^{-r(T-\tau)}, 0)$ is the payoff of a call option with maturity date τ and strike price $Ke^{-r(T-\tau)}$.
From (6), we see that under risk-neutral evaluation, the original European chooser option can be replicated by
1. a put option with strike price K and maturity date T, and
2. a call option with strike price $Ke^{-r(T-\tau)}$ and maturity date τ.
Notice that, if we replace $P_{\tau,T,K}$ using put-call parity instead of $C_{\tau,T,K}$ in (1), we would get a different result in the end – we can also replicate the chooser option by
1. a call option with strike price K and maturity date T, and
2. a put option with strike price $Ke^{-r(T-\tau)}$ and maturity date τ.
This also makes intuitive sense because the call option and put option are symmetric in the original definition of the chooser option.
Notice that, we made no assumptions on the evolution path of the stock price to replicate the chooser option. So the formula and replication are correct independent of any model assumptions.

3 Under Black-Scholes

Under the black-scholes setting, we just plug in the BSM formula for the replicating put and call option into (6) and we can get the price of the chooser option. Since no other difficult derivation is needed, we just state the BSM formula here, and the leave the rest of the calculation to the audience who are interested.
$C(t, t', k) = N(d_1)S_t - N(d_2)Ke^{-r(t'-t)}$
$P(t, t', k) = N(-d_2)Ke^{-r(t'-t)} - N(-d_1)S_t$
where
$d_1 = \frac{ln(\frac{S_t}{k}) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$
$d_2 = d_1 - \sigma\sqrt{t}$
$N(.)$ = cumulative distribution function(CDF) of the standard normal distribution

S_t = price of the stock at time t
σ = volatility of the stock

Notice that, we can price the chooser option under any other setting – we just need prices for the put and call options.