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Equilibrium selection and bounded rationality in symmetric normal-form games ☆

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Abstract

We develop and test a model of initial play for symmetric normal-form games with multiple Nash equilibria. First, we specify an encompassing model that incorporates both equilibrium selection principles and boundedly rational behavioral models. We then design experimental games that can identify a variety of equilibrium selection principles. Model comparisons and hypothesis tests indicate that (1) boundedly rational behavior is prevalent in initial-period play, (2) homogeneous population models can be strongly rejected in favor of heterogeneous population models, and (3) deductive selection principles add no statistically significant contribution to explaining the data.

JEL classification: B41; C14; C51; C72; C90

Keywords: Equilibrium selection; Bounded rationality; Heterogeneity

1. Introduction

Recent advances in experimental economics characterize the heterogeneity in one-shot games with a unique Nash equilibrium by allowing for different hierarchies of sophistication. This line of research is useful in understanding behavior in a wide set of games and in adapting

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the rigid super-rational prescriptions of game theory to human behavior. The main innovation of models of hierarchical bounded rationality over traditional game theory is that they recognize that some humans are either incapable of identifying and playing Nash equilibrium strategies or are skeptical of others' ability to do so. As such, some human players apply simpler, often naïve rules to reach their decisions while more sophisticated people believe others use such rules. This category of models includes Stahl and Wilson (1994, 1995, henceforth, SW), Stahl (1993, 1996), Haruvy et al. (2001, henceforth, HSW), Haruvy (2002), Nagel (1995), Costa-Gomes et al. (2001), Bosch-Domènech et al. (2002), Camerer et al. (2004), and Gneezy (2002).

While Nash equilibrium play is not the dominant mode of behavior in such studies it nevertheless plays an important role in two ways. First, in all of the above studies, a non-negligible proportion of the population plays Nash equilibrium strategies. Such players could be classified as 'Nash types'. Second, a large population segment, which we refer to as 'Worldly types,' incorporates Nash equilibrium into beliefs. Members of this segment anticipate that Nash equilibrium strategies will be played by some in the population (SW, 1995; HSW 2001). Unfortunately, in the absence of a unique solution concept in games of multiple equilibria, existing models of hierarchical bounded rationality are limited to games of unique Nash equilibrium.

As important as games of single equilibrium are, games of multiple equilibria are of great interest to economists. Prominent works in experimental economics include Cooper et al. (1990), Van Huyck et al. (1990, 1991, henceforth, VHBB), Van Huyck et al. (1994, 1997, henceforth, VHCB), Straub (1995), McKelvey and Palfrey (1995), Anderson et al. (2001), and Haruvy and Stahl (2004). The above examples enable conclusions about the relative saliency of various deductive principles. The general conclusion in experimental studies of long-term behavior is that deductive equilibrium selection principles are poor predictors of long-run outcomes. However, the extent to which human players take deductive equilibrium selection principles into consideration has not yet been gauged. In this paper, we combine the two strands of hierarchical bounded rationality and deductive equilibrium selection by developing a model of one-shot play for two-player, symmetric normal-form games with multiple Nash equilibria.

The problem of multiple equilibria presents a difficulty as well as an opportunity to reexamine the explanatory power of equilibrium selection principles. Our approach is to specify an encompassing econometric model that incorporates equilibrium selection principles as well as boundedly rational models of behavior. This constitutes an extension of the SW model (i) to embrace games with multiple Nash equilibria and (ii) to fit the population distribution of choices. We design an experiment and then estimate the model using the experimental data. The implicit nesting of hypotheses in such a model allows for statistically powerful tests.

In Section 2, we review three deductive equilibrium selection principles: payoff dominance, risk dominance, and security. Section 3 develops the model, beginning with a generic model of probabilistic choice functions (or *behavioral rules*). We demonstrate how a small set of archetypal behavioral rules can be used in a mixture model to represent heterogeneous populations. Section 4 explains the experimental design. Section 5 presents a description of the experimental data, estimation results, model comparisons, hypothesis tests and robustness tests.

Our model comparisons and hypotheses tests indicate that (1) boundedly rational, in particular level-1 thinking, is prevalent in initial-period play, (2) homogeneous population models can be strongly rejected, and (3) deductive selection principles add no statistically significant contribution. These results are robust for a holdout sample of games and predict the initial play quite well for multiple-period sessions of these games. Section 6 concludes.

2. Deductive equilibrium selection principles

The *payoff dominance* (PD) principle relies on the idea that “rational individuals will cooperate in pursuing their common interests if the conditions permit them to do so” (Harsanyi and Selten, 1988, p. 356). In a symmetric normal-form game, the PD equilibrium is the symmetric Pareto-optimal Nash equilibrium. Experimental studies by Cooper et al. (1990, 1992), VHBB (1990, 1991) and Straub (1995) on coordination games provide substantial evidence that players often fail to coordinate their actions to obtain a Pareto-optimal equilibrium in experimental settings.

Risk-based solution concepts differ in what taken to be the best proxy for “risk.” The *security* selection principle is based on Von Neumann and Morgenstern’s (1953) maximin criterion. A *secure action* is that action that maximizes the minimum possible payoff (VHBB, 1990). VHBB found security to be a good equilibrium selection principle. However, in their game, all actions were equilibrium actions. For security to be used as an equilibrium selection principle in games with non-equilibrium actions, one must restrict the security criterion to equilibrium actions by defining the *secure equilibrium action* as the equilibrium action that satisfies:

$$\underset{k \in \mathrm{NE},\, j \in \mathrm{NE}}{\arg\max \min}\, U_{kj} \tag{1}$$

where U is a $J \times J$ matrix of game payoffs for the row player in a given game and NE denotes the set of Nash equilibrium actions. This specification applies the security criterion to the game after the deletion of non-equilibrium actions. In accordance with this restriction, the security (SEC) selection principle is an equilibrium selection principle that predicts the maximin action after restricting attention to the set of equilibrium actions.

We define risk dominance in accordance with Harsanyi and Selten’s *heuristic justification*, in which selection of an equilibrium results from postulating an initial state of uncertainty where the players have uniformly distributed second order beliefs (i.e., each player believes that the other players’ beliefs are uniformly distributed on the relevant space of priors). Briefly,[1] given a symmetric $n \times n$ game with payoff matrix U, let NE denote the set of Nash equilibrium actions, and let B denote the simplex on NE. For each $j \in \mathrm{NE}$, define q_j^{RD} as the relative proportion of B for which action j is the best response to some belief in B. Then the action $k \in \mathrm{NE}$ that maximizes $U_k q^{\mathrm{RD}}$ (where U_k is the kth row of the payoff matrix) is the risk-dominant NE action. This solution concept closely follows Harsany and Selten’s heuristic definition, coincides with their pairwise definition in games with two pure equilibria, and ensures transitivity of the risk-dominance relation in symmetric $n \times n$ games.

3. Specification of the encompassing econometric model

The basic component of our econometric model is a probabilistic choice function that is based on evidence. Let y be a $J \times 1$ real vector of evidence, with the implication that $y_j > y_k$ means that there is more evidence in favor of choosing action j than there is for choosing action k. We suppose that the decision-maker measures this evidence with some error or considers other latent aspects

[1] There is a simpler definition involving pairwise comparisons of the Nash product for games with two pure Nash equilibria. Since this definition does not satisfy transitivity and other desirable properties for games with more equilibria, we use the heuristic definition. The two approaches coincide in games of two equilibria. They are both due to Harsanyi and Selten. See Haruvy and Stahl (1998) for more detail.

of the actions so that the probability of choosing action j given evidence y is

$$P_j(y) \equiv \frac{\exp(y_j)}{\sum_\lambda \exp(y_\lambda)}. \tag{2}$$

$P(y)$ is a logit probabilistic choice function based on evidence y. We will represent each type of behavior as a logit probabilistic choice function based on specific evidence.

One major advantage of this approach is that even when there is overwhelming (but finite) evidence in favor of a particular action, the choice probabilities will be strictly positive for all actions, and the small probabilities on the less favored actions can be interpreted as trembles or idiosyncratic noise. Moreover, since the choice probabilities respect the ranking of actions according to the evidence vector, the tremble probabilities are “proper” in the sense that the actions with less favorable evidence are less likely to be chosen.

In the following subsection, we apply this approach to derive behavioral rules based on three equilibrium selection principles. Next, we extend this approach to include the SW level-n theory of bounded rationality as well as optimistic behavior and hybrid (“Worldly”) behavior.

3.1. *Nash equilibrium selection evidence*

When confronting real choice data, it is virtually certain that choices inconsistent with Nash equilibrium selection principles will be observed. Therefore, it necessary to supplement these pure selection principles with a model of errors or trembles. To ensure that our results are not artifacts of a particular error model, we explore two alternative approaches for modeling errors in the context of equilibrium selection: (i) prior-based, and (ii) uniform trembles. Both can be represented as an evidence-based logit probabilistic choice function.

3.1.1. *Games with a unique Nash equilibrium*

Since any theory developed for games with multiple Nash equilibria should also apply to games with a unique equilibrium, we begin with the latter case first. Let p^{NE} denote the unique Nash equilibrium expressed as a probability distribution over the available actions. Since game theory specifies the belief of a Nash player to be p^{NE}, it is natural to take $\nu U p^{\mathrm{NE}}$ as the evidence vector, where $\nu > 0$ is a scalar that is inversely proportional to the variance of calculation errors and noise induced by latent idiosyncratic factors. The *prior-based behavioral rule* of a Nash player in a game with a unique Nash equilibrium is then defined as the probabilistic choice function in Eq. (2) with $y = \nu U p^{\mathrm{NE}}$. Thus, the tremble probabilities to non-Nash actions are monotonic in the expected utility of those actions given belief p^{NE}.

An alternative specification of the potential choice errors is one in which non-Nash actions all have an equal but small probability. To distinguish this *uniform-error model* from the logit model, we let Φ^{NE} denote the probability choice vector. Then, given an $\varepsilon \in [0, 1]$:

$$\Phi^{\mathrm{NE}}(\varepsilon) = (1-\varepsilon)p^{\mathrm{NE}} + \varepsilon P^0, \tag{3}$$

where P^0 denotes the uniform distribution over the J possible actions. The shortcoming of this commonly employed specification is that non-Nash actions with low expected payoff given belief p^{NE} are just as likely to be chosen as non-Nash actions with high expected payoff.

3.1.2. *Games with multiple Nash equilibria*

In games with multiple Nash equilibria, we have multiple candidates for the evidence vector: $\nu U p^k$ for each $k \in \mathrm{NE}$. An equilibrium selection principle can be used to single out one of these candidates. We also explore how to incorporate evidence derived from the ranking criteria of each selection principle.

For games with a unique Nash equilibrium that corresponds to a particular selection principle (PD, RD, SEC), there is a unique prior (or belief) corresponding to that selection principle (denoted $p^{\mathrm{PD}}, p^{\mathrm{RD}}, p^{\mathrm{SEC}}$). This prior assigns a probability of one to the Nash action selected by that principle. It would then seem natural to use νUp^{PD} as the evidence for PD selection principle.

However, such a specification entails an undesirable discontinuity in behavior. To illustrate, consider a symmetric game with two Nash equilibria (actions 1 and 2) and $U_{11} > U_{22}$. In this case, p^{PD} would assign a probability of one to action 1 no matter how small the payoff difference, but when $U_{11} = U_{22}$, presumably both equilibria are equally likely.

This discontinuity can be smoothed by specifying a probabilistic choice between alternative equilibrium beliefs that depends monotonically on the ranking of the equilibria. Specifically, define the ranking-based PD evidence as $\{r_j^{\mathrm{PD}} \equiv U_{jj}, j \in \mathrm{NE}\}$, and define the corresponding *selection probabilities*:

$$\psi_j(\gamma; r^{\mathrm{PD}}) \equiv \frac{\exp(\gamma r_j^{\mathrm{PD}})}{\left[\sum_{k \equiv \mathrm{NE}} \exp(\gamma r_k^{\mathrm{PD}})\right]}, \quad j \equiv \mathrm{NE}. \tag{4}$$

$\psi(\gamma; r^{\mathrm{PD}})$ is a probability distribution whose support is the set of NE actions. For large values of γ, a probability of virtually 1 will be assigned to the PD action, and for $\gamma \approx 0$, the probability will be spread equally over all NE actions.

We can interpret γ as the strength of the PD selection criterion in determining which Nash equilibrium an individual player believes will be played by others. Under this "individual-prior" interpretation, the evidence vector would be

$$y^{\mathrm{PD}}(\gamma) \equiv U\left[\sum_{k \equiv \mathrm{NE}} \psi_k(\gamma; r^{\mathrm{PD}}) p^k\right], \tag{5}$$

and the probabilistic choice function would be given by Eq. (2) with $y = \nu y^{\mathrm{PD}}(\gamma)$.[2]

A similar approach can be applied to the risk dominance and the security selection principles by using their respective ranking criteria in place of payoffs. Specifically, for the risk dominance principle, the ranking-based evidence is $\{r_j^{\mathrm{RD}} = U_j q^{\mathrm{RD}}, j \in \mathrm{NE}\}$, where q^{RD} was defined in Section 2. For the security selection principle, the ranking-based evidence is $\{r_j^{\mathrm{SEC}} = \min_{k \in \mathrm{NE}} U_{jk}, j \in \mathrm{NE}\}$. Note that since Eq. (5) are confined to the set of Nash equilibria, we need not define ranking-based evidence for non-Nash actions.

3.2. Boundedly rational models of behavior

Recent experimental research on initial-period play sheds some light on what hitherto has been thought of as 'accidental' initial distribution of play. Theory derived from such experimental work has generated several classes of bounded rationality models, two of which we consider in this

[2] Note that it is possible that the best response to such a belief is a non-Nash action. Since a theory of equilibrium selection ought to have players choosing some NE action, we entertained a "subpopulation mixture" interpretation, in which each Nash player has an extreme belief that just one of the NE will be played by all other players, and $\psi_k(\gamma; r^{\mathrm{PD}})$ represents the proportion of the Nash subpopulation that believes in $k \in \mathrm{NE}$. However, in a nested model with both the individual-prior and the subpopulation types, we were able to reject the latter (HS99).

essay. The first class is hierarchical in nature and is based on works by Stahl (1993) and Nagel (1995); the second class is based on models of optimistic and pessimistic behaviors.

3.2.1. *Level-n bounded rationality*

We embrace SW's conjecture that different boundedly rational[3] behaviors are due to different depths of reasoning by a self-referential process starting with a uniform prior over other players' strategies. In particular, level-1 bounded rationality postulates that a given player, due to insufficient reason, holds a uniform prior, P^0, over other players' actions and hence has a prior-based evidence vector of UP^0.[4] Then the *level-1 behavioral rule* is the probabilistic choice function defined by Eq. (2) with $y = \nu_1 UP^0$, which we will denote hereafter as

$$P^1(\nu_1) \equiv P(\nu_1 UP^0). \tag{6}$$

Level-2 bounded rationality postulates a prior of $b^1(P^0)$, where $b^1(\cdot)$ puts equal probability on best responses to $(\cdot)$ and zero probability on inferior responses. Hence, a level-2 player has the prior-based evidence vector $Ub^1(P^0)$. The *level-2 behavioral rule* is the probabilistic choice function defined by Eq. (2) with $y = \nu_2 Ub^1(P^0)$, which we will denote hereafter as

$$P^2(\nu_2) \equiv P(\nu_2 Ub^1(P^0)). \tag{7}$$

The *level-0 behavioral rule* is uniformly random over all the actions in the game and so is denoted by P^0, which is equivalent to having an evidence vector of zeros since, from Eq. (2), $P(0) = P^0$.

3.2.2. *Optimistic and pessimistic behavior*

Haruvy et al. (1999) investigate the existence of optimistic and pessimistic behavior in player populations; they find optimistic behavior to be significant and to describe best a non-negligible portion of the player sample, but they find pessimistic behavior to be insignificant. An "optimistic" player, also known as a maximax type, is one who tends to choose the action that can potentially give him the highest payoff in the game. In other words, the optimistic player can be thought of as one who believes his opponents will act in his best interest, whichever action he chooses. Hence, the natural optimistic evidence is

$$y_{\text{opt}}(\nu_{\text{opt}}) \equiv \nu_{\text{opt}}\{\max_k U_{jk},\ j = 1, \ldots, J\}, \tag{8}$$

and the corresponding *optimistic behavioral rule* is $P^{\text{opt}}(\nu_{\text{opt}}) \equiv P[y_{\text{opt}}(\nu_{\text{opt}})]$. A pessimistic player can be analogously defined, but given the HSW finding, it is not pursued here.

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[3] The self-referential nature of the concept of "rationality" in interactive situations raises the question as to whether limits on computational powers of human beings impose fundamental constraints on their ability to arrive at rational strategies (Binmore, 1987, 1988).

[4] In logit models, the predicted choice frequencies are sensitive to the addition of redundant actions and dominated actions. Thus, we caution the reader that duplicating some action many times and/or adding an "obviously bad" action is likely to make the level-1 prediction empirically less representative. Humans are good at filtering out "unimportant" information and concentrating on pertinent information, and it is likely that even a minimally rational player (like level-1 types) will take this into account when formulating beliefs. However, the study of such refinements is beyond the scope of this paper. Recall that equilibrium refinements such as perfection and properness are also sensitive to these factors.

3.2.3. Worldly behavioral rules

If a reader of this article were to participate in one of our experimental sessions, he or she would undoubtedly identify the Nash equilibria, but would be well advised not to assume that all of the other players would identify and choose a Nash action. A savvy ("Worldly") participant would have a belief that includes some participants playing Nash equilibria and others making boundedly rational choices. To capture this kind of behavior in games with a unique Nash equilibrium, SW specified the *Worldly archetype* as a player whose prior is given by

$$q_j^{\mathrm{WNE}} = (\mu_{\mathrm{w}}, \varepsilon_{\mathrm{w}}) \equiv \varepsilon_{\mathrm{w}} P^1(\mu_{\mathrm{w}}) + (1 - \varepsilon_{\mathrm{w}}) p^{\mathrm{NE}} \tag{9}$$

where $P^1(\cdot)$ is defined by Eq. (6) and p^{NE} is the Nash prior. The proportion of the population believed to be boundedly rational is $\varepsilon_{\mathrm{w}} \in [0.1,0.9]$, and the choice probabilities of the boundedly rational subpopulation are believed to come from level-1 types with precision μ_{w}. The evidence generated by this prior is $y^{\mathrm{WNE}}(\mu_{\mathrm{w}},\varepsilon_{\mathrm{w}}) \equiv Uq^{\mathrm{WNE}}(\mu_{\mathrm{w}},\varepsilon_{\mathrm{w}})$, and the probabilistic choice function is given by Eq. (2) with $y = y^{\mathrm{WNE}}(\mu_{\mathrm{w}},\varepsilon_{\mathrm{w}})$. Since SW considered only games with unique symmetric NE, this specification was adequate. In the current setting, however, the Worldly specification must be extended to games with multiple NE.

Our approach to specifying Worldly behavior in the presence of multiple NE parallels our approach to specifying the naïve Nash archetypes in Section 3.1.2. We suppose an individual's prior on the Nash equilibria (conditional on some NE) is given by the PD selection probabilities $\psi(\gamma;r^{\mathrm{PD}})$, Eq. (4). Then the complete prior of that Worldly PD (WPD) type would be[5]

$$q_j^{\mathrm{WPD}}(\gamma_{\mathrm{w}}, \mu_{\mathrm{w}}, \varepsilon_{\mathrm{w}}) \equiv \varepsilon_{\mathrm{w}} P^1(\mu_{\mathrm{w}}) + (1 - \varepsilon_{\mathrm{w}}) \sum_{k \equiv \mathrm{NE}} \psi_k(\gamma_{\mathrm{w}}; r^{\mathrm{PD}}) p^k, \tag{10}$$

and the probabilistic choice probabilities, denoted $P^{\mathrm{WPD}}(\nu,\gamma_{\mathrm{w}},\mu_{\mathrm{w}},\varepsilon_{\mathrm{w}})$, would be computed in the usual manner. We interpret γ_{w} as the strength of the PD selection criteria in the mind of this Worldly individual-prior PD type. We similarly define priors for Worldly individual-prior RD and SEC types. Note that as ε_{w} decreases, the choice probability of a Nash action would not necessarily increase; indeed, the best response to a mixture over the NE could be a non-Nash action.

3.3. Homogeneous population models

In a homogeneous population model all individuals are assumed to use one and only one behavioral rule, such as the payoff-dominance Nash rule. Of course, the specific probabilistic choice function will depend on the game (the actions and payoffs) because the evidence vector depends on the game. We will make this dependence explicit by using "g" where needed to denote a particular game in some group of games G. For example, U^g will denote the payoff matrix in game g, and $p^{\mathrm{PD}}(g)$ will denote the payoff-dominant Nash equilibrium belief for game g; hence the evidence vector is written as $y^g(\nu) = \nu U^g p^{\mathrm{PD}}(g)$.

[5] This specification of the Worldly prior makes a minimal departure from SW, which allows us to attribute any change in results to the treatment of multiple equilibria. As noted in SW, a belief that the boundedly rational players are level-0 and level-1 types can be approximated by a single level-1 type with a lower precision (μ_{w}). We excluded level-2 types from the Worldly model of others because (a) level-2 types are not empirically significant, and (b) when we did include them, they never received any weight. Similarly, we also tried optimistic evidence, and found that it did not receive any weight in the estimated prior.

Further, let $a(i,g)$ denote the action of player i in game g. Then, given a behavioral rule and the evidence $y^g(\nu)$ that defines that rule, the probability of the population's joint choices for a group of G games is

$$\prod_i \prod_{g \in G} P_{a(i,g)}[y^g(\nu)]. \tag{11}$$

To fit the model to the data, we maximize the log of this joint probability. In our simple behavioral models, there will typically be only one or two parameters over which to maximize.

3.4. Heterogeneous population models

We next construct an econometric model that allows for sub-populations of players using different behavioral rules. Such a model allows us (1) to test conditionally for the existence of a sub-population that uses a given behavioral rule, (2) to measure the extent of use of one rule versus another in first-period play if neither is rejected (i.e., measure the relative size of a sub-population), and (3) to test alternative methods of modeling a specific behavioral type.

We associate a sub-population of potential individuals with each behavioral rule t in a class of behavioral rules T. We let $\alpha_t \geq 0$ denote the proportion of the total population that comprises the sub-population that uses behavioral rule t. Obviously, we require that $\sum_{t \in T} \alpha_t = 1$. For example, we might specify T to be $\{0, 1, 2, \text{opt}, \text{PD}\}$, meaning that we include the level-0, level-1, level-2 and optimistic boundedly rational rules as well as the payoff-dominant individual-prior Nash rule.

Letting y^{gt} denote the evidence associated with rule t for game g, the probabilistic choice function is $P(y^{gt})$, given by Eq. (2), and the aggregate probabilistic choice function for the population in game g is

$$\bar{P}(\alpha; g) \equiv \sum_{t \in T} \alpha_t P(y^{gt}). \tag{12}$$

Assuming that the subpopulation proportions (α_t) are the same for all games, the log-likelihood function for observed population choices is then proportional to

$$\mathrm{LL}(\alpha) = \sum_{g \in G} \sum_{j \in J} n_j^g \log[\bar{P}_j(\alpha; g)], \tag{13}$$

where n_j^g denotes the number of individuals choosing action j in game g. Of course, this log-likelihood function also depends on the scaling and weighting parameters implicit in the evidence for each rule.

Note that we do not identify individual players with any specific behavioral rule (or type). This point highlights a major difference between the econometric methodology and mixture approach taken here and that of SW. In SW, each player's conditional likelihood over choices in the entire set of games was conditional on his type and each player's unconditional likelihood was a mixture over conditional likelihoods. Then the product of these over all players gave the likelihood of the data. In this paper, the mixture over the choice probabilities of each type is taken first; the likelihood of the data is then the product of these mixed likelihoods over games and players. The former imposes the assumption that a player's type is constant over all games, while the latter does not. One interpretation is that players redraw their type for each game. Another interpretation is that while individual players may experiment with different behavioral rules in different games, the proportion of the population using a particular rule is constant across games.

We choose the population approach because (i) unchanging population proportions are more likely a better approximation of reality than unchanging individuals, and (ii) there are many economic applications for which only population data is available or only population predictions are needed (e.g., marketing, public policy impact studies, portfolio hedging, etc.). For a comparison of the two approaches to mixture models, see Appendix A of Haruvy and Stahl (1999, henceforth HS99).

4. Experimental design

Twenty symmetric 3×3 game matrices were selected so the various equilibrium selection theories as well as boundedly rational behaviors predict identifiably different patterns of choices across the games. Fourteen of these games are coordination games with two or three symmetric pure Nash equilibria (six taken from Cooper et al., 1990). For these coordination games, no two equilibrium selection principles predict the same action for all 14 games. Two of the games are specifically designed to separate out Harsanyi and Selten's pairwise risk dominance from our extension of risk dominance. The six non-coordination games were designed to separate out various boundedly rational types. The payoff matrices for the row player are presented in Fig. 1; the transposes of these matrices give the payoffs for the column player.

Each participant "plays the field"; that is, each player faces the empirical distribution of choices of all other participants, also known as "mean matching". In the coordination literature, the mean of players' actions is a common example of an abstract market process (Cooper and John, 1988). As the number of players increases, the influence of an individual player on the mean goes to zero, and in the limit an individual player cannot influence the market outcome. The payoffs are in terms of probability points of winning $ 2 in each game.[6]

The experiment was conducted in a computer laboratory,[7] and each participant was assigned a computer terminal. During the experiment, each game matrix was presented on the computer screen (see Fig. 2) once a participant clicked on a button corresponding to that game. Participants had an on-screen calculator available to them to calculate hypothetical payoffs. On the main screen, players viewed the game matrices, entered hypotheses regarding the other players' distribution of play, calculated payoffs to each action given a hypothesis, and chose a row action for each game matrix. Since equilibrium selection theories presuppose that players possess beliefs and compute expected payoffs given those beliefs, the on-screen calculator was provided to reduce noise in the data due to participant calculation errors. We address possible framing effects of the calculator and the robustness of our results to those effects in Section 5.5.

In addition to the main screen, players had available to them an instruction screen containing a copy of the instructions. Each player played each of the 20 games with no feedback until all

[6] In two sessions limited to advanced business students, the lottery prize was US$ 3 per game. Roth and Malouf (1979) were first to use such payment schemes in experiments in order to induce risk neutrality. Further, the independence axiom is not required to deduce predictions. Since then, studies have shown that binary lotteries induce behavior virtually indistinguishable from that resulting from direct payment (e.g., Cox et al., 1985, 1988; Walker et al., 1990; Cox and Oaxaca, 1995).

[7] Complete experiment instructions to accompany this essay are in HS99, Appendix B.

Below are the game matrices of the experiment sorted in to categories but numbered according to the order in which they were presented to participantson the screen. Above each column of eachmatrix is the number of players who chose the corresponding row of that matrix.

Key: PD = Payoff dominant Nash equilibrium strategy
RD = Risk dominant Nash equilibrium strategy
PRD = Pair-wise risk dominant strategy (only indicated when distinct from RD)
SEC = Security Nash equilibrium strategy
L1 = Level-1 Strategy
L2 = Level-2 Strategy (only indicated when distinct from L1)
DOM = Dominated strategy

Game 1

116	7	24	
70	60	90	L1, SEC, RD
60	80	50	PRD
40	20	100	PD

Game 2

1	10	136	
60	0	0	DOM
0	55	25	PD, RD
100	35	35	L1, SEC

Game 3

65	76	6	
25	30	100	L1
40	45	65	Unique NE
31	0	40	DOM

Game 4

21	118	8	
70	30	20	PD
60	60	30	L1, RD
45	45	40	SEC

Game 5

134	4	9	
35	70	35	L1, SEC
0	60	0	DOM
25	0	55	PD, RD

Game 6

32	5	110	
40	15	70	Unique NE
22	80	0	DOM
30	100	55	L1

Game 7

124	5	18	
60	30	60	L1, RD, SEC
50	25	55	DOM
30	20	70	PD

Game 8

4	139	4	
35	35	70	SEC
25	55	100	L1, RD,PD
0	0	60	DOM

Game 9

120	19	8	
30	50	100	L1
40	45	10	L2
35	60	0	

Game 10

3	7	137	
25	45	55	DOM
35	50	35	(NE)
30	45	60	L1, SEC, RD, PD

Game 11

98	46	3	
55	25	65	L1,RD,PD
35	35	70	SEC
0	0	60	DOM

Game 12

96	13	38	
30	100	22	L1
35	0	45	
51	50	20	L2

Game 13

102	15	30	
60	60	30	L1
30	70	20	PD, RD
70	25	35	L2, SEC

Game 14

4	128	15	
50	0	0	DOM
70	35	35	L1, SEC
0	25	55	PD, RD

Game 15

43	77	27	
75	40	45	Unique NE
70	15	100	L1
70	60	0	L2

Game 16

110	22	15	
20	0	60	L1 (NE)
0	60	0	PD, RD
10	25	25	DOM

Game 17

4	10	133	
50	0	0	DOM
0	55	25	PD, RD
100	35	35	L1, SEC

Game 18

72	28	47	
25	30	100	L1
60	31	51	Unique NE
95	30	0	L2

Game 19

29	97	21	
80	60	50	PRD
60	70	90	L1, SEC, RD
0	0	100	PD

Game 20

2	11	134	
55	0	25	PD, RD
50	50	30	
35	100	35	L1, SEC

Fig. 1. The game matrices and player choices. Below are the game matrices of the experiment sorted into categories but numbered according to the order in which they were presented to participants on the screen. Above each column of each matrix is the number of players who chose the corresponding row of that matrix. Key: PD = payoff dominant Nash equilibrium strategy; RD = risk dominant Nash equilibrium strategy; PRD = pairwise risk dominant strategy (only indicated when distinct from RD); SEC = security Nash equilibrium strategy; L1 = level-1 strategy; L2 = level-2 strategy (only indicated when distinct from L1); DOM = dominated strategy.

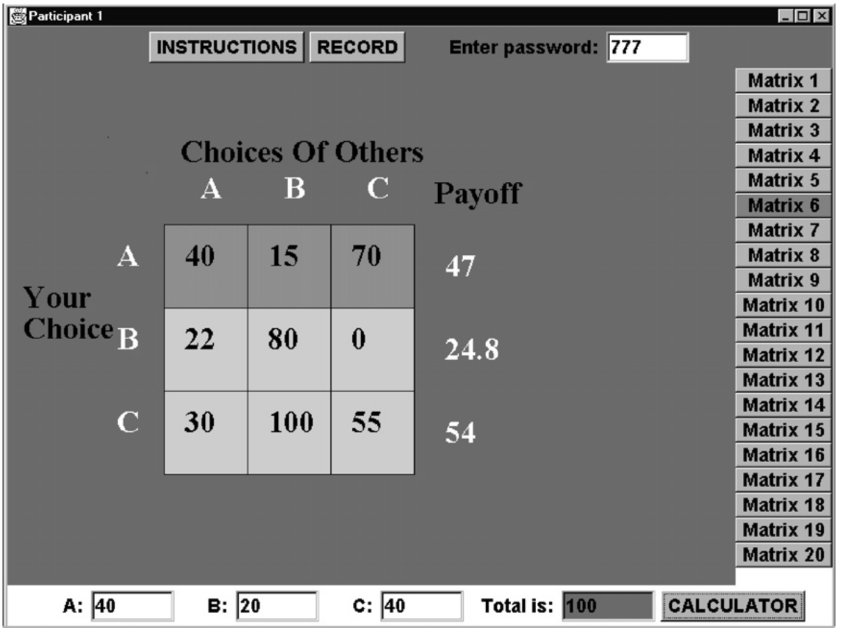

Fig. 2. The computer screen.

of the games have been played. The total amount of time allotted for all 20 games was 30 min. Within the time allotted, players could revisit any game and revise their choices.

Six sessions were conducted, with 23 subjects in the first session, 24 subjects in the second session, and 25 subjects in the remaining four sessions. The subject pool for the first four sessions consisted of upper division and graduate students in the fields of social sciences and natural sciences (economics graduate students were not permitted to participate) at the University of Texas. The subject pool for the last two sessions consisted of fourth and fifth year accounting and finance students at the University of Texas. Furthermore, all payoff matrices in the last two sessions were rescaled to 0–100 when necessary and rounded off to the nearest multiple of 5.[8] Each session lasted 1 h and 30 min. Average payments for the four sessions were US$ 22.48, 23.46, 23.16, 22.84, 30.72, and 31.80, with standard deviations of US$ 4.44, 4.62, 3.46, and 3.34, 6.38, and 6.48, respectively.[9]

Allowing participants to visit and revisit games in any temporal order they desired was intended to attenuate any effects of our arbitrary ordering of the games. However, after conducting the first two sessions, we observed that many participants visited the games in the order in which they were listed with few revisits. To neutralize whatever order effects may be present in the first two sessions, we exactly reversed the order of the games for the next two sessions of the experiment. The fifth session had all participants face the game in increasing order and the sixth session had the participants face the games in the reverse order.

[8] Note two exceptions: (1) in game 1, cell (1,1) was rescaled to 62.5 yet rounded off to 60 instead of 65; (2) in game 11, cell (1,3) was rescaled to 92.9 yet rounded off to 90.

[9] The differences between the first four sessions and the last two sessions are due to different lottery prizes; the general student body faced binary lotteries worth US$ 2, and the business school pool faced binary lotteries worth US$ 3.

5. Experiment results and econometric testing

Subjects in sessions 5 and 6 with rescaled payoffs behaved indistinguishably from the subjects in session 1–4 without rescaled payoffs. Therefore, prior to parameter estimation, all game payoffs were rescaled to [0,100], which increased the maximized likelihood of the data.

5.1. Description of the experimental data

An initial inspection of players' choices (presented above the game matrices in Fig. 1) reveals a significant amount of coordination failure in first-period play. We observe that in some games, players display a great degree of heterogeneity in actions (game matrices 3, 15, 18 and 19 in Fig. 1) whereas in other games there seems to be almost a consensus (game matrices 2, 5, 8, 10, 14 and 17 in Fig. 1).[10]

We define the "hit rate" of a particular selection principle as the percentage of players who choose the action predicted by that selection principle. The hit rates for payoff dominance, risk dominance and security selection are 26%, 44%, and 65%, respectively, for the entire set of fourteen coordination games.

Looking at the hit rates for the deductive selection principles in each of the coordination games (Fig. 3), we notice that no one deductive selection principle can explain the majority of choices in more than ten games. We define the "success rate" of a selection principle as the percentage of games for which the rule successfully predicts the action played by the majority of players (i.e., the percentage of games for which the hit rate was greater than 50%). We have success rates of 27%, 50% and 71% for payoff-dominance, risk-dominance and security selection, respectively. These statistics are unsatisfactory for real predictive ability. Furthermore, each of the deductive selection principles has at least one game for which it predicted correctly 0% of the actions taken by players. Yet, if we are forced to choose a unique deductive selection principle on the basis of which we make predictions, it would seem that security is the strongest candidate in terms of hit rates and success rates out of all of the deductive selection principles.

Given the disappointing performance of deductive equilibrium principles, we turn to bounded rationality theories as a possible prediction tool for initial conditions. The simplest model of bounded rationality is a homogeneous model consisting of only the level-1 rule. The hit rate of level-1 bounded rationality is 83%. The success rate of level-1 bounded rationality in predicting the action played by the majority of players is 100%. The lowest hit rate for level-1 occurs in game 19 with a hit rate of 66%. This is significantly better than the lowest hit rate for any selection principle.

5.2. Testing of homogeneous population models

Though hit rates and success rates are indicators of predictive power, a more rigorous comparison must be made. Recall that econometric specification of equilibrium selection principles (Section 3.1) is problematic due to the need for a model of errors. Two alternatives were discussed; namely, the prior-based and the uniform tremble approaches. Further, we specified two prior-based

[10] Games 14 and 17 show substantially different behavior from that found by Cooper et al. (1990). However, Cooper et al. were interested only in limit behavior in a repeated setting, using a pairwise matching protocol. Therefore, even if we had the initial period data from Cooper et al., there are substantial reasons initial play might differ.

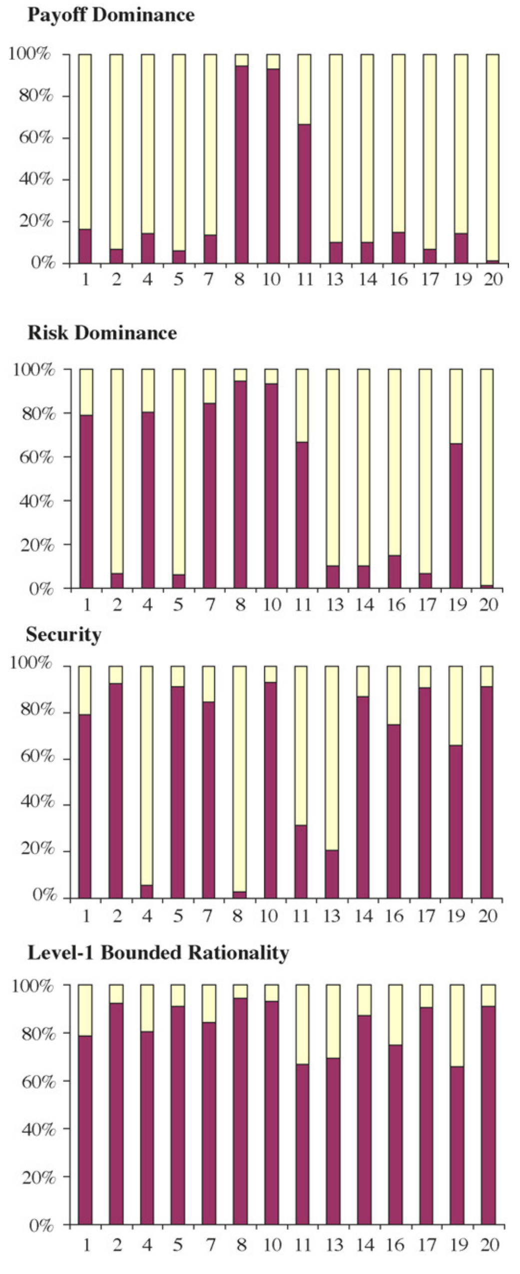

Fig. 3. Performance of pure behavioral rules. The following charts show the hit rate per game of the three Nash equilibrium selection rules of payoff dominance, risk-dominance, and security, compared to the bounded rationality rule of SW's level-1. The x-axis lists the 14 coordination games. On the y-axis is the hit rate of a given rule per game. The hit rate is the percentage of players who chose the action in a given game that the person using a given rule without error would choose with probability 1.

Table 1
Estimation of homogeneous models

Behavioral rule	ν estimate	γ estimate	Log-likelihood
Models using prior-based evidence			
PD and RD	0.041	0.000	−2902.60
Security	0.038	0.850	−2846.13
Behavioral rule	**ε estimate**	**γ estimate**	**Log-likelihood**
Models with uniform trembles			
PD and RD	0.843	0.000	−3171.39
Security	0.644	0.177	−2921.97
Behavioral rule	**ν estimate**		**Log-likelihood**
Bounded rational model			
Level-1	0.0952		−2002.01

For comparison, the log-likelihood for the homogeneous model with random behavior is −3229.92.

models: one with an individual-prior interpretation and the other with the subpopulation mixture interpretation.

Table 1 gives the results for the homogeneous equilibrium selection models. The payoff-dominance and the risk-dominance rules are listed together because the results are identical.[11] The maximum likelihood estimate of γ was exactly 0 in both models, implying that neither equilibrium selection principle can help to explain the data better than (what we will henceforth call) the *uniform Nash equilibrium* (UNE) model in which each Nash equilibrium is equally likely. As can be seen, only the security principle improves the fit of the model over the UNE model.

The second group of results in Table 1 is for the uniform-error approach. We see that all selection principles under the uniform tremble approach perform a great deal worse than their counterparts under the prior-based approach. Furthermore, all selection principles under the uniform tremble approach perform worse than the UNE model. Therefore, we can reject the uniform tremble approach for homogeneous population models.

The last result shown in Table 1 is for the level-1 bounded rationality rule. The log-likelihood of −2002.01 is an enormous improvement over the next best log-likelihood of −2777.54 produced by the SEC model. This is not surprising given the hit rate analysis (Fig. 3). While the security selection model is best among the homogeneous equilibrium selection models undoubtedly because of its attention to risk, the homogeneous level-1 model is far superior.

The combined evidence of hit rates and econometric fit demonstrate that no reasonable characterization of first-period play should be made without allowing for bounded rationality behavioral rules, especially the level-1 rule. Moreover, the success of the level-1 rule sheds some light on our comparison of deductive selection rules. The property of being the maximin (over NE) and the property of having the greatest row sum are highly correlated, and hence the predictions of SEC and level-1 often coincide. The PD rule (with $\gamma = \infty$), which is not closely related to level-1, performs the worst, whereas SEC, which most closely coincides with the level-1 predictions, performs the best.[12]

[11] The results are also the same for pairwise risk dominance.

[12] The security NE and L1 actions differ in four of the coordination games (4, 8, 11, 13) and in all non-coordination games (3, 6, 8, 12, 15, 18). In addition, in game 16, two actions (A and B) are secure whereas only A is L1. In the nine

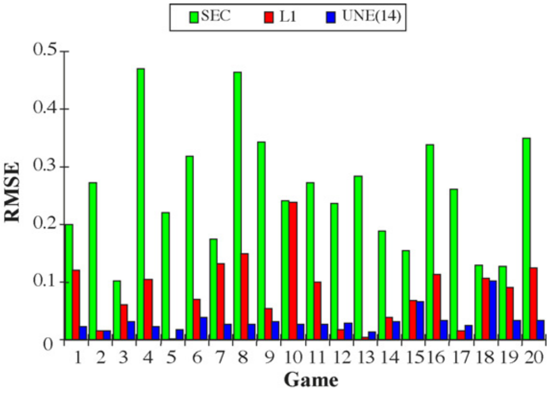

Fig. 4. Root-mean-squared-error comparisons.

5.3. Testing of heterogeneous population models

By allowing for heterogeneity in the population of players, we are able to obtain a substantial improvement in fit since players display diverse behavior in several of the games. Heterogeneous models of behavior cannot be compared using hit rates and success rates as was done previously since more than one action may be predicted for a given game. However, econometric techniques can be used to compare goodness of fit.

5.3.1. The base model

We begin by describing a model with a set of eight rules: $T = \{0, 1, 2, \text{opt}, \text{UNE}, \text{and W}\}$. Namely, we have the level-0, level-1, and level-2 boundedly rational rules, the optimistic rule, a UNE rule, and a Worldly rule. This initial model has 12 parameters: five ν scaling parameters (one for each rule except level-0 of course), a pair of Worldly parameters (μ_{W},ε_{W}), and five population proportion parameters (α_t).[13]

The maximized log-likelihood value of this 12-parameter model is −1809.207, and the parameter estimates are shown in Table 2. Compared to the best homogeneous models (Table 1), our base model increases the log-likelihood value by almost 200 points, which even with the loss of 13 degrees of freedom is strongly statistically significant ($p < 10^{-40}$). In other words, the heterogeneity allowed by the heterogeneous population model enormously improves the fit of the data. Fig. 4 shows the root mean squared error comparisons of the base heterogeneous model (labeled "UNE" in the figure), and the best two homogeneous models (labeled "level-1 only" and

games where they do coincide, the two behavioral rules are still different by virtue of the different action propensities they imply due to the use of very different evidence vectors.

[13] When estimating mixture models, identification can be difficult without some restrictions on the parameter values. For instance, without appropriate lower bounds on the ν_k scaling factors, essentially level-0 type behavior can be falsely identified as some other type with very low precision. To avoid this mis-identification, we require that each scaling factor $\nu_k \geq 0.1$. Further, as the magnitude of these scaling factors becomes large, the probabilistic choice function tends to put unit probability on one action, and the likelihood function becomes virtually flat in these parameters. To ensure quick convergence of the maximization algorithm and to prevent tiny differences in essentially zero probability choices from having unwarranted influence, we imposed an upper bound of 1.0309 on each ν scaling factor. At this upper bound, a 10-point difference in payoffs causes the higher-payoff action to be a 1000 times more likely than the other action, and it makes no practical difference in the predicted choice probabilities for the games used in our experiment.

Table 2
Estimation of heterogeneous population models

	UNE	PD	RD	SEC
ν_1	0.560	0.560	0.559	0.557
ν_2	0.480	0.489	0.488	1.031
ν_{opt}	0.331	0.331	0.330	0.317
ν_{NE}	0.918	0.918	0.918	1.031
γ	–	0.000	0.000	0.039
ν_{w}	0.248	0.248	0.248	0.283
ε_{w}	0.845	0.845	0.845	0.840
μ_{w}	0.045	0.045	0.045	0.043
γ_{w}	–	0.000	0.000	0.015
α_0	0.079	0.079	0.079	0.080
α_1	0.310	0.310	0.310	0.321
α_2	0.039	0.039	0.039	0.054
α_{opt}	0.127	0.127	0.127	0.128
α_{NE}	0.023	0.023	0.023	0.040
α_{W}	0.422	0.422	0.422	0.377
Log-likelihood	−1809.207	−1809.207	−1809.207	−1807.515

In each column, the unique Nash evidence enters the model through both the archetypal Nash rule (the row with ν_{NE}) and the Worldly rule (the row with ν_{w}). The UNE column is the base model in which each Nash equilibrium is equally likely. The other columns represent the Nash equilibrium selection principles of payoff dominance (PD), risk dominance (RD), and security (SEC). Since numbers are represented to the third decimal point, 0.000 can be thought of as some number smaller than 0.001. ν_k is the scaling parameter for evidence vector k in archetypal rule k, γ is the strength parameter for the specific equilibrium selection principle (of the corresponding column) for the archetypal Nash rules, μ_{w} is the precision parameter for the boundedly rational types in the Worldly rules, ε_{w} is the mixture parameter in the Worldly type's prior, γ_{w} is the strength parameter for the specific equilibrium selection principle (of the corresponding column) for the Worldly rules, and α_t is the proportion of population using rule t, where $t \in \{0,1,2\}$ denotes the level-t rule, t = opt denotes the archetypal optimistic rule, t = NE denotes the Nash rule, and t = = W denotes the Worldly rule.

"SEC only"). Thus, the models can be ranked as UNE being the best and SEC the worst (among these three) not only by log-likelihood values, but also by comparisons of root-mean-squared-error.

5.3.2. Testing equilibrium selection principles

We enhance the 12-parameter base UNE model by incorporating evidence from equilibrium selection principles. For the naïve Nash rules we add one parameter (γ) to the model (the same parameter for both the individual-prior and the mixture specifications). For the Worldly rules, we add one parameter (γ_{w}) to the model (the same parameter for both the individual-prior and the mixture specifications). If any equilibrium selection principle matters in a statistically significant way, then some of these additional parameters will be significantly positive.

We considered each of the three equilibrium selection principles separately, adding one principle to the naïve Nash rules and the Worldly rules simultaneously. The estimation results are shown in Table 2. It is noteworthy that the parameter estimates vary little across the alternative models. The greatest improvement in the log-likelihood value is 1.692 for the security (SEC) selection principle, but since two additional parameters are introduced, this is not statistically significant at the 5% level (the χ^2 p-value is 0.184, and the bootstrapped p-value is 0.136). Therefore, we

Table 3
Confidence intervals for the UNE model

Parameter	MLE estimate	95% confidence interval	
ν_1	0.560	0.253	1.031
ν_2	0.480	0.100	1.031
ν_{opt}	0.331	0.134	1.031
ν_{NE}	0.902	0.100	1.031
ν_{w}	0.248	0.209	0.375
ε_{w}	0.845	0.755	0.900
μ_{w}	0.045	0.031	0.060
α_0	0.079	0.048	0.101
α_1	0.310	0.203	0.437
α_2	0.039	0.000	0.112
α_{opt}	0.127	0.100	0.163
α_{NE}	0.023	0.000	0.061
α_{W}	0.422	0.280	0.508

cannot reject the hypothesis that *none of the deductive equilibrium selection principles makes a statistically significant contribution* to explaining the data.[14,15]

While SEC was dramatically better than UNE among the homogeneous equilibrium selection models (Table 1), once boundedly rational behavior is admitted into a heterogeneous model, SEC no longer makes a statistically significant contribution. This likelihood ratio test has asymptotic power of 1, and Monte Carlo simulations show that for our sample size, it has a power of 0.9 or higher against alternatives with γ and γ_{w} that are large enough to generate prediction frequencies that are distinguishable from the UNE predictions using the Pearson χ^2 test.[16]

5.3.3. UNE model parameter estimates

Table 3 presents the bootstrapped confidence intervals for the 12-parameter UNE model. To generate these confidence intervals we use the standard parametric bootstrap procedure in which 999 pseudo-data sets of 147×20 choices are generated under the hypothesis that the UNE model with the ML estimates is the true data generation process. For each pseudo-data set we find the ML estimates of the parameters, sort the 999 pseudo estimates and use the 25th and 975th ordered estimates for the confidence interval. The width of these confidence intervals indicates that even with 147×20 choice observations, the precision of the ML estimates for this model (under the null hypothesis) is not great. We attribute this imprecision to the considerable correlation among the behavioral rules in the 20 games considered, despite our effort to select games for which each type predicts different choices. Nevertheless, the predicted choice probabilities of the model are fairly insensitive to this parameter imprecision (otherwise, the likelihood function would be sensitive, thereby yielding tighter confidence intervals).

[14] We obtained a similar negative result using pairwise risk-dominance.

[15] Estimating heterogeneous mixture models in which uniform trembles are specified for the equilibrium-selection types, we find that none of these alternative models yields a maximized likelihood value greater than that for the corresponding logit specification (given in Table 2). While non-nested, both specifications entail the same number of parameters. We, therefore, also conclude that none of the equilibrium selection principles with the uniform tremble specification helps explain the empirical data.

[16] While only about 2% of behavior can be classified as pure Nash behavior, 41% of behavior is characterized as Worldly behavior, and Nash evidence is a crucial part of the Worldly specification, driving the power of the test.

Table 4
Predicted choice frequencies in game 19 by rule

Rule	*A*	*B*	*C*
Level-1	0.004	0.996	0.000
Level-2	0.008	0.992	0.000
Nash	0.000	1.000	0.000
Optimistic	0.001	0.035	0.964
Worldly	0.303	0.697	0.000

Examining the subpopulation proportion parameters (α_t), we see that the most prevalent behavioral rule is the Worldly rule (42%), and the next most prevalent rule is the level-1 rule (31%); together, these two rules account for almost three-fourths of the observed behavior. The least prevalent rules are the level-2 rule (4%) and the individual-prior Nash rule (2%), and these have questionable statistical significance since their confidence intervals contain 0. The remaining empirically significant rules are the level-0 rule (8%) and the optimistic rule (13%).

To gauge the behavioral effect of the other parameters, we present in Table 4 the predicted choice frequencies in game 19 for each behavioral rule. As is apparent, the precision parameters (ν_1, ν_2, ν_{NE}, ν_{opt}) for the level-1, level-2, Nash and the optimistic rules are sufficiently high to put virtually all probability mass on just one action. With three Nash equilibria in this game, the level-1, and Nash evidences are all the same (63.33, 73.33, 33.33), and hence action B is predicted; the level-2 evidence is (60, 70, 0), also leading to action B.

The Worldly rule, on the other hand, is the only rule that puts significant mass on action A.[17] To see how this happens, we first look at the Worldly type's model of the boundedly rational types. With a low precision of $\mu_{\mathrm{w}} = 0.045$, the predicted choice probabilities for these types is (0.354, 0.555, 0.092), which gives an evidence vector of (66.2, 68.3, 9.2). Giving this evidence a weight of 0.845 and giving the Nash evidence a weight of 0.155 yields a combined Worldly evidence vector of (65.7, 69.1, 12.9), which given precision $\nu_{\mathrm{w}} = 0.248$ yields the choice probabilities shown.

5.4. Goodness-of-fit

As a benchmark for assessing goodness-of-fit, suppose our predicted frequencies of choices for all 20 games exactly matched the actual empirical frequencies in the data; the resulting log-likelihood of −1773.705 would be the maximum possible for the experimental data. In other words, our fitted model, with a log-likelihood of −1809.207, comes remarkably close to the maximum possible for this data set. As another measure of goodness-of-fit, we computed the statistic:

$$\sum_{g \in G} \sum_{j \in J} \frac{(n_j^g - n\bar{P}_j(\alpha, g))^2}{n\bar{P}_j(\alpha, g)}, \tag{14}$$

where n is the total number of individuals.[18] The computed value is 37.77, which is distributed χ^2 with 40 degrees of freedom and has a p-value of 0.571. Hence, we cannot reject, at any

[17] Since 20% of the actual choices in game 19 are A's, the Worldly rule plays an essential explanatory role.

[18] Since in sessions 5 and 6, 50 participants faced games with the payoffs rescaled and rounded off, there are very slight differences in the predicted probabilities for 6 of the 20 games. Accordingly, in Eq. (14), is the average predicted probability vector weighted by sample size (97 and 50, respectively). Separate subsample tests yield the same conclusion.

commonly accepted level of significance, the hypothesis that the data is generated by the fitted model.

5.5. Robustness tests

To test whether the on-screen calculator significantly affects behavior and our results, we conducted two additional sessions with 24 subjects each *without the calculator*. In these sessions, as expected, there are more choices of dominated strategies (17.8% versus 3.3%) and fewer level-1 choices (60.1% versus 76.4%). These differences are statistically significant, so we cannot pool the *calculator* and *no-calculator* data sets.[19] Nash choices increase in some games and decrease in other games, but in the aggregate the percentage of choices that were consistent with some Nash equilibrium remain about the same (82.8% and 80.8%, respectively).[20] We estimated the heterogeneous population models with and without equilibrium selection principles (as in Sections 5.3.1 and 5.3.2), and again we found that *we cannot reject the hypothesis that none of the deductive equilibrium selection principles makes a statistically significant contribution* to explaining the data.

To test the predictive power of the above model out-of-sample, we estimate the model's parameters on a subset of games and use these estimates to predict the behavior in another subset of games. For this purpose, the composition of the two subsets has to be similar, and the subset used to estimate the parameters for prediction must be large enough to have a reasonable efficiency of parameter estimates. For the predicted subset (subset II), we picked six representative games (11, 17, 1, 3, 9, and 15 in Fig. 1) out of the 20 games selected for the experiment. These six games were picked a priori as representative of the entire set of 20 games. One game was picked from each of the four categories of games: dominance-solvable games, mixed Nash equilibrium games, SW litmus-test games, and pairwise versus extended risk-dominance games; two were picked from the Cooper et al. coordination games. Furthermore, we ensured that the games in each subset provided a satisfactory temporal sampling. We refer to the remaining subset of 14 games as subset I.

To test the robustness of the UNE model with respect to the games used, we conduct likelihood ratio tests using subsets I and II. The likelihood-ratio statistic of subset I relative to the full set of games is $2(1196.86 - 1196.06) = 1.60$, and the likelihood-ratio of subset II relative to the full set is $2(612.34 - 609.01) = 6.66$; these are distributed χ^2 with 12 degrees of freedom and p-values exceeding 0.85. Thus, we cannot reject the null hypothesis that the parameter estimates from the full set of games are valid for subsets I and II. Further, the likelihood-ratio statistic for predicting subset II from subset I is $2(616.10 - 609.01) = 14.18$. This statistic is distributed χ^2 with 12 degrees of freedom and has a p-value of 0.289. Thus, we cannot reject the null hypothesis that the parameter estimates from subset I are valid for subset II. In other words, the parameter estimates

[19] For the six games from SW (which also had no calculator), the choice behavior in our no-calculator treatment is statistically indistinguishable from that of the original SW data.

[20] Consistent with the decrease in level-1 behavior, the proportion of Nash choices that were not also level-1 choices increased without the calculator. Reducing the computational complexity of identifying the level-1 action (by use of the calculator) leads to more level-1 choices, perhaps because (when identified) level-1 actions are preferred to the other Nash actions, or because the presence of the calculator somehow biases choices towards the level-1 action. Investigating this effect would be interesting to pursue in future research. Nonetheless, if we want to apply what we learn in the laboratory to real-world games, we should recognize that calculators as well as staff advisors are often used in the decision process, which makes the calculator treatment relevant.

Table 5
Goodness of prediction of initial period in repeated play

Measure	Uniform	Level-1	UNE	Actual
LL	−136.40	−111.44	−101.80	−90.84
PCS	17.61	17.74	5.19	0
RMSE	0.281	0.133	0.116	0

are stable across these subsets of games, demonstrating the out-of-sample predictive power of the model.

As another test of out-of-sample predictive power, we compute the goodness-of-fit statistic, Eq. (14), for subset II based on the parameter estimates from subset I. The computed statistic of 14.52 is distributed χ^2 with 12 degrees of freedom and has a p-value of 0.269. Thus, we cannot reject at any commonly accepted level of significance the hypothesis that the data for subset II are generated by the model with estimates obtained using subset I.

Last but not least, we test how well the UNE model can predict the initial play for multiple-period sessions of a game. We use data gathered to study learning theories (Haruvy and Stahl, 2000). In those experiments, a session consisted of one game being played for 12 periods using a mean-matching protocol (comparable to that used here) with population feedback after each period. We ran 5 sessions of game 1, 7 of game 13, 5 of game 14, 10 of game 16, and 5 of game 19. We compute three goodness of prediction measures: log-likelihood (LL), root-mean-squared-error (RMSE), and Pearson's χ^2 (PCS).

Table 5 reports the *per-session* measures summed over the five games. For comparison, we also compute the measures for uniform and level-1 initials.[21] We observe from Table 5 that the UNE(12) model predictions are best by each measure, with level-1 and uniform initials second and third, respectively.[22] Thus, the UNE model can serve as a reliable theory of initial conditions for learning dynamics.

6. Conclusions

SW (1994, 1995), Nagel (1995), Bosch-Domènech et al. (2002), and Costa-Gomes et al. (2001) have all demonstrated that non-Nash behavioral rules are more salient than Nash behavior.[23] However, in games with unique Nash equilibrium, these same studies have also found a significant amount of Nash behavior. Hence, Nash behavior must be included in empirically accurate models. Unfortunately, Nash behavior is ambiguous in games with multiple equilibria. We extend the SW model to games with multiple equilibria by postulating Nash subtypes corresponding to equilibrium selection principles (PD, RD and SEC), as well as a UNE type in which the prior belief is uniform over pure Nash equilibria.

We allow for heterogeneous population mixtures that entail boundedly rational rules as well as equilibrium selection principles. By relaxing the homogeneity assumption we improve the fit significantly. Moreover, the estimated parameters are shown to be robust over subsets of games.

[21] Level-1 initials are the predictions of the homogeneous level-1 model of Table 1.
[22] By RMSE, level-1 is slightly better than uniform period 0. These results also hold game-by-game.
[23] Nagel does not report much Nash play in guessing (later known as beauty contest) games. However, with the same games, Bosch-Domènech et al. (2002) report significantly more Nash play (choice of 0 or 1) in newspaper experiments. They also report that Nash equilibrium is the largest mode in experiments with game theorists and economists.

We find that in the presence of boundedly rational rules and a simple uniform Nash equilibrium (UNE) rule, none of the equilibrium selection principles have significant explanatory power.

Our results can be combined with adaptive dynamics to arrive at a prediction of the dynamic path of play, thereby forming a complete dynamic theory of equilibrium selection. Indeed, the UNE model predicted the initial periods of out-of-sample multi-period sessions far better than a simple level-1 model and uniform initials. However, one should note that the number of parameters entailed in these initial-condition models is 12, 1 and 0, respectively. Tradeoffs of parsimony and accuracy are often important considerations in the choice of a model.

Finally, one should be cautious when applying these results to different subject pools, games, and decision framing. Bosch-Domènech et al. (2002) have shown that the general finding that the population is made up of level-*n* types is robust, whereas the proportions of each type are not robust. Likewise, Camerer et al. compare estimates across different subject pools and note that there are substantial differences in level of sophistication between high school students, community college students, cal-tech students, econ PhDs, game theorists, and other populations.

In this study, we used two different subject pools: (i) engineering and science majors and (ii) business majors, and found no behavioral differences. On the other hand, if we were to recruit high-school dropouts or experienced subjects, we would expect at least different proportions of types. This is a natural and desirable property of heterogeneous models. With regard to games, we established robustness over 20 symmetric 3×3 normal-form games with binary lottery payoffs, which gives us the confidence to expect that our model and results will apply to a fairly wide class of similar games.[24] Last, but not least, the on-screen calculator has a discernible effect. On the other hand, the aggregate amount of Nash behavior was unaffected, so we expect our results regarding equilibrium selection principles to be robust to this framing effect.

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