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Level-*n* bounded rationality and dominated strategies in normal-form games

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Abstract

Dominated strategies play a crucial role in game theory and its solution concepts. While empirical studies confirm that humans generally avoid dominated strategies, they also suggest that humans seldom believe others will avoid such strategies. Hence, the iterated dominance solution is not likely to be a good predictor of one-shot behavior. We investigate how the salience of a dominated strategy affects the extent to which players believe that others will recognize and avoid it. Level-*n* theory serves as a useful tool in this empirical investigation, as it is able to classify behavior into levels of bounded rationality and provide clear statistical tests for model comparisons. We find that even the most obviously dominated strategies do not induce consistently significant behavioral differences in a variety of one-shot games. Nevertheless, the fit of the Level-*n* model can be improved by hypothesizing that the Level-0 choices and Level-1 beliefs are tilted slightly away from the uniform distribution to the extent that the average payoff of a strategy falls below a threshold.

JEL classification: C14; C51; C72; C90

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1. Introduction

Experimental investigations have found that subjects generally avoid dominated strategies, but seldom iteratively eliminate them.[1] This paper explores the conjecture that some dominated

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[1] Nagel (1995), Stahl (1996), Stahl and Wilson (1994, 1995), Costa-Gomes et al. (2001), Sefton and Yavas (1996), and Katok et al. (2002).

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strategies are more obvious than others and that subjects will iteratively eliminate only the most obvious ones. We test this conjecture by manipulating the salience of dominated strategies in an experiment.

Dominated strategies also impact theories of bounded rationality. In particular, while the Stahl and Wilson (1994) Level-n theory and its extensions have been reasonably robust in a wide class of symmetric normal-form games,[2] it is not necessarily robust when games have obviously dominated strategies. To see this, consider game 18 of Appendix A. Recall that a *Level-1* type is assumed to believe that the other players are equally likely to choose among the available strategies, and to choose a logit best-reply to that belief. Surely strategy B is obviously so bad that no reasonable player would believe it is as likely to be chosen as A or C. Nonetheless, the pure Level-1 type believes B is as likely and, therefore, the Level-1 type is predicted to be most likely to choose A.

It is reasonable to hypothesize a modified Level-1 type that is able to do one round of deleting obviously dominated strategies and best responds to a uniform prior over the remaining decisions. Such a type, called D1, was proposed by Costa-Gomes et al. (2001), but the games were not selected to distinguish this type sufficiently from some of the other types, nor were they designed to investigate the degrees of obviousness of a dominated strategy. Selten et al. (2003) likewise discovered a strategy that involved reducing a 3×3 game to a 2×2 through a process they call “best-reply cascade” (similar to Level-n iterations) and then choosing the action with the maximum own row payoff sum in the reduced game. Such a strategy could also be viewed as a modified Level-1 type.

In Section 2, we present a benchmark population model of bounded rationality that follows the literature on hierarchical thinking (Stahl and Wilson, 1994, 1995; Costa-Gomes et al., 2001; Haruvy et al., 2001; Selten et al., 2003; Camerer et al., 2004) in positing that the population of participants is made up of several types, some of which use iterative models of other players. The parameters of this model are estimated on three existing data sets.

A new experiment was designed to test: (i) several measures of the obviousness of a dominated strategy and (ii) how the benchmark model should be modified for games with obviously dominated strategies. The estimated model of Section 2 is used to predict the choice frequencies for the new experiment. Potential model innovations that incorporate measures of obviousness of dominated strategies are tested against this benchmark prediction. The new experiment, model innovations and results are presented in Section 3.

2. A bounded rationality population model

Let U denote the $J \times J$ payoff matrix of the row player of a symmetric normal-form game, and let U_j denote the jth row. Let p^0 denote the uniform probability distribution over the J strategies, and let p^{NE} denote a uniform probability distribution over the symmetric pure-strategy Nash equilibria. Define $\mathrm{br}_k(y, \nu) \equiv \exp(\nu y_k)/\sum_{j=1}^{J}\exp(\nu y_j)$, for $k = 1, \ldots, J$, the logit best-reply to expected payoff vector y with precision ν.

We begin with the diverse-prior model of Level-n bounded rationality (Haruvy et al.), which entails several types of players. A Level-0 type has no understanding of the game and is equally likely to choose each strategy. A Maximax type has probabilistic choice function $\mathrm{br}(m, \nu)$ where $m_j \equiv \max_k U_{jk}$.

[2] Stahl and Wilson (1995), Haruvy and Stahl (in press), and Haruvy et al. (2001).

All other types are distinguished by a two-parameter prior belief:

$$q(\mu, \varepsilon_1) \equiv \varepsilon_1 \mathrm{br}(Up^0, \mu) + (1 - \varepsilon_1) p^{\mathrm{NE}}. \tag{1}$$

A Level-1 type believes all other players are Level-0 types; hence, $\varepsilon_1 = 1$ and $\mu = 0$. A Level-2 type believes all others are Level-1 types; hence, $\varepsilon_1 = 1$ and $\mu > 0$. A Nash type believes all others will play p^{NE}; hence, $\varepsilon_1 = 0$. Finally a Worldly type believes all others are noisy Level-1 or Nash types; hence, $0 < \varepsilon_1 < 1$ and $\mu > 0$.

To allow for diversity in the population of participants, we assume that the actual beliefs of an individual player come from a distribution that is a convex combination of the uniform distribution over the simplex (with weight ε_0) and a truncated normal distribution with type-dependent mean $q(\mu, \varepsilon_1)$ and standard deviation σ. Letting $f(z|\mu, \varepsilon_0, \varepsilon_1, \sigma)$ denote this distribution of priors, the expected population choice probabilities are:

$$P^e(\nu, \mu, \varepsilon_0, \varepsilon_1, \sigma) \equiv \int \mathrm{br}(Uz, \nu) f(z|\mu, \varepsilon_0, \varepsilon_1, \sigma)\,\mathrm{d}z. \tag{2}$$

Letting α_n denote the proportion of the population that is Level-n for $n = 0$, 1, and 2; α_m denote the proportion that is Maximax; α_{NE} denote the proportion that is Nash; α_w denote the proportion that is Worldly, the combined probabilistic choice function is

$$\begin{aligned} P^*(\nu, \mu_2, \mu_w, \varepsilon_0, \varepsilon_1, \sigma, \underline{\alpha}) \equiv\ & \alpha_0 p^0 + \alpha_m \mathrm{br}(\underline{m}, \nu) + \alpha_1 P^e(\nu, 0, \varepsilon_0, 1, \sigma) \\ & + \alpha_2 P^e(\nu, \mu_2, \varepsilon_0, \varepsilon_1, \sigma) + \alpha_{\mathrm{NE}} P^e(\nu, \cdot, \varepsilon_0, 0, \sigma) \\ & + \alpha_w P^e(\nu, \mu_w, \varepsilon_0, \varepsilon_1, \sigma). \end{aligned} \tag{3}$$

Three experimental data sets are used to estimate the parameters of this model. The first is from Haruvy et al. and entails 15 symmetric 3×3 games and 58 participants. The second data set is from Haruvy and Stahl and entails 20 symmetric 3×3 games and 50 participants. The third experiment was conducted for a preliminary study and entailed 15 games and 47 participants; 12 games were selected from the previous two experiments, and 3 new games had obviously dominated strategies. We exclude the 3 new games with obviously dominated strategies from the estimation exercise, leaving a 47-game data set. Of these 47 games, 35 are distinct while 12 are duplicates.

To estimate the parameters we maximize the log-likelihood (LL) function for the 47-game data set. Appendix B presents the parameter estimates, the t-ratios, and the variance–covariance matrix of these estimates. We will refer to the fitted model as the *boundedly rational population* (BRP) model. The maximized LL value is −1838.78. The pseudo-R^2 (entropy/LL) is 0.973. The root mean squared error (RMSE) is 0.052. The aggregated Pearson Chi-square statistic (PCS) is 91.90, which has a p-value of 0.541 (94 d.f.). Therefore, we cannot reject the hypothesis that the fitted BRP model is the data generating process for the 47-game data set.

3. Confronting obviously dominated strategies

A fourth experiment was designed to test a variety of conjectures about what makes a dominated strategy "obvious". The design has two dimensions of variation: (1) the maximum payoff a dominated strategy offers, and (2) the maximin level of the game. One conjecture is that a strategy is obviously dominated to the extent that its maximum payoff is less than the maximin payoff of the game. This experiment entailed 15 symmetric 3×3 games and 75 participants. The row player's payoffs for these 15 games are given in Appendix A along with the aggregate choices. In addition, the three new games of the third experiment are listed as games 16–18. The experimental protocols for the new experiments were exactly as in Haruvy and Stahl.

To gauge the robustness of our BRP model to obviously dominated strategies, we use the estimated BRP parameters to predict choice frequencies for the new 18-game data set and compute three goodness-of-fit measures. The LL of the data is −912.55, giving a pseudo-R^2 of 0.952. The game-averaged RMSE is a disappointing 0.084. The aggregated PCS statistic is 82.74 (p-value = 0.000015), rejecting the hypothesis that this BRP model generated the data. Five of the 18 games fail the individual PCS test (1, 9, 10, 16, and 18).

Game 1 has only 35% choosing the Level-1 action (A); deleting the dominated strategy (C), the Level-1 choice would be action B. Thus, it appears that C was "obvious enough" for Level-1 types to believe others will avoid it. However, the same logic does not appear to extend to game 2. Though strategy A appears obviously dominated to us, the BRP model predicts the choices almost perfectly (PCS = 0.508; RMSE = 0.010), due to the Worldly type. Therefore, it is not clear how the BRP model should be modified to explain this data better.

A cursory look at the new data provides little support for the conjecture that a strategy is obviously dominated to the extent that its maximum payoff is less than the maximin payoff of the game. The eight games for which the maximum payoff of the dominated strategy is less than the maximin payoff (2, 6, 10, 12, 14, 16, 17, and 18) do not manifest dramatically different behavior: the original Level-1 prediction still accounts for 292 out of 516 choices (56.6%), and it is the modal choice in six of those eight games. For the other six games with strictly dominated strategies (1, 4, 7, 8, 9, and 15), the Level-1 prediction accounts for 233 out of 450 choices (51.8%), and it is the modal choice in three of those six games.

A natural and parsimonious modification of the BRP model is to tilt the Level-0 distribution away from dominated strategies. We define the logit best-reply function with precision μ and tilt q^0 as $\mathrm{BR}_k(y, \mu, q^0) \equiv q_k^0 \exp(\mu y_k)/\sum_{j=1}^{J} q_j^0 \exp(\mu y_j)$. We then assume that Level-0 and Level-1 choice frequencies are q^0 and $\mathrm{br}_k(Uq^0, \nu)$ respectively and that the belief of a Worldly type is

$$q(\mu, \varepsilon_1, q^0) \equiv \varepsilon_1 \mathrm{BR}(Uq^0, \mu, q^0) + (1 - \varepsilon_1)p^{\mathrm{NE}}. \tag{4}$$

To test the motivating conjecture of the experiment design, let $z_j \equiv \min\{0, m_j - M\}$, where m_j is the maximum payoff of strategy j and M is the maximin payoff of the game, and assume $q^0 = \mathrm{br}(z, \gamma)$, a noisy best response to z with precision γ. All strategies whose maximum payoff is at least as great as the maximin payoff will have equal probabilities more than $1/J$, while any strategy whose maximum payoff is less than M will have a probability less than $1/J$. The parameter γ gauges how sensitive Level-0 choice probabilities are to such shortfalls. Fixing the eight parameters of the BRP model at the MLE values from Section 2 and maximizing the log-likelihood of the new 18-game data set with respect to the one new parameter, we find $\hat{\gamma} = 0.020$, LL = −908.67, PCS = 75.12, and RMSE = 0.076. While the increase in LL is statistically significant, the PCS test still fails badly (p = 0.00014).

Instead of the maximum payoff (m_j) of a strategy, we consider the average payoff of strategy j ($a_j \equiv U_j p^0$), and instead of the maximin payoff we introduce a free parameter Z. Then, $z_j \equiv \min\{0, a_j - Z\}$ and $q^0 = \mathrm{br}(z, \gamma)$. We call this the *average-payoff* (AP)-tilt model. We find $\hat{\gamma} = 0.0080$, $\hat{Z} = 38.3$, LL = −899.66, PCS = 56.85, and RMSE = 0.062. The increase in the LL is clearly statistically significant as is the decrease in the PCS, but the aggregate PCS test still fails as badly (p = 0.014). An insightful way to interpret the estimated AP tilt is that the Level-1 type believes that about 15% of the population will avoid the dominated strategy while the remaining 85% are as likely to choose the dominated strategy as any other strategy.

To illustrate the behavioral effect of the Level-0 tilt, we present here the predicted choice frequencies of both the BRP and the AP-tilt models for games 1 and 12.[3] In the first column, p^1, p^m, and p^w denote the predicted choice frequencies for Level-1, Maximax, and Worldly types respectively; p^* denotes the model prediction, and Freq denotes the actual choice frequencies. The last row (labeled GOF) gives LL less entropy, PCS, and RMSE in that order.

	BRP model			AP-tilt model		
	A	B	C	A	B	C
Game 1						
q^0	0.333	0.333	0.333	0.358	0.358	0.285
p^1	0.787	0.213	0.000	0.587	0.413	0.000
p^m	1.000	0.000	0.000	1.000	0.000	0.000
p^w	0.160	0.840	0.000	0.161	0.839	0.000
p^*	0.503	0.470	0.026	0.415	0.562	0.023
Freq	0.347	0.653	0.000	0.347	0.653	0.000
GOF	−6.17	10.97	0.140	−2.61	3.65	0.067
Game 12						
q^0	0.333	0.333	0.333	0.369	0.282	0.349
p^1	0.897	0.000	0.103	0.874	0.000	0.126
p^m	0.950	0.000	0.050	0.950	0.000	0.050
p^w	0.409	0.001	0.590	0.265	0.000	0.735
p^*	0.653	0.027	0.320	0.586	0.023	0.391
Freq	0.720	0.000	0.280	0.720	0.000	0.280
GOF	−2.41	2.90	0.048	−3.95	6.36	0.101

For game 1, we can see that the tilt of q^0 away from the dominated strategy C causes a shift in the Level-1 prediction away from A towards B, thereby improving all three goodness-of-fit measures. This tilt has no effect on the Worldly prediction because the effect of q^0 in $\mathrm{BR}_k(y, \mu, q^0)$ offsets the effect of q^0 on the belief $q^w(\mu, \varepsilon_1, q^0)$. For game 12, the tilt away from the dominated strategy B causes a shift in both the Level-1 and Worldly prediction away from A towards C, thereby deteriorating all three goodness-of-fit measures.

Since there are games in the first three data sets with strictly dominated strategies, the modified model will have an effect there as well. Fixing the parameters at the values just used, all three goodness-of-fit measures remain essentially the same. Estimating all ten parameters on the four data sets pooled (65 games) produces an insignificant LL increase (5.385, 10 d.f., and p-value = 0.836). In other words, the parameters of the AP-tilt model are robust across data sets. Furthermore, the aggregated PCS statistic of the ten-parameter model is 148.75, which has a p-value of 0.125 (130 d.f.); therefore, we cannot reject the hypothesis that the fitted AP-tilt model is the data generating process for the whole 65-game data set.

4. Conclusions

We began this investigation with the intuition that we could find an empirical criteria to determine when a strategy is so obviously dominated that no reasonable human would believe that

[3] Similar results for all 18 games are presented in Appendix C (available on the JEBO website).

others are as likely to choose it as an undominated strategy. The data from our experiments do not support the conjecture that a strategy is obviously dominated to the extent that its maximum possible payoff is less than the maximin level of the game. On the other hand, we did find that the fit of the BRP model could be improved by hypothesizing that a Level-0 type is exponentially less likely to choose a strategy to the extent that its average payoff falls below a threshold of about 40% between the minimum and maximum payoffs of the game, as well as hypothesizing a corresponding tilt in the beliefs of the more sophisticated Level-1 and Worldly types.

Interestingly, the magnitude of the tilt is equivalent to only 15% of the Level-0 types avoiding the dominated strategy. One reason that such a small tilt is adequate to explain the data is that the Worldly type in the BRP model already incorporates much of the effect. Moreover, this finding suggests that Level-*n* thinking is very deeply rooted in subjects' belief formation and approach to strategic games to the point that it can represent a real bias in behavior and generate 'obviously unreasonable' beliefs.

Although we have found that our human subjects in one-shot games seldom believe others will recognize and avoid even obviously dominated strategies, it does not follow that such dominated strategies will survive over time. Since humans do succeed in avoiding dominated strategies, when that empirical evidence is available to the players, any learning dynamic for which the beliefs are responsive to the history will eventually drive out iterated strictly dominated strategies. Nonetheless, the presence of dominated strategies at the start of the dynamic process can substantively alter the path of play, thereby affecting long-run behavior.

Appendix A

The new games and choices (underlined)

	A	B	C
1)	26	49	0
A	0	35	100
B	55	40	20
C	30	0	0

	A	B	C
2)	1	36	38
A	1	0	0
B	100	10	5
C	5	5	90

	A	B	C
3)	64	8	3
A	30	50	100
B	40	45	10
C	35	60	0

4)	62	12	1
A	55	10	100
B	5	90	5
C	15	0	0

5)	53	4	18
A	80	80	20
B	20	100	0
C	100	10	30

6)	1	41	33
A	10	5	5
B	100	30	35
C	0	80	30

7)	36	1	38
A	80	10	5
B	25	0	10
C	5	100	20

8)	0	42	33
A	15	0	0
B	0	90	10
C	100	0	20

9)	44	27	4
A	35	0	100
B	1	100	1
C	10	40	40

10) 0	12	63	22
A	0	0	38
B	55	25	40
C	35	35	43

11)	17	52	6
A	80	60	50
B	60	70	90
C	0	0	100

12)	54	0	21
A	20	100	20
B	5	5	5
C	0	5	90

13)	39	14	22
A	70	90	38
B	100	0	40
C	88	48	43

14)	1	43	31
A	10	10	10
B	15	80	15
C	100	0	30

15)	30	44	1
A	20	0	100
B	10	90	0
C	0	0	5

16)	36	11	0
A	75	10	100
B	5	90	5
C	0	1	1

17)	0	29	18
A	1	0	0
B	10	90	10
C	100	5	20

18)	13	0	34
A	10	100	10
B	0	0	0
C	5	5	90

Appendix B

Parameter estimates and statistics of BRP model[4]

	MLE	T-ratio	Variance–covariance matrix ($\times 10^{-4}$)								
			σ	ε_0	ν	μ	ε_1	α_0	α_1	α_m	α_w
σ	0.0157	2.816	0.3100								
ε_0	0.285	5.183	0.1600	30.2000							
ν	0.295	8.744	−0.4490	8.1100	11.4000						
μ	0.645	18.733	−0.0287	−0.1690	−0.1100	0.1190					
ε_1	0.843	34.844	0.2130	−0.0666	−0.5890	−0.0971	5.8500				
α_0	0.079	4.822	−0.1660	−0.4900	2.6600	−0.0568	0.1430	2.6800			
α_1	0.451	14.135	−0.1860	−5.0700	−3.1800	0.5330	0.2340	−0.1480	10.2000		
α_m	0.0552	4.128	0.0868	−2.1100	−1.4400	−0.0296	−0.4140	−0.8130	−1.1700	1.7900	
α_w	0.415	12.857	0.2650	7.6700	1.9600	−0.4470	0.0371	−1.7200	−8.8800	0.1940	10.4000

Appendix C. Supplementary data

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.jebo.2006.04.002.

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[4] We cannot reject the hypothesis of a single precision parameter ν and a single σ. In addition, we cannot reject the hypothesis that there are no Level-2 types ($\alpha_2 = 0$) and no Nash types ($\alpha_{NE} = 0$).