10. 2D Integration using the Trapezoidal and Simpson Rules

Background

The trapezoidal rule and Simpson's rule for ordinary integrals can be extended to multiple integrals.

Theorem (Trapezoidal 2D Rule) Consider $z = f(x, y)$ over the rectangle $\mathbf{R} = \{(x, y) : a \le x \le b,\ c \le y \le d\}$. Given that the interval $[a, b]$ is subdivided into m subintervals $\{[x_{i-1}, x_i]\}_{i=1}^{m}$ of equal width $h = \dfrac{b-a}{m}$ by using the equally spaced sample points $x_i = x_0 + i h$ for $i = 0, 1, 2, \ldots, m$. Also, assume that the interval $[c, d]$ is subdivided into n subintervals $\{[y_{j-1}, y_j]\}_{j=1}^{n}$ of equal width $k = \dfrac{b-a}{n}$ by using the equally spaced sample points $y_j = y_0 + j h$ for $j = 0, 1, 2, \ldots, n$.
The composite Trapezoidal rule is

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx \approx T2D(f, h, k)$$

where

$$\begin{aligned} T2D(f, h, k) = \frac{1}{4} h k \Bigg(& f(a, c) + f(b, c) + f(a, d) + f(b, d) \\ & + 2 \sum_{i=1}^{m-1} f(x_i, c) + 2 \sum_{i=1}^{m-1} f(x_i, d) + 2 \sum_{j=1}^{n-1} f(a, y_j) + 2 \sum_{j=1}^{n-1} f(b, y_j) \\ & + 4 \sum_{j=1}^{n-1} \left(\sum_{i=1}^{m-1} f(x_i, y_j) \right) \Bigg) \end{aligned}$$

It can be shown that the error term is of the form $E_{T2D}(f, h, k) = O(h^2) + O(k^2)$, that is

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = T2D(f, h, k) + O(h^2) + O(k^2).$$

Remark. The Trapezoidal rule had the pattern of weights $1, 2, 2, 2, \ldots, 2, 2, 1$ and the Trapezoidal 2D rule extends this pattern to a grid in the rectangle **R**.

$$\begin{array}{cccccccc}
1 & 2 & 2 & 2 & \cdots & 2 & 2 & 1 \\
2 & 4 & 4 & 4 & \cdots & 4 & 4 & 2 \\
2 & 4 & 4 & 4 & \cdots & 4 & 4 & 2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
2 & 4 & 4 & 4 & \cdots & 4 & 4 & 2 \\
2 & 4 & 4 & 4 & \cdots & 4 & 4 & 2 \\
1 & 2 & 2 & 2 & \cdots & 2 & 2 & 1
\end{array}$$

Theorem (Simpson's 2D Rule) Consider $z = f(x, y)$ over the rectangle $\mathbf{R} = \{(x, y) : a \le x \le b,\ c \le y \le d\}$. Given that the interval $[a, b]$ is subdivided into $2m$ subintervals $\{[x_{i-1}, x_i]\}_{i=1}^{2m}$ of equal width $h = \frac{b-a}{2m}$ by using the equally spaced sample points $x_i = x_0 + i h$ for $i = 0, 1, 2, \ldots, 2m$. Also, assume that the interval $[c, d]$ is subdivided into $2n$ subintervals $\{[y_{j-1}, y_j]\}_{j=1}^{2n}$ of equal width $k = \frac{b-a}{2n}$ by using the equally spaced sample points $y_j = y_0 + j h$ for $j = 0, 1, 2, \ldots, 2n$.

The composite Simpson's rule is

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx \approx S2D(f, h, k)$$

where

$$\begin{aligned} S2D(f, h, k) = \frac{1}{9} h k \Bigg(& f(a, c) + f(a, d) + f(b, c) + f(b, d) \\
& + 4\sum_{j=1}^{n} f(a, y_{2j-1}) + 2\sum_{j=1}^{n-1} f(a, y_{2j}) + 4\sum_{j=1}^{n} f(b, y_{2j-1}) + 2\sum_{j=1}^{n-1} f(b, y_{2j}) \\
& + 4\sum_{i=1}^{m} f(x_{2i-1}, c) + 2\sum_{i=1}^{m-1} f(x_{2i}, c) + 4\sum_{i=1}^{m} f(x_{2i-1}, d) + 2\sum_{i=1}^{m-1} f(x_{2i}, d) \\
& + 16\sum_{j=1}^{n}\left(\sum_{i=1}^{m} f(x_{2i-1}, y_{2j-1})\right) + 8\sum_{j=1}^{n-1}\left(\sum_{i=1}^{m} f(x_{2i-1}, y_{2j})\right) \\
& + 8\sum_{j=1}^{n}\left(\sum_{i=1}^{m-1} f(x_{2i}, y_{2j-1})\right) + 4\sum_{j=1}^{n-1}\left(\sum_{i=1}^{m-1} f(x_{2i}, y_{2j})\right)\Bigg) \end{aligned}$$

It can be shown that the error term is of the form $E_{S2D}(f, h, k) = O(h^4) + O(k^4)$, that is

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = S2D(f, h, k) + O(h^4) + O(k^4).$$

Remark. Simpson's rule had the pattern of weights $1, 4, 2, 4, 2, 4, 2, \ldots, 2, 4, 1$ and Simpson's 2D rule extends this pattern to a grid in the rectangle **R**.

$$\begin{matrix}
1 & 4 & 2 & 4 & 2 & 4 & 2 & \cdots & 2 & 4 & 1 \\
4 & 16 & 8 & 16 & 8 & 16 & 8 & \cdots & 8 & 16 & 4 \\
2 & 8 & 4 & 8 & 4 & 8 & 4 & \cdots & 4 & 8 & 2 \\
4 & 16 & 8 & 16 & 8 & 16 & 8 & \cdots & 8 & 16 & 4 \\
2 & 8 & 4 & 8 & 4 & 8 & 4 & \cdots & 4 & 8 & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
2 & 8 & 4 & 8 & 4 & 8 & 4 & \cdots & 4 & 8 & 2 \\
4 & 16 & 8 & 16 & 8 & 16 & 8 & \cdots & 8 & 16 & 4 \\
1 & 4 & 2 & 4 & 2 & 4 & 2 & \cdots & 2 & 4 & 1
\end{matrix}$$

Example 1. Compare the 2D Trapezoidal and 2D Simpson rule approximations to

$\int_0^1 \int_0^1 8\, e^{-x^2-y^4}\, dy\, dx$?

Solution 1.

Example 1. Compare the 2D Trapezoidal and 2D Simpson rule approximations to $\int_0^1 \int_0^1 8 e^{-x^2-y^4} dy\, dx$ that were calculated in Examples 1 and 4?

Solution 1.

The `true value` or analytic solution.

$$\int_0^1 \int_0^1 8 e^{-x^2-y^4} dy dx = 5.04756680717$$

Trapezoidal rule approximation using and m = 10 by n = 10 grid.

$$\int_0^1 (\int_0^1 (8 e^{-x^2-y^4}) dy) dx \approx 5.0360837636$$

Trapezoidal rule approximation using and m = 20 by n = 20 grid.

$$\int_0^1 (\int_0^1 (8 e^{-x^2-y^4}) dy) dx \approx 5.04469842165$$

Simpson's rule approximation using and m = 5 by n = 5 grid.

$$\int_0^1 (\int_0^1 (8 e^{-x^2-y^4}) dy) dx \approx 5.04764458608$$

Simpson's rule approximation using and m = 10 by n = 10 grid.

$$\int_0^1 (\int_0^1 (8 e^{-x^2-y^4}) dy) dx \approx 5.04757148312$$

We can compare the error in these approximations.

$$\int_0^1 (\int_0^1 (8 e^{-x^2-y^4}) dy) dx - TR[10,10] = 0.0114830435646$$

$$\int_0^1 (\int_0^1 (8 e^{-x^2-y^4}) dy) dx - TR[20,20] = 0.0028683855208$$

$$\int_0^1 (\int_0^1 (8 e^{-x^2-y^4}) dy) dx - SR[5,5] = -0.0000777789085494$$

$$\int_0^1 \left(\int_0^1 \left(8\, e^{-x^2-y^4} \right) dy \right) dx - SR[10,10] = -4.67594893744 \times 10^{-6}$$

The error for the Trapezoidal 2D rule has the form $E_{T2D}(f, h, k) = O(h^2) + O(k^2)$ where h and k are the step sizes for the variables x and y, respectively.
The error for Simpson's 2D rule has the form $E_{S2D}(f, h, k) = O(h^4) + O(k^4)$ where h and k are the step sizes for the variables x and y, respectively.
For the above examples, we have the following results.

$$E_{T2D}\left(f \frac{1}{10} \frac{1}{10}\right) = 0.0114830435646$$

$$E_{T2D}\left(f \frac{1}{20} \frac{1}{20}\right) = 0.0028683855208$$

$$E_{S2D}\left(f \frac{1}{5} \frac{1}{5}\right) = -0.0000777789085494$$

$$E_{S2D}\left(f \frac{1}{10} \frac{1}{10}\right) = -4.67594893744 \times 10^{-6}$$

Remark. Since both the step sizes were reduced by a factor of $\frac{1}{2}$ the remainder term $E_{T2D}(f, h, k)$ should be reduced by approximately $\left(\frac{1}{2}\right)^2 = 0.25$.

Remark. Since both the step sizes were reduced by a factor of $\frac{1}{2}$ the remainder term $E_{S2D}(f, h, k)$ should be reduced by approximately $\left(\frac{1}{2}\right)^4 = 0.0625$.

$$E_{T2D}\left(f \frac{1}{20} \frac{1}{20}\right) / E_{T2D}\left(f \frac{1}{10} \frac{1}{10}\right) = 0.249793$$

$$E_{T2D}\left(f \frac{1}{20} \frac{1}{20}\right) / E_{T2D}\left(f \frac{1}{10} \frac{1}{10}\right) \approx 0.25$$

$$E_{S2D}\left(f \frac{1}{10} \frac{1}{10}\right) / E_{S2D}\left(f \frac{1}{5} \frac{1}{5}\right) = 0.0601185$$

$$E_{S2D}(f\frac{1}{10}\frac{1}{10})/E_{S2D}(f\frac{1}{5}\frac{1}{5}) \approx 0.0625$$

Therefore, both the 2D Trapezoidal and 2D Simpson's rules is behaving as predicted.