

# CEE6501 — Lecture 4.3

## Matrix Bandwidth and Extending the DSM to 3D

# Learning Objectives

By the end of this lecture, you will be able to:

- Explain why global stiffness matrices for trusses are typically **sparse**
- Define **half-bandwidth** and relate it to storage and computational cost
- Show how **node numbering** changes bandwidth without changing physics
- Extend the Direct Stiffness Method (DSM) from 2D to **3D trusses**
- Write the **3D truss element** stiffness in local and global form
- State the **support constraints** needed to prevent rigid body motion in 3D

# Agenda

1. Sparsity and bandwidth in the global stiffness matrix
2. DSM in 3D: DOFs, rotations, and supports

# Part 1 — Sparsity and Bandwidth

## Sparsity (local physical connectivity)

In truss and frame models, each node connects to only a small number of neighboring elements.

As a result, each equilibrium equation involves only a few degrees of freedom. Entries in the stiffness matrix only exist where DOFs are linked by members.

This locality leads to a **sparse** global stiffness matrix  $\mathbf{K}$ : most entries are exactly zero.

- Sparsity enables memory-efficient storage
- It also enables specialized **sparse solvers** that avoid unnecessary operations

Sparsity is a direct consequence of the *physics and topology* of the structure.

## Bandwidth (connectivity meets indexing)

Sparsity describes *how many* nonzero entries exist. **Bandwidth** describes *how far from the diagonal* those nonzeros extend.

For a symmetric matrix  $\mathbf{K}$ , the **half-bandwidth** is defined as

$$NHB = \max\{|i - j| : K_{ij} \neq 0\}.$$

A smaller half-bandwidth means that nonzero entries are clustered closer to the diagonal.

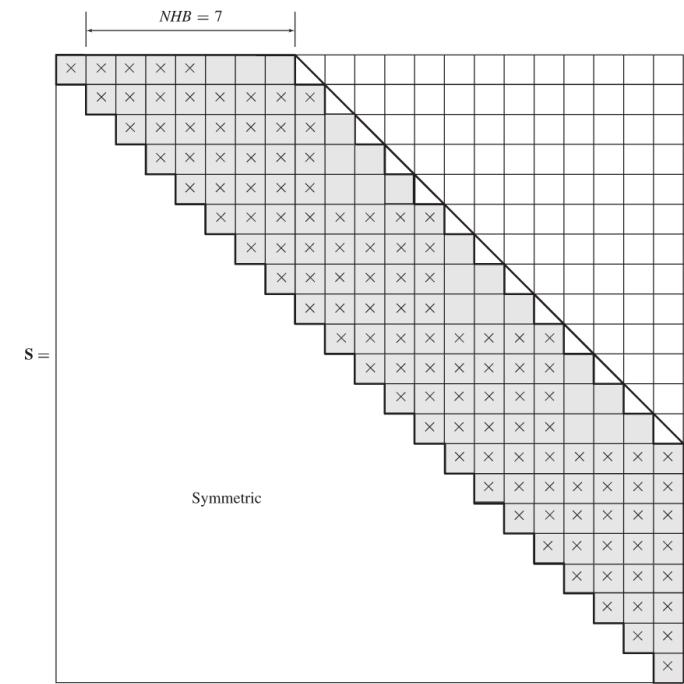
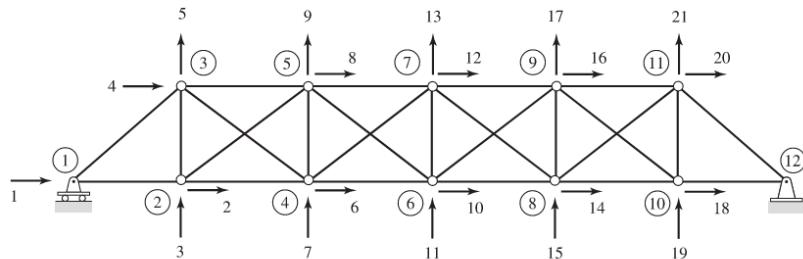
## Why node numbering matters

The physical structure is unchanged, but the **algebraic representation** of  $\mathbf{K}$  depends on the ordering of degrees of freedom.

- Poor numbering places strongly coupled DOFs far apart → **large bandwidth**
- Good numbering keeps coupled DOFs close → **small bandwidth**

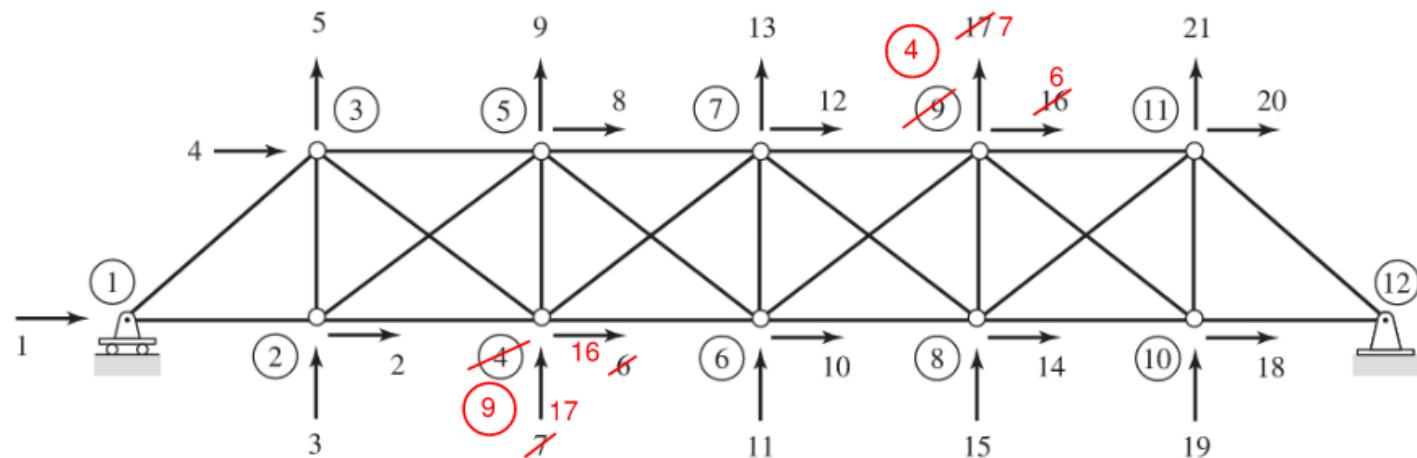
Bandwidth affects the cost of matrix factorization, even though it has no effect on the underlying physics.

# Example 1: A well-numbered system

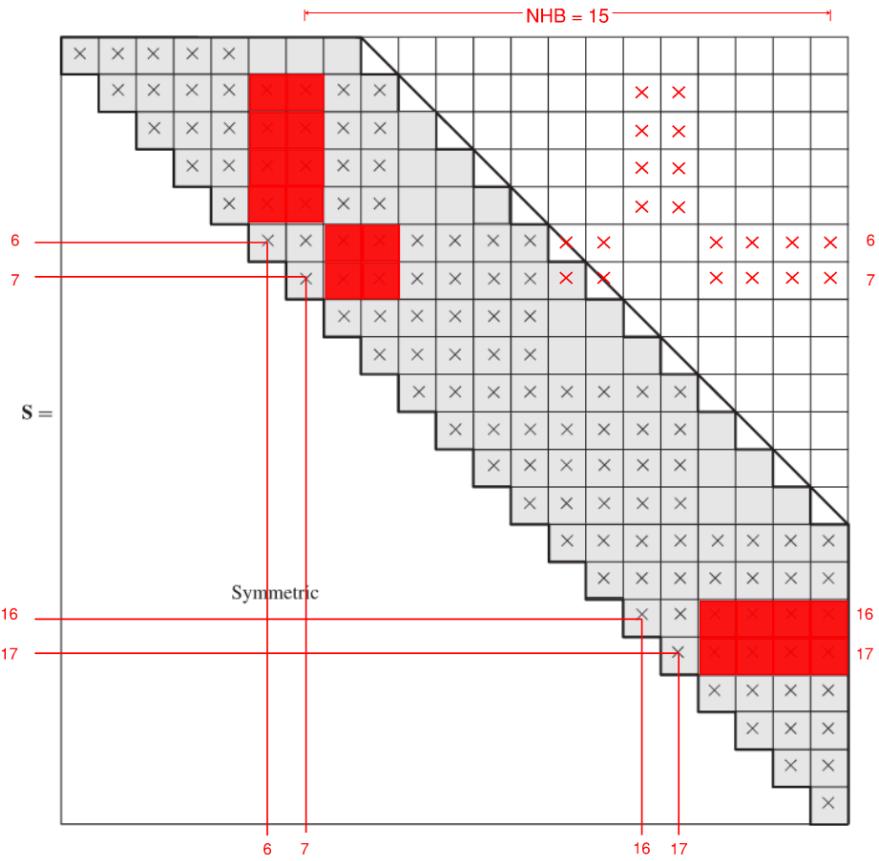
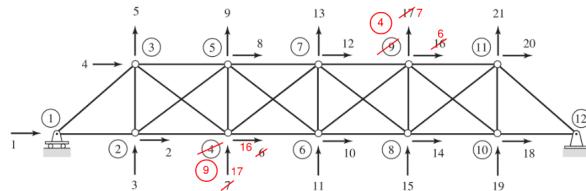


## Example 2: A poorly-numbered system

Swap Node 6 and 9, number DOFs accordingly



# Half-Bandwidth Increases from 7 to 15



## Timing Demo — Same system, different bandwidth

Goal: compare solve times for the **same underlying SPD system** under two different degree-of-freedom orderings.

We construct matrices with:

- identical size  $n$
- identical physical connectivity and sparsity pattern
- but very different **effective half-bandwidths**

This isolates the computational impact of bandwidth and ordering.

# Solver types used in the timing demo

We solve the **same linear system** using three strategies that make different assumptions about matrix structure.

## 1) Dense solve (NumPy)

```
np.linalg.solve(K, B)
```

- Treats  $\mathbf{K}$  as fully dense (LU-type)
- Ignores sparsity, bandwidth, and symmetry

## 2) Sparse LU (no reordering)

```
splu(K, perm_c_spec="NATURAL")
```

- Sparse storage, original DOF ordering
- Strongly affected by bandwidth and fill-in

## 3) Sparse LU (with reordering)

```
splu(K, perm_c_spec="COLAMD")
```

- Applies fill-reducing reordering
- Reduces bandwidth and factorization cost

*Details of sparse factorization are beyond this lecture (see Kassimali §9.9).*

Switch to demo code

## What this demo shows

- The two systems represent the **same physics**, differing only in DOF ordering
- Dense solvers ignore sparsity and bandwidth entirely (unaffected)
- Larger bandwidth typically leads to higher factorization cost for sparse solvers
- Sparse solvers are sensitive to ordering, but can reduce its impact via reordering

**Takeaway:** DOF ordering changes the algebraic structure of  $\mathbf{K}$ , which can strongly influence computational cost even though the physics is unchanged.

# Part 2 — Extending the DSM to 3D Trusses

*Same DSM workflow, but with 3 translational DOFs per node and a 3D orientation.*

## Degrees of freedom in 3D

For a 3D truss node:

$$\mathbf{d}_i = [u_{xi} \quad u_{yi} \quad u_{zi}]^T, \quad \mathbf{d}_j = [u_{xj} \quad u_{yj} \quad u_{zj}]^T$$

Element displacement vector in global coordinates:

$$\mathbf{d}_e = [u_{xi} \quad u_{yi} \quad u_{zi} \quad u_{xj} \quad u_{yj} \quad u_{zj}]^T \equiv \text{DOFs } [1, 2, 3, 4, 5, 6]$$

## 3D direction cosines

Let the element connect nodes  $i$  and  $j$  with coordinates  $(x_i, y_i, z_i)$  and  $(x_j, y_j, z_j)$ .

$$L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$$

$$l = \frac{x_j - x_i}{L}, \quad m = \frac{y_j - y_i}{L}, \quad n = \frac{z_j - z_i}{L}$$

These are the direction cosines of the element axis in global coordinates.

## 3D Truss Element Matrices

We will write:

- the **local** element stiffness  $\mathbf{k}_{local}$  ( $6 \times 6$ )
- the **transformation** matrix  $\mathbf{T}$  ( $6 \times 6$ )
- the **global** element stiffness  $\mathbf{k}_{global}$  ( $6 \times 6$ )

## Local stiffness matrix (6×6)

A truss element carries **axial force only** and therefore has stiffness only in the direction of its axis.

In a local coordinate system where the element axis is the local  $x'$  direction, only the axial DOFs are active.

Then the 6×6 local stiffness is:

$$\mathbf{k}_{local} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Only axial coupling appears; transverse DOFs have zero stiffness in a truss model.

## Transformation Matrix: Local axes and direction cosines

A 3D truss element carries **axial force only**, but we still need a consistent mapping between global and local translation components.

Let the element run from node  $i$  to node  $j$  with coordinates  $(x_i, y_i, z_i)$  and  $(x_j, y_j, z_j)$ .

Define the element differences and length:

$$\Delta x = x_j - x_i, \quad \Delta y = y_j - y_i, \quad \Delta z = z_j - z_i,$$

$$L = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}.$$

The **axial direction cosines** (local  $x'$  axis) are:

$$l = \frac{\Delta x}{L}, \quad m = \frac{\Delta y}{L}, \quad n = \frac{\Delta z}{L}.$$

## Why we need *two more* directions in 3D

In 2D, specifying the element axis automatically defines the perpendicular direction. In 3D, the axis  $(l, m, n)$  alone does **not** uniquely define a coordinate system: the element can still rotate about its own axis.

So we complete an orthonormal local basis:

- $\hat{\mathbf{e}}_{x'}$  along the member (given by  $(l, m, n)$ )
- $\hat{\mathbf{e}}_{y'}$  transverse
- $\hat{\mathbf{e}}_{z'}$  transverse

These transverse directions do **not** add stiffness in a truss (still axial-only), but they make the transformation well-defined.

## Constructing $\hat{\mathbf{e}}_{y'}$ and $\hat{\mathbf{e}}_{z'}$

Pick a reference vector  $\mathbf{a}$  that is **not** parallel to the bar axis. A simple robust rule:

$$\mathbf{a} = \begin{cases} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, & |n| < 0.9 \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, & |n| \geq 0.9 \quad (\text{bar nearly vertical}) \end{cases}$$

Define the local unit vectors:

$$\hat{\mathbf{e}}_{x'} = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

$$\hat{\mathbf{e}}_{y'} = \frac{\mathbf{a} \times \hat{\mathbf{e}}_{x'}}{\|\mathbf{a} \times \hat{\mathbf{e}}_{x'}\|}$$

$$\hat{\mathbf{e}}_{z'} = \hat{\mathbf{e}}_{x'} \times \hat{\mathbf{e}}_{y'}$$

Write the transverse direction cosines as:

$$\hat{\mathbf{e}}_{y'} = \begin{bmatrix} l_y \\ m_y \\ n_y \end{bmatrix} \quad \hat{\mathbf{e}}_{z'} = \begin{bmatrix} l_z \\ m_z \\ n_z \end{bmatrix}$$

**Meaning:**  $l_y$  is the  $x$ -component of the local  $y'$  axis,  $m_y$  is the  $y$ -component, etc.

## 3×3 rotation matrix $\mathbf{R}$ (direction cosines)

Collect the three local unit vectors (written in global components) into:

$$\mathbf{R} = \begin{bmatrix} l & m & n \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix}$$

By construction, the basis is orthonormal, so  $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ .

# 6×6 transformation matrix $\mathbf{T}$

Same definition as in 2D:

$$\mathbf{d}_{local} = \mathbf{T} \mathbf{d}_{global}$$

with the block-diagonal transformation:

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$$

Explicitly (in direction cosines):

$$\mathbf{T} = \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ l_y & m_y & n_y & 0 & 0 & 0 \\ l_z & m_z & n_z & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \\ 0 & 0 & 0 & l_y & m_y & n_y \\ 0 & 0 & 0 & l_z & m_z & n_z \end{bmatrix}$$

# Global element stiffness (6×6)

Compute:

$$\mathbf{k}_{global} = \mathbf{T}^T \mathbf{k}_{local} \mathbf{T}$$

For a 3D truss element, the result can be written directly as:

$$\mathbf{k}_{global} = \frac{EA}{L} \begin{bmatrix} l^2 & lm & ln & -l^2 & -lm & -ln \\ lm & m^2 & mn & -lm & -m^2 & -mn \\ ln & mn & n^2 & -ln & -mn & -n^2 \\ -l^2 & -lm & -ln & l^2 & lm & ln \\ -lm & -m^2 & -mn & lm & m^2 & mn \\ -ln & -mn & -n^2 & ln & mn & n^2 \end{bmatrix}$$

This is the standard 3D truss element stiffness in global coordinates.

Even though  $\mathbf{T}$  is a full 3D rotation, a truss remains axial-only. That is why the closed-form  $\mathbf{k}_{global}$  only contains  $l, m, n$  terms.

In [6]: `import sympy as sp`

```
# --- symbols ---
EA_L = 1
l, m, n = sp.symbols("l m n", real=True)
ly, my, ny = sp.symbols("l_y m_y n_y", real=True)
lz, mz, nz = sp.symbols("l_z m_z n_z", real=True)

# --- T matrix (6x6) ---
T = sp.Matrix([
    [l, m, n, 0, 0, 0],
    [ly, my, ny, 0, 0, 0],
    [lz, mz, nz, 0, 0, 0],
    [0, 0, 0, l, m, n],
    [0, 0, 0, ly, my, ny],
    [0, 0, 0, lz, mz, nz],
])

# --- local stiffness (6x6) ---
k_local = EA_L * sp.Matrix([
    [1, 0, 0, -1, 0, 0],
    [0, 0, 0, 0, 0, 0],
    [0, 0, 0, 0, 0, 0],
    [-1, 0, 0, 1, 0, 0],
    [0, 0, 0, 0, 0, 0],
    [0, 0, 0, 0, 0, 0],
])
])
```

```
In [7]: # --- global stiffness via transformation ---
k_global = sp.simplify(T.T * k_local * T)

# Pretty output (in notebook)
sp.pprint(k_global)
```

$$\begin{bmatrix} 2 & & & & & \\ 1 & 1 \cdot m & 1 \cdot n & -1 & 2 & -1 \cdot m & -1 \cdot n \\ & & 2 & & & & \\ 1 \cdot m & m & m \cdot n & -1 \cdot m & -m & -m \cdot n & \\ & & & 2 & & & \\ 1 \cdot n & m \cdot n & n & -1 \cdot n & -m \cdot n & -n & 2 \\ & & & & & & \\ 2 & & & & & & \\ -1 & -1 \cdot m & -1 \cdot n & 1 & 2 & 1 \cdot m & 1 \cdot n \\ & & & & & & \\ & & 2 & & & & \\ -1 \cdot m & -m & -m \cdot n & 1 \cdot m & m & m \cdot n & \\ & & & & & & \\ & & 2 & & & & \\ -1 \cdot n & -m \cdot n & -n & 1 \cdot n & m \cdot n & n & 2 \end{bmatrix}$$

## Supports and stability in 3D

A free rigid body in **3D** has **6 rigid body modes**:

- 3 translations
- 3 rotations

In a **3D truss model**, only **translations** are included as DOFs, but the structure can still undergo rigid body motion unless enough nodal translations are constrained.

- In **2D trusses**, at least **3 independent translational restraints** are required to prevent rigid body motion.
- In **3D trusses**, at least **6 independent translational restraints** are required.

Typical sufficient constraint patterns include:

- Fixing all three translations at one node, two translations at a second node, and one translation at a third node (provided the geometry itself is not degenerate)

# What does “degenerate geometry” mean?

A constraint set is **degenerate** if it does *not* eliminate all rigid body motion, even though the **number** of restrained DOFs is sufficient.

In other words: you may have *six* restraints in 3D, but their **locations or directions** do not block all translations and rotations.

Examples of degenerate support layouts:

- All constrained nodes lie on a **single line** → rotation about that line remains
- All constrained nodes lie in a **single plane**, allowing rotation out of the plane
- Constrained directions are **collinear** or redundant
- A “one-DOF” restraint acts in a direction that does not resist any remaining rigid motion

## Key idea:

It is not just *how many* DOFs you constrain, but *where* and *in what directions*.

If constraints are degenerate, the global stiffness matrix **K** is **singular**.