CEE6501 — Lecture 3.2

The Direct Stiffness Method (DSM) for Trusses (PART 1)

Learning Objectives

By the end of this lecture, you will be able to:

- Construct the **local-to-global transformation** for a truss member
- Compute element stiffness in global coordinates:

$$\boldsymbol{k} = \boldsymbol{T}^{\top} \boldsymbol{k}' \boldsymbol{T}$$

- Assemble the **global stiffness matrix** $\boldsymbol{K}$ using scatter-add
- Explain why an **unsupported structure** leads to a **singular stiffness matrix**
- Apply boundary conditions through **DOF partitioning** and solve for displacements
- Recover **member axial forces** from global displacements for design

Agenda

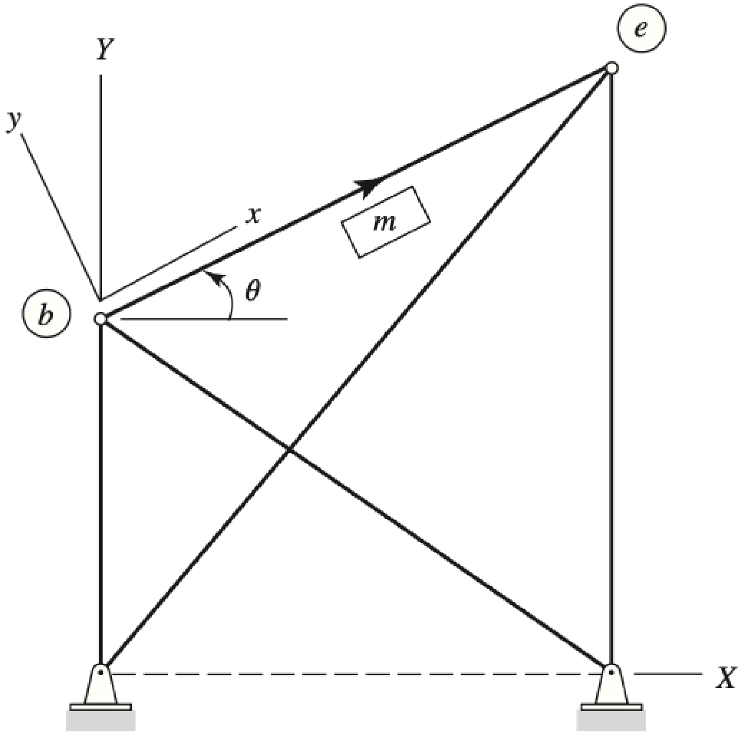
1. Local-to-global transformation using direction cosines
2. Member stiffness relations in the global coordinate system
3. Assembly of the global stiffness matrix $\boldsymbol{K}$ (manual)

Big idea:

- A truss is a network of axial springs.
- Each element contributes stiffness to shared DOFs.
- Assembly is adding contributions into the right global rows/columns.

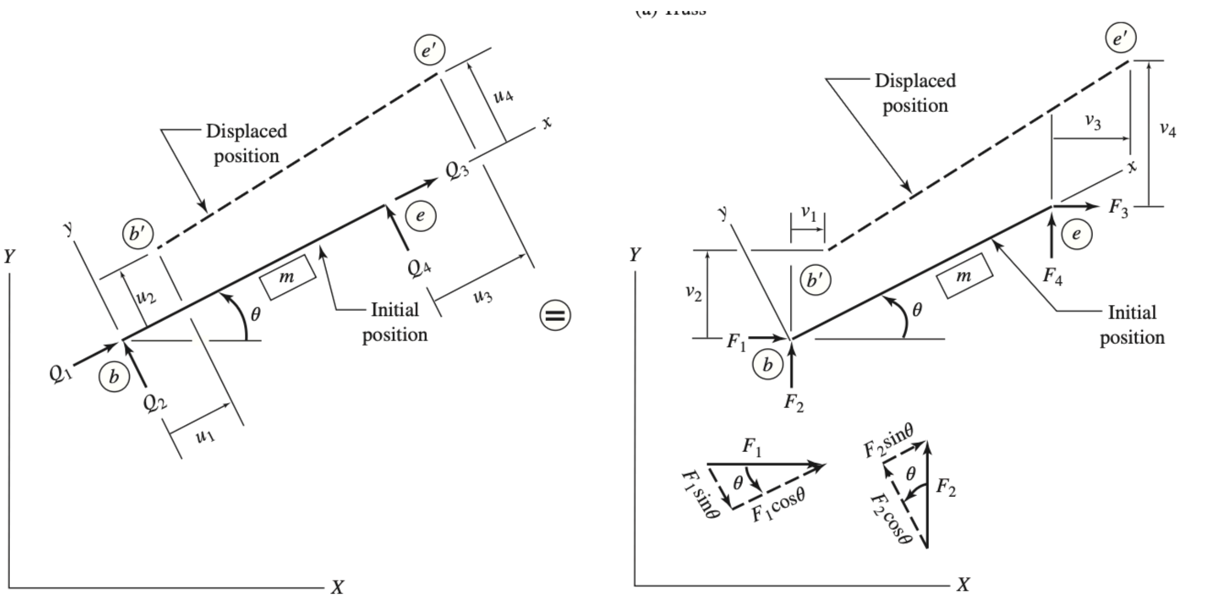
Part 1 — Local to Global Transformation

Generic Truss Element in a Structure



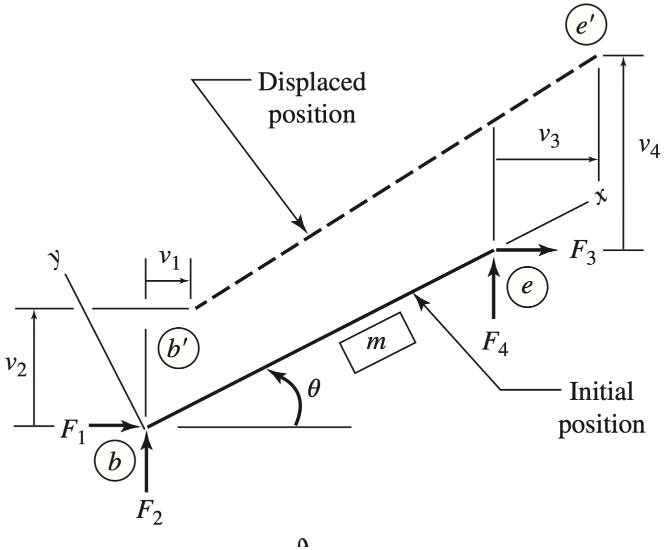
- A truss member is embedded in a **global coordinate system** (X, Y)
- The element stiffness was derived in a **local coordinate system** (x, y) aligned with the member
- The member orientation is defined by an angle θ, measured **counterclockwise** from global X to local x
- Structural assembly requires **transforming forces and displacements** between local and global coordinates

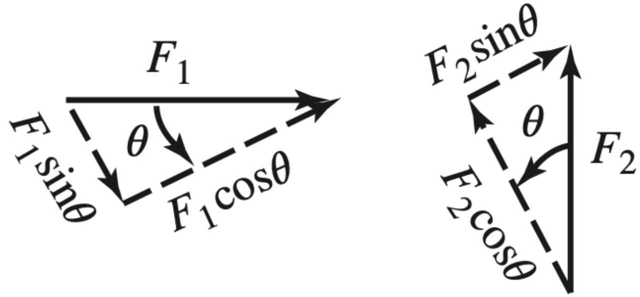
Local/Global Perspective



- **Local coordinate system (left):** forces $\boldsymbol{Q}$, displacements $\boldsymbol{u}$
- **Global coordinate system (right):** forces $\boldsymbol{F}$, displacements $\boldsymbol{v}$

Global ($\boldsymbol{F}$) → Local ($\boldsymbol{Q}$) Force Transformation: Trigonometry





At node b (start node):

- $Q_1 = F_1 \cos\theta + F_2 \sin\theta$
- $Q_2 = -F_1 \sin\theta + F_2 \cos\theta$

At node e (end node):

- $Q_3 = F_3 \cos\theta + F_4 \sin\theta$
- $Q_4 = -F_3 \sin\theta + F_4 \cos\theta$

Global ($\boldsymbol{F}$) → Local ($\boldsymbol{Q}$) Force Transformation: Matrix Form

- Local member forces $\boldsymbol{Q}$ are obtained by **rotating** global nodal forces $\boldsymbol{F}$ into the member's local coordinate system
- Each 2×2 block applies a **rotation by** θ at a node
- The transformation changes **direction only**, not force magnitude
- This operation is a **pure coordinate transformation**

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{\boldsymbol{T}} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$\boldsymbol{Q} = \boldsymbol{T}\boldsymbol{F}$$

Direction Cosines: Rotation Terms

- Direction cosines define the **orientation of a truss member** in the global (X, Y) coordinate system
- The angle θ is measured **counterclockwise** from the global X axis to the local x axis
- Computed directly from the **nodal coordinates** of the element (b = start node, e = end node)

$$\cos\theta = \frac{X_e - X_b}{\sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}}, \qquad \sin\theta = \frac{Y_e - Y_b}{\sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}}$$

- The denominator is the **member length** L
- Once computed, $\cos\theta$ and $\sin\theta$ are **reused throughout the element formulation**

Global ($\boldsymbol{v}$) → Local ($\boldsymbol{u}$) Displacements

- Nodal displacements are transformed using the **same rotation matrix** as forces
- Displacements and forces transform identically because they are defined along the **same directions**
- This is a **pure coordinate rotation**, not a change in deformation

$$\boldsymbol{u} = \boldsymbol{T}\boldsymbol{v}$$

- $\boldsymbol{v}$: global displacement vector
- $\boldsymbol{u}$: local displacement vector

Local ($\boldsymbol{Q}$) $\longrightarrow$ Global ($\boldsymbol{F}$) Force: Transformation Equations

- This is the **reverse of the global → local process**
- Local member forces are **rotated back** into the global (X, Y) directions

At node b (start node):

$$F_1 = Q_1 \cos\theta - Q_2 \sin\theta, \qquad F_2 = Q_1 \sin\theta + Q_2 \cos\theta$$

At node e (end node):

$$F_3 = Q_3 \cos\theta - Q_4 \sin\theta, \qquad F_4 = Q_3 \sin\theta + Q_4 \cos\theta$$

Local ($\boldsymbol{Q}$) → Global ($\boldsymbol{F}$) Force: Matrix Form

- Local member forces are mapped to global nodal forces using the **transpose** of the global→local transformation

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix}}_{\boldsymbol{T}^\top} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

$$\boldsymbol{F} = \boldsymbol{T}^\top \boldsymbol{Q}$$

Recall (Global → Local):

$$\boldsymbol{T} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Properties of the Transformation Matrix

$$\boldsymbol{T}^{-1} = \boldsymbol{T}^{T}$$

The transformation matrix is **orthogonal**, which greatly simplifies operations involving the stiffness transformations.

Summary — Local ↔ Global Transformations

Direction cosines (member orientation):

$$\cos\theta = \frac{X_e - X_b}{L}, \qquad \sin\theta = \frac{Y_e - Y_b}{L}$$

Global → Local (forces or displacements):

$$\boldsymbol{Q} = \boldsymbol{T}\boldsymbol{F}, \qquad \boldsymbol{u} = \boldsymbol{T}\boldsymbol{v}$$

Local → Global (forces or displacements):

$$\boldsymbol{F} = \boldsymbol{T}^{\mathsf{T}}\boldsymbol{Q}, \qquad \boldsymbol{v} = \boldsymbol{T}^{\mathsf{T}}\boldsymbol{u}$$

$$\boldsymbol{T} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

- $\boldsymbol{T}$ is a **pure rotation matrix** (no translation)
- $\boldsymbol{T}^{-1} = \boldsymbol{T}^{\mathsf{T}}$ (orthogonal)

Part 2 — Member Stiffness in the Global Coordinate System

Goal

- We have derived the **local stiffness relation** (Lecture 3.1 today):

$$\boldsymbol{Q} = \boldsymbol{k}\boldsymbol{u}$$

- We also know how to **transform forces and displacements** between local and global systems
- Objective: express the **member stiffness relation entirely in global coordinates**

Transformation Chain — From Local to Global (Step-by-Step)

Step 1 — Local force–displacement relation

$$\boldsymbol{Q} = \boldsymbol{k}\,\boldsymbol{u}$$

The element stiffness matrix $\boldsymbol{k}$ relates **local nodal displacements** $\boldsymbol{u}$ to the corresponding **local nodal forces** $\boldsymbol{Q}$.

Step 2 — Transform local forces to global forces

$$\boldsymbol{F} = \boldsymbol{T}^\top \boldsymbol{Q}$$

Global nodal forces are obtained by rotating the local force vector into the global coordinate system.

Substitute the local stiffness relation from Step 1, $\boldsymbol{Q} = \boldsymbol{k}\,\boldsymbol{u}$, into the force transformation:

$$\boldsymbol{F} = \boldsymbol{T}^\top \boldsymbol{Q} \implies \boldsymbol{F} = \boldsymbol{T}^\top (\boldsymbol{k}\,\boldsymbol{u})$$

Step 3 — Transform global displacements to local displacements

$$\boldsymbol{u} = \boldsymbol{T}\,\boldsymbol{v}$$

Substitute the displacement transformation into the previous expression:

$$\boldsymbol{F} = \boldsymbol{T}^{\mathsf{T}}\boldsymbol{k}\,\boldsymbol{u} \implies \boldsymbol{F} = \boldsymbol{T}^{\mathsf{T}}\boldsymbol{k}\big(\boldsymbol{T}\,\boldsymbol{v}\big)$$

Step 4 — Rearrange into global stiffness form

$$\boldsymbol{F} = \left(\boldsymbol{T}^{\mathsf{T}}\,\boldsymbol{k}\,\boldsymbol{T}\right)\boldsymbol{v}$$

Step 5 — Final global stiffness relation

Define the global element stiffness matrix:

$$\boldsymbol{K} = \boldsymbol{T}^{\mathsf{T}}\,\boldsymbol{k}\,\boldsymbol{T}$$

This is the element stiffness relation used directly in global assembly for the Direct Stiffness Method.

textbook notation:

$$\boxed{\boldsymbol{F} = \boldsymbol{K}\,\boldsymbol{v}}$$

our notation (interchangeable):

$$\boxed{\boldsymbol{f} = \boldsymbol{K}\,\boldsymbol{u}}$$

Calculating the Global Stiffness Matrix, $\boldsymbol{K}$

The global element stiffness matrix is obtained by **rotating the local axial stiffness** into the global coordinate system:

$$\boldsymbol{K} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Carrying out the matrix multiplication yields the **closed-form global stiffness matrix**:

$$\boldsymbol{K} = \frac{EA}{L} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

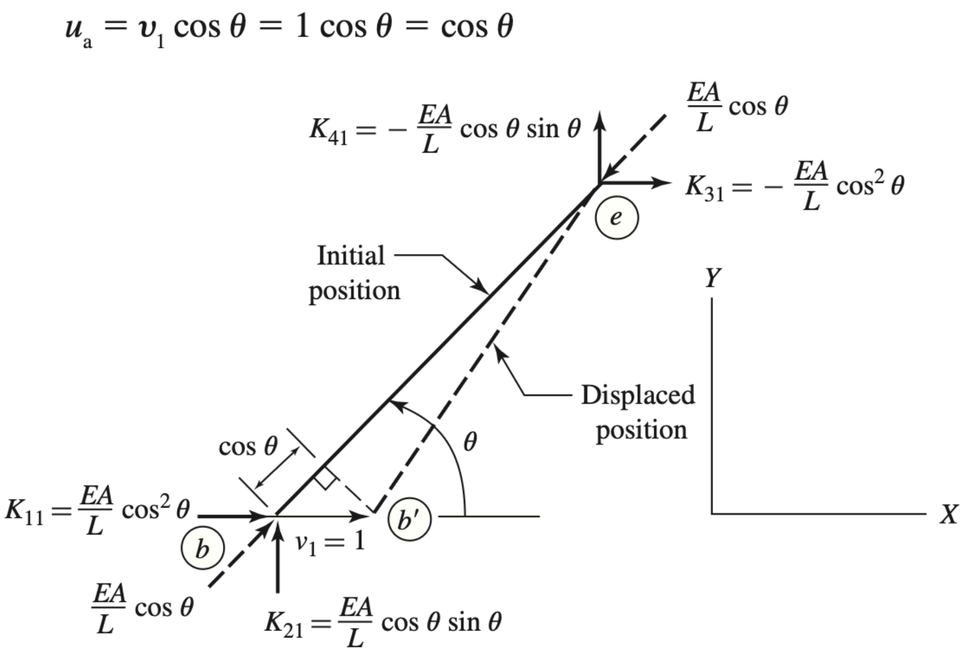
Key Observations — Global Stiffness Matrix, $\boldsymbol{K}$

- The member global stiffness matrix $\boldsymbol{K}$ is **symmetric**, just like the local stiffness matrix
- $\boldsymbol{K}$ represents the same physical behavior, but expressed in the **global** (X, Y) **coordinate system**
- Each coefficient K_{ij} is the **force at global DOF** i required to produce a **unit displacement at global DOF** j, with all other displacements fixed

Direct Calculation of Global Stiffness Matrix, $\boldsymbol{K}$

- One *could* derive $\boldsymbol{K}$ directly by:
  - Applying **unit global displacements** to a generic inclined truss member
  - Evaluating the **global end forces** required to produce each unit displacement in global coordinates
- The j**th column of** $\boldsymbol{K}$ gives the global nodal force pattern caused by $v_j = 1$
- This approach is **theoretically equivalent** to the transformation-based derivation and provides a clear physical interpretation of $\boldsymbol{K}$.
- However, it is **significantly more labor-intensive**, and is mainly useful as a **verification tool**, rather than for routine analysis.

Example: First Column of $\boldsymbol{K}$



(a) First Column of **K** ($v_1 = 1, v_2 = v_3 = v_4 = 0$)

- Impose $v_1 = 1$, all other global displacements zero
- Project the resulting axial deformation onto the member axis
- Resolve the axial force back into global components

You recover the **first column of $\boldsymbol{K}$**, which should exactly match $\boldsymbol{T}^\top \boldsymbol{k} \boldsymbol{T}$.

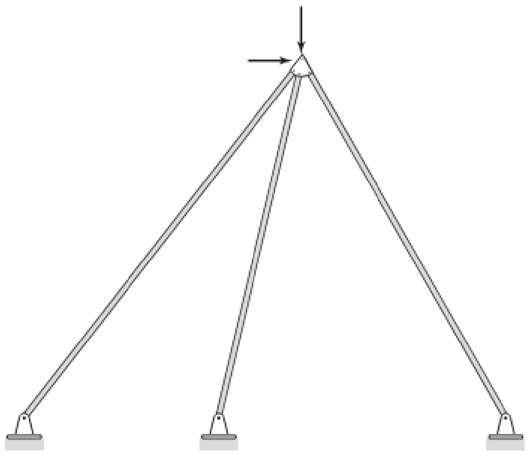
Part 3 — Assembling the Global Structure Stiffness Matrix

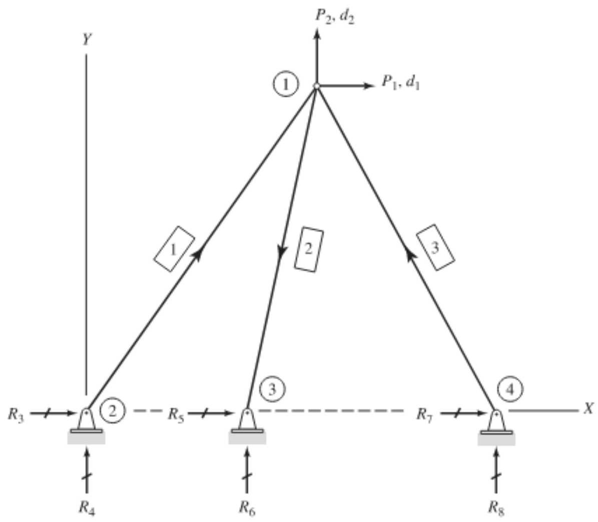
> *Manual Method*

Example Structure

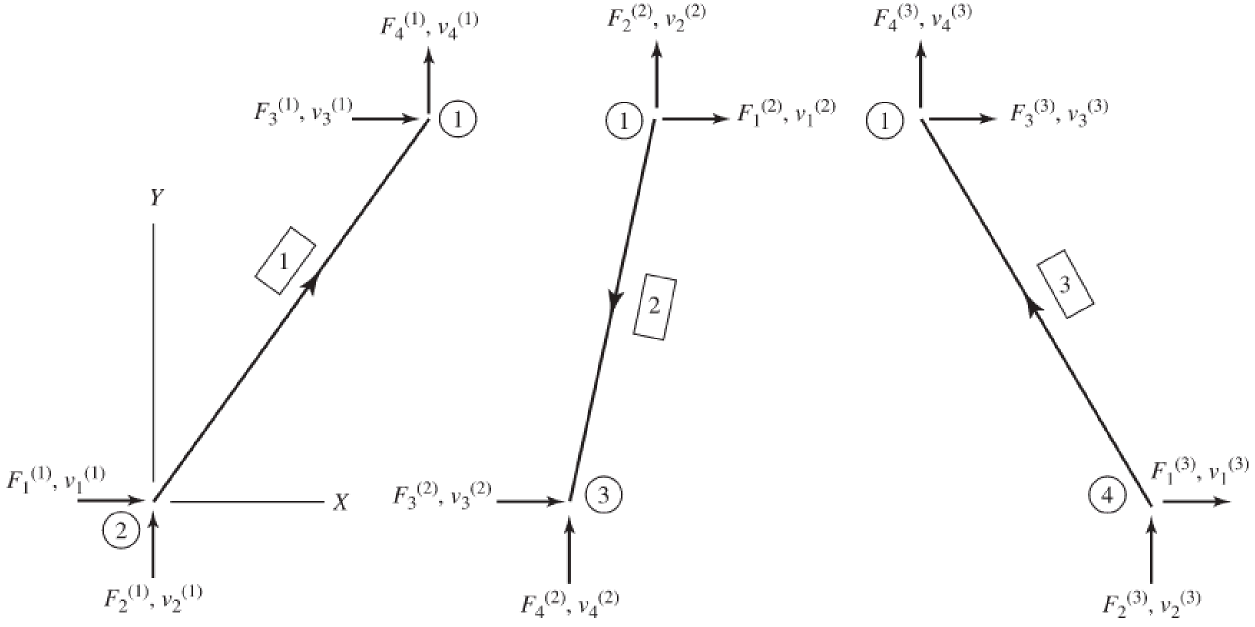
We now move from **element-level** to a **complete truss structure** behavior.

- Structure composed of **3 axial truss elements**
- **4 nodes**, each with (X, Y) translational DOFs, node 1 at the top
- Total system size: **8 global degrees of freedom**
    - node 1: DOF (1,2)
    - node 2: DOF (3,4)
    - node 3: DOF (5,6)
    - node 4: DOF (7,8)





Element-Level View of the Structure



(c)

- Nodes **2, 3, and 4** are **pinned** (no displacement)
- **Node 1** is free to move
- All three elements are connected at **node 1**
- A displacement at node 1 induces forces in **all connected members**

Element Forces and Notation

Local forces are defined **per element**:

- Superscript (e) → element number
- Subscript (i) → local DOF index

Examples:

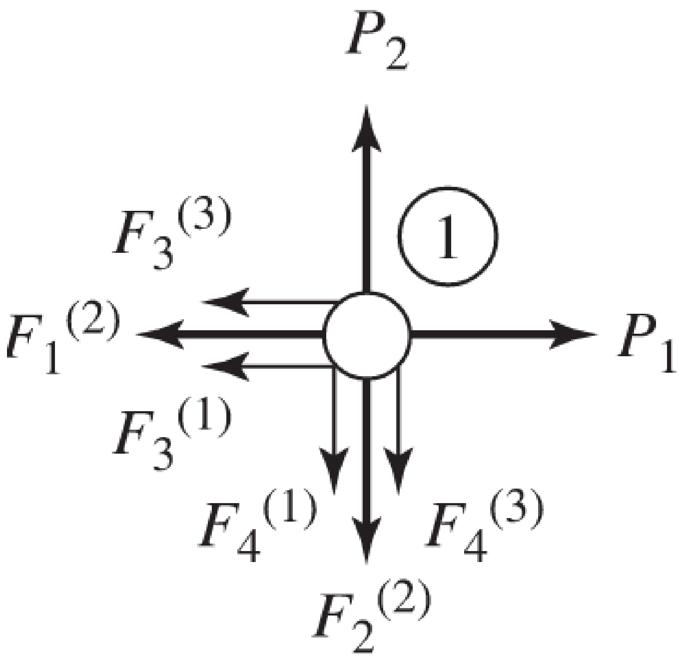
- $F_3^{(1)}$ → force at local DOF 3 in **element 1**
- $F_1^{(2)}$ → force at local DOF 1 in **element 2**

Equilibrium Equations at Node 1

Because all elements share **node 1**, equilibrium at this node couples the response of all members.

P = external applied loaded

F = internal forces

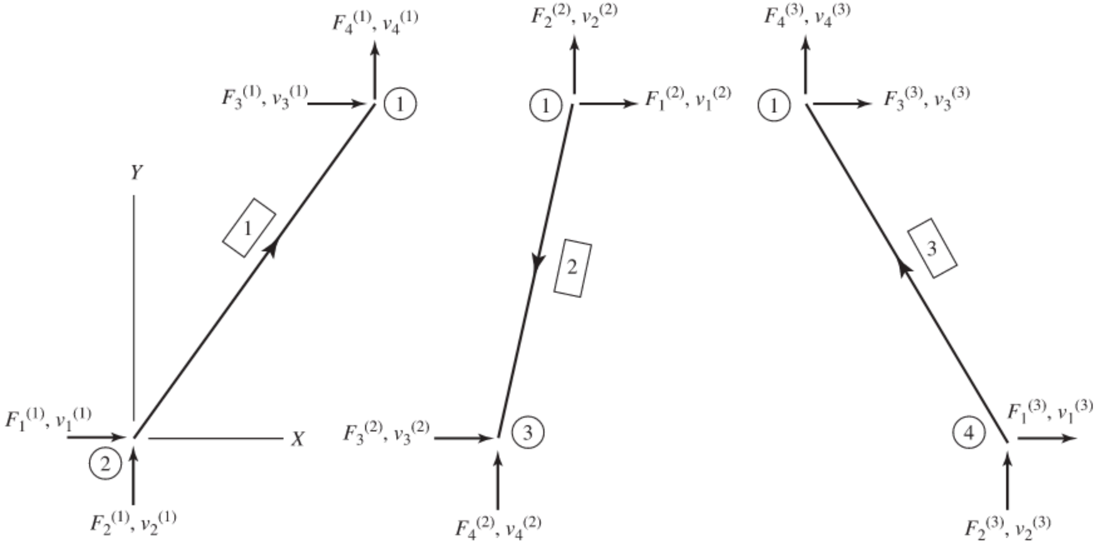


Force equilibrium at node 1:

$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)}$$

$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)}$$

Compatability Equations



(c)

d_1 and d_2 are global displacements in X and Y at node 1

$$\text{Member (1)}: \quad v_1^{(1)} = v_2^{(1)} = 0, \qquad v_3^{(1)} = d_1, \qquad v_4^{(1)} = d_2$$

$$\text{Member (2)}: \quad v_1^{(2)} = d_1, \qquad v_2^{(2)} = d_2, \qquad v_3^{(2)} = v_4^{(2)} = 0$$

$$\text{Member (3)}: \quad v_1^{(3)} = v_2^{(3)} = 0, \qquad v_3^{(3)} = d_1, \qquad v_4^{(3)} = d_2$$

Member 1 — Force–Displacement Relations

Compatibility (Member 1): $v_1^{(1)} = v_2^{(1)} = 0,\ v_3^{(1)} = d_1,\ v_4^{(1)} = d_2$

Element-level global stiffness expression:

$$\begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)} \\ F_4^{(1)} \end{Bmatrix} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & \boxed{K_{33}^{(1)}} & \boxed{K_{34}^{(1)}} \\ K_{41}^{(1)} & K_{42}^{(1)} & \boxed{K_{43}^{(1)}} & \boxed{K_{44}^{(1)}} \end{bmatrix} \begin{Bmatrix} v_1^{(1)} = 0 \\ v_2^{(1)} = 0 \\ v_3^{(1)} = d_1 \\ v_4^{(1)} = d_2 \end{Bmatrix}$$

The forces acting at **end node of member 1** (Global node 1) are:

$$F_3^{(1)} = K_{33}^{(1)} d_1 + K_{34}^{(1)} d_2$$

$$F_4^{(1)} = K_{43}^{(1)} d_1 + K_{44}^{(1)} d_2$$

Member 2 — Force–Displacement Relations

Compatibility (Member 2): $v_1^{(2)} = d_1,\ v_2^{(2)} = d_2,\ v_3^{(2)} = v_4^{(2)} = 0$

Element-level global stiffness expression:

$$\begin{Bmatrix} F_1^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \\ F_4^{(2)} \end{Bmatrix} = \begin{bmatrix} \boxed{K_{11}^{(2)}} & \boxed{K_{12}^{(2)}} & K_{13}^{(2)} & K_{14}^{(2)} \\ \boxed{K_{21}^{(2)}} & \boxed{K_{22}^{(2)}} & K_{23}^{(2)} & K_{24}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} \end{bmatrix} \begin{Bmatrix} v_1^{(2)} = d_1 \\ v_2^{(2)} = d_2 \\ v_3^{(2)} = 0 \\ v_4^{(2)} = 0 \end{Bmatrix}$$

The forces acting at **start node of member 2** (Global node 1) are:

$$F_1^{(2)} = K_{11}^{(2)} d_1 + K_{12}^{(2)} d_2$$

$$F_2^{(2)} = K_{21}^{(2)} d_1 + K_{22}^{(2)} d_2$$

Member 3 — Force–Displacement Relations

Compatibility (Member 3): $v_1^{(3)} = v_2^{(3)} = 0,\ v_3^{(3)} = d_1,\ v_4^{(3)} = d_2$

Element-level global stiffness expression:

$$\begin{Bmatrix} F_1^{(3)} \\ F_2^{(3)} \\ F_3^{(3)} \\ F_4^{(3)} \end{Bmatrix} = \begin{bmatrix} K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & \boxed{K_{33}^{(3)}} & \boxed{K_{34}^{(3)}} \\ K_{41}^{(3)} & K_{42}^{(3)} & \boxed{K_{43}^{(3)}} & \boxed{K_{44}^{(3)}} \end{bmatrix} \begin{Bmatrix} v_1^{(3)} = 0 \\ v_2^{(3)} = 0 \\ v_3^{(3)} = d_1 \\ v_4^{(3)} = d_2 \end{Bmatrix}$$

The forces acting at **end node of member 3** (Global node 1) are:

$$F_3^{(3)} = K_{33}^{(3)} d_1 + K_{34}^{(3)} d_2$$

$$F_4^{(3)} = K_{43}^{(3)} d_1 + K_{44}^{(3)} d_2$$

Global Stiffness Matrix for Free DOFs

$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)}$$
$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)}$$

Substituting element-level force expressions (from last 3 slides) into the node 1 force equilibrium gives:

$$P_1 = \left(K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)}\right)d_1 + \left(K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)}\right)d_2$$

$$P_2 = \left(K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)}\right)d_1 + \left(K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)}\right)d_2$$

These equations can be written compactly as:

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)} \end{bmatrix}}_{\boldsymbol{K}_\mathrm{s}} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$\boxed{\boldsymbol{P} = \boldsymbol{K}_\mathrm{s}\,\boldsymbol{d}}$$

Physical Interpretation of the Structure Stiffness Matrix

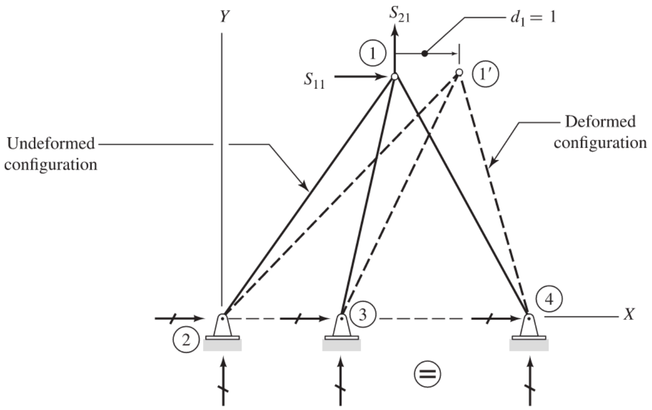
- The **structure stiffness matrix $\boldsymbol{K}_s$** has the same physical meaning as an element stiffness matrix, but at the **structure (joint) level**
- A stiffness coefficient $K_{s,ij}$ represents:
  - The **joint force** at DOF i
  - Required to cause a **unit displacement** at DOF j
  - While **all other joint displacements are zero**
- Equivalently:
  - Each **column** of $\boldsymbol{K}_s$ corresponds to a unit displacement pattern
  - The column entries are the **resulting joint forces** needed to enforce that displacement

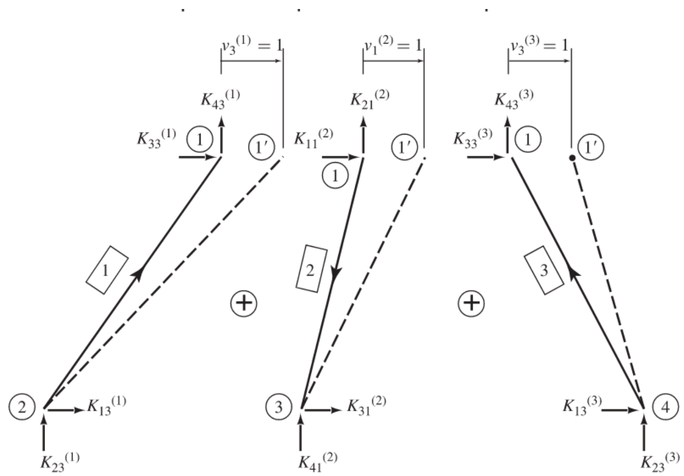
Example — First Column of $\boldsymbol{K}_s$

- To obtain the **first column** of $\boldsymbol{K}_s$:
    - Impose a **unit displacement** at the first free DOF:

$$d_1 = 1, \qquad d_2 = 0$$

    - All other joint displacements are held fixed
- The resulting joint forces (DOF 1, 2) define the **first column** of $\boldsymbol{K}_s$





Wrap-Up

Today you started building the DSM pipeline for trusses:

- local bar stiffness $\rightarrow$ transformation $\rightarrow$ global element stiffness $\rightarrow$ assembling global stiffness matrix for a structure

Next Lecture: continue building the DSM pipeline for trusses and implement DSM in Python for a worked truss example and discuss efficiency (sparsity/bandedness).