CEE6501 — Lecture 3.1

Local Behavior of an Axial Element

Learning Objectives

By the end of this lecture, you will be able to:

- Define truss DOFs and a consistent sign convention
- Explain local vs global coordinates for an axial member
- Derive the 2×2 local stiffness matrix for an axial element
- Interpret stiffness matrix properties (symmetry, rigid-body mode)
- Describe the idea of flexibility as the inverse relation (preview)
- Formulate the local 4×4 element stiffness matrix (local-only; not transformed)

Agenda

1. Definitions & concepts (DOFs, global vs local)
2. Planar trusses as a model system (DOFs, nodal vectors)
3. Truss stability & determinacy (why statics is not enough)
4. Global and local coordinate systems (notation + sign conventions)
5. Axial element kinematics (local)
6. Axial statics + constitutive relation $\left(\frac{EA}{L}\right)$
7. Local stiffness matrix (2×2)
8. Flexibility formulation (preview)
9. Local 4×4 element stiffness matrix (local; not transformed)
10. Concept checks / mini-examples (McGuire §2.6)

Part 1 — Introductory Definitions & Concepts

Degrees of Freedom (DOFs)

A **degree of freedom (DOF)** is an **independent displacement component** used to describe a structure's motion.

The set of DOFs is the **minimum set of joint displacement components** needed to uniquely describe the deformed configuration under arbitrary loading.

What DOFs Represent

In matrix analysis, DOFs are:

- the **unknown displacement components** we solve for
- the **locations/directions** where nodal loads are applied
- the coordinates used to describe the structure’s deformation

Once DOFs are defined, the response is written compactly in matrix form.

DOFs Depend on the Structural Model

The structural model determines **what motion is allowed**:

- **Trusses:** joint translations only
- **Frames:** joint translations **and rotations**
- **Higher-order models:** additional deformation modes

So “DOFs” are not universal — they depend on the assumptions built into the model.

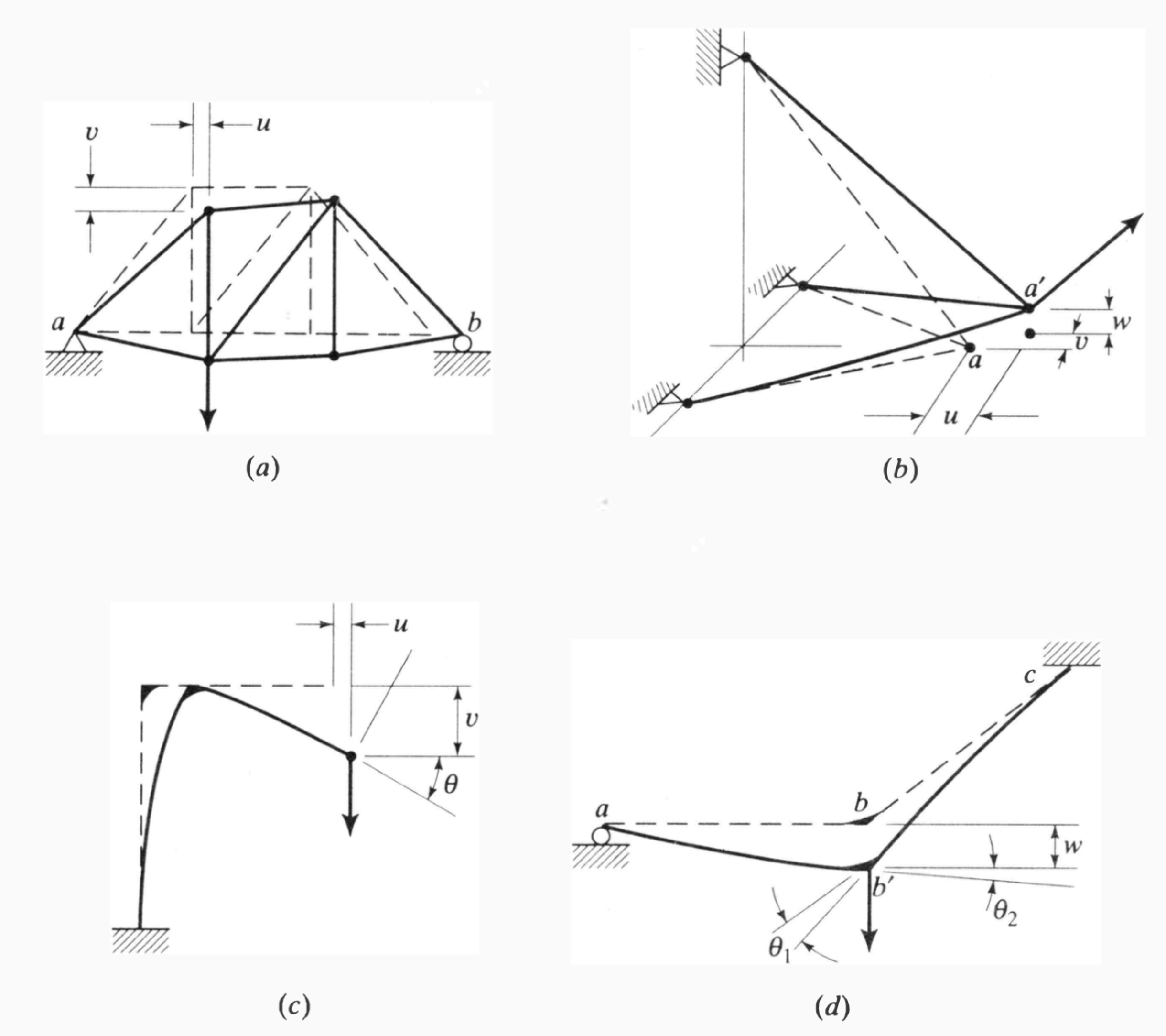


Figure 2.1 Joint displacements. (a) Pin-jointed plane truss. (b) Pin-jointed space truss. (c) Plane frame (in-plane loading). (d) Plane frame (out-of-plane loading).

Global vs Local Viewpoint

In structural analysis we constantly switch between two perspectives:

- **Global:** how the *entire structure* moves and equilibrates
- **Local:** how an *individual element* deforms internally

This distinction is fundamental to the matrix stiffness method.

Global–Local Workflow

The stiffness method follows a consistent pattern:

local element behavior
$\longrightarrow$ assemble to a **global system**
$\longrightarrow$ solve for **global displacements**
$\longrightarrow$ recover **local deformations**
$\longrightarrow$ compute **member forces / stresses**

We solve the structure **globally**, but evaluate and design **locally**.

Notation: Local vs Global

Notation: Global vs Local

We use notation to distinguish viewpoints:

- **Global (structure):** $\mathbf{u}$, $\mathbf{f}$, $\mathbf{K}$
- **Local (element):** $\mathbf{u}'$, $\mathbf{f}'$, $\mathbf{k}'$

The prime $(')$ indicates quantities defined in an **element's local coordinate system**.

Part 2 — Planar Trusses as a Structural System

What Is a Plane Truss?

A **plane truss** is a two-dimensional framework of straight members that:

- lie entirely in a single plane
- connect through **frictionless pin joints**
- carry **axial force only** (no bending or shear)
- are loaded only at the joints

Key consequence: each member is in **tension** or **compression** only.

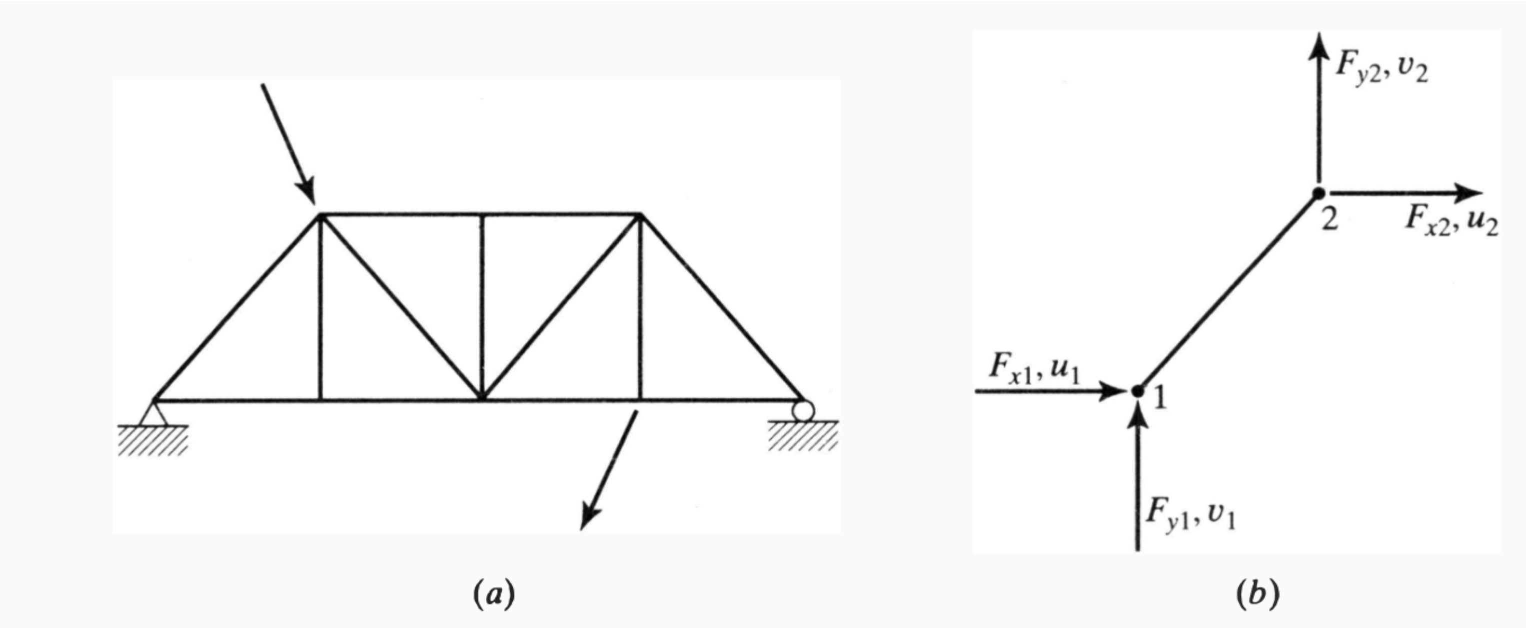


Figure. (a) Idealized planar truss (pin-jointed members). (b) Typical truss member carrying axial force.

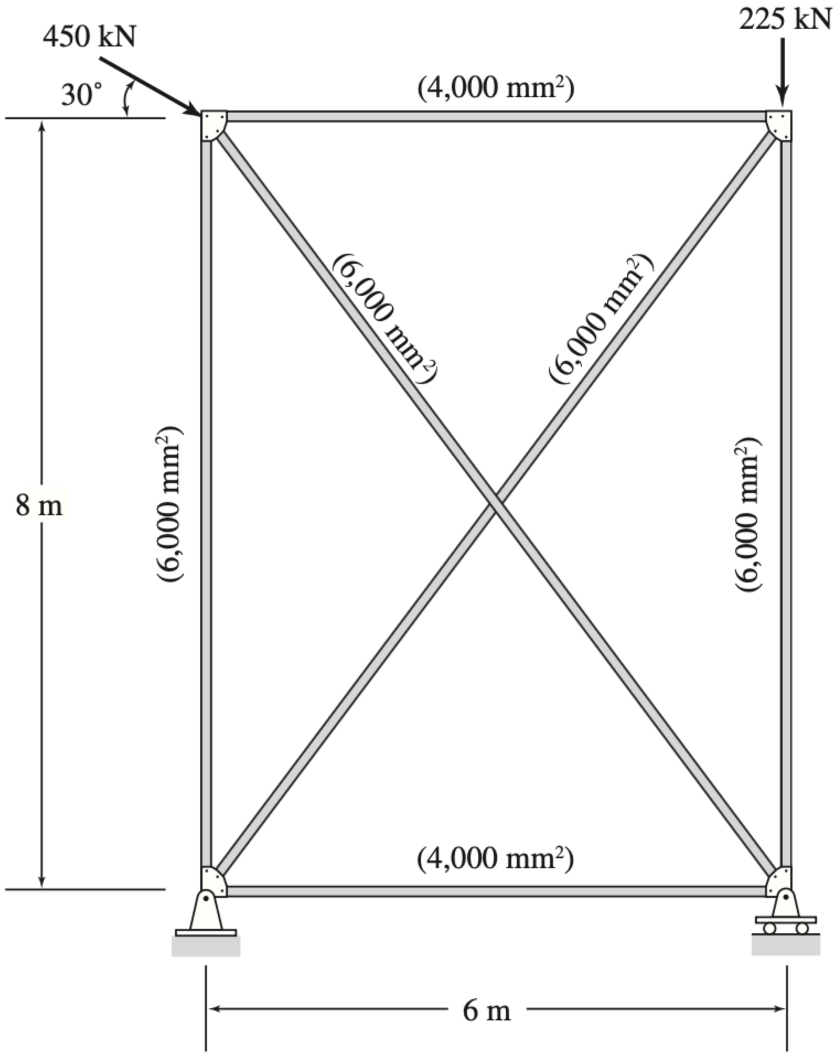


Figure. Idealized planar truss showing member properties, loads, and supports.

Why Trusses Are Ideal for Matrix Analysis

Because members carry **axial force only**:

- Each member behaves like a **1D axial element**
- Force–displacement relations are **linear and simple**
- The local behavior can be derived cleanly and assembled into a system

This makes trusses an excellent first application of the **matrix stiffness method**.

Components of a Truss Analysis Model

A truss analysis (analytical) model has four ingredients:

- **Nodes (joints):** where displacements are defined
- **Elements (members):** connect nodes and carry axial force
- **Supports:** restrain specific displacement components
- **Applied forces:** external loads acting at nodes

What we identify on the line diagram of a truss:

- **Nodes** (circled numbers)
- **Members** (element connectivity between nodes)
- **Applied loads** (known forces at DOF locations)
- **Supports** (restrained displacement components)
- **Free DOFs** (unknown joint translations)

Next we make the DOFs explicit and store them in a displacement vector.

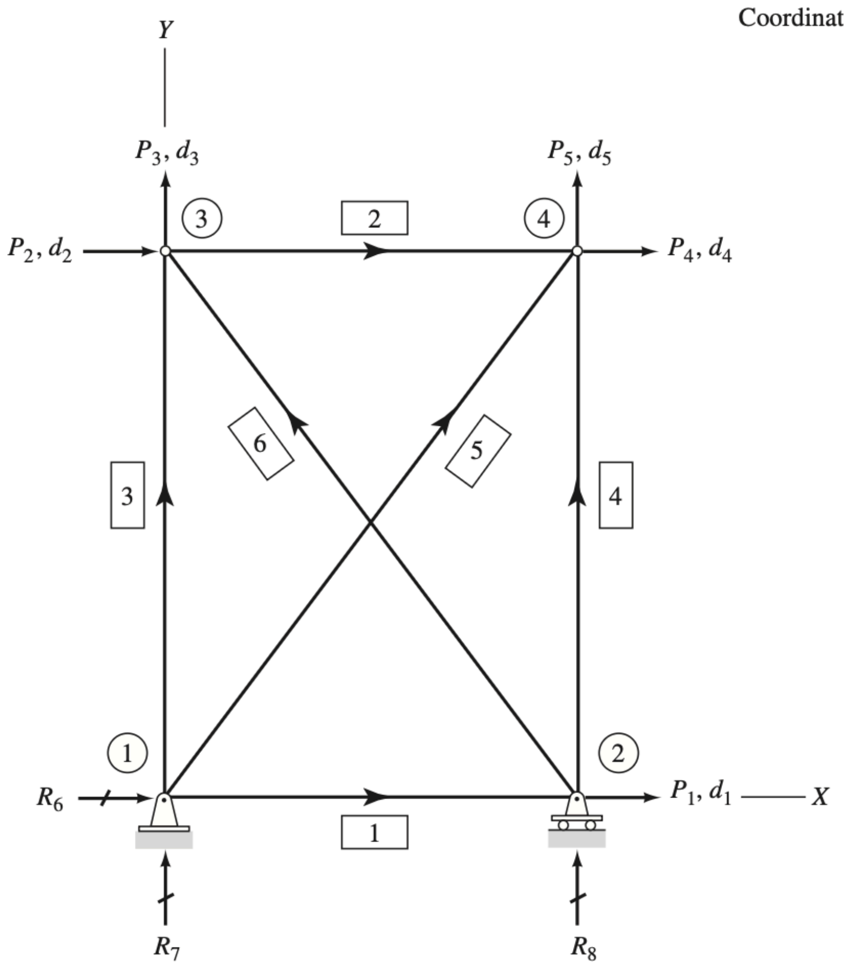


Figure. Components of a planar truss analysis model.

DOFs for a 2D Truss Node

For a planar truss joint, the possible displacement components are:

- u_x — translation in global $+x$
- u_y — translation in global $+y$

Each **free joint** therefore has **two DOFs**.

A truss with j joints has up to $2j$ joint displacement components *(before supports are applied)*.

Free vs Restrained Joints

Not all joints are free to move.

- **Free joints**
  → displacements are **unknown**
  → contribute DOFs

- **Restrained joints (supports)**
  → displacements are **prescribed** (often zero)
  → remove DOFs

Supports eliminate specific displacement components in x and/or y.

Restrained Degrees of Freedom

Each support restrains one or more displacement components:

- **Pin support**
  → restrains u_x and u_y
  → contributes **2** restraints

- **Roller support**
  → restrains **one** displacement component
  → contributes **1** restraint

We count **restrained displacement components**, not supports.

Let r be the total number of restrained joint displacement components.

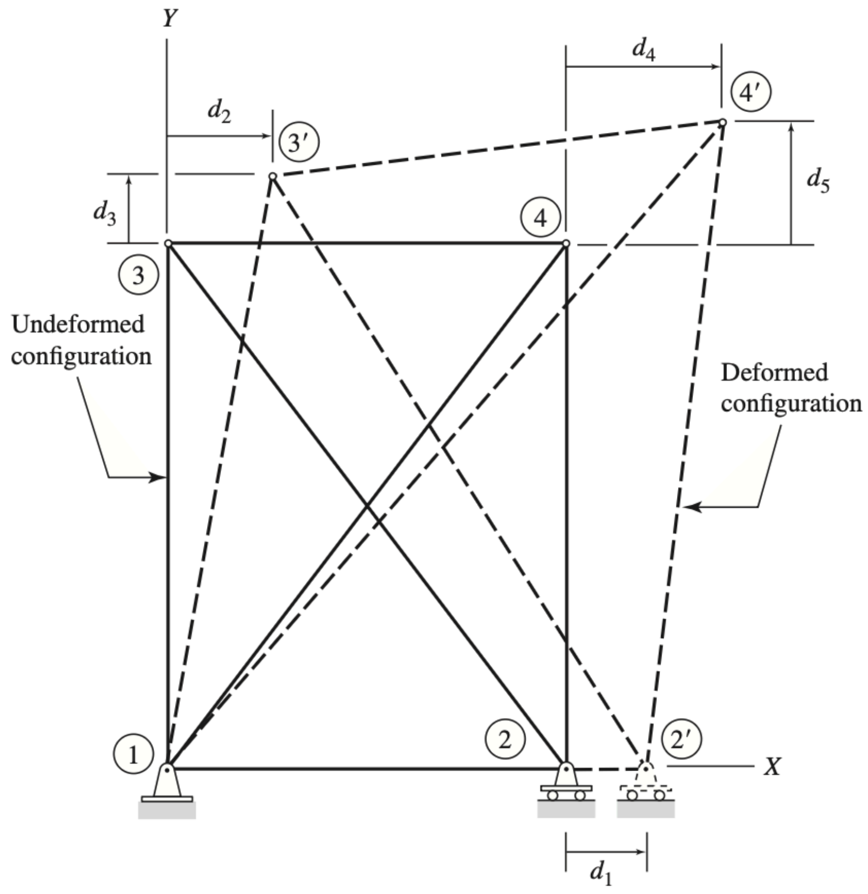


Figure. Truss joint displacements defined in the global coordinate system.

The DOFs are the **independent joint translations** that describe the deformed shape.

- d_1–d_5 are defined in the global x–y system
- Positive directions follow the global axes
- Collect them into a displacement vector $\{d\}$

Structural Vectors: Displacements and Loads

Recall the global equilibrium relation:

$$\mathbf{K}\,\mathbf{u} = \mathbf{f}$$

In matrix form, we collect the structure's unknowns and knowns into vectors.

Displacement vector

- $\mathbf{u}$ — global displacement vector
    - $\mathbf{d}$ → **free degrees of freedom** (unknown joint displacements)
    - 0 → **restrained degrees of freedom** (prescribed displacements)

Force vector

- $\mathbf{f}$ — global force vector
    - $\mathbf{P}$ → **applied nodal loads** at free DOFs
    - $\mathbf{R}$ → **reaction forces** at restrained coordinates

Applied load vector (free DOFs)

$$\mathbf{P} = \{P_1,\ P_2,\ P_3,\ P_4,\ P_5\}$$
$$= \{0,\ 389.7,\ -225,\ 0,\ -225\}$$

- $P_1 = 0$
- $P_2 = 389.7$ kN (positive x)
- $P_3 = -225$ kN (negative y)
- $P_4 = 0$
- $P_5 = -225$ kN (negative y)

Each load component corresponds directly to a numbered **degree of freedom**.

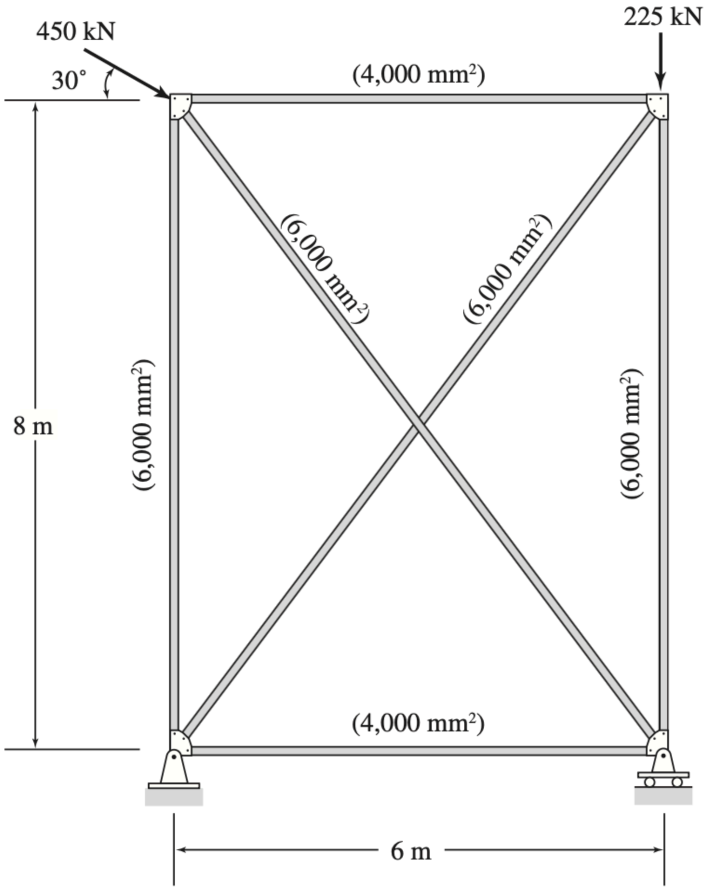
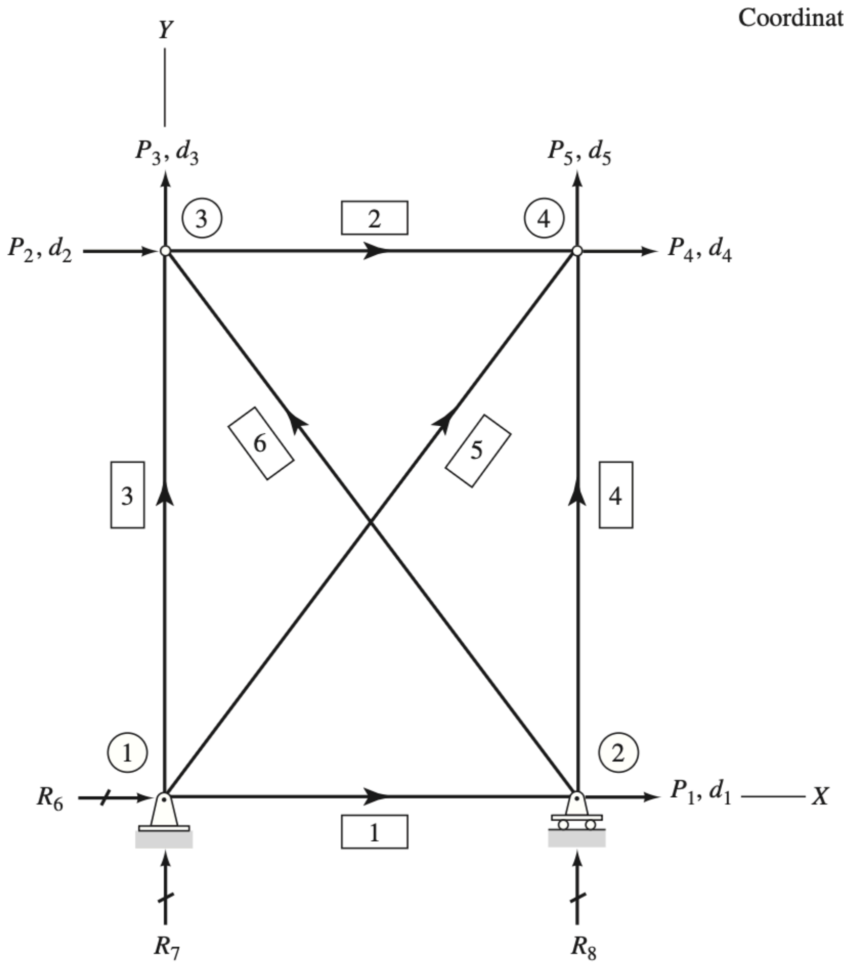


Figure. Applied nodal loads.

The global equilibrium is written as:

$$\mathbf{K}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 389.7 \\ -225 \\ 0 \\ -225 \\ R_6 \\ R_7 \\ R_8 \end{Bmatrix} \text{kN}$$



Components of a planar truss analysis model.

In-Class Exercise - Structural Vectors

1. **Number the joints** (nodes) on the figure.
2. **Number the members** (elements).
3. **Assign global DOF numbers** at each node.
4. **Identify DOF types**:
    - *Circle* free (unknown) DOFs
    - *Box* DOFs with applied loads
    - *Underline* restrained DOFs (reaction forces)
5. **Write the global displacement vector** $\{u\}$.
6. **Write the global force vector** $\{f\}$.

(Based on your DOF numbering; use variables where unknown and insert known values where prescribed.)

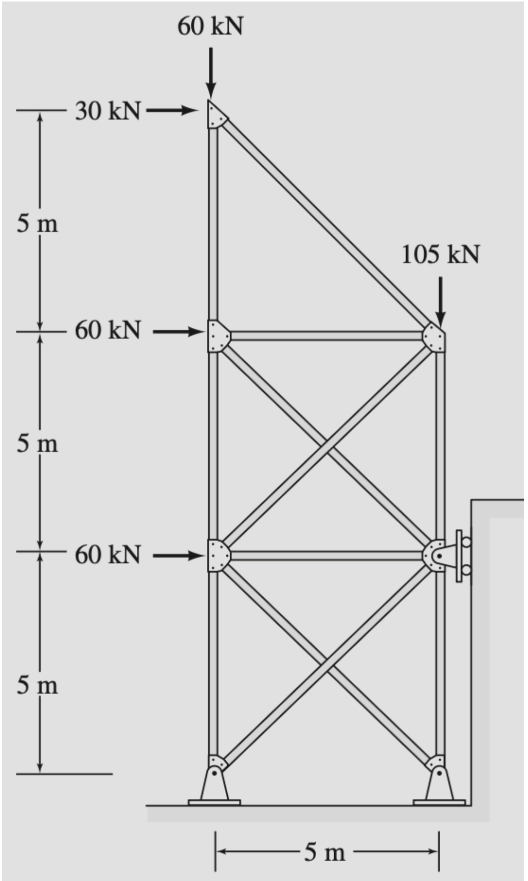


Figure. Truss structure.

Answers — Structural Vectors

$$\boldsymbol{u} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_5 \\ u_6 \\ 0 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \end{Bmatrix} \qquad \boldsymbol{f} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ 60 \\ 0 \\ r_7 \\ 0 \\ 60 \\ 0 \\ 0 \\ -105 \\ 30 \\ 60 \end{Bmatrix}$$

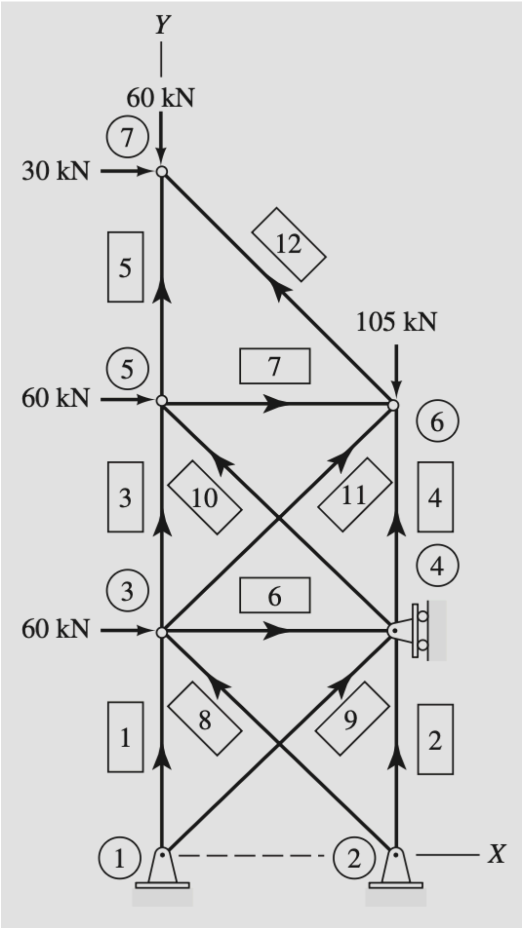


Figure. Truss structure numbered.

Part 3 - Truss Stability and Determinacy

Counting Degrees of Freedom

Let:

- j = number of joints
- r = number of restrained joint displacement components
- N_{CJT} = DOFs per free joint

Then the number of structural degrees of freedom is:

$$N_{\mathrm{DOF}} = N_{\mathrm{CJT}}\, j - r$$

DOFs for a Planar Truss

For a planar truss, each free joint has two translational DOFs:

$$N_{\mathrm{CJT}} = 2 \quad (u_x,\ u_y)$$

So:

$$\boxed{N_{\mathrm{DOF}} = 2j - r}$$

This is the number of **independent joint displacements** that must be solved for.

Stability vs Static Determinacy

- **Stability:** does the structure prevent rigid-body motion?
- **Static determinacy:** can forces be found from equilibrium alone?

These are related, but **not the same**.

We will separate:

- **stability** (rigid-body motion)
- **external determinacy** (reactions from global equilibrium)
- **internal determinacy** (member forces from joint equilibrium)

Stability Requirement (Rigid-Body Motion)

A free planar structure has **three rigid-body motions**:

- translation in x
- translation in y
- rotation in the plane

To prevent rigid-body motion, the supports must restrain at least:

$$r \geq 3$$

This is a **stability requirement**, not a determinacy condition.

- **Stability** asks whether the structure can resist rigid-body motion and admit an equilibrium configuration.
- **Determinacy** is a separate question, asked **only after stability is ensured**, and concerns whether forces can be found from equilibrium alone.

Global Equilibrium (What Statics Provides)

For a planar structure, global equilibrium provides **three equations**:

- $\sum F_x = 0$
- $\sum F_y = 0$
- $\sum M = 0$

These equations apply to the **entire structure** treated as a rigid body.

They govern **reaction forces only** (external equilibrium).

External (Global) Static Indeterminacy

Let r be the number of reaction components.

Global equilibrium provides **3 equations** in 2D.
So if:

$$r > 3$$

then reactions cannot be determined from statics alone.

The structure is **externally statically indeterminate**.

Internal Static Determinacy for a Planar Truss (Counting)

At each joint, equilibrium gives two equations:

$$\sum F_x = 0, \qquad \sum F_y = 0$$

Across j joints, that is $2j$ **joint equilibrium equations**.

Unknown forces are:

- m member axial forces
- r reaction components

A necessary counting condition for solving forces by equilibrium is thus:

$$m + r = 2j$$

Interpreting the Internal Counting Condition

The comparison

$$m + r \quad \text{vs.} \quad 2j$$

compares **unknown forces** to **joint equilibrium equations**:

- $m + r < 2j$
  → **unstable / mechanism** (not enough constraints; a deformation mode exists)
- $m + r = 2j$
  → **statically determinate by counting** *(if geometry is stable)*
- $m + r > 2j$
  → **statically indeterminate by counting** (redundant member/support forces)

This is a **counting test** — it does not guarantee stability.

What the Counting Condition Does *Not* Guarantee

Counting compares equations and unknowns, but it does **not**:

- detect rigid-body motion
- detect geometric mechanisms
- guarantee a unique solution

So passing a counting test is **necessary**, not sufficient.

Possible Outcomes

A structure can therefore be:

- **Stable but statically indeterminate**
  - stable (no rigid-body motion)
  - but has redundancies: $m + r > 2j$ or $r > 3$
  - forces depend on deformation compatibility
- **Unstable but satisfy** $m + r = 2j$
  - the count matches, but the geometry forms a mechanism
  - equilibrium equations exist, but the structure can move without resistance

Why the Direct Stiffness Method is Powerful

Equilibrium alone does **not** enforce compatibility.

The Direct Stiffness Method solves by enforcing:

- equilibrium at joints
- compatibility of joint displacements
- member force–deformation relations

We can still solve indeterminate cases when:

$$r > 3 \quad \text{or} \quad m + r > 2j$$

If a structure is **unstable**, it appears as a **singular stiffness matrix $\mathbf{K}$**.

In-Class Exercise — Stability & Determinacy

Questions (work in pairs):

1. How many **joints** (j) does this structure have?
2. How many **members** (m) are present?
3. How many **reaction components** (r) are provided by the supports?
4. Does the structure satisfy the **stability requirement**?
5. Based on counting, is the structure **statically determinate or indeterminate**?
6. If indeterminate, is the indeterminacy **external, internal, or both**?

No force calculations — focus on counting and concepts.

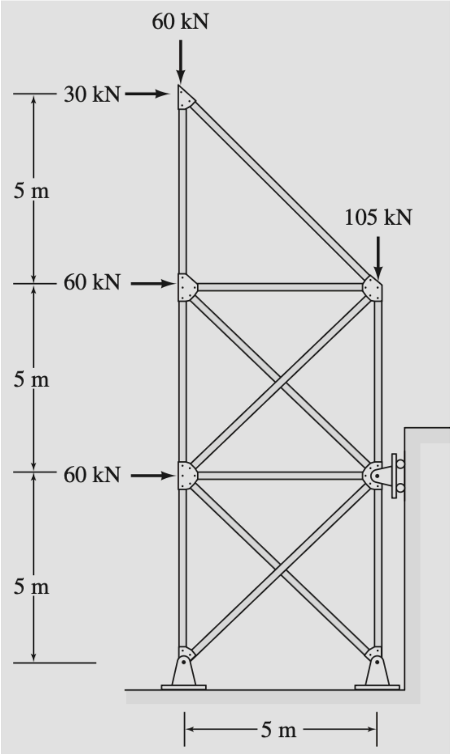


Figure. Truss used for stability and determinacy assessment.

✅ Answers — Stability & Determinacy

- **Number of joints**: $j = 7$:
- **Number of members**: $m = 12$:
- **Number of reaction components** $r = 2 + 2 + 1 = 5$:
- **Stability requirement:**
  For a planar structure, external stability requires

$$r \geq 3$$

Since $5 \geq 3$, the structure **is stable**.

- **Static determinacy (counting test):**
  For a planar truss, compare

$$m + r \quad \text{and} \quad 2j$$

$$m + r = 12 + 5 = 17, \qquad 2j = 2(7) = 14$$

  Since (17 > 14), the structure is **statically indeterminate**.

- **Type of indeterminacy:**
  - Total indeterminacy:

$$D_t = m + r - 2j = 3$$

  - External indeterminacy:

$$D_e = r - 3 = 2$$

  - Internal indeterminacy:

$$D_i = D_t - D_e = 1$$

👉 The structure is **indeterminate both externally and internally**.

Internal indeterminacy means the structure contains **more members than are required to satisfy equilibrium alone** —
there is at least one **redundant element** whose force cannot be found using statics only.

✅ This is **not a problem** for the **Direct Stiffness Method**:
stiffness-based formulations naturally handle redundancy by enforcing **compatibility and equilibrium simultaneously**.

Part 4 — Global and Local Coordinate Systems

Why Local Coordinates?

Local coordinates simplify element behavior:

- axial deformation occurs **along the member axis**
- stress–strain relations are simplest in that direction
- local coordinates separate **element behavior** from **global geometry**

Coordinate and Sign Conventions

- Global axes: $+x$ to the right, $+y$ upward
- Positive axial force: **tension**
- Local $+x'$ axis: defined from **start node** to **end node**

Consistent conventions are essential for correct assembly.

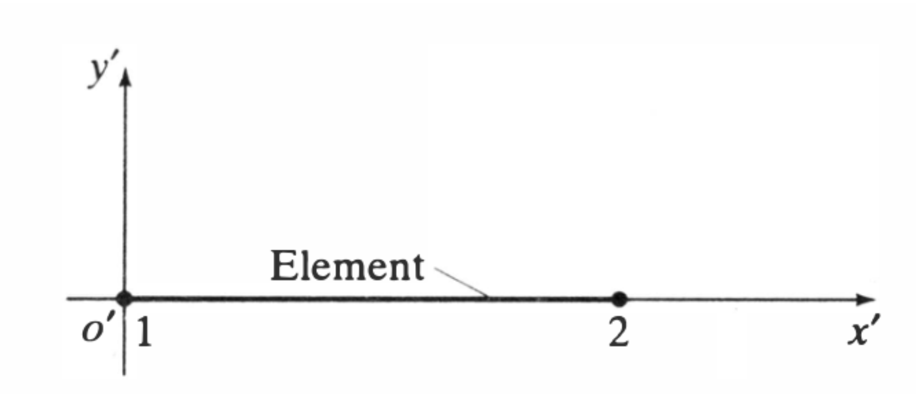


Figure. Definition of local element axis.

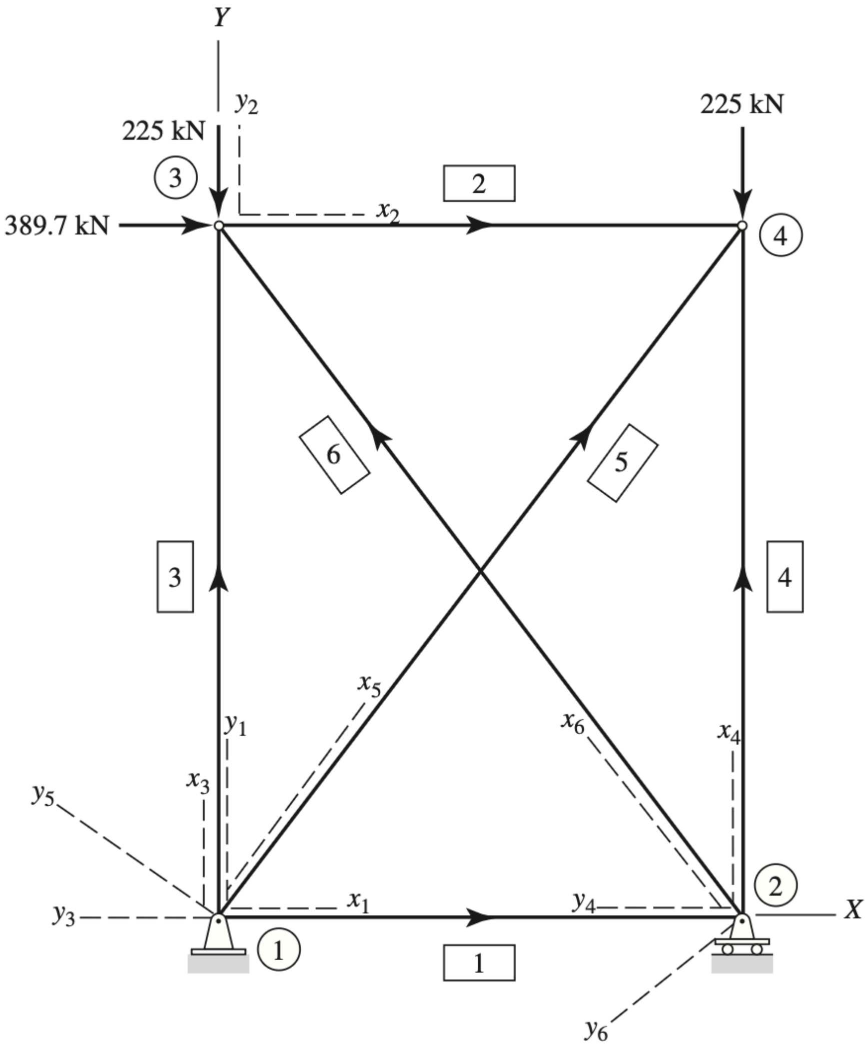


Figure. Local element axes superimposed on a truss structure.

Each truss member is assigned its own **local coordinate system**.

- The local axis x' is aligned with the member
- The positive direction is defined from start node to end node
- Axial deformation and force are expressed in this system

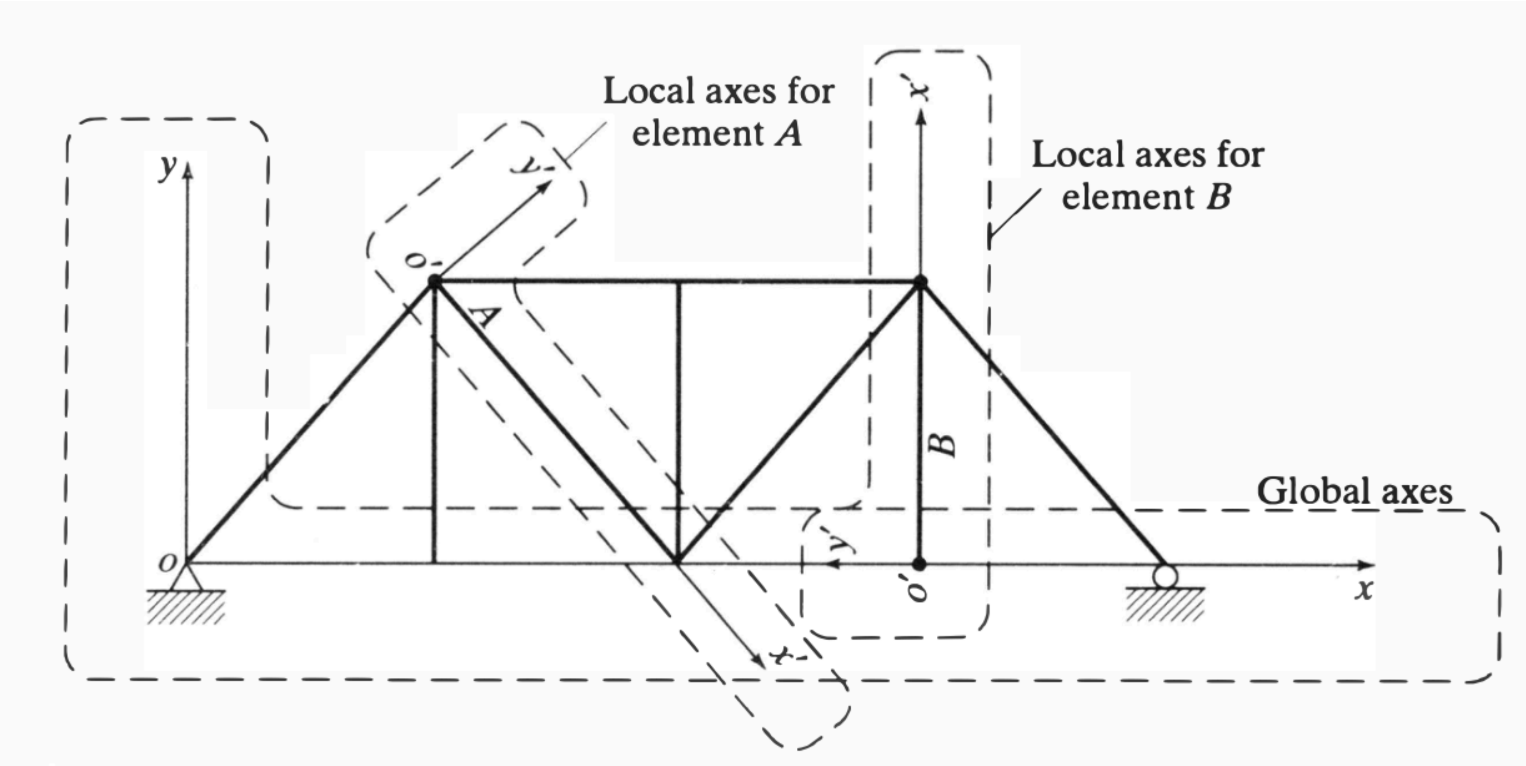


Figure. Relationship between global and local coordinate systems.

Part 5 — Axial Element Kinematics (Local)

> ***Draft note (delete later):*** *the axial force element (focus of Lecture 3.1)*

Consider a 2-node axial element in local coordinates:

- Local nodal displacements: u_1', u_2'
- Axial deformation:

$$\delta = u_2' - u_1'$$

- Axial strain:

$$\varepsilon = \frac{\delta}{L}$$

Part 6 — Axial Statics + Constitutive (EA/L)

> ***Draft note (delete later):*** *axial element statics, setting it up, Young's modulus,* EA/L

Hooke's law (uniaxial):

$$\sigma = E\varepsilon$$

Axial force:

$$N = A\sigma = EA\varepsilon = \frac{EA}{L}(u_2' - u_1')$$

End forces in local coordinates:

- Node 1: $f_1' = -N$
- Node 2: $f_2' = +N$

Part 7 — Local Stiffness Matrix (2×2)

> ***Draft note (delete later):*** *stiffness matrix form, basics ($k_{ij} = k_{ji}$), end with the 2×2 matrix*

Collect the local relation into matrix form:

$$\begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}$$

Key properties:

- Symmetric: $k_{ij} = k_{ji}$
- Rigid-body mode: if $u_1' = u_2'$, then $\delta = 0$ and $\{f'\} = \{0\}$

Physical Interpretation

If $u_1' = u_2'$:

- $\delta = 0$ so $N = 0$ and $f_1' = f_2' = 0$
- The element does not resist rigid translation along its axis

If $u_2' = -u_1'$:

- Large extension/compression for a given magnitude

Part 8 — Flexibility Formulation (Preview)

> ***Draft note (delete later):*** *flexibility formulation, why important, will we use?*

Stiffness viewpoint:

$$\{f\} = [k]\{u\}$$

Flexibility viewpoint (inverse mapping):

$$\{u\} = [f]\{f\}$$

Why it matters:

- Leads to compatibility-based methods
- Useful for interpretation

In this course, we primarily use stiffness-based methods.

Part 9 — Local 4×4 Stiffness Matrix (Local-Only)

> ***Draft note (delete later):*** *formulate the 4×4 matrix in local (many zeros); more generic case (Kassimali §3.3); full derivation*

For a planar truss member, describe the element DOFs in the local coordinate system:

$$\{u\}_e^{(local)} = [u_{1x'},\ u_{1y'},\ u_{2x'},\ u_{2y'}]^T$$

In an ideal truss member, only the axial DOFs (x' direction) create axial strain. That means the element stiffness only couples $u_{1x'}$ and $u_{2x'}$.

Embed the 2×2 axial stiffness into a 4-DOF local description:

$$[k]_e^{(local)} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Interpretation:

- The zeros reflect that the truss element provides no stiffness in y'.
- The 4×4 form matches the 2 DOFs per node bookkeeping used later for assembly.

Part 10 — Concept Checks / Mini-Examples (McGuire §2.6)

> ***Draft note (delete later):*** *examples of non-matrix structural analysis truss problems using these concepts (McGuire §2.6)*

Check A: Given E, A, L and u'_1, u'_2, compute δ, N, f'_1, f'_2.

Check B: If EA/L doubles, what happens to end forces for the same $\{u'\}$?

Check C: What displacement pattern produces zero force, and why?

Check D: In the local 4×4 matrix, which DOFs are inactive, and what physical assumption causes that?

In [27]:
```
# (Optional) Mini-example code stub
# Define E, A, L, and local nodal displacements u1p, u2p

import numpy as np

E = None
A = None
L = None
u1p = None
u2p = None

# Local 2×2 stiffness
# k2 = (E*A/L) * np.array([[1, -1], [-1, 1]])
k2 = None

# TODO: compute f' = k2 * u'
u_local = None
f_local = None

k2, u_local, f_local
```

Out[27]:
```
(None, None, None)
```

Looking Ahead

➡️ Next (Lecture 3.2):

- Build the transformation matrix from local to global
- Rotate element stiffness into global coordinates
- Assemble the global stiffness matrix for a truss
- Apply supports, solve for displacements, and recover member forces