CEE6501 — Lecture 3.2

The Direct Stiffness Method (DSM) for Trusses

Learning Objectives

By the end of this lecture, you will be able to:

- Build a local-to-global transformation for a truss member
- Compute element stiffness in global coordinates: $[k]_g = [T]^T[k'][T]$
- Assemble the global stiffness matrix $[K]$ by scatter-add
- Explain why an unsupported structure yields a singular stiffness matrix
- Apply boundary conditions via partitioning and solve for displacements
- Recover member axial forces from global displacements

Agenda

**Part 2 (today): Global behavior and the Direct Stiffness Method**

1. Transformation from local to global (direction cosines)
2. Member stiffness relations in the global coordinate system
3. Nodal equilibrium and why assembly works
4. Assemble the global stiffness matrix $[K]$
5. First attempt: no supports (singular $[K]$)
6. Constraints and supports
7. Partitioning into free vs restrained DOFs
8. Solve for global displacements
9. Second attempt: with supports
10. Recover element forces in local coordinates
11. DSM summary: step-by-step procedure
12. Stiffness matrix features + indeterminacy

Big idea:

- A truss is a network of axial springs.
- Each element contributes stiffness to shared DOFs.
- Assembly is adding contributions into the right global rows/columns.

Part 1 — Local to Global Transformation

Truss Element in a Structure
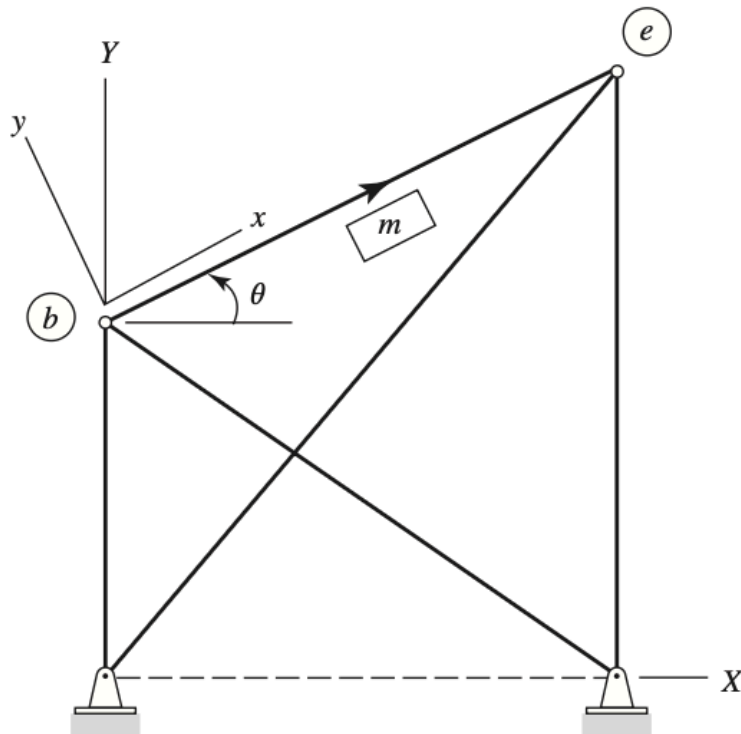
- A truss member is embedded in a **global coordinate system** (X, Y)
- The element stiffness was derived in a **local coordinate system** (x, y) aligned with the member
- The member orientation is defined by an angle θ, measured **counterclockwise** from global X to local x
- Structural assembly requires **transforming forces and displacements** between local and global coordinates

Transformation Perspectives
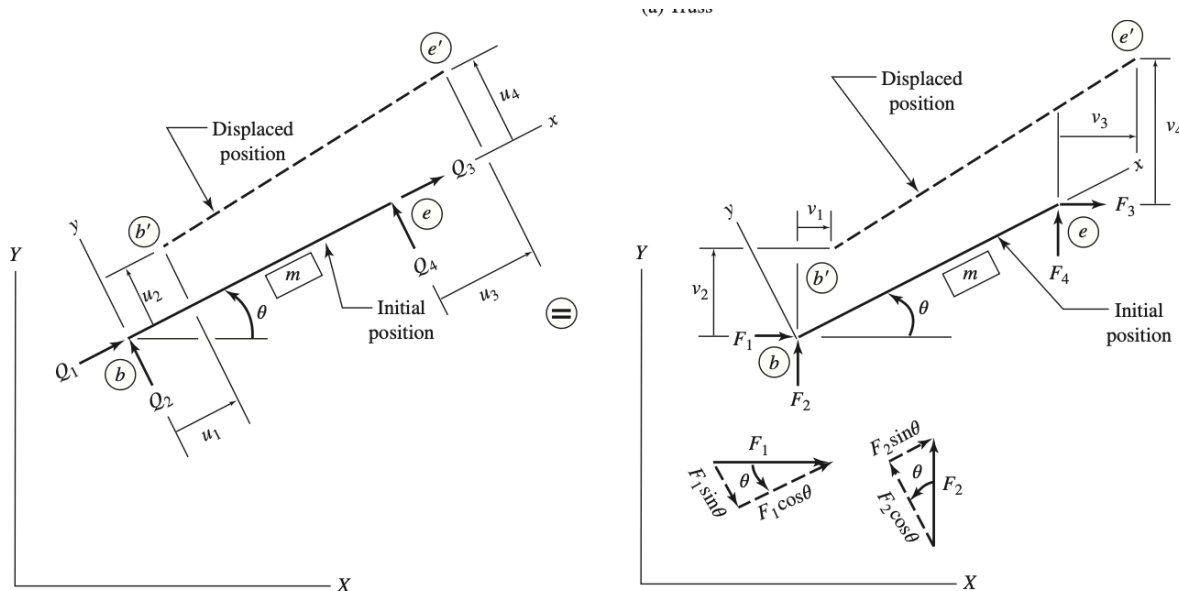
- **Local coordinate system (left):** forces Q, displacements u
- **Global coordinate system (right):** forces F, displacements v
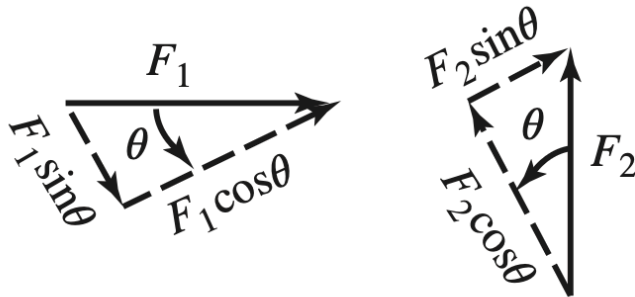
Global → Local Forces (Trigonometry)
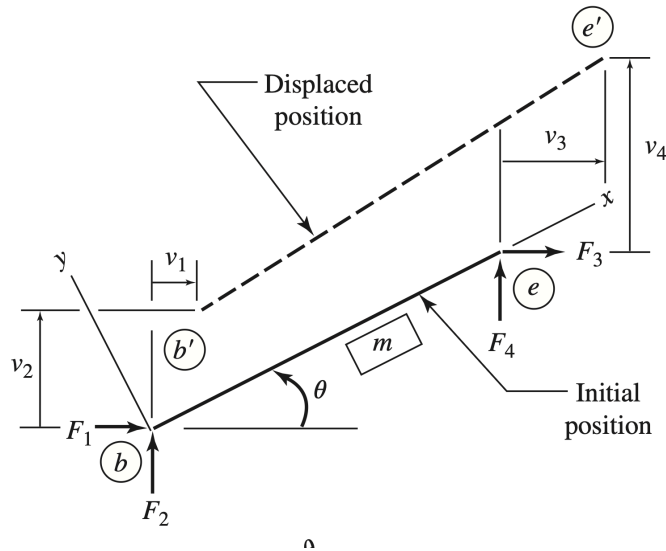
At node b (start node):

- $Q_1 = F_1 \cos\theta + F_2 \sin\theta$

- $Q_2 = -F_1 \sin\theta + F_2 \cos\theta$

At node e (end node):

- $Q_3 = F_3 \cos\theta + F_4 \sin\theta$

- $Q_4 = -F_3 \sin\theta + F_4 \cos\theta$

Global $\longrightarrow$ Local Force Transformation (Matrix Form)

- Local member forces $\boldsymbol{Q}$ are obtained by **rotating** global nodal forces $\boldsymbol{F}$ into the member's local coordinate system
- Each 2×2 block applies a **rotation by** θ at a node
- The transformation changes **direction only**, not force magnitude
- This operation is a **pure coordinate transformation**

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{\boldsymbol{T}} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$\boldsymbol{Q} = \boldsymbol{TF}$$

Direction Cosines (Rotation Terms)

- Direction cosines define the **orientation of a truss member** in the global (X, Y) coordinate system
- The angle θ is measured **counterclockwise** from the global X axis to the local x axis
- Computed directly from the **nodal coordinates** of the element (b = start node, e = end node)

$$\cos\theta = \frac{X_e - X_b}{\sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}}, \qquad \sin\theta = \frac{Y_e - Y_b}{\sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2}}$$

- The denominator is the **member length** L
- Once computed, $\cos\theta$ and $\sin\theta$ are **reused throughout the element formulation**

Global → Local Displacements

- Nodal displacements are transformed using the **same rotation matrix** as forces
- Displacements and forces transform identically because they are defined along the **same directions**
- This is a **pure coordinate rotation**, not a change in deformation

$$\boldsymbol{u} = \boldsymbol{T}\boldsymbol{v}$$

- $\boldsymbol{v}$: global displacement vector
- $\boldsymbol{u}$: local displacement vector

Local → Global Force Transformation

- This is the **reverse of the global** → **local process**
- Local member forces are **rotated back** into the global (X, Y) directions

At node b (start node):

$$F_1 = Q_1 \cos \theta - Q_2 \sin \theta, \qquad F_2 = Q_1 \sin \theta + Q_2 \cos \theta$$

At node e (end node):

$$F_3 = Q_3 \cos \theta - Q_4 \sin \theta, \qquad F_4 = Q_3 \sin \theta + Q_4 \cos \theta$$

Local → Global Force Transformation (Matrix Form)

- Local member forces are mapped to global nodal forces using the **transpose** of the global→local transformation

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix}}_{\boldsymbol{T}^{\mathsf{T}}} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

$$\boldsymbol{F} = \boldsymbol{T}^{\mathsf{T}}\boldsymbol{Q}$$

Recall (Global → Local):

$$\boldsymbol{T} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Properties of the Transformation Matrix

$$\boldsymbol{T}^{-1} = \boldsymbol{T}^T$$

The transformation matrix is **orthogonal**, which greatly simplifies operations involving the stiffness transformations.

Summary — Local ↔ Global Transformations

Direction cosines (member orientation):

$$\cos\theta = \frac{X_e - X_b}{L}, \qquad \sin\theta = \frac{Y_e - Y_b}{L}$$

Global → Local (forces or displacements):

$$\boldsymbol{Q} = \boldsymbol{T}\boldsymbol{F}, \qquad \boldsymbol{u} = \boldsymbol{T}\boldsymbol{v}$$

Local → Global (forces or displacements):

$$\boldsymbol{F} = \boldsymbol{T}^{\mathsf{T}}\boldsymbol{Q}, \qquad \boldsymbol{v} = \boldsymbol{T}^{\mathsf{T}}\boldsymbol{u}$$

$$\boldsymbol{T} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

- $\boldsymbol{T}$ is a **pure rotation matrix**
- $\boldsymbol{T}^{-1} = \boldsymbol{T}^{\mathsf{T}}$ (orthogonal)

Part 2 — Member Stiffness in the Global Coordinate System

Goal

- We have derived the **local stiffness relation** (Lecture 3.1 today):

$$Q = ku$$

- We also know how to **transform forces and displacements** between local and global systems
- Objective: express the **member stiffness relation entirely in global coordinates**

Transformation Chain — From Local to Global (Step-by-Step)

Step 1 — Local force–displacement relation

$$Q = k\,u$$

The element stiffness matrix k relates **local nodal displacements u** to the corresponding **local nodal forces Q**.

Step 2 — Transform local forces to global forces

$$F = T^{\mathsf{T}}\,Q$$

Global nodal forces are obtained by rotating the local force vector into the global coordinate system.

Substitute the local stiffness relation from Step 1, $Q = k\,u$, into the force transformation:

$$F = T^{\mathsf{T}} Q \implies F = T^{\mathsf{T}}\left(k\,u\right)$$

Step 3 — Transform global displacements to local displacements

$$u = T\,v$$

Substitute the displacement transformation into the previous expression:

$$F = T^{\mathsf{T}} k\,u \implies F = T^{\mathsf{T}} k\,(T\,v)$$

Step 4 — Rearrange into global stiffness form

$$F = \left(T^{\mathsf{T}} k\,T\right) v$$

Step 5 — Final global stiffness relation

Define the global element stiffness matrix:

$$K = T^\top k\, T$$

This is the element stiffness relation used directly in global assembly for the Direct Stiffness Method.

textbook notation:

$$\boxed{F = K\, v}$$

our notation (interchangeable):

$$\boxed{f = K\, u}$$

Calculating the Global Stiffness Matrix, $\boldsymbol{K}$

The global element stiffness matrix is obtained by **rotating the local axial stiffness** into the global coordinate system:

$$\boldsymbol{K} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Carrying out the matrix multiplication yields the **closed-form global stiffness matrix**:

$$\boldsymbol{K} = \frac{EA}{L} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$
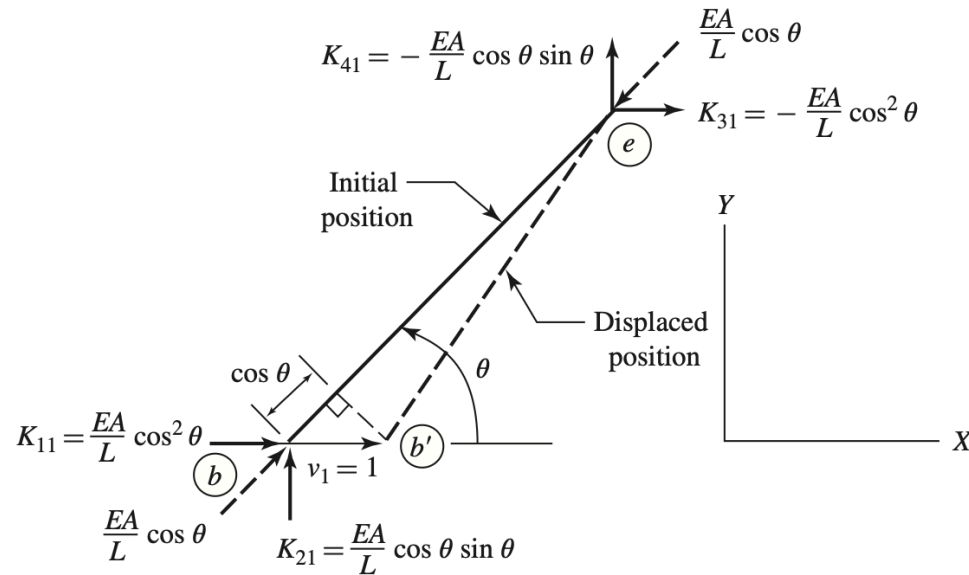
Key Observations — Global Stiffness Matrix, $\boldsymbol{K}$

- The member global stiffness matrix $\boldsymbol{K}$ is **symmetric**, just like the local stiffness matrix

- $\boldsymbol{K}$ represents the same physical behavior, but expressed in the **global** (X, Y) **coordinate system**

- Each coefficient K_{ij} is the **force at global DOF** i required to produce a **unit displacement at global DOF** j, with all other displacements fixed

Direct Calculation of Global Stiffness Matrix, $\boldsymbol{K}$

- One *could* derive $\boldsymbol{K}$ directly by:

  - Applying **unit global displacements** to a generic inclined truss member
  - Evaluating the **global end forces** required to produce each unit displacement in global coordinates

- The j**th column of** $\boldsymbol{K}$ gives the global nodal force pattern caused by $v_j = 1$

- This approach is **theoretically equivalent** to the transformation-based derivation and provides a clear physical interpretation of $\boldsymbol{K}$.

- However, it is **significantly more labor-intensive**, and is mainly useful as a **verification tool**, rather than for routine analysis.

Example: First Column of K

$$u_a = v_1 \cos\theta = 1\cos\theta = \cos\theta$$

$K_{41} = -\dfrac{EA}{L}\cos\theta\sin\theta$

$\dfrac{EA}{L}\cos\theta$

$K_{31} = -\dfrac{EA}{L}\cos^2\theta$

e

Initial position

Y

$\cos\theta$

θ

Displaced position

$K_{11} = \dfrac{EA}{L}\cos^2\theta$

b'

X

b

$v_1 = 1$

$\dfrac{EA}{L}\cos\theta$

$K_{21} = \dfrac{EA}{L}\cos\theta\sin\theta$

(a) First Column of $\mathbf{K}$ ($v_1 = 1, v_2 = v_3 = v_4 = 0$)

- Impose $v_1 = 1$, all other global displacements zero
- Project the resulting axial deformation onto the member axis
- Resolve the axial force back into global components

You recover the **first column of K**, which should exactly match $T^{\mathsf{T}}kT$.

Part 3 — Assembling the Global Structure Stiffness Matrix (Manual Method)
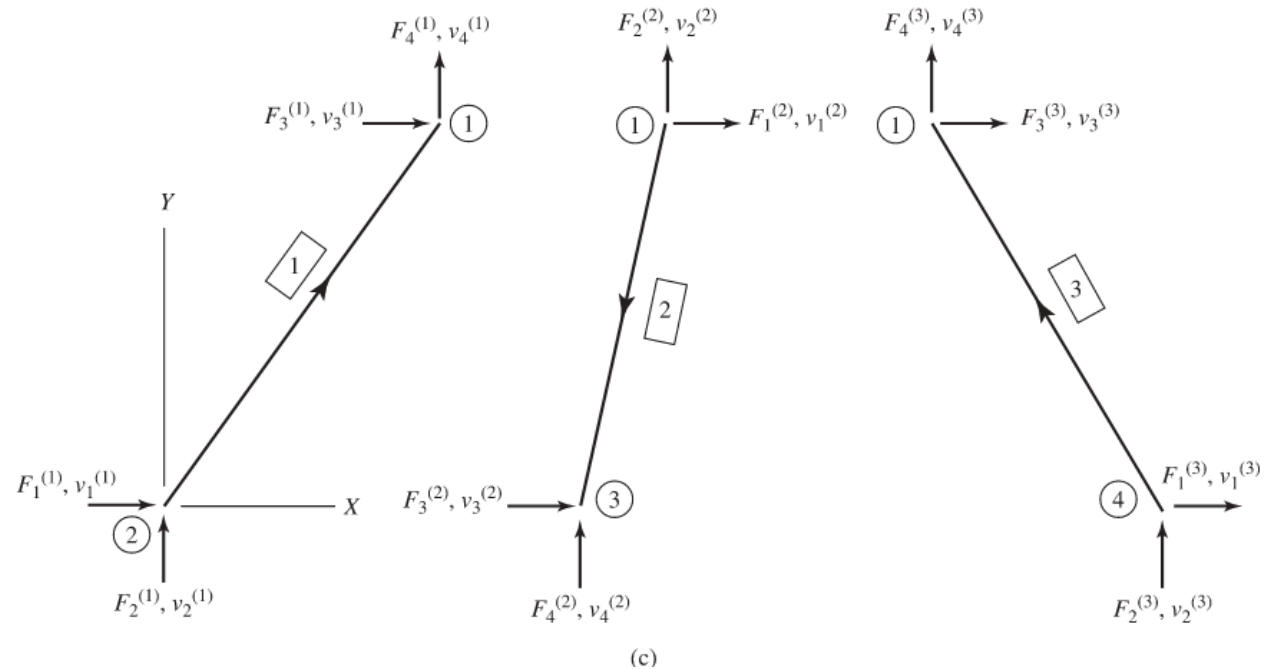
Example Structure

We now move from **element-level** to a **complete truss structure** behavior.

- Structure composed of **3 axial truss elements**
- **4 nodes**, each with (X, Y) translational DOFs, node 1 at the top
- Total system size: **8 global degrees of freedom**
  - node 1: DOF (1,2)
  - node 2: DOF (3,4)
  - node 3: DOF (5,6)
  - node 4: DOF (7,8)
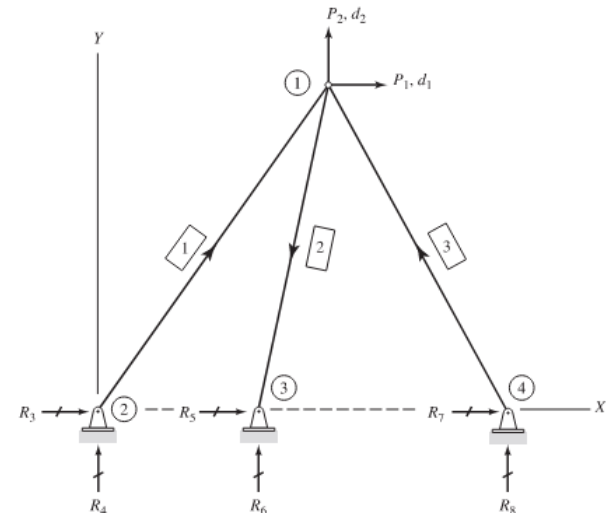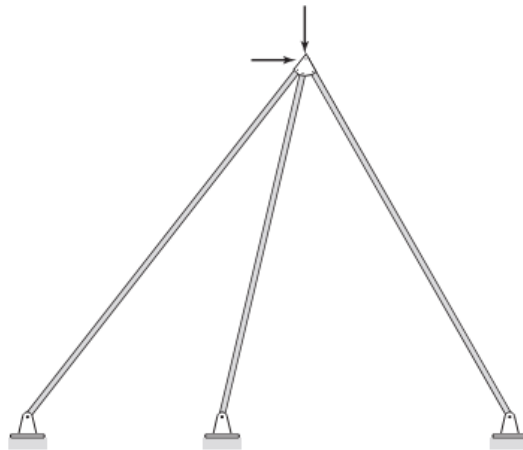
Element-Level View of the Structure

(c)

- Nodes **2, 3, and 4** are **pinned** (no displacement)
- **Node 1** is free to move
- All three elements are connected at **node 1**
- A displacement at node 1 induces forces in **all connected members**

Element Forces and Notation

Local forces are defined **per element**:

- Superscript (e) → element number
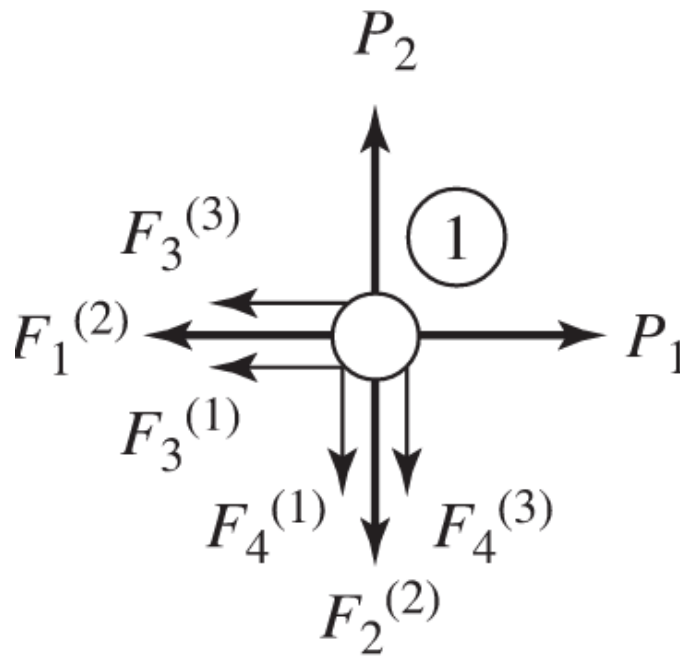- Subscript (i) → local DOF index

Examples:

- $F_3^{(1)}$ → force at local DOF 3 in **element 1**
- $F_1^{(2)}$ → force at local DOF 1 in **element 2**

Equilibrium Equations at Node 1

Because all elements share **node 1**, equilibrium at this node couples the response of all members.
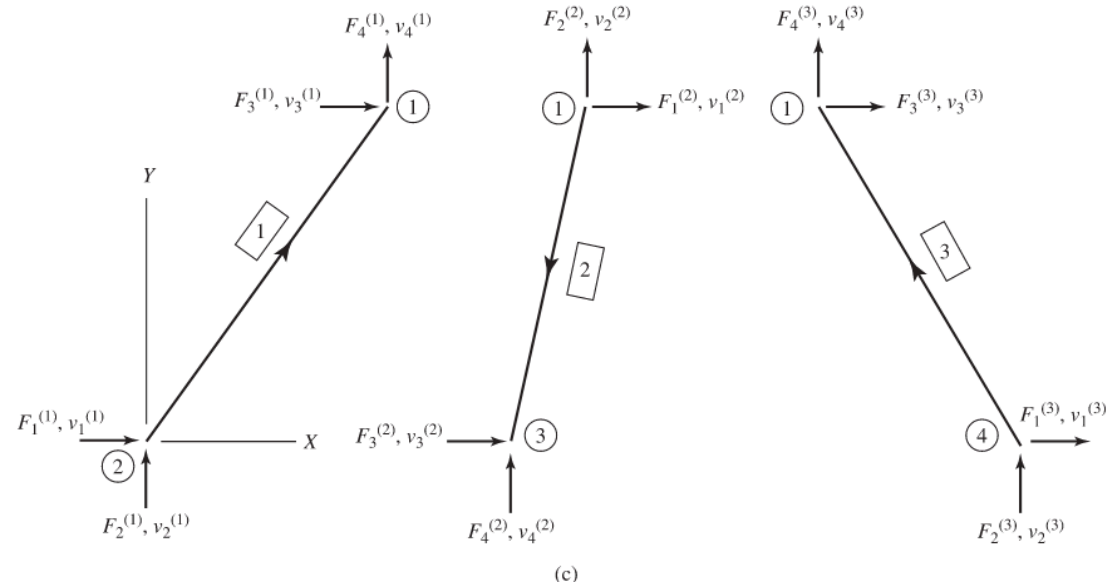
P = external applied loaded

F = internal forces

Force equilibrium at node 1:

$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)}$$

$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)}$$

Compatability Equations

(c)

d_1 and d_2 are global displacements in X and Y at node 1

$$\text{Member (1):} \quad v_1^{(1)} = v_2^{(1)} = 0, \qquad v_3^{(1)} = d_1, \qquad v_4^{(1)} = d_2$$

$$\text{Member (2):} \quad v_1^{(2)} = d_1, \qquad\qquad v_2^{(2)} = d_2, \qquad v_3^{(2)} = v_4^{(2)} = 0$$

$$\text{Member (3):} \quad v_1^{(3)} = v_2^{(3)} = 0, \qquad v_3^{(3)} = d_1, \qquad v_4^{(3)} = d_2$$

Member 1 — Force–Displacement Relations

Compatibility (Member 1): $v_1^{(1)} = v_2^{(1)} = 0, \; v_3^{(1)} = d_1, \; v_4^{(1)} = d_2$

Element-level global stiffness expression:

$$
\left\{
\begin{array}{c}
F_1^{(1)} \\
F_2^{(1)} \\
F_3^{(1)} \\
F_4^{(1)}
\end{array}
\right\}
=
\begin{bmatrix}
K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\
K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\
K_{31}^{(1)} & K_{32}^{(1)} & \boxed{K_{33}^{(1)}} & \boxed{K_{34}^{(1)}} \\
K_{41}^{(1)} & K_{42}^{(1)} & \boxed{K_{43}^{(1)}} & \boxed{K_{44}^{(1)}}
\end{bmatrix}
\left\{
\begin{array}{c}
v_1^{(1)} = 0 \\
v_2^{(1)} = 0 \\
v_3^{(1)} = d_1 \\
v_4^{(1)} = d_2
\end{array}
\right\}
$$

The forces acting at **end node of member 1** (Global node 1) are:

$$
F_3^{(1)} = K_{33}^{(1)} d_1 + K_{34}^{(1)} d_2
$$

$$
F_4^{(1)} = K_{43}^{(1)} d_1 + K_{44}^{(1)} d_2
$$

Member 2 — Force–Displacement Relations

Compatibility (Member 2): $v_1^{(2)} = d_1,\ v_2^{(2)} = d_2,\ v_3^{(2)} = v_4^{(2)} = 0$

Element-level global stiffness expression:

$$
\begin{Bmatrix} F_1^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \\ F_4^{(2)} \end{Bmatrix}
=
\begin{bmatrix}
\boxed{K_{11}^{(2)}} & \boxed{K_{12}^{(2)}} & K_{13}^{(2)} & K_{14}^{(2)} \\
\boxed{K_{21}^{(2)}} & \boxed{K_{22}^{(2)}} & K_{23}^{(2)} & K_{24}^{(2)} \\
K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\
K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)}
\end{bmatrix}
\begin{Bmatrix} v_1^{(2)} = d_1 \\ v_2^{(2)} = d_2 \\ v_3^{(2)} = 0 \\ v_4^{(2)} = 0 \end{Bmatrix}
$$

The forces acting at **start node of member 2** (Global node 1) are:

$$F_1^{(2)} = K_{11}^{(2)} d_1 + K_{12}^{(2)} d_2$$

$$F_2^{(2)} = K_{21}^{(2)} d_1 + K_{22}^{(2)} d_2$$

Member 3 — Force–Displacement Relations

Compatibility (Member 3): $v_1^{(3)} = v_2^{(3)} = 0,\ v_3^{(3)} = d_1,\ v_4^{(3)} = d_2$

Element-level global stiffness expression:

$$\left\{ \begin{array}{c} F_1^{(3)} \\ F_2^{(3)} \\ F_3^{(3)} \\ F_4^{(3)} \end{array} \right\} = \begin{bmatrix} K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & \boxed{K_{33}^{(3)}} & \boxed{K_{34}^{(3)}} \\ K_{41}^{(3)} & K_{42}^{(3)} & \boxed{K_{43}^{(3)}} & \boxed{K_{44}^{(3)}} \end{bmatrix} \left\{ \begin{array}{c} v_1^{(3)} = 0 \\ v_2^{(3)} = 0 \\ v_3^{(3)} = d_1 \\ v_4^{(3)} = d_2 \end{array} \right\}$$

The forces acting at **end node of member 3** (Global node 1) are:

$$F_3^{(3)} = K_{33}^{(3)} d_1 + K_{34}^{(3)} d_2$$

$$F_4^{(3)} = K_{43}^{(3)} d_1 + K_{44}^{(3)} d_2$$

Global Stiffness Matrix for Free DOFs

$$P_1 = F_3^{(1)} + F_1^{(2)} + F_3^{(3)}$$

$$P_2 = F_4^{(1)} + F_2^{(2)} + F_4^{(3)}$$

Substituting element-level force expressions into the node 1 force equilibrium gives:

$$P_1 = \left(K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)}\right)d_1 + \left(K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)}\right)d_2$$

$$P_2 = \left(K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)}\right)d_1 + \left(K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)}\right)d_2$$

These equations can be written compactly as:

$$\left\{\begin{array}{c} P_1 \\ P_2 \end{array}\right\} = \underbrace{\begin{bmatrix} K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} \\ K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)} \end{bmatrix}}_{\boldsymbol{K}_{\mathrm{s}}} \left\{\begin{array}{c} d_1 \\ d_2 \end{array}\right\}$$

$$\boxed{\boldsymbol{P} = \boldsymbol{K}_{\mathrm{s}}\,\boldsymbol{d}}$$

Physical Interpretation of the Structure Stiffness Matrix

- The **structure stiffness matrix** K_s has the same physical meaning as an element stiffness matrix, but at the **structure (joint) level**
- A stiffness coefficient $K_{s,ij}$ represents:
  - The **joint force** at DOF i
  - Required to cause a **unit displacement** at DOF j
  - While **all other joint displacements are zero**
- Equivalently:
  - Each **column** of K_s corresponds to a unit displacement pattern
  - The column entries are the **resulting joint forces** needed to enforce that displacement

Example — First Column of K_s

- To obtain the **first column** of K_s:
  - Impose a **unit displacement** at the first free DOF:

$$d_1 = 1, \qquad d_2 = 0$$

  - All other joint displacements are held fixed
- The resulting joint forces (DOF 1, 2) define the **first column** of K_s
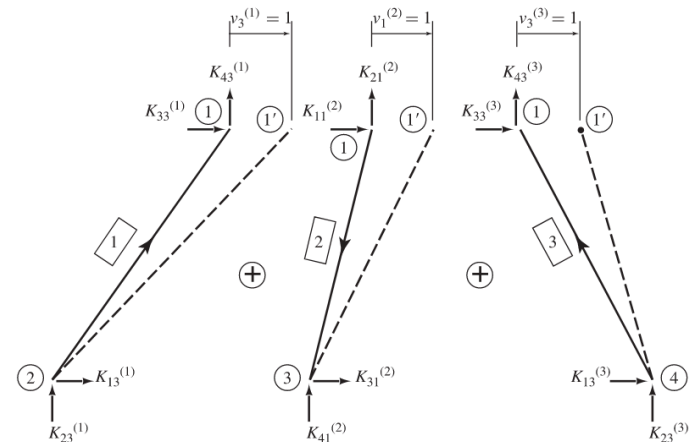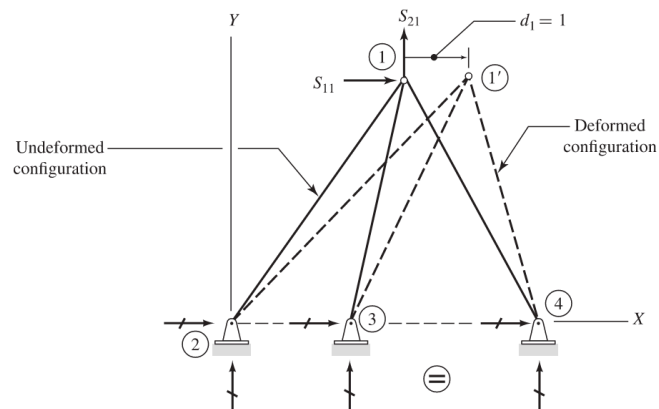
# Part 4 — Assembling the Global Structure Stiffness Matrix

Direct Stiffness Method

Why Not the Manual (Equilibrium + Compatibility) Approach?

- Becomes **tedious** very quickly
- Requires manually tracking:
    - Free vs constrained DOFs
    - Compatibility relationships
- Hard to implement **programmatically**
- Does **not scale** to larger or changing structures

Key idea: We want a method that is systematic, scalable, and algorithmic.

Reminder — Global Degrees of Freedom
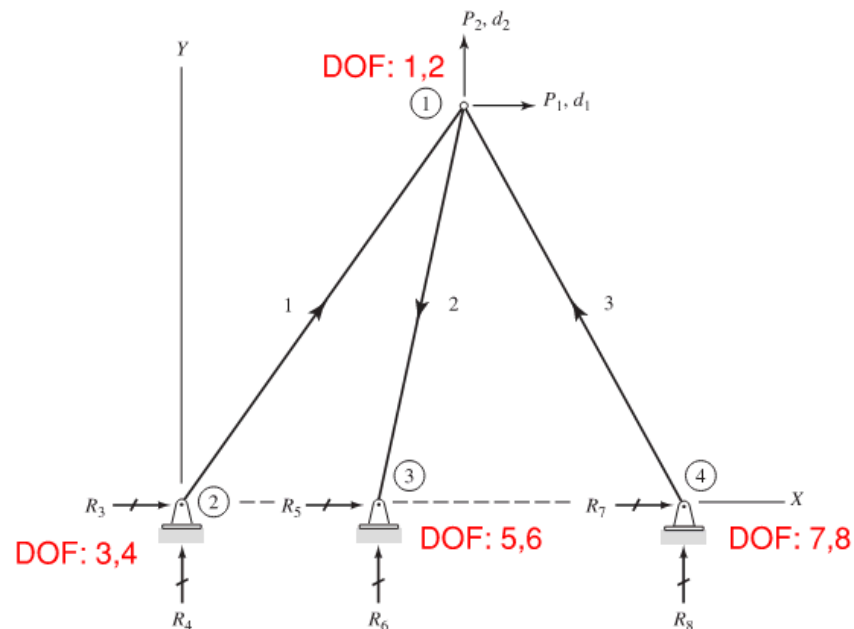
- Each node contributes two DOFs: (X, Y)
- 4 nodes → **8 global DOFs**
- Consistent numbering is essential for assembly

Map Element DOFs to Global DOFs

$$\mathbf{K}_1 = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)} \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix}$$

(columns: 3 4 1 2)

$$\mathbf{K}_2 = \begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix}$$

(columns: 1 2 5 6)

$$\mathbf{K}_3 = \begin{bmatrix} K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} \\ K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix} \begin{matrix} 7 \\ 8 \\ 1 \\ 2 \end{matrix}$$

(columns: 7 8 1 2)

Each element has **4 local DOFs** $(1, 2, 3, 4)$, which must be mapped to the correct **global DOFs** before assembly.

For this structure:

- **Element 1:** local $(1, 2, 3, 4)$ $\rightarrow$ global $(3, 4, 1, 2)$
- **Element 2:** local $(1, 2, 3, 4)$ $\rightarrow$ global $(1, 2, 5, 6)$
- **Element 3:** local $(1, 2, 3, 4)$ $\rightarrow$ global $(7, 8, 1, 2)$

This mapping determines **which rows and columns** of the global stiffness matrix each element stiffness contributes to.

Method 1: Assemble a Structure Matrix for the Free DOFs

In this example, only **global DOFs 1 and 2** are free; all other DOFs are constrained.

- Contributions from **each element** that map to DOFs 1 and 2 are **summed**
- This produces a reduced **structure stiffness matrix** for the free DOFs
- The result matches the stiffness matrix obtained earlier using the **manual equilibrium and compatibility** approach

For example, the $(1, 1)$ entry of the structure stiffness matrix is (matches what we did before):

$$K_{s,11} = K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)}$$

Method 2: Assemble the Full Global Stiffness Matrix

When assembling the **global structure stiffness matrix**, do **not** worry yet about which DOFs are constrained or free.

- **Assemble all DOFs first** into a single global matrix
- **Apply boundary conditions later** (supports, prescribed displacements)
- This separation is what makes the method **general, systematic, and scalable**

For the structure shown, there are **8 global DOFs**, so 8×8 stiffness matrix:

$$
\boldsymbol{K} = \begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} \\
K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} \\
K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} \\
K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88}
\end{bmatrix}
$$

Each entry K_{ij} represents the **combined stiffness contribution** from all elements that connect global DOF j to global DOF i.

Element 1 — Contribution to Global Stiffness Matrix

Local → Global DOF mapping (Element 1):

$$(1, 2, 3, 4) \;\rightarrow\; (3, 4, 1, 2)$$

$$
\mathbf{K}_1 =
\begin{bmatrix}
K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\
K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\
K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\
K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)}
\end{bmatrix}
\begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix}
$$

$$
\boldsymbol{K} =
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} \\
K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} \\
K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} \\
K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88}
\end{bmatrix}
$$

Element 2 — Contribution to Global Stiffness Matrix

Local → Global DOF mapping (Element 2):

$$(1, 2, 3, 4) \;\to\; (1, 2, 5, 6)$$

$$
\mathbf{K}_2 =
\begin{bmatrix}
K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\
K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\
K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\
K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)}
\end{bmatrix}
\begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix}
$$

with column labels $1 \quad 2 \quad 5 \quad 6$

$$
\mathbf{K} =
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} \\
K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} \\
K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} \\
K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88}
\end{bmatrix}
$$

Element 3 — Contribution to Global Stiffness Matrix

Local → Global DOF mapping (Element 3):

$$(1, 2, 3, 4) \rightarrow (7, 8, 1, 2)$$

$$\mathbf{K}_3 = \begin{bmatrix} K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} \\ K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix} \begin{matrix} 7 \\ 8 \\ 1 \\ 2 \end{matrix}$$

$$\boldsymbol{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} \end{bmatrix}$$

Final Global Structure Stiffness Matrix

- Each K_{ij} may include **multiple element contributions**
- The matrix is **symmetric**
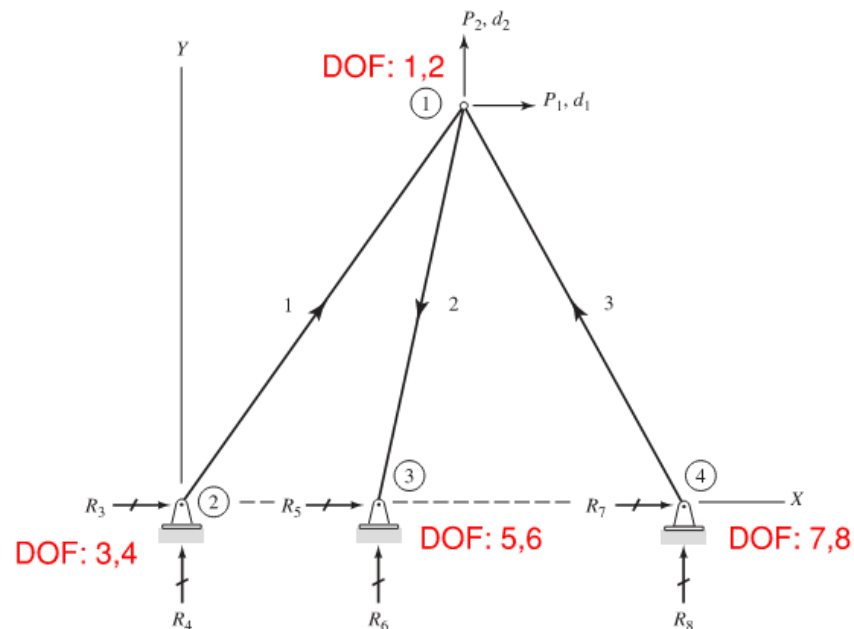- Diagonal terms, K_{ii}, must be non-zero and positive for stability

$$\boldsymbol{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\ K_{31} & K_{32} & K_{33} & K_{34} & 0 & 0 & 0 & 0 \\ K_{41} & K_{42} & K_{43} & K_{44} & 0 & 0 & 0 & 0 \\ K_{51} & K_{52} & 0 & 0 & K_{55} & K_{56} & 0 & 0 \\ K_{61} & K_{62} & 0 & 0 & K_{65} & K_{66} & 0 & 0 \\ K_{71} & K_{72} & 0 & 0 & 0 & 0 & K_{77} & K_{78} \\ K_{81} & K_{82} & 0 & 0 & 0 & 0 & K_{87} & K_{88} \end{bmatrix}$$

Interpreting Zeros in the Stiffness Matrix

- Zero entry → no stiffness coupling
- Example: $K_{37} = 0$
- DOF 3 and DOF 7 share no connecting element

Part 5 — Constraints and Supports

Try Inverting

You might think: *we have $\boldsymbol{K}$, so we can just solve*

$$\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f}$$

and we're done.

But if you try to compute $\boldsymbol{u} = \boldsymbol{K}^{-1}\boldsymbol{f}$ **right now**, you'll get an error:

- $\boldsymbol{K}$ is **singular** (not invertible)
- the structure still has **rigid-body motion** because we haven't applied supports / boundary conditions yet

Why do we get a singular stiffness matrix?

When the global stiffness matrix $\mathbf{K}$ is assembled **without supports**, the structure is free to undergo **rigid-body motion**. These motions do not stretch or compress members and therefore produce **no internal strain energy**.

Mathematically, an unconstrained structure admits non-zero displacement vectors $\mathbf{u} \neq \mathbf{0}$ such that

$$\mathbf{Ku} = \mathbf{0}$$

These displacement patterns are **rigid-body (zero-energy) modes**, meaning $\mathbf{K}$ has a **non-trivial nullspace** and is therefore **singular**.

Bottom line: without supports, $\mathbf{K}$ cannot uniquely map loads to displacements.

What does this mean in practice?

Consider the assembled 8×8 global stiffness matrix $\mathbf{K}$. At this stage, it contains **no information about how the structure is attached to the ground**.

As a result, the structure can move without inducing member deformation:

- **Rigid translation in** X
- **Rigid translation in** Y
- **Rigid-body rotation** (geometry-dependent)

These motions are **rigid-body modes**, not structural deformations.

Because rigid-body modes produce zero internal force, $\mathbf{K}$ cannot uniquely map applied loads to displacements. **Supports (boundary conditions)** remove these modes and render $\mathbf{K}$ invertible.

Boundary conditions: how supports enter the model

Supports and constraints are enforced by specifying **known displacements** (a.k.a. prescribed DOFs).

Common idealized supports (2D truss):

- **Pin support** at node i: Prescribes both translations: $u_{ix} = 0$ and $u_{iy} = 0$
- **Roller support** at node i: Prescribes one translation and allows the other:

Prescribed displacement (settlement / actuation):

- You can prescribe a non-zero value, e.g., $u_{iy} = -2 \text{ mm}$ at a support.

Why this fixes singularity:

- Constraints remove rigid-body modes by preventing global translations/rotations.
- After constraints, the remaining free DOFs correspond to true structural deformation.

What we do next (workflow)

We do **not** modify the element --> global matrix formulation to handle supports.

Instead, the Direct Stiffness Method proceeds like this:

1. Assemble the **full** global matrix K and global load vector F
2. Identify which DOFs are **free** vs **restrained**
3. Enforce boundary conditions by **partitioning** (or equivalently, by eliminating restrained DOFs)
4. Solve only for the unknown (free) displacements
5. Recover reactions at restrained DOFs

So supports are handled cleanly at the **system equation** level — via partitioning (next section).

Part 5 — Partitioning the Matrix (Free vs Restrained)

Core idea

Separate DOFs into:

- **Free DOFs** f: unknown displacements (to solve for)
- **Restrained DOFs** r: prescribed displacements (known from supports/constraints)

Reorder the global displacement vector so all free DOFs come first:

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_f \\ \boldsymbol{u}_r \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_f \\ \boldsymbol{F}_r \end{bmatrix}$$

Then the global system becomes a block system:

$$\begin{bmatrix} \boldsymbol{K}_{ff} & \boldsymbol{K}_{fr} \\ \boldsymbol{K}_{rf} & \boldsymbol{K}_{rr} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_f \\ \boldsymbol{u}_r \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_f \\ \boldsymbol{F}_r \end{bmatrix}$$

Interpretation:

- $\boldsymbol{K}_{ff}$ couples free DOFs to free DOFs
- $\boldsymbol{K}_{fr}$ couples restrained DOFs into the free equations
- $\boldsymbol{K}_{rf}$ and $\boldsymbol{K}_{rr}$ are used to recover reactions

How do we identify free vs restrained DOFs?

For each node in a 2D truss, we typically have two DOFs:

- u_x (global X translation)
- u_y (global Y translation)

A DOF is **restrained** if its displacement is prescribed:

- Pin support at node i: restrains both u_{ix} and u_{iy}
- Roller at node i (vertical): restrains u_{iy} only
- Any imposed displacement: restrains that DOF to a specified value

A DOF is **free** if its displacement is unknown and must be solved from equilibrium.

In implementation, you typically store:

- a list of free DOF indices, e.g. `free = [ ... ]`
- a list of restrained DOF indices, e.g. `rest = [ ... ]`

Then you form submatrices by indexing into K.

What the submatrices mean (physical interpretation)

Start from the full equilibrium statement:

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}$$

After partitioning, the **top block row** is:

$$\boldsymbol{K}_{ff}\boldsymbol{u}_f + \boldsymbol{K}_{fr}\boldsymbol{u}_r = \boldsymbol{F}_f$$

This is the equation we solve for unknown displacements $\boldsymbol{u}_f$.

The **bottom block row** is:

$$\boldsymbol{K}_{rf}\boldsymbol{u}_f + \boldsymbol{K}_{rr}\boldsymbol{u}_r = \boldsymbol{F}_r$$

This equation is used to compute **support reactions** once $\boldsymbol{u}_f$ is known.

Important notes:

- $\boldsymbol{F}_r$ are not usually prescribed loads — they are typically **unknown reactions**
- $\boldsymbol{u}_r$ are known (often zero), so they act like a known input

Non-zero prescribed displacements (settlement / actuation)

Partitioning also handles cases where restrained displacements are not zero.

Example:

- A support settles by δ: $u_{iy} = -\delta$ is prescribed

The free-DOF equation is still:

$$\boldsymbol{K}_{ff}\boldsymbol{u}_f + \boldsymbol{K}_{fr}\boldsymbol{u}_r = \boldsymbol{F}_f$$

Rearranged into a solve-ready form:

$$\boldsymbol{K}_{ff}\boldsymbol{u}_f = \boldsymbol{F}_f - \boldsymbol{K}_{fr}\boldsymbol{u}_r$$

So, prescribed support motion enters as an *effective load term* on the right-hand side.

Part 6 — Solving for Global Displacements and Forces

Solve the partitioned system

Start from the partitioned equilibrium equations:

$$\boldsymbol{K}_{ff}\boldsymbol{u}_f + \boldsymbol{K}_{fr}\boldsymbol{u}_r = \boldsymbol{F}_f$$

$$\boldsymbol{K}_{rf}\boldsymbol{u}_f + \boldsymbol{K}_{rr}\boldsymbol{u}_r = \boldsymbol{F}_r$$

Typically, $\boldsymbol{u}_r$ is known (often $\boldsymbol{0}$). Solve for the unknown displacements:

$$\boxed{\boldsymbol{u}_f = \boldsymbol{K}_{ff}^{-1}\left(\boldsymbol{F}_f - \boldsymbol{K}_{fr}\boldsymbol{u}_r\right)}$$

Then compute reactions at the supports (restrained DOFs):

$$\boxed{\boldsymbol{F}_r = \boldsymbol{K}_{rf}\boldsymbol{u}_f + \boldsymbol{K}_{rr}\boldsymbol{u}_r}$$

Interpretation:

- $\boldsymbol{u}_f$ gives the structural deformation
- $\boldsymbol{F}_r$ gives the support reactions required to enforce the boundary conditions

Algorithmic view (what you do in code)

Given:

- full global stiffness matrix $\boldsymbol{K}$
- full global load vector $\boldsymbol{F}$
- sets of indices `free` and `rest`
- prescribed displacements $\boldsymbol{u}_r$ (often zeros)

Steps:

1. Extract submatrices:

- $\boldsymbol{K}_{ff} = \boldsymbol{K}[f, f]$
- $\boldsymbol{K}_{fr} = \boldsymbol{K}[f, r]$
- $\boldsymbol{K}_{rf} = \boldsymbol{K}[r, f]$
- $\boldsymbol{K}_{rr} = \boldsymbol{K}[r, r]$

2. Extract subvectors:

- $\boldsymbol{F}_f = \boldsymbol{F}[f]$
- $\boldsymbol{F}_r = \boldsymbol{F}[r]$ (usually unknown reactions; can initialize as zeros)

3. Solve:

$$\boldsymbol{u}_f = \boldsymbol{K}_{ff}^{-1}(\boldsymbol{F}_f - \boldsymbol{K}_{fr}\boldsymbol{u}_r)$$

4. Recover reactions:

$$\boldsymbol{F}_r = \boldsymbol{K}_{rf}\boldsymbol{u}_f + \boldsymbol{K}_{rr}\boldsymbol{u}_r$$

Practical note:

- In numerical computing, you rarely compute $\boldsymbol{K}_{ff}^{-1}$ explicitly.
- You solve $\boldsymbol{K}_{ff}\boldsymbol{u}_f = \boldsymbol{b}$ using a linear solver.

Sanity checks after solving

1) Dimensions and indexing

- $\boldsymbol{K}_{ff}$ should be $n_f \times n_f$
- $\boldsymbol{u}_f$ should be length n_f

2) Stability

- If the structure is properly constrained, $\boldsymbol{K}_{ff}$ should be non-singular
- If it is still singular, you likely have:
  - insufficient supports (rigid-body mode remains), or
  - a disconnected node / element connectivity error

3) Equilibrium

- Reconstruct the full displacement vector $\boldsymbol{u}$ by inserting $\boldsymbol{u}_f$ and $\boldsymbol{u}_r$
- Compute $\boldsymbol{K}\boldsymbol{u}$ and compare to $\boldsymbol{F}$
- Differences should be near numerical tolerance (floating point error)

4) Reactions

- Check reaction directions and magnitudes against intuition
- Sum of reactions should balance applied loads (global equilibrium)

Part 7 — Recover Element Forces (Back to Local)

> **Draft note (delete later):** *now go back to local, for element forces etc. Section 3.8 Kassimalu explains, in the step by step guide*

For each member:

1. Extract element global displacement vector $\{u\}_e$
2. Transform to local: $\{u'\} = [T]\{u\}_e$
3. Compute local end forces: $\{f'\} = [k']\{u'\}$
4. Axial force: $N = \frac{EA}{L}\left(u'_2 - u'_1\right)$

Part 8 — DSM Summary (Explicit Steps)

> ***Draft note (delete later):*** *the direct stiffness method summary (what we just did), steps to solve, explicit steps*

1. Define geometry (nodes, members)

2. Number DOFs

3. For each member:
   - compute $L,\ \theta,\ c,\ s$
   - build $[T]$
   - compute $[k]_g = [T]^T [k'][T]$

4. Assemble $[K]$

5. Apply boundary conditions (partition into f, r)

6. Solve for u_f

7. Recover reactions F_r

8. Recover member forces/stresses in local coordinates

Part 9 — Features of the Stiffness Matrix + Indeterminacy

> ***Draft note (delete later):*** *some features of the stiffness matrix + indeterminacy (mcGuire 3.3, 3.4)*

Typical properties (for stable, properly constrained trusses):

- Symmetric
- Sparse
- Positive definite on free DOFs

Indeterminacy (conceptual):

- More members than needed for determinacy: stiffness method still works
- The matrix system enforces compatibility and equilibrium automatically

Part 10 — Worked Example: Second Attempt (With Supports)

Draft note (delete later): *now solve McGuire 3.2, same structure as 3.1 but now with supports*

Now apply supports and solve for:

- global displacements $\{u\}$
- reactions at restrained DOFs

In [ ]:
```python
# (Optional) DSM code scaffold stub
# Placeholder for the hands-on notebook (or Lecture 3 lab section).

import numpy as np

# TODO: define nodes, connectivity, E, A
nodes = None          # e.g., np.array([[x1,y1],[x2,y2],...])
members = None        # e.g., list of (n1, n2)
E = None
A = None

# TODO: DOF numbering, element k_g, assembly, partitioning, solve, recovery
K = None
F = None
u = None

K, F, u
```

Wrap-Up

Today you built the DSM pipeline for trusses:

- local bar stiffness $\rightarrow$ transformation $\rightarrow$ global element stiffness
- assembly $\rightarrow$ constraints $\rightarrow$ solve $\rightarrow$ member force recovery

Next: implement DSM in Python for a worked truss example and discuss efficiency (sparsity/bandedness).