CEE6501 — Lecture 4.3

Matrix Bandwidth and Extending the DSM to 3D

Learning Objectives

By the end of this lecture, you will be able to:

- Explain why global stiffness matrices for trusses are typically **sparse**
- Define **half-bandwidth** and relate it to storage and computational cost
- Show how **node numbering / DOF ordering** changes bandwidth without changing physics
- Demonstrate (with timing) why exploiting structure can accelerate linear solves
- Extend the Direct Stiffness Method (DSM) from 2D to **3D trusses**
- Write the **3D truss element** stiffness in local form, the 3D transformation matrix, and the assembled global element stiffness
- State the **support constraints** needed to prevent rigid body motion in 3D

Agenda

1. Sparsity in the global stiffness matrix
2. Half-bandwidth and why node numbering matters
3. A timing demo: same size, different bandwidth
4. DSM in 3D: DOFs, rotations, and supports
5. 3D truss element matrices: $\mathbf{k}_{local}$, $\mathbf{T}$, $\mathbf{k}_{global}$

Part 1 — Sparsity and Bandwidth

> *In large structural systems, performance depends not only on how many nonzeros we have, but on where they appear in the matrix.*

Sparsity (local physical connectivity)

In truss and frame models, each node connects to only a small number of neighboring elements. As a result, each equilibrium equation involves only a few degrees of freedom.

This locality leads to a **sparse** global stiffness matrix $\mathbf{K}$: most entries are exactly zero.

- Sparsity enables memory-efficient storage
- It also enables specialized **sparse solvers** that avoid unnecessary operations

Sparsity is a direct consequence of the *physics and topology* of the structure.

Bandwidth (connectivity meets indexing)

Sparsity describes *how many* nonzero entries exist. **Bandwidth** describes *how far from the diagonal* those nonzeros extend.

For a symmetric matrix $\mathbf{K}$, the **half-bandwidth** is defined as

$$NHB = \max\{|i-j| : K_{ij} \neq 0\}.$$

A smaller half-bandwidth means that nonzero entries are clustered closer to the diagonal.

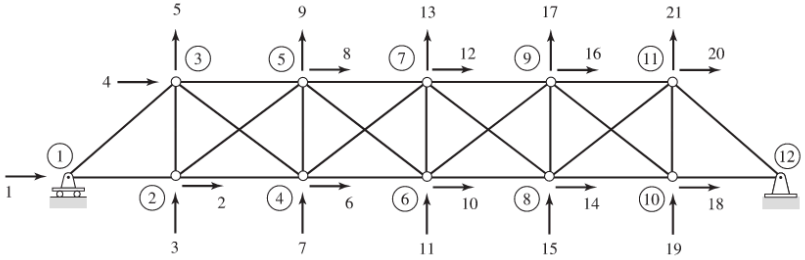
Why node numbering matters

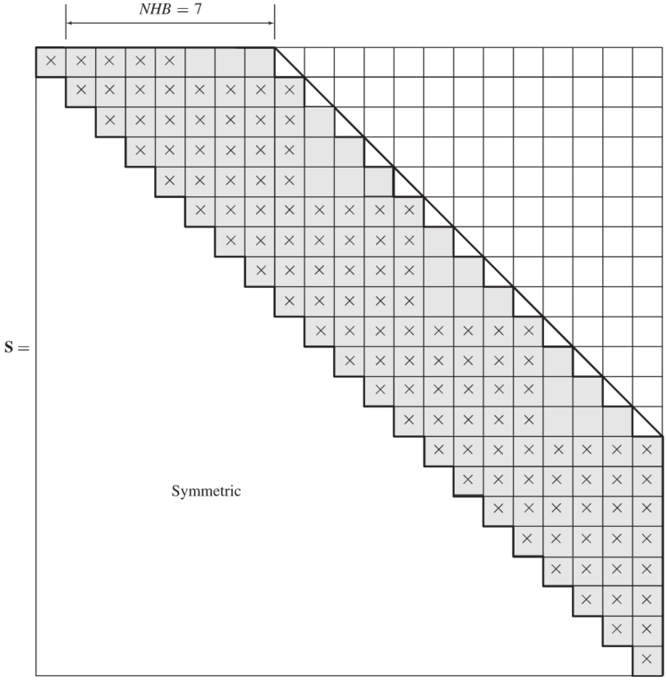
The physical structure is unchanged, but the **algebraic representation** of $\mathbf{K}$ depends on the ordering of degrees of freedom.

- Poor numbering places strongly coupled DOFs far apart → **large bandwidth**
- Good numbering keeps coupled DOFs close → **small bandwidth**

Bandwidth affects the cost of matrix factorization, even though it has no effect on the underlying physics.

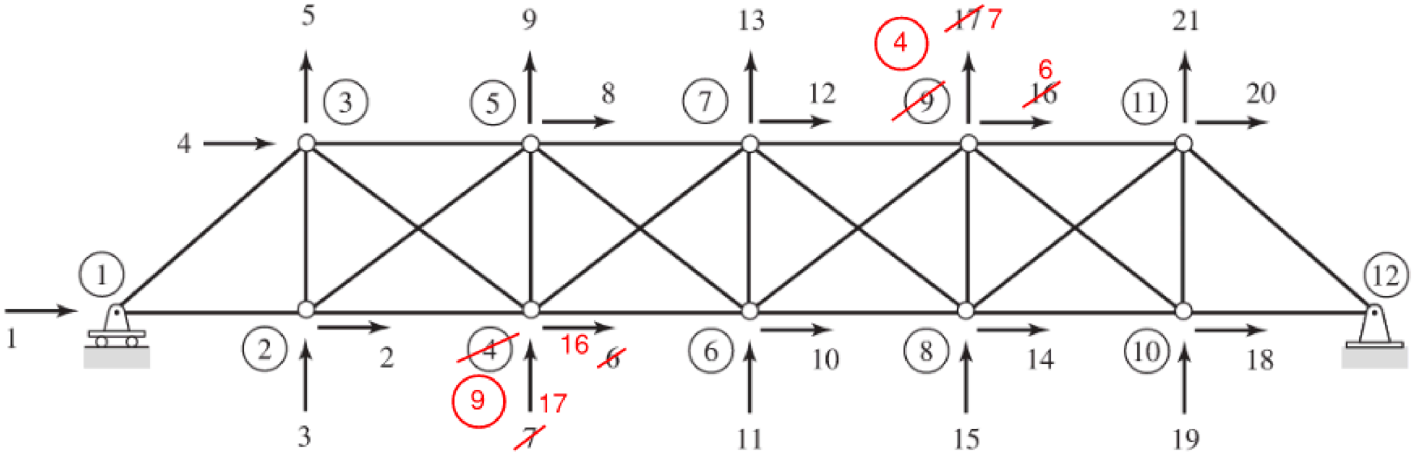
Example 1: A well-numbered system



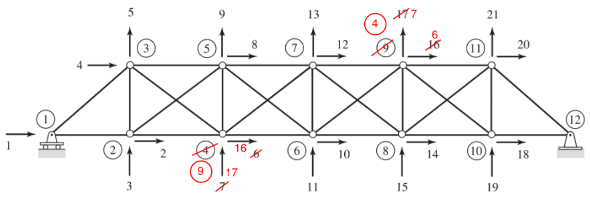


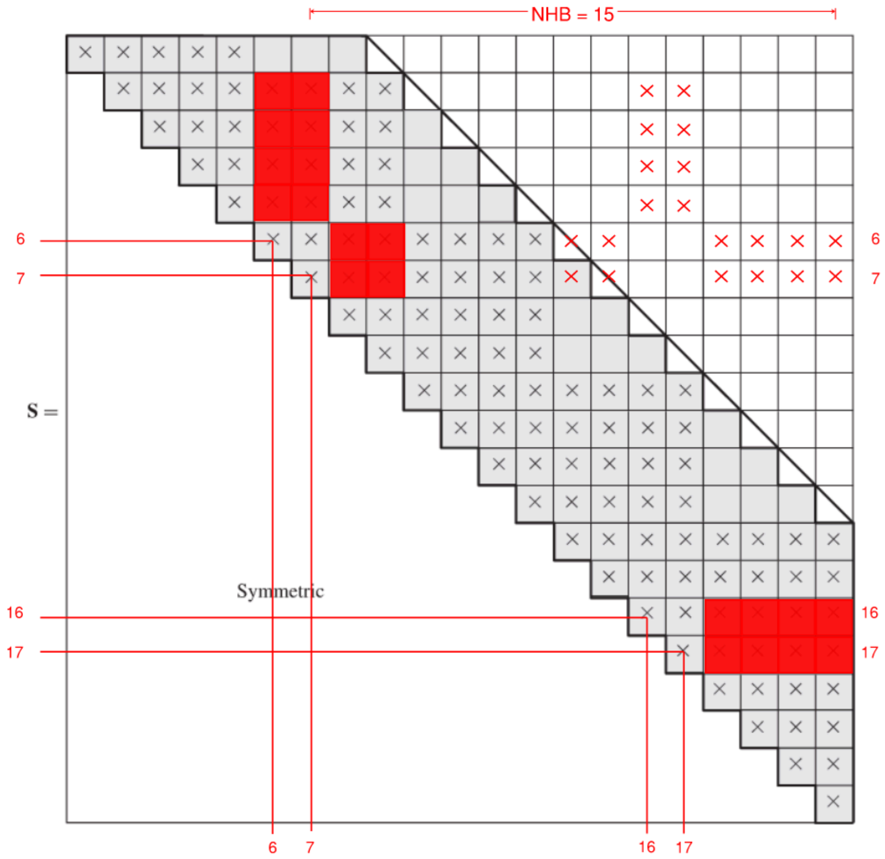
Example 2: A poorly-numbered system

Swap Node 6 and 9, number DOFs accordingly



Half-Bandwidth Increases from 7 to 15





Timing Demo — Same system, different bandwidth

Goal: compare solve times for the **same underlying SPD system** under two different degree-of-freedom orderings.

We construct matrices with:

- identical size n
- identical physical connectivity and sparsity pattern
- but very different **effective half-bandwidths**

This isolates the computational impact of bandwidth and ordering.

Solver types used in the timing demo

We solve the **same linear system** using three strategies that make different assumptions about matrix structure.

1) Dense solve (NumPy)

```
np.linalg.solve(K, B)
```

- Treats $\mathbf{K}$ as fully dense (LU-type)
- Ignores sparsity, bandwidth, and symmetry

2) Sparse LU (no reordering)

```
splu(K, permc_spec="NATURAL")
```

- Sparse storage, original DOF ordering
- Strongly affected by bandwidth and fill-in

3) Sparse LU (with reordering)

```
splu(K, permc_spec="COLAMD")
```

- Applies fill-reducing reordering
- Reduces bandwidth and factorization cost

Details of sparse factorization are beyond this lecture (see Kassimali §9.9).

Switch to demo code

What this demo shows

- The two systems represent the **same physics**, differing only in DOF ordering
- Dense solvers ignore sparsity and bandwidth entirely (unaffected)
- Larger bandwidth typically leads to higher factorization cost for sparse solvers
- Sparse solvers are sensitive to ordering, but can reduce its impact via reordering

Takeaway: DOF ordering changes the algebraic structure of $\mathbf{K}$, which can strongly influence computational cost even though the physics is unchanged.

Part 2 — Extending the DSM to 3D Trusses

> *Same DSM workflow, but with 3 translational DOFs per node and a 3D orientation.*

Degrees of freedom in 3D

For a 3D truss node:

$$\mathbf{d}_i = \begin{bmatrix} u_{xi} & u_{yi} & u_{zi} \end{bmatrix}^T, \quad \mathbf{d}_j = \begin{bmatrix} u_{xj} & u_{yj} & u_{zj} \end{bmatrix}^T$$

Element displacement vector in global coordinates:

$$\mathbf{d}_e = \begin{bmatrix} u_{xi} & u_{yi} & u_{zi} & u_{xj} & u_{yj} & u_{zj} \end{bmatrix}^T \equiv \text{DOFs } [1, 2, 3, 4, 5, 6]$$

3D direction cosines

Let the element connect nodes i and j with coordinates (x_i, y_i, z_i) and (x_j, y_j, z_j).

$$L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$$

$$l = \frac{x_j - x_i}{L}, \quad m = \frac{y_j - y_i}{L}, \quad n = \frac{z_j - z_i}{L}$$

These are the direction cosines of the element axis in global coordinates.

3D Truss Element Matrices

We will write:

- the **local** element stiffness $\mathbf{k}_{local}$ (6×6)
- the **transformation** matrix $\mathbf{T}$ (6×6)
- the **global** element stiffness $\mathbf{k}_{global}$ (6×6)

Local stiffness matrix (6×6)

A truss element carries **axial force only** and therefore has stiffness only in the direction of its axis.

In a local coordinate system where the element axis is the local x' direction, only the axial DOFs are active.

Then the 6×6 local stiffness is:

$$\mathbf{k}_{local} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Only axial coupling appears; transverse DOFs have zero stiffness in a truss model.

Transformation matrix $\mathbf{T}$ (6×6)

For a truss, we only need the mapping from global translations to **axial** direction. A convenient 6×6 transformation (acting on translations only) is:

$$\mathbf{d}_{local} = \mathbf{T}\,\mathbf{d}_{global}$$

with

$$\mathbf{T} = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix} = \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ y_x & y_y & y_z & 0 & 0 & 0 \\ z_x & z_y & z_z & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \\ 0 & 0 & 0 & y_x & y_y & y_z \\ 0 & 0 & 0 & z_x & z_y & z_z \end{bmatrix}.$$

This version is sufficient because $\mathbf{k}_{local}$ only acts on the axial rows/cols.

> *Note: A full 3D rotation basis would include two additional transverse unit vectors. For a truss, those transverse directions do not contribute stiffness.*

Global element stiffness (6×6)

Compute:

$$\mathbf{k}_{global} = \mathbf{T}^T \, \mathbf{k}_{local} \, \mathbf{T}.$$

For a 3D truss element, the result can be written directly as:

$$\mathbf{k}_{global} = \frac{EA}{L} \begin{bmatrix} l^2 & lm & ln & -l^2 & -lm & -ln \\ lm & m^2 & mn & -lm & -m^2 & -mn \\ ln & mn & n^2 & -ln & -mn & -n^2 \\ -l^2 & -lm & -ln & l^2 & lm & ln \\ -lm & -m^2 & -mn & lm & m^2 & mn \\ -ln & -mn & -n^2 & ln & mn & n^2 \end{bmatrix}.$$

This is the standard 3D truss element stiffness in global coordinates.

Supports and stability in 3D

A free 3D structure has **3 rigid body translations**.

For a 3D **truss** model (translations only), you must prevent rigid body motion by constraining enough nodal translations.

Typical sufficient constraint patterns include (examples):

- Fix all three translations at one node, and two at a second node, and one at a third node (if geometry allows)
- Or use support conditions consistent with the physical supports (pins/rollers) but ensuring no global drift

If constraints are insufficient, $\mathbf{K}$ is singular → the model has a mechanism.