CEE6501 — Lecture 3.1

Local Behavior of an Axial Element

Learning Objectives

By the end of this lecture, you will be able to:

- Define truss DOFs and a consistent sign convention
- Explain local vs global coordinates for an axial member
- Derive the 2×2 local stiffness matrix for an axial element
- Interpret stiffness matrix properties (symmetry, rigid-body mode)
- Describe the idea of flexibility as the inverse relation (preview)
- Formulate the local 4×4 element stiffness matrix (local-only; not transformed)

Agenda

1. Matrix structural analysis: definitions & concepts
2. Truss DOFs and sign conventions
3. Local vs global coordinates
4. Structure idealization (trusses)
5. Axial element kinematics
6. Axial statics + constitutive (EA/L)
7. Local stiffness matrix (2×2)
8. Flexibility formulation (preview)
9. Local 4×4 stiffness matrix (local level; not transformed)
10. Concept checks / mini-examples (McGuire §2.6)

Part 1 — Structural Viewpoint and Coordinates

Degrees of Freedom (DOFs)

A **degree of freedom (DOF)** is an independent displacement component of a structure.

The complete set of DOFs represents the **minimum number of joint displacements** required to uniquely describe the deformed configuration of a structure under arbitrary loading.

What DOFs Represent

From a matrix analysis perspective, DOFs are:

- The **unknown quantities** we solve for
- The locations where external forces are applied
- The coordinates used to describe structural deformation

Once the DOFs are defined, the structural response can be written compactly in matrix form.

DOFs Depend on the Structural Model

- Trusses: joint translations only
- Frames: joint translations and rotations
- Higher-order models: additional deformation modes

The chosen model determines **which displacements are allowed** and **which are restrained**.

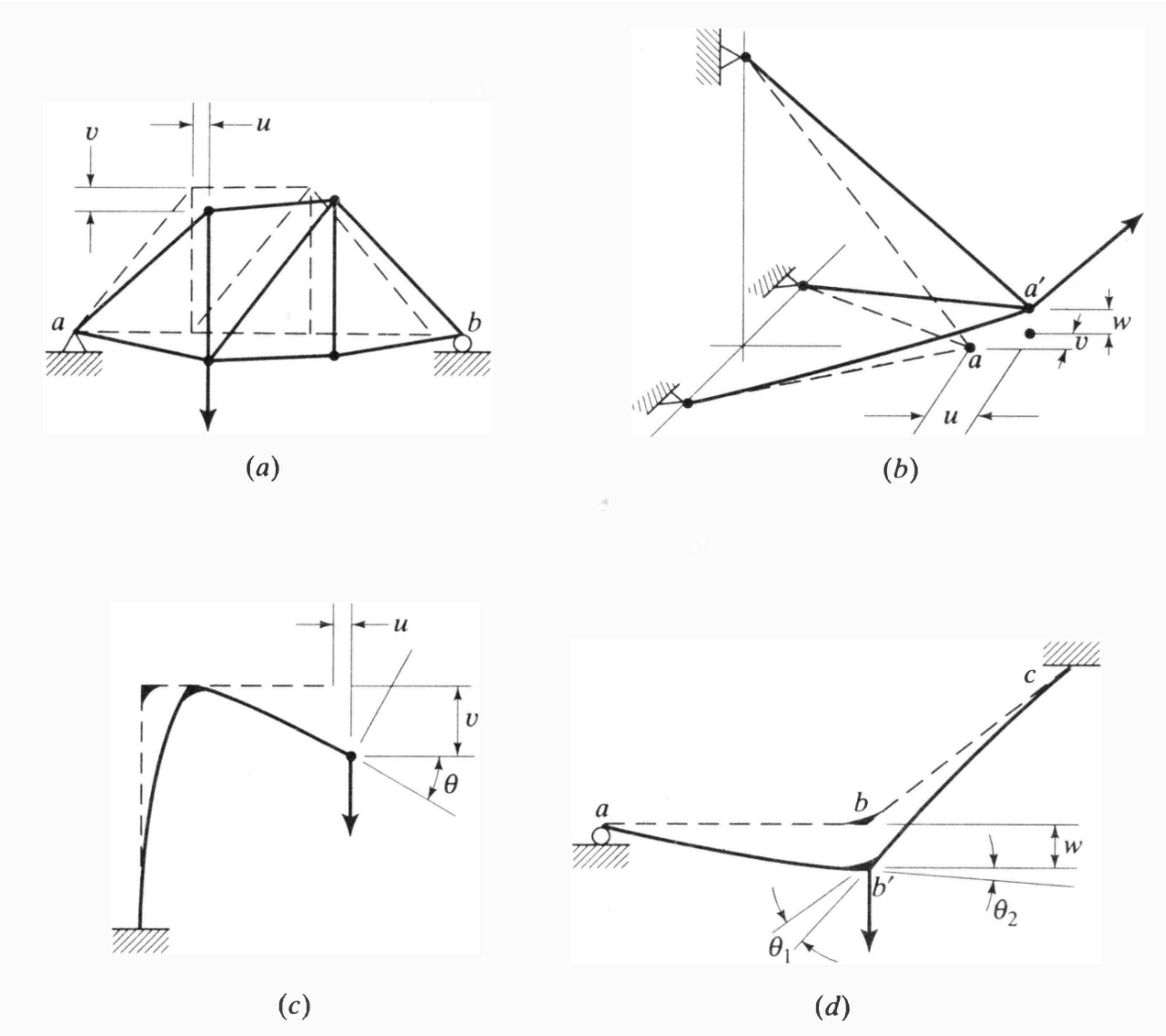


Figure 2.1 Joint displacements. (a) Pin-jointed plane truss. (b) Pin-jointed space truss. (c) Plane frame (in-plane loading). (d) Plane frame (out-of-plane loading).

Global vs Local Viewpoint

In structural analysis we constantly switch between two perspectives:

- **Global:** how the *entire structure* moves and equilibrates
- **Local:** how an *individual element* deforms internally

This distinction is fundamental to the matrix stiffness method.

Global–Local Workflow

The stiffness method follows a consistent pattern:

Local stiffness → global stiffness → solve for global displacements → recover local deformations → compute forces and stresses

We solve the structure **globally**, but evaluate and design **locally**.

Notation: Local vs Global

Throughout this course, notation distinguishes the viewpoint:

- Global quantities: $\{u\}$, $\{f\}$, $[K]$
- Local (element) quantities: $\{u'\}$, $\{f'\}$, $[k']$

The prime (′) indicates quantities defined in an **element’s local coordinate system**.

Part 2 — Planar Trusses as a Model System

What Is a Plane Truss?

A **plane truss** is a two-dimensional framework of straight, prismatic members that:

- Lie entirely in a single plane
- Are connected by frictionless pin joints
- Carry **axial force only** (no bending or shear)
- Are loaded only at the joints

As a result, truss members experience **tension or compression only**.

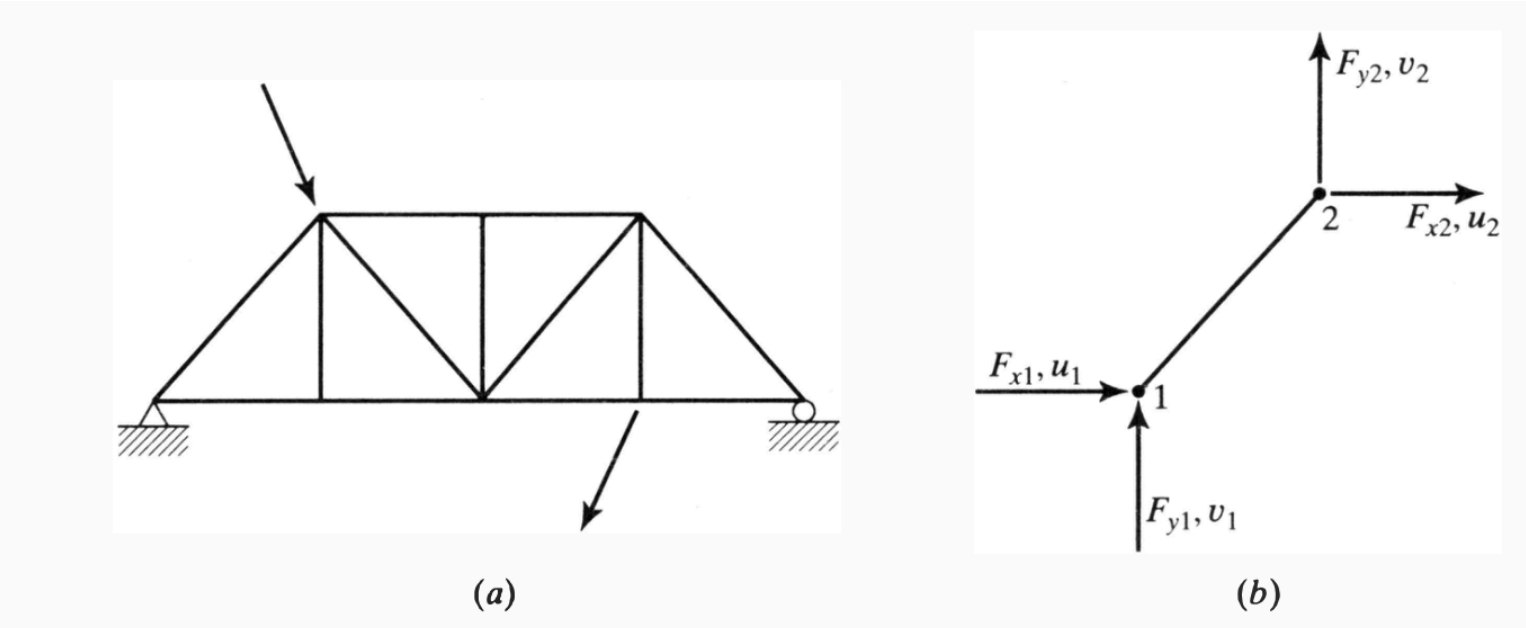


Figure. a) Idealized planar truss b) Typical truss member.

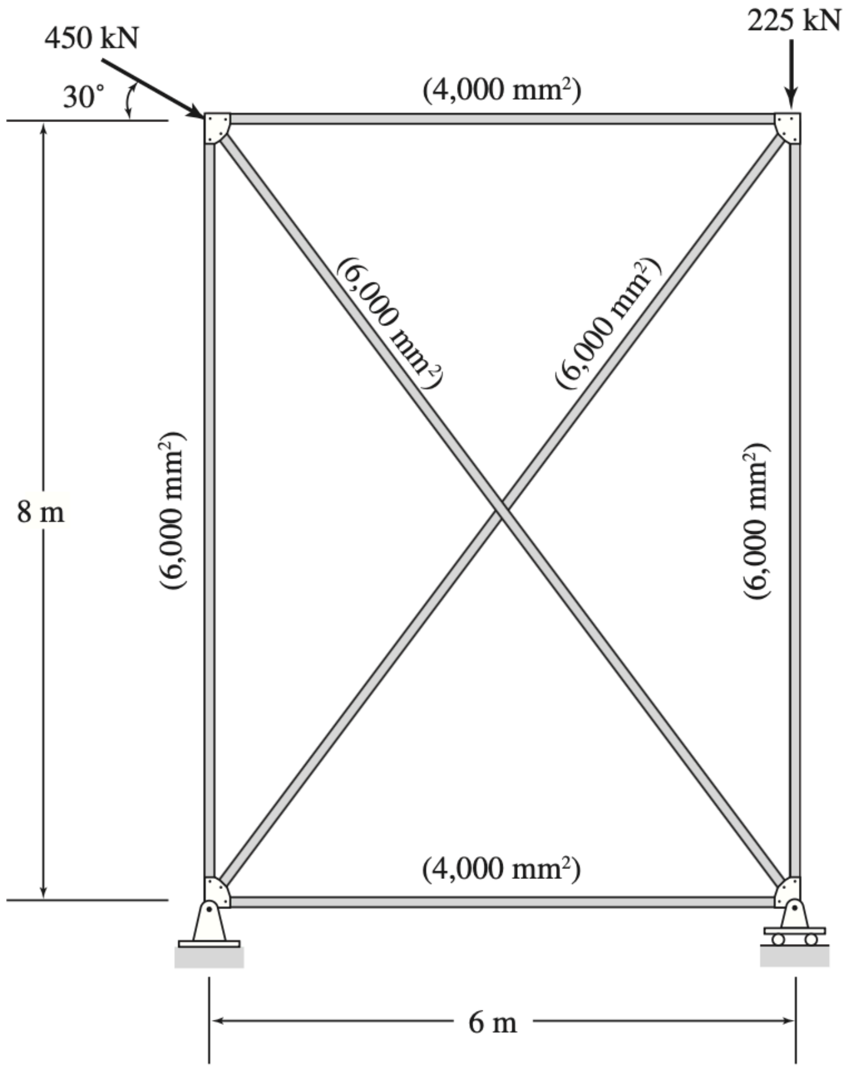
Components of a Truss Model

- Nodes (joints)
- Elements (members)
- Supports (boundary conditions)
- Applied forces

A **planar truss** is an idealized structural system used to carry loads through axial forces.

- Members are straight and prismatic
- Connections are modeled as frictionless pins
- All members lie in a single plane
- Loads are applied only at the joints

As a result, truss members carry **tension or compression only**.

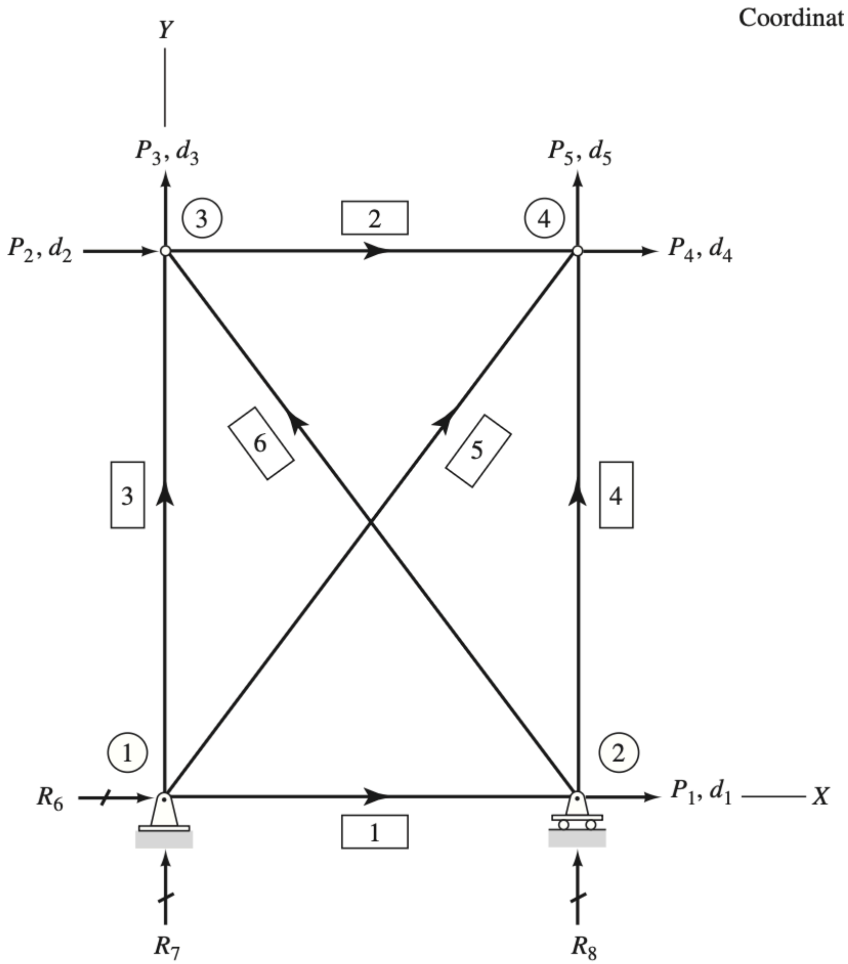


Idealized planar truss structure.

In matrix structural analysis, a truss is represented by a **discrete model** consisting of:

- **Nodes** (joints)
- **Elements** (members)
- **Supports** (boundary conditions)
- **Applied forces**

Each of these components contributes directly to the formulation of the global stiffness matrix.



Components of a planar truss analysis model.

DOFs for a 2D Truss Node

For a planar truss node:

- u_x — displacement in global $+x$
- u_y — displacement in global $+y$

Each free node has **2 degrees of freedom**. A truss with n nodes has up to $2n$ DOFs before supports are applied.

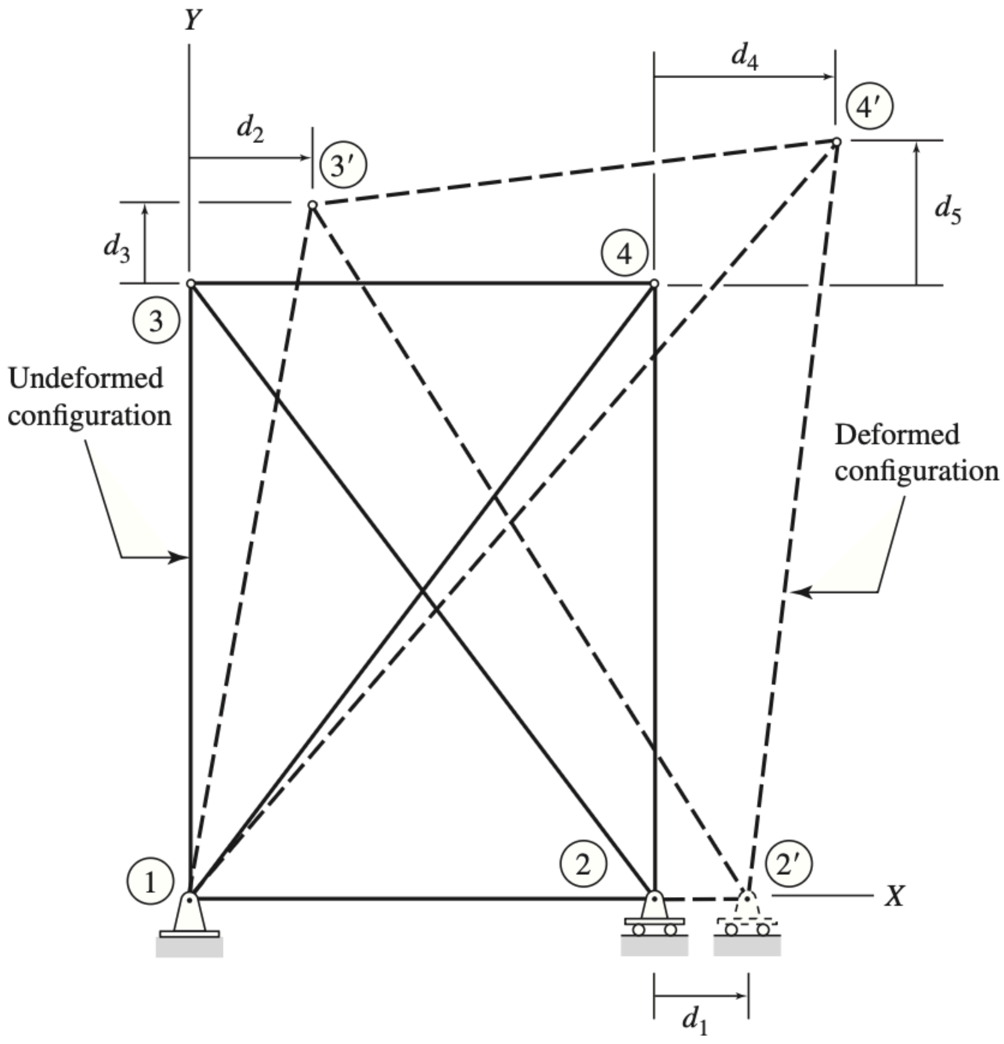


Figure. Truss joint displacements defined in the global coordinate system.

Joint displacements are defined with respect to the **global coordinate system**.

- Each free joint has translations in x and y
- Displacements are positive along the global axes
- These displacements form the global vector $\{u\}$

Part 3 — Global and Local Coordinate Systems

Why Local Coordinates?

- Axial deformation occurs **along the member axis**
- Constitutive relations are simplest in that direction
- Local coordinates isolate element behavior from global geometry

Coordinate and Sign Conventions

- Global axes: $+x$ to the right, $+y$ upward
- Positive axial force: **tension**
- Local $+x'$ axis: defined from start node to end node

Consistent conventions are essential for correct assembly.

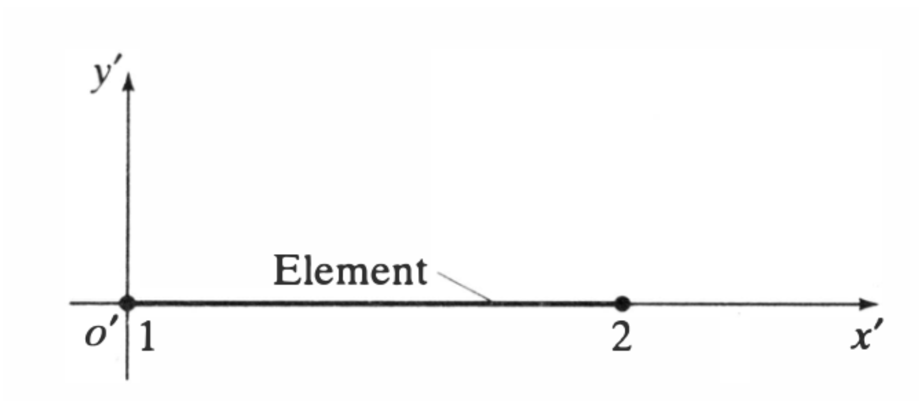


Figure. Definition of local element axis.

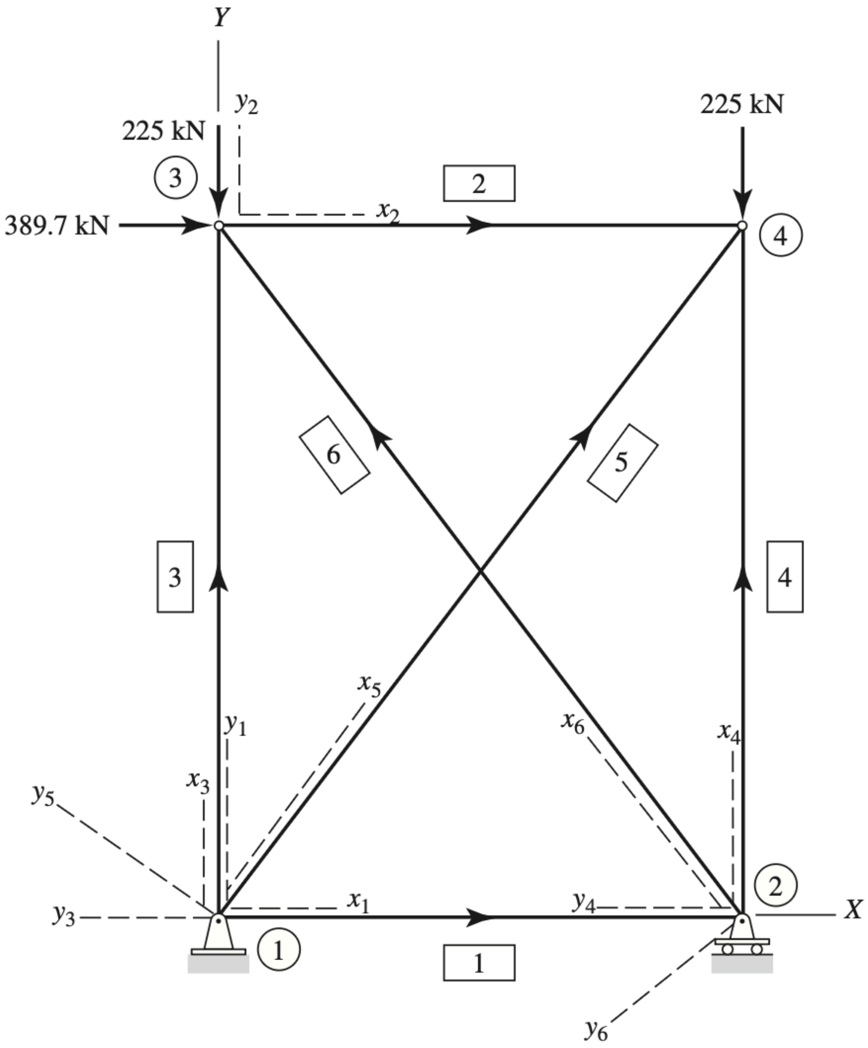


Figure. Local element axes superimposed on a truss structure.

Each truss member is assigned its own **local coordinate system**.

- The local axis x' is aligned with the member
- The positive direction is defined from start node to end node
- Axial deformation and force are expressed in this system

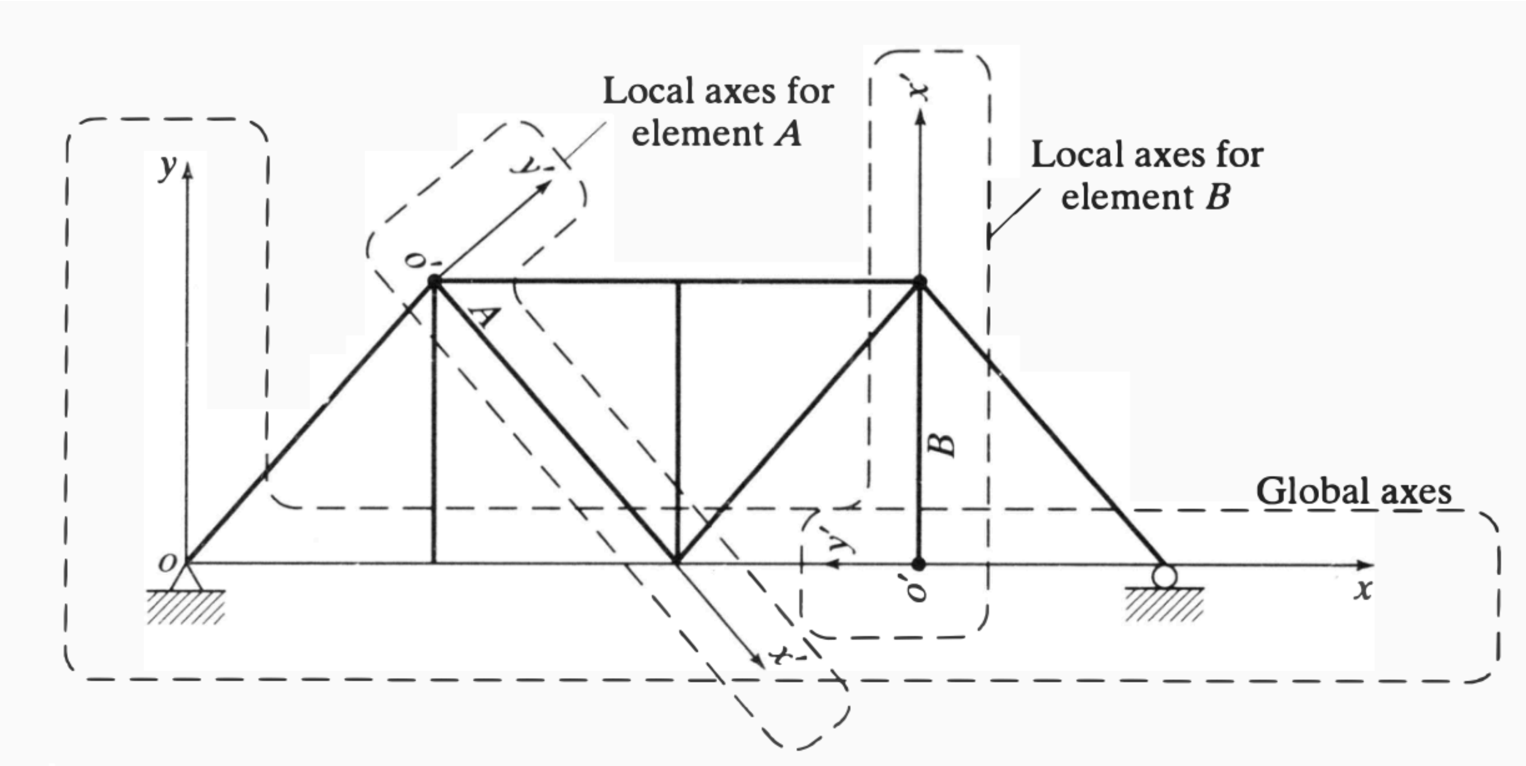


Figure. Relationship between global and local coordinate systems.

Part 5 — Axial Element Kinematics (Local)

> ***Draft note (delete later):*** *the axial force element (focus of Lecture 3.1)*

Consider a 2-node axial element in local coordinates:

- Local nodal displacements: u'_1, u'_2
- Axial deformation:

$$\delta = u'_2 - u'_1$$

- Axial strain:

$$\varepsilon = \frac{\delta}{L}$$

Part 6 — Axial Statics + Constitutive (EA/L)

> ***Draft note (delete later):*** *axial element statics, setting it up, Young's modulus,* EA/L

Hooke's law (uniaxial):

$$\sigma = E\varepsilon$$

Axial force:

$$N = A\sigma = EA\varepsilon = \frac{EA}{L}(u'_2 - u'_1)$$

End forces in local coordinates:

- Node 1:
  $f'_1 =$
  $-N$

- Node 2:
  $f'_2 =$
  $+N$

Part 7 — Local Stiffness Matrix (2×2)

> ***Draft note (delete later):*** *stiffness matrix form, basics ($k_{ij} = k_{ji}$), end with the 2×2 matrix*

Collect the local relation into matrix form:

$$\begin{bmatrix} f'_1 \\ f'_2 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix}$$

Key properties:

- Symmetric: $k_{ij} = k_{ji}$
- Rigid-body mode: if $u'_1 = u'_2$, then $\delta = 0$ and $\{f'\} = \{0\}$

Physical Interpretation

If $u'_1 = u'_2$:

- $\delta = 0$ so $N = 0$ and $f'_1 = f'_2 = 0$
- The element does not resist rigid translation along its axis

If $u'_2 = -u'_1$:

- Large extension/compression for a given magnitude

Part 8 — Flexibility Formulation (Preview)

> ***Draft note (delete later):*** *flexibility formulation, why important, will we use?*

Stiffness viewpoint:

$$\{f\} = [k]\{u\}$$

Flexibility viewpoint (inverse mapping):

$$\{u\} = [f]\{f\}$$

Why it matters:

- Leads to compatibility-based methods
- Useful for interpretation

In this course, we primarily use stiffness-based methods.

Part 9 — Local 4×4 Stiffness Matrix (Local-Only)

> ***Draft note (delete later):*** *formulate the 4×4 matrix in local (many zeros); more generic case (Kassimali §3.3); full derivation*

For a planar truss member, describe the element DOFs in the local coordinate system:

$$\{u\}_e^{(local)} = [u_{1x'},\ u_{1y'},\ u_{2x'},\ u_{2y'}]^T$$

In an ideal truss member, only the axial DOFs (x' direction) create axial strain. That means the element stiffness only couples $u_{1x'}$ and $u_{2x'}$.

Embed the 2×2 axial stiffness into a 4-DOF local description:

$$[k]_e^{(local)} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Interpretation:

- The zeros reflect that the truss element provides no stiffness in y'.

- The 4×4 form matches the 2 DOFs per node bookkeeping used later for assembly.

Part 10 — Concept Checks / Mini-Examples (McGuire §2.6)

> ***Draft note (delete later):*** *examples of non-matrix structural analysis truss problems using these concepts (McGuire §2.6)*

Check A: Given E, A, L and u_1', u_2', compute δ, N, f_1', f_2'.

Check B: If EA/L doubles, what happens to end forces for the same $\{u'\}$?

Check C: What displacement pattern produces zero force, and why?

Check D: In the local 4×4 matrix, which DOFs are inactive, and what physical assumption causes that?

In [ ]:
```
# (Optional) Mini-example code stub
# Define E, A, L, and local nodal displacements u1p, u2p

import numpy as np

E = None
A = None
L = None
u1p = None
u2p = None

# Local 2×2 stiffness
# k2 = (E*A/L) * np.array([[1, -1], [-1, 1]])
k2 = None

# TODO: compute f' = k2 * u'
u_local = None
f_local = None

k2, u_local, f_local
```

Out[ ]:
```
(None, None, None)
```

Looking Ahead

➡️ Next (Lecture 3.2):

- Build the transformation matrix from local to global
- Rotate element stiffness into global coordinates
- Assemble the global stiffness matrix for a truss
- Apply supports, solve for displacements, and recover member forces