CEE6501 — Lecture 3.1

Local Behavior of an Axial Element

Learning Objectives

By the end of this lecture, you will be able to:

- Define truss DOFs and a consistent sign convention
- Explain local vs global coordinates for an axial member
- Derive the 2×2 local stiffness matrix for an axial element
- Interpret stiffness matrix properties (symmetry, rigid-body mode)
- Describe the idea of flexibility as the inverse relation (preview)
- Formulate the local 4×4 element stiffness matrix (local-only; not transformed)

Agenda

1. Definitions & concepts (DOFs, global vs local)
2. Planar trusses as a model system (DOFs, nodal vectors)
3. Truss stability & determinacy (why statics is not enough)
4. Global and local coordinate systems (notation + sign conventions)
5. Axial element kinematics (local)
6. Axial statics + constitutive relation $\left(\frac{EA}{L}\right)$
7. Local stiffness matrix (2×2)
8. Flexibility formulation (preview)
9. Local 4×4 element stiffness matrix (local; not transformed)
10. Concept checks / mini-examples (McGuire §2.6)

Part 1 — Introductory Definitions & Concepts

Degrees of Freedom (DOFs)

A **degree of freedom (DOF)** is an **independent displacement component** used to describe a structure’s motion.

The set of DOFs is the **minimum set of joint displacement components** needed to uniquely describe the deformed configuration under arbitrary loading.

What DOFs Represent

In matrix analysis, DOFs are:

- the **unknown displacement components** we solve for
- the **locations/directions** where nodal loads are applied
- the coordinates used to describe the structure's deformation

Once DOFs are defined, the response is written compactly in matrix form.

DOFs Depend on the Structural Model

The structural model determines **what motion is allowed**:

- **Trusses:** joint translations only
- **Frames:** joint translations **and rotations**
- **Higher-order models:** additional deformation modes

So “DOFs” are not universal — they depend on the assumptions built into the model.

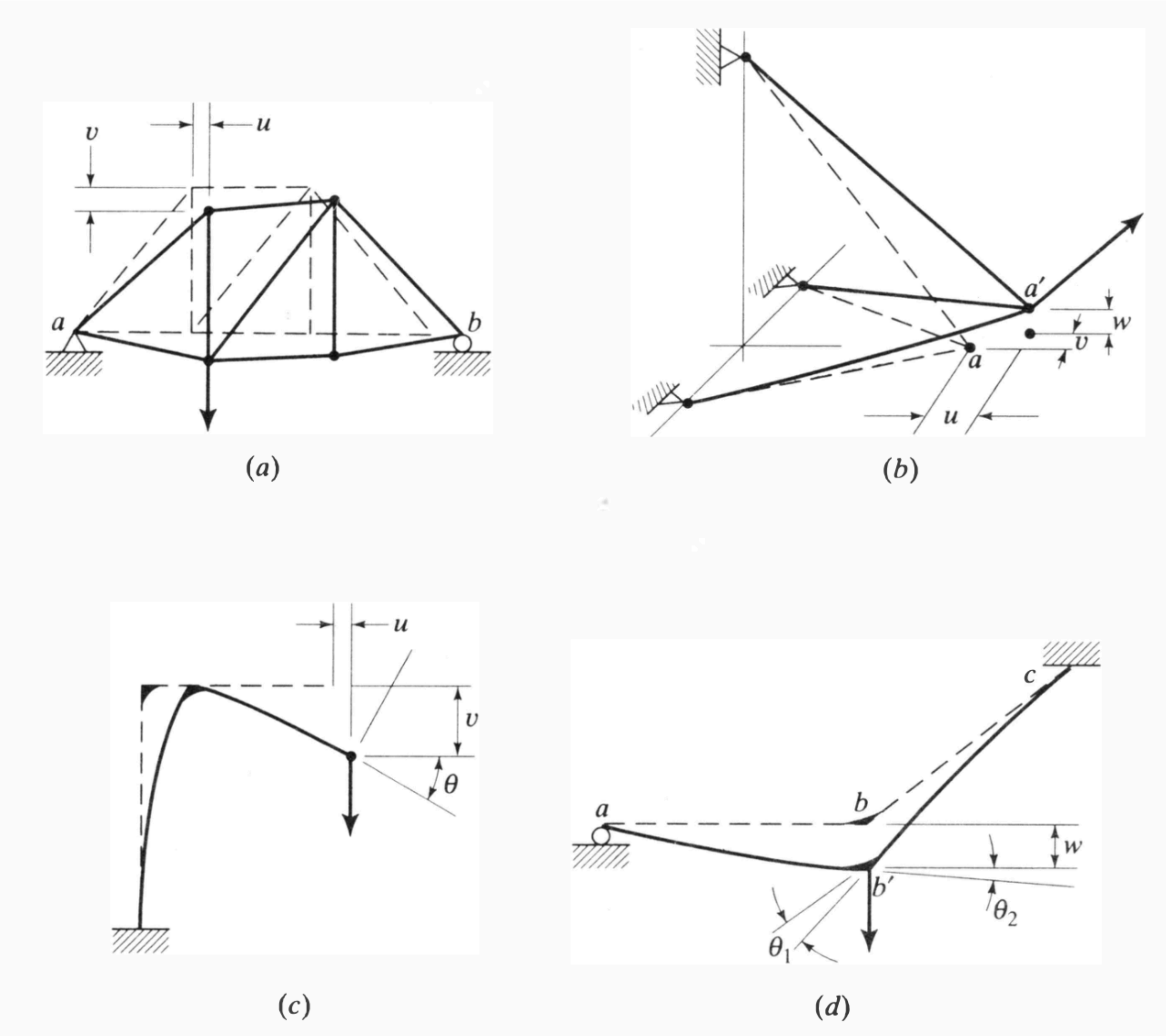


Figure 2.1 Joint displacements. (a) Pin-jointed plane truss. (b) Pin-jointed space truss. (c) Plane frame (in-plane loading). (d) Plane frame (out-of-plane loading).

Global vs Local Viewpoint

In structural analysis we constantly switch between two perspectives:

- **Global:** how the *entire structure* moves and equilibrates
- **Local:** how an *individual element* deforms internally

This distinction is fundamental to the matrix stiffness method.

Global–Local Workflow

The stiffness method follows a consistent pattern:

local element behavior
$\longrightarrow$ assemble to a **global system**
$\longrightarrow$ solve for **global displacements**
$\longrightarrow$ recover **local deformations**
$\longrightarrow$ compute **member forces / stresses**

We solve the structure **globally**, but evaluate and design **locally**.

Notation: Local vs Global

Notation: Global vs Local

We use notation to distinguish viewpoints:

- **Global (structure):** $\mathbf{u}$, $\mathbf{f}$, $\mathbf{K}$
- **Local (element):** $\mathbf{u}'$, $\mathbf{f}'$, $\mathbf{k}'$

The prime $(')$ indicates quantities defined in an **element's local coordinate system**.

Part 2 — Planar Trusses as a Structural System

What Is a Plane Truss?

A **plane truss** is a two-dimensional framework of straight members that:

- lie entirely in a single plane
- connect through **frictionless pin joints**
- carry **axial force only** (no bending or shear)
- are loaded only at the joints

Key consequence: each member is in **tension** or **compression** only.

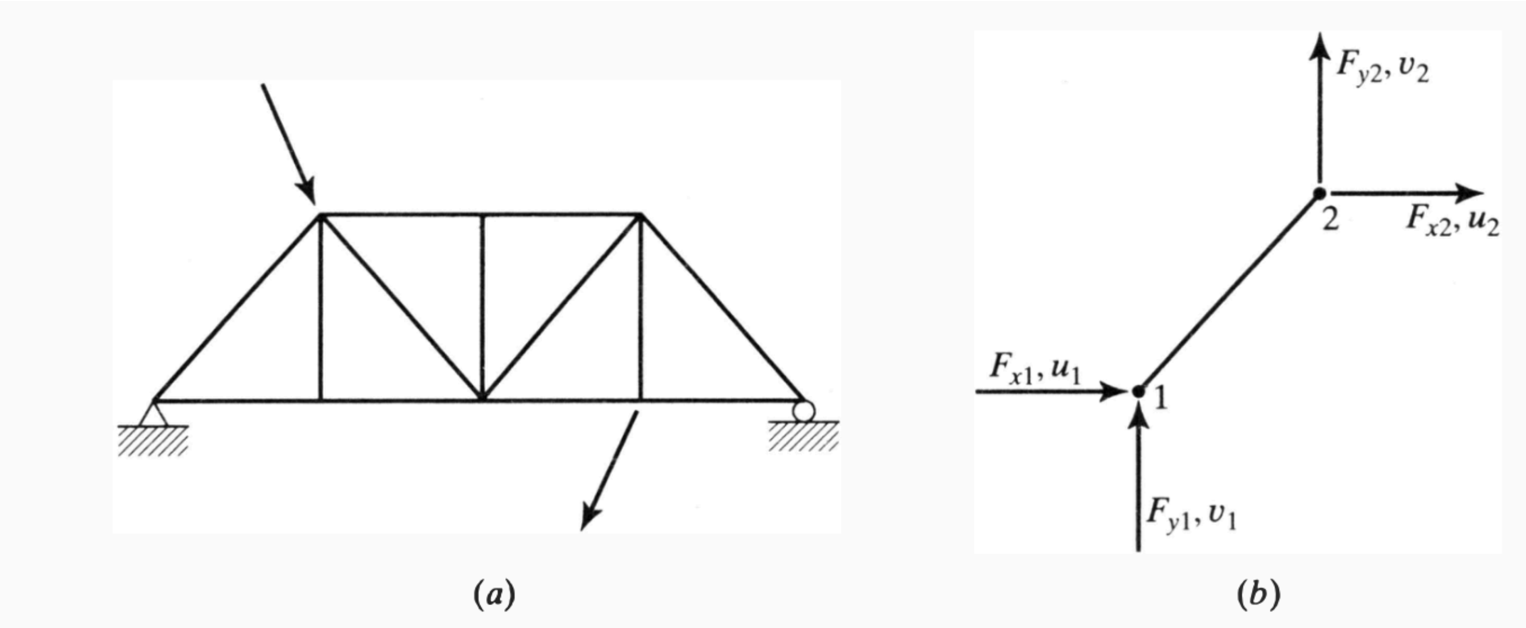


Figure. (a) Idealized planar truss (pin-jointed members). (b) Typical truss member carrying axial force.

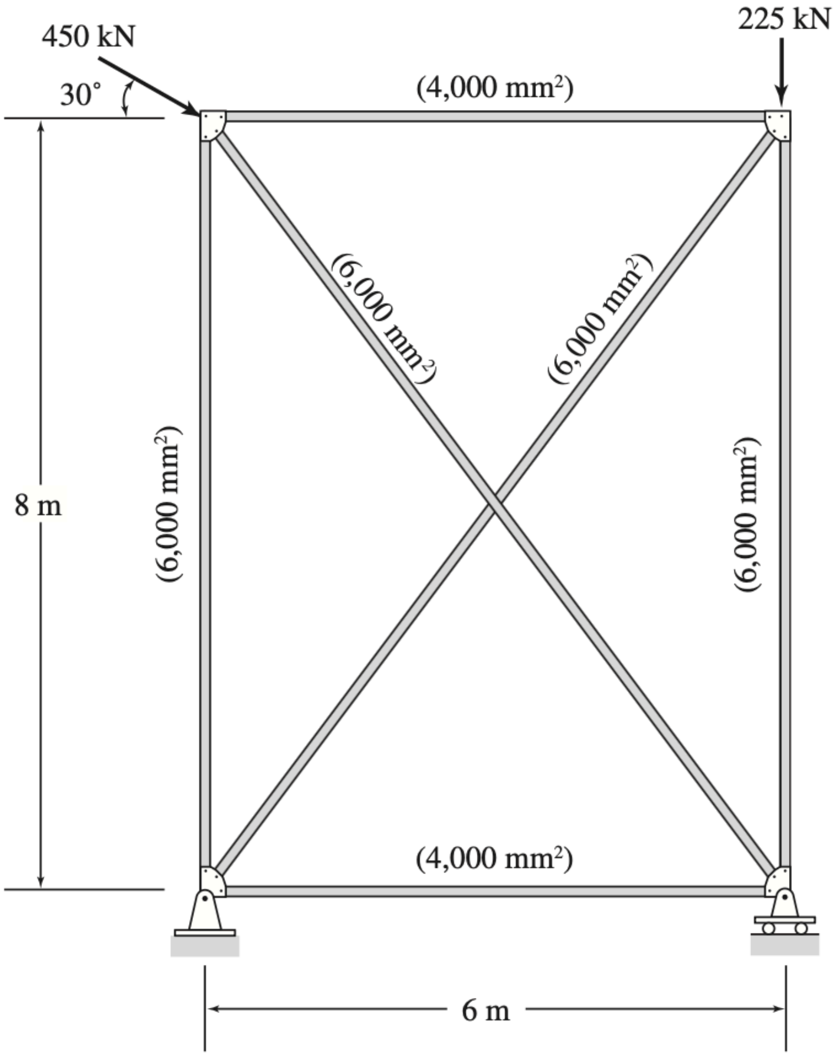


Figure. Idealized planar truss showing member properties, loads, and supports.

Why Trusses Are Ideal for Matrix Analysis

Because members carry **axial force only**:

- Each member behaves like a **1D axial element**
- Force–displacement relations are **linear and simple**
- The local behavior can be derived cleanly and assembled into a system

This makes trusses an excellent first application of the **matrix stiffness method**.

Components of a Truss Analysis Model

A truss analysis (analytical) model has four ingredients:

- **Nodes (joints):** where displacements are defined
- **Elements (members):** connect nodes and carry axial force
- **Supports:** restrain specific displacement components
- **Applied forces:** external loads acting at nodes

What we identify on the line diagram of a truss:

- **Nodes** (circled numbers)
- **Members** (element connectivity between nodes)
- **Applied loads** (known forces at DOF locations)
- **Supports** (restrained displacement components)
- **Free DOFs** (unknown joint translations)

Next we make the DOFs explicit and store them in a displacement vector.

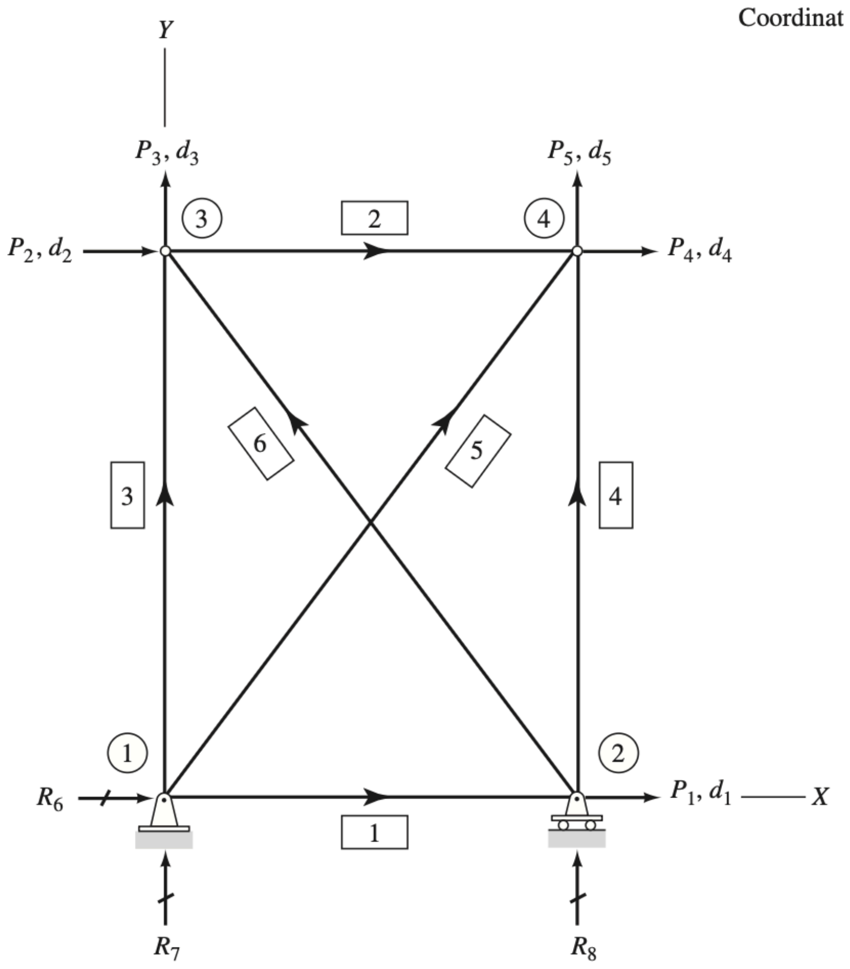


Figure. Components of a planar truss analysis model.

DOFs for a 2D Truss Node

For a planar truss joint, the possible displacement components are:

- u_x — translation in global $+x$
- u_y — translation in global $+y$

Each **free joint** therefore has **two DOFs**.

A truss with j joints has up to $2j$ joint displacement components *(before supports are applied)*.

Free vs Restrained Joints

Not all joints are free to move.

- **Free joints**
  → displacements are **unknown**
  → contribute DOFs

- **Restrained joints (supports)**
  → displacements are **prescribed** (often zero)
  → remove DOFs

Supports eliminate specific displacement components in x and/or y.

Restrained Degrees of Freedom

Each support restrains one or more displacement components:

- **Pin support**
  → restrains u_x and u_y
  → contributes **2** restraints

- **Roller support**
  → restrains **one** displacement component
  → contributes **1** restraint

We count **restrained displacement components**, not supports.

Let r be the total number of restrained joint displacement components.

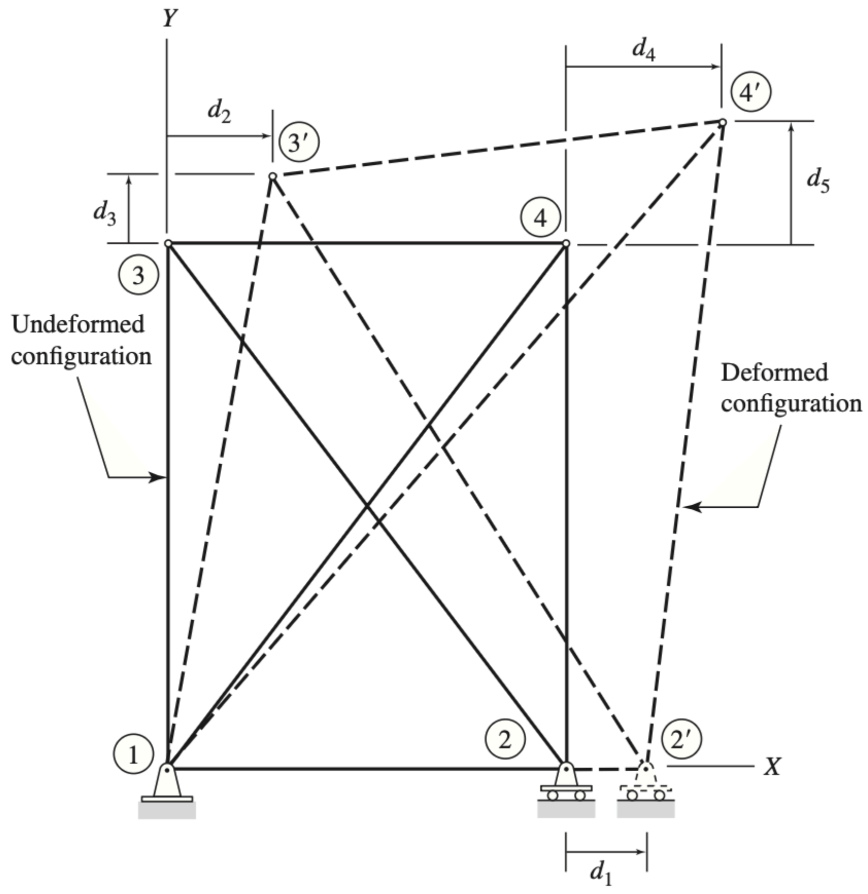


Figure. Truss joint displacements defined in the global coordinate system.

The DOFs are the **independent joint translations** that describe the deformed shape.

- d_1–d_5 are defined in the global x–y system
- Positive directions follow the global axes
- Collect them into a displacement vector $\{d\}$

Structural Vectors: Displacements and Loads

Recall the global equilibrium relation:

$$\mathbf{K}\,\mathbf{u} = \mathbf{f}$$

In matrix form, we collect the structure's unknowns and knowns into vectors.

Displacement vector

- $\mathbf{u}$ — global displacement vector
    - $\mathbf{d}$ → **free degrees of freedom** (unknown joint displacements)
    - 0 → **restrained degrees of freedom** (prescribed displacements)

Force vector

- $\mathbf{f}$ — global force vector
    - $\mathbf{P}$ → **applied nodal loads** at free DOFs
    - $\mathbf{R}$ → **reaction forces** at restrained coordinates

Applied load vector (free DOFs)

$$\mathbf{P} = \{P_1,\ P_2,\ P_3,\ P_4,\ P_5\}$$
$$= \{0,\ 389.7,\ -225,\ 0,\ -225\}$$

- $P_1 = 0$
- $P_2 = 389.7$ kN (positive x)
- $P_3 = -225$ kN (negative y)
- $P_4 = 0$
- $P_5 = -225$ kN (negative y)

Each load component corresponds directly to a numbered **degree of freedom**.

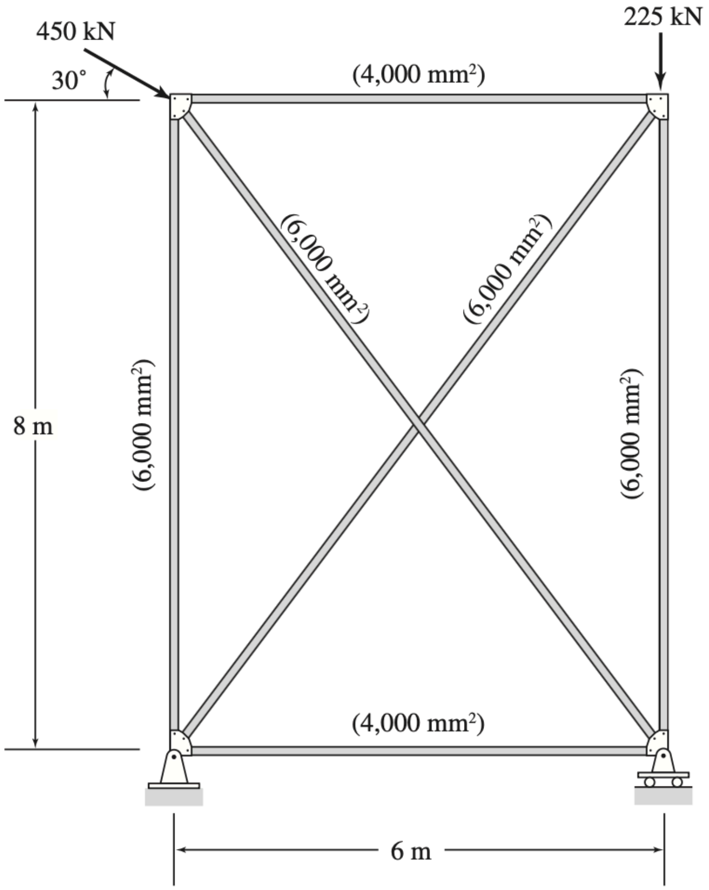
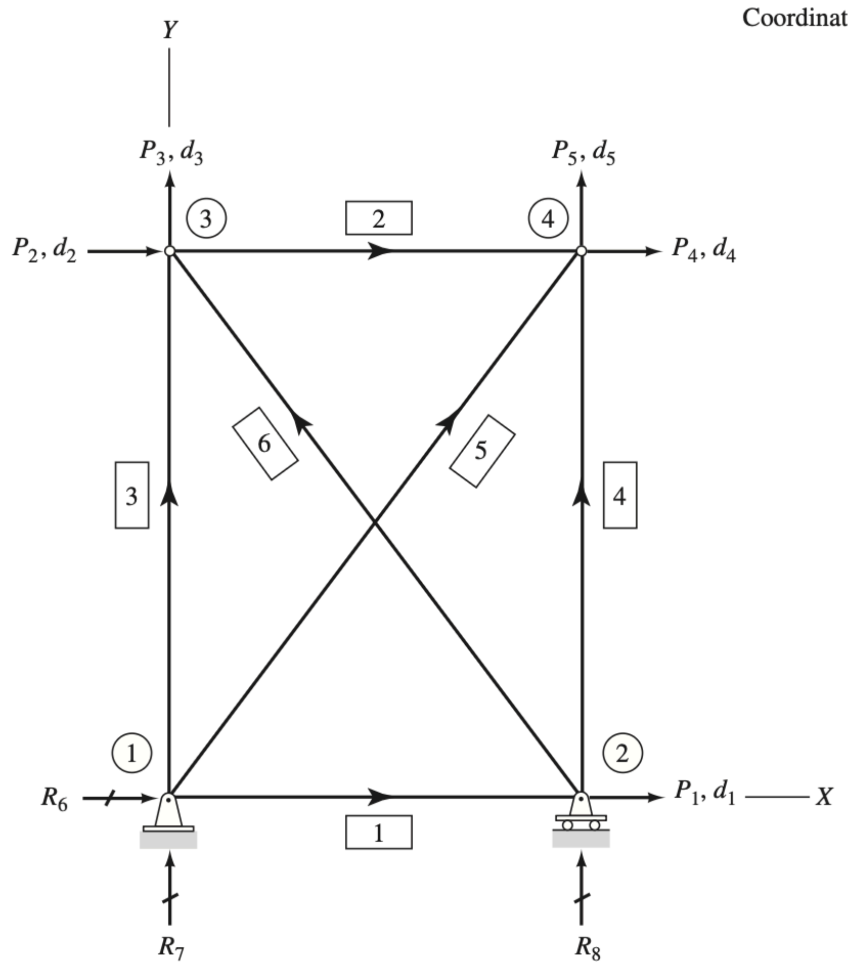


Figure. Applied nodal loads.

The global equilibrium is written as:

$$\mathbf{K}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 389.7 \\ -225 \\ 0 \\ -225 \\ R_6 \\ R_7 \\ R_8 \end{Bmatrix} \text{kN}$$



Components of a planar truss analysis model.

In-Class Exercise - Structural Vectors

1. **Number the joints** (nodes) on the figure.
2. **Number the members** (elements).
3. **Assign global DOF numbers** at each node.
4. **Identify DOF types**:
    - *Circle* free (unknown) DOFs
    - *Box* DOFs with applied loads
    - *Underline* restrained DOFs (reaction forces)
5. **Write the global displacement vector** $\{u\}$.
6. **Write the global force vector** $\{f\}$.

(Based on your DOF numbering; use variables where unknown and insert known values where prescribed.)

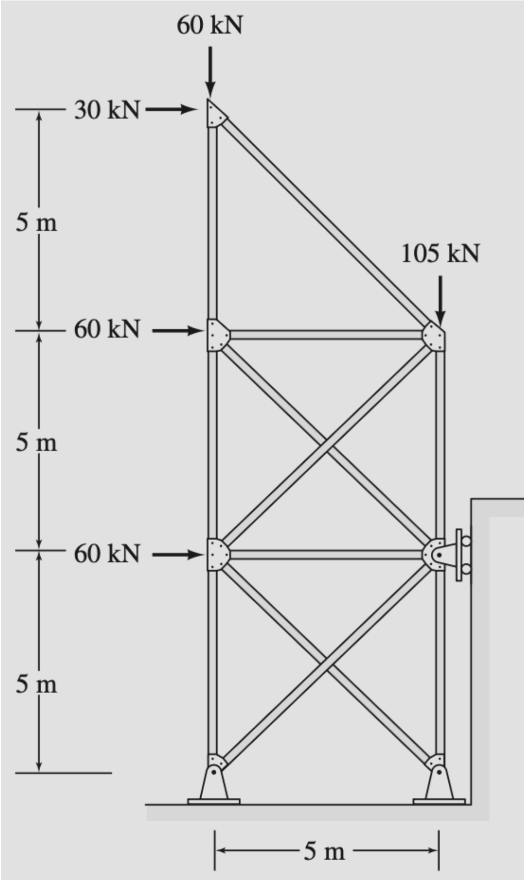


Figure. Truss structure.

Answers — Structural Vectors

$$
\boldsymbol{u} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_5 \\ u_6 \\ 0 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \end{Bmatrix} \qquad \boldsymbol{f} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ 60 \\ 0 \\ r_7 \\ 0 \\ 60 \\ 0 \\ 0 \\ -105 \\ 30 \\ 60 \end{Bmatrix}
$$

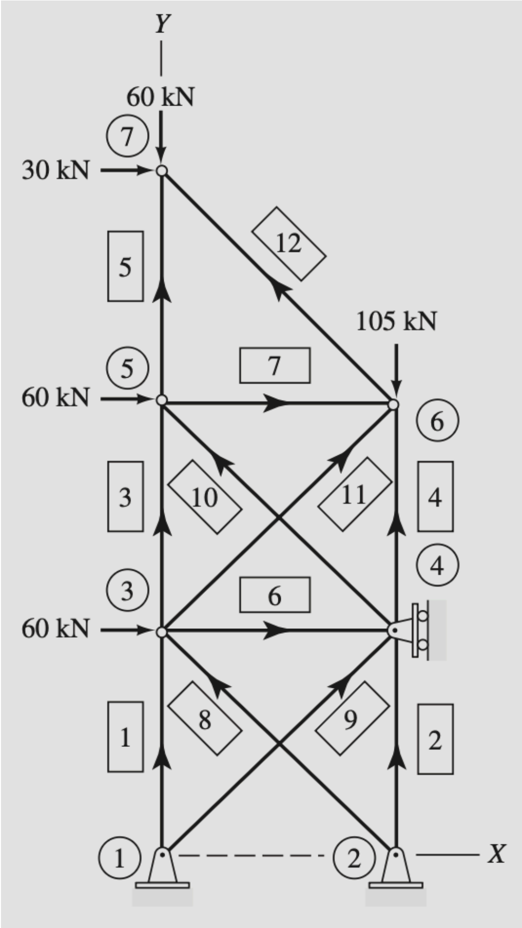


Figure. Truss structure numbered.

Part 3 - Truss Stability and Determinacy

Counting Degrees of Freedom

Let:

- j = number of joints
- r = number of restrained joint displacement components
- N_{CJT} = DOFs per free joint

Then the number of structural degrees of freedom is:

$$N_{\mathrm{DOF}} = N_{\mathrm{CJT}}\, j - r$$

DOFs for a Planar Truss

For a planar truss, each free joint has two translational DOFs:

$$N_{\mathrm{CJT}} = 2 \quad (u_x,\ u_y)$$

So:

$$\boxed{N_{\mathrm{DOF}} = 2j - r}$$

This is the number of **independent joint displacements** that must be solved for.

Stability vs Static Determinacy

- **Stability:** does the structure prevent rigid-body motion?
- **Static determinacy:** can forces be found from equilibrium alone?

These are related, but **not the same**.

We will separate:

- **stability** (rigid-body motion)
- **external determinacy** (reactions from global equilibrium)
- **internal determinacy** (member forces from joint equilibrium)

Stability Requirement (Rigid-Body Motion)

A free planar structure has **three rigid-body motions**:

- translation in x
- translation in y
- rotation in the plane

To prevent rigid-body motion, the supports must restrain at least:

$$r \geq 3$$

This is a **stability requirement**, not a determinacy condition.

- **Stability** asks whether the structure can resist rigid-body motion and admit an equilibrium configuration.
- **Determinacy** is a separate question, asked **only after stability is ensured**, and concerns whether forces can be found from equilibrium alone.

Global Equilibrium (What Statics Provides)

For a planar structure, global equilibrium provides **three equations**:

- $\sum F_x = 0$
- $\sum F_y = 0$
- $\sum M = 0$

These equations apply to the **entire structure** treated as a rigid body.

They govern **reaction forces only** (external equilibrium).

External (Global) Static Indeterminacy

Let r be the number of reaction components.

Global equilibrium provides **3 equations** in 2D.
So if:

$$r > 3$$

then reactions cannot be determined from statics alone.

The structure is **externally statically indeterminate**.

Internal Static Determinacy for a Planar Truss (Counting)

At each joint, equilibrium gives two equations:

$$\sum F_x = 0, \qquad \sum F_y = 0$$

Across j joints, that is $2j$ **joint equilibrium equations**.

Unknown forces are:

- m member axial forces
- r reaction components

A necessary counting condition for solving forces by equilibrium is thus:

$$m + r = 2j$$

Interpreting the Internal Counting Condition

The comparison

$$m + r \quad \text{vs.} \quad 2j$$

compares **unknown forces** to **joint equilibrium equations**:

- $m + r < 2j$
  → **unstable / mechanism** (not enough constraints; a deformation mode exists)
- $m + r = 2j$
  → **statically determinate by counting** *(if geometry is stable)*
- $m + r > 2j$
  → **statically indeterminate by counting** (redundant member/support forces)

This is a **counting test** — it does not guarantee stability.

What the Counting Condition Does *Not* Guarantee

Counting compares equations and unknowns, but it does **not**:

- detect rigid-body motion
- detect geometric mechanisms
- guarantee a unique solution

So passing a counting test is **necessary**, not sufficient.

Possible Outcomes

A structure can therefore be:

- **Stable but statically indeterminate**
    - stable (no rigid-body motion)
    - but has redundancies: $m + r > 2j$ or $r > 3$
    - forces depend on deformation compatibility
- **Unstable but satisfy** $m + r = 2j$
    - the count matches, but the geometry forms a mechanism
    - equilibrium equations exist, but the structure can move without resistance

Why the Direct Stiffness Method is Powerful

Equilibrium alone does **not** enforce compatibility.

The Direct Stiffness Method solves by enforcing:

- equilibrium at joints
- compatibility of joint displacements
- member force–deformation relations

We can still solve indeterminate cases when:

$$r > 3 \quad \text{or} \quad m + r > 2j$$

If a structure is **unstable**, it appears as a **singular stiffness matrix $\mathbf{K}$**.

In-Class Exercise — Stability & Determinacy

Questions (work in pairs):

1. How many **joints** (*j*) does this structure have?
2. How many **members** (*m*) are present?
3. How many **reaction components** (*r*) are provided by the supports?
4. Does the structure satisfy the **stability requirement**?
5. Based on counting, is the structure **statically determinate or indeterminate**?
6. If indeterminate, is the indeterminacy **external, internal, or both**?

No force calculations — focus on counting and concepts.

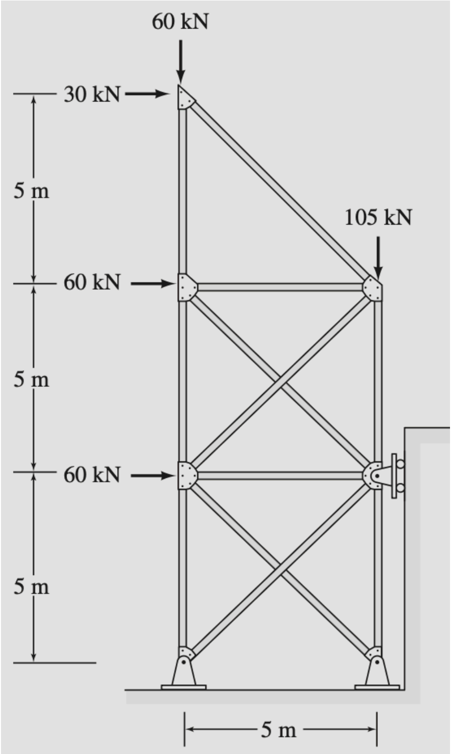


Figure. Truss used for stability and determinacy assessment.

✅ Answers — Stability & Determinacy

- **Number of joints**: $j = 7$:
- **Number of members**: $m = 12$:
- **Number of reaction components** $r = 2 + 2 + 1 = 5$:
- **Stability requirement:**
  For a planar structure, external stability requires

$$r \geq 3$$

Since $5 \geq 3$, the structure **is stable**.

- **Static determinacy (counting test):**
  For a planar truss, compare

$$m + r \quad \text{and} \quad 2j$$

$$m + r = 12 + 5 = 17, \qquad 2j = 2(7) = 14$$

  Since (17 > 14), the structure is **statically indeterminate**.

- **Type of indeterminacy:**

  - Total indeterminacy:

$$D_t = m + r - 2j = 3$$

  - External indeterminacy:

$$D_e = r - 3 = 2$$

  - Internal indeterminacy:

$$D_i = D_t - D_e = 1$$

👉 The structure is **indeterminate both externally and internally**.

External indeterminacy
There are **more reaction components than required for global equilibrium**, so the support reactions cannot be determined using statics alone.

Internal indeterminacy
The structure contains **more members than are required for equilibrium**, meaning at least one **redundant load path** exists and some member forces cannot be found using statics alone.

Another way to think about indeterminacy:
Indeterminacy often means there is **more than one load path** available for forces to travel through the structure.
How the load is shared between these paths depends on **stiffness and deformation compatibility**, not just equilibrium.

✅ This is **not a problem** for the **Direct Stiffness Method**:
stiffness-based formulations resolve redundancy automatically by enforcing **compatibility and equilibrium simultaneously**.

Part 4 — Global and Local Coordinate Systems

Why Local Coordinates?

Local coordinates simplify element behavior:

- axial deformation occurs **along the member axis**
- stress–strain relations are simplest in that direction
- local coordinates separate **element behavior** from **global geometry**

Coordinate and Sign Conventions

- Global axes: $+x$ to the right, $+y$ upward
- Positive axial force: **tension**
- Local $+x'$ axis: defined from **start node** to **end node**

Consistent conventions are essential for correct assembly.

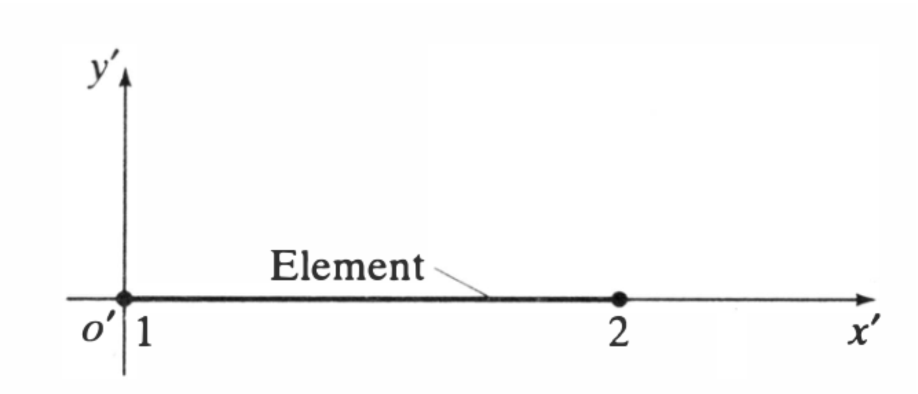


Figure. Definition of local element axis.

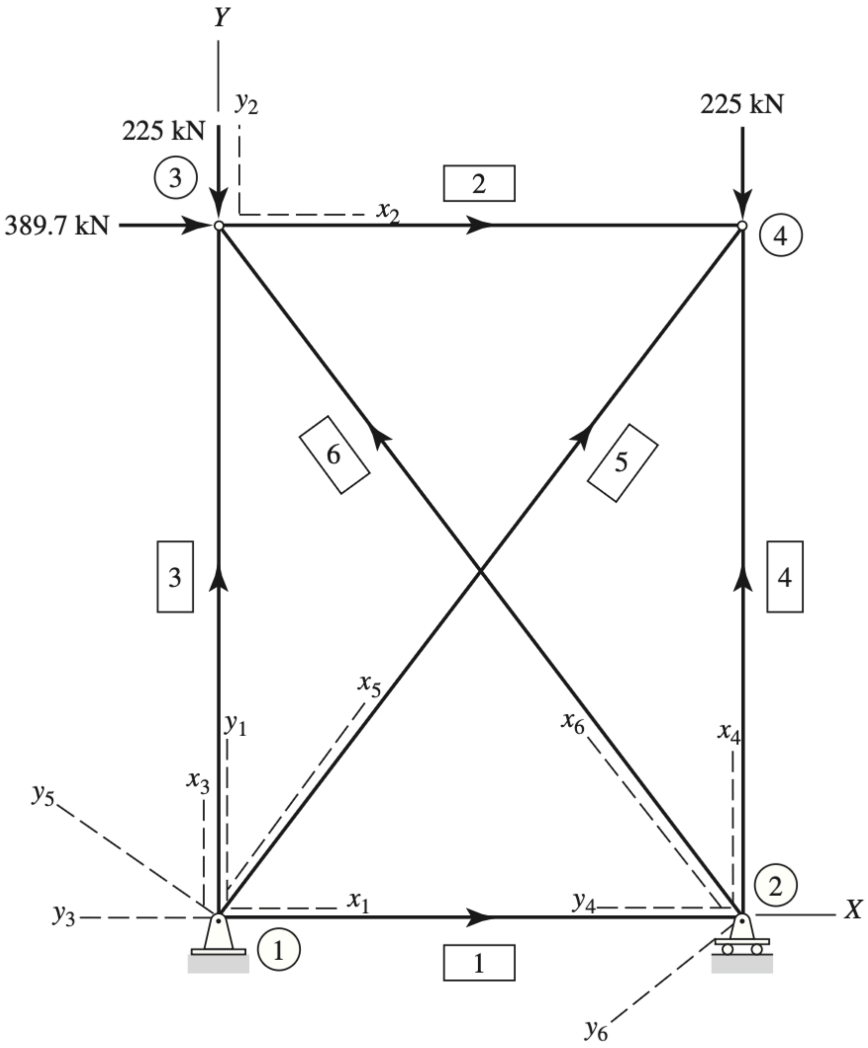


Figure. Local element axes superimposed on a truss structure.

Each truss member is assigned its own **local coordinate system**.

- The local axis x' is aligned with the member
- The positive direction is defined from start node to end node
- Axial deformation and force are expressed in this system

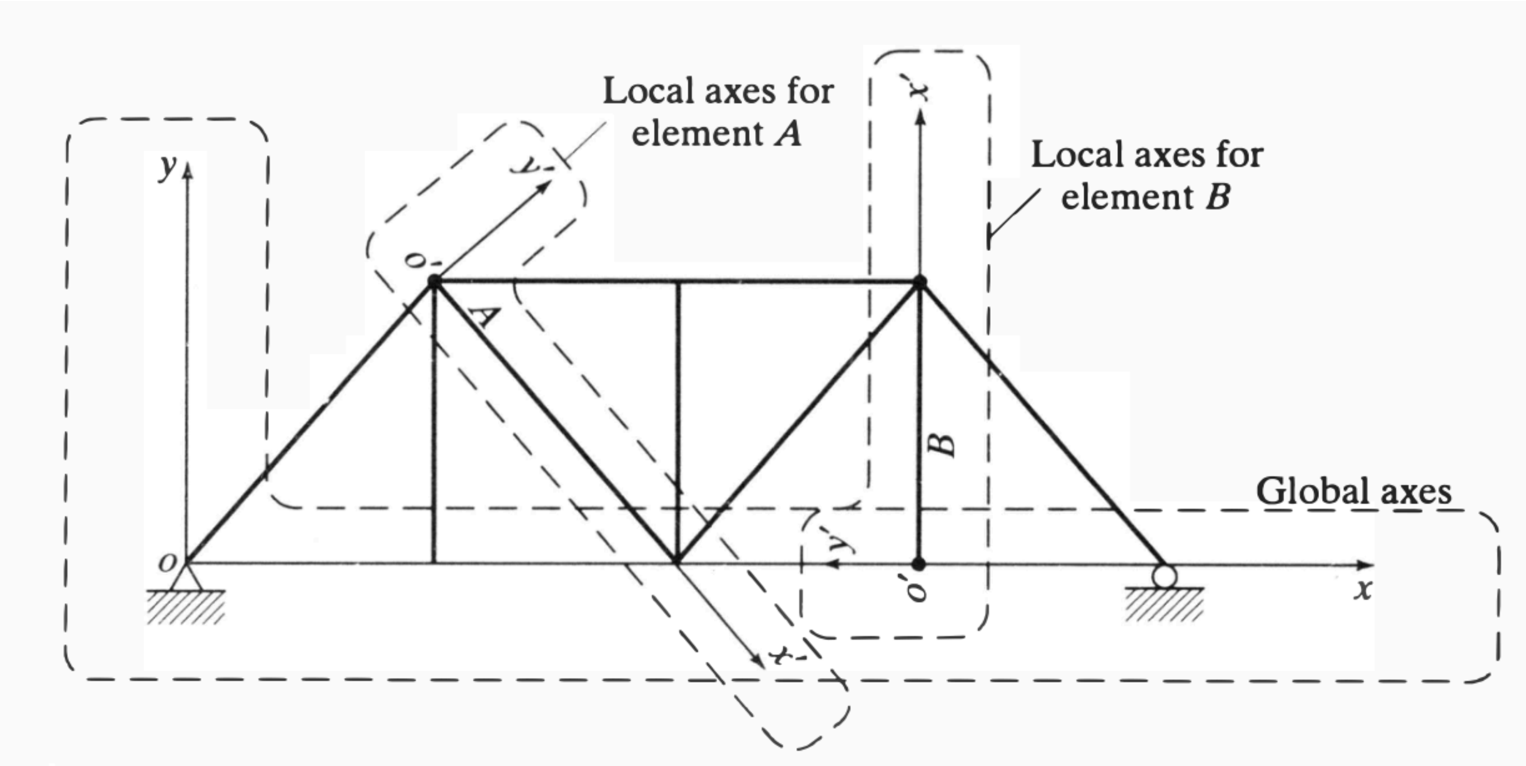


Figure. Relationship between global and local coordinate systems.

Part 5 — Axial Element Kinematics

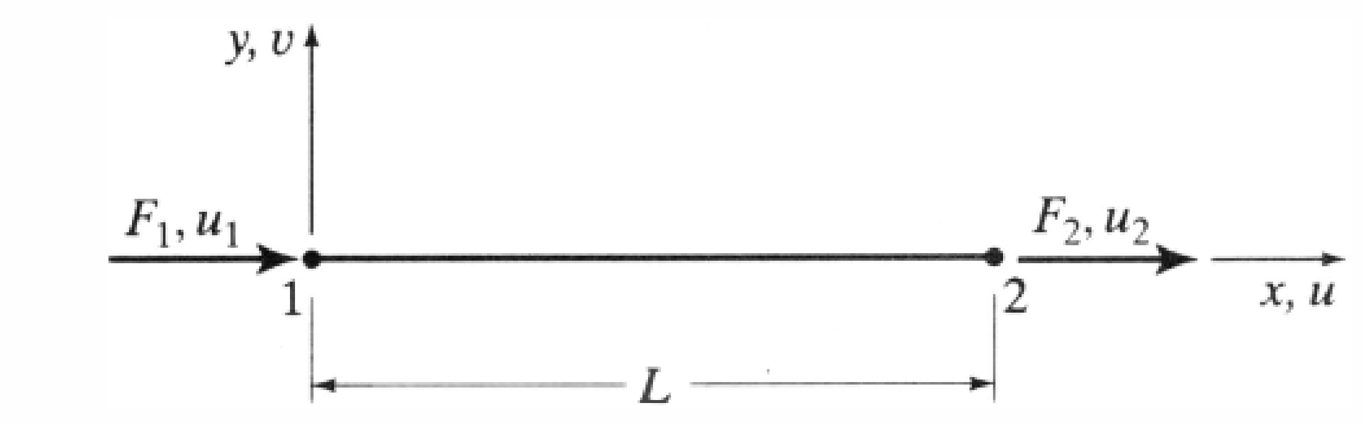
Axial truss element (local view)

- Straight, prismatic truss member
- Length L, area A, Young's modulus E
- Two nodes define the element: node 1 and node 2
- Two **local axial DOFs**: u_1 and u_2 (one at each node)
- Local x axis is aligned with the member (positive to the right)

TO-DO

Note that in the following section it is assumed we are only talking about local level behavior, hence the $'$ notation will not be used to distinguish between local and global matrices

Local degrees of freedom



- Axial displacement at node 1: u_1
- Axial displacement at node 2: u_2

In this section: axial DOFs only (u_1, u_2).

> *Note: transverse DOFs (v_1, v_2) appear later when we build the full 4×4 local truss matrix.*

Axial kinematics

The axial deformation is the relative displacement of the ends:

$$\delta = u_2 - u_1$$

- If $u_2 > u_1$ then $\delta > 0$ → **elongation (tension)**
- If $u_2 < u_1$ then $\delta < 0$ → **shortening (compression)**

Free-body diagram (axial)

- Axial end force at node 1: F_1
- Axial end force at node 2: F_2

Equilibrium along the member axis:

$$\sum F_x = 0$$

➡️ The end forces are equal and opposite.

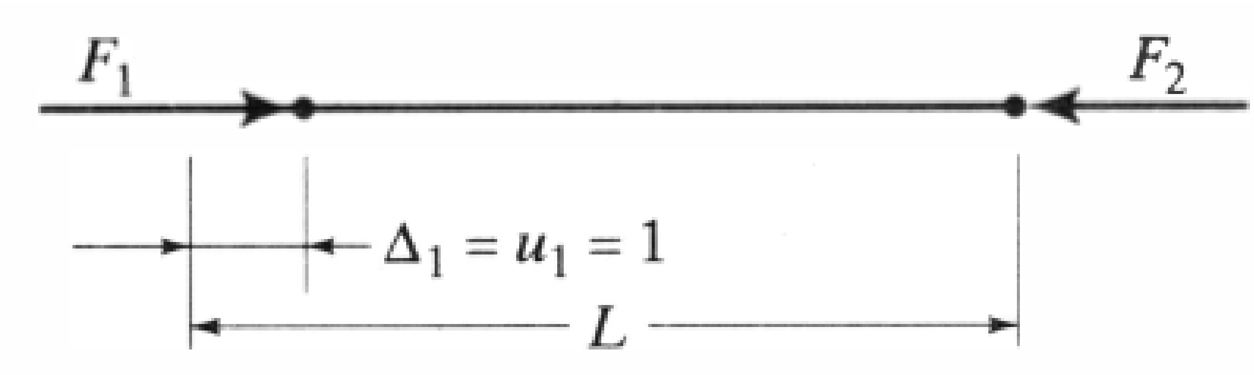
Kinematics examples (what do δ and F look like?)

Compression example: $u_1 = 1, u_2 = 0$

$$\delta = 0 - 1 = -1 < 0 \quad \Rightarrow \text{compression}$$

Take positive as pointing to the right:

$$+F_1 - F_2 = 0 \quad \Rightarrow \quad F_1 = F_2$$

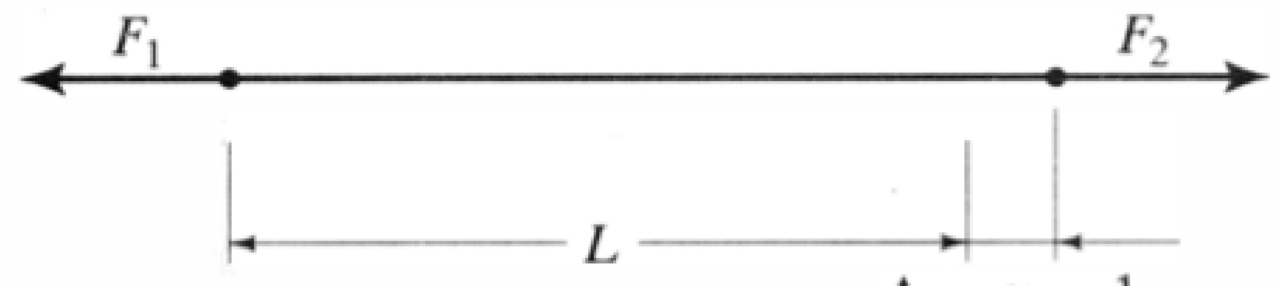


Tension example: $u_1 = 0, u_2 = 1$

$$\delta = 1 - 0 = 1 > 0 \quad \Rightarrow \text{tension}$$

Take positive as pointing to the right:

$$-F_1 + F_2 = 0 \quad \Rightarrow \quad F_1 = F_2$$



(c)

Part 6 — Deriving Local 2x2 Stiffness Matrix

Axial constitutive law (linear elasticity)

Hooke's law (uniaxial, linear elastic):

$$E = \frac{\sigma}{\varepsilon}$$

Strain (Egnineering strain, small deformation, $L = 0$):

$$\varepsilon = \frac{\delta}{L} = \frac{u_2 - u_1}{L}$$

Stress definition:

$$\sigma = \frac{F}{A}$$

Axial force:

$$F = \frac{EA}{L}(u_2 - u_1)$$

> *Notes: In geometric nonlinearity, the "L" and kinematics change; in material nonlinearity, the σ–ε law changes.*

End forces in terms of u_1, u_2

Using the same sign convention (positive to the right):

$$F_1 = \frac{EA}{L}(u_1 - u_2)$$

- If $u_1 > u_2$ (compression), then $F_1 > 0$ ($F_2 < 0$)

$$F_2 = \frac{EA}{L}(u_2 - u_1)$$

- If $u_2 > u_1$ (tension), then $F_2 > 0$ ($F_1 < 0$)

Local axial stiffness matrix (2×2)

Multiply out the end-force equations:

$$F_1 = \frac{EA}{L} \cdot u_1 - \frac{EA}{L} \cdot u_2$$

$$F_2 = -\frac{EA}{L} \cdot u_1 + \frac{EA}{L} \cdot u_2$$

Collect in matrix form:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Scenario 1 (unit displacement): $u_1 = 1,\ u_2 = 0$

$$\delta = u_2 - u_1 = 0 - 1 = -1 \quad \text{(compression)}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \frac{EA}{L} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

So:

$$F_1 = +EA/L$$

$$F_2 = -EA/L$$

> *This matches the "compression example" we looked at in Part 5*

Scenario 2: $u_1 = 5,\ u_2 = 3$

$$\delta = u_2 - u_1 = 3 - 5 = -2 \quad \text{(compression)}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 5 \\ 3 \end{Bmatrix} = \frac{EA}{L} \begin{Bmatrix} 2 \\ -2 \end{Bmatrix}$$

So:

$$F_1 = +2EA/L$$

$$F_2 = -2EA/L$$

> *Double the amount of strain, double the amount of force.*

Scenario 3: $u_1 = 1,\ u_2 = -1$

$$\delta = u_2 - u_1 = -1 - 1 = -2 \quad \text{(compression)}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \frac{EA}{L} \begin{Bmatrix} 2 \\ -2 \end{Bmatrix}$$

So:

$$F_1 = +2EA/L$$

$$F_2 = -2EA/L$$

> *Same relative deformation ($\delta = -2$) ⇒ same end-force pattern.*

Scenario 4: $u_1 = -2,\ u_2 = -1$

$$\delta = u_2 - u_1 = -1 - (-2) = +1 \quad \text{(tension)}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -2 \\ -1 \end{Bmatrix} = \frac{EA}{L} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

So:

$$F_1 = -EA/L$$

$$F_2 = +EA/L$$

> *Same example we looked at in Part 5, tension where* $u = 1$.

Scenario 5 (rigid-body translation): $\boldsymbol{u_1 = 3,\ u_2 = 3}$

$$\delta = u_2 - u_1 = 3 - 3 = 0 \quad \text{(no deformation)}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 3 \\ -3 \end{Bmatrix} = \frac{EA}{L}\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

So:

$$F_1 = 0$$

$$F_2 = 0$$

> *Rigid translation produces **no internal axial force**.*

In [43]:

```python
# Axial 2×2 stiffness: one editable scenario
import numpy as np

# Choose stiffness scale (set EA_over_L = 1 for clean numbers)
EA_over_L = 1.0
k = EA_over_L * np.array([[1.0, -1.0],
                          [-1.0, 1.0]])

# --- Edit these ---
u1 = 1.0
u2 = 0.0

u = np.array([u1, u2])
f = k @ u
delta = u2 - u1

state = "tension" if delta > 0 else ("compression" if delta < 0 else "ri

# print("k =\n", k)
print(f"u = [u1, u2] = {u}")
print(f"delta = u2 - u1 = {delta}  ->  {state}")
print(f"f = [F1, F2] = {f}")
```

```
u = [u1, u2] = [1. 0.]
delta = u2 - u1 = -1.0  ->  compression
f = [F1, F2] = [ 1. -1.]
```

IMPORTANT: Definition of a stiffness coefficient k_{ij}

$$k_{ij} = \text{force at DOF } i \text{ due to a unit displacement at DOF } j,$$
$$\text{with all other DOFs held fixed.}$$

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

> *Here we only have 2 DOFs. Later (beams / frames), this interpretation becomes more important.*

Building the 2×2 stiffness matrix (unit displacement method)

Each column of k is built by:

- Impose a **unit displacement** at one DOF
- Hold all the other DOF fixed
- Record the resulting nodal force pattern

This suppresses rigid-body motion while defining the columns.

Column 1: impose $u_1 = 1$ ($u_2 = 0$)

$$\begin{bmatrix} \boxed{k_{11}} & k_{12} \\ \boxed{k_{21}} & k_{22} \end{bmatrix}$$

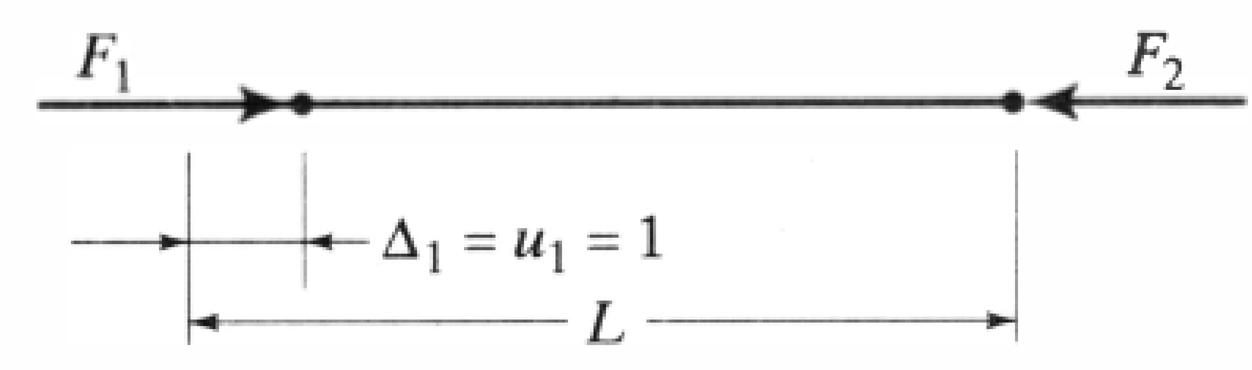
k_{11} $(i = 1, j = 1)$:

- $i = 1$: force at DOF 1
- $j = 1$: due to a unit displacement at DOF 1 ($u_1 = 1$)
- all other displacements are 0 (i.e., $u_2 = 0$)

k_{21} $(i = 2, j = 1)$:

- $i = 2$: force at DOF 2
- $j = 1$: due to a unit displacement at DOF 1 ($u_1 = 1$)
- all other displacements are 0 (i.e., $u_2 = 0$)

Column 1: impose $u_1 = 1$ ($u_2 = 0$)



The resulting nodal force vector is:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}_{u_1=1,\ u_2=0} = \frac{EA}{L}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

This force vector **defines the first column** of the local stiffness matrix:

$$\begin{Bmatrix} k_{11} \\ k_{21} \end{Bmatrix} = \frac{EA}{L}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Column 2: impose $u_2 = 1$ ($u_1 = 0$)

$$\begin{bmatrix} k_{11} & \boxed{k_{12}} \\ k_{21} & \boxed{k_{22}} \end{bmatrix}$$

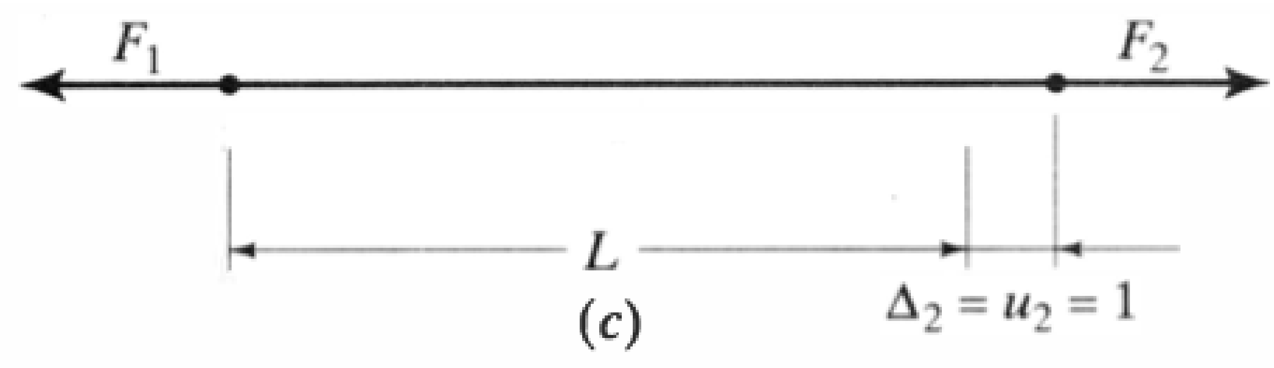
k_{12} $(i = 1, j = 2)$:

- $i = 1$: force at DOF 1
- $j = 2$: due to a unit displacement at DOF 2 ($u_2 = 1$)
- all other displacements are 0 (i.e., $u_1 = 0$)

k_{22} $(i = 2, j = 2)$:

- $i = 2$: force at DOF 2
- $j = 2$: due to a unit displacement at DOF 2 ($u_2 = 1$)
- all other displacements are 0 (i.e., $u_1 = 0$)

Column 2: impose $u_2 = 1$ ($u_1 = 0$)



The resulting nodal force vector is:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}_{u_1=0,\ u_2=1} = \frac{EA}{L}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

This force vector **defines the first column** of the local stiffness matrix:

$$\begin{Bmatrix} k_{12} \\ k_{22} \end{Bmatrix} = \frac{EA}{L}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Equilibrium and column sums

$$\boldsymbol{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Check each column is self-equilibrated:

$$\sum F_x = 0$$

➡️ The entries in each column sum to zero.

Symmetry (reciprocity)

$$k_{12} = k_{21} = -\frac{EA}{L}$$

- **Case 1 (k_{12}):** response to $u_2 = 1$ (with $u_1 = 0$)
  → the element is **stretched**, producing a negative force (left) at DOF 1

- **Case 2 (k_{21}):** response to $u_1 = 1$ (with $u_2 = 0$)
  → the element is **compressed**, producing a negative force (left) at DOF 2

In both cases, the magnitude of the resisting force is the same.

Mathematically, this follows from **energy reciprocity**:
the work done by DOF 1 on DOF 2 equals the work done by DOF 2 on DOF 1.

➡️ **Symmetry reflects that the element's response depends only on relative deformation, not on which end is displaced.**

Why the element stiffness matrix is singular?

$$\det(\boldsymbol{k}) = 0$$

Matrix meaning: there is some **nonzero/non-trivial displacement vector** that produces **zero force**.

Physical meaning: the element alone is not anchored — it can undergo **rigid-body translation** (we showed this in scenario 5)

This is not a problem at the local level: stability is enforced only when we assemble the **full structure** and apply supports in the form of DOF restraints.

Rigid-body motion (worked out)

Let $u_1 = u_2 = c$:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} c \\ c \end{Bmatrix} = \frac{EA}{L} \begin{Bmatrix} c - c \\ -c + c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

➡️ No internal force unless there is **relative displacement**, translation to the right.

Key takeaways

- The 2×2 axial stiffness matrix enforces **equilibrium + compatibility** at the element level
- Symmetry reflects **reciprocity** ($k_{12} = k_{21}$)
- Singularity reflects **rigid-body freedom** (no force if $u_1 = u_2$)
- Supports/constraints are applied at the **global structure level**, not the element level

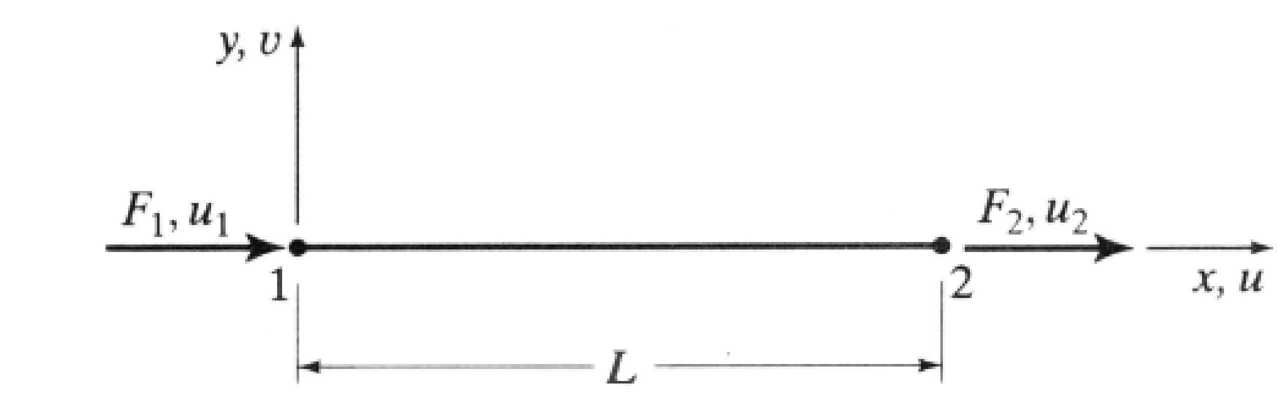
➡️ This 2×2 matrix is the fundamental building block of truss stiffness assembly.

Part 7 — Deriving Local 4×4 Stiffness Matrix

Why do we need a 4×4 matrix?

In a 2D truss, each node has **two translational degrees of freedom**:

- u : displacement in the local x direction (along the member axis)
- v : displacement in the local y direction (perpendicular to the member)

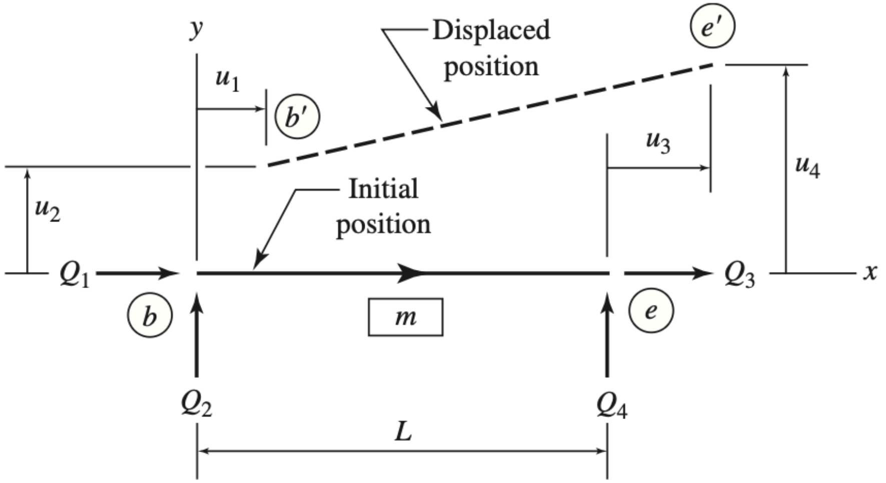


In **Part 6**, we derived a **2×2 stiffness matrix** by considering only the axial (x-direction) behavior of the element.

To describe the **complete local behavior** of a truss element, we must account for **both translations at both nodes**.

This leads to a **4×4 local stiffness relation**, with **2 DOFs per node** and **4 DOFs total**.

Generic displacement for a 2D truss element



Local DOF numbering (for 2x2):

- **DOF 1:** u_1 — node 1, local x
- **DOF 2:** u_2 — node 2, local x

Local DOF numbering (for 4x4):

- **DOF 1:** u_1 — node 1, local x
- **DOF 2:** u_2 — node 1, local y
- **DOF 3:** u_3 — node 2, local x
- **DOF 4:** u_4 — node 2, local y

Local displacement and force vectors

Local displacement vector:

$$\boldsymbol{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Local nodal force vector (using Q to stay consistent with figures):

$$\boldsymbol{F} = \boldsymbol{Q} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

We seek:

$$\boldsymbol{Q} = \boldsymbol{k}\,\boldsymbol{u}$$

Four equations (one per DOF)

$$Q_1 = k_{11}u_1 + k_{12}u_2 + k_{13}u_3 + k_{14}u_4$$

$$Q_2 = k_{21}u_1 + k_{22}u_2 + k_{23}u_3 + k_{24}u_4$$

$$Q_3 = k_{31}u_1 + k_{32}u_2 + k_{33}u_3 + k_{34}u_4$$

$$Q_4 = k_{41}u_1 + k_{42}u_1 + k_{43}u_3 + k_{44}u_4$$

Each equation expresses **force equilibrium at a single local degree of freedom**.

For a linear elastic element, the force at any DOF is a **linear combination of all DOF displacements**:

- displacing one DOF can induce forces at *all* DOFs
- the proportionality constants are the stiffness coefficients k_{ij}

Same equations in matrix form

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Building the 4×4 stiffness matrix (unit displacement method)

Recall:

$$k_{ij} = \text{force at DOF } i \text{ due to a unit displacement at DOF } j,$$
$$\text{with all other DOFs held fixed.}$$

Each column of k is built by:

- Impose a **unit displacement** at one DOF
- Hold all the other DOF fixed
- Record the resulting nodal force pattern

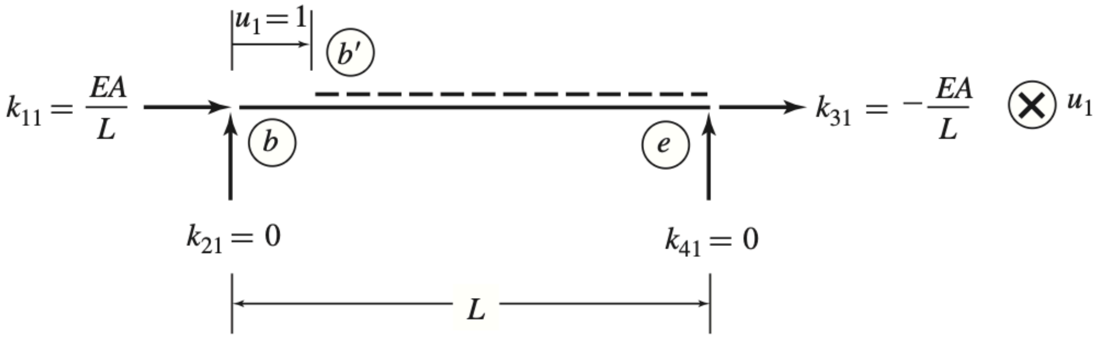
We will do this for DOFs 1–4.

Column 1: impose $u_1 = 1$ $(u_2 = u_3 = u_4 = 0)$

$$\begin{bmatrix} \boxed{k_{11}} & k_{12} & k_{13} & k_{14} \\ \boxed{k_{21}} & k_{22} & k_{23} & k_{24} \\ \boxed{k_{31}} & k_{32} & k_{33} & k_{34} \\ \boxed{k_{41}} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

- k_{11} $(i = 1,\ j = 1)$
  force at **DOF 1** due to unit displacement at **DOF 1**
- k_{21} $(i = 2,\ j = 1)$
  force at **DOF 2** due to unit displacement at **DOF 1**
- k_{31} $(i = 3,\ j = 1)$
  force at **DOF 3** due to unit displacement at **DOF 1**
- k_{41} $(i = 4,\ j = 1)$
  force at **DOF 4** due to unit displacement at **DOF 1**

Column 1: impose $u_1 = 1$ ($u_2 = u_3 = u_4 = 0$)



Axial equilibrium

$$\sum F_x = 0 \;\Rightarrow\; k_{11} + k_{31} = 0 \;\Rightarrow\; k_{31} = -k_{11}$$

Transverse and moment equilibrium

$$\sum F_y = 0 \;\Rightarrow\; k_{21} + k_{41} = 0, \qquad \sum M_e = 0 \;\Rightarrow\; k_{21}L = 0$$

Since $L \neq 0$:

$$k_{21} = 0 \;\Rightarrow\; k_{41} = 0$$

Column 1: stiffness terms

The resulting nodal force vector Q is:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}_{u_1=1,\ u_{2,3,4}=0} = \frac{EA}{L}\begin{Bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} k_{11} \\ k_{21} \\ k_{31} \\ k_{41} \end{Bmatrix}$$

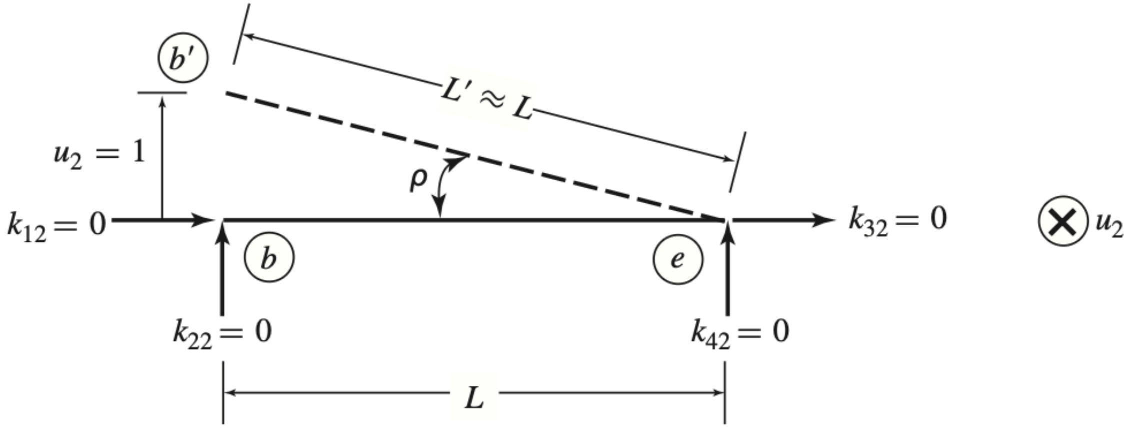
This force vector **defines the first column** of the local stiffness matrix:

Column 2: impose $u_2 = 1$ ($u_1 = u_3 = u_4 = 0$)

$$\begin{bmatrix} k_{11} & \boxed{k_{12}} & k_{13} & k_{14} \\ k_{21} & \boxed{k_{22}} & k_{23} & k_{24} \\ k_{31} & \boxed{k_{32}} & k_{33} & k_{34} \\ k_{41} & \boxed{k_{42}} & k_{43} & k_{44} \end{bmatrix}$$

- k_{12} $(i = 1,\ j = 2)$
  force at **DOF 1** due to unit displacement at **DOF 2**
- k_{22} $(i = 2,\ j = 2)$
  force at **DOF 2** due to unit displacement at **DOF 2**
- k_{32} $(i = 3,\ j = 2)$
  force at **DOF 3** due to unit displacement at **DOF 2**
- k_{42} $(i = 4,\ j = 2)$
  force at **DOF 4** due to unit displacement at **DOF 2**

Column 2: impose $u_2 = 1$ ($u_1 = u_3 = u_4 = 0$)



Because the displacement is **small compared to the member length** ($u_2 \ll L$), the change in length is negligible:

$$L' \approx L \quad \Rightarrow \quad \delta \approx 0$$

Thus a transverse unit displacement produces **no axial strain**.

Axial equilibrium

$$\sum F_x = 0 \;\Rightarrow\; k_{12} + k_{32} = 0$$

$$\delta = 0 \;\Rightarrow\; k_{12} = 0 \;\Rightarrow\; k_{32} = 0$$

Vertical and moment equilibrium

$$\sum F_y = 0 \;\Rightarrow\; k_{22} + k_{42} = 0$$

$$\Rightarrow\; k_{22} = -k_{42}$$

A nonzero pair $(k_{22},\ k_{42})$ would form a **force couple**, violating:

$$\sum M_e = 0$$

Therefore:

$$k_{22} = k_{42} = 0$$

Column 2: stiffness terms

The resulting nodal force vector $\boldsymbol{Q}$ is:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}_{u_2=1,\ u_{1,3,4}=0} = \frac{EA}{L} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} k_{12} \\ k_{22} \\ k_{32} \\ k_{42} \end{Bmatrix}$$

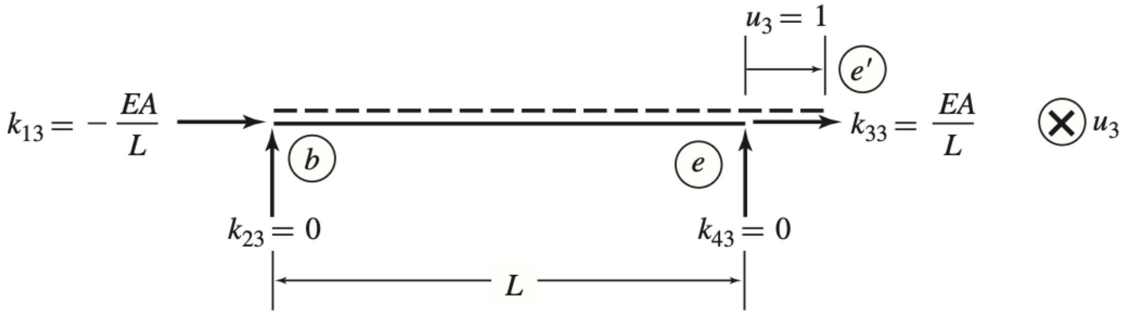
This force vector **defines the second column** of the local stiffness matrix.

Column 3: impose $u_3 = 1$ ($u_1 = u_2 = u_4 = 0$)

$$\begin{bmatrix} k_{11} & k_{12} & \boxed{k_{13}} & k_{14} \\ k_{21} & k_{22} & \boxed{k_{23}} & k_{24} \\ k_{31} & k_{32} & \boxed{k_{33}} & k_{34} \\ k_{41} & k_{42} & \boxed{k_{43}} & k_{44} \end{bmatrix}$$

- k_{13} $(i = 1,\ j = 3)$
  force at **DOF 1** due to unit displacement at **DOF 3**
- k_{23} $(i = 2,\ j = 3)$
  force at **DOF 2** due to unit displacement at **DOF 3**
- k_{33} $(i = 3,\ j = 3)$
  force at **DOF 3** due to unit displacement at **DOF 3**
- k_{43} $(i = 4,\ j = 3)$
  force at **DOF 4** due to unit displacement at **DOF 3**

Column 3: impose $u_3 = 1$ ($u_1 = u_2 = u_4 = 0$)



The resulting nodal force vector $\boldsymbol{Q}$ is:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}_{u_3=1,\ u_{1,2,4}=0} = \frac{EA}{L}\begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} k_{13} \\ k_{23} \\ k_{33} \\ k_{43} \end{Bmatrix}$$

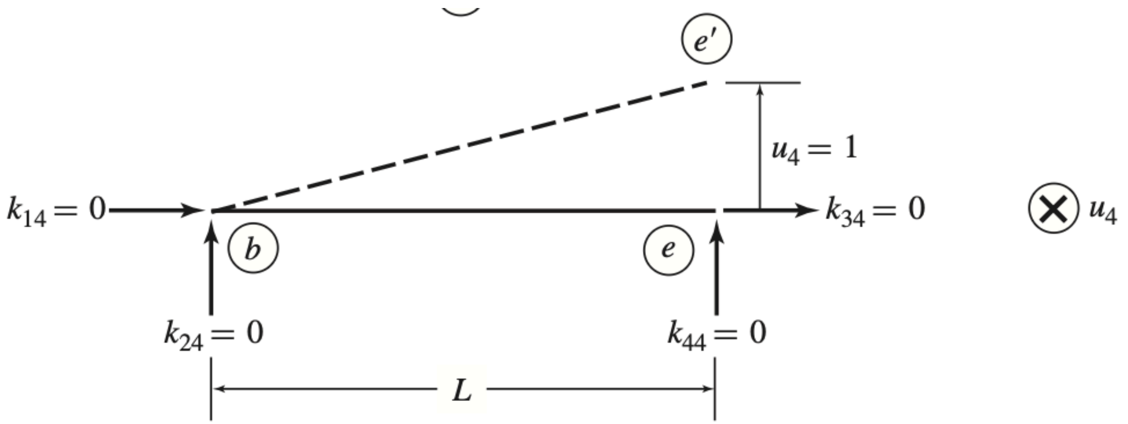
This force vector **defines the third column** of the local stiffness matrix.

Column 4: impose $u_4 = 1$ ($u_1 = u_2 = u_3 = 0$)

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & \boxed{k_{14}} \\ k_{21} & k_{22} & k_{23} & \boxed{k_{24}} \\ k_{31} & k_{32} & k_{33} & \boxed{k_{34}} \\ k_{41} & k_{42} & k_{43} & \boxed{k_{44}} \end{bmatrix}$$

- k_{14} $(i = 1,\ j = 4)$
  force at **DOF 1** due to unit displacement at **DOF 4** ($u_4 = 1$)
- k_{24} $(i = 2,\ j = 4)$
  force at **DOF 2** due to unit displacement at **DOF 4**
- k_{34} $(i = 3,\ j = 4)$
  force at **DOF 3** due to unit displacement at **DOF 4**
- k_{44} $(i = 4,\ j = 4)$
  force at **DOF 4** due to unit displacement at **DOF 4**

Column 4: impose $u_4 = 1$ ($u_1 = u_2 = u_3 = 0$)



The resulting nodal force vector $\boldsymbol{Q}$ is:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}_{u_4=1,\ u_{1,2,3}=0} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} k_{14} \\ k_{24} \\ k_{34} \\ k_{44} \end{Bmatrix}$$

This force vector **defines the fourth column** of the local stiffness matrix.

Summary — Local 4×4 Truss Stiffness Matrix

For a 2D truss element in **local coordinates**, the full element stiffness matrix is:

$$\boldsymbol{k} = \begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Key observations

- Only the **axial DOFs** (u_1, u_3) contribute stiffness
- Transverse DOFs (u_2, u_4) produce **no internal force** for a truss element
- The matrix is:
  - **symmetric** (reciprocity)
  - **singular** (rigid-body motion possible at the element level)

➡️ This local matrix is the **building block** for assembling the global stiffness matrix, after transformation to global coordinates.

In-Class Exercise — Local Truss Stiffness (Python)

Work in pairs. Goal: **compute** the local nodal force vector and verify the key properties of the **local 4×4** stiffness matrix.

Given

Use:

- $E = 200\,\mathrm{GPa}$
- $A = 4{,}000\,\mathrm{mm}^2$
- $L = 2.0\,\mathrm{m}$

Local DOF order and stiffness matrix (as per derivation in class):

$$\boldsymbol{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}, \qquad \boldsymbol{f} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} \qquad \boldsymbol{k} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Tasks

1. **Build $\boldsymbol{k}_e$** in Python (careful with units).
2. For each displacement case below, compute:
    - $\boldsymbol{Q}_e = \boldsymbol{k}_e \boldsymbol{u}_e$
    - the axial deformation $\delta = u_2 - u_1$
3. Check:
    - which entries of $\boldsymbol{Q}_e$ are always zero
    - whether $\boldsymbol{k}_e$ is symmetric
    - whether $\det(\boldsymbol{k}_e) = 0$

Displacement cases (local)

- **Case A (axial extension):** $u_1 = 0,\ u_2 = 1\,\mathrm{mm}$, all $v = 0$
- **Case B (rigid translation):** $u_1 = u_2 = 3\,\mathrm{mm}$, all $v = 0$
- **Case C (transverse motion):** $v_1 = 1\,\mathrm{mm}$, all $u = 0$, $v_2 = 0$

Python starter (edit values only)

```python
import numpy as np

# --- material + geometry (use consistent units!) ---
E = 200e9           # Pa = N/m^2
A = 4000e-6         # mm^2 -> m^2
L = 2.0             # m

k0 = E*A/L

k = k0 * np.array([
    [ 1, 0, -1, 0],
    [ 0, 0,  0, 0],
    [-1, 0,  1, 0],
    [ 0, 0,  0, 0],
], dtype=float)

def run_case(name, u1=0.0, v1=0.0, u2=0.0, v2=0.0):
    u = np.array([u1, v1, u2, v2], dtype=float)  # meters
    Q = k @ u
    delta = u2 - u1
    print(f"\n{name}")
    print("u  =", u)
    print("Q  =", Q)
    print("δ  =", delta)

# --- cases (mm -> m) ---
mm = 1e-3
```

```
run_case("Case A: axial extension", u1=0*mm, v1=0, u2=1*mm, v2=0)
run_case("Case B: rigid translation", u1=3*mm, v1=0, u2=3*mm, v2=0)
run_case("Case C: transverse motion", u1=0, v1=1*mm, u2=0, v2=0)

print("\nChecks")
print("k symmetric? ", np.allclose(k, k.T))
print("det(k) =", np.linalg.det(k))
```

Looking Ahead

➡️ Next (Lecture 3.2):

- Build the transformation matrix from local to global
- Rotate element stiffness into global coordinates
- Assemble the global stiffness matrix for a truss
- Apply supports, solve for displacements, and recover member forces