Axial Anomaly via Point-Splitting Regularization in 2D and 4D

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1 What is the Axial Anomaly

Suppose a certain classical theory possesses a symmetry. Upon quantizing the theory, it may turn out that this symmetry is broken. There are several types of anomalies, depending on which symmetry was broken. We will focus our attention on the so-called chiral (axial) symmetry. This type of anomaly, like some others, manifests itself at short distances.

The subject of our consideration is an integral part of QFT, which we "do not know" yet. Therefore, within the framework of my presentation, I will try as much as possible to avoid using any knowledge from this science. However, as you understand, the purely quantum nature of the phenomenon implies that sooner or later we will have to apply some knowledge from QFT.

Let's understand the word "chiral". We say "chiral" when something relates to "right" and "left". Which objects are divided into right and left? These are spinors, i.e., fermions. So the anomaly is related to fermions. Let's finally get to the heart of the matter.

2 Classical Fermion Field and Its Symmetries

2.1 Dirac Lagrangian

Let's start with the classical description of the fermion field. As we know, it is described by the Dirac Lagrangian

$$\mathcal{L}_D = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi, \tag{1}$$

where γ^μ are the Dirac matrices in the chiral representation, i.e., they are a representation of the Clifford algebra: an algebra with commutation relations of the form

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{2}$$

Let me give their values (these are 4x4 matrices):

$$\gamma^0 = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{3}$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

I will immediately introduce another Dirac matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}. \tag{5}$$

It possesses some important properties:

$$(\gamma^5)^\dagger = \gamma^5, \tag{6}$$

$$\{\gamma^5, \gamma^\mu\} = 0. \tag{7}$$

Also note that the object $\gamma^\mu\gamma^5$ transforms as a pseudovector, meaning it transforms as a vector under Lorentz transformations but changes sign under parity transformations, i.e., when the sign of the spatial coordinates is reversed. For more on parity and pseudovectors.

The bar over the spinor ψ means Dirac conjugation, i.e.,

$$\bar{\psi} = \psi^\dagger\gamma^0. \tag{8}$$

It can be seen that the equation has a block-diagonal form. This means this representation is reducible into two 2-dimensional representations, i.e.,

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{9}$$

2.2 Noether Currents

So we have split the spinor into two bispinors: right and left. Chiral symmetry precisely relates these two types of spinors. Indeed, let's try to find some conserved current in the Lagrangian related to ψ_L and ψ_R. A Noether current must have the following properties: it must be bilinear in the fields and be a vector quantity. We can construct two such objects:

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \qquad j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi. \tag{10}$$

It is easy to check that the first current is indeed conserved, while for the second one we have

$$\partial_\mu j^{\mu 5} = 2im\bar{\psi}\gamma^5\psi, \tag{11}$$

meaning it is also a conserved current, but only for massless fermion fields. This current is called the axial current (while the first one is the vector current). And indeed, it corresponds to the chiral symmetry transformation

$$\psi_R \to e^{i\alpha}\psi_R, \qquad \psi_L \to e^{-i\alpha}\psi_L. \tag{12}$$

3 Point-Splitting Method for Computing the Axial Anomaly

3.1 Fermion Field in an External EM Field

Now we are ready to consider the violation of this symmetry (i.e., the current $j^{\mu 5}$ will no longer be conserved). Consider a massless fermion field in an external EM field. The corresponding Lagrangian is:

$$\mathcal{L} = \bar{\psi}(i\not{D})\psi - \frac{1}{4}(F_{\mu\nu})^2, \tag{13}$$

where the slashed symbol means contraction with Dirac matrices: $\not{D} = \gamma^\mu D_\mu$. The covariant derivative: $D_\mu = \partial_\mu + ieA_\mu$. Writing the Euler-Lagrange equations successively for the fields ψ and $\bar{\psi}$ (treated as independent fields), we obtain the equations of motion:

$$\not{\partial}\psi = -ie\not{A}\psi, \qquad \partial_\mu\bar{\psi}\gamma^\mu = ie\bar{\psi}\not{A}. \tag{14}$$

To determine the axial current, we use the following method: we separate the fermion field into two independent sources ψ and $\bar{\psi}$, placing them at a distance ε from each other, and then take this number to zero. By performing this trick, we will explicitly see that the product of fermion operators is singular at small distances, and this singularity will manifest itself in the non-conservation of the axial current. Then the axial current is defined as

$$j^{\mu 5} = \bar{\psi}(x + \frac{\varepsilon}{2})\gamma^\mu\gamma^5 \exp\left[-ie\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} dz A(z)\right]\psi(x - \frac{\varepsilon}{2}). \tag{15}$$

The exponential in this expression is called a Wilson line and is introduced so that the operator is locally gauge invariant.

So, now we can carefully differentiate this current using the equations of motion. We get

$$\partial_\mu j^{\mu 5} = \bar{\psi}(x + \frac{\varepsilon}{2})\big[-ie\gamma^\mu\epsilon^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu)\big]\gamma^5\psi(x - \frac{\varepsilon}{2}). \tag{16}$$

It might seem that everything vanishes when $\varepsilon \to 0$, but this is not so, because after quantizing the Dirac field, the product of local operators is singular. That is, we need to consider in detail the object $\psi(y)\bar{\psi}(z)$. In QFT, such an object is called a two-point correlation function or propagator. I will not dwell on why the propagator is exactly this way and not another, because this is a topic for the QFT course. For now, I will only give the answer, you will have to take my word for it. At this point, we need to choose the dimension of the space we are working in.

3.2 Two Dimensions

Let's start with the simpler case $D = 2$. In it, the fermion propagator is written as:

$$\psi(y)\bar{\psi}(z) = \int \frac{d^2k}{(2\pi)^2} e^{-ik(y-z)} \frac{i\not{k}}{k^2} = -\not{\partial}\left[\frac{i}{4\pi}\ln(y-z)^2\right] = -\frac{i}{2\pi}\frac{\gamma^\alpha(y-z)_\alpha}{(y-z)^2}. \tag{17}$$

Applied to our situation

$$\bar{\psi}(x + \frac{\varepsilon}{2})\Gamma\psi(x - \frac{\varepsilon}{2}) = -\frac{i}{2\pi}\operatorname{Tr}\left[\frac{\gamma^\alpha\varepsilon_\alpha}{\varepsilon^2}\Gamma\right]. \tag{18}$$

Substituting, we get

$$\partial_\mu j^{\mu 5} = -\frac{i}{2\pi}\operatorname{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^5\right]\frac{\varepsilon_\alpha}{\varepsilon^2}(-ie\varepsilon^\nu F_{\mu\nu}). \tag{19}$$

In $D = 2$ the trace is $\operatorname{Tr}[\gamma^\alpha\gamma^\mu\gamma^5] = 2\epsilon^{\alpha\mu}$, therefore, taking the limit $\varepsilon \to 0$:

$$\partial_\mu j^{\mu 5} = \frac{e}{2\pi}\epsilon^{\mu\alpha}F_{\nu\alpha} 2\lim_{\varepsilon\to 0}\left[\frac{\varepsilon_\mu\varepsilon^\nu}{\varepsilon^2}\right]. \tag{20}$$

Here the limit is taken collectively, i.e., it reduces to averaging the product of two unit vectors over all directions. This is a standard problem for which I recall the answer:

$$\langle n^\alpha n^\beta \rangle = \frac{1}{d}\eta^{\alpha\beta}. \tag{21}$$

Finally, we obtain

$$\partial_\mu j^{\mu 5} = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{22}$$

We see that the conservation law, i.e., a certain symmetry, was broken after quantizing the system, which is called an anomaly. Recalling our calculations, we understand that the non-zero term came from the singularity of local fermion operators, which is a purely quantum property. Let's not stop there and move on to studying the case of space dimension $D = 4$.

3.3 Four Dimensions

All calculations up to the propagator equation remain valid, but the propagator itself changes:

$$\psi(y)\bar{\psi}(z) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(y-z)} \frac{i\not{k}}{k^2} = -\not{\partial}\left[\frac{i}{4\pi}\frac{1}{(y-z)^2}\right] = -\frac{i}{2\pi^2}\frac{\gamma^\alpha (y-z)_\alpha}{(y-z)^4}. \tag{23}$$

This term is highly singular; however, after taking the trace with $\gamma^\mu\gamma^5$ it will unfortunately yield zero. This means we must account for higher orders in perturbation theory. They will give less singular contributions but may give a non-zero term in the current divergence. Indeed, let's consider the second term in the perturbation series. The integral corresponding to this diagram:

$$\int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4} e^{-i(k+p)y} e^{ikz} \frac{i(\not{k}+\not{p})}{(k+p)^2}(-ie\not{A}(p))\frac{i\not{k}}{k^2}. \tag{24}$$

Taking the trace with $\gamma^\mu\gamma^5$, we get

$$\langle \psi(x+\frac{\varepsilon}{2})\gamma^\mu\gamma^5\psi(x-\frac{\varepsilon}{2})\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4} e^{ik\varepsilon} e^{-ipx}\,\mathrm{Tr}\left[(-\gamma^\mu\gamma^5)\frac{i(\not{k}+\not{p})}{(k+p)^2}(-ie\not{A}(p))\frac{i\not{k}}{k^2}\right] = \tag{25}$$

$$= \text{Expanding in large k} = \tag{26}$$

$$= -2e\epsilon^{\alpha\beta\mu\gamma}F_{\alpha\beta}\frac{i}{8\pi^2}\frac{\varepsilon_\gamma}{\varepsilon^2}. \tag{27}$$

Substituting into the expression for the current and taking the limit, we obtain

$$\partial_\mu j^{\mu 5} = -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}. \tag{28}$$

The result obtained is known as the Adler-Bell-Jackiw anomaly. In conclusion, I would like to note that this anomaly plays a major role in the study of chiral symmetries not only in QED but also in QCD.