Quantum Chromodynamics (QCD) from different perspectives

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Diagrammatic technique

$$\mathcal{L}_{\mathsf{QCD}} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \frac{\xi}{2}(\partial^\mu A^a_\mu)^2 + \bar{\psi}(iD - m)\psi + \bar{c}^a(\partial^\mu D^{ac}_\mu)c^c, \quad (1)$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu, \quad (2)$$

$$D_\mu = \partial_\mu - igA^a_\mu t^a_r. \quad (3)$$

Fermion propagator

$$\langle\psi(x)\bar{\psi}(y)\rangle = \frac{1}{Z}\int D\bar{\psi}D\psi\ \psi(x)\psi(y)\exp i\int d^4x\ \bar{\psi}(i\partial - m)\psi, \tag{4}$$

$$Z = \int D\bar{\psi}D\psi\ \exp i\int_{-T}^{T} d^4x\ \bar{\psi}(i\partial - m)\psi. \tag{5}$$

Integrals over Grassmann numbers

We consider integrals of the form $\int\limits_{-\infty}^{+\infty} dx$.

Taylor expansion: $f(\theta) = A + B\theta$.

$$\int d\theta f(\theta) = \int d\theta (A + B\theta). \tag{6}$$

Perform the shift $\theta \to \theta + \eta$. This yields

$$\int d\theta (A + B\theta) = \int d\theta ((A + B\eta) + B\theta). \tag{7}$$

Define $\int d\theta (A + B\theta) = B$.

Sign convention: $\int d\theta \int d\eta = +1$.
Define complex conjugation: $(\theta\eta)^* = \eta^*\theta^*$.
For complex Grassmann variables,

$$\theta = \frac{\theta_1 + i\theta_2}{\sqrt{2}}, \qquad \theta^* = \frac{\theta_1 - i\theta_2}{\sqrt{2}}, \tag{8}$$

such that $d\theta_1 d\theta_2 = d\theta d\theta^*$. Define

$$\int d^{2n}\theta\ f(\theta) = \int d\theta_1 d\theta_1^* d\theta_2 d\theta_2^* \dots d\theta_n d\theta_n^*\ f(\theta). \tag{9}$$

Gaussian integral:

$$\int d\theta^* d\theta\ e^{-\theta^* b\theta} = \int d\theta^* d\theta (1 - \theta^* b\theta) = \int d\theta^* d\theta (1 + \theta\theta^* b) = b. \tag{10}$$

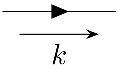
Now compute a general Gaussian integral with a Hermitian matrix B with eigenvalues b_i:

$$\int d^{2n}\theta \; e^{-\theta^* B\theta} = \int d^{2n}\theta' \; e^{\sum -\theta_i'^* b_i \theta_i'} = \prod b_i = \det B.$$

Similarly,

$$\int d^{2n}\theta \; \theta_k \theta_l^* e^{-\theta^* B\theta} = (\det B)(B^{-1})_{kl}.$$

Propagators



$$= \langle \psi_{i\alpha}(x)\bar{\psi}(y)_{j\beta}\rangle =$$

$$= \int \frac{d^4k}{(2\pi)^4}\left(\frac{i}{k-m}\right)_{\alpha\beta}\delta_{ij}e^{-ik(x-y)} =$$

$$= \frac{i(k+m)}{k^2-m^2+i}\delta_{ij}\delta(x-y). \quad (11)$$

Expressions for the remaining propagators (gluons and ghosts):

1. Gluon:

$$a,\mu \; \underset{k}{\overset{\sim\!\sim\!\sim\!\sim}{\longrightarrow}} \; b,\nu \;=\; \langle A^a_\mu(x) A^b_\nu(y) \rangle =$$

$$= \int \frac{d^4k}{(2\pi)^4} \, \frac{-i}{k^2} \left(g_{\mu\nu} - (1-\zeta) \frac{k_\mu k_\nu}{k^2 + i} \right) \delta^{ab} e^{-ik(x-y)}, \quad (12)$$

2. Ghost:

$$a \; \underset{k}{\overset{- - \blacktriangleright - -}{\longrightarrow}} \; b \;=\; \langle c^a(x) \bar{c}^b(y) \rangle = \int \frac{d^4k}{(2\pi)^4} \, \frac{i\delta^{ab}}{k^2 + i} e^{-ik(x-y)}, \quad (13)$$

where α, β are Dirac spinor indices, and i, j are symmetry group indices.

$$\mathcal{L}_{QCD} = \mathcal{L}_0 + gA^a_\mu \bar{\psi}\gamma^\mu t^a \psi - gf^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} - \\ - \frac{1}{4}g^2(f^{eab}A^a_\mu A^b_\nu)(f^{ecd}A^c_\mu A^d_\nu) - \bar{c}^a g\partial_\mu f^{abc}A^b_\mu c^c. \quad (14)$$

For perturbation theory we expand e^S:

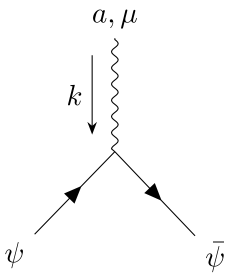
$$\exp i\int \mathcal{L} = \exp i\int \mathcal{L}_0[1 + i(\text{nonlinear terms}) + \ldots]. \quad (15)$$

Note that the expansion introduces the imaginary unit:

$$\int [\mathcal{D}\phi]e^{iS_0 + iS_{\text{int}}} = \int [\mathcal{D}\phi]e^{iS_0}\sum_{n=0}^{\infty}\frac{(iS_{\text{int}})^n}{n!}. \quad (16)$$

Interaction vertices

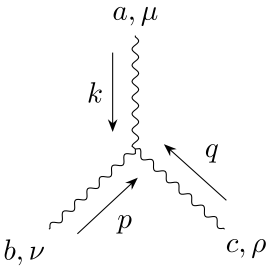
$$\mathcal{L}_{QCD} = \mathcal{L}_0 + gA^a_\mu \bar{\psi}\gamma^\mu t^a \psi - gf^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} - \\ - \frac{1}{4}g^2(f^{eab}A^a_\mu A^b_\nu)(f^{ecd}A^c_\mu A^d_\nu) - \bar{c}^a g\partial_\mu f^{abc}A^b_\mu c^c. \quad (17)$$



$$= \quad ig\gamma^\mu t^a. \quad (18)$$

$$\mathcal{L}_{QCD} = \mathcal{L}_0 + gA^a_\mu \bar{\psi}\gamma^\mu t^a \psi - gf^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} - \\ - \frac{1}{4}g^2(f^{eab}A^a_\mu A^b_\nu)(f^{ecd}A^c_\mu A^d_\nu) - \bar{c}^a g \partial_\mu f^{abc} A^b_\mu c^c. \quad (19)$$

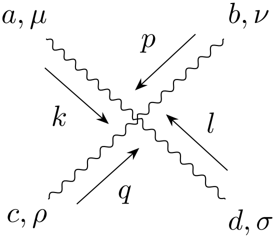
One contraction (as in the picture): $-igf^{abc}(-ik^\nu)g^{\mu\rho}$.



$$= \quad gf^{abc}[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\nu + g^{\rho\mu}(q-k)^\nu]. \quad (20)$$

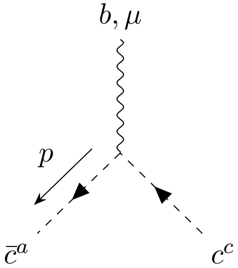
$$\mathcal{L}_{QCD} = \mathcal{L}_0 + gA^a_\mu \bar{\psi}\gamma^\mu t^a \psi - gf^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} - \\ - \frac{1}{4}g^2(f^{eab}A^a_\mu A^b_\nu)(f^{ecd}A^c_\mu A^d_\nu) - \bar{c}^a g \partial_\mu f^{abc} A^b_\mu c^c. \quad (21)$$

One contraction (as in the picture): $-ig^2 f^{eab} f^{ecd} g^{\mu\rho} g^{\nu\sigma}$.



$$= \quad -ig^2 \Big[f^{abe} f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + \\ + f^{ace} f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + \\ + f^{ade} f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}) \Big]. \quad (22)$$

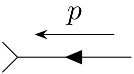
$$\mathcal{L}_{QCD} = \mathcal{L}_0 + gA^a_\mu \bar{\psi}\gamma^\mu t^a \psi - gf^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} - \\ - \frac{1}{4}g^2(f^{eab}A^a_\mu A^b_\nu)(f^{ecd}A^c_\mu A^d_\nu) - \bar{c}^a g \partial_\mu f^{abc} A^b_\mu c^c. \quad (23)$$

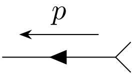


$$= \quad -gf^{abc}p^\mu. \quad (24)$$

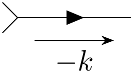
External fermion lines

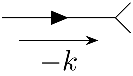
$$\psi(x)\left|p,s\right\rangle = \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\sum_{s'} a_{p'}^{s'} u^{s'}(p')e^{-ip'x}\sqrt{2E_p}a_p^{s\dagger}\left|0\right\rangle =$$
$$= e^{-ipx}u^s(p)\left|0\right\rangle . \quad (25)$$

$\psi \underbrace{\left|p,s\right\rangle}_{\text{fermion}} =$ 

$= u^s(p),\ \underbrace{\left\langle p,s\right|}_{\text{fermion}} \bar\psi =$ 

$= \bar u^s(p),$ (26)

$\bar\psi \underbrace{\left|k,s\right\rangle}_{\text{antifermion}} =$ 

$= \bar v^s(k),\ \underbrace{\left\langle k,s\right|}_{\text{antifermion}} \psi =$ 

$= v^s(k)$ (27)

External gluon line

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=0}^{3} \left(a_p^r \varepsilon_\mu^r(p) e^{-ipx} + a_p^{r\dagger} \varepsilon_\mu^{r*}(p) e^{ipx} \right). \tag{28}$$

$$\varepsilon_\mu(p) a^\alpha \qquad \text{incoming line,} \tag{29}$$

$$\varepsilon_\mu^*(p) a^{\alpha*} \qquad \text{outgoing line.} \tag{30}$$

Feynman rules

1. Diagrams consist of vertices and lines connecting them. The number of vertices corresponds to the perturbative order at which the diagram appears. Each vertex carries a momentum flowing through it.
2. Each vertex contributes the factor $g(2\pi)^D\delta(\sum_i k_i)$, where k_i are the momenta entering the diagram. The delta function imposes momentum conservation.
3. Integrate over each momentum not fixed by conservation: $\int \frac{d^4q}{(2\pi)^4}$. The number of integrals equals the number of loops.
4. Each diagram element contributes the corresponding expression derived earlier.