The No-Hair Theorem
Spaces Admitting Killing Vectors

Ivan Poluboyarinov

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This report will introduce the formalism of Killing fields, which will be actively used in subsequent reports. Today we will not deal with physics, but only introduce the corresponding mathematical definitions and formulate important statements that will be used in the next reports.

1 Killing Fields

Throughout the report, a torsion-free connection ∇ will be considered. The metric g has the Minkowski space signature (1, 3).

Definition 1. *A vector field K on a manifold (M, g) is called a Killing field if*

$$L_K g = 0. \qquad (\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0) \tag{1}$$

Each vector field corresponds to a conjugate vector field (1-form). The corresponding Killing 1-form will be denoted by the same letter K.

The following important properties follow directly from the definition:

Lemma 1. *For a Killing field K and an arbitrary form α, the following holds:*

$$d^\dagger K = 0, \tag{2}$$

$$L_k \star \alpha = \star L_K \alpha, \tag{3}$$

$$d^\dagger (K \wedge \alpha) + K \wedge d^\dagger \alpha = -L_K \alpha, \tag{4}$$

where the notation for the codifferential is introduced $d^\dagger = \star d \star$: $\Lambda_p \to \Lambda_{p-1}$.

Remark. *Note that the notation $d^\dagger$ for the codifferential is not accidental. It turns out that this operator is indeed adjoint to the usual exterior derivative operator with respect to the scalar product*

$$\langle ., . \rangle = \int_M (.|.)\eta, \qquad \textit{where } \eta = \sqrt{|g|} dx^1 \wedge \ldots \wedge dx^4 \quad \textit{(volume form)} \tag{5}$$

$$\langle d\alpha, \beta \rangle = \langle \alpha, d^\dagger \beta \rangle. \tag{6}$$

Proof. 1): A direct consequence of the definition, if we write out the action of the codifferential operator on 1-forms: $d^\dagger K = \nabla^\mu K_\mu$.

2) On one hand

$$L_K(\alpha \wedge * \beta) = L_K[(\alpha|\beta)\eta] = L_K[(\alpha|\beta)]\eta + (\alpha|\beta) L_K \eta = \tag{7}$$

$$= (L_K \alpha|\beta)\eta + (\alpha|L_K \beta)\eta = (L_K \alpha) \wedge * \beta + \alpha \wedge *(L_K \beta). \tag{8}$$

But on the other hand

$$L_K(\alpha \wedge * \beta) = (L_K \alpha) \wedge * \beta + \alpha \wedge (L_K * \beta), \tag{9}$$

from which

$$\alpha \wedge *(L_K \beta) = \alpha \wedge (L_K * \beta). \tag{10}$$

3) Using the Cartan identity $L_K = di_K + i_K d$ for the expression $L_K * \alpha$ we get (actively using properties (43)-(46))

$$L_K * \alpha = di_K * \alpha + i_K d * \alpha = d * (\alpha \wedge K) - *(K \wedge *d * \alpha) = \tag{11}$$

$$= (-1)^p d * (K \wedge \alpha) - *(K \wedge *d * \alpha) = - * (K \wedge d^\dagger \alpha) - *d^\dagger (K \wedge \alpha). \tag{12}$$

Using (3) and applying another Hodge star, we get

$$-(-1)^p L_K \alpha = (-1)^p (K \wedge d^\dagger \alpha) + (-1)^p d^\dagger (K \wedge \alpha), \tag{13}$$

which gives us the required statement. ■

Definition 2. *For a vector field X, one can introduce a 1-form, called the Ricci form, as*

$$R(X)_\mu = R_{\mu\nu} X^\nu. \tag{14}$$

Definition 3. *The Laplace-de Rham operator is the differential operator*

$$\Delta = -(d^\dagger d + d d^\dagger). \tag{15}$$

Lemma 2 (w/o proof)**.** *For a Killing field, the following holds:*

$$\Delta K = -2R(K). \tag{16}$$

2 Norm and Torsion Form

For an arbitrary Killing field, the following objects play an important role:

Definition 4. *For a Killing field K, the 0-form N and the 1-form ω*

$$N = (K|K), \qquad \omega = \frac{1}{2} \star (K \wedge dK) \tag{17}$$

are called the norm and the torsion (torsion form) respectively.

Remark. *If $N > 0$, then K is called spacelike. Other signs — by analogy.*

Let us prove important statements about these objects.

Lemma 3.

$$dK = -\frac{1}{N} \left[2 * (K \wedge \omega) + K \wedge dN \right]. \tag{18}$$

Proof. Consider the object $-2 * (K \wedge \omega)$, insert a clever unit $** = 1$ and use the properties (43)-(46):

$$-2 * (K \wedge \omega) = 2(- * (K \wedge * * \omega)) = 2 i_K * \omega = 2 i_K \frac{1}{2} (K \wedge dK) = i_K (K \wedge dK) = \tag{19}$$

$$= \text{Leibniz rule} = i_K(K) \wedge dK - K \wedge i_K(dK). \tag{20}$$

From the Cartan identity $0 = L_K K = i_K dK + d i_K K$ we have $i_K dK = -d i_K K = -dN$. In total

$$-2 * (K \wedge \omega) = N dK + K \wedge dN. \tag{21}$$

■

Lemma 4.

$$N(dK|dK) = (dN|dN) - 4(\omega|\omega). \tag{22}$$

Proof. Easily proven by direct calculation using the previous lemma. ■

These two lemmas allow us to prove the following fundamental statements expressing $R(K, K)$ and $K \wedge R(K)$ in terms of N and ω.

Theorem 1.

$$d\omega = *(K \wedge R(K)), \tag{23}$$

$$d^\dagger \omega = -\frac{2}{N}(\omega|dN), \tag{24}$$

$$(\omega|\omega) = \frac{1}{4}\left[(dN|dN) - N\Delta N - 2NR(K,K)\right]. \tag{25}$$

Proof. 1) Apply the exterior derivative to $\omega = \frac{1}{2} * (K \wedge dK)$:

$$2d\omega = d * (K \wedge dK) = - * * d * (K \wedge dK) = - * d^\dagger(K \wedge dK) = - * ((d^\dagger K) \wedge dK - K \wedge d^\dagger dK) = \tag{26}$$

$$= \text{the first term vanishes due to (2), replace the second with the Laplacian} = \tag{27}$$

$$= - * (K \wedge \Delta K) = (16) = - * (K \wedge -2R(K)) = 2 * (K \wedge R(K)). \tag{28}$$

2) w/o proof

3) Recall that $\Delta N = -d^\dagger dN$, $dN = -i_K dK = *(K \wedge *dK)$. Then

$$-\Delta N = *d * *(K \wedge *dK) = *d(K \wedge *dK) = *(dK \wedge *dK) - *(K \wedge d * dK) = \tag{29}$$

$$= \{\alpha \wedge *\beta = (\alpha|\beta)\eta\} = *(dK|dK)\eta - *(K \wedge * * d * dK) = *(dK|dK)\eta - *(K|d^\dagger dK)\eta = \tag{30}$$

$$= \Big[(dK|dK) + (K|\Delta K)\Big] * \eta = \{*\eta = -1\} = -(dK|dK) - (K|\Delta K). \tag{31}$$

Substituting the expression for dK, we get

$$-\Delta N = -\frac{1}{N}\Big[(dN|dN) - 4(\omega|\omega)\Big] - (K|\Delta K). \tag{32}$$

∎

3 Ernst Potential

Definition 5. *Introduce the 1-form*

$$\mathcal{E} = -dN + 2i\omega. \tag{33}$$

From Theorem 1 it immediately follows that

$$d\mathcal{E} = 2i * (K \wedge R(K)), \tag{34}$$

$$d^\dagger \mathcal{E} - \frac{(\mathcal{E}|\mathcal{E})}{N} = -2(K|R(K)). \tag{35}$$

In gravity applications, it often turns out that the first equation is $d\mathcal{E} = 0$, which means we can introduce a potential

$$dE = \mathcal{E}. \tag{36}$$

This potential is called the Ernst potential.

Corollary 1. *Then the second equation will give (since the scalar product in the second equation will also give zero; $N = -\operatorname{Re} E$)*

$$\Delta E = 2\frac{(dE|dE)}{E + \bar{E}}. \tag{37}$$

This equation is called the Ernst equation.

4 Example. Kerr Solution

It can be shown, and this will be done to some extent in the next report, that in axially symmetric stationary systems, i.e., where there are two Killing fields: one corresponding to invariance under time translation, the second — under the rotation group $SO(2)$, the Ernst potential is closely related to the metric. Specifically: the metric in the corresponding coordinates can be expressed through the components of the Ernst potential. I will give without proof the corresponding theorem:

Theorem 2 (w/o proof)**.** *For an asymptotically flat, stationary, and axially symmetric vacuum spacetime, the metric can be written (in appropriate coordinates) in terms of certain functions X, A, h as follows:*

$$g = -\frac{\rho^2}{X}dt^2 + X(d\varphi + Adt) + \frac{1}{X}e^{2h}(d\rho^2 + dz^2) \tag{38}$$

where these functions are expressed through the Ernst potential in a certain way (not important right now), and the potential itself satisfies the following equation

$$\frac{1}{\rho}\nabla(\rho\nabla E) + \frac{(\nabla E|\nabla E)}{X} = 0, \qquad \nabla = (\partial_\rho, \partial_z). \tag{39}$$

Thus, instead of the extremely complex solution of the Einstein equations, we have reduced the problem to finding a solution to a second-order differential equation. This task is significantly simpler and can be solved by several coordinate changes, after which the solution can be easily guessed simply by the form of the equation.

So, if we perform the necessary coordinate changes, we arrive at the equation

$$((x^2-1)\epsilon_{,x})_{,x} + ((1-y^2)\epsilon_{,y})_{,y} = -2\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}\left[(x^2-1)\epsilon_{,x} + (1-y^2)\epsilon^2_{,y}\right], \quad \text{where } \epsilon = \frac{1+E}{1-E}, \tag{40}$$

which has a simple solution $\epsilon = px + iqy$, where $p^2 + q^2 = 1$.

Substituting this solution into the expression for the metric (which is not written out, but take my word for it), and after another series of variable changes, we get

$$g = \frac{1}{\Sigma}\left[-(\Delta - a^2\sin^2\theta)dt^2 + 2a\sin^2\theta(\Delta - (r^2+a^2))dtd\phi + \sin^2\theta((r^2+a^2)^2 - \Delta a^2\sin^2\theta)d\phi^2\right] + \tag{41}$$

$$+\Sigma\left(\frac{1}{\Delta}dr^2 + d\theta^2\right), \qquad \text{where} \quad \Delta = r^2 - 2mr + a^2, \quad \Sigma = r^2 + a^2\cos^2\theta. \tag{42}$$

This is nothing but the Kerr solution for a rotating uncharged black hole with mass $M = m$ and angular momentum $J = am$.

Reference Formulas

Properties of the Hodge star:

$$*^{-1} = -(-1)^p *, \tag{43}$$

$$*^2 = -(-1)^p, \tag{44}$$

$$i_X \alpha = - * (X \wedge * \alpha), \tag{45}$$

$$i_X * \alpha = *(\alpha \wedge X), \tag{46}$$

Various Leibniz rules:

$$P \wedge Q = (-1)^{pq} Q \wedge P, \tag{47}$$

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^s \alpha \wedge (d\beta), \tag{48}$$

$$L_v(\alpha \wedge \beta) = (L_v \alpha) \wedge \beta + \alpha \wedge (L_v \beta), \tag{49}$$

$$i_v(\alpha \wedge \beta) = (i_v \alpha) \wedge \beta + (-1)^s \alpha \wedge (i_v \beta) \tag{50}$$