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Exact solutions for time-fractional diffusion-wave equations by decomposition method

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Abstract
The time-fractional diffusion-wave equation is considered. The time-fractional diffusion equation is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0, 2]$. The fractional derivative is described in the Caputo sense. This paper presents the analytical solutions of the fractional diffusion equations by an Adomian decomposition method. By using initial conditions, the explicit solutions of the equations have been presented in the closed form and then their numerical solutions have been represented graphically. Four examples are presented to show the application of the present technique. The present method performs extremely well in terms of efficiency and simplicity.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

A time-fractional diffusion-wave equation is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0, 2]$ [1, 2]. Now-a-days, the fractional diffusion equation plays important roles in modelling anomalous diffusion and subdiffusion systems, the description of fractional random walk and the unification of diffusion and wave propagation phenomena, see, e.g. the reviews in [1, 3–6], and references therein.

In this paper, we shall consider the time-fractional diffusion equation [1]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < \alpha \leqslant 2, \tag{1.1}$$

where $\partial^\alpha(\bullet)/\partial t^\alpha$ is the Caputo derivative of order α.

In this paper, we use the Adomian decomposition method [7, 8] to obtain a solution of a fractional diffusion equation (1.1). Large classes of linear and nonlinear differential equations, both ordinary as well as partial, can be solved by the Adomian decomposition method [7–14]. A reliable modification of the Adomian decomposition method has been done by Wazwaz [15]. The decomposition method provides an effective procedure for obtaining analytical solutions of a wide and general class of dynamical systems representing real physical problems [8–11]. Recently, the implementation of the Adomian decomposition method for solution of the generalized regularized long-wave (RLW) and Korteweg–de Vries (KdV) equations has been well established by notable researchers [16–19]. This method efficiently works for initial-value or boundary-value problems and for linear or nonlinear, ordinary or partial differential equations and even for stochastic systems. Moreover, we have the advantage of a single global method for solving ordinary or partial differential equations as well as many other types of equations. Recently, solutions of fractional differential equations have been obtained through Adomian decomposition method by researchers [20–27]. The application of the Adomian decomposition method for the solution of nonlinear fractional differential equations has also been established by Shawagfeh [23], Saha Ray and Bera [26].

2. Mathematical aspects

2.1. Mathematical definition

The mathematical definition of fractional calculus has been the subject of several different approaches [28, 29]. The most frequently encountered definition of an integral of fractional order is the Riemann–Liouville integral, in which the fractional order integral is defined as

$$D_t^{-q} f(t) = \frac{\mathrm{d}^{-q} f(t)}{\mathrm{d}t^{-q}} = \frac{1}{\Gamma(q)} \int_0^t \frac{f(x)\,\mathrm{d}x}{(t-x)^{1-q}}, \tag{2.1.1}$$

while the definition of a fractional order derivative is

$$\begin{aligned} D_t^q f(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{\mathrm{d}^{-(n-q)} f(t)}{\mathrm{d}t^{-(n-q)}}\right) \\ &= \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{f(x)\,\mathrm{d}x}{(t-x)^{1-n+q}}, \end{aligned} \tag{2.1.2}$$

where q ($q > 0$ and $q \in \mathbb{R}$) is the order of the operation and n is an integer that satisfies $n-1 \leqslant q < n$.

The fractional derivative of $f(t)$ in the Caputo sense is defined by

$${}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)\,\mathrm{d}x}{(t-x)^{\alpha+1-n}}, \tag{2.1.3}$$

where α ($\alpha > 0$ and $\alpha \in \mathbb{R}$) is the order of the operation and n is an integer that satisfies $n-1 < \alpha < n$.

Furthermore,

$$D_t^{-\mu} t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu+1)} t^{\nu+\mu}, \quad \mu > 0, \quad \nu > -1, \quad t > 0$$

$$D_t^{-\mu}({}^{C}D_t^{\mu} f(t)) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0+)\frac{t^k}{k!}, \quad m-1 < \mu \leqslant m, \quad m \in \mathbb{N}.$$

2.2. Definition—Mittag–Leffler function

A one-parameter function of the Mittag–Leffler type is defined by the series expansion [28]

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad (\alpha > 0). \tag{2.2.1}$$

3. The time-fractional diffusion-wave equation model and the solution

We consider the equation

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{3.1}$$

Here, $\partial^{\alpha}(\bullet)/\partial t^{\alpha}$ is the Caputo derivative of order α.

We further consider the following boundary conditions:

$$u(0,t) = u(L,t) = 0, \quad t \geqslant 0$$

and initial conditions:

$$u(x,0) = f(x), \quad 0 < x < L,$$
$$u_t(x,0) = 0, \quad 0 < x < L \quad \text{for } 1 < \alpha \leqslant 2.$$

The last initial condition is assumed to ensure the continuous dependence of the solution on the parameter α in the transition from $\alpha = 1^-$ to $\alpha = 1^+$, [1, 2].

We adopt the Adomian decomposition method for solving equation (3.1). In the light of this method, we assume

$$u = \sum_{n=0}^{\infty} u_n, \tag{3.2}$$

to be the solution of equation (3.1).

Now, equation (3.1) can be written as

$$L_t u(x,t) = D_t^{(m-\alpha)}(L_{xx} u(x,t)), \quad m-1 < \alpha < m \tag{3.3}$$

where $L_t \equiv \partial^m/\partial t^m$ which is an easily invertible linear operator, $D_t^{(m-\alpha)}(\bullet)$ is the Riemann–Liouville derivative of order $(m-\alpha)$, $L_{xx} = \partial^2/\partial x^2$.

Therefore, by the Adomian decomposition method, we can write,

$$u(x,t) = \phi + L_t^{-1}(D_t^{(m-\alpha)}(L_{xx} u(x,t))), \tag{3.4}$$

with $L_t\phi = 0$ where $u_0 = \phi$, $u_1 = L_t^{-1}(D_t^{(m-\alpha)}(L_{xx}u_0))$, $u_2 = L_t^{-1}(D_t^{(m-\alpha)}(L_{xx}u_1))$, $u_3 = L_t^{-1}(D_t^{(m-\alpha)}(L_{xx}u_2))$ and so on.

The decomposition series (3.2) solution generally converges very rapidly in real physical problems [8]. The rapidity of this convergence means that few terms are required. Convergence of this method has been rigorously established by Cherruault [30], Abbaoui and Cherruault [31, 32] and Himoun, *et al* [33]. The practical solution will be the n-term approximation ϕ_n

$$\phi_n = \sum_{i=0}^{n-1} u_i(x,t), \quad n \geqslant 1, \tag{3.5}$$

with

$$\lim_{n\to\infty} \phi_n = u(x,t)$$

4. Implementation of the present method

Example 1. Let us consider initial conditions:

$$u(x,0) = \sin(x), \quad u_t(x,0) = 0,$$
$$0 < x < \pi \quad \text{for } 1 < \alpha \leqslant 2$$

and boundary conditions:

$$u(0,t) = u(\pi,t) = 0, \quad t \geqslant 0$$

for equation (3.1), as taken in [34]. We will then obtain from a recursive scheme for the Adomian decomposition method

$$\begin{aligned} u_0 &= u(x,0) + tu_t(x,0) = \sin(x) \\ u_1 &= L_t^{-1}[D_t^{(m-\alpha)}(L_{xx}u_0)] = \frac{-t^{\alpha}}{\Gamma(\alpha+1)} \sin(x) \\ u_2 &= L_t^{-1}[D_t^{(m-\alpha)}(L_{xx}u_1)] = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \sin(x) \\ u_3 &= L_t^{-1}[D_t^{(m-\alpha)}(L_{xx}u_2)] = \frac{-t^{3\alpha}}{\Gamma(3\alpha+1)} \sin(x) \\ u_4 &= L_t^{-1}[D_t^{(m-\alpha)}(L_{xx}u_3)] = \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} \sin(x) \end{aligned}$$

and so on.

Therefore, the solution is

$$u(x,t) = \sum_{r=0}^{\infty} \frac{(-t^{\alpha})^r}{\Gamma(r\alpha+1)} \sin(x) = E_{\alpha}(-t^{\alpha}) \sin(x), \qquad (4.1)$$

where $E_{\lambda}(z)$ is the Mittag–Leffler function in one parameter.

The solution (4.1) can be verified through substitution in equation (3.1).

Example 2. Let us consider initial conditions:

$$u(x,0) = f(x), \quad 0 < x < 2, \qquad (4.2)$$

$$u_t(x,0) = 0, \quad 0 < x < 2 \quad \text{for } 1 < \alpha \leqslant 2, \qquad (4.3)$$

where

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant 1 \\ 2-x, & 1 \leqslant x \leqslant 2 \end{cases}, \qquad (4.4)$$

and boundary conditions:

$$u(0,t) = u(2,t) = 0, \quad t \geqslant 0, \qquad (4.5)$$

for the equation (3.1), as taken in [3].

We see that $f(x)$ is a periodic function with period 2. The Fourier sine series of $f(x)$ in [0, 2] can be obtained as

$$f(x) = \sum_{n=1}^{\infty} \left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right] \sin\left(\frac{(2n-1)\pi x}{2}\right). \qquad (4.6)$$

Therefore, after considering $f(x)$ as a Fourier sine series, we can take

$$u(x,0) = \sum_{n=1}^{\infty} \left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right] \sin\left(\frac{(2n-1)\pi x}{2}\right), \qquad (4.7)$$

because of the fact that Fourier sine series is well adapted to functions which are zero at the end points $x = 0$ and $x = 2$ of the interval [0, 2], since all the basis functions $\sin((2n-1)\pi x/2)$ have this property.

We will then obtain from a recursive scheme for the Adomian decomposition method

$$\begin{aligned} u_0 &= u(x,0) + tu_t(x,0) \\ &= \sum_{n=1}^{\infty} \left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right] \sin\left(\frac{(2n-1)\pi x}{2}\right) \end{aligned}$$

$$\begin{aligned} u_1 = L_t^{-1}[D_t^{(m-\alpha)}(L_{xx}u_0)] &= \frac{-t^{\alpha}}{\Gamma(\alpha+1)} \sum_{n=1}^{\infty} \left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right] \\ &\times \left(\frac{(2n-1)\pi}{2}\right)^2 \sin\left(\frac{(2n-1)\pi x}{2}\right) \end{aligned}$$

$$\begin{aligned} u_2 = L_t^{-1}[D_t^{(m-\alpha)}(L_{xx}u_1))] &= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \sum_{n=1}^{\infty} \left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right] \\ &\times \left(\frac{(2n-1)\pi}{2}\right)^4 \sin\left(\frac{(2n-1)\pi x}{2}\right) \end{aligned}$$

$$\begin{aligned} u_3 = L_t^{-1}[D_t^{(m-\alpha)}(L_{xx}u_2)] &= \frac{-t^{3\alpha}}{\Gamma(3\alpha+1)} \sum_{n=1}^{\infty} \left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right] \\ &\times \left(\frac{(2n-1)\pi}{2}\right)^6 \sin\left(\frac{(2n-1)\pi x}{2}\right) \end{aligned}$$

$$\begin{aligned} u_4 = L_t^{-1}[D_t^{(m-\alpha)}(L_{xx}u_3)] &= \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} \sum_{n=1}^{\infty} \left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right] \\ &\times \left(\frac{(2n-1)\pi}{2}\right)^8 \sin\left(\frac{(2n-1)\pi x}{2}\right) \end{aligned}$$

and so on.

Therefore, the solution is

$$\begin{aligned} u(x,t) &= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n-1}}{(2n-1)^2}\right] \\ &\quad \times \sum_{k=0}^{\infty} \frac{\left(-\frac{(2n-1)^2\pi^2 t^{\alpha}}{4}\right)^k}{\Gamma(\alpha k+1)} \sin\left(\frac{(2n-1)\pi x}{2}\right) \\ &= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n-1}}{(2n-1)^2}\right] E_{\alpha}\left(-\frac{(2n-1)^2\pi^2 t^{\alpha}}{4}\right) \\ &\quad \times \sin\left(\frac{(2n-1)\pi x}{2}\right), \end{aligned} \qquad (4.8)$$

where $E_{\lambda}(z)$ is the Mittag–Leffler function in one parameter. The same solution has been obtained by Agrawal [3].

Example 3. Let us consider the equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = D_t^{\beta} u(x,t), \quad 0 < \beta < 2, \qquad (4.9)$$

where $D_t^{\beta}(\bullet)$ is the Caputo derivative of order β.

With initial conditions

$$u(x,0)=\delta(x), \quad u_t(x,0)=0. \tag{4.10}$$

Taking the Fourier transform of equation (4.9) and (4.10), we obtain

$$D_t^{\beta}\bar{u}(\omega,t)=-\omega^2\bar{u}(\omega,t), \tag{4.11}$$

$$\bar{u}(\omega,0)=\frac{1}{\sqrt{2\pi}}, \tag{4.12}$$

where $\bar{u}(\omega,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{+\mathrm{i}\omega x}\bar{u}(x,t)\,\mathrm{d}x,\ \omega\in\mathbb{R}$.

Applying $D_t^{-\beta}$ to eq. (4.11), we obtain

$$\begin{aligned}\bar{u}(\omega,t)&=\bar{u}(\omega,0)-\omega^2 J_t^{\beta}\bar{u}(\omega,t) && \text{for } 0<\beta\leqslant 1\\ \bar{u}(\omega,t)&=\bar{u}(\omega,0)+t\bar{u}_t(\omega,0)-\omega^2 J_t^{\beta}\bar{u}(\omega,t) && \text{for } 1<\beta<2\\ &=\bar{u}(\omega,0)-\omega^2 J_t^{\beta}\bar{u}(\omega,t),\end{aligned}$$

where $J_t^{\beta}\equiv D_t^{-\beta}$ is the Riemann–Liouville integral of order β.

Following the Adomian decomposition method, we have

$$\bar{u}(\omega,t)=\sum_{n=0}^{\infty}\bar{u}_n(\omega,t)$$

where

$$\begin{aligned}\bar{u}_0&=\frac{1}{\sqrt{2\pi}},\\ \bar{u}_1&=-\omega^2 J_t^{\beta}\bar{u}_0=-\frac{\omega^2}{\sqrt{2\pi}}\frac{t^{\beta}}{\Gamma(\beta+1)}\\ \bar{u}_2&=-\omega^2 J_t^{\beta}\bar{u}_1=\frac{\omega^4}{\sqrt{2\pi}}\frac{t^{2\beta}}{\Gamma(2\beta+1)}\\ \bar{u}_3&=-\omega^2 J_t^{\beta}\bar{u}_2=-\frac{\omega^6}{\sqrt{2\pi}}\frac{t^{3\beta}}{\Gamma(3\beta+1)}\end{aligned}$$

and so on.

$$\begin{aligned}\bar{u}(\omega,t)&=\frac{1}{\sqrt{2\pi}}\left[1-\frac{\omega^2 t^{\beta}}{\Gamma(\beta+1)}\right.\\ &\quad\left.+\frac{\omega^4 t^{2\beta}}{\Gamma(2\beta+1)}-\frac{\omega^6 t^{3\beta}}{\Gamma(3\beta+1)}+\ldots..\right]\\ &=\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-\omega^2)^k t^{k\beta}}{\Gamma(k\beta+1)}=\frac{1}{\sqrt{2\pi}}E_{\beta}(-\omega^2 t^{\beta}).\end{aligned} \tag{4.13}$$

Taking the inverse Fourier transform of equation (4.13) we obtain the solution

$$u(x,t)=\tfrac{1}{2}t^{-\beta/2}M_{\beta/2}(|x|/t^{\beta/2}), \quad -\infty<x<+\infty, \quad t\geqslant 0, \tag{4.14}$$

where $u(x,t)=1/\sqrt{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\omega x}\bar{u}(\omega,t)\,\mathrm{d}\omega,\ x\in\mathbb{R}$ and

$$M_{\beta/2}\left(|x|/t^{\beta/2}\right)=\sum_{n=0}^{\infty}\frac{\left(-|x|/t^{\beta/2}\right)^n}{n!\Gamma\left[-\frac{n\beta}{2}+\left(1-\frac{\beta}{2}\right)\right]}, \quad 0<\frac{\beta}{2}<1$$

Here, $M_{\beta/2}$ denotes the so-called M function of order $\beta/2$, which is a special case of the Wright function [2, 35]. The same solution has been obtained by Mainardi *et al* [35].

Example 4. Let us consider $(1+1)$ dimensional nonlinear fractional equation

$$D_t^{\alpha}u-\gamma^2 u_{xx}+c^2u-\sigma u^3=0, \quad 1<\alpha\leqslant 2 \tag{4.15}$$

with initial conditions

$$u(x,0)=\varepsilon\cos kx, \quad u_t(x,0)=0. \tag{4.16}$$

Applying $D_t^{-\alpha}$ to equation (4.15), we obtain

$$u(x,t)=u(x,0)+tu_t(x,0)+\gamma^2 J_t^{\alpha}u_{xx}-J_t^{\alpha}Nu$$

where $J_t^{\alpha}\equiv D_t^{-\alpha}$ is the Riemann–Liouville integral of order α and $Nu=c^2u-\sigma u^3=\sum_{n=0}^{\infty}A_n$.

In this case, the Adomian polynomials A_n are as follows

$$A_0=c^2u_0-\sigma u_0^3$$

$$A_1=u_1(c^2-3\sigma u_0^2), \quad A_2=u_2(c^2-3\sigma u_0^2)+\frac{u_1^2}{2!}(-6\sigma u_0)$$

and so on.

Following the Adomian decomposition method, we have

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)$$

where

$$u_0=\varepsilon\cos kx$$

$$\begin{aligned}u_1&=\gamma^2 J_t^{\alpha}u_{0_{xx}}-J_t^{\alpha}A_0=-\frac{\gamma^2\varepsilon k^2 t^{\alpha}\cos(kx)}{\Gamma(\alpha+1)}\\ &\quad-\frac{\left(c^2\varepsilon\cos kx-\sigma\varepsilon^3\cos^3(kx)\right)t^{\alpha}}{\Gamma(\alpha+1)}\end{aligned}$$

$$\begin{aligned}u_2&=\gamma^2 J_t^{\alpha}u_{1_{xx}}-J_t^{\alpha}A_1=\frac{\varepsilon\gamma^4k^4\cos(kx)t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &\quad-\gamma^2\left[-c^2\varepsilon k^2\cos(kx)+\frac{3k^2\sigma\varepsilon^3}{4}\cos(kx)\right.\\ &\quad\left.+\frac{9k^2\sigma\varepsilon^3}{4}\cos(3kx)\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{c^2\varepsilon\gamma^2k^2\cos(kx)t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &\quad+\frac{c^2(c^2\varepsilon\cos kx-\sigma\varepsilon^3\cos^3(kx))t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &\quad-\frac{3\sigma\varepsilon^3\gamma^2k^2\cos^3(kx)t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &\quad-\frac{3\sigma\varepsilon^2\cos^2(kx)(c^2\varepsilon\cos kx-\sigma\varepsilon^3\cos^3(kx))t^{2\alpha}}{\Gamma(2\alpha+1)}\end{aligned}$$

and so on.

Therefore, the three-term decomposition method solution is

$$u=\sum_{i=0}^{2}u_i. \tag{4.17}$$

5. Alternative approach for 2nd example

We can obtain the same solution (4.8) in another way. First we take the finite Fourier sine transform of equation (3.1), integrating the second term of the resulting equation by parts, and then applying the boundary conditions, we obtain

$$\frac{d^{\alpha}\bar{u}(n,t)}{dt^{\alpha}} + \frac{n^2\pi^2}{4}\bar{u}(n,t) = 0, \tag{5.1}$$

where n is a wavenumber, and

$$\bar{u}(n,t) = \int_0^2 u(x,t)\sin\left(\frac{n\pi x}{2}\right) dx, \tag{5.2}$$

is the finite sine transform of $u(x,t)$.

Taking the finite sine transform of equation (4.2) and equation (4.3), we obtain

$$\bar{u}(n,0) = \int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right) dx, \tag{5.3}$$

and $\bar{u}_t(n,0) = 0$ for $1 < \alpha \leqslant 2$.

Let us rewrite the equation (5.1) in an operator form

$$L_t\bar{u} = -\frac{n^2\pi^2}{4}D_t^{(m-\alpha)}(\bar{u}), \quad m-1<\alpha<m, \tag{5.4}$$

where $L_t \equiv d^m/dt^m$ which is an easily invertible linear operator, $D_t^{(m-\alpha)}(\bullet)$ is the Riemann–Liouville derivative of order $(m-\alpha)$. Applying the inverse operator L_t^{-1} to equation (5.4) yields

$$\bar{u}(n,t) = \phi - \frac{n^2\pi^2}{4}L_t^{-1}[D_t^{(m-\alpha)}(\bar{u}(n,t))]$$

with $L_t\phi = 0$. The function ϕ can be identified from the given initial conditions.

The Adomian decomposition method [7, 8] assumes an infinite series solution for the unknown function $\bar{u}(n,t)$ in the form

$$\bar{u}(n,t) = \sum_{k=0}^{\infty}\bar{u}_k(n,t), \tag{5.5}$$

Therefore, by the Adomian decomposition method, we can write,

$$\bar{u}_0 = \phi$$

$$\bar{u}_1 = -\frac{n^2\pi^2}{4}L_t^{-1}[D_t^{(m-\alpha)}(\bar{u}_0)]$$

$$\bar{u}_2 = -\frac{n^2\pi^2}{4}L_t^{-1}[D_t^{(m-\alpha)}(\bar{u}_1)]$$

$$\bar{u}_3 = -\frac{n^2\pi^2}{4}L_t^{-1}[D_t^{(m-\alpha)}(\bar{u}_2)]$$

and so on.

Therefore, the entire components $\bar{u}_0$, $\bar{u}_1$, $\bar{u}_2, \ldots$ are identified and the series solution thus entirely determined. In this case, the exact solution in a closed form may be obtained. The practical solution will be the n-term approximation ϕ_n

$$\phi_n = \sum_{i=0}^{n-1}\bar{u}_i(n,t), \quad n \geqslant 1, \tag{5.6}$$

with

$$\lim_{n\to\infty}\phi_n = \bar{u}(n,t)$$

Now, let us consider $\alpha = 3/2$ and $m = 2$. Then from recursive relations of the Adomian decomposition method with initial conditions, equation (5.3) gives

$$\bar{u}_0 = \bar{u}(n,0) + t\bar{u}_t(n,0) = \int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right) dx$$
$$= \frac{8}{\pi^2}\left[\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\right],$$

$$\bar{u}_1 = -\frac{n^2\pi^2}{4}L_t^{-1}[D_t^{1/2}(\bar{u}_0)]$$
$$= \frac{8}{\pi^2}\left[\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\right]\left[-\frac{n^2\pi^2}{4}\frac{t^{3/2}}{\Gamma\left(\frac{5}{2}\right)}\right],$$

$$\bar{u}_2 = -\frac{n^2\pi^2}{4}L_t^{-1}[D_t^{1/2}(\bar{u}_1)]$$
$$= \frac{8}{\pi^2}\left[\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\right]\left[\left(\frac{n^2\pi^2}{4}\right)^2\frac{t^3}{\Gamma(4)}\right],$$

$$\bar{u}_3 = -\frac{n^2\pi^2}{4}L_t^{-1}[D_t^{1/2}(\bar{u}_2)]$$
$$= \frac{8}{\pi^2}\left[\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\right]\left[-\left(\frac{n^2\pi^2}{4}\right)^3\frac{t^{9/2}}{\Gamma\left(\frac{11}{2}\right)}\right],$$

$$\bar{u}_4 = -\frac{n^2\pi^2}{4}L_t^{-1}[D_t^{1/2}(\bar{u}_3)]$$
$$= \frac{8}{\pi^2}\left[\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\right]\left(\frac{n^2\pi^2}{4}\right)^4\frac{t^6}{\Gamma(7)}$$

and so on.

Therefore, the series $\bar{u}(n,t)$ becomes

$$\bar{u}(n,t) = \frac{8}{\pi^2}\left[\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\right]\sum_{k=0}^{\infty}\frac{\left(-\frac{n^2\pi^2 t^{3/2}}{4}\right)^k}{\Gamma\left(\frac{3k}{2}+1\right)}. \tag{5.7}$$

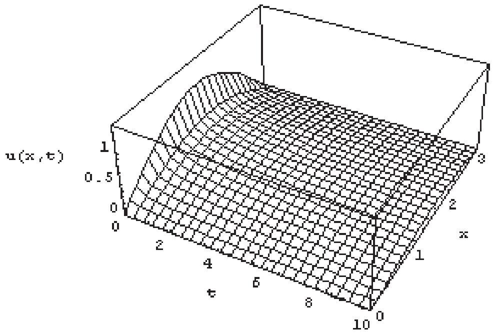

Figure 1. Decomposition method solution of equation (3.1) for $\alpha = 1/2$.

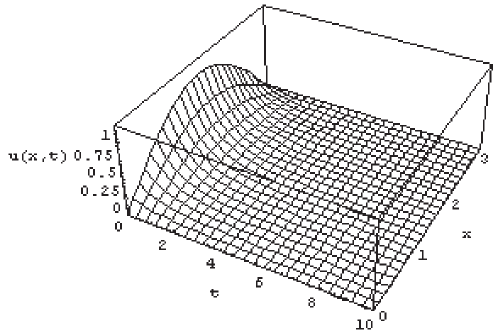

Figure 2. Decomposition method solution of equation (3.1) for $\alpha = 1$.

Taking the inverse finite sine transform of equation (5.7), we obtain the solution

$$
\begin{aligned}
u(x,t) &= \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\right]\sum_{k=0}^{\infty}\frac{\left(-\frac{n^2\pi^2 t^{3/2}}{4}\right)^k}{\Gamma\left(\frac{3k}{2}+1\right)} \\
&\times \sin\left(\frac{n\pi x}{2}\right) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n-1}}{(2n-1)^2}\right]\sum_{k=0}^{\infty}\frac{\left(-\frac{(2n-1)^2\pi^2 t^{3/2}}{4}\right)^k}{\Gamma\left(\frac{3k}{2}+1\right)} \\
&\times \sin\left(\frac{(2n-1)\pi x}{2}\right) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n-1}}{(2n-1)^2}\right] \\
&\times E_\alpha\left(-\frac{(2n-1)^2\pi^2 t^{3/2}}{4}\right)\sin\left(\frac{(2n-1)\pi x}{2}\right),
\end{aligned}
\tag{5.8}
$$

where $E_\lambda(z)$ is the Mittag–Leffler function in one parameter. The two solutions (4.8) and (5.8) are same for $\alpha = 3/2$.

6. Numerical results and discussions

Figures 1–5 show the evolution results for $\alpha = 1/2$, 1, 3/2, 7/4 and 2 respectively. Figures 2 and 5 (α =1 and 2) show the diffusion and wave solutions respectively. Comparison of figures 1 and 2 shows that the 1/2 order derivative system exhibits fast diffusion in the beginning and slow diffusion later. Figure 3 shows the combined diffusion and wave behaviour. Increasing the parameter α results in an increased wave behaviour, as shown in figure 4 ($\alpha = 7/4$). The classical wave solution ($\alpha = 2$) is plotted in figure 5. The same behaviours were also described by Diethelm and Weilbeer [34].

Figures 6–9 have been drawn using equation (4.8) or equation (5.8) for $\alpha = 0.5$, 1, 1.5 and 2. Figures 6 and 7 show fast diffusion and figures 8 and 9 show slow diffusion. So, as value of α increases, diffusion behaviour decreases.

Equation (4.14) has been used to draw figures 10–12. From figures 10–12, we observe that diffusion behaviour becomes wave behaviour with an increase of β. Figure 11 shows one wave whereas figure 12 shows two waves.

In the present numerical analysis we assume $\gamma = -1$, $c = -1$, $\sigma = 0.75$, $k = 0.2$, $\varepsilon = 0.015$ for the fourth example.

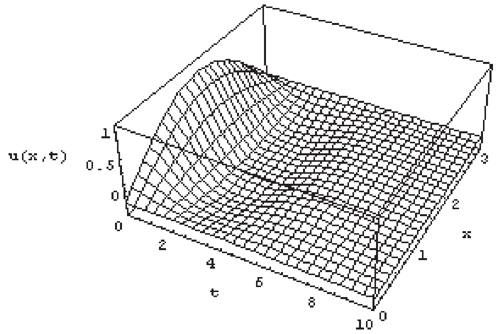

Figure 3. Decomposition method solution of equation (3.1) for $\alpha = 3/2$.

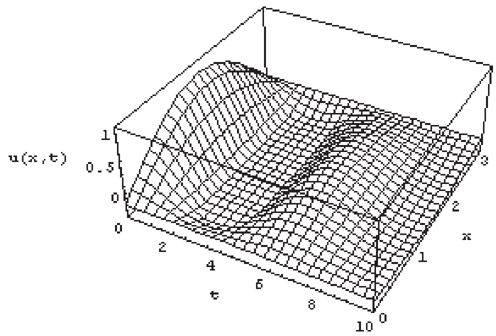

Figure 4. Decomposition method solution of equation (3.1) for $\alpha = 7/4$.

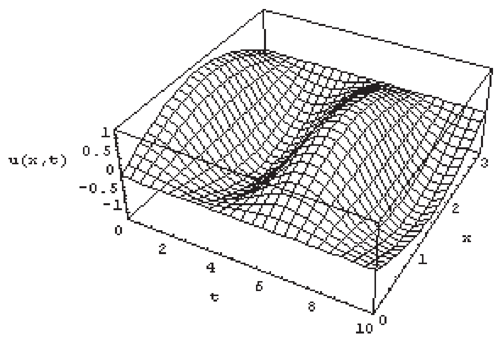

Figure 5. Decomposition method solution of equation (3.1) for $\alpha = 2$.

Equation (4.17) has been used to draw figure 13, which shows fast diffusion as t increases.

Figures 1–13 have been drawn using the *Mathematica* software [36].

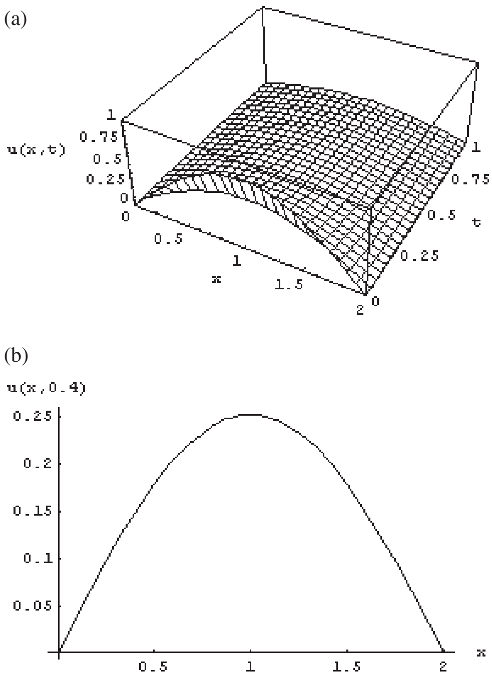

Figure 6. Second example presenting (a) decomposition method solution for $\alpha = 0.5$; (b) the decomposition method solution at $t = 0.4$ for $\alpha = 0.5$.

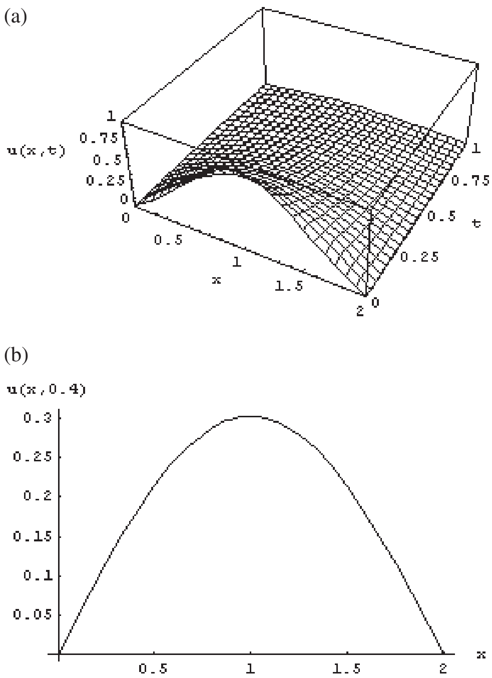

Figure 7. Second example presenting (a) decomposition method solution for $\alpha = 1$; (b) the decomposition method solution at $t = 0.4$ for $\alpha = 1$.

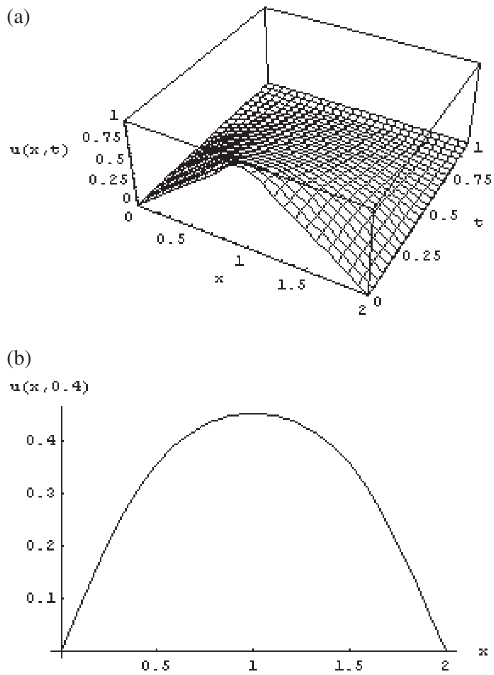

Figure 8. Second example presenting (a) decomposition method solution for $\alpha = 1.5$; (b) the decomposition method solution at $t = 0.4$ for $\alpha = 1.5$.

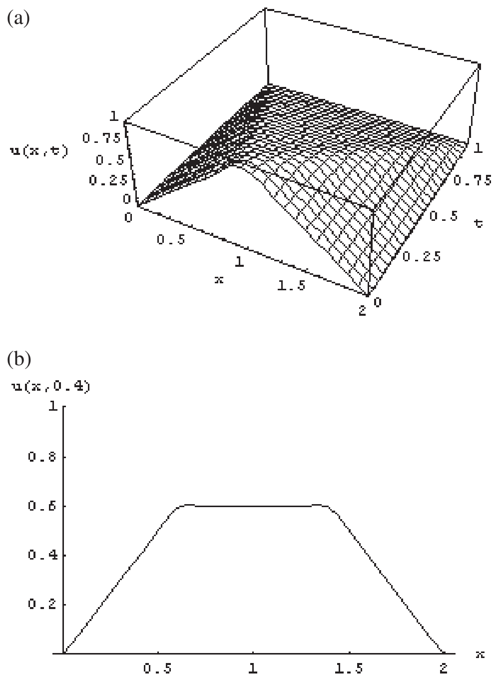

Figure 9. Second example presenting (a) decomposition method solution for $\alpha = 2$; (b) the decomposition method solution at $t = 0.4$ for $\alpha = 2$.

7. Conclusion

This paper presents an analytical scheme to obtain the solution of a time-fractional diffusion-wave equation. Four typical examples have been discussed as demonstrations. The first example has been solved in order to obtain the behaviours of equation (3.1) for different values of α and compared with the results of Diethelm and Weilbeer [34]. In the second example, the Adomian decomposition method has been successfully applied in two different ways. In both ways, the same result has been achieved. In the third example, we have obtained the same result as Mainardi *et al* [35]. A nonlinear fractional equation has been solved in the 4th example. The physical significance of all the examples has been presented in this paper. In our previous papers [24–27], we have already successfully exhibited the applicability of the Adomian decomposition method to obtain a solution for a dynamic system containing fractional derivatives of order

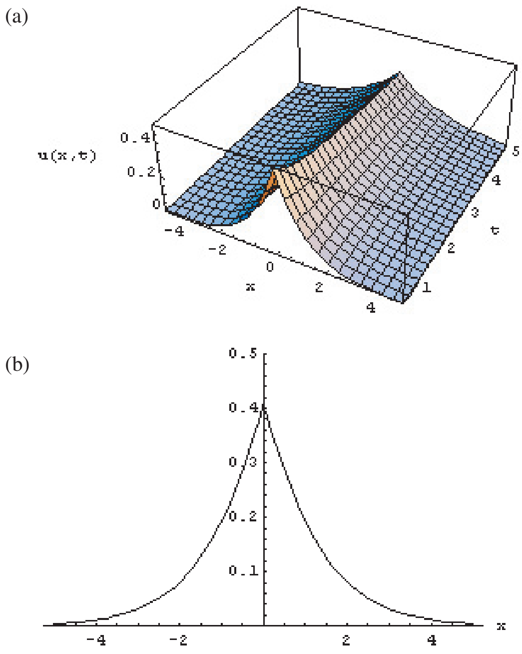

Figure 10. Third example presenting (a) decomposition method solution for $\beta = 1/2$; (b) the decomposition method solution at $t = 1$ for $\beta = 1/2$.

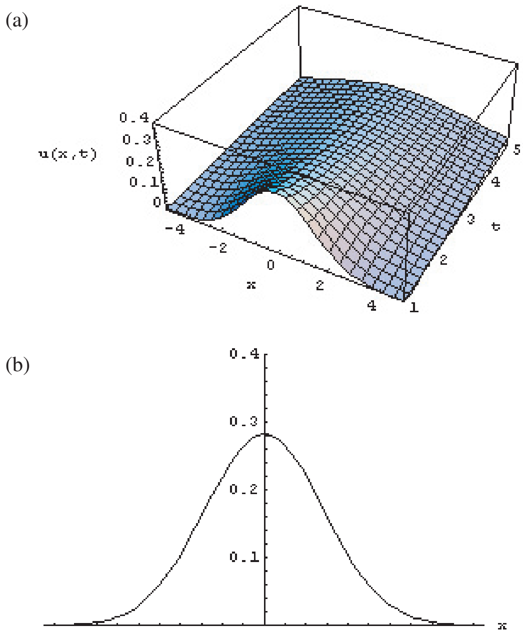

Figure 11. Third example presenting (a) decomposition method solution for $\beta = 1$; (b) the decomposition method solution at $t = 1$ for $\beta = 1$.

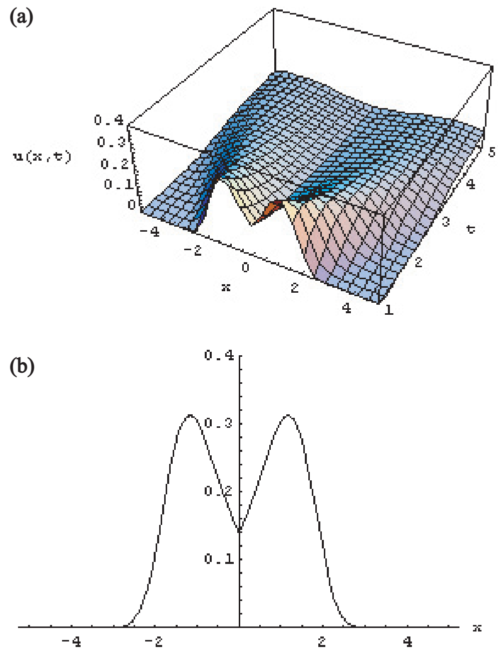

Figure 12. Third example presenting (a) decomposition method solution for $\beta = 3/2$; (b) the decomposition method solution at $t = 1$ for $\beta = 3/2$.

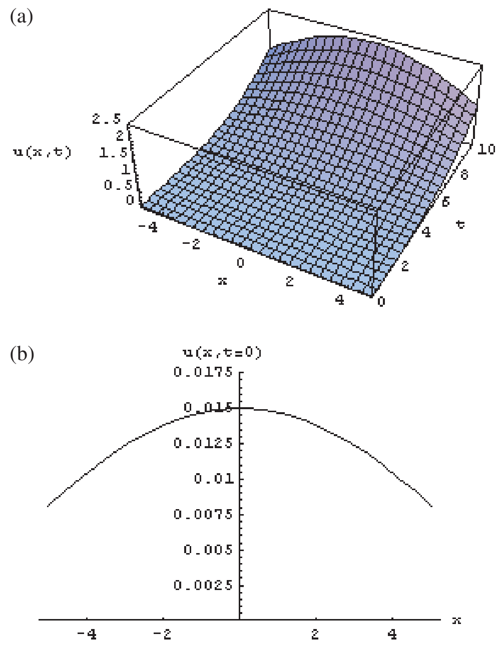

Figure 13. Fourth example presenting (a) decomposition method solution for $\alpha = 3/2$; (b) the decomposition method solution at $t = 0$ for $\alpha = 3/2$.

$1/2$ and $3/2$. In this study, we demonstrate that this method is also well suited to solving time-fractional diffusion-wave and nonlinear fractional equations. The decomposition method is straightforward, without restrictive assumptions and the components of the series solution can be easily computed using any mathematical symbolic package. Moreover, this method does not change the problem into a convenient one for the use of linear theory. It, therefore, provides more realistic series solutions that generally converge very rapidly in real physical problems. When solutions are computed numerically, the rapid convergence is obvious. Moreover, no linearization or perturbation is required. It can avoid the difficulty of finding the inverse of the Laplace transform and can reduce the labour of perturbation method.

Furthermore, as the decomposition method does not require discretization of the variables, i.e., time and space, it is not affected by computational round off errors and one is not faced with necessity of large computer memory and time. Consequently, the computational load will be reduced.

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