Hackenbush & Nim

Intro to Combinatorial Game Theory

December 24, 2024

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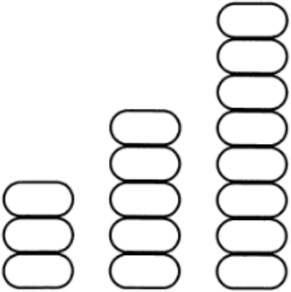
Main Question 1.

Given some state of our game, who wins under optimal play?

CGT seeks to answer this questions with a mix of various algebraic and combinatorial techniques

Nim



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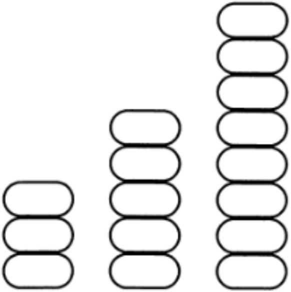

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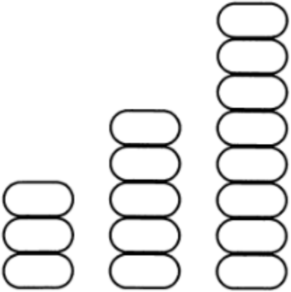

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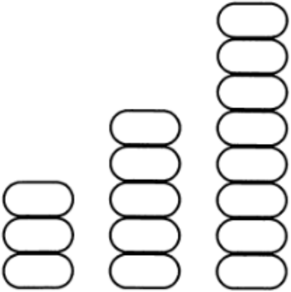

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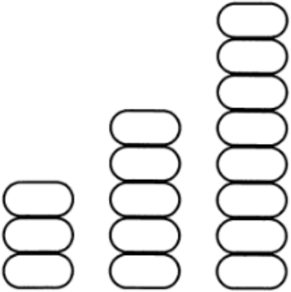

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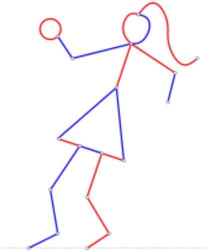

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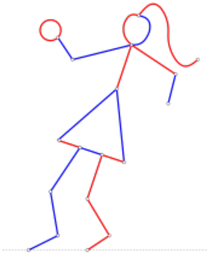

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  - There exists a vertex we denote *ground*

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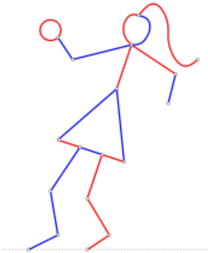

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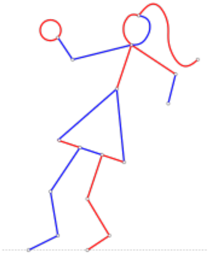

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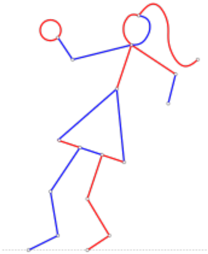

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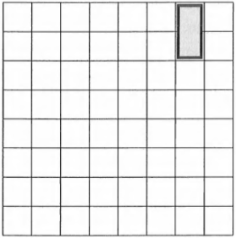

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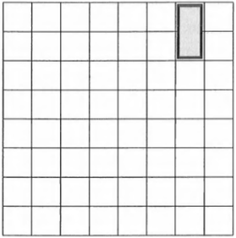

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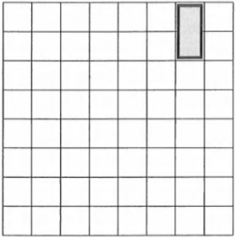

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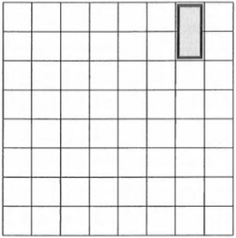

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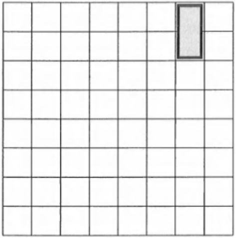

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Vocab

Definition 1 (Types of Play).

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Examples: Hackenbush and Domineering are partizan where was Nim is impartial

Winning Nim Strategy

Theorem 3.

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If G has zero Nim value, the second player can always win. Why?

More Vocab

Definition 4.

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ruleset a system of playable rules

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Let G and H be combinatorial games. H is a **left option** (resp. **right option**) of G if Left (resp. Right) can move directly from G to H

Definition 6.

Let G and H be combinatorial games. H is a **subposition** of G if there exists a sequence of consecutive moves leading from G to H

Last Vocab Slide (I promise)

Let G be a game.

Definition 7.

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1. A **run** of G of length k is a sequence of positions

$$G_0, G_1, G_2, \ldots, G_k$$

such that $G_0 = G$ and G_i is an option of G_{i+1} Note that k can be ∞, in which case there is no last game G_k

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2. An alternating run is a run in which moves alternative between left and right: that is,

$$G_{i+1} \text{ is a left option of } G_i \implies G_{i+2} \text{ is a right option of } G_{i+1}$$

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3. A **play** is alternating run that's infintely long or whose last game has no options for either player

Outcome Classes

Theorem 8 (Fundamental Theorem of Short Games).

Let G be a short game and assume normal play.

1. *Either Left can force a win by playing first on G or else Right can force a win playing second*
2. *Either Right can force a win by playing first on G or else Left can force a win playing second*

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Every short game therefore belongs to one of the four outcome classes

- $\mathcal{N}$ first player (the **N**ext player) can force a win
- $\mathcal{P}$ second player (the **P**revious player) can force a win
- $\mathcal{L}$ Left can force a win, no matter who moves first
- $\mathcal{R}$ Right can force a win, no matter who moves first

Adding Two Games

Definition 9.

Given two games G and H, the games $G + H$ is as follows:

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1. Games G and H are placed side by side
2. A player must move in exactly one of G or H (not both!)

Fundamental Equivalence

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For short games G and H, we write $G = H$ if

$$o(G + X) = o(H + X) \text{ for all short games } X$$

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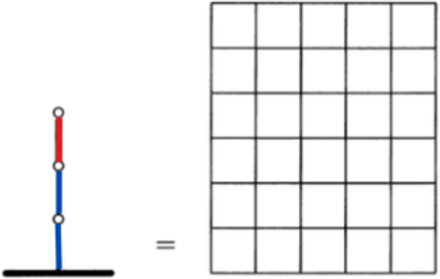

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Where to Learn More!

Thank you for playing and listening!

Further reading:

- This great Hackenbush videogame.
  - itch.io link: https://fi-le.itch.io/hackenbush
  - Github: https://github.com/lennart-finke/hackenbush
- This Youtube video about Hackenbush.
  - HACKENBUSH: a window to a new world of math
  - One of the best math-tube videos ever IMO!
- The CGT Bible: Winning Ways for your Mathematical Plays