Square Coloring up to Rotations

An Introduction to Groups and Burnside's Lemma

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Math Circle

Overview

The Plan

Big Question 1.

Consider a cube and 6 arbitrary, distinct colours. We'll colour the sides of our cube using these six colours. Suppose two colorings of our cube are considered the same if we can rotate one colored cube onto the other. How many such cubes exists? What if we swap 6 with n?

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Related Question 2 (AIME 1996).

Two of the squares of a 7×7 checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by applying a rotation in the plane of the board. How many inequivalent color schemes are possible?

These problems motivate all further discussion. Broadly, there are two lines of attack for them: case bashing and case bashing with a bit of elegance. These lectures will be on this *bit of elegance*.

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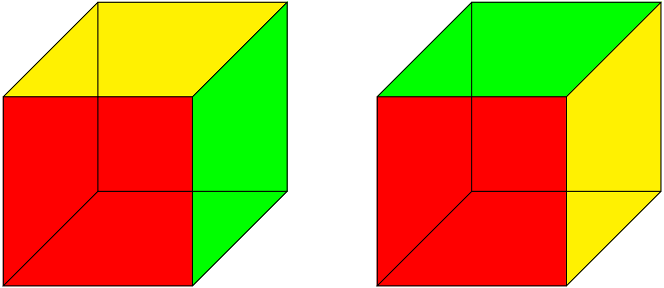

(a) (b)

Figure: two of the same colorings (opposite sides are the same colour here)

Part 1

Groups: What are they?

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Definition 1 (Group).

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Example 2 (Prototypical).

$\mathbb{Z}$ is a group! Remember that the data of a group is both an operation and a set. We'll let our operation be standard addition.

The Integers Modulo n

Example 3 ($\mathbb{Z}/n\mathbb{Z}$).

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$$a + n\mathbb{Z} = \{m \in \mathbb{Z} \mid m = a + kn, k \in \mathbb{Z}\}$$

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for all integer a. There can only be finitely many such sets. This is because there are only finitely many residues modulo n, and, if a and b differ by a multiple of n (i.e. $a \equiv b \pmod{n}$), then $a + n\mathbb{Z} = b + n\mathbb{Z}$.

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Following my inner lazy mathematician, I'll use this as a PSET problem :)

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I'll elaborate on how cyclic notation works in class with more examples.

Checking the Group Axioms of the Symmetric Group

(1) We hope to show that, for permutations $f, g : X \to X$, $f \circ g : X \to X$ is a permutation. Suppose $\exists x \in X$ such that $f \circ g(x) = f \circ g(x')$. Then,

$$f(g(x)) = f(g(x')) \implies g(x) = g(x') \implies x = x'$$

by the injectivity of f and g. We conclude $f \circ g$ is injective. Consider $x \in X$. By the surjectivity of f and g, $\exists y, z \in X$ such that $f(y) = x$ and $g(z) = y$. Then, $f \circ g(z) = x$, i.e. $f \circ g$ is surjective.

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(4) Function composition is associative. For $f, g, h \in S_n$ and $x \in X$, $f \circ (g \circ h)(x) = f(g(h(x))) = (f \circ g) \circ h(x)$.

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(4) Inverses in a group are unique. Suppose h, h' in G were inverses of g. Then, by associativity, $h = hgh' = h'$.

Part 2

Introduction to Group Actions & Burnside's Lemma

Burnside's Lemma

Theorem 5.

Let a finite group G act on a finite set X. Write X/G for the set of orbits in X, and X^g for the set of elements of X fixed by $g \in G$, i.e. $X^g = \{x \in X \mid g \cdot x = x\}$. Then,

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Fun Fact 2.

Burnside's Lemma was not found by Burnside. It was originally a theorem of Cauchy that was misattributed to Burnside. That's why you'll find that this result is often jokingly referred to as 'not Burnside's Lemma' or 'the theorem that is not Burnside's'

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Example 7.

(1) For any group G, the subset $\{e\}$ is a subgroup. It's called the *trivial group*. Similarly, the entire group G is also technically a subgroup.

(2) For any group G and element $g \in G$, $\langle g \rangle := \{g^n \mid n \in \mathbb{Z}\}$ is a subgroup of G. $\langle g \rangle$ is called the cyclic group generated by g.

We say $H \leq G$ when H is a subgroup of G.

Group Actions

Definition 8.

Let X be a set and G a group. An action of G on X is a function $\phi : G \times X \to X$ such that

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Example 9 (Prototypical).

Remember the symmetric group S_n? Well, consider the obvious group action on the set $X = \{1, 2, \ldots, n\}$. Since $f \in S_n$ is a permutation of X by definition, we can define our action on X such that, for $x \in X$, $fx = f(x)$.

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(3) Let G be an arbitrary group with group operation $*$. We can make G act on its underlying set (i.e. itself) by left multiplication. That is, for $g \in G$ and $x \in X = G$, $gx := g * x$.

Preliminary Definitions

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Definition 11 (Orbit).

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Definition 13 (Coset).

A coset of a subgroup $H \leq G$ is a set of the form $gH := \{gh \mid h \in H\}$, where $g \in G$.

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We'll omit the proofs from this presentation. We do this not because they are beyond the viewer but because they lead to group theoretical rabbit holes. For the sake of time and clarity, we'll assume their validity. However, do make sure to see their proofs in the write-up 'Group Theory Proofs' on your own time.

Guess Who's Back, Back Again.

We'll prove Burnside's Lemma now. To recap, we're trying to show $|G| \times |X/G| = \sum_{g \in G} X^g$, where X/G are the set of orbits of X under the action of G and $X^g = \{x \in X \mid g \cdot x = x\}$.

Guess Who's Back, Back Again.

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Proof.

By the orbit-stabilizer Theorem,

$$\begin{aligned}
\sum_{g \in G} |X^g| &= \sum_{g \in G} |\{x \in X : \; gx = x\}| = |\{(x, g) \in X \times G : \; gx = x\}| \\
&= \sum_{x \in X} |g \in G : \; gx = x| = \sum_{x \in X} |G_x| = |G| \sum_{x \in X} \frac{1}{|\mathcal{O}(x)|} \\
&= |G| \sum_{k \in |X/G|} 1 \\
&= |G| \times |X/G|
\end{aligned}$$

$\blacksquare$

Part 3
Using Burnside's Lemma

The Rotations of a Cube

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Now, let G be the group of rotations of the cube and X the total number of colored cubes (we can use n colors), where two cubes are considered distinct even if they can be rotated onto one another. Clearly, $|X| = n^6$. We'll let G act on X in the standard way: a rotation in $\rho \in G$ sends a given cube to the cube obtained when it's rotated by ρ. Perhaps you can see that answering our question now boils down to determining $|X/G|$ in our group action. This is where Burnside's Lemma comes into play.

Big Question 1

Proposition 16.

$|G| = 24.$

Proposition 17.

$\sum_{g \in G} |X^g| = n^6 + 3n^4 + 12n^3 + 8n^2.$

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Proof.

From inspection, there are $3 \times 3 = 9$ rotations of about an axis passing through the centres of opposite faces, $4 \times 2 = 8$ rotations about an axis passing through two vertices, $6 \times 1 = 6$ rotations about an axis passing through the mid-points of edges, and 1 identity rotation. Use a Rubik's cube (or any other (preferably colored) cube) or your imagination to verify this. Then, we note $9 + 8 + 6 + 1 = 24$. ∎

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We'll leave this for the PSET :) Using these results, we conclude, by Burnside's Lemma, that there are $\frac{n^6+3n^4+12n^3+8n^2}{24}$ distinct cubes.