APPLYING THE BORSUK-ULAM THEOREM

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Abstract. This writeup acts as a sketch of a presentation on topological combinatorics given for Math 2240 at Cornell. We hope to present one potential answer to the question of "why topology?" by connecting two seemingly disparate areas of mathematics: combinatorics (where we work with the discrete) and topology (where we work with the continuous).

For the sake avoiding repetition and our own carelessness, assume all maps are continuous unless stated otherwise.

Much of the material from this talk comes from [1].

1. Borsuk-Ulam & Alternate Formulations

Naturally, let's start with vocab.

Definition 1.1. The n-sphere $\mathbb{S}^n$ is the subset

$$\{\mathbf{x} \in \mathbb{R}^{n+1} \mid |x| = 1\}$$

of $\mathbb{R}^{n+1}$. Similarly, B^n is the unit ball centered around $\mathbf{0}$, i.e. $\{\mathbf{x} \in \mathbb{R}^n \mid |x| \leq 1\}$.

Definition 1.2. Two points $\mathbf{x}, \mathbf{y} \in \mathbb{S}^n$ are antipodal if $\mathbf{x} = -\mathbf{y}$, i.e. they're on diametrically opposite ends of the sphere. A map $f : \mathbb{S}^n \to X$ is antipodal (or odd) if $f(-x) = -f(x)$.

Note that the superscripts of $\mathbb{S}^n$ and B^n denote the dimensions of our sets, not the dimension of the space they live in. I.e. the boundary of B^n is S^{n-1}, not S^n.

Theorem 1.3 (Borsuk-Ulam). *Given a continuous map $f : \mathbb{S}^n \to \mathbb{R}^n$, f identifies two antipodal points: i.e. $\exists \mathbf{x} \in \mathbb{S}^n$ such that $f(\mathbf{x}) = f(-\mathbf{x})$.*

While originally formualted by Stanislaw Ulam, the first proof of Theorem 1.3 was given by Karol Borsuk. Since then Borsuk-Ulam has found a number of equivalent formulations, proofs, and applications both at the heart and entirely away from topology. Ultimately, we only hope to present of small slice this. For a comprehensive treatment, see [2].

Example 1.4. The one dimension case is boils down to IVT. For $f : \mathbb{S}^1 \to \mathbb{R}$, consider $F(x) = f(x) - f(-x)$. Notice that our problem reduces to looking for the zeros of F. Now, pick some $x \in \mathbb{S}^1$. If $F(x) = 0$, we're done. Otherwise, $F(x)$ and $F(-x)$ have differing signs by the antipodality of F. Hence, IVT gives the existence of $y \in \mathbb{S}^1$ such that $F(y) = 0 \implies f(y) = f(-y)$.

By law, all talks about Borsuk-Ulam must present the following example.

Example 1.5. Since temprature and pressure vary continuously across the earth, mapping each place to it's temprature-pressure pair on the plane gives a map $f : \mathbb{S}^2 \to \mathbb{R}^2$. Hence, there exists two antipodal points on the earth with the same temprature and pressure.

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Let's see some equivalent forms!

Theorem 1.6 (Equivalent Formulations)**.** *For $n \geq 0$, the following are equivalent and true:*

(1) *Borusk-Ulam, Theorem 1.3.*
(2) *For antipodal $f : \mathbb{S}^n \to \mathbb{R}^n$, there exists $\mathbf{x} \in \mathbb{S}^n$ such that $f(\mathbf{x}) = 0$.*
(3) *There does not exist an antipodal map $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$.*
(4) *There does not exist $f : B^n \to \mathbb{S}^n$ that is antipodal on the boundary.*
(5) *For any cover $F_1, \ldots, F_{n+1}$ of the sphere $\mathbb{S}^n$ by $n+1$ closed sets, there is at least one set containing a pair of antipodal points.*
(6) *For any cover $U_1, \ldots, U_{n+1}$ of the sphere $\mathbb{S}^n$ by $n+1$ open sets, there is at least one set containing a pair of antipodal points.*

Proof. We'll proceed by (1) $\iff$ (2) $\iff$ (3) $\iff$ (4).

(1) $\implies$ (2) For antipodal f, there exists $\mathbf{x}$ such that

$$f(\mathbf{x}) = f(-\mathbf{x}) = -f(\mathbf{x}) \implies f(\mathbf{x}) = 0.$$

(2) $\implies$ (1) For continuous f, $F(x) = f(x) - f(-x)$ is antipodal so there exists $\mathbf{x}$ such that

$$F(\mathbf{x}) = 0 \implies f(\mathbf{x}) = -f(-\mathbf{x}).$$

□

Finally, we'll prove the $n = 2$ case. While the higher dimensional cases aren't excessively technical, we'll skip it to stay within time. See [1, ch 2] (or the appendix if I add it later) for the full proof.

Proof of Borsuk-Ulam for $n = 2$. □

An easy yet powerful consequence of Borsuk-Ulam is the Brower fixed point theorem:

Theorem 1.7. *For any convex compact $K \subseteq \mathbb{R}^n$, a map $K \to K$ has a fixed point.*

We don't actually have to work at the leval of generality of Theorem 1.7.

Lemma 1.8. *Every convex compact subset of $\mathbb{R}^n$ is homeomorphic to the unit ball, B^n.*

2. Hex Game

The game of Hex is played between two players, say Red and Green. Each of them takes turns coloring a finite hexagonal grid. After all hexagons are colored, Red wins if there's a red path connecting the top and bottom, and Green wins if there's a green path connecting the left and right.

Remark. Play a game after introducing Hex give a feel for the problem (if time permits).

Theorem 2.1. *Hex can never end in a tie.*

We won't assume that

3. Fair Division

Suppose you and a friend steal an open necklace, engraved with m precious stones. There are d kinds of stones (labelled $1, \dots, d$), an even number of each kind. Neither you nor your co-conspirator know the values of the different stone types. Hence, you decide to split the stones between each of you such that you both have the same number of jewels of each kind.

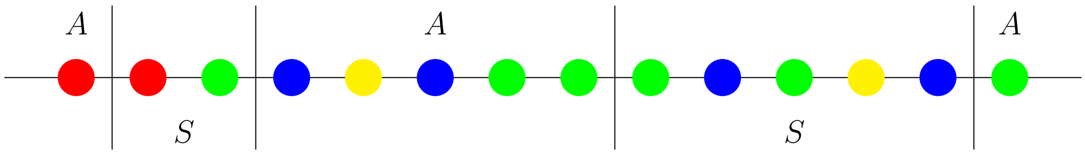

Figure 1. Example necklace and cut: segments labelled A go to Akash and those labelled S go to Sid. Our necklace has 14 stones of 4 types and was split using 4 cuts.

Instead of removing the individual stones one by one, you decide to split the jewels by making as few cuts as possible to the necklace, divvying out the remaining strands between yourselves.

Theorem 3.1. *The minimal number of cuts needed is at most d.*

Proof. At its current state, there isn't any good way to apply Borsuk-Ulam on our puzzle. Let's change that! We'll consider the interval $[0, m]$ as our neckalce. Break $[0, m]$ into the union of subintervals

$$\bigcup_{k=0}^{m-2} [k, k+1) \cup [m-1, m] .$$

The kth stone corresponds to the kth subinterval. Then, define characteristic functions for the stones as $f_i : [0, m] \to \{0, 1\}$ for $i \in [d]$ such that

$$f_i(x) = \begin{cases} 1 & x \in [k-1, k) \text{ and the } k\text{th stone is of type } i \\ 0 & \text{otherwise} \end{cases} .$$

Fix $0 = z_0 \leq z_1 \leq z_2 \leq \cdots \leq z_d \leq m = z_{d+1}$ to act as the cuts on our string. If we wish to assign the part $[z_i, z_{i-1}]$ $(i = 1, \dots, d+1)$ to the first thief, set $x_i = \left(\sqrt{z_i - z_{i-1}}\right)/m$ and $x_i = -\left(\sqrt{z_i - z_{i-1}}\right)/m$ otherwise. The tupple $(x_1, x_2, \dots, x_{d+1}) \in \mathbb{S}^d$ encodes a cut of the necklace. Noting that $z_j = \sum_{i=1}^{j} m^2 x_i^2$, consider the continuous map

$$(x_1, x_2, \dots, x_{d+1}) \to \left(\sum_{j=1}^{d+1} \operatorname{sign}(x_j) \int_{[z_j, z_{j-1}]} f_1(x)\, \mathrm{d}x, \dots, \sum_{j=1}^{d+1} \operatorname{sign}(x_j) \int_{[z_j, z_{j-1}]} f_d(x)\, \mathrm{d}x \right) .$$

Since f is antipodal, theorems 1.3 and **??** tell us $\exists \mathbf{x} \in \mathbb{S}^n$ such that $f(\mathbf{x}) = 0$. The cut associated $\mathbf{x}$ is a fair division.

However, we might run into a problem translating this back into the discrete case! What if some z_i associated with $\mathbf{x}$ lies in $(k, k+1)$ for some $k \in \mathbb{Z}$? (I.e. we cut partially into one of the jewels.) Given a non-integral cut subdividing a stone of type i, where a portion of stone i is assigned to the first thief, we know our cut is either unneccesary or there exists at least one other partial cut into stones of type i assigned to thief one since the sums of lengths of stone i intervals is an integer. In the latter case, we can move the first "non-integral" cut to the right and the remaining to the left, without changing the loot for each thief. $\square$

Some generalizations blah blah

References

[1] Jiří Matoušek, Anders Björner, Günter M Ziegler, et al. *Using the Borsuk-Ulam theorem: lectures on topological methods in combinatorics and geometry.* Springer, 2003.

[2] Heinrich Steinlein. Borsuk's antipodal theorem and its generalizations and applications: A survey, méthodes topologiques en analyse non linéaire. *Sem. Math. Sup.*, 95:166–235, 1985.

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