

Exeter Mathematics School Team Mathematical Olympiad
Round 1 : Monday 26 April 2021

Time allowed One week school, commencing 09:00 Monday 26th

Scoring rules • Questions 1-5 will be marked out of 5, with the final answer given 5 marks (Credit will **not** be given for incomplete solutions).

- Questions 6-9 will be marked out of 10 (Credit **will** be given for incomplete solutions).
- Question 10 will be marked out of 15 (Credit **will** be given for incomplete solutions).

Instructions • Full written solutions for questions 6-10, *not just answers*, are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt (On paper or in \LaTeX). Rough work *can* be handed in, but should be clearly marked.

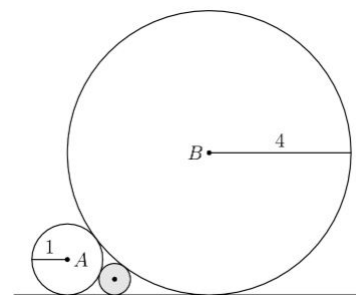
- One or two complete solutions (For questions 6-10) will gain more credit than partial attempts at all problems.
- Do not communicate to other teams regarding the ETMO until the round is over.
- The use of the Internet or other sources to assist in the solving of any problems is strictly prohibited
- The use of rulers and compasses is allowed, but calculators and protractors are not only forbidden, but unadvised as they will not give you any meaningful advantage.
- If using paper, staple all the pages neatly together in the top left hand corner, with questions in numerical order. Return solutions to me, by teams or in real life, Ben Folland by Friday 30th 12:00.

Exeter Mathematics School Team Mathematical Olympiad
Round 1, 2021, Questions 1-5

1. Ben writes down 6 different primes p, q, r, s, t, u all less than 20, such that $p + q = r + s = t + u$. What is the value of $p + q$?
2. How many 0s are at the end of $100!$
3. Find the sum of all integer values x such that

$$(x^2 - 17x + 71)^{(x^2 - 34x + 240)} = 1$$

4. Circles A and B, with radius 1 and 4 respectively, are tangent to a line and touch at a single point. A third smaller circle is tangent to the line and touches A and B at single points. Find the radius of the smaller circle.



5. Find the sum of all (not necessarily real) roots of the equation:

$$x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$$

Exeter Mathematics School Team Mathematical Olympiad
Round 1, 2021, Questions 6-9

6. A Folland-Aizawa number is a positive integer which equals 13 times the sum of its digits. Find, with proof, all Folland-Aizawa numbers.

7. Prove that $\forall a, b, c, d > 0$,

$$(a + b + c + d)(a^3 + b^3 + c^3 + d^3) \geq (a^2 + b^2 + c^2 + d^2)^2$$

8. The AM-GM inequality states: for all positive real numbers x_1, x_2, \dots, x_n ,

$$\frac{x_1 + x_2 + x_3 + \dots + x_n}{n} \geq \sqrt[n]{x_1 x_2 x_3 \dots x_n}. \quad (1)$$

a) Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3 \quad (2)$$

$\forall a, b, c \in \mathbb{R}^+$ and determine when the equality holds.

b) Find the minimum value of

$$\frac{a^2}{b} + \frac{b}{c^2} + \frac{c}{a} \quad (3)$$

For $a, b, c \in \mathbb{R}^+$

9. Solve for positive real x :

$$x + \sqrt{x(x+1)} + \sqrt{x(x+2)} + \sqrt{(x+1)(x+2)} = 2 \quad (4)$$

Exeter Mathematics School Team Mathematical Olympiad
Round 1, 2021, Question 10

10. Prove, that in acute triangle ABC ,

$$\tan(A)\tan(B)\tan(C) \geq 3\sqrt{3}$$

This is the end of the question paper.