Modern Portfolio Theory and Asset Allocation

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Returns and Log Returns

Let S_t denote the price of an underlying at time t. Then the one-period simple (arithmetic) return is given by

$$r_{t,t+1} = \frac{S_{t+1} - S_t}{S_t} = \frac{S_{t+1}}{S_t} - 1$$

and the one-period log-return is given by

$$\tilde{r}_{t,t+1} = \log(1 + r_{t+1}) = \log\left(\frac{S_{t+1}}{S_t}\right).$$

We also define the T-period cumulative (simple) return as

$$r_{t,t+T} = \frac{S_{t+T}}{S_t} - 1.$$

Analogously, the T-period cumulative log return is defined as

$$\tilde{r}_{t,t+T} = \log\left(\frac{S_{t+T}}{S_t}\right) = \log\left(\prod_{i=1}^{T} \frac{S_{t+i}}{S_{t+i-1}}\right) = \sum_{i=1}^{T} \log\left(\frac{S_{t+i}}{S_{t+i-1}}\right) = \sum_{i=1}^{T} \tilde{r}_{t,t+i}$$

Notation (Cont.)

So, from the previous slide, we see that the T-period cumulative log return is the sum of one-period log returns. This makes log-returns favorable to work with, but in practice, $r_{t,t+1} = \tilde{r}_{t,t+1}$ if $r_{t,t+1}$ is small (first order Taylor expansion of $\log(1+x)$).

Finally, the volatility of an underlying is defined as $\sigma_{t+1} = \sqrt{\mathrm{Var}(\tilde{r}_{t,t+1})}$ where $\tilde{r}_{t,t+1}$ is a random variable as of time t since the next-period price S_{t+1} is unknown.

There are a couple of natural questions to ask:

- Why bother using returns instead of prices?
- What time-interval / scale do we use to calculate returns? Hourly, daily, weekly, monthly, annually?

Why Use Returns?

There are a few reasons investor consider returns instead of prices:

- Returns are normalized by price /scale invariant. This makes it easier to compare between two stocks which trade at very different prices.
- Returns are approximately stationary. What this means is that, if we view r_t as a stochastic process, then $\mu_t = E(r_t)$ is constant over time, and the **autocovariance** $\text{Cov}(r_t, r_{t-h})$ only depends on the lag h, not on t, i.e. on how much time has **elapsed** rather than what point in time we are right now. So, certain properties of returns remain stable over time, which is useful when relying on time-series/econometric techniques for asset return prediction.
- Returns are approximately normally distributed, especially at longer time scales. But this is not quite true and shouldn't always be assumed...

Stylized Facts

Here we present a few stylized facts about returns (i.e. broad generalizations about patterns that roughly hold). We focus specifically on log-returns $\tilde{r}_{t,t+1}$.

- **Lagged autocorrelations** $\gamma(h) = \text{Cov}(\tilde{r}_{t,t+1}, \tilde{r}_{t-h,t-h+1})$ are small unless we observe prices and returns at time scales where market microstructure becomes relevant (e.g. intraday)
- **Heavy tails** - The distributions of returns, especially unconditional returns, exhibit *heavy tails*. This means that the probability of large positive or negative returns is higher than what we would expect with a thin tailed distribution, e.g. a Gaussian distribution.
- **Second moments** $\tilde{r}^2_{t,t+1}$ and absolute returns $|\tilde{r}_{t,t+1}|$ show the strongest autocorrelation.
- **Aggregational Gaussianity** - At longer time scales, e.g. weekly, monthly, the distribution of returns becomes closer to Gaussian.

What is a Portfolio?

Now that we have looked at individual returns, we can consider a portfolio. A portfolio is simply a collection of assets:

- n assets, with prices $P_{i,t}$
- A collection of N_i shares/units of each asset i (in cases of short sales we have $N_i < 0$)
- Total value of portfolio $\Pi_t = \sum_{i=1}^{n} N_i P_{i,t}$
- We can also define portfolio weights $w_i = \frac{N_i P_{i,t}}{\sum_{i=1}^{n} N_i P_{i,t}}$ such that a portfolio can be defined by its weights $\{w_1, \ldots, w_n\}$ where $\sum_{i=1}^{n} w_i = 1$. So weights can be negative if we are short a particular stock.
- An **equal weighted** portfolio has $w_i = 1/n$ for all i. A **cash/dollar neutral** strategy is defined such that $\Pi_t = 0$ and $\sum_i w_i = 0$, i.e. long and short positions offset each other.

Diversification

It can be seen that the portfolio return $r_{p,t+1}$ equals $\sum_{i=1}^{n} w_i r_{i,t+1}$. Dropping time subscripts, we have

- **Expected portfolio return** $E(r_p) = \sum_{i=1}^{n} w_i E(r_i)$
- **Portfolio variance** $\text{Var}(r_p) = \text{Var}\left(\sum_{i=1}^{n} w_i E(r_i)\right) = \sum_{i,j} w_i w_j \rho_{ij} \sigma_i \sigma_j$ where $\rho_{ij} = \text{Corr}(r_i, r_j)$ and σ_i is the volatility of asset i.

Importantly, if the assets are **not** perfectly correlated, i.e. $\rho_{ij} < 1$, then

$$\text{Var}(r_p) < \sum_{i,j} w_i w_j \sigma_i \sigma_j = (w_1\sigma_1 + \ldots + w_n\sigma_n)^2$$

and hence

$$\sigma_p = \sqrt{\text{Var}(r_p)} < w_1\sigma_1 + \ldots + w_n\sigma_n.$$

So the portfolio volatility is always less than the (weighted) average volatilities of individual assets, unless all assets are perfectly correlated!

Key takeaway: Diversification always reduces risk!

A Visualization

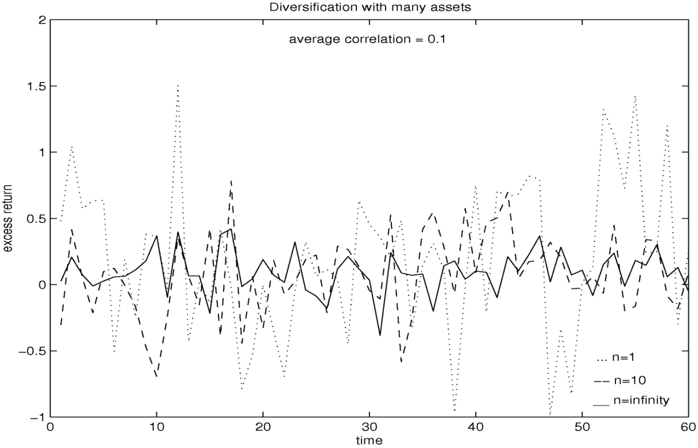


Figure 1: Diversification reduces risk!

Two Kinds of Risk

We can divide risks into two categories:

- **idiosyncratic** risk which is specific to single-assets (e.g. a pharmaceutical company facing a lawsuit due to harmful side effects of one of its drugs)
- **systematic** risk which is inherent to the entire economy (e.g. interest rate changes affecting cost of capital, inflation risk, 2008 financial crisis, COVID-19, etc.)

Idiosyncratic risks can be diversified away by holding a mix of assets across sectors, regions, valuations, etc. Systematic risk **cannot** be diversified away since it is present in all assets.

Two Kinds of Risk (Cont.)

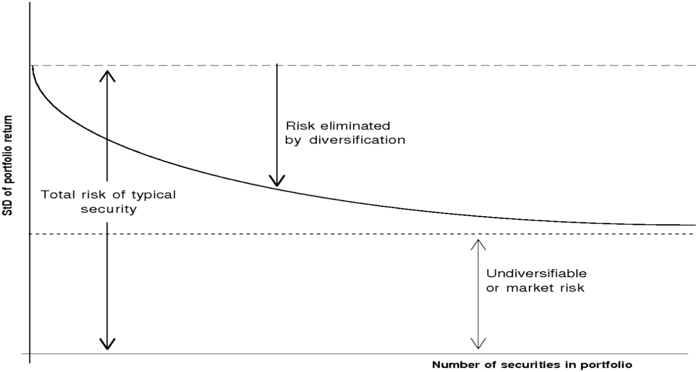


Figure 2: Idiosyncratic versus Systematic Risk

A Simple Example

To analyze the above assertion, consider an equal weighted portfolio with n assets. Then $w_i = 1/n$ for all i. We have

$$\begin{aligned}
\sigma_p^2 = \sum_{i,j} w_i w_j \rho_{ij}\sigma_i\sigma_j &= \sum_{i=1}^{n}\left(\frac{1}{n}\right)^2 \sigma_i^2 + \sum_{i\neq j}\left(\frac{1}{n}\right)^2 \rho_{ij}\sigma_i\sigma_j \\
&= \frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\right) + \frac{n^2-n}{n^2}\left(\frac{1}{n^2-n}\sum_{i\neq j}\mathrm{Cov}(r_i, r_j)\right) \\
&= \frac{1}{n}(\text{Average asset variance}) + \left(1-\frac{1}{n}\right)(\text{Average covariance})
\end{aligned}$$

As $n \to \infty$, the first term $\frac{1}{n}$(Average asset variance) disappears, and so we are left with $\sigma_p^2 \approx$ Average covariance. So, the idiosyncratic risk embedded in individual variances disappear, but the systematic risk which is common to all assets persists through the covariance term.

Why Portfolios?

Before moving on to optimal portfolio choice, we summarize previous discussions as to why not pick the "best" asset instead of forming a portfolio:

- We don't know which stock is "best"! Moreover, individual stocks have idiosyncratic noise/variance which makes predicting returns harder.
- Portfolios provide diversification, reducing unnecessary risk.
- By carefully choosing weights, portfolios can customize and manage risk/reward trade-offs

A "good" portfolio is one that optimally trades off between reward and risk. Riskier assets (higher variances) require investors to be compensated for that risk with a higher return, so how do we balance the two?

Investor's Optimization Problem

Tradeoff between Risk and Return

Portfolio Return:

$$E(r_p) = \sum_{i=1}^{n} w_i E(r_i)$$

Portfolio Risk (Variance):

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j$$

Optimization Problem:

- Maximize $E(r_p)$ subject to $\sigma_p \leq \sigma_{\text{target}}$

Key Tradeoff

- **Higher Returns:** Usually involve higher risks due to increased volatility.
- **Lower Risk:** Leads to more stable but potentially lower returns.
- **Efficient Frontier:** Represents portfolios offering the best possible return for a given level of risk.

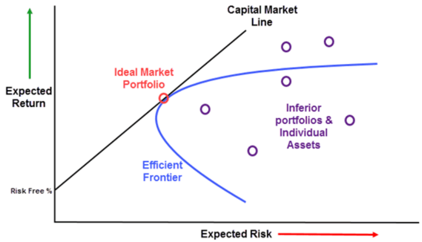
Portfolio Frontier and Efficient Frontier

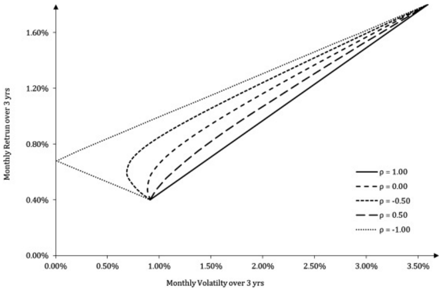
Portfolio frontier:

- Represents all possible risk-return combinations for a set of portfolios constructed from given assets

Efficient frontier:

- A subset of portfolio frontier that offer the highest expected return for a given level of risk or the lowest risk for a given level of return





Minimum Variance Portfolio

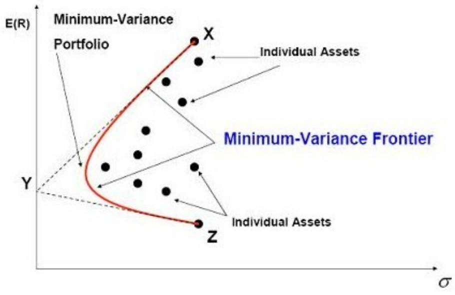
- The Minimum Variance Portfolio minimizes risk (variance) for a given set of assets
- Idea: Diversification reduces risk
- Located on the leftmost point of the Efficient Frontier

Formula:

$$w_{MVP} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}$$

Where:

- Σ: Covariance matrix of asset returns
- $\mathbf{1}$: Vector of ones



Capital Market Line (CML) w/ risk-free asset

Concept

- The Capital Market Line represents portfolios combining the risk-free asset and the market portfolio
- It is a straight line tangent to the Efficient Frontier, showing the highest Sharpe ratio
- Investors can leverage or lend at the risk-free rate to adjust risk-return preferences

Formula

$$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m} \cdot \sigma_p$$

Where:

- r_f: Risk-free rate
- $E(R_m)$: Expected return of the market portfolio
- σ_m: Standard deviation of the market portfolio
- σ_p: Portfolio risk

Performance Evaluation

- **Sharpe Ratio**: Measures the excess return per unit of total risk:

$$\frac{E[r_p] - r_f}{\sigma_p}$$

- **Sortino Ratio**: Measures the excess return per unit of downside risk:

$$\frac{E[r_p] - r_f}{\sigma_d}$$

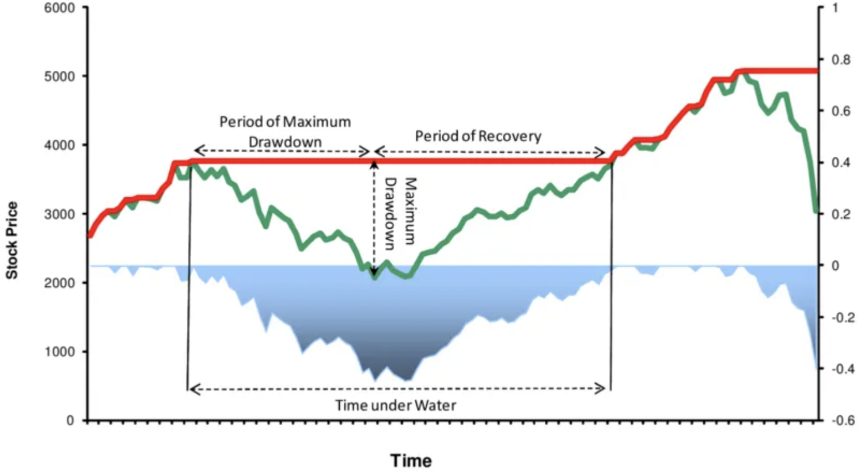
- **Volatility**: Measures the total risk of a portfolio:

$$\hat{\sigma}_p = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(r_i - \bar{r})^2}$$

- **Max Drawdown**: Measures the largest peak-to-trough decline in portfolio value:

$$\frac{\text{Trough Value} - \text{Peak Value}}{\text{Peak Value}}$$

Max Drawdown



More Portfolio Statistics

- The **beta** of a portfolio measures its sensitivity to market movements. Formally,

$$\beta = \frac{\mathrm{Cov}(r_p, r_m)}{\mathrm{Var}(r_m)}$$

 where r_m is the return on the market portfolio (e.g S&P 500). It can be calculated by a linear regression of historical portfolio returns $r_{p,t}$ on historical market returns $r_{m,t}$.

- The **alpha** of a portfolio is its **excess** return above a benchmark. Typically, the benchmark is market return, in which case *Jensen's* alpha is calculated as

$$\alpha = (r_p - r_f) - \beta(r_m - r_f)$$

 where β is the portfolio beta and r_f is the risk free rate. The **information ratio** of a portfolio/strategy is the Sharpe ratio of its alpha. Alpha is often referred to as **active return** since this component of return is **uncorrelated** with the benchmark.

What Are Factor Models?

Definition

- A statistical model that uses explanatory variables to assess the risk and return of financial assets
- Recognize that market risk is not the only determinant of returns

Formula

$$E[r_i] = r_f + \beta_{i,1}F_1 + \beta_{i,2}F_2 + \cdots + \beta_{i,k}F_k$$

- $E[r_i]$: Expected return of asset i
- r_f: Risk-free rate
- $\beta_{i,k}$: Sensitivity of asset i to factor k (Factor loading)
- F_k: Risk premium associated with factor k (Factor return)

Motivation

- Real-world returns are influenced by many variables. Factor models reduce the dimensionality by summarizing risks into key factors
- Helps investors and portfolio managers understand what drives returns and build diversified portfolios

Snippets of a Conservation with Giuseppe (Gappy) Paleologo Explaining Factors

- "The ideas behind factor investing are actually very, very simple. The first assumption, the first philosophical assumption underlying factor investing, is that the world is sparse. Okay, so the fact that the world is sparse means that there are few causes or few drivers behind any major event."
- "The world is sparse because the world is heavy-tailed. [This] means that a few terms in everything that you know adds up to something. The few terms, the bigger terms, dominate the sum. This is really the definition of a tail—that, you know, especially when you have large, large values of a sum of something, it's really only a few terms, or even just one term that dominates the whole sum."

Snippets (Cont.)

- "Factor modeling is a way to reduce the complexity of a large number of investable assets into the sum of two things: a very small set of drivers—things that affect everything a little. There are like 3, 4, 5, 10 things that affect 50,000 assets in the world and then you have a little bit of noise that you cannot describe by this very, very synthetic description of the world. And this little noise is what we call idiosyncratic return in the language of factor models."

Capital Asset Pricing Model (CAPM)

Definition

- Explains the relationship between the expected return of an asset and its risk, measured by its sensitivity to market movements

Formula

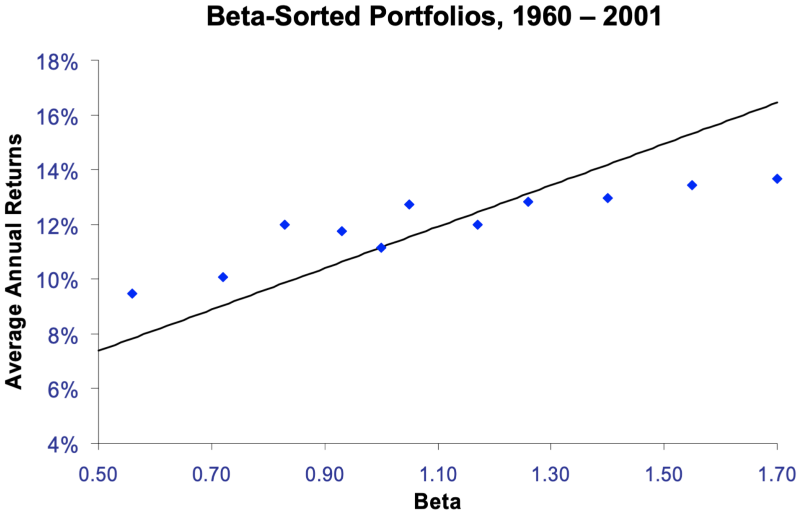
$$E[r_i] = r_f + \beta_i \big(E[r_m] - r_f\big)$$

- $E[r_i]$: Expected return of asset i
- r_f: Risk-free rate
- β_i: Sensitivity of the asset's return to market return
- $E[r_m]$: Expected market return
- $E[r_m] - r_f$: Market risk premium

Motivation

- Dimensionality Reduction: CAPM explains returns using one factor: market risk
- Real-World Insights: evaluate whether an asset is over- or under-priced based on its risk-adjusted returns

Does the CAPM Work?



Fama-French 3-Factor Model

Expands CAPM by incorporating two additional factors

- Size (SMB): Small Minus Big, capturing the size premium
- Value (HML): High Minus Low, capturing the value premium

Formula

$$E[r_i] = r_f + \beta_i \left(E[r_m] - r_f \right) + \mathrm{SMB}_i \cdot \mathrm{SMB} + \mathrm{HML}_i \cdot \mathrm{HML}$$

- $E[r_i]$: Expected return of asset i.
- r_f: Risk-free rate.
- $E[r_m]$: Expected return of the market portfolio.
- β_i: Sensitivity of asset i to the market risk premium.
- SMB_i: Sensitivity of asset i to the size premium (Small Minus Big).
- HML_i: Sensitivity of asset i to the value premium (High Minus Low).
- SMB: The return difference between small-cap and large-cap stocks.
- HML: The return difference between high book-to-market and low book-to-market stocks.