Applied Computational Finance

Lesson 3 - Finite Difference Methods for Heston Model

Giovanni Della Lunga

Laurea Magistrale in Quantitative Finance

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Stochastic Volatility

- As you alredy know, one of the limitations of using the famous Black-Scholes model is the assumption of a constant volatility;
- A major modeling step away from the assumption of constant volatility in asset pricing, was made by modeling the volatility/variance as a diffusion process;
- The resulting models are the stochastic volatility (SV) models;
- The idea to model volatility as a random variable is confirmed by practical financial data which indicates the variable and unpredictable nature of the stock price's volatility;

Stochastic Volatility

- The most significant argument to consider the volatility to be stochastic is the implied volatility smile/skew, which is present in the financial market data, and can be accurately recovered by SV models, especially for options with a medium to long time to the maturity date T.
- With an additional stochastic process, which is correlated to the asset price process S_t, we deal with a system of SDEs, for which option valuation is more expensive than for a scalar asset price process;
- The most popular SV model is the **Heston Model**

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The Heston Model

- The Heston model assumes that the underlying stock price, S_t, follows a Black-Scholes–type stochastic process, but with a stochastic variance v_t that follows a Cox, Ingersoll, and Ross (1985) process.
- Hence, the Heston model is represented by the bivariate system of stochastic differential equations (SDEs) (where $\mathbb{E}^P[dW_{1,t}, dW_{2,t}] = \rho\, dt$):

$$\boxed{\begin{aligned} dS_t &= \mu S_t\, dt + \sqrt{v_t} S_t\, dW_{1,t} \\ dv_t &= \kappa(\theta - v_t)\, dt + \sigma \sqrt{v_t}\, dW_{2,t} \end{aligned}}$$

- We will sometimes drop the time index and write $S = S_t$, $v = v_t$, $W_1 = W_{1,t}$ and $W_2 = W_{2,t}$ for notational convenience;

- The parameters of the model are:

  μ the drift of the process for the stock;
  $\kappa > 0$ the mean reversion speed for the variance;
  $\theta > 0$ the mean reversion level for the variance;
  $\sigma > 0$ the volatility of the variance;
  $v_0 > 0$ the initial (time zero) level of the variance;
  $\rho \in [-1, 1]$ the correlation between the two Brownian motions W_1 and W_2;

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The Heston PDE

- In this section, we explain how to derive the PDE for the Heston model;
- This derivation is a special case of a PDE for general stochastic volatility models, described in books by Gatheral (2006), Lewis (2000), Musiela and Rutkowski (2011), Joshi (2008), and others.
- The argument is similar to the hedging argument that uses a single derivative to derive the Black-Scholes PDE;
- In the Black-Scholes model, a portfolio is formed with the underlying stock, plus a single derivative which is used to hedge the stock and render the portfolio riskless.

The Heston PDE

- In the Heston model, however, an additional derivative is required in the portfolio, to hedge the volatility;
- Hence, we form a portfolio consisting of one option $V = V(S, v, t)$, Δ units of the stock, and ϕ units of another option $U(S, v, t)$ for the volatility hedge.
- The portfolio has value

$$\Pi = V + \Delta s + \phi U$$

and its variation is

$$d\Pi = dV + \Delta dS + \phi dU$$

The Heston PDE

- The strategy is similar to that for the Black-Scholes case. We apply Ito's lemma to obtain the processes for U and V, which allows us to find the process for Π.
- We then find the values of Δ and ϕ that makes the portfolio riskless, and we use the result to derive the Heston PDE.
- Consider a generic function of two stochastic variables and time, $f = f(S, v, t)$, we want to Taylor expand in the first order in dt:

$$
\begin{aligned}
df = & \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial v} dv + \frac{\partial f}{\partial t} dt \\
& + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial v^2} (dv)^2 + \frac{\partial^2 f}{\partial S \partial v} dS \, dv
\end{aligned}
$$

The Heston PDE

Apply Ito's calculus means

$$(dS)^2 = (rS\,dt + \sqrt{v}S\,dW_1)^2 \sim vS^2\,(dW_1)^2 \sim vS^2\,dt$$

$$(dv)^2 = [\kappa(\theta - v)\,dt + \sigma\sqrt{v}\,dW_2]^2 \sim \sigma^2 v(dW_2)^2 \sim \sigma^2 v\,dt$$

$$\begin{aligned} dS\,dv &= (rS\,dt + \sqrt{v}S\,dW_1)[\kappa(\theta - v)\,dt + \sigma\sqrt{v}\,dW_2] \\ &\sim \sigma S v\,dW_1\,dW_2 = \sigma S v \rho \end{aligned}$$

Where the last passage must be interpreted in a probabilistic way remembering that

$$\mathbb{E}[dW_1\,dW_2] = \rho$$

The Heston PDE

The result is that dV follows the process

$$dV = \frac{\partial V}{\partial S}\, dS + \frac{\partial V}{\partial v}\, dv + \frac{\partial V}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} v S^2\, dt$$
$$+ \frac{1}{2}\frac{\partial^2 V}{\partial v^2}\sigma^2 v\, dt + \frac{\partial^2 V}{\partial S \partial v}\sigma S v \rho\, dt$$

Applying Ito's lemma to the second derivative of $U(S, v, t)$, produces an expression identical but in terms of U

$$dU = \frac{\partial U}{\partial S}\, dS + \frac{\partial U}{\partial v}\, dv + \frac{\partial U}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 U}{\partial S^2} v S^2\, dt$$
$$+ \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\sigma^2 v\, dt + \frac{\partial^2 U}{\partial S \partial v}\sigma S v \rho\, dt$$

The Heston PDE

Substituting these two expressions the change in portfolio value can be written as

$$
\begin{aligned}
d\Pi &= dV + \phi dU + \Delta dS \\
&= \frac{\partial V}{\partial S}\, dS + \frac{\partial V}{\partial v}\, dv \\
&+ \frac{\partial V}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} v S^2\, dt + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}\sigma^2 v\, dt + \frac{\partial^2 V}{\partial S \partial v}\sigma S v \rho\, dt \\
&+ \phi \left[\frac{\partial U}{\partial S}\, dS + \frac{\partial U}{\partial v}\, dv\right] \\
&+ \phi \left[+\frac{\partial U}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 U}{\partial S^2} v S^2\, dt + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\sigma^2 v\, dt + \frac{\partial^2 U}{\partial S \partial v}\sigma S v \rho\, dt\right] \\
&+ \Delta dS
\end{aligned}
$$

A simple rearrangement of terms gives us

$$\begin{aligned}
d\Pi &= dV + \phi dU + \Delta dS \\
&= \left[\frac{\partial V}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} v S^2\, dt + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}\sigma^2 v\, dt + \frac{\partial^2 V}{\partial S \partial v}\sigma S v \rho\, dt \right] \\
&+ \phi \left[\frac{\partial U}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 U}{\partial S^2} v S^2\, dt + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\sigma^2 v\, dt + \frac{\partial^2 U}{\partial S \partial v}\sigma S v \rho\, dt \right] \\
&+ \left[\frac{\partial V}{\partial S} + \phi \frac{\partial U}{\partial S} + \Delta \right] dS + \left[\frac{\partial V}{\partial v} + \phi \frac{\partial U}{\partial v} \right] dv
\end{aligned}$$

The Heston PDE

In order for the portfolio to be hedged against movements in both the stock and volatility, the two terms multiplying dS and dv must be zero:

$$\left[\frac{\partial V}{\partial S} + \phi\frac{\partial U}{\partial S} + \Delta\right] = 0$$
$$\left[\frac{\partial V}{\partial v} + \phi\frac{\partial U}{\partial v}\right] = 0$$

This implies that the hedge parameters must be

$$\phi = -\frac{\partial V}{\partial v}\bigg/\frac{\partial U}{\partial v}$$
$$\Delta = -\left[\frac{\partial V}{\partial S} + \phi\frac{\partial U}{\partial S}\right]$$

The Heston PDE

Substitute these values of Δ and ϕ gives us

$$
\begin{aligned}
d\Pi &= dV + \phi dU + \Delta dS \\
&= \left[\frac{\partial V}{\partial t}\,dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} vS^2\,dt + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}\sigma^2 v\,dt + \frac{\partial^2 V}{\partial S \partial v}\sigma S v \rho\,dt\right] \\
&+ \phi\left[\frac{\partial U}{\partial t}\,dt + \frac{1}{2}\frac{\partial^2 U}{\partial S^2} vS^2\,dt + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\sigma^2 v\,dt + \frac{\partial^2 U}{\partial S \partial v}\sigma S v \rho\,dt\right]
\end{aligned}
\tag{1}
$$

The condition that the portfolio earn the risk-free rate, r, implies that the change in portfolio value is:

$$
d\Pi = r\left(V + \Delta S + \phi U\right)\,dt \tag{2}
$$

The Heston PDE

Now equate Equation 1 with 2, substitute for ϕ and Δ, drop the dt term and re-arrange. This yields

$$\frac{\left[\dfrac{\partial V}{\partial t}+\dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}vS^2+\dfrac{1}{2}\dfrac{\partial^2 V}{\partial v^2}\sigma^2 v+\dfrac{\partial^2 V}{\partial S\partial v}\sigma S v\rho\right]-rV+rS\dfrac{\partial V}{\partial S}}{\dfrac{\partial V}{\partial v}}$$

$$=\frac{\left[\dfrac{\partial U}{\partial t}+\dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}vS^2+\dfrac{1}{2}\dfrac{\partial^2 U}{\partial v^2}\sigma^2 v+\dfrac{\partial^2 U}{\partial S\partial v}\sigma S v\rho\right]-rU+rS\dfrac{\partial U}{\partial S}}{\dfrac{\partial U}{\partial v}}$$

The Heston PDE

The left-hand side of the last equation is a function of V only, and the right-hand side is a function of U only. This implies that both sides can be written as a function $f(S, v, t)$:

$$\frac{\left[\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} v S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}\sigma^2 v + \frac{\partial^2 V}{\partial S \partial v}\sigma S v \rho\right] - rV + rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial v}}$$

$$= -f(S, v, t) \Rightarrow$$

$$\Rightarrow \left[\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} v S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}\sigma^2 v + \frac{\partial^2 V}{\partial S \partial v}\sigma S v \rho\right] - rV$$

$$= -rS\frac{\partial V}{\partial S} - f(S, v, t)\frac{\partial V}{\partial v}$$

The Heston PDE

Comparing

$$\left[\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} v S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}\sigma^2 v + \frac{\partial^2 V}{\partial S \partial v}\sigma S v \rho\right] - rV$$
$$= -rS\frac{\partial V}{\partial S} - f(S, v, t)\frac{\partial V}{\partial v}$$

with

$$dS = rS\,dt + \sqrt{v}S\,dW_1$$
$$dv = \kappa(\theta - v)\,dt + \sigma\sqrt{v}\,dW_2$$

we see that the function $f(S, v, t)$ can be chosen as the drift of the variance process;

The Heston PDE

Since $r = \mu - \lambda\sigma$, where λ is the *market price of risk*, we can justify a form similar for f

$$\boxed{f(S, v, t) = \kappa(\theta - v) - \lambda(S, v, t)}$$

It's possible to verify that the factor λ can be always re-absorbed by a change of measure, this is the reason why in the following we assume simply $\lambda = 0$.

The Heston PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma v S \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2} \sigma^2 v \frac{\partial^2 V}{\partial v^2} - rV + rS \frac{\partial V}{\partial S} + \kappa(\theta - v) \frac{\partial V}{\partial v} = 0$$

If we use t to represents maturity, the sign of the derivative $\partial V / \partial t$ is the opposite of what it would be if t represented time:

$$\frac{\partial V}{\partial t} = \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma v S \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2} \sigma^2 v \frac{\partial^2 V}{\partial v^2} - rV + rS \frac{\partial V}{\partial S} + \kappa(\theta - v) \frac{\partial V}{\partial v}$$

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Semi-Analytical Solution

- The time-t price of a European call on a non-dividend-paying stock with spot price S_t, when the strike is K and the time to maturity is $\tau = T - t$ in the Heston Model can be written as:

$$C(K) = S_t P_1 - K e^{-r\tau} P_2 \tag{3}$$

where P_1 and P_2 are two probabilities under two different measures:

$$C(K) = S_t \mathbb{Q}^S(S_T > K) - K e^{-r\tau} \mathbb{Q}(S_T > K) \tag{4}$$

- The measure $\mathbb{Q}$ uses the bond B_t as the numeraire, while the measure $\mathbb{Q}^S$ uses the stock price S_t.

Semi-Analytical Solution

- In the Heston model, it can be shown that P_1 and P_2 can be written as:

$$P_j = \mathsf{Pr}_j\left(\ln S_T - \ln K\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathsf{Re}\left[\frac{e^{-i\phi \ln K} f_j(\phi; x, v)}{i\phi}\right] d\phi$$

- Heston postulates that these characteristic functions are of the log linear form

$$f_j(\phi; x_t, v_t) = \exp\left(C_j(\tau, \phi) + D_j(\tau, \phi) v_t + i\phi x_t\right)$$

where C_j and D_j are constant coefficients and $\tau = T - t$ is the time to maturity.

Semi-Analytical Solution

- The coefficients can be shown to be

$$D_j(\tau,\phi) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right)$$

and

$$C_j(\tau,\phi) = r i\phi\tau + \frac{a}{\sigma^2}\left[(b_j - \rho\sigma i\phi + d_j)\tau - 2\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right]$$

where $a = \kappa\theta$ and

$$d_j = \sqrt{(\rho\sigma i\phi - b_j)^2 - \sigma^2(2u_j i\phi - \phi^2)}$$

$$g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j}$$

Semi-Analytical Solution

... and with

$$u_1 = \frac{1}{2} \quad u_2 = -\frac{1}{2}$$

$$b_1 = \kappa + \lambda - \rho\sigma \quad b_2 = \kappa + \lambda$$

Let’s code ...

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Explicit Scheme

- Now, having a challenge algorithm, we can start to work on the main problem of this lesson: the solution of the Heston PDE equation in **Two Dimension** with Finite Difference Method.
- We first show how to construct uniform and nonuniform grids for the discretization of the stock price and the volatility, and present formulas for finite difference approximations to the derivatives in the Heston PDE.
- Next, we explain the boundary conditions of the PDE for a European call and then we will implement an Explicit scheme.

Building a Uniform Grid

- We denote the maximum values of S, v, and t as $Smax$, $Vmax$ and $Tmax = \tau$ (the maturity), and the minimum values as $Smin$, $Vmin$, and $Tmin = 0$.
- We denote by $U_{i,j}^n = U(S_i, v_j, t_n)$ the value of an option at time t_n when the stock price is S_i and the volatility is v_j. We use $NS + 1$ points for the stock price, $NV + 1$ points for the volatility, and $NT + 1$ points for the maturity. For convenience, sometimes we write simply $U(S_i, v_j)$ for $U_{i,j}^n$.

Building a Uniform Grid

- Using the minimum values $Smin = Vmin = 0$ a uniform grid for (S, v, t) can be constructed as

$$
\begin{aligned}
S_i &= i \times ds & i &= 0, 1, \ldots, NS \\
v_j &= j \times dv & j &= 0, 1, \ldots, NV \\
t_n &= n \times dt & n &= 0, 1, \ldots, NT
\end{aligned}
$$

Approximation of Derivatives

$$\frac{\partial^2 U}{\partial S^2} \sim \frac{U^n_{i+1,j} - 2U^n_{i,j} + U^n_{i-1,j}}{(ds)^2}$$

$$\frac{\partial^2 U}{\partial S \partial v} \sim \frac{U^n_{i+1,j+1} + U^n_{i-1,j-1} - U^n_{i-1,j+1} - U^n_{i+1,j-1}}{4\, ds\, dv}$$

$$\frac{\partial^2 U}{\partial v^2} \sim \frac{U^n_{i,j+1} - 2U^n_{i,j} + U^n_{i,j-1}}{(dv)^2}$$

$$\frac{\partial U}{\partial S} \sim \frac{U^n_{i+1,j} - U^n_{i-1,j}}{2\, ds}$$

$$\frac{\partial U}{\partial v} \sim \frac{U^n_{i,j+1} - U^n_{i,j-1}}{2\, dv}$$

Explicit Scheme

The explicit scheme produces an expression for the PDE that is very simple when the grids are uniform. Lets' re-write the PDE in the following form:

$$U_{i,j}^{n+1} = U_{i,j}^{n} + dt\left[\frac{1}{2}vS^2\frac{\partial^2}{\partial S^2} + \rho\sigma vS\frac{\partial^2}{\partial v\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2}{\partial v^2} - r \right.$$
$$\left. + (r-q)S\frac{\partial}{\partial S} + \kappa(\theta - v)\frac{\partial}{\partial v}\right]U_{i,j}^{n}$$

Explicit Scheme

Now we substitute the approximations to the derivatives under a uniform grid to obtain

$$
\begin{aligned}
U_{i,j}^{n+1} = U_{i,j}^{n} + dt \Bigg[& \frac{1}{2} v_j S_i^2 \frac{U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n}}{(ds)^2} \\
& + \rho \sigma v_j S_i \frac{U_{i+1,j+1}^{n} + U_{i-1,j-1}^{n} - U_{i-1,j+1}^{n} - U_{i+1,j-1}^{n}}{4\, ds\, dv} \\
& + \frac{1}{2} \sigma^2 v_j \frac{U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}}{(dv)^2} \\
& + (r - q) S_i \frac{U_{i+1,j}^{n} - U_{i-1,j}^{n}}{2\, ds} \\
& + \kappa(\theta - v) \frac{U_{i,j+1}^{n} - U_{i,j-1}^{n}}{2\, dv} \Bigg] - r\, dt\, U_{i,j}^{n}
\end{aligned}
$$

Explicit Scheme

$$
\begin{aligned}
U_{i,j}^{n+1} = & \left[1 - dt\left(v_j \frac{S_i^2}{(ds)^2} + \sigma^2 \frac{v_j}{(dv)^2} + r\right)\right] U_{i,j}^n \\
& + dt \left[\frac{v_j S_i^2}{2(ds)^2} - \frac{(r-q)S_i}{2\,ds}\right] U_{i-1,j}^n \\
& + dt \left[\frac{v_j S_i^2}{2(ds)^2} + \frac{(r-q)S_i}{2\,ds}\right] U_{i+1,j}^n \\
& + dt \left[\frac{\sigma^2 v_j}{2(dv)^2} - \frac{\kappa(\theta - v_j)}{2\,dv}\right] U_{i,j-1}^n \\
& + dt \left[\frac{\sigma^2 v_j}{2(dv)^2} + \frac{\kappa(\theta - v_j)}{2\,dv}\right] U_{i,j+1}^n \\
& + dt \frac{\rho \sigma v_j S_i}{4\,ds\,dv} \left(U_{i+1,j+1}^n + U_{i-1,j-1}^n - U_{i-1,j+1}^n - U_{i+1,j-1}^n\right)
\end{aligned}
$$

Explicit Scheme

$$C_1 = \left[1 - dt\left(j\,dv\frac{i^2\,(ds)^2}{(ds)^2} + \sigma^2\frac{j\,dv}{(dv)^2} + r\right)\right] = \left[1 - dt\left(i^2 j\,dv + \frac{\sigma^2 j}{dv} + r\right)\right]$$

$$C_2 = dt\left[\frac{j\,dv\,i^2\,(ds)^2}{2(ds)^2} - \frac{(r-q)i\,ds}{2\,ds}\right] = \left[\frac{i\,dt}{2}(ij\,dv - r + q)\right]$$

$$C_3 = dt\left[\frac{j\,dv\,i^2\,(ds)^2}{2(ds)^2} + \frac{(r-q)i\,ds}{2\,ds}\right] = \left[\frac{i\,dt}{2}(ij\,dv + r - q)\right]$$

$$C_4 = dt\left[\frac{\sigma^2 j\,dv}{2(dv)^2} - \frac{\kappa(\theta - j\,dv)}{2\,dv}\right] = \left[\frac{dt}{2\,dv}\left(\sigma^2 j - \kappa(\theta - j\,dv)\right)\right]$$

$$C_5 = dt\left[\frac{\sigma^2 j\,dv}{2(dv)^2} + \frac{\kappa(\theta - j\,dv)}{2\,dv}\right] = \left[\frac{dt}{2\,dv}\left(\sigma^2 j + \kappa(\theta - j\,dv)\right)\right]$$

$$C_6 = dt\,\frac{\rho\sigma j\,dvi\,ds}{4\,ds\,dv} = \frac{\rho\sigma j\,i\,dt}{4}$$

Explicit Scheme

With these positions we obtain

$$
\begin{aligned}
U_{i,j}^{n+1} &= C_1 U_{i,j}^{n} + C_2 U_{i-1,j}^{n} + C_3 U_{i+1,j}^{n} + C_4 U_{i,j-1}^{n} \\
&\quad + C_5 U_{i,j+1}^{n} + C_6 \left(U_{i+1,j+1}^{n} + U_{i-1,j-1}^{n} - U_{i-1,j+1}^{n} - U_{i+1,j-1}^{n} \right)
\end{aligned}
$$

Boundary Conditions

Boundary Condition at Maturity.
At maturity $(t = 0)$ the value of the call is its intrinsic value (the payoff)

$$U(S_i, v_j, 0) = max(0, S_i - K)$$

This implies that the boundary condition for $t = 0$ is

$$U^0_{i,j} = (S_i - K)^+ \quad \text{for } i = 0, 1, \ldots, N_S \text{ and } j = 0, 1, \ldots, N_V$$

Boundary Conditions

Boundary Contition for $S = S_{min}$
When $S = Smin = 0$, the call is worthless. Hence, we have $U(0, v_j, t_n) = 0$ and the boundary condition is

$$U^n_{0,j} = 0 \quad \text{for } n = 0, \ldots, N_T \text{ and } j = 0, 1, \ldots, N_V$$

Boundary Conditions

Boundary Contition for $S = S_{max}$
As S becomes large, delta for the call option approaches one. Hence, for $S = Smax$, we have

$$\frac{\partial U}{\partial S}(Smax, v_j, t_n) = 1$$

The boundary condition for $Smax$ is, therefore,

$$U^n_{N_S,j} = Smax - Ke^{-rT} \qquad \text{for } n = 0, \ldots, N_T \text{ and } j = 0, ..., N_V$$

Boundary Conditions

Boundary Contition for $v = v_{max}$

When v becomes large, we have $U(S_i, vmax, t_n) = S_i$. The boundary condition for $vmax$ is, therefore:

$$U^n_{i,N_V} = S_i \qquad \text{for } n = 0, \dots, N_T, \text{ and } i = 0, \dots, N_S$$

Since at $vmax$ we have $U = S_i$ and, therefore, $\partial U / \partial S = 1$.

Boundary Conditions

Boundary Contition for $v = v_{min}$.
When $v = v_{min} = 0$, the boundary condition is a little more complicated. When $v = 0$ the PDE becomes

$$\frac{\partial U}{\partial t} = -rU + (r - q)S\frac{\partial U}{\partial S} + \kappa\theta\frac{\partial U}{\partial v}$$

We can use central difference for $\partial U / \partial S$

$$\frac{\partial U}{\partial S}(i, 0, t_n) \simeq \frac{U(t, i+1, 0) - U(t, i-1, 0)}{dS}$$

and forward difference for $\partial U / \partial v$

$$\frac{\partial U}{\partial v}(i, 0, t_n) \simeq \frac{U(t, i, 1) - U(t, i, 0)}{dv}$$

Boundary Conditions

Boundary Contition for $v = v_{min}$.

$$\frac{U(t+1,i,0)-U(t,i,0)}{dt} = -rU(t,i,0) + (r-q)S\frac{U(t,i+1,0)-U(t,i-1,0)}{2dS} + \kappa\theta\frac{U(t,i,1)-U(t,i,0)}{dv}$$

$$\begin{aligned}
U(t+1,i,0) &= U(t,i,0) - rU(t,i,0)dt + \frac{1}{2}(r-q)S\frac{dt}{dS}\left(U(t,i+1,0)-U(t,i-1,0)\right) \\
&+ \kappa\theta\frac{dt}{dv}\left(U(t,i,1)-U(t,i,0)\right) \\
&= U(t,i,0)\left(1 - rdt - \kappa\theta\frac{dt}{dv}\right) \\
&+ \frac{1}{2}(r-q)S\frac{dt}{dS}\left(U(t,i+1,0)-U(t,i-1,0)\right) \\
&+ \kappa\theta\frac{dt}{dv}U(t,i,1)
\end{aligned}$$

Let’s code ...

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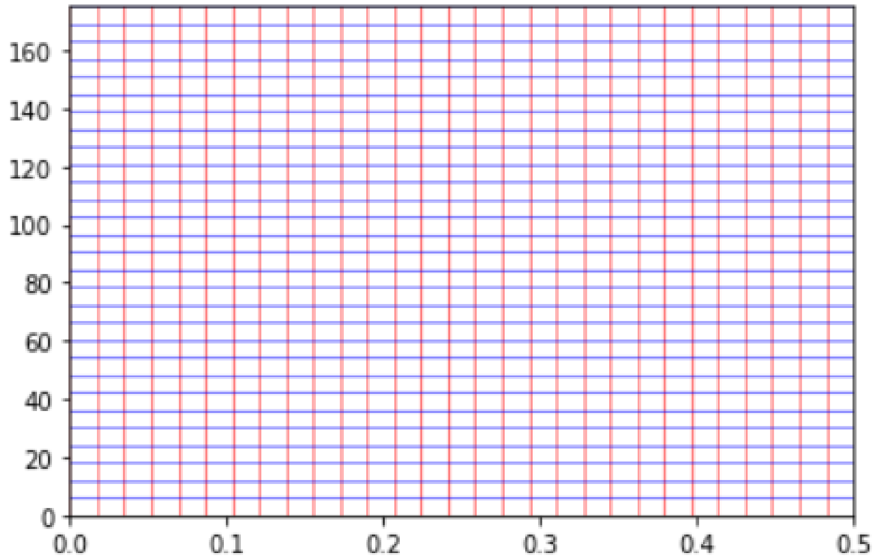
Non Uniform Grid

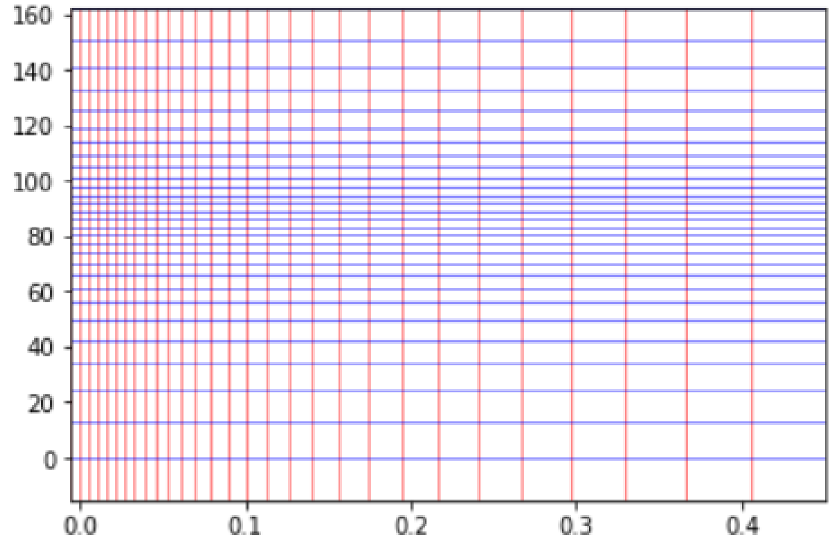
- Uniform grids are those that have equally spaced increments for the two state variables.



These grids have two advantages: first, they are easy to construct, and second, since the increments are equal, the finite difference approximations to the derivatives in the PDE take on a simple form.

Non Uniform Grid

- Non-uniform grids are more complicated to construct, and the finite difference approximations to the derivatives are more complicated.
- These grids, however, can be made finer around certain points, in particular, around the region $(S, v) = (K, 0)$ where option prices are often required.



Hence, non-uniform grids are often preferable since they produce more accurate prices with fewer grid points, and consequently, with less computation time.

Non Uniform Grid

In'T Hout and Foulon (2010) describe a non-uniform grid that is finer around the strike price K and around the spot volatility $v_0 = 0$. Their grid of size $N_S + 1$ for the stock price is

$$S_i = K + c \sinh(\xi_i), \quad i = 0, \ldots, N_S$$

where we select $c = K/5$ and where $\xi_i = \sinh^{-1}(-K/c) + i\Delta\xi$ with

$$\Delta\xi = \frac{1}{N_S - 1}\left[\sinh^{-1}\left(\frac{S_{max} - K}{c}\right) - \sinh^{-1}\left(\frac{-K}{c}\right)\right]$$

The grid of size $N_v + 1$ for the volatility is

$$v_j = d \sinh\left(j\Delta\eta\right), \quad j = 0, \ldots, N_V$$

with $\Delta\eta = \sinh^{-1}(V_{max}/d)/(N_V - 1)$. The BuildGrid function is used throughout to construct uniform and nonuniform grids.

Non Uniform Grid

Derivatives Approximations for Non Uniform Grid

$$\frac{\partial U}{\partial S}(S_i, v_j) = \left(\frac{U^n_{i+1,j} - U^n_{i-1,j}}{S_{i+1} - S_{i-1}} \right)$$

$$\frac{\partial U}{\partial v}(S_i, v_j) = \left(\frac{U^n_{i,j+1} - U^n_{i,j-1}}{v_{j+1} - v_{j-1}} \right)$$

$$\frac{\partial^2 U}{\partial S^2}(S_i, v_j) = \left(\frac{U^n_{i+1,j} - U^n_{i,j}}{S_{i+1} - S_i} - \frac{U^n_{i,j} - U^n_{i-1,j}}{S_i - S_{i-1}} \right) \frac{1}{S_{i+1} - S_i}$$

Non Uniform Grid

Derivatives Approximations for Non Uniform Grid

$$\frac{\partial^2 U}{\partial v^2}(S_i, v_j) = \left(\frac{U^n_{i,j+1} - U^n_{i,j}}{v_{j+1} - v_j} - \frac{U^n_{i,j} - U^n_{i,j-1}}{v_j - v_{j-1}} \right) \frac{1}{v_{j+1} - v_j}$$

$$\frac{\partial^2 U}{\partial S \partial v}(S_i, v_j) = \left(\frac{U^n_{i+1,j+1} - U^n_{i-1,j+1} - U^n_{i+1,j-1} + U^n_{i-1,j-1}}{(S_{i+1} - S_{i-1})(v_{j+1} - v_{j-1})} \right)$$

Let's code ...