Applied Computational Finance

Lesson 1 - Pricing with Trees

Giovanni Della Lunga

Laurea Magistrale in Quantitative Finance

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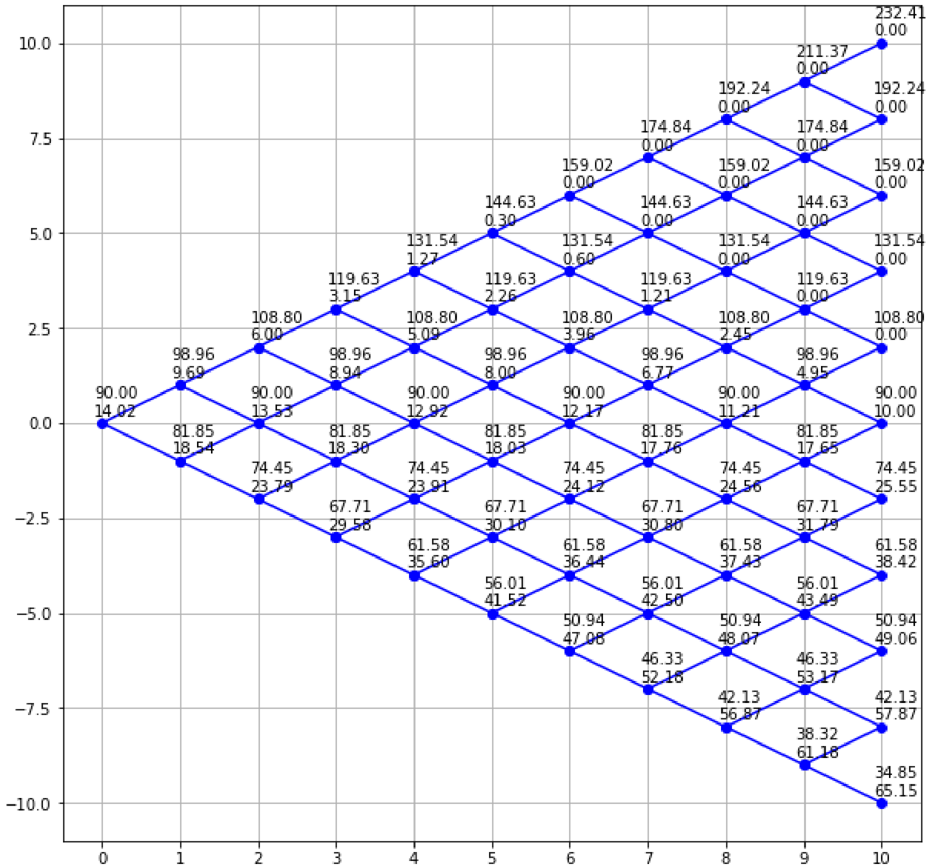
Outline

Introduction

- The binomial pricing model traces the evolution of the option's key underlying variables in discrete-time.
- This is done by means of a binomial lattice (Tree), for a number of time steps between the valuation and expiration dates.
- Each node in the lattice represents a possible price of the underlying at a given point in time.
- Valuation is performed iteratively, starting at each of the final nodes (those that may be reached at the time of expiration), and then working backwards through the tree towards the first node (valuation date).
- The value computed at each stage is the value of the option at that point in time.

Introduction



Option valuation using this method is a three-step process:

- Price tree generation,
- Calculation of option value at each final node,
- Sequential calculation of the option value at each preceding node.

Create the Binomial Price Tree

- The tree of prices is produced by working forward from valuation date to expiration.
- At each step, it is assumed that the underlying instrument will move up (with probability p) or down (with probability $q = 1 - p$) by a specific factor u or d per step of the tree (where, by definition, $u \geq 1$ and $0 < d \leq 1$)
- So, if S is the current price, then in the next period the price will either be $S_{up} = S \cdot u$ or $S_{down} = S \cdot d$.
- The up and down factors are calculated using the underlying volatility, σ , and the time duration of a step, Δt, measured in years (using the day count convention of the underlying instrument).

Create the Binomial Price Tree

- With respect to the definition of p, u and d some important assumptions about the behavior of the stochastic process of the underlying stock have to be done. In the following we'll follow the framework set up by Cox, Ross and Rubinstein.
- Let's assume that the stochastic process is continuous as $n \to \infty$. The parameters p, u and d must be chosen in a way to determine the right values of the stock expected return and variance at the end of each interval Δt.

Create the Binomial Price Tree

- For each step, the expected future stock price is

$$E_\pi[S_{t+\Delta t}] = pS_t u + (1-p)S_t d \Rightarrow \frac{E_\pi[S_{t+\Delta t}]}{S_t} = pu + (1-p)d \quad (1)$$

and, conditional to S_t

$$E_\pi\left[\frac{S_{t+\Delta t}}{S_t}|S_t\right] = pu + (1-p)d \quad (2)$$

The variance of this return is given by

$$\begin{aligned} Var\left[\frac{S_{t+\Delta t}}{S_t}|S_t\right] &= E_\pi\left[\left(\frac{S_{t+\Delta t}}{S_t}\right)^2 |S_t\right] - \left(E_\pi\left[\frac{S_{t+\Delta t}}{S_t}|S_t\right]\right)^2 \\ &= [pu^2 + (1-p)d^2] - [pu + (1-p)d]^2 \end{aligned}$$

Create the Binomial Price Tree

- The Cox, Ross and Rubinstein binomial tree model assume that in the limit $n \to \infty$, the discrete process must converge to a geometric brownian motion for the underlying stock price.
- So the idea is to find values for u and d in order to have

$$Var\ [X_t|S_t] = \sigma^2 \Delta t$$

where $X_t = \frac{S_{t+\Delta t}}{S_t}$

- We add an additional assumption in order to obtain a tree that recombines, so that the effect of a down movement followed by an up movement is the same as the effect of an up movement, followed by a down movement.

$$u = \frac{1}{d} \tag{3}$$

Create the Binomial Price Tree

- With a bit of trivial algebra we find

$$
\begin{aligned}
Var\ [X_t|S_t] &= pu^2 + (1-p)d^2 - p^2u^2 - (1-p)^2d^2 - 2p(1-p)ud \\
&= pu^2(1-p) + p(1-p)d^2 - 2p(1-p)ud \\
&= p(1-p)(u^2 + d^2 - 2ud) \\
&= p(1-p)(u-d)^2
\end{aligned}
\tag{4}
$$

- Now it's time to say something more about the probability p
- For this we will resort to the hypothesis of absence of arbitrage...

Create the Binomial Price Tree

- For our purposes it's enough to consider only one period.
- Let's say that at time $t + \Delta t$ an option on S can assume only two possibile values: either C_u or C_d
- If we have a portfolio with a long position of α shares of the stock and a short position in the option, the value of our portfolio in the two states will be:

$$\alpha \cdot S_t \cdot u - C_u$$

$$\alpha \cdot S_t \cdot d - C_d$$

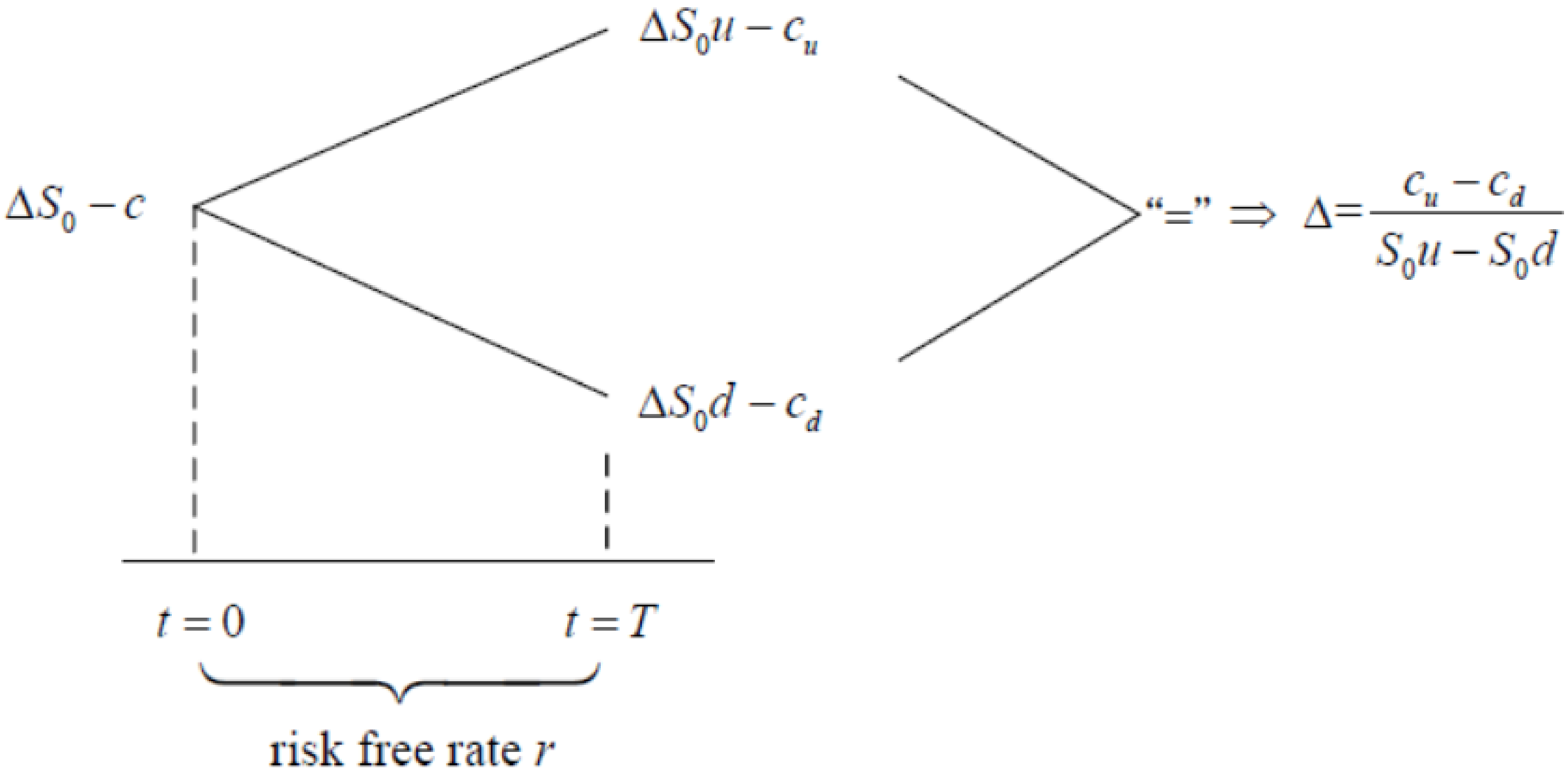
Create the Binomial Price Tree

- It is easy to check that if we choose α as

$$\alpha = \frac{C_u - C_d}{(u-d)S_t} \sim \frac{\partial C}{\partial S} \tag{5}$$

our portfolio will have the same value in both states so it will be a **risk free** portfolio.

Create the Binomial Price Tree



Create the Binomial Price Tree

- To preclude the existence of arbitrage the return of our portfolio must be the risk free rate, so in Δt we have

$$P_{t+\Delta t} = e^{r\Delta t} P_t$$

- If c_t is the option value at time t we can write

$$\frac{P_{t+\Delta t}}{P_t} = \frac{S_t\alpha \cdot u - c_u}{S_t\alpha - c_t} = e^{rT} \tag{6}$$

from this we can obtain the value for c_t:

$$\begin{aligned} c_t &= \frac{S_t \cdot \alpha \cdot e^{r\Delta t} + c_u - u \cdot S_t \cdot \alpha}{e^{r\Delta t}} \\ &= \frac{S_t \cdot \alpha \cdot (e^{r\Delta t} - u) + c_u}{e^{r\Delta t}} \end{aligned}$$

Create the Binomial Price Tree

- Using the previous value for α

$$
\begin{aligned}
c_t &= \frac{\frac{c_u - c_d}{u-d}(e^{r\Delta t} - u) + c_u}{e^{r\Delta t}} \\
&= \frac{c_u e^{r\Delta t} - c_u u - c_d e^{r\Delta t} + c_d u + c_u u - c_u d}{e^{r\Delta t}(u - d} \\
&= e^{-r\Delta t} \frac{c_u(e^{r\Delta t} - d) + c_d(u - e^{r\Delta t})}{(u - d)} \\
&= e^{-r\Delta t} \left[c_u \frac{e^{r\Delta t} - d}{u - d} + c_d \frac{u - e^{r\Delta t}}{u - d} \right] \\
&= e^{-r\Delta t} \left[p \cdot c_u + (1 - p) \cdot c_d \right] \qquad (7)
\end{aligned}
$$

Create the Binomial Price Tree

- In the previous equation

$$p \equiv \frac{e^{rT} - d}{u - d} \qquad 1 - p \equiv \frac{u - e^{rT}}{u - d} \tag{8}$$

if we assume $d \leq r \leq d$, p is always greater than zero and less than one showing the basic properties of a probability measure.

- This is the so called **risk neutral probability**
- Now we can compute the variance

$$Var\ [X_t|S_t] = p(1-p)(u-d)^2 = \sigma^2 \Delta t \tag{9}$$

Create the Binomial Price Tree

Using the value for p, we obtain

$$\begin{aligned} Var\ [X_t|S_t] &= p(1-p)(u-d)^2 \\ &= \frac{e^{r\Delta t}-d}{u-d}\left[\frac{u-e^{r\Delta t}}{u-d}\right](u-d)^2 \\ &= (e^{r\Delta t}-d)(u-e^{r\Delta t}) \\ &= e^{r\Delta t}(u+d)-ud-e^{2r\Delta t} \end{aligned} \tag{10}$$

from the last row, keeping the lower order terms we have

$$\begin{aligned} Var\ [X_t|S_t] &= e^{r\Delta t}(u+d)-ud-e^{2r\Delta t} \\ &\sim (1+r\Delta t)(u+d)-1-1-2r\Delta t \\ &= (1+r\Delta t)(u+d-2) \end{aligned} \tag{11}$$

Create the Binomial Price Tree

If we choose

$$u = e^{\sigma\sqrt{\Delta t}}, \qquad d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}} \tag{12}$$

up to the first order in Δt we have

$$u \sim 1 + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t, \qquad d \sim 1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t \tag{13}$$

Create the Binomial Price Tree

and finally

$$\begin{aligned}
Var\ [X_t|S_t] &= (1+r\Delta t)(u+d-2) \\
&\sim (1+r\Delta t)\Big(1+\sigma\sqrt{\Delta t}+\frac{1}{2}\sigma^2\Delta t+1-\sigma\sqrt{\Delta t}+\frac{1}{2}\sigma^2\Delta t-2\Big) \\
&= (1+r\Delta t)\Big(\sigma^2\Delta t\Big) \\
&\sim \sigma^2\Delta t+O\left((\Delta t)^2\right) \qquad (14)
\end{aligned}$$

Create the Binomial Price Tree

- As we have already said, the CRR method ensures that the tree is recombinant, i.e. if the underlying asset moves up and then down (u, d), the price will be the same as if it had moved down and then up (d, u) — here the two paths merge or recombine.
- This property reduces the number of tree nodes, and thus accelerates the computation of the option price.
- This property also allows that the value of the underlying asset at each node can be calculated directly via formula, the node-value will be:

$$S_n = S_0 \cdot u^{N_u - N_d} = S_0 \cdot u^{N_u} \cdot d^{N_d}$$

Where N_u is the number of up ticks and N_d is the number of down ticks.

Find option value at each final node

At each final node of the tree — i.e. at expiration of the option—the option value is simply its intrinsic, or exercise, value:

$$Max[(S_n - K), 0], \quad \text{for a call option}$$

$$Max[(K - S_n), 0], \quad \text{for a put option}$$

Where K is the strike price and S_n is the spot price of the underlying asset at the $n-th$ period.

Find option value at earlier nodes

- Once the above step is complete, the option value is then found for each node, starting at the penultimate time step, and working back to the first node of the tree (the valuation date) where the calculated result is the value of the option.
- The "binomial value" is found at each node, using the risk neutrality assumption. Under this assumption,the expected value is calculated using the option values from the later two nodes (Option up and Option down) weighted by their respective risk neutral probabilities — p of an up move in the underlying, and $(1-p)$ of a down move.

Find option value at earlier nodes

- The expected value is then discounted at r, the risk free rate corresponding to the life of the option.
- We finally obtain the iteration formula:

$$c_{t-\Delta t,i} = e^{-r\Delta t}\left[p \cdot c_{t,i} + (1-p)\cdot c_{t,i+1}\right] \tag{15}$$

- If exercise is permitted at the node, then the model takes the greater of binomial and exercise value at the node.

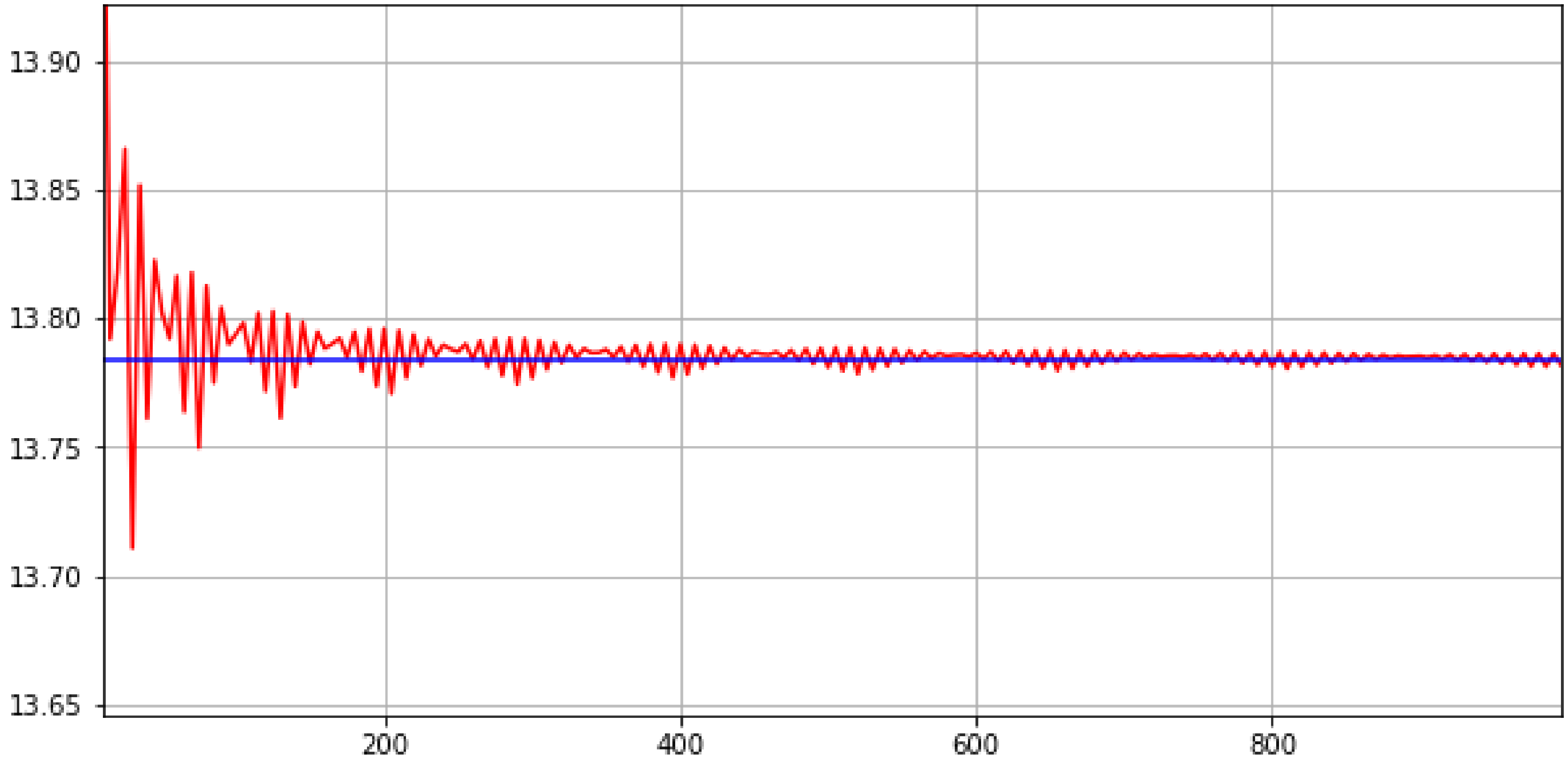
$$c_{t-\Delta t,i} = \max\left[e^{-r\Delta t}\left(p \cdot c_{t,i} + (1-p)\cdot c_{t,i+1}\right), S_t - K\right] \tag{16}$$

for a call option, and

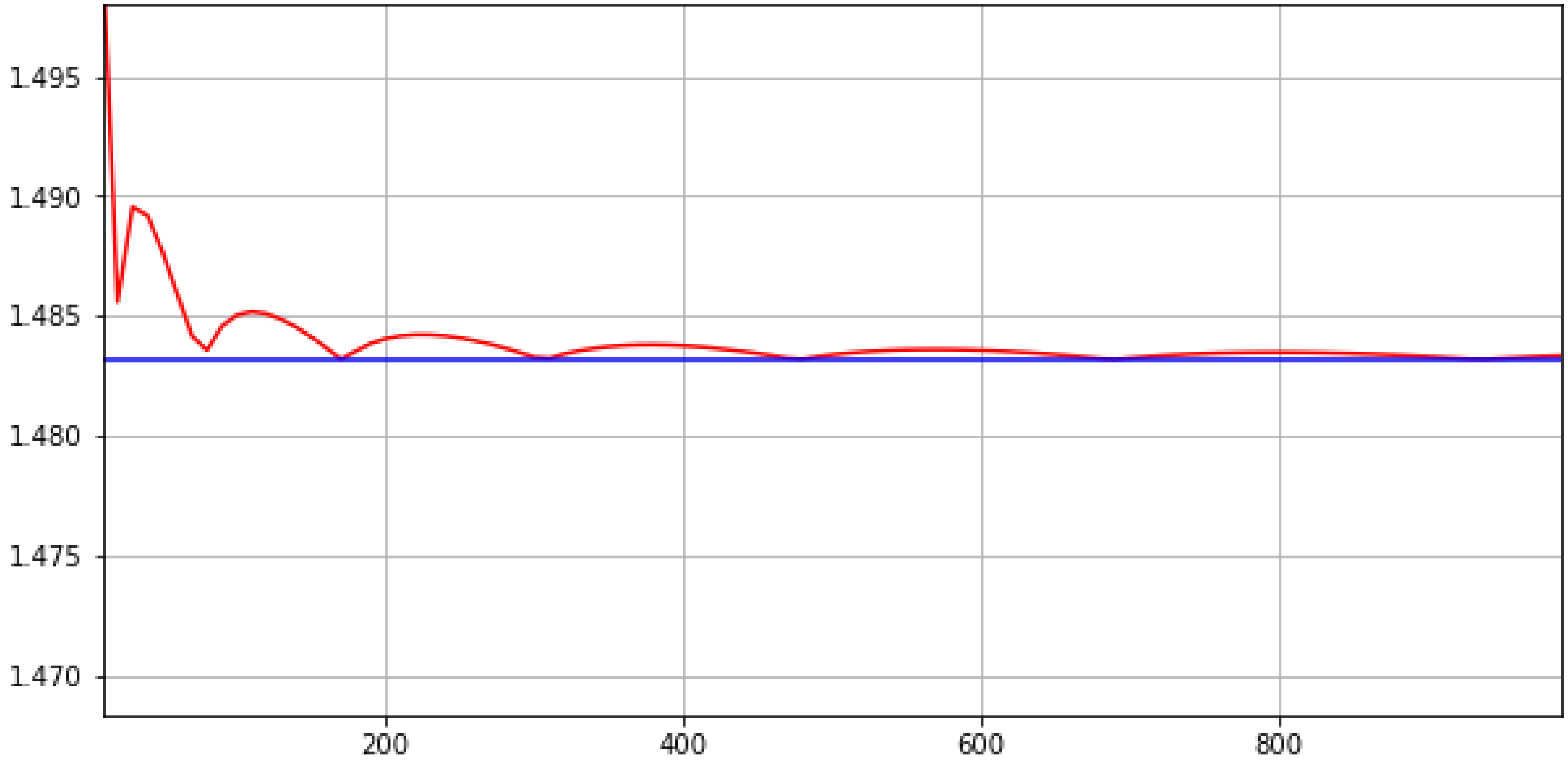
$$c_{t-\Delta t,i} = \max\left[e^{-r\Delta t}\left(p \cdot c_{t,i} + (1-p)\cdot c_{t,i+1}\right), K - S_t\right] \tag{17}$$

for a put option.

Convergency Rate



Convergency Rate



Outline

Trinomial Tree

- Under the trinomial method, the underlying stock price is modeled as a recombining tree, where, at each node the price has three possible paths: an up, down and stable or middle path.
- These values are found by multiplying the value at the current node by the appropriate factor u, d or m where

$$u = e^{\sigma\sqrt{2\Delta t}},\ d = e^{-\sigma\sqrt{2\Delta t}},\ m = 1$$

and the corresponding probability are:

$$p_u = \left(\frac{e^{(r-q)\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2,\ p_d = \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{(r-q)\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2$$

and $p_m = 1 - (p_u + p_d)$

Trinomial Tree

- As with the binomial model, these factors and probabilities are specified so as to ensure that the price of the underlying evolves as a martingale, while the moments are matched to those of the log-normal distribution;
- Note that for p_u , p_d and p_m to be in the interval $(0, 1)$ the following condition on Δt has to be satisfied:

$$\Delta t < 2\frac{\sigma^2}{(r-q)^2}.$$

Convergency Rate