Lesson 3 - Finite Difference in Option Pricing

March 21, 2021

TO BE DONE

- Capire il comportamento del metodo alle differenze finite esplicito con griglia uniforme rispetto al parametro di correlazione rho. Perché viene tolto?
- INSERIRE IL METODO WEIGHTED PER LA RISOLUZIONE DELL'EQUAZIONE DIFFERENZIALE

1 Finite Difference Methods for Heston PDE

[223]:
```
%matplotlib inline

import math
import numpy              as np
import pandas             as pd
import scipy.stats        as ss
import matplotlib
import pylab

from pylab import *
from matplotlib import pyplot as pl
```

1.1 Introduction

In this notebook, we present methods to obtain the European call price by solving the Heston PDE along a two-dimensional grid representing the stock price and the volatility. We first show how to construct uniform and nonuniform grids for the discretization of the stock price and the volatility, and present formulas for finite difference approximations to the derivatives in the Heston PDE.

As you alredy known, one of the limitations of using the famous Black-Scholes model is the assumption of a constant volatility σ. A major modeling step away from the assumption of constant volatility in asset pricing, was made by modeling the volatility/variance as a diffusion process. The resulting models are the stochastic volatility (SV) models. The idea to model volatility as a random variable is confirmed by practical financial data which indicates the variable and unpredictable nature of the stock price's volatility. The most significant argument to consider the volatility to be stochastic is the implied volatility smile/skew, which is present in the financial market data, and can be accurately recovered by SV models, especially for options with a medium to long time to the maturity date T. With an additional stochastic process, which is correlated to the asset price process St, we deal with a system of SDEs, for which option valuation is more expensive than for a scalar asset price process.

The most popular SV model is the Heston model, for which the system of stochastic equations under the risk-neural measure reads:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dZ_t^1$$

with the variance following the square-root diffusion

$$dv_t = \kappa_v(\theta_v - v_t)dt + \sigma_v \sqrt{v_t} dZ_t^2$$

The two Brownian motions are instantaneously correlated with $dZ_t^1 dZ_t^2 = \rho$. The second equation models a mean reversion process for the variance, with the parameters, r the risk-free interest rate, θ_v the long term variance, κ_v the reversion speed; σ_v is the volatility of the variance, determining the volatility of v_t. There is an additional parameter v_0, the t_0-value of the variance.

1.1.1 The Heston PDE

The Heston partial differential equation (PDE) for the value $U(S,,v,t)$ of an option on a dividend-paying stock, when the spot price is S, the volatility is v and the maturity is t, is

$$\frac{\partial U}{\partial t} = \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho \sigma v S \frac{\partial^2 U}{\partial v \partial S} + \frac{1}{2} \sigma^2 v \frac{\partial^2 U}{\partial v^2} - rU + (r-q) S \frac{\partial U}{\partial S} + \kappa(\theta - v) \frac{\partial U}{\partial v} \tag{1}$$

In a compact notation, we can express the PDE as

$$\frac{\partial U}{\partial t} = LU(t) \tag{2}$$

As we have already seen, to implement finite differences, we first need a discretization grid for the **two state variables (the stock price and the variance)**, and a discretization grid for the maturity. These grids can have equally or unequally spaced increments. Second, we need **discrete approximations to the continuous derivatives** that appear in the PDE. Finally, we need a **finite difference methodology** to solve the PDE.

INSERIRE DEDUZIONE PDE HESTON

1.2 How to check the PDE solution?

As we have already said, it is extremely important to have procedures for verifying and controlling the developed algorithms. For this reason we will now develop a simple Python code to calculate the price of a European option in the Heston model by implementing the well-known semi-analytical solution.

The time-t price of a European call on a non-dividend-paying stock with spot price S_t, when the strike is K and the time to maturity is $\tau = T - t$ in the Heston Model can be written as:

$$C(K) = S_t P_1 - K e^{-r\tau} P_2 \tag{3}$$

where P_1 and P_2 are two probabilities under two different measures:

$$C(K) = S_t \mathbb{Q}^S(S_T > K) - Ke^{-r\tau}\mathbb{Q}(S_T > K) \tag{4}$$

The measure $\mathbb{Q}$ uses the bond B_t as the numeraire, while the measure $\mathbb{Q}^S$ uses the stock price S_t.

1.2.1 Calculation of Probability for Heston Model

In the Heston model, it can be shown that P_1 and P_2 can be written as:

$$P_j = \Pr_j(\ln S_T - \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \ln K} f_j(\phi; x, v)}{i\phi}\right] d\phi \tag{5}$$

Heston postulates that these characteristic functions are of the log linear form

$$f_j(\phi; x_t, v_t) = \exp\left(C_j(\tau, \phi) + D_j(\tau, \phi) v_t + i\phi x_t\right) \tag{6}$$

where C_j and D_j are constant coefficients and $\tau = T - t$ is the time to maturity. The coefficients can be shown to be

$$D_j(\tau, \phi) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right) \tag{7}$$

and

$$C_j(\tau, \phi) = ri\phi\tau + \frac{a}{\sigma^2}\left[(b_j - \rho\sigma i\phi + d_j)\tau - 2\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right] \tag{8}$$

where $a = \kappa\theta$ and

$$d_j = \sqrt{(\rho\sigma i\phi - b_j)^2 - \sigma^2(2u_j i\phi - \phi^2)} \tag{9}$$

$$g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j} \tag{10}$$

and with

$$u_1 = \frac{1}{2} \quad u_2 = -\frac{1}{2} \tag{11}$$

$$b_1 = \kappa + \lambda - \rho\sigma \quad b_2 = \kappa + \lambda \tag{12}$$

We will use two functions to implement the Heston call price using the *trapezoidal rule* to solve the integral: *HestonProb* and *HestonPriceTrapz*. The first function calculates the characteristic functions and returns the real part of the integral. Since this is the most complicated calculation, we are going to describe all the single steps befor collect all into the single block function. First of all we define model and option parameters:

[224]:
```
'''
Model Parameters

Heston parameters:
    kappa  = volatility mean reversion speed parameter
    theta  = volatility mean reversion level parameter
    lambda = risk parameter
    rho    = correlation between two Brownian motions
    sigma  = volatility of variance
    v0     = initial variance

    phi    = integration variable
    Pnum   = 1 or 2 (for the probabilities)
'''
kappa  = 1.5
theta  = 0.04
sigma  = 0.3
rho    = 1
lambd  = 0.0
v0     = 0.05412

Pnum   = 1
phi    = 0.00000001
'''
Option features

    K = strike price
    tau = maturity
    r = risk free rate
    q = dividend yield
    S = spot price
'''
K      = 100.0
tau    = 0.15
r      = 0.02
q      = 0.05
S      = 101.52
```

[225]:
```
import math
import cmath

x = math.log(S)
a = kappa*theta
```

```
# Parameters "u" and "b" are different for P1 and P2 (see equations 7 and 8).
if(Pnum == 1):
    u = 0.5
    b = kappa + lambd - rho * sigma
else:
    u = -0.5
    b = kappa + lambd
```

Now we can start calculating the factors: $i\rho\sigma\phi$, $2ui\phi$ and $i\phi\tau$

[226]:
```
rhosigmaphi = complex(0, rho * sigma * phi)
twouphi     = complex(0, 2 * u * phi)
phitaui     = complex(0, phi * tau)

one         = complex(1, 0)
two         = complex(2, 0)
```

which can be used to calculate:

$$d_j = \sqrt{(\rho\sigma i\phi - b_j)^2 - \sigma^2(2u_j i\phi - \phi^2)} \tag{13}$$

[227]:
```
d1 = (rhosigmaphi - complex(b, 0))**2
d2 = (twouphi     - complex(phi * phi, 0)) * complex(sigma*sigma, 0)
d  = cmath.sqrt(d1 - d2)
d
```

[227]: `(1.2-3.375e-09j)`

and ...

$$g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j} \tag{14}$$

[228]:
```
g = (complex(b, 0) - rhosigmaphi + d)/(complex(b, 0) - rhosigmaphi - d)
g
```

[228]: `(-16.999999999999996-6399999999.999999j)`

Now we compute the ratio ...

$$\frac{1 - g_j e^{d_j\tau}}{1 - g_j} \tag{15}$$

[229]:
```
BigG = (one - g*cmath.exp(d*complex(tau, 0)))/(one - g)
BigG
```

[229]: (1.1972173631218102-5.752760770926336e-10j)

… and finally we can calculate C_j:

$$C_j(\tau, \phi) = (r - q)i\phi\tau + \frac{a}{\sigma^2}\left[(b_j - \rho\sigma i\phi + d_j)\tau - 2\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right] \tag{16}$$

[230]:
```
term2 = (complex(b, 0) - rhosigmaphi + d)
term3 = complex(r - q, 0) * phitaui
c1    = complex(tau, 0) * term2
c2    = two * cmath.log(BigG)
c3    = c1 - c2
term4 = complex(a/sigma**2, 0) * c3
bigC  = term3 + term4
bigC
```

[230]: -4.181870595598485e-11j

… and D_j:

$$D_j(\tau, \phi) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right) \tag{17}$$

[231]:
```
# (bj - i*rho*sigma*phi + dj) is term2
term5 = term2/complex(sigma * sigma, 0)
b1    = cmath.exp(d*complex(tau, 0))
term6 = (one - b1)/(one - g*b1)
bigD  = term5*term6
bigD
```

[231]: (1.6723187497605994e-18+6.863741191196999e-10j)

Now we can calculate the *characteristic function*:

$$f_j(\phi; x_t, v_t) = \exp\left(C_j(\tau, \phi) + D_j(\tau, \phi)v_t + i\phi x_t\right) \tag{18}$$

[232]:
```
# characteristic function

phixi = complex(0, phi * x)
f     = cmath.exp(bigC + bigD * complex(v0, 0) + phixi)
f
```

[232]: (0.9999999999999989+4.619788609543208e-08j)

and the integrand in the formula for probability:

$$\mathrm{Re}\left[\frac{e^{-i\phi \ln K} f_j(\phi; x, v)}{i\phi}\right] \tag{19}$$

[233]:
```
philogKi   = complex(0, -math.log(K) * phi)
e1         = cmath.exp(philogKi)
e2         = e1 * f
E          = e2/complex(0, phi)
HestonProb = E.real
HestonProb
```

[233]: 0.014618423555117976

1.2.2 Probability Density Function

Below we simply put all the pieces together in a single function:

[234]:
```
def HestonProb(phi,kappa,theta,lambd,rho,sigma,tau,K,S,r,q,v0,Pnum):

    x      = math.log(S)

    a = kappa*theta
    # Parameters "u" and "b" are different for P1 and P2.
    if(Pnum == 1):
        u = 0.5
        b = kappa + lambd - rho * sigma
    else:
        u = -0.5
        b = kappa + lambd

    rhosigmaphi = complex(0, rho * sigma * phi)
    twouphi     = complex(0, 2 * u * phi)
    phitaui     = complex(0, phi * tau)

    d1   = (rhosigmaphi - complex(b, 0))**2
    d2   = (twouphi     - complex(phi * phi, 0)) * complex(sigma*sigma, 0)
    d    = cmath.sqrt(d1 - d2)

    g    = (complex(b, 0) - rhosigmaphi + d)/(complex(b, 0) - rhosigmaphi - d)
    BigG = (1 - g*cmath.exp(d*complex(tau, 0)))/(1 - g)

    term2 = (complex(b, 0) - rhosigmaphi + d)
    term3 = complex(r - q, 0) * phitaui
    c1    = complex(tau, 0) * term2
    c2    = 2 * cmath.log(BigG)
    c3    = c1 - c2
    term4 = complex(a/sigma**2, 0) * c3
    bigC  = term3 + term4
```

```
    term5 = term2/complex(sigma * sigma, 0)
    b1    = cmath.exp(d*complex(tau, 0))
    term6 = (1 - b1)/(1 - g*b1)
    bigD  = term5*term6

    dv0   = bigD * complex(v0, 0)
    phixi = complex(0, phi * x)
    f     = cmath.exp(bigC + dv0 + phixi)

    philogKi   = complex(0, -math.log(K) * phi)
    e1         = cmath.exp(philogKi)
    e2         = e1 * f
    E          = e2/complex(0, phi)
    HestonProb = E.real

    return HestonProb
```

[235]:
```
print(HestonProb(phi,kappa,theta,lambd,rho,sigma,tau,K,S,r,q,v0,Pnum))
```

```
0.014618423555117976
```

1.3 Pricing an European Call Option

$$C(K) = S_t P_1 - K e^{-r\tau} P_2 \tag{20}$$

[236]:
```
def␣
↪HestonPriceTrapz(PutCall,kappa,theta,lambd,rho,sigma,T,K,S,r,q,v0,trap,Lphi,Uphi,N):
↪
    '''
    ' Heston (1993) price of a European option.
    ' Uses the original formulation by Heston
    ' Heston parameters:
    '    kappa  = volatility mean reversion speed parameter
    '    theta  = volatility mean reversion level parameter
    '    lambda = risk parameter
    '    rho    = correlation between two Brownian motions
    '    sigma  = volatility of variance
    '    v0     = initial variance
    ' Option features.
    '    PutCall = 'C'all or 'P'ut
    '    K = strike price
    '    S = spot price
    '    r = risk free rate
    '    q = dividend yield
    '    T = maturity
    ' Integration features
```

```
    '    Lphi = lower limit
    '    Uphi = upper limit
    '    N    = Number of grid points
    '''
```

[237]:
```
    N = 500
    T = tau

    Lphi = 0.00000001
    Uphi = 150

    dphi = (Uphi - Lphi) / (N - 1)
    phi  = np.arange(Lphi, Uphi+dphi, dphi)

    # Weights for trapezoidal rule
    W = np.full(N, dphi)
    W[0] = W[N-1] = 0.5*dphi

    int1 = np.array([HestonProb(ph, kappa, theta, lambd, rho, sigma, T, K, S,␣
↪r, q, v0, 1) for ph in phi])*W
    int2 = np.array([HestonProb(ph, kappa, theta, lambd, rho, sigma, T, K, S,␣
↪r, q, v0, 2) for ph in phi])*W

    I1 = np.sum(int1)
    I2 = np.sum(int2)

    P1 = 0.5 + 1 / np.pi * I1
    P2 = 0.5 + 1 / np.pi * I2

    # The call price
    HestonC = S * math.exp(-q * T) * P1 - K * math.exp(-r * T) * P2

    # The put price by put-call parity
    HestonP = HestonC - S * math.exp(-q * T) + K * math.exp(-r * T)
```

[238]:
```
print('Call price : ' + str(HestonC))
print('Put  price : ' + str(HestonP))
```

```
Call price : 4.0490387831701184
Put  price : 2.9880402082684867
```

Of course we could have used directly the 'trapz' method of Numpy library. Below the simple code to obtain the same result as above...

[239]:
```
    y1 = np.array([HestonProb(ph, kappa, theta, lambd, rho, sigma, T, K, S, r,␣
↪q, v0, 1) for ph in phi])
    y2 = np.array([HestonProb(ph, kappa, theta, lambd, rho, sigma, T, K, S, r,␣
↪q, v0, 2) for ph in phi])
```

```
    # call trapz method of numply
    I1np = np.trapz(y1, phi)
    I2np = np.trapz(y2, phi)

    P1np = 0.5 + 1 / np.pi * I1np
    P2np = 0.5 + 1 / np.pi * I2np

    # The call price
    HestonC2 = S * math.exp(-q * T) * P1np - K * math.exp(-r * T) * P2np

    # The put price by put-call parity
    HestonP2 = HestonC - S * math.exp(-q * T) + K * math.exp(-r * T)
```

[240]:
```
print('Call price : ' + str(HestonC2))
print('Put  price : ' + str(HestonP2))
```

```
Call price : 4.0490387831701184
Put  price : 2.9880402082684867
```

1.4 Finite Differences for Heston Model

Now, having a challenge algorithm, we can start to work on the main problem of this lesson: the solution of the Heston PDE equation in Two Dimension with Finite Difference Method. We first show how to construct uniform and nonuniform grids for the discretization of the stock price and the volatility, and present formulas for finite difference approximations to the derivatives in the Heston PDE. Next, we explain the boundary conditions of the PDE for a European call and then we will implement an Explicit scheme.

1.4.1 Explicit Scheme

[207]:
```
# Heston Parameters
params = (kappa, theta, sigma, v0, rho, lambd)
```

[208]:
```
# Boundary Values for Stock Price, Volatility, and Maturity

Smin = 0.0000
Smax = 2 * K
Vmin = 0
Vmax = 0.5
Tmin = 0
Tmax = 0.15

# Number of Steps in Each Dimension

NS = 30
NV = 30
```

```
NT = 1500
```

[209]:
```
# Option Features
Strike = K
```

[210]:
```
'''
' Finite differences for the Heston PDE for a European Call
' Reference:
' In 'T Hout and Foulon "ADI Finite Difference Schemes for Option Pricing in␣
↪the Heston Model with Correlation"
' Int J of Num Analysis and Modeling, 2010.
' INPUTS
'    params     = 6x1 vector of Heston parameters
'    Strike     = Strike price
'    r          = risk free rate
'    q          = Dividend yield
'    Smax, Smin = Max and Min values of stock price
'    Vmax, Vmin = Max and Min values of volatility
'    Tmax, Tmin = Max and Min values of maturity
'    NS, NV, NT = Number of points on stock price, volatility, and maturity␣
↪grids
'    GridType   = Type of Grid ("Uniform" or "NonUniform")
'    Si         = Value of Spot price at which to interpolate on U(S,V)
'    Vi         = Value of Volatility at which to interpolate on U(S,V)
' OUTPUT
'    U(Si,Vi)   = Interpolated value of U(S,V) and points (Si,Vi)
'''
# Heston parameters. Note: only kappa, theta, and sigma are needed
kappa = params[0]
theta = params[1]
sigma = params[2]

# Increment for Stock Price, Volatility, and Maturity
ds = (Smax - Smin) / (NS - 1)
dv = (Vmax - Vmin) / (NV - 1)
dt = (Tmax - Tmin) / (NT - 1)

print(ds, dv, dt)
```

```
6.896551724137931 0.017241379310344827 0.00010006671114076051
```

Building a Uniform Grid

[211]:
```
# Building a Uniform Grid
Mat  = np.arange(Tmin, Tmax + dt, dt)
Spot = np.arange(Smin, Smax + ds, ds)
Vol  = np.arange(Vmin, Vmax + dv, dv)
```

```python
# Make sure the array have the right dimension
Mat  = Mat[:NT]
Spot = Spot[:NS]
Vol  = Vol[:NV]
# Sanity check
print(Mat.shape, Spot.shape, Vol.shape)
```

```
(1500,) (30,) (30,)
```

Approximation of Derivatives Recall that we use the notation $U^n_{i,j}, U(S_i, v_j, t_n)$, or $U(S_i, v_j)$ to represent the value of a European option at time t_n when the stock price is S_i and the volatility is v_j. Whenever possible throughout this notebook, we use central difference approximations to the first- and second-order derivatives in the S and v directions in the PDE for $U^n_{i,j}$. In general, as we have already seen in the previous notebook, first- and second-order derivatives of $U^n_{i,j}$ at a point (S_i, v_j) on the grid can be written in terms of sums of values of U at points adjacent to (S_i, v_j). Using a uniform grid we can write:

$$
\begin{aligned}
\frac{\partial^2 U}{\partial S^2} &\sim \frac{U^n_{i+1,j} - 2U^n_{i,j} + U^n_{i-1,j}}{(ds)^2} && (21) \\
\frac{\partial^2 U}{\partial S \partial v} &\sim \frac{U^n_{i+1,j+1} + U^n_{i-1,j-1} - U^n_{i-1,j+1} - U^n_{i+1,j-1}}{4\, ds\, dv} && (22) \\
\frac{\partial^2 U}{\partial v^2} &\sim \frac{U^n_{i,j+1} - 2U^n_{i,j} + U^n_{i,j-1}}{(dv)^2} && (23) \\
\frac{\partial U}{\partial S} &\sim \frac{U^n_{i+1,j} - U^n_{i-1,j}}{2\, ds} && (24) \\
\frac{\partial U}{\partial v} &\sim \frac{U^n_{i,j+1} - U^n_{i,j-1}}{2\, dv} && (25)
\end{aligned}
$$

Discretization Scheme The explicit scheme produces an expression for the PDE that is very simple when the grids are uniform. Lets' re-write the PDE in the following form:

$$
U^{n+1}_{i,j} = U^n_{i,j} + dt \left[\frac{1}{2} v S^2 \frac{\partial^2}{\partial S^2} + \rho \sigma v S \frac{\partial^2}{\partial v \partial S} + \frac{1}{2} \sigma^2 v \frac{\partial^2}{\partial v^2} - r + (r-q) S \frac{\partial}{\partial S} + \kappa(\theta - v) \frac{\partial}{\partial v} \right] U^n_{i,j} \quad (26)
$$

Now we substitute the approximations to the derivatives under a uniform grid to obtain

$$
\begin{aligned}
U_{i,j}^{n+1} = U_{i,j}^{n} + dt\Bigg[&\frac{1}{2}v_j S_i^2 \frac{U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n}}{(ds)^2} \\
&+ \rho\sigma v_j S_i \frac{U_{i+1,j+1}^{n} + U_{i-1,j-1}^{n} - U_{i-1,j+1}^{n} - U_{i+1,j-1}^{n}}{4\,ds\,dv} \\
&+ \frac{1}{2}\sigma^2 v_j \frac{U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}}{(dv)^2} \\
&+ (r-q)S_i \frac{U_{i+1,j}^{n} - U_{i-1,j}^{n}}{2\,ds} \\
&+ \kappa(\theta - v)\frac{U_{i,j+1}^{n} - U_{i,j-1}^{n}}{2\,dv}\Bigg] - r\,dt\,U_{i,j}^{n}
\end{aligned}
\tag{27}
$$

Grouping common terms

$$
\begin{aligned}
U_{i,j}^{n+1} = &\left[1 - dt\left(v_j \frac{S_i^2}{(ds)^2} + \sigma^2 \frac{v_j}{(dv)^2} + r\right)\right] U_{i,j}^{n} \\
&+ dt\left[\frac{v_j S_i^2}{2(ds)^2} - \frac{(r-q)S_i}{2\,ds}\right] U_{i-1,j}^{n} \\
&+ dt\left[\frac{v_j S_i^2}{2(ds)^2} + \frac{(r-q)S_i}{2\,ds}\right] U_{i+1,j}^{n} \\
&+ dt\left[\frac{\sigma^2 v_j}{2(dv)^2} - \frac{\kappa(\theta - v_j)}{2\,dv}\right] U_{i,j-1}^{n} \\
&+ dt\left[\frac{\sigma^2 v_j}{2(dv)^2} + \frac{\kappa(\theta - v_j)}{2\,dv}\right] U_{i,j+1}^{n} \\
&+ dt\,\frac{\rho\sigma v_j S_i}{4\,ds\,dv}\left(U_{i+1,j+1}^{n} + U_{i-1,j-1}^{n} - U_{i-1,j+1}^{n} - U_{i+1,j-1}^{n}\right)
\end{aligned}
\tag{28}
$$

Noting that $S_i = i \times ds$ and $v_j \times dv$ we obtain

$$
\begin{aligned}
C_1 &= \left[1 - dt\left(j\,dv\frac{i^2\,(ds)^2}{(ds)^2} + \sigma^2\frac{j\,dv}{(dv)^2} + r\right)\right] = \left[1 - dt\left(i^2 j\,dv + \frac{\sigma^2 j}{dv} + r\right)\right] \\
C_2 &= dt\left[\frac{j\,dv\,i^2\,(ds)^2}{2(ds)^2} - \frac{(r-q)i\,ds}{2\,ds}\right] = \left[\frac{i\,dt}{2}(ij\,dv - r + q)\right] \\
C_3 &= dt\left[\frac{j\,dv\,i^2\,(ds)^2}{2(ds)^2} + \frac{(r-q)i\,ds}{2\,ds}\right] = \left[\frac{i\,dt}{2}(ij\,dv + r - q)\right] \\
C_4 &= dt\left[\frac{\sigma^2 j\,dv}{2(dv)^2} - \frac{\kappa(\theta - j\,dv)}{2\,dv}\right] = \left[\frac{dt}{2\,dv}\left(\sigma^2 j - \kappa(\theta - j\,dv)\right)\right] \\
C_5 &= dt\left[\frac{\sigma^2 j\,dv}{2(dv)^2} + \frac{\kappa(\theta - j\,dv)}{2\,dv}\right] = \left[\frac{dt}{2\,dv}\left(\sigma^2 j + \kappa(\theta - j\,dv)\right)\right] \\
C_6 &= dt\,\frac{\rho\sigma j\,dvi\,ds}{4\,ds\,dv} = \frac{\rho\sigma j\,i\,dt}{4}
\end{aligned} \tag{29}
$$

$$
U_{i,j}^{n+1} = C_1 U_{i,j}^n + C_2 U_{i-1,j}^n + C_3 U_{i+1,j}^n + C_4 U_{i,j-1}^n + C_5 U_{i,j+1}^n + C_6\left(U_{i+1,j+1}^n + U_{i-1,j-1}^n - U_{i-1,j+1}^n - U_{i+1,j-1}^n\right) \tag{30}
$$

Boundary Conditions for the PDE DA FINIRE

Boundary Condition at Maturity.

Boundary Contition for $S = S_{min}$

Boundary Contition for $S = S_{max}$

Boundary Contition for $v = v_{max}$

Boundary Contition for $v = v_{min}$. When $v = v_{min} = 0$, the boundary condition is a little more complicated. When $v = 0$ the PDE becomes

$$
\frac{\partial U}{\partial t} = -rU + (r-q)S\frac{\partial U}{\partial S} + \kappa\theta\frac{\partial U}{\partial v} \tag{31}
$$

We can use central difference for $\partial U/\partial S$

$$
\frac{\partial U}{\partial S}(i,0,t_n) \simeq \frac{U(t,i+1,0) - U(t,i-1,0)}{dS} \tag{32}
$$

and forward difference for $\partial U/\partial v$

$$
\frac{\partial U}{\partial v}(i,0,t_n) \simeq \frac{U(t,i,1) - U(t,i,0)}{dv} \tag{33}
$$

$$\frac{U(t+1,i,0)-U(t,i,0)}{dt} = -rU(t,i,0) + (r-q)S\frac{U(t,i+1,0)-U(t,i-1,0)}{2dS} + \kappa\theta\frac{U(t,i,1)-U(t,i,0)}{dv} \tag{34}$$

$$\begin{aligned}
U(t+1,i,0) &= U(t,i,0) - rU(t,i,0)dt + \frac{1}{2}(r-q)S\frac{dt}{dS}\left(U(t,i+1,0)-U(t,i-1,0)\right) \\
&\quad + \kappa\theta\frac{dt}{dv}\left(U(t,i,1)-U(t,i,0)\right) \\
&= U(t,i,0)\left(1 - rdt - \kappa\theta\frac{dt}{dv}\right) \\
&\quad + \frac{1}{2}(r-q)S\frac{dt}{dS}\left(U(t,i+1,0)-U(t,i-1,0)\right) \\
&\quad + \kappa\theta\frac{dt}{dv}U(t,i,1)
\end{aligned}$$

[212]:

```
# Initialize the 2-D grid with zeros
U = np.zeros((NS, NV))

# Temporary grid for previous time steps
u = np.zeros((NS, NV))

# Boundary condition for Call Option at t = Maturity

for j in arange(NV):
    U[:,j] = np.maximum(Spot - Strike, 0)

# loop on maturity
for tt in np.arange(NT):
    # Boundary condition for Smin and Smax
    U[0,:]    = 0
    U[NS-1,:] = np.max(Smax - Strike, 0)
    # Boundary condition for Vmax
    U[:,NV-1] = np.maximum(Spot - Strike, 0)
    # Update the temporary grid u(s,t) with the boundary conditions
    u = U
    # Boundary condition for Vmin.
    # Previous time step values are in the temporary grid u(s,t)
    for ss in np.arange(1, NS-1):
        U[ss, 0] = u[ss, 0] * (1 - r * dt - kappa * theta * (dt/dv))      \
               + dt * 0.5 * (r - q) * (ss+1) * (u[ss + 1, 0] - u[ss - 1, 0])  \
               + kappa * theta * (dt/dv) * u[ss, 1]

    # Update the temporary grid u(s,t) with the boundary conditions
    u = U
```

```
    # Interior points of the grid (non boundary).
    # As usual previous time step values are in the temporary grid u(s,t)
    for ss in np.arange(1, NS - 1):
        for vv in np.arange(1, NV - 1):
            C1 = (1 - dt * ss * ss * vv * dv - sigma * sigma * vv * dt / dv - r␣
↪* dt)
            C2 = (1 / 2 * dt * ss * ss * vv * dv - 1 / 2 * dt * (r - q) * ss)
            C3 = (1 / 2 * dt * ss * ss * vv * dv + 1 / 2 * dt * (r - q) * ss)
            C4 = (1 / 2 * dt * sigma * sigma * vv / dv - 1 / 2 * dt * kappa *␣
↪(theta - vv * dv) / dv)
            C5 = (1 / 2 * dt * sigma * sigma * vv / dv + 1 / 2 * dt * kappa *␣
↪(theta - vv * dv) / dv)
            C6 = 1 / 4 * dt * sigma * ss * vv
            # The PDE
            U[ss, vv] = C1 * u[ss, vv] + C2 * u[ss - 1, vv] + C3 * u[ss + 1,␣
↪vv] \
                    + C4 * u[ss, vv - 1] + C5 * u[ss, vv + 1] \
                    + C6 * (u[ss + 1, vv + 1] + u[ss - 1, vv - 1] - u[ss - 1, vv␣
↪+ 1] - u[ss + 1, vv - 1])
```

[213]:
```
df = pd.DataFrame(U, index = Spot.round(4), columns=Vol.round(4))

df = df.round(4)
df.to_csv('out.txt', sep=';')
```

[214]:
```
import scipy.interpolate

f = scipy.interpolate.interp2d(Vol, Spot, U)

Si = S
vi = v0

print("European Option Price (Heston Model) = " + str(f(vi, Si)[0]))
```

```
European Option Price (Heston Model) = 4.2767840265270465
```

1.4.2 Non Uniform Grid

DA FINIRE

We'll describe a nonuniform grid that is finer around the strike price K and around the spot volatility $v_0 = 0$. The grid of size $N_S + 1$ for stock price is

$$S_i = K + c\sinh(\xi_i), \quad i = 0, \ldots, N_S$$

where we select $c = K/5$ and where $\xi_i = \sinh^{-1}(-K/c) + i\Delta\xi$ with

$$\Delta\xi = \frac{1}{N_S - 1}\left[\sinh^{-1}\left(\frac{S_{max} - K}{c}\right) - \sinh^{-1}\left(\frac{-K}{c}\right)\right]$$

The grid of size $N_v + 1$ for the volatility is

$$v_j = d\,\sinh\left(j\Delta\eta\right), \quad j = 0, \ldots, N_V$$

with $\Delta\eta = \sinh^{-1}(V_{max}/d)/(N_V - 1)$

The BuildGrid function is used throughout to construct uniform and nonuniform grids.

[215]:
```
def BuildGrid(grid_type, p_grid, p_option):

    K     = p_option['Strike']
    Smax  = p_grid['Smax']
    Smin  = p_grid['Smin']
    Vmax  = p_grid['Vmax']
    Vmin  = p_grid['Vmin']
    NS    = p_grid['NS']
    NV    = p_grid['NV']

    if grid_type == 'uniform':
        # Increment for Stock Price, Volatility, and Maturity
        ds = (Smax - Smin) / (NS - 1)
        dv = (Vmax - Vmin) / (NV - 1)
        # Building a Uniform Grid
        Spot = np.arange(Smin, Smax + ds, ds)
        Vol  = np.arange(Vmin, Vmax + dv, dv)
        # Make sure the array have the right dimension
        Spot = Spot[:NS]
        Vol  = Vol[:NV]
    else:
        C = K / 5
        dz = 1 / (NS - 1) * (math.asinh((Smax - K) / C) - math.asinh(-K / C))
        # The Spot Price Grid
        Z    = np.zeros(NS)
        Spot = np.zeros(NS)
        for i in range(NS):
            Z[i] = math.asinh(-K / C) + (i - 1) * dz
            Spot[i] = K + C * math.sinh(Z[i])

        # The volatility grid
        d = Vmax / 10
        dn = math.asinh(Vmax / d) / (NV - 1)
        N = np.zeros(NV)
        Vol = np.zeros(NV)
        for j in range(NV):
```

```python
            N[j] = (j - 1) * dn
            Vol[j] = d * math.sinh(N[j])

    return (Spot, Vol)
```

Below the result of the previous function, the grid for the stock price is represented by blue lines and the volatility by red lines. Note that in the stock price dimension, the grid is finest around the strike price, while in the volatility dimension, the grid becomes finer as we progress toward zero.

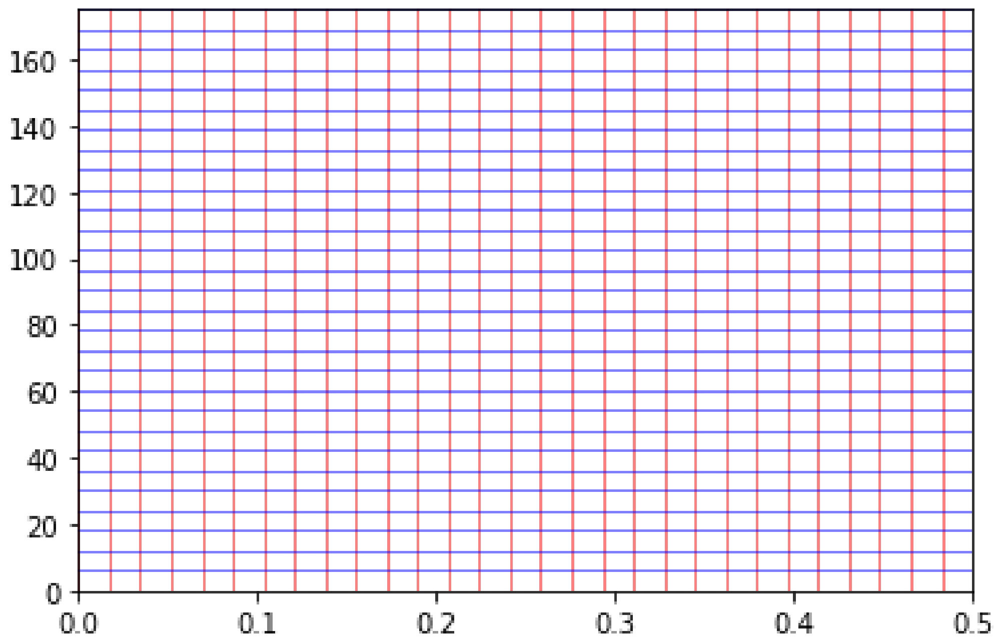
[216]:
```python
    # Option features
    p_option = {'Strike':87.5
               ,'r':0.02
               ,'q':0.05
               }
    # Grid parameters
    p_grid   = {'NS':30
               ,'NV':30
               ,'NT':1500
               ,'Smin':0
               ,'Smax':2 * p_option['Strike']
               ,'Vmin':0
               ,'Vmax':0.5
               ,'Tmin':0
               ,'Tmax':0.15
               }

(Spot, Vol) = BuildGrid('uniform', p_grid, p_option)
```

[217]:
```python
for xc in Vol:
    plt.axvline(x=xc, color='red', linestyle='-', linewidth = .5)

for yc in Spot:
    plt.axhline(y=yc, color='blue', linestyle='-', linewidth = .5)

plt.ylim(min(Spot), max(Spot))
plt.xlim(min(Vol), max(Vol))
plt.show()
```

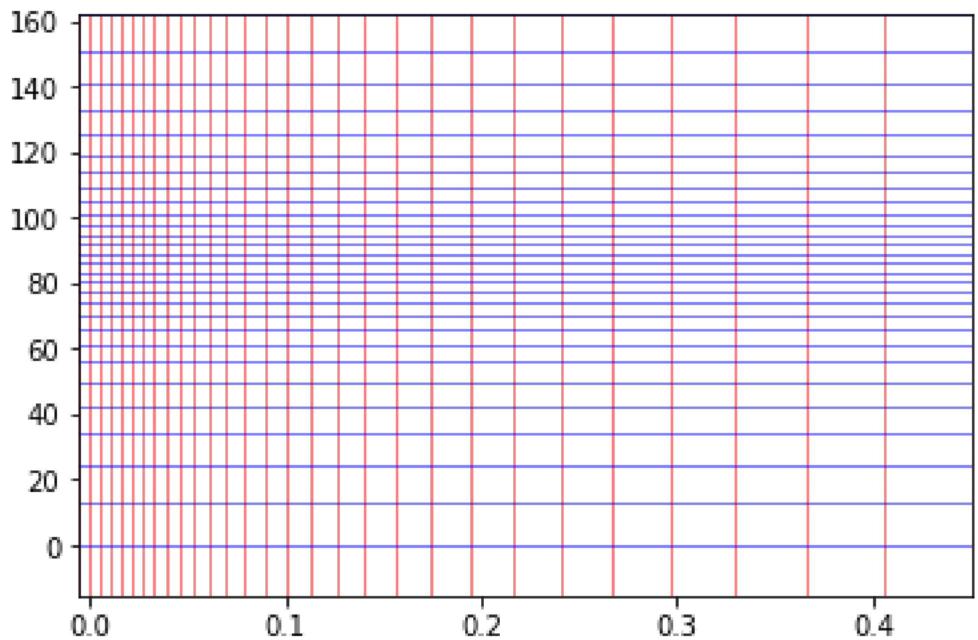


[218]:
```
(Spot, Vol) = BuildGrid('non-uniform', p_grid, p_option)
```

[219]:
```
for xc in Vol:
    plt.axvline(x=xc, color='red', linestyle='-', linewidth = .5)

for yc in Spot:
    plt.axhline(y=yc, color='blue', linestyle='-', linewidth = .5)

plt.ylim(min(Spot), max(Spot))
plt.xlim(min(Vol), max(Vol))
plt.show()
```



1.4.3 Derivatives Approximations for Non Uniform Grid

$$\frac{\partial U}{\partial S}(S_i, v_j) = \left(\frac{U^n_{i+1,j} - U^n_{i-1,j}}{S_{i+1} - S_{i-1}}\right)$$

$$\frac{\partial U}{\partial v}(S_i, v_j) = \left(\frac{U^n_{i,j+1} - U^n_{i,j-1}}{v_{j+1} - v_{j-1}}\right)$$

$$\frac{\partial^2 U}{\partial S^2}(S_i, v_j) = \left(\frac{U^n_{i+1,j} - U^n_{i,j}}{S_{i+1} - S_i} - \frac{U^n_{i,j} - U^n_{i-1,j}}{S_i - S_{i-1}}\right)\frac{1}{S_{i+1} - S_i}$$

$$\frac{\partial^2 U}{\partial v^2}(S_i, v_j) = \left(\frac{U^n_{i,j+1} - U^n_{i,j}}{v_{j+1} - v_j} - \frac{U^n_{i,j} - U^n_{i,j-1}}{v_j - v_{j-1}}\right)\frac{1}{v_{j+1} - v_j}$$

$$\frac{\partial^2 U}{\partial S \partial v}(S_i, v_j) = \left(\frac{U^n_{i+1,j+1} - U^n_{i-1,j+1} - U^n_{i+1,j-1} + U^n_{i-1,j-1}}{(S_{i+1} - S_{i-1})(v_{j+1} - v_{j-1})}\right)$$

[220]:
```
def HestonExplicitPDE_NU(Si, vi, p_heston, p_option, p_grid):

    start = time.time()
    print('program starts...')
    #
    # Number of Steps in Each Dimension
    #
    NS = p_grid['NS']
```

```python
NV = p_grid['NV']
NT = p_grid['NT']
#
# Option Features
#
Strike = p_option['Strike']
r      = p_option['r']
q      = p_option['q']
#
# Boundary Values for Stock Price, Volatility, and Maturity
#
Smax = p_grid['Smax']
Tmin = p_grid['Tmin']
Tmax = p_grid['Tmax']
#
# Heston parameters. Note: only kappa, theta, and sigma are needed
#
kappa = p_heston['kappa']
theta = p_heston['theta']
sigma = p_heston['sigma']
rho   = p_heston['rho']
#
# Building grid
#
(Spot, Vol) = BuildGrid('non-uniform', p_grid, p_option)

# Increment for Maturity
dt = (Tmax - Tmin) / (NT - 1)

# Initialize the 2-D grid with zeros
U = np.zeros((NS, NV))

# Temporary grid for previous time steps
u = np.zeros((NS, NV))

# Boundary condition for Call Option at t = Maturity
for j in range(NV):
    U[:,j] = np.maximum(Spot - Strike, 0)

for tt in range(NT):
    # Boundary condition for Smin and Smax
    U[0,:]    = 0
    U[NS-1,:] = np.max(Smax - Strike, 0)
    # Boundary condition for Vmax
    U[:,NV-1] = np.maximum(Spot - Strike, 0)
    # Update the temporary grid u(s,t) with the boundary conditions
    u = U
```

```
        # Boundary condition for Vmin.
        # Previous time step values are in the temporary grid u(s,t)
        for ss in range(1, NS-1):
            DerV     = (u[ss, 1] - u[ss, 0]) / (Vol[1] - Vol[0])
            DerS     = (u[ss + 1, 0] - u[ss - 1, 0]) / (Spot[ss + 1] - Spot[ss - 1])
            LHS      = -r * u[ss, 0] + (r - q) * Spot[ss] * DerS + kappa * theta * DerV
            U[ss, 0] = LHS * dt + u[ss, 0]
        # Update the temporary grid u(s,t) with the boundary conditions
        u = U
        # Interior points of the grid (non boundary).
        # As usual previous time step values are in the temporary grid u(s,t)
        for s in range(1, NS - 1):
            for v in range(1, NV - 1):
                DerS = (u[s + 1, v] - u[s - 1, v]) / (Spot[s + 1] - Spot[s - 1])       # Central difference for dU/dS
                DerV = (u[s, v + 1] - u[s, v - 1]) / (Vol[v + 1] - Vol[v - 1])        # Central difference for dU/dV
                DerSS = ((u[s + 1, v] - u[s, v]) / \
                        (Spot[s + 1] - Spot[s]) - (u[s, v] - u[s - 1, v]) / \
                        (Spot[s] - Spot[s - 1])) / (Spot[s + 1] - Spot[s])            # d2U/dS2
                DerVV = ((u[s, v + 1] - u[s, v]) / \
                        (Vol[v + 1] - Vol[v]) - (u[s, v] - u[s, v - 1]) / \
                        (Vol[v] - Vol[v - 1])) / (Vol[v + 1] - Vol[v])                # d2U/dV2
                DerSV = (u[s + 1, v + 1] - u[s - 1, v + 1] - u[s + 1, v - 1] + u[s - 1, v - 1]) / \
                        (Spot[s + 1] - Spot[s - 1]) / (Vol[v + 1] - Vol[v - 1])       # d2U/dSdV
                L = 0.5 * Vol[v] * Spot[s] * Spot[s] * DerSS + rho * sigma * Vol[v] * Spot[s] * DerSV \
                  + 0.5 * sigma * sigma * Vol[v] * DerVV - r * u[s, v] \
                  + (r - q) * Spot[s] * DerS + kappa * (theta - Vol[v]) * DerV
                # The PDE
                U[s, v] = L * dt + u[s, v]

    U = U.transpose()

    f = scipy.interpolate.interp2d(Spot, Vol, U)
    end = time.time()
    print('program ends...')
    return (f(Si, vi), end - start)
```

[221]:
```python
def HestonExplicitPDE(Si, vi, p_heston, p_option, p_grid):
    '''
    ' Finite differences for the Heston PDE for a European Call
    ' Reference:
    ' In 'T Hout and Foulon "ADI Finite Difference Schemes for Option Pricing␣
↪in the Heston Model with Correlation"
    ' Int J of Num Analysis and Modeling, 2010.
    ' INPUTS
    '    params     = 6x1 vector of Heston parameters
    '    Strike     = Strike price
    '    r          = risk free rate
    '    q          = Dividend yield
    '    Smax, Smin = Max and Min values of stock price
    '    Vmax, Vmin = Max and Min values of volatility
    '    Tmax, Tmin = Max and Min values of maturity
    '    NS, NV, NT = Number of points on stock price, volatility, and maturity␣
↪grids
    '    GridType   = Type of Grid ("Uniform" or "NonUniform")
    '    Si         = Value of Spot price at which to interpolate on U(S,V)
    '    Vi         = Value of Volatility at which to interpolate on U(S,V)
    ' OUTPUT
    '    U(Si,Vi)   = Interpolated value of U(S,V) and points (Si,Vi)
    '''
    start = time.time()
    print('program starts...')
    #
    # Number of Steps in Each Dimension
    #
    NS = p_grid['NS']
    NV = p_grid['NV']
    NT = p_grid['NT']
    #
    # Option Features
    #
    Strike = p_option['Strike']
    r      = p_option['r']
    q      = p_option['q']
    #
    # Boundary Values for Stock Price, Volatility, and Maturity
    #
    Smin = p_grid['Smin']
    Smax = p_grid['Smax']
    Vmin = p_grid['Vmin']
    Vmax = p_grid['Vmax']
    Tmin = p_grid['Tmin']
    Tmax = p_grid['Tmax']
    # Heston parameters. Note: only kappa, theta, and sigma are needed
```

```python
kappa = p_heston['kappa']
theta = p_heston['theta']
sigma = p_heston['sigma']
# Increment for Stock Price, Volatility, and Maturity
ds = (Smax - Smin) / (NS - 1)
dv = (Vmax - Vmin) / (NV - 1)
dt = (Tmax - Tmin) / (NT - 1)
# Building a Uniform Grid
Mat  = np.arange(Tmin, Tmax + dt, dt)
Spot = np.arange(Smin, Smax + ds, ds)
Vol  = np.arange(Vmin, Vmax + dv, dv)
# Make sure the array have the right dimension
Mat  = Mat[:NT]
Spot = Spot[:NS]
Vol  = Vol[:NV]

# Initialize the 2-D grid with zeros
U = np.zeros((NS, NV))

# Temporary grid for previous time steps
u = np.zeros((NS, NV))

# Boundary condition for Call Option at t = Maturity
for j in range(NV):
    U[:,j] = np.maximum(Spot - Strike, 0)

# loop on maturity
c1 = (1 - r * dt - kappa * theta * dt / dv)
c2 = dt * 0.5 * (r - q)/ds
c3 = kappa * theta * (dt/dv)
for tt in range(NT):
    # Boundary condition for Smin and Smax
    U[0,:]    = 0
    U[NS-1,:] = np.max(Smax - Strike, 0)
    # Boundary condition for Vmax
    U[:,NV-1] = np.maximum(Spot - Strike, 0)
    # Update the temporary grid u(s,t) with the boundary conditions
    u = U
    # Boundary condition for Vmin.
    # Previous time step values are in the temporary grid u(s,t)

    U[1:NS-2,0] = c1 * u[1:NS-2,0] \
                + c2 * Spot[1:NS-2] * (u[2:NS-1, 0] - u[0:NS-3,0]) \
                + c3 * u[1:NS-2,1]

    # Update the temporary grid u(s,t) with the boundary conditions
    u = U
```

```
        # Interior points of the grid (non boundary).
        # As usual previous time step values are in the temporary grid u(s,t)
        for i in range(1, NS - 1):
            for j in range(1, NV - 1):
                C1 = (1 - dt * i * i * j * dv - sigma * sigma * j * dt / dv - r␣
↪* dt)
                C2 = (0.5 * dt * i * i * j * dv - 0.5 * dt * (r - q) * i)
                C3 = (0.5 * dt * i * i * j * dv + 0.5 * dt * (r - q) * i)
                C4 = (0.5 * dt * sigma * sigma * j / dv - 0.5 * dt * kappa *␣
↪(theta - j * dv) / dv)
                C5 = (0.5 * dt * sigma * sigma * j / dv + 0.5 * dt * kappa *␣
↪(theta - j * dv) / dv)
                C6 = 0.25 * dt * sigma * i * j
                # The PDE
                U[i, j] = C1 * u[i, j] + C2 * u[i - 1, j] + C3 * u[i + 1, j] \
                       + C4 * u[i, j - 1] + C5 * u[i, j + 1] \
                       + C6 * (u[i + 1, j + 1] + u[i - 1, j - 1] - u[i - 1, j +␣
↪1] - u[i + 1, j - 1])

    U = U.transpose()

    f = scipy.interpolate.interp2d(Spot, Vol, U)
    end = time.time()
    print('program ends...')
    return (f(Si, vi), end - start)
```

[ ]:

[222]:

```
    # Heston Parameters
    p_heston = {'kappa' :kappa
               ,'theta' :theta
               ,'sigma' :sigma
               ,'v0'    :v0
               ,'rho'   :rho
               ,'lambda':0.00000}
    # Option features
    p_option = {'Strike':K
               ,'r'     :r
               ,'q'     :q
               }
    # Grid parameters
    p_grid   = {'NS':30
               ,'NV':30
               ,'NT':1500
               ,'Smin':0
               ,'Smax':2 * p_option['Strike']
               ,'Vmin':0
```

```
            ,'Vmax':0.5
            ,'Tmin':0
            ,'Tmax':tau
            }

Si = S
vi = p_heston['v0']

call_price_1 =  HestonExplicitPDE(Si, vi, p_heston, p_option, p_grid)
call_price_2 =  HestonExplicitPDE_NU(Si, vi, p_heston, p_option, p_grid)

print(call_price_1[0], call_price_1[1])
print(call_price_2[0], call_price_2[1])
```

```
program starts…
program ends…
program starts…
program ends…
[4.27674443] 10.416719913482666
[4.04314205] 27.161454916000366
```

1.5 Weighted Method

[ ]:

```

```

2 References

Rouah F. D., **The Heston Model and its Extensions in VBA**, Wiley Financial Series

Rouah F. D., **The Heston Model and its Extensions in Matlab and C#**, Wiley Financial Series