Applied Computational Finance

Lesson 2 - Variance Reduction Methods

Giovanni Della Lunga

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Outline

Variance Reduction Methods

- In this section we briefly discuss techniques for improving on the speed and efficiency of a simulazion, usually called *variance reduction techniques*;
- If we do nothing about efficiency, the number of MC replications we need to achieve acceptable pricing acccuracy may be surprisingly large;
- As a result in many cases variance reduction techiques are a practical requirement;
- From a general point of view these methods are based on two principal strategies for reducing variance:
    - Taking advantage of tractable features of a model to adjust or correct simulation output
    - Reducing the variability in simulation input

Variance Reduction Methods

- From the first section we remember that the variance of the estimator is

$$var\left(\tilde{I}_n\right) = \frac{var(f(U_i))}{n}$$

- So, the standard error of the sample mean is the standard deviation or

$$SE\left(\tilde{I}_n\right) = \frac{\sigma_f}{\sqrt{n}}$$

where $\sigma_f^2 = var(f(U_i))$

Variance Reduction Methods

- The most commonly used strategies for variance reduction are the following:
  - **Antithetic variates**
  - **Control variates**
  - **Moment Matching**
  - Stratified Sampling
  - Importance Sampling
  - Low-discrepancy sequences

Variance Reduction Methods - Antithetic Variates

- In this case we construc the estimator by using two brownian trajectories that are mirror images of each other;
- This causes cancellation of dispersion;
- This method tends to reduce the variance modestly but it is extremely easy to implement and as a result very commonly used;
- For the antithetic method to work we need V^+ and V^- to be negatively correlated;
- this will happen if the payoff function is a monotonic function of Z;

Variance Reduction Methods - Antithetic Variates

To analyze the variance reduction produced by the method imagine we have to estimate an expectation $E[Y]$ and that using antithetic sampling we produce a series of pairs of observations $(Y_1, \tilde{Y}_1), \ldots, (Y_n, \tilde{Y}_n)$. The procedure presents the following characteristics:

- the pairs $(Y_1, \tilde{Y}_1), \ldots, (Y_n, \tilde{Y}_n)$ are i.i.d.;
- For each i, Y_i and $\tilde{Y}_i$ have the same distribution, though they are not independent.

Variance Reduction Methods - Antithetic Variates

The antithetic variates estimator is defined as

$$\hat{Y}_{AV} = \frac{1}{2n}\Big(\sum_{i=1}^{n} Y_i + \sum_{i=1}^{n} \tilde{Y}_i\Big) = \frac{1}{n}\sum_{i=1}^{n}\Big(\frac{Y_i + \tilde{Y}_i}{2}\Big)$$

$$\frac{\hat{Y}_{AV} - E[Y]}{\sigma_{AV}/\sqrt{n}} \Rightarrow N(0,1)$$

with

$$\sigma^2_{AV} = \mathit{var}\Big[\frac{Y_i + \tilde{Y}_i}{2}\Big]$$

Variance Reduction Methods - Antithetic Variates

Antithetic variates reduce variance if:

$$\mathit{var}\left[\hat{Y}_{AV}\right] < \mathit{var}\left[Y_i\right]$$

where Y_i is the estimator of a normal montecarlo simulation.

$$\begin{aligned}
&\mathit{var}\left[Y_i + \tilde{Y}_i\right] < 2\mathit{var}\left[Y_i\right] \\
&\Rightarrow 2\mathit{var}\left[Y_i\right] + 2\mathit{cov}\left[Y_i, \tilde{Y}_i\right] < 2\mathit{var}\left[Y_i\right] \\
&\Rightarrow \mathit{cov}\left[Y_i, \tilde{Y}_i\right] < 0
\end{aligned}$$

Remark

The condition requires that the negative dependence of the input random variables produces negative covariance between the estimates of two paired replications.

Variance Reduction Methods - Antithetic Variates

Remark

The condition requires that the negative dependence of the input random variables produces negative covariance between the estimates of two paired replications.

- A sufficient condition ensuring this is monotonicity of the mapping function from inputs to outputs defined in the algorithm.
- The method of antithetic variates increases efficiency in pricing options that depend monotonically on inputs (e.g. European, American or Asian options).
- In the case of non-monotome payoffs, the method does not necessarily provide better performance than the standard Monte Carlo.

Variance Reduction Methods - Antithetic Variates

We would like to estimate

$$I = \int_0^1 \frac{1}{1+x}\,\mathrm{d}x$$

using Monte Carlo integration. This integral is the expected value of $f(U)$, where

$$f(U) = \frac{1}{1+U}$$

and U follows a uniform distribution $[0,1]$. Using a sample of size n denote the points in the sample as $u_1, \cdots, u_n$. Then the estimate is given by

$$I \approx \frac{1}{n}\sum_i f(u_i)$$

Variance Reduction Methods - Antithetic Variates

```
n = 1000

u = np.random.uniform(0,1,n)

f = 1.0 /(1.0 + u)

print('\nNormal Simulation : \n')
print('True value           = {}'.format(round(np.log(2),6)))
print('Integral estimation = {}'.format(round(np.average(f),6)))
print('Estimation variance = {}'.format(round(np.std(f)**2,6)))

u1 = np.random.uniform(0,1,n/2)
u2 =1 - u1

f1 = 1.0 / (1.0 + u1)
f2 = 1.0 / (1.0 + u2)
f  = 0.5 * (f1 + f2)

print('\nAntithetic Variate : \n')
print('Integral estimation = {}'.format(round(np.average(f),6)))
print('Estimation variance = {}'.format(round(np.std(f)**2,6)))
```

Variance Reduction Methods - Antithetic Variates

- To apply the antithetic variate technique, we generate standard normal random numbers Z and define two set of samples of the undelying price

$$S_T^+ = S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z} \quad S_T^- = S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}(-Z)}$$

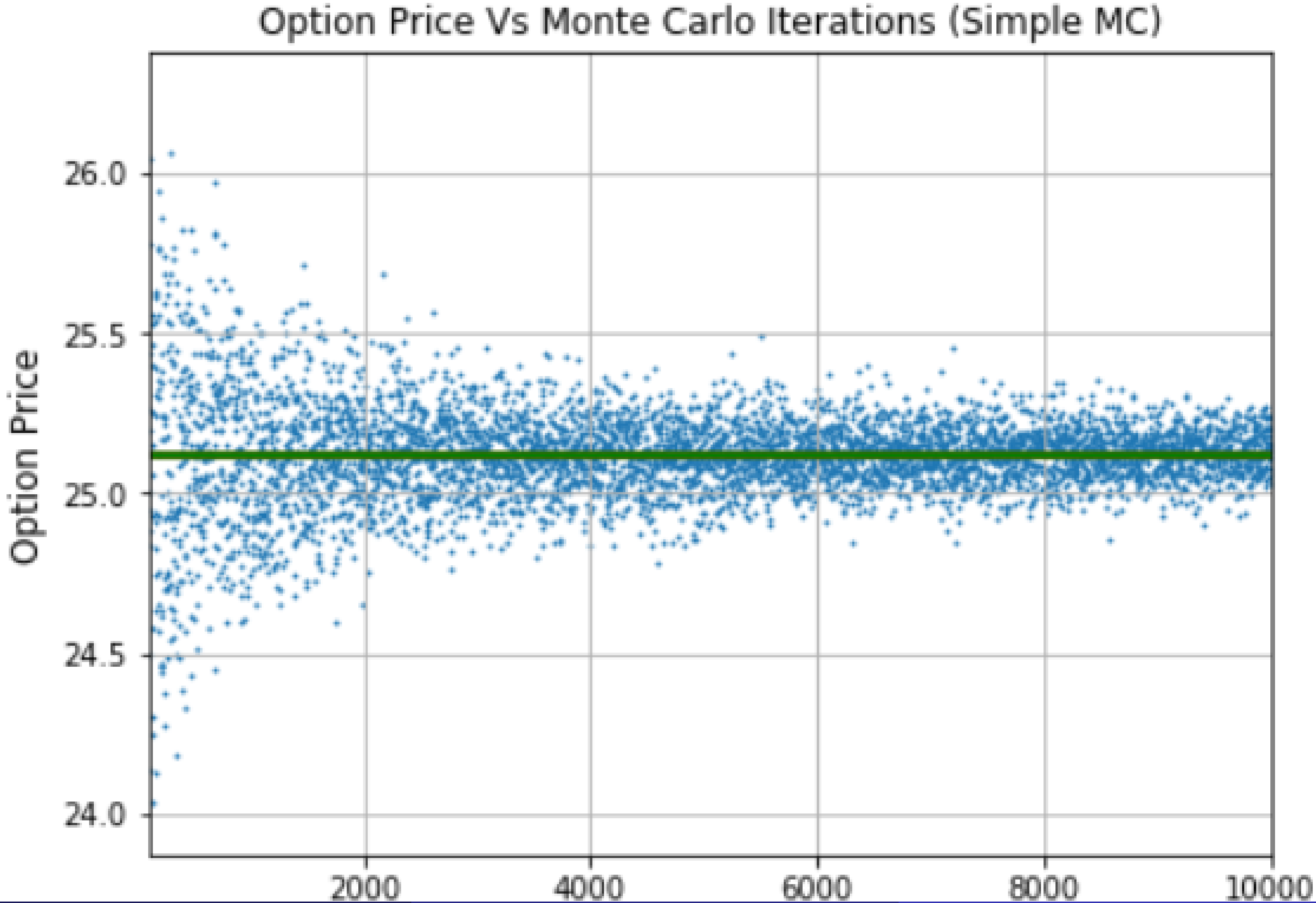
- Similarly we define two sets of discounted payoff samples ...

$$V_T^+ = \max[S^+(T) - K, 0] \quad V_T^- = \max[S^-(T) - K, 0]$$

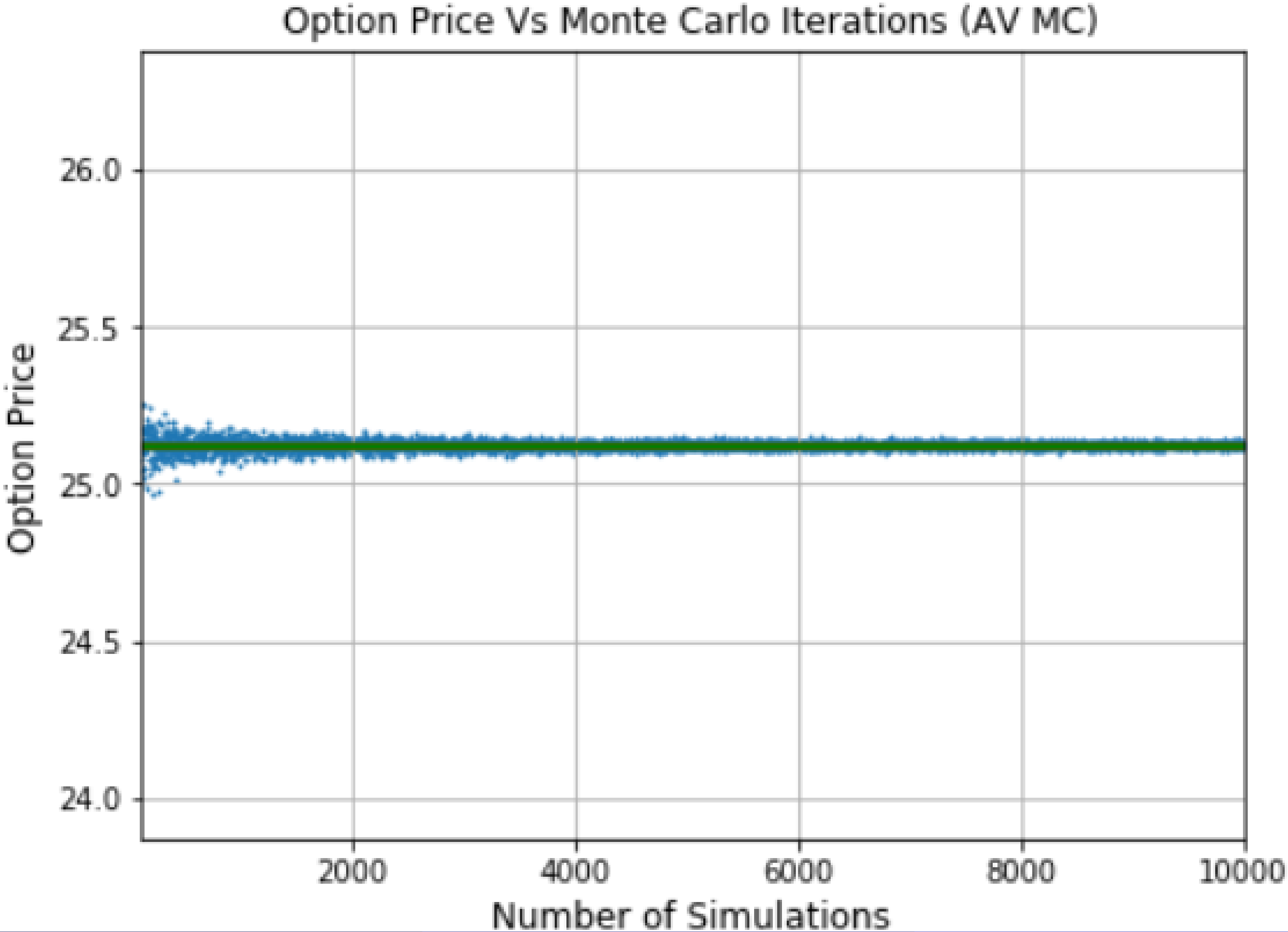
- ... and at last we construct our mean estimator by averaging these samples

$$\bar{V}_0 = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{2}\left(V_j^+ + V_j^-\right)$$

Variance Reduction Methods - Antithetic Variates



Variance Reduction Methods - Antithetic Variates



Variance Reduction Methods - Control Variate

- The method of control variate is among the most effective of the variance reduction techniques.
- It generally exploits information about the errors in estimates of known quantities to reduce the error in an estimate of an unknown quantity.
- Suppose again we want to estimate the derivative price α, the Monte Carlo estimator from n independent and identically distributed replications $\alpha_1, \ldots, \alpha_n$ is

$$\hat{\alpha} = \frac{\alpha_1 + \cdots + \alpha_n}{n}$$

.

Variance Reduction Methods - Control Variate

Imagine now, that for each replication it is possible to calculate another output X_i along with α_i. The pairs $(X_i, \alpha_i), i = 1, \ldots, n$ are i.i.d. and suppose that the expectation $E[X]$ is known. Thus for any fixed b it is possible to calculate

$$\alpha_i(b) = \alpha_i - b\,(X_i - E[X])$$

for each replication i. The control variate Monte Carlo estimator would then be

$$\hat{\alpha}(b) = \hat{\alpha} - b\left(\hat{X} - E[X]\right) = \frac{1}{n}\sum_{i=1}^{n}(\alpha_i - b\,(X_i - E[X]))$$

and the observed error $\hat{X} - E[X]$ is used to control the estimate $E[\alpha]$.

Variance Reduction Methods - Control Variate

As before we would like to estimate

$$I = \int_0^1 \frac{1}{1+x}\,\mathrm{d}x$$

using Monte Carlo integration. This integral is the expected value of $f(U)$, where

$$f(U) = \frac{1}{1+U}$$

But now we introduce $g(U) = 1 + U$ as a **control variate** with a **known expected value**

$$\mathbb{E}\left[g\left(U\right)\right] = \int_0^1 (1+x)\,\mathrm{d}x = \tfrac{3}{2}$$

Variance Reduction Methods - Control Variate

The variance of each replication $\alpha_i(b)$ is

$$\begin{aligned} \mathit{var}\left[\alpha_i(b)\right] &= \mathit{var}\left[\alpha_i - b(X_i - E[X])\right] \\ &= \mathit{var}\left(\alpha\right) + \mathit{var}\left(bX_i\right) - 2\mathit{cov}\left(\alpha, bX_i\right) \\ &= \sigma_\alpha^2 + b^2\sigma_X^2 - 2b\sigma_\alpha\sigma_X\rho \end{aligned} \tag{1}$$

The optimal coefficient that minimizes the variance is

$$\frac{\partial}{\partial b}\mathit{var}\left[\alpha_i(b)\right] = 2b\sigma_X^2 - 2\sigma_\alpha\sigma_X\rho = 0$$

$$\Rightarrow \boxed{b^\star = \frac{\sigma_\alpha}{\sigma_X}\rho = \frac{\mathit{cov}\left[X, \alpha\right]}{\mathit{var}\left[X\right]}}$$

Variance Reduction Methods - Control Variate

Plugging this value into (**??**) and arranging it is possible to find the ratio of the variance of the optimally controlled estimator to the one of the uncontrolled estimator

$$\frac{\mathit{var}\left[\hat{\alpha} - b^{\star}(\hat{X} - E[X])\right]}{\mathit{var}\left[\hat{\alpha}\right]} = 1 - \rho^2 \qquad (2)$$

Remark

From equation (**??**) it is possible to observe that with the optimal coefficient $b^{\star}$, the effectiveness of a control variate is determined by the size of the correlation between α and X. Moreover the sign of the correlation is irrelevant.

Variance Reduction Methods - Control Variate

In the previous example let's take a new estimator

$$I \approx \frac{1}{n}\sum_i f(u_i) + \beta\left(\frac{1}{n}\sum_i g(u_i) - 3/2\right)$$

```
I = f - beta*(g - g_expected)

print('True value            = {}'.format(round(np.log(2),6)))
print('Integral estimation = {}'.format(round(np.average(I),6)))
print('Estimation Variance = {}'.format(round(np.std(I)**2,6)))
```

```
True value          = 0.693147
Integral estimation = 0.693068
Estimation Variance = 0.000641
```

Variance Reduction Methods - Control Variate

A Classical Application: Asian Option Pricing

- Asian options are options whose final value depends on the arithmetic average of the prices of the underlying asset, recorded on predetermined dates:
- For example

$$C(T) = \max\left(A(0,T) - K, 0\right)$$

where A denotes the average price for the period $[0,T]$, and K is the strike price. The equivalent put option is given by

$$P(T) = \max\left(K - A(0,T), 0\right)$$

- Asian options are cheaper than ordinary options as the calculation of the average actually decreases the volatility of the underlying.

Variance Reduction Methods - Control Variate

A Classical Application: Asian Option Pricing

The problem with most Asian options is that they are written on arithmetic means of the underlying observed on different dates of detection.

There is no analytical solution to this problem.

Evaluation techniques:

- Moment matching (Turnbull and Wakeman): the distribution of the mean is approximated with a log-normal distribution with equal mean and variance.
- Monte Carlo method: scenarios are generated for the sampling dates, the pay-offs are calculated for each path and the average is calculated

Variance Reduction Methods - Control Variate

A Classical Application: Asian Option Pricing

- If it is assumed that the price of the underlying asset, S, is distributed in a log-normal way and that S_{avg} is a geometric mean of S, we can use analytical formulas to evaluate the European-type Asian options.
- This depends on the fact that the geometric mean of a set of variables distributed in a log-normal way is also log-normal.
- It's remarkable that it's possible to find a closed-form solution for the geometric Asian option
- When used in conjunction with control variates in Monte Carlo simulations, the formula is useful for deriving fair values for the arithmetic Asian option.

- In the standard montecarlo simulation the option price, for each path, is calculated as

$$C^i = e^{-rT} \max\left(A^i - K, 0\right)$$

where A is the discrete sampled arithmetic average calculated on a discrete set of points and n is the number of simulations.

- This leads to the usual estimator

$$\hat{C} = \sum_{i=1}^{n} C^i$$

- Using the method of the control variable in addition to the previously described variables we must calculate the geometric mean for each simulation

$$G^i = \left(\prod_{j=1}^{m} S^i_{t_j} \right)^{\frac{1}{m}}$$

and the sampled value of the Asian geometric average option

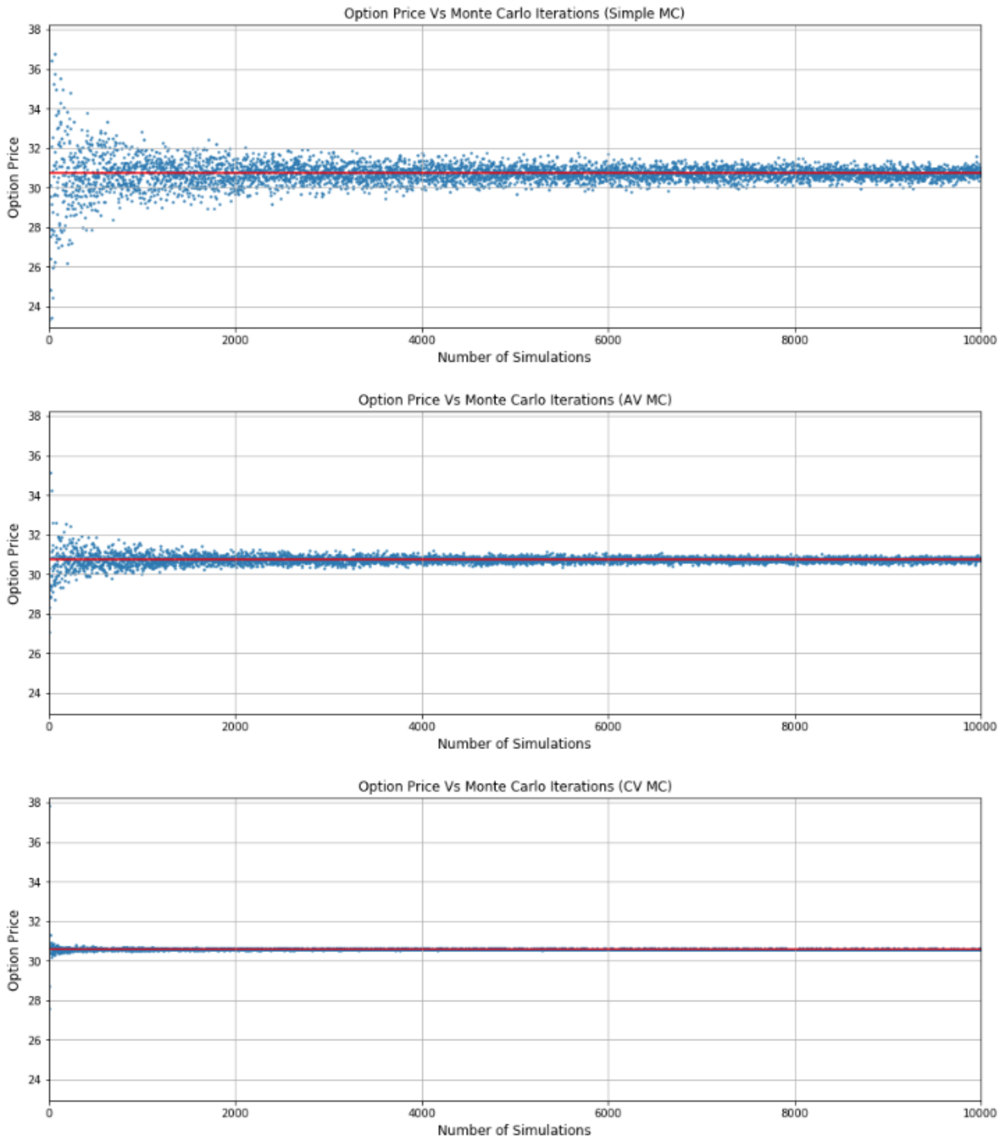
$$C^i_G = exp^{-rT} \max \left(G^i - K, 0 \right)$$

- finally we use the estimator

$$\hat{C} = \sum_{i=1}^{n} \left[C^i - \beta \left(C^i_G - C_G \right) \right]$$

Variance Reduction Methods - Control Variate

A Classical Application: Asian Option Pricing



Variance Reduction Methods - Control Variate

A Classical Application: Asian Option Pricing

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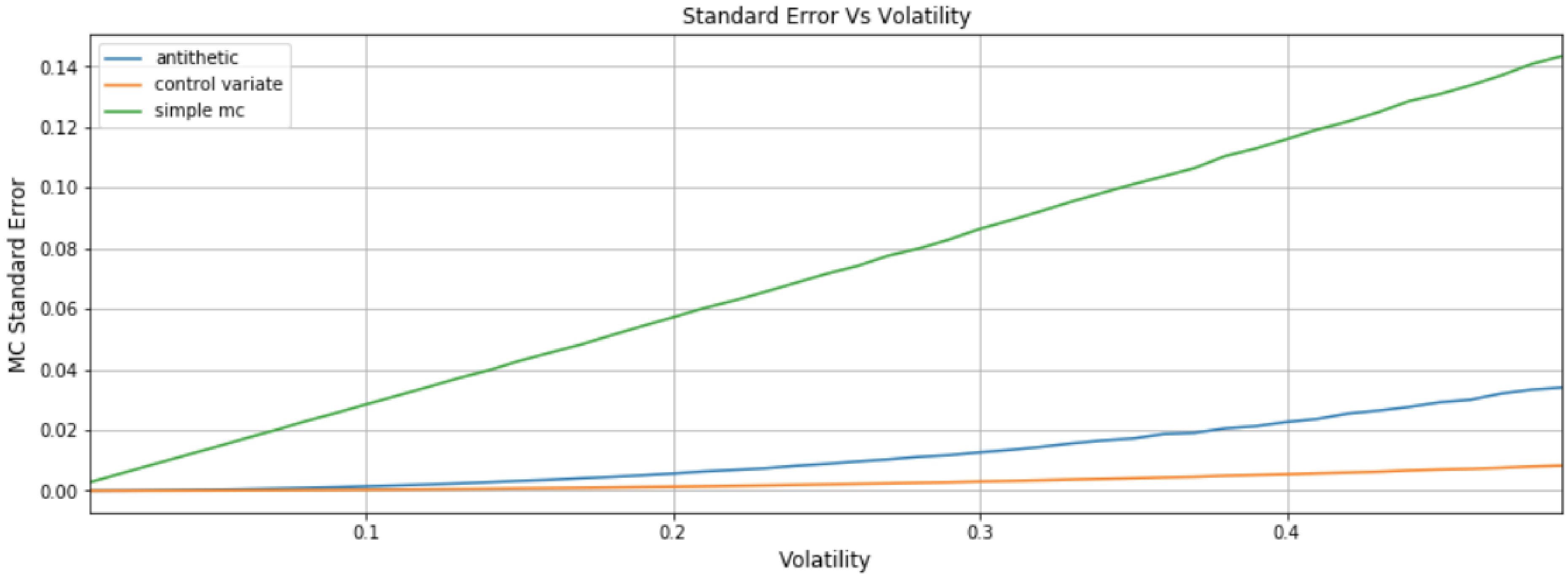
A Classical Application: Asian Option Pricing



Variance Reduction Methods - Moment Matching

- Let $z_i, i = 1, ..., n,$ denote an independent standard normal random vector used to drive a simulation.
- The sample moments will not exactly match those of the standard normal. The idea of moment matching is to transform the z_i to match a finite number of the moments of the underlying population.
- For example, the first and second moment of the normal random number can be matched by defining

$$\tilde{z}_i = (z_i - \tilde{z})\frac{\sigma_z}{s_z} + \mu_z, i = 1, ...n \tag{3}$$

where $\tilde{z}$ is the sample mean of the z_i and σ_z is the population standard deviation, s_z is the sample standard deviation of z_i, and μ_z s the population mean.