ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

DIPARTIMENTO DI
SCIENZE STATISTICHE "PAOLO FORTUNATI"

3 - Introduction to Deep Learning

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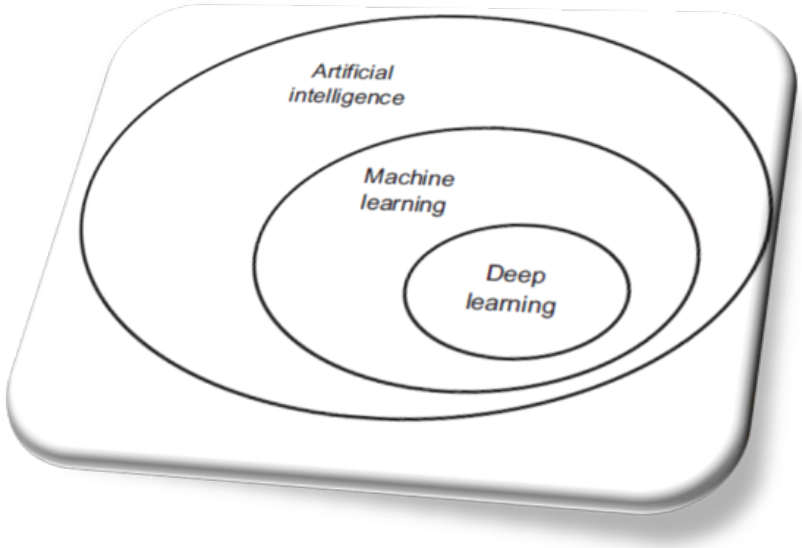
Subsection 1

Machine Learning and Deep Learning

Introduction: Machine Learning and Deep Learning

- As we know to do machine learning we need three things:
- Input data points
- Examples of the expected output
- A way to measure whether the algorithm is doing a good job

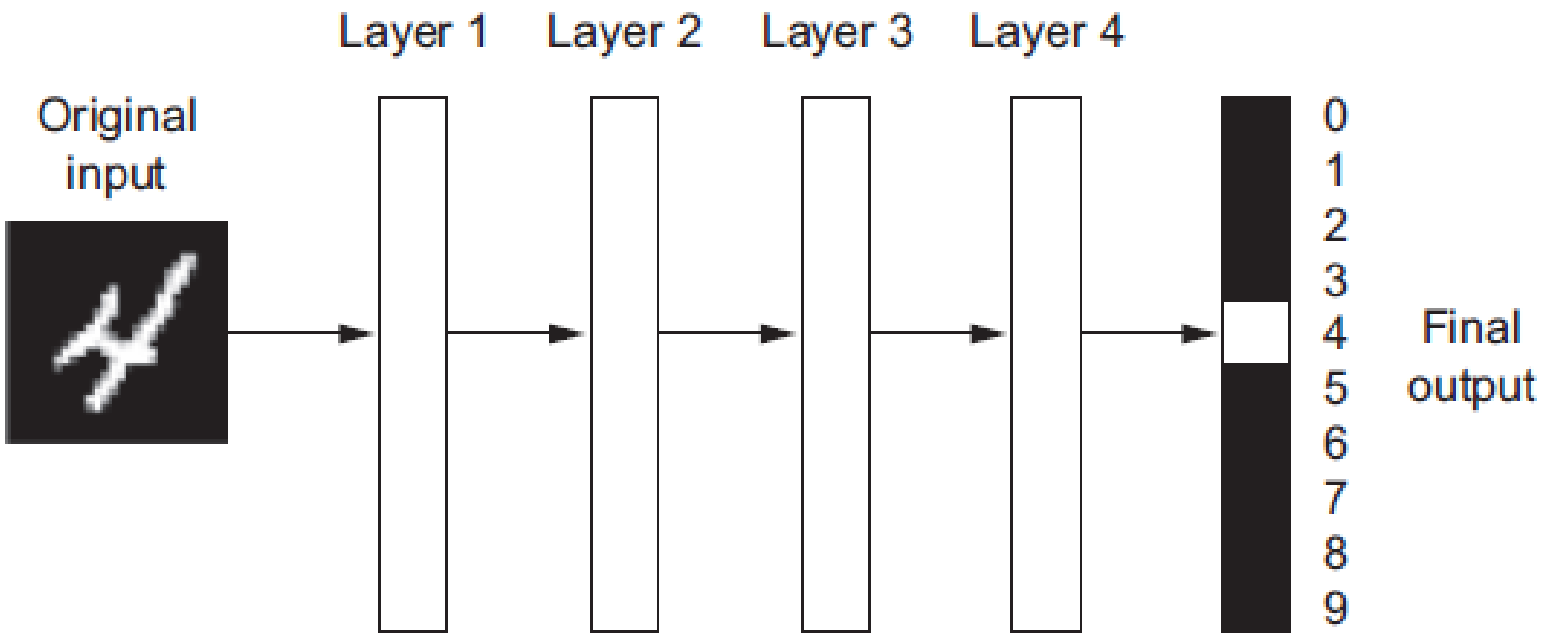


Introduction: Machine Learning and Deep Learning

- A machine-learning model transforms its input data into meaningful outputs, a process that is **learned** from exposure to known examples of inputs and outputs.
- Therefore, the central problem in machine learning and deep learning is to **meaningfully transform data**: in other words, **to learn useful representations of the input data at hand, representations that get us closer to the expected output**.
- What is a representation? At its core, it is simply a different way to look at data, to represent or encode data.

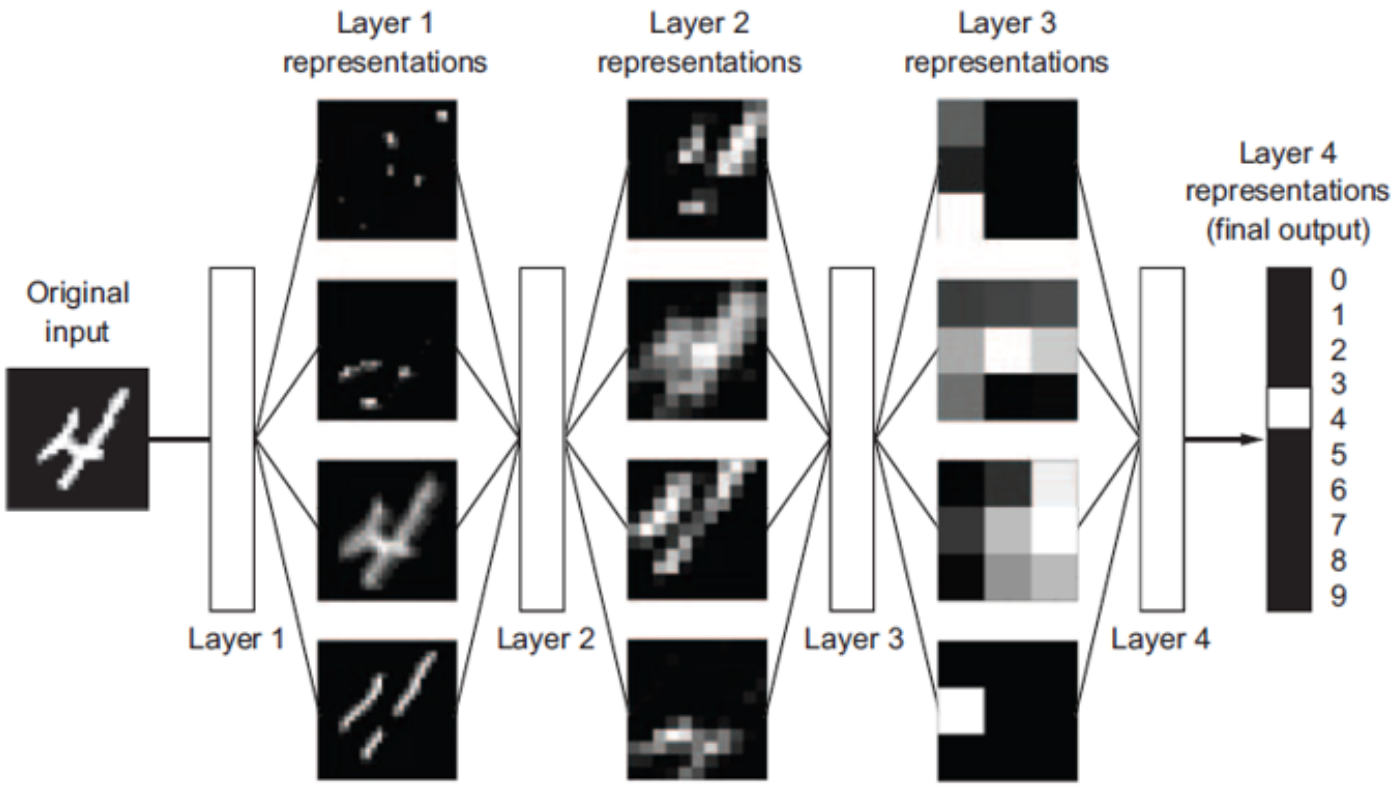
Introduction: Machine Learning and Deep Learning

Deep learning is a specific subfield of machine learning: a new take on learning representations from data that puts an emphasis on learning successive layers of increasingly meaningful representations.



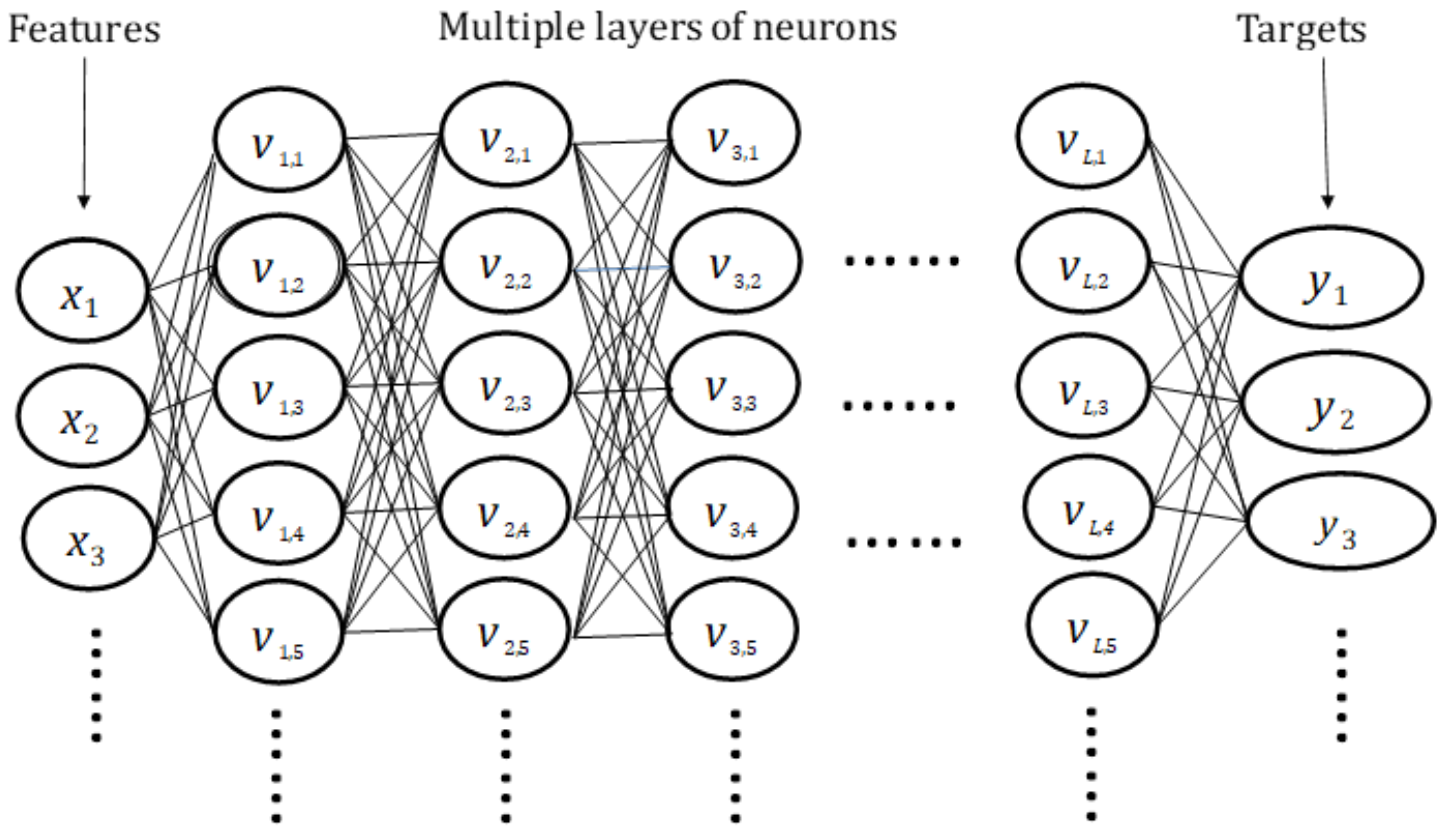
Introduction: Machine Learning and Deep Learning

Deep learning is a specific subfield of machine learning: a new take on learning representations from data that puts an emphasis on learning successive layers of increasingly meaningful representations.



Introduction: Machine Learning and Deep Learning

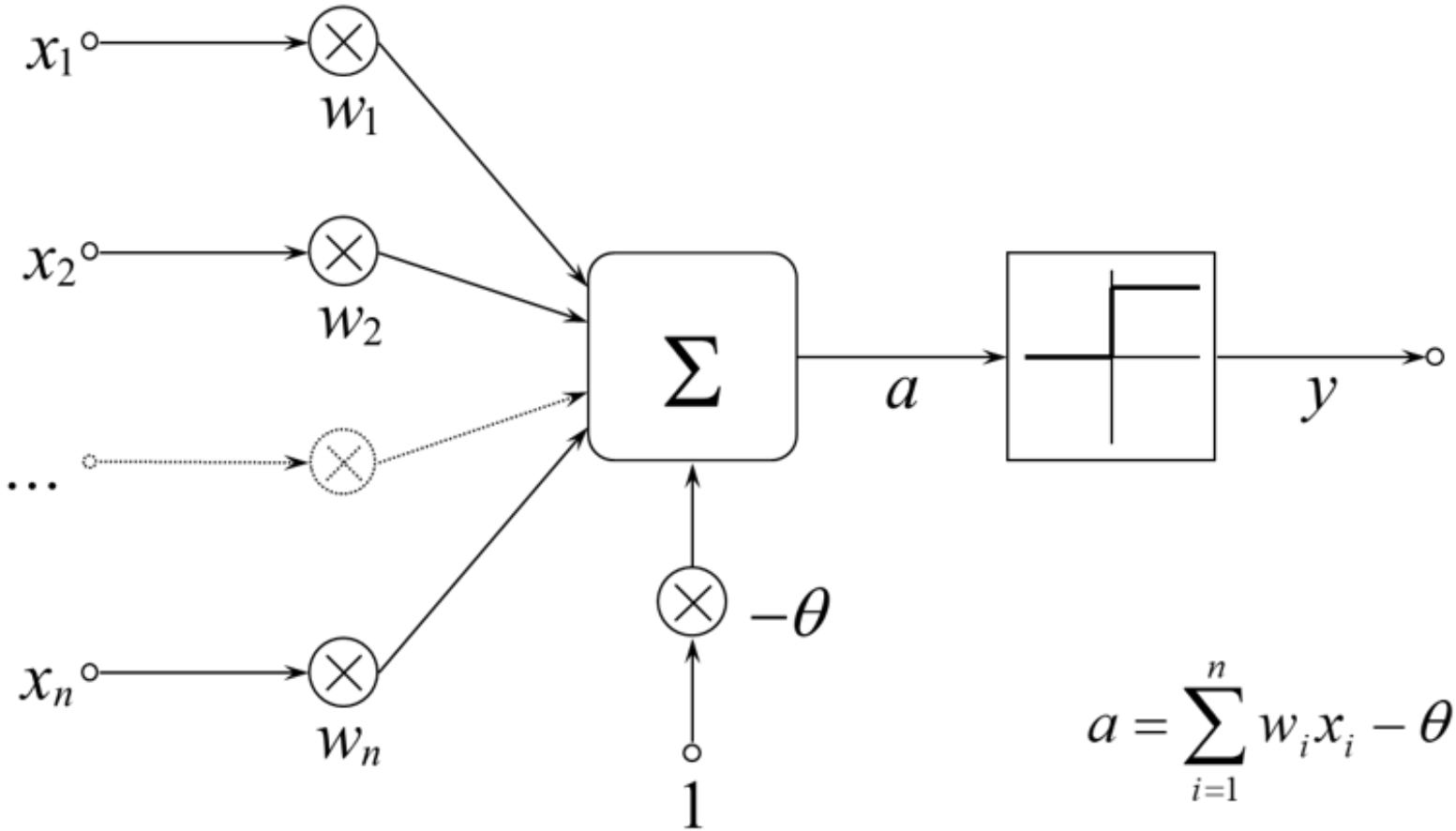
In deep learning, these layered representations are (almost always) learned via models called **neural networks**, structured in literal layers stacked on top of each other.



Subsection 2

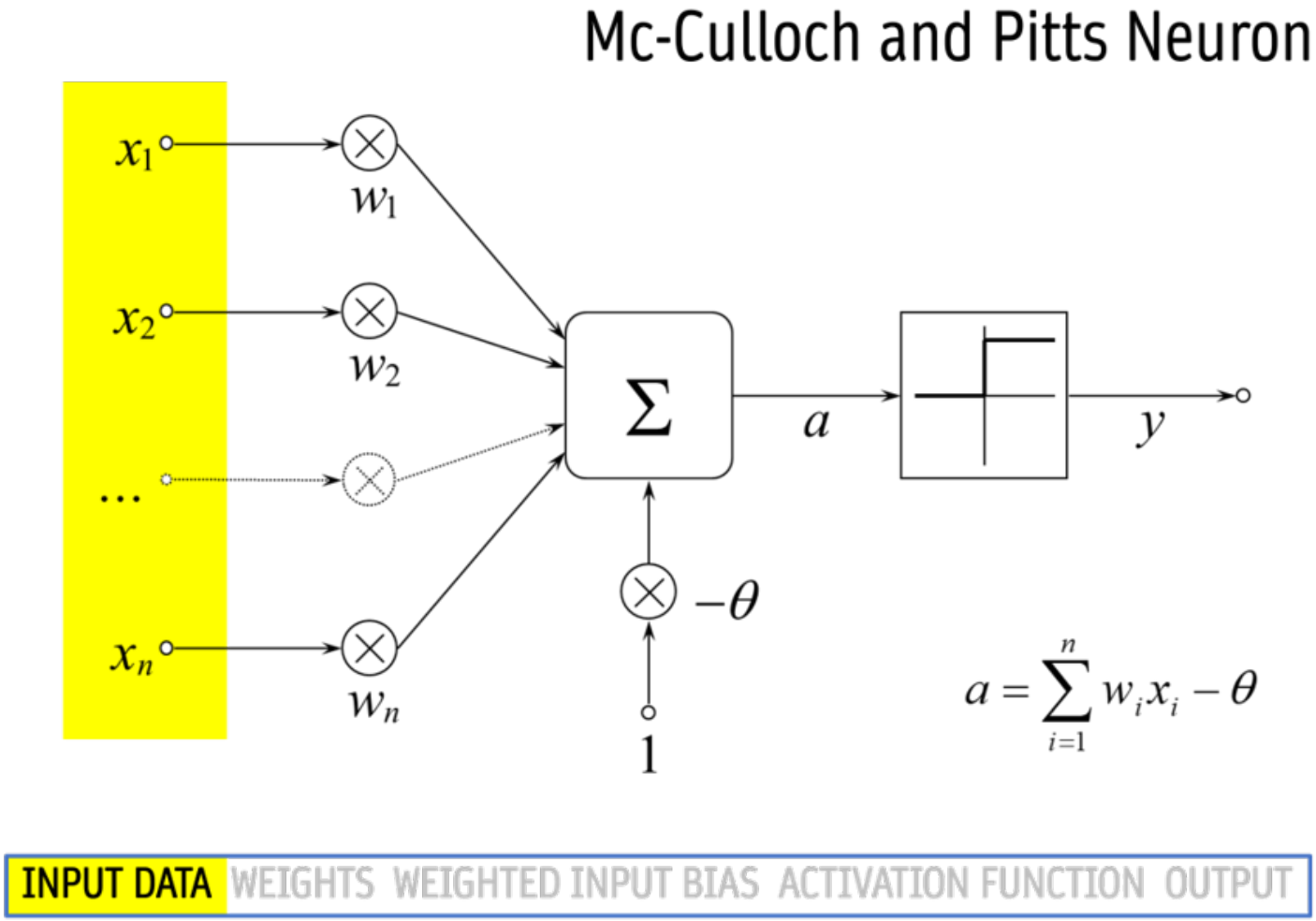
The McCulloch-Pitts Neuron

Mc-Culloch and Pitts Neuron



$$a = \sum_{i=1}^{n} w_i x_i - \theta$$

NN Data Flow


Mc-Culloch and Pitts Neuron
x_1
x_2
...
x_n
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Σ
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$-\theta$
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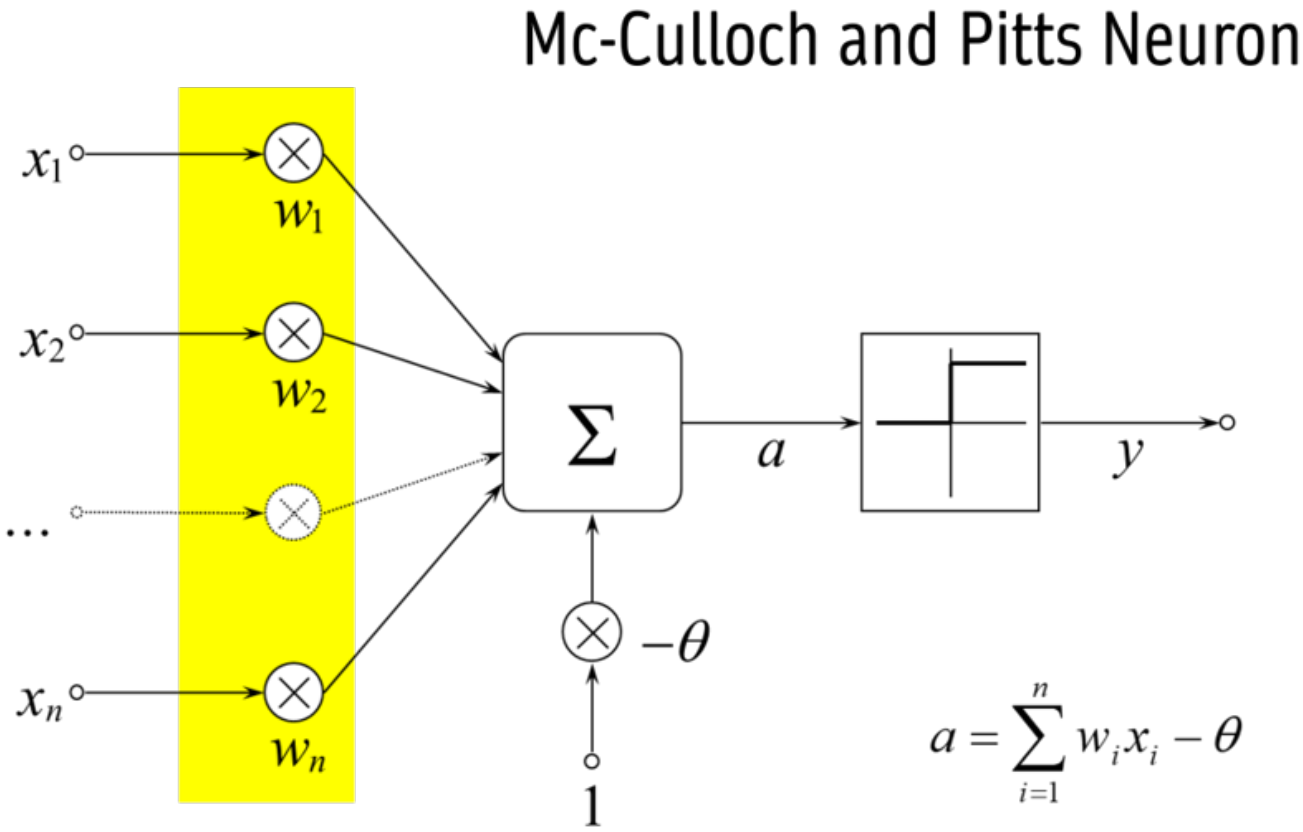
INPUT DATA
WEIGHTS
WEIGHTED INPUT
BIAS
ACTIVATION FUNCTION
OUTPUT

NN Data Flow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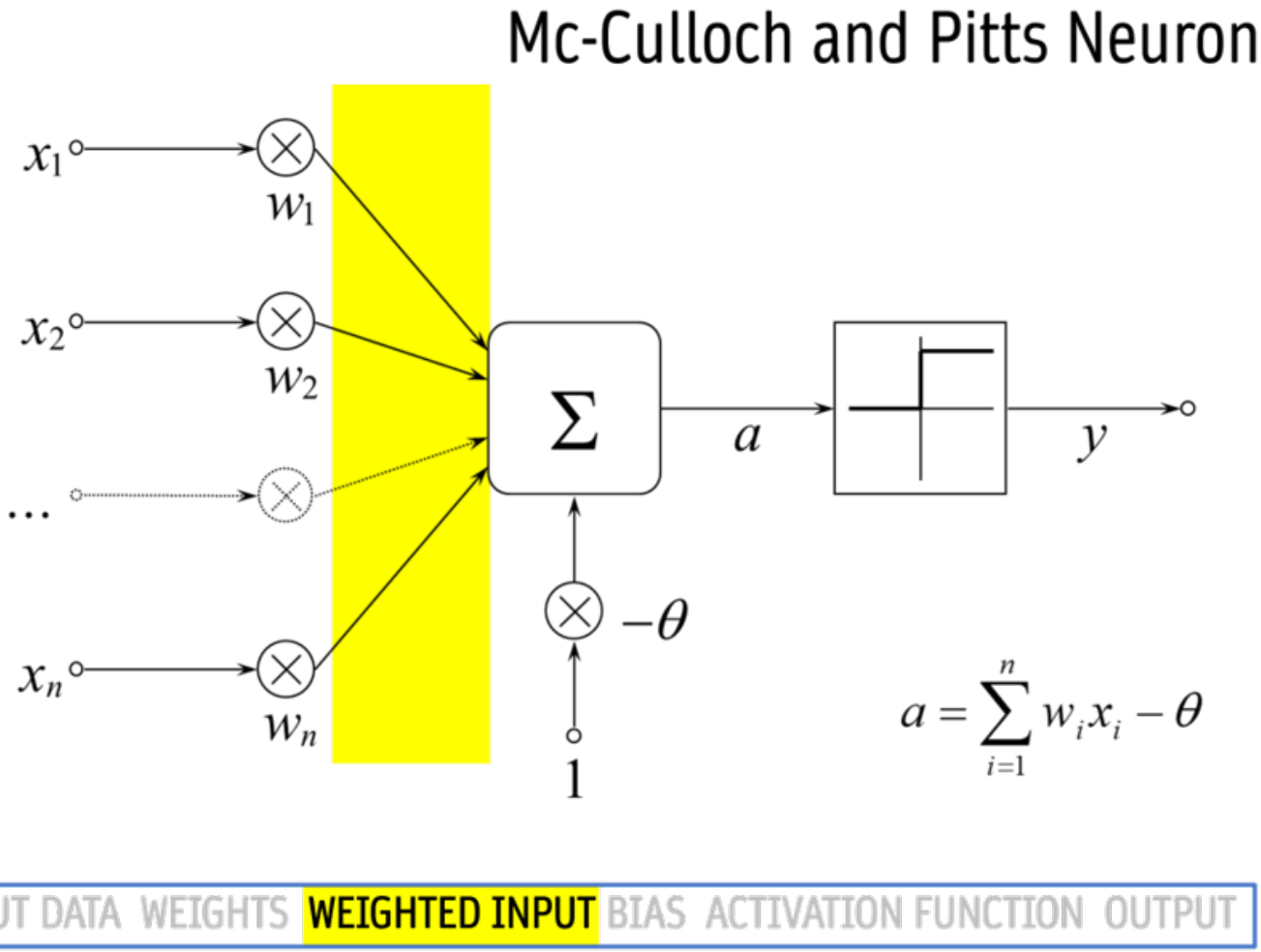
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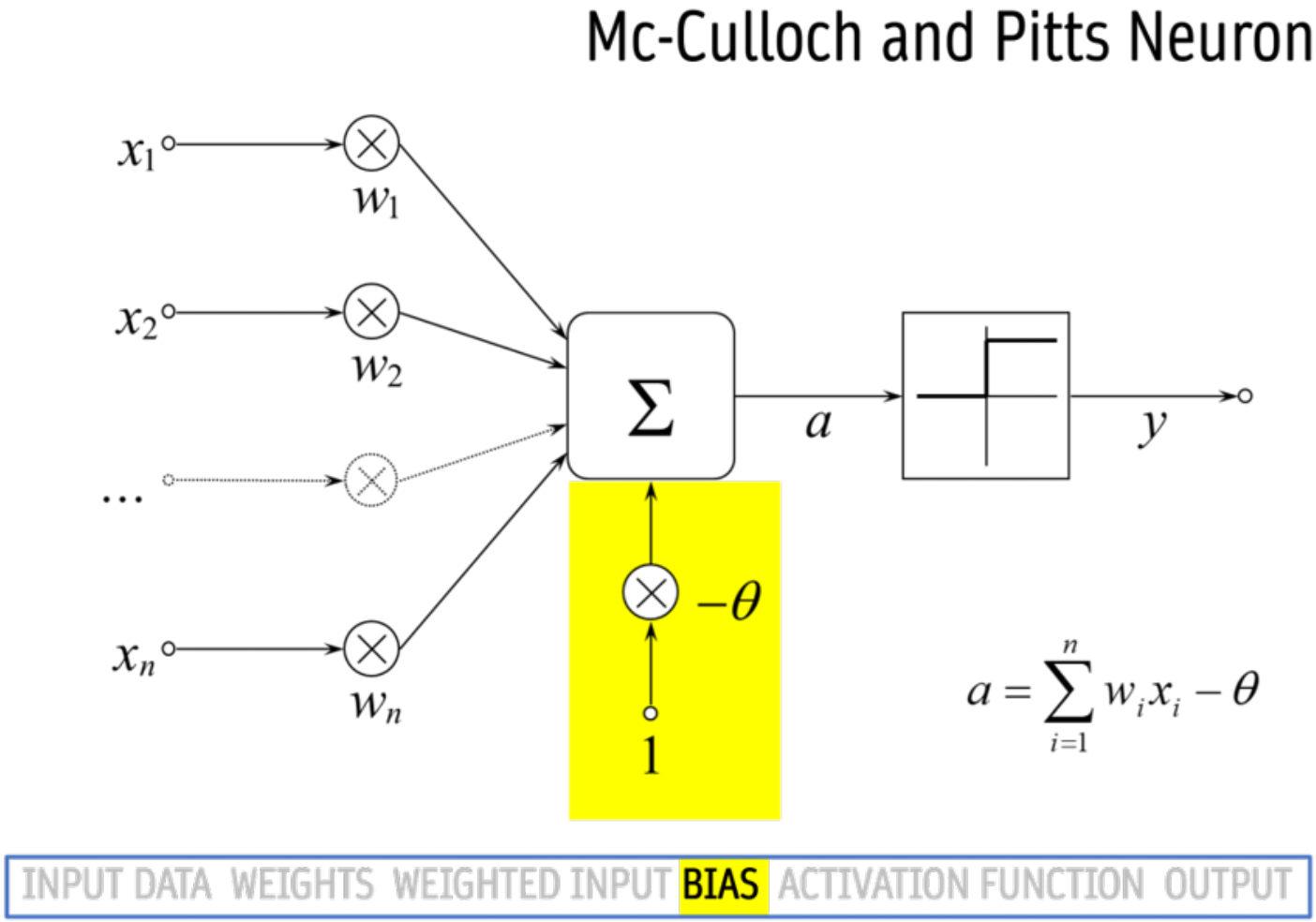
Mc-Culloch and Pitts Neuron

From a functional point of view

- an input signal formally present but associated with a zero weight is equivalent to an absence of signal;
- the threshold can be considered as an additional synapse, connected in input with a fixed weight equal to 1;

NN Data Flow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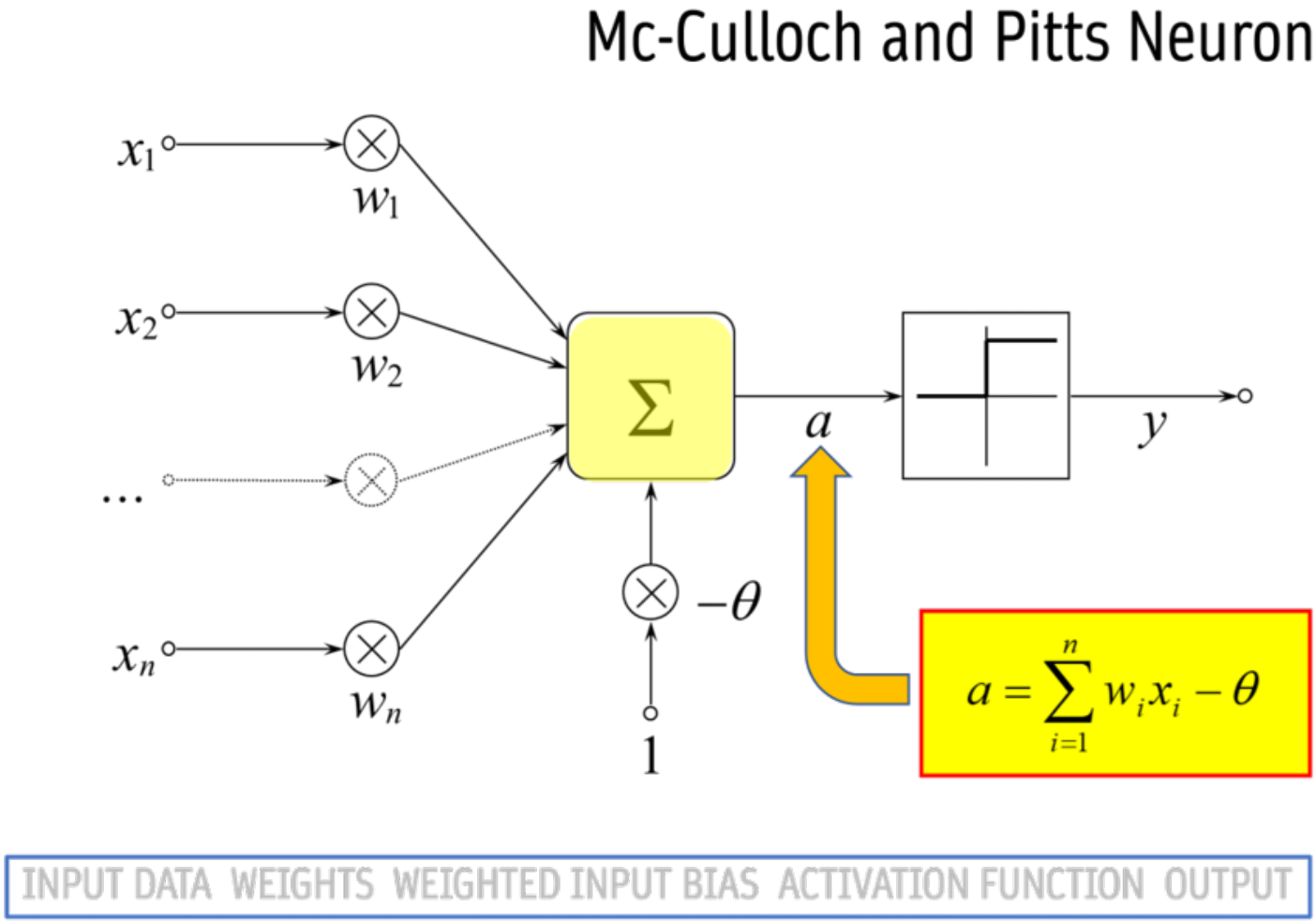
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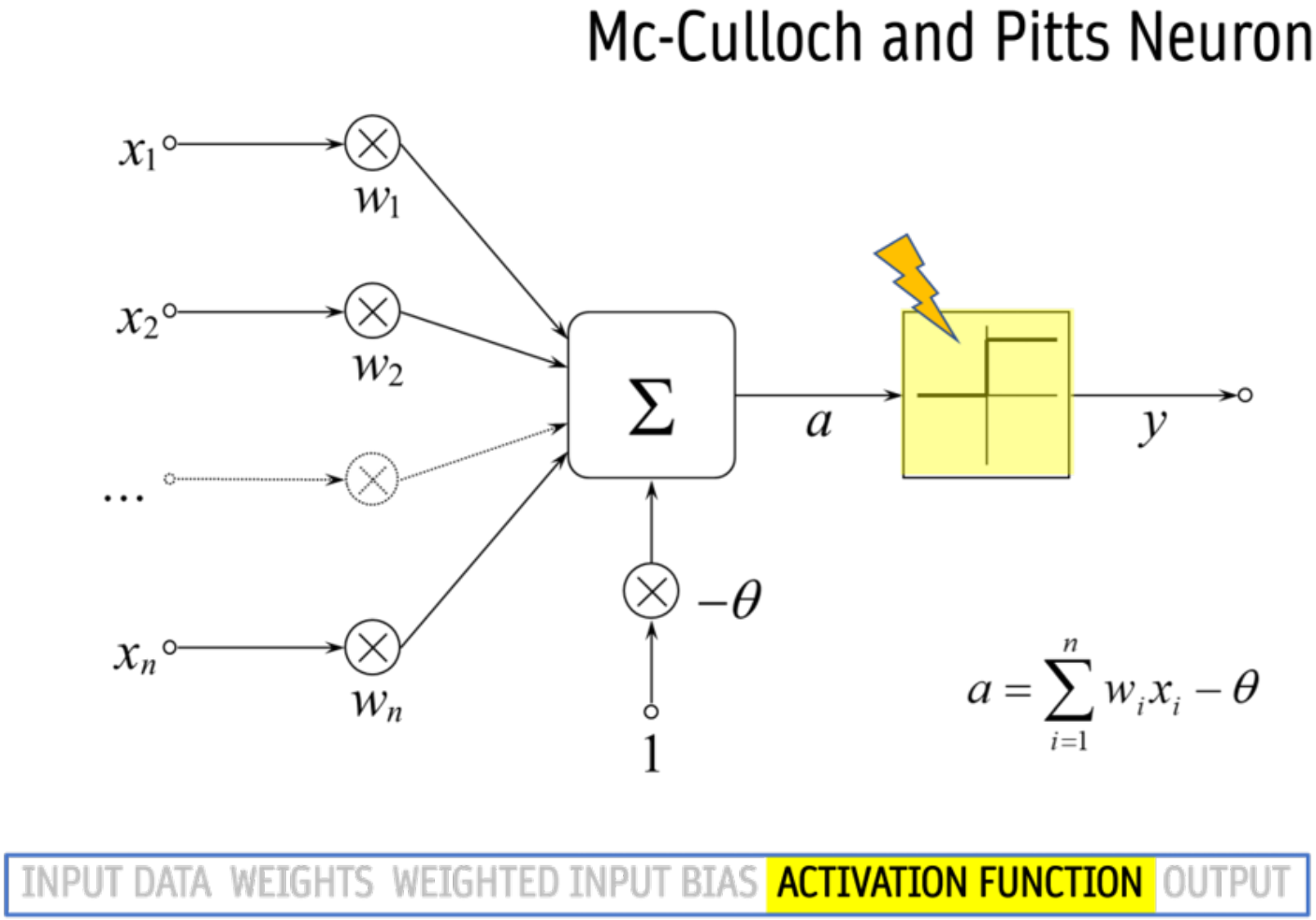
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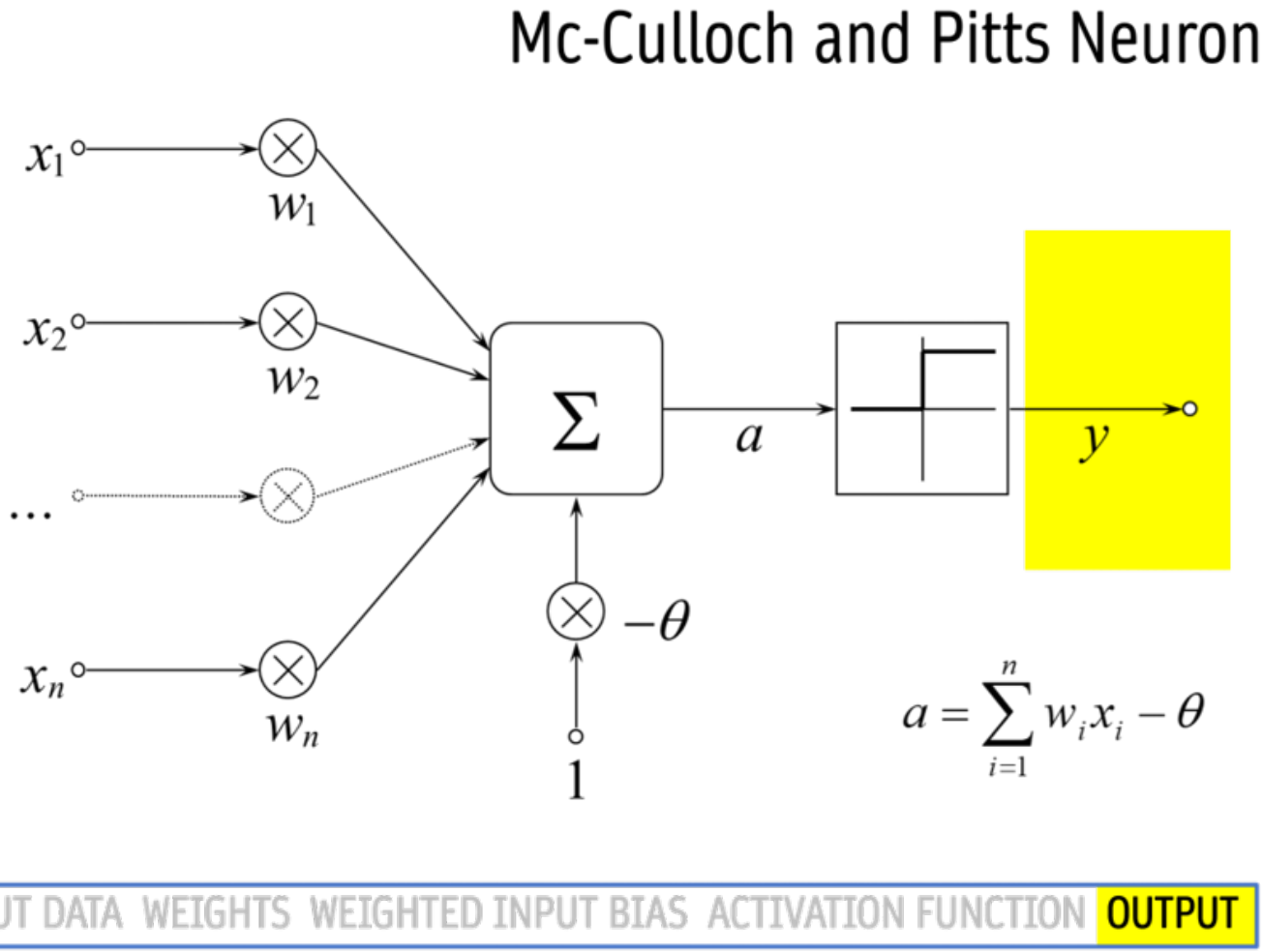
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Activation Function

$$a = \sum_{i=1}^{n} w_i \, x_i - \theta \tag{1}$$

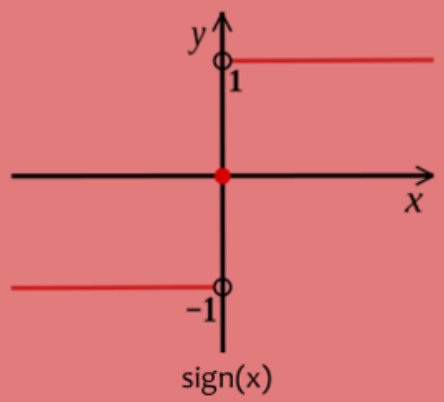
$$y = f(a) = \begin{cases} 0, & \text{if } a \leq 0 \\ 1, & \text{if } a > 0 \end{cases} \tag{2}$$

- The function f is called the response or activation function:
- in the McCulloch and Pitts neuron f is simply the step function, so the answer is binary: it is 1 if the weighted sum of the stimuli exceeds the internal threshold; 0 otherwise.
- Other models of artificial neurons predict continuous response functions

Activation Function

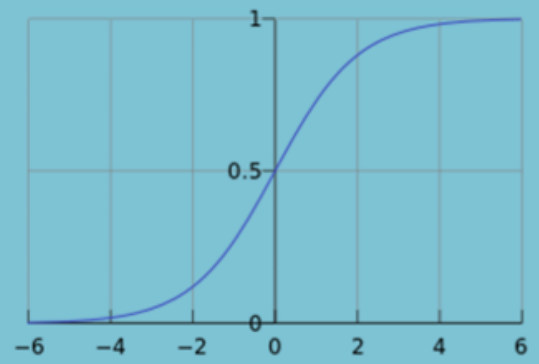
This decision function isn't differentiable:

$$h(\mathbf{x}) = \text{sign}(\boldsymbol{\theta}^T \mathbf{x})$$



Use a differentiable function instead:

$$p_{\boldsymbol{\theta}}(y = 1|\mathbf{x}) = \frac{1}{1 + \exp(-\boldsymbol{\theta}^T \mathbf{x})}$$



$$\text{logistic}(u) \equiv \frac{1}{1+e^{-u}}$$

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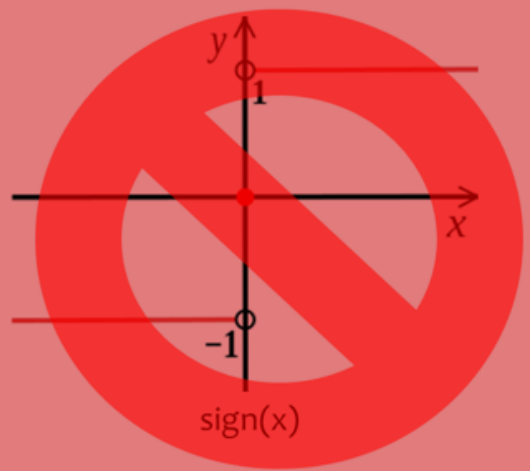

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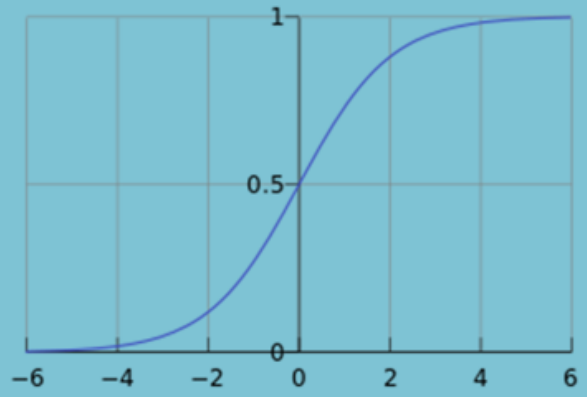


$$\text{logistic}(u) \equiv \frac{1}{1+e^{-u}}$$

Neural Network Basic Constituents

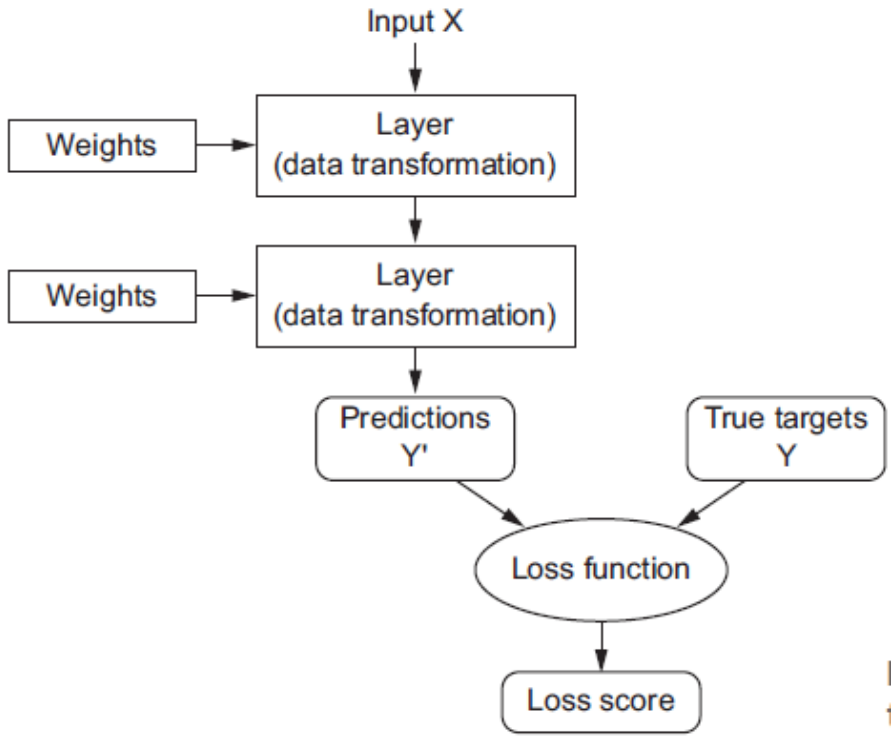
- A neural network consists of:
- A set of nodes (neurons), or units connected by links.
- A set of **weights** associated with links.
- A set of thresholds or activation levels.
- Neural network design requires:
- 1. The choice of the number and type of units.
- 2. The determination of the morphological structure.
- 3. Coding of training examples, in terms of network inputs and outputs.
- 4. Initialization and training of weights on interconnections, through the set of learning examples.

Neural Network Basic Constituents

- The specification of what a layer does to its input data is stored in the layer s weights, which in essence are a bunch of numbers.
- In technical terms, we could say that the transformation implemented by a layer is parameterized by its weights (Weights are also sometimes called the parameters of a layer.)
- In this context, **learning means finding a set of values for the weights of all layers in a network, such that the network will correctly map example inputs to their associated targets**.

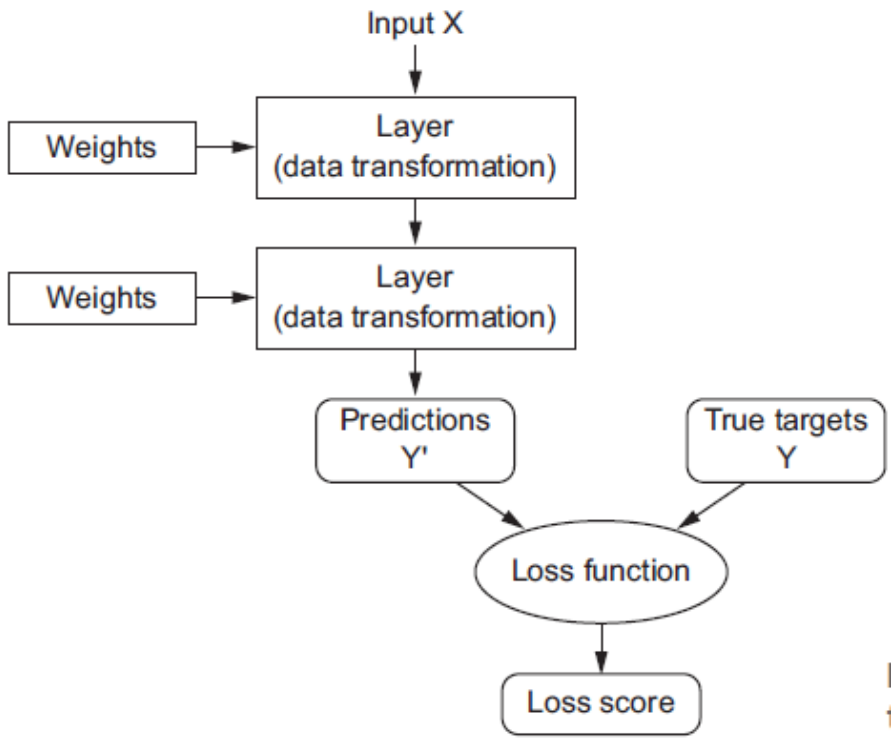
Loss Function

- To control the output of a neural network, you need to be able to measure how far this output is from what you expected.
- This is the job of the **loss function** of the network, also called the objective function.

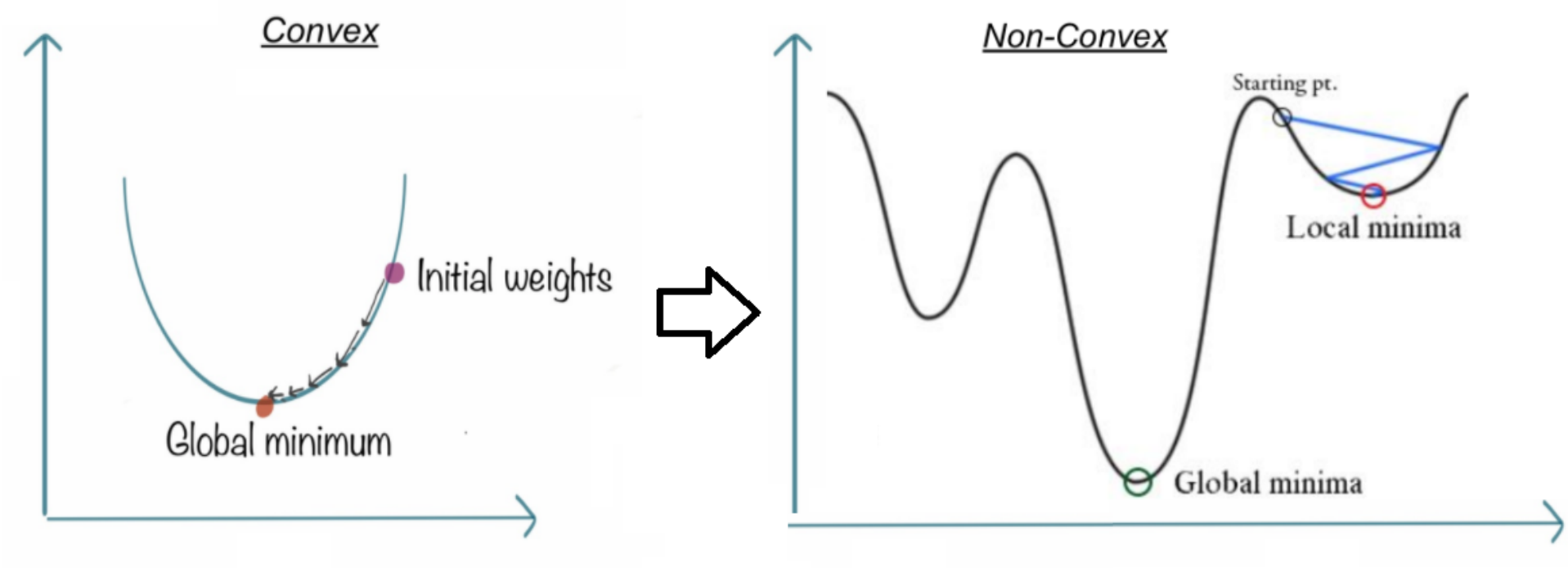


Loss Function

- The loss function takes the predictions of the network and the true target (what you wanted the network to output) and **computes a distance score, capturing how well the network has done on this specific example**

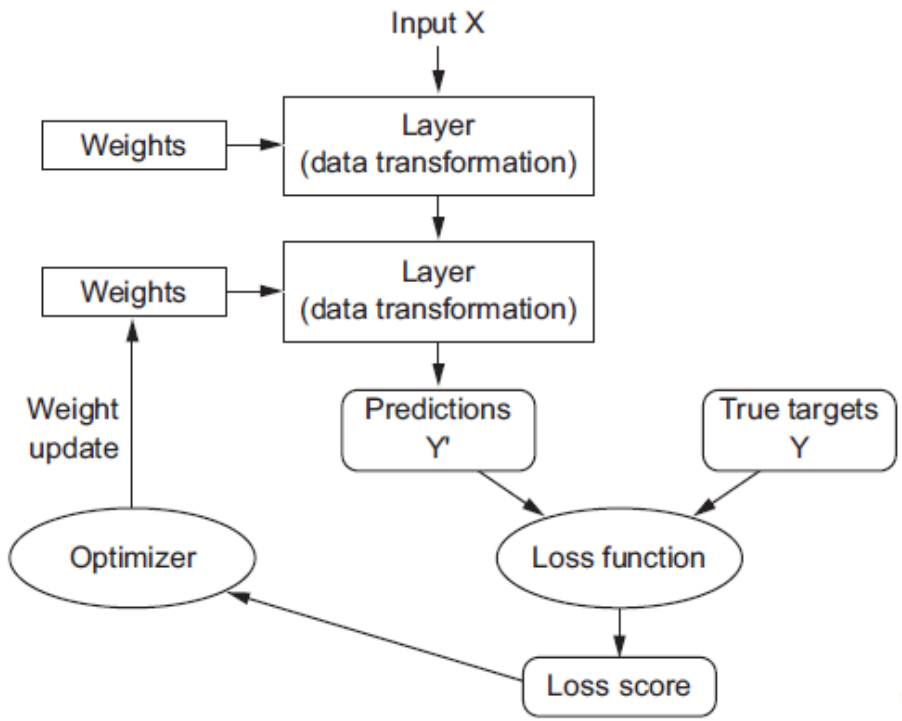


Loss Function



Backpropagation

- The fundamental trick in deep learning is to use this score as a feedback signal to adjust the value of the weights a little, in a direction that will lower the loss score for the current example.
- This adjustment is the job of the optimizer, which implements what is called the Backpropagation algorithm: the central algorithm in deep learning.

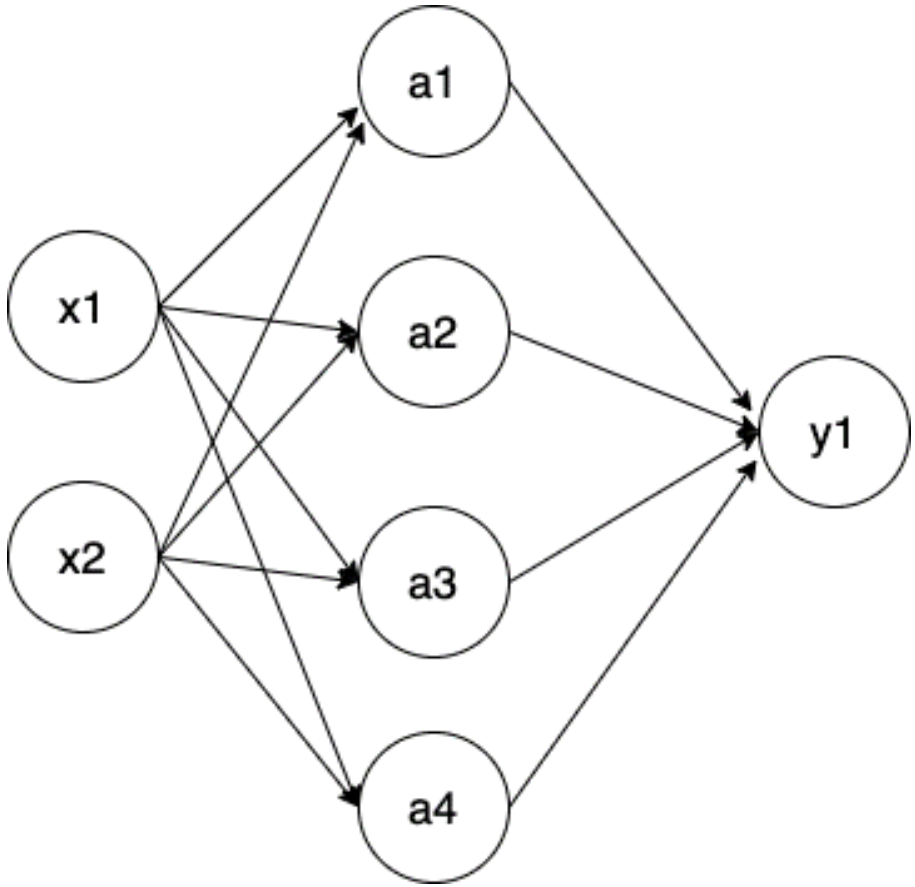


Implementing a Single Layer NN

In each hidden unit, take a_1 as example, a linear operation followed by an activation function, f, is performed. So given input $x = (x_1, x_2)$, inside node a_1, we have:

$$z_1 = w_{11}x_1 + w_{12}x_2 + b_1$$

$$a_1 = f(w_{11}x_1 + w_{12}x_2 + b_1) = f(z_1)$$



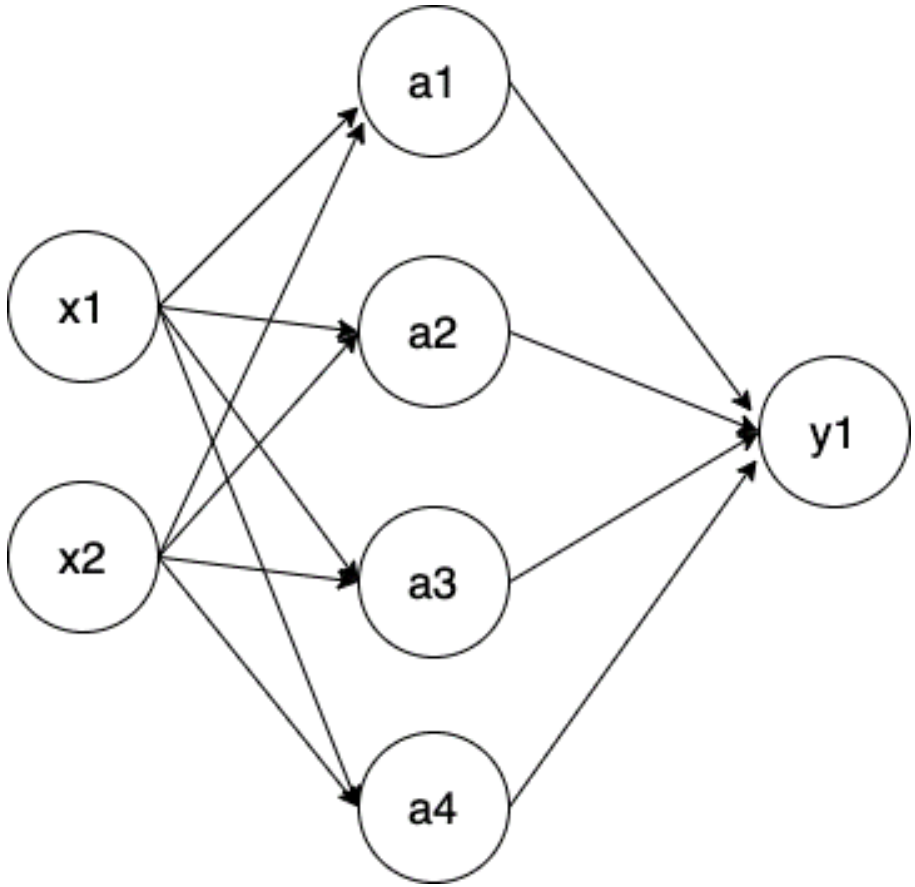
Implementing a Single Layer NN

Same for node a_2, it would have:

$$z_2 = w_{21}x_1 + w_{22}x_2 + b_2$$

$$a_2 = f(w_{21}x_1 + w_{22}x_2 + b_2) = f(z_2)$$

And same for a_3 and a_4 and so on



Implementing a single Layer NN

We can also write in a more compact form

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \\ w_{31} & w_{32} \\ w_{41} & w_{42} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \Rightarrow Z^{[1]} = W^{[1]} \cdot X + B^{[1]} \quad (3)$$

Implementing a single Layer NN

Let's assume that the first activation function is the tanh and the output activation function is the *sigmoid*. So the result of the hidden layer is:

$$A^{[1]} = \tanh Z^{[1]}$$

This result is applied to the output node which will perform another linear operation with a different set of weights, $W^{[2]}$:

$$Z^{[2]} = W^{[2]} \cdot A^{[1]} + B^{[2]}$$

and the final output will be the result of the application of the output node activation function (the sigmoid) to this value:

$$\hat{y} = \sigma(Z^{[2]})$$

Logistic Loss Function

Logistic Loss Function

$$L(y, \hat{y}) = \begin{cases} -\log \hat{y} & \text{when } y = 1 \\ -\log(1 - \hat{y}) & \text{when } y = 0 \end{cases} \tag{4}$$

$$L(y, \hat{y}) = -[y \log \hat{y} + (1 - y) \log (1 - \hat{y})]$$

Delta Rule

Given a generic actual value y, we want to minimize the loss L, and the technic we are going to apply here is gradient descent, basically what we need to do is to apply derivative to our variables and move them slightly down to the optimum. Here we have 2 variables, W and b, and for this example, the update formula of them would be:

$$W = W - \frac{\partial L}{\partial W}$$

$$b = b - \frac{\partial L}{\partial b}$$

Delta Rule

The delta rule algorithm works by computing the gradient of the loss function with respect to each weight. In order to get the derivative of our targets, chain rules would be applied:

$$\frac{\partial L}{\partial W} = \frac{\partial L}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial Z} \frac{\partial Z}{\partial W}$$

$$\frac{\partial L}{\partial b} = \frac{\partial L}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial Z} \frac{\partial Z}{\partial b}$$

Gradient Calculation

$$L(y, \hat{y}) = -[y \log \hat{y} + (1-y) \log (1-\hat{y})] \Rightarrow \frac{\partial L}{\partial \hat{y}} = -\frac{y}{\hat{y}} + \frac{1-y}{1-\hat{y}} = \frac{\hat{y}-y}{\hat{y}(1-\hat{y})}$$

$$Z = W^{[i]} \cdot x^{[i]} + B^{[i]} \Rightarrow \frac{\partial Z}{\partial W} = x \quad \frac{\partial Z}{\partial b} = 1$$

Gradient Calculation

Hidden Layer Activation Function (Hyperbolic Tangent)

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \Rightarrow \frac{d}{dx}\tanh x = 1 - (\tanh x)^2 \tag{5}$$

Output Layer Activation Function (Sigmoid Function)

$$\sigma(x) = \left[\frac{1}{1 + e^{-x}}\right] \Rightarrow \frac{d}{dx}\sigma(x) = \sigma(x) \cdot (1 - \sigma(x)) \tag{6}$$

Gradient Calculation

Output Layer

$$\frac{\partial L}{\partial \hat{y}} = \frac{\hat{y} - y}{\hat{y}(1 - \hat{y})}$$
$$\frac{\partial Z}{\partial W_O} = x$$
$$\frac{\partial \hat{y}}{\partial Z} = \frac{\partial \sigma}{\partial Z} = \sigma(Z) \cdot (1 - \sigma(Z)) = (\hat{y})(1 - \hat{y}) \tag{7}$$

So the complete gradient is:

$$\frac{\partial L}{\partial W_O} = \frac{\partial L}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial Z} \frac{\partial Z}{\partial W_O} = (\hat{y} - y) \cdot x$$
$$\frac{\partial L}{\partial b} = (\hat{y} - y) \tag{8}$$

Gradient Calculation

Hidden Layer Now we have to calculate

$$\frac{\partial Z}{\partial W_H}$$

Remember that

$$Z = W_O \cdot tanh\left(W_H \cdot X + b_H\right) + b_O$$

and

$$\frac{\partial Z}{\partial W_H} = W_O \cdot \frac{\partial\, tanh(\dots)}{\partial W_H} \cdot X = W_O \cdot \left(1 - tanh^2(\dots)\right) \cdot X$$

and finally

$$\frac{\partial L}{\partial W_H} = (\hat{y} - y) \cdot W_O \cdot X \cdot \left(1 - tanh^2(\dots)\right) \tag{9}$$

Weights Update

Defining

$$\Delta^{[2]} = A^{[2]} - Y \tag{10}$$

$$\Delta^{[1]} = \Delta^{[2]} \cdot W^{[2]T} \cdot (1 - A^{[1]^2}) \tag{11}$$

We have

Output Layer

$$dW^{[2]} = \frac{1}{m}\left[A^{[2]} - Y\right] A^{[1]^T} = \frac{1}{m}\Delta^{[2]} A^{[1]^T} \tag{12}$$

$$db^{[2]} = \frac{1}{m} np.sum(dZ^{[2]}, axis = 1, keepdims = True) \tag{13}$$

Weights Update

Hidden Layer

$$
\begin{aligned}
dW^{[1]} &= \frac{1}{m}\left[A^{[2]} - Y\right] \cdot X^T \cdot W^{[2]T} \cdot (1 - A^{[1]^2}) \\
&= \frac{1}{m}\Delta^{[2]} \cdot W^{[2]T} \cdot (1 - A^{[1]^2}) \cdot X^T \\
&= \frac{1}{m}\Delta^{[1]} \cdot X^T
\end{aligned} \tag{14}
$$

$$
db^{[1]} = \frac{1}{m} np.sum(dZ^{[1]}, axis = 1, keepdims = True) \tag{15}
$$

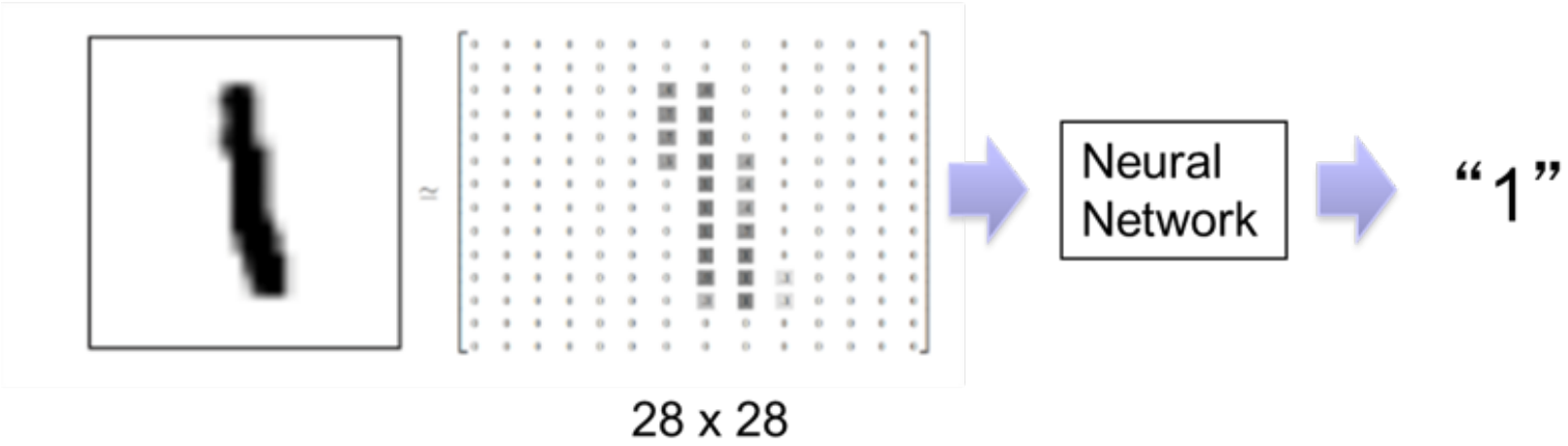
Example

- Using **03-introduction-to-deep-learning** Notebook
- Par. 3.3

Subsection 3

Keras

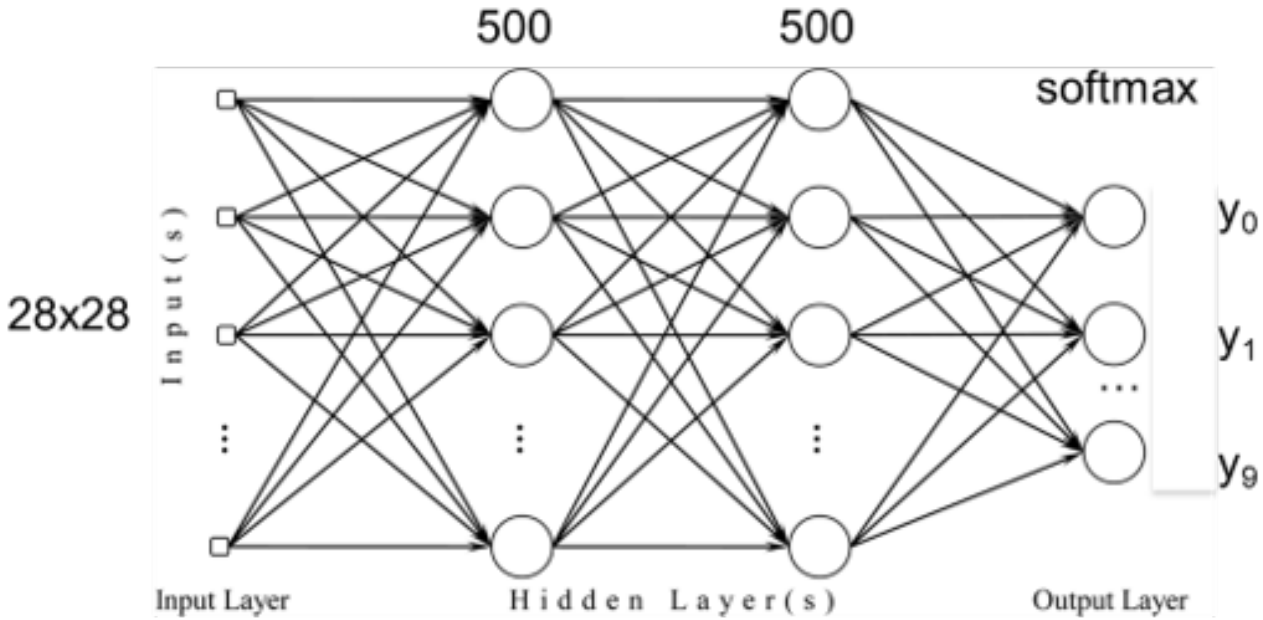
Introduction to keras



- 0-9 handwritten digit recognition:
- MNIST Data maintained by Yann LeCun: http://yann.lecun.com/exdb/mnist/
- Keras provides data sets loading function at http://keras.io/datasets

Implementing in Keras

Fully Connected NN




```
model = sequential()    # layers are sequentially added
model.add( Dense(input_dim=28*28, output_dim=500))
model.add(Activation('sigmoid'))  #: softplus, softsign,relu,tanh, hard_sigmoid
model.add(Dense( output_dim = 500))
model.add (Activation('sigmoid'))
Model.add(Dense(output_dim=10))
Model.add(Activation('softmax'))
model.compile(loss='categorical_crossentropy', optimizer='adam', metrics=['accuracy'])
model.fit(x_train, y_train, batch_size=100, nb_epoch=20)
```

Implementing in Keras

Training