Chapter 1 Module

§ 1.1 Definition and Propeties of Module

Assume R to be a unital commutative ring with unit 1_R (or simply 1). We also assume $0_R \neq 1_R$ to avoid some bad propety of the ring R.

<u>Def 1.1</u> Let $(R,+,\cdot)$ be a ring, $(M,+)$ be an Abelian group, if there is a map $R\times M \to M$, $(a,m)\mapsto a\cdot m$ such that:

(1) $a(x+y) = ax + ay$

(2) $(a+b)x = ax + bx$

(3) $(ab)x = a(bx)$

(4) $1x = x$

then M is called a left R module.

<u>Remark.</u> The right module can be defined similaly. When R is a commutative ring, a left R-module is also a right R-module. (Def. 1.2)

<u>Example</u> 1.1 Vector spaces are modules over fields.

<u>Example</u> 1.2 Any Abelian group M is a $\mathbb{Z}$-module with action given by

$$m\cdot x := \underbrace{x+\cdots+x}_{m},\ 0_{\mathbb{Z}}\cdot x := 0_M,\ (-1)\cdot x = -x.$$

<u>Example</u> 1.3 Let V be a vector space over field $\mathbb{F}$ and T be a linear operator $T: V\to V$. Consider polynomial ring $\mathbb{F}[\lambda]$, we have $\mathbb{F}[\lambda]\cong\mathbb{F}[T]$, via the action of T on V, we obtain a $\mathbb{F}[\lambda]$ module structure over V,

$$\mathbb{F}[\lambda]\times V \to V,$$
$$(p(\lambda),\ x) \mapsto p(T)\cdot x.$$

<u>Example</u> 1.4 Ring R is a module over itself. If S is a subring of R, then R is a S-module, but S is not necessarilly a R-module. When S is a ideal of R (all ideals are subrings), S is an R-module. Notice R is a subring of $R[x_1,\cdots,x_n]$, thus $R[x_1,\cdots,x_n]$ is an R-module. Similarly, the ring of formal power series $R[[x]]$ is an R-module.

<u>Example</u> 1.5 Let R,S be ring and $\psi: R\to S$ be a ring homomorphism, M be a S-module, then M is an R-module with action $a\cdot x := \psi(a)\cdot x$.

<u>Claim</u> Let M be an R-module, we have:

(1) $a\cdot 0 = 0$ (1') $0_R x = 0_M$

(2) $a\cdot(-x) = -(ax)$ (2') $(-a)\cdot x = -(a\cdot x)$

(3) $a\cdot \Sigma_i x_i = \Sigma_i a\cdot x_i$ (3') $(\Sigma_i a_i)\cdot x = \Sigma_i (a_i\cdot x)$

Proof. (1) $a\cdot x = a\cdot(x+0) = a\cdot x + a\cdot 0 \Rightarrow a\cdot 0 = 0$.

(1') $a\cdot x = (a+0)\cdot x = a\cdot x + 0\cdot x \Rightarrow 0\cdot x = 0$

(2) $a\cdot(-x) + a\cdot x = a\cdot 0 = 0 \Rightarrow a\cdot(-x) = -a\cdot x$

(2') $(-a)\cdot x + a\cdot x = [(-a)+a]\cdot x = 0 \Rightarrow (-a)\cdot x = -a\cdot x$

(3) and (3') are obvious.

Mainly use the cancellation property of group.

Example 1.6 Zero module $M=\{0\}$.

Def. 1.3 Let M be an R-module and $N\neq\emptyset$ a subset of M. N is called a submodule of M if

(1) N is a subgroup of M;

(2) $ax\in N$, $\forall a\in R$, $\forall x\in N$.

Prop 1.1 The non-empty subset $N\subseteq M$ is a submodule iff

(1) $\forall y_1, y_2\in N$, $y_1+y_2\in N$

(2) $\forall a\in R$, $\forall x\in N$, $a\cdot x\in N$

Proof. "$\Rightarrow$" is obvious

"$\Leftarrow$": Choose $a=0$ in (2), and $y\in N$, based on claim above, we obtain $0\cdot y = 0 \in N$.

Similarly, choose $a=-1$ and $y\in N$, $(-1)\cdot y = -y\in N$. Thus N is a subgroup.

Example 1.7 Let M be a $\mathbb{Z}$-module, N is a submodule iff N is an addtive subgroup.

Example 1.8 Let V be a vector space over $\mathbb{F}$, W is a submodule iff W is a subspace.

Example 1.9 Let V be a vector space over field $\mathbb{F}$, T be a linear map $T: V\to V$. Regard V as a $\mathbb{F}[T]$ module, a submodule of V is an invariant subspace.

Proof. "$\Rightarrow$" N is a submodule of V, show N is invariant subspace.

N is a subgroup, since $\mathbb{F}\subseteq \mathbb{F}[T]$, N is closed under the action of $\mathbb{F}$, thus N is a subspace.

Since $T\in\mathbb{F}[T]$, N should be closed under the action of T.

"$\Leftarrow$" N is an invariant subspace of V, show that N is a submodule.

N is a subgroup of V. N is invariant under the action of T, thus it is also invariant under the action of T^m $(m=1,2,3,\cdots)$, this implies N is invariant under the action of $p(T)\in\mathbb{F}[T]$.

Example 1.10 Regard ring R as R-module, N is a submodule iff N is an ideal.

This is clear from definition.

Example 1.11 Let $\{N_i \mid i \in I\}$ be a family of submodules, then $\bigcap_{i\in I} N_i$ is a submodule.

Example 1.12 Zero submodule $\{0\}$.

Example 1.13 For ring R and R module M, let $x \in M$, we define annihilator of x as

$$\mathrm{Ann}_R(x) := \{a \in R \mid a \cdot x = 0\}.$$

(1) $\mathrm{Ann}_R(x)$ is an ideal of R

Proof. Step 1. Show $\mathrm{Ann}_R(x)$ is a subgroup.

Suppose $a, b \in \mathrm{Ann}_R(x)$, $a \cdot x = b \cdot x = 0$, then $(a-b)\cdot x = 0$. This implies $a-b \in \mathrm{Ann}_R(x)$, $\mathrm{Ann}_R(x)$ is a subgroup.

Step 2. Show $R \cdot \mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R(x)$.

$\forall r \in R$ and $a \in \mathrm{Ann}_R(x)$, $(r \cdot a)\cdot x = r \cdot (a \cdot x) = r \cdot 0 = 0$, $\Rightarrow r \cdot a \in \mathrm{Ann}_R(x)$.

(2) If $\mathrm{Ann}_R(x) \neq 0$, x is called a torsion element.

(3) If R is an integral domain (commutative ring such that, $\forall a, b \in R$, $a \cdot b = 0$ implies $a = 0$ or $b = 0$), the set of all torsion elements $T(M)$ is a submodule, called torsion submodule. If $M = T(M)$, M is called a torsion module. M is called torsion free if $T(M) = 0$.

Proof. Step 1. Show $T(M)$ is a subgroup.

If $x \in T(M)$, $\exists\, a \in R$ $a \neq 0$ s.t. $a \cdot x = 0$, this implies that $a \cdot (-x) = -a \cdot x = 0 \Rightarrow -x \in T(M)$. $0 \in T(M)$ is obvious.

If $x, y \in T(M)$, there exist nonzero $a, b \in R$ such that $a \cdot x = b \cdot y = 0$. Set $r = a \cdot b$, we have $r \cdot x = r \cdot y = 0$, where $r \neq 0$ since $a, b \neq 0$. (Property of integral domain)

Step 2. Show $\forall a \in R$, $\forall x \in T(M)$, $a \cdot x \in T(M)$.

Since $x \in T(M)$, $\exists\, b \in R$, $b \neq 0$ s.t. $b \cdot x = 0$. This implies that $b \cdot (ax) = a \cdot (bx) = a \cdot 0 = 0 \Rightarrow a \cdot x \in T(M)$.

(4) If G is finite group, when regarded as $\mathbb{Z}$-module, we have $T(G) = G$.

Proof. Since all elements in finite group are of finite order.

(5) For vector space V, when regarded as $\mathbb{F}[K]$-module for some linear map $K: V \to V$, we have

$$T(V) = V.$$

Proof. Suppose $\dim V = d$, for $v \in V$, if $v \in \operatorname{Ker} K$, $Tv = 0 \Rightarrow v \in T(V)$.

If $v \notin V$, $v, Tv, \cdots, T^d v, T^{d+1} v$ must be linear dependent, there exist nonzero $(\alpha_0, \alpha_1, \cdots, \alpha_{d+1})$ such that $\sum_i \alpha_i T^i v = 0$, meaning $p(T) = \sum_i \alpha_i T^i \neq 0$ and $p(T) v = 0$. Thus $v \in T(V)$.

<u>Def</u>. 1.4 If nonzero R module M only have submodules $\{0\}$ and M, M is called a simple module, or irreducible module.

Let M be an R module, $S \subseteq R$ be a subset of R, $X \subseteq M$ a subset of M, then we define S-linear combinations of X as

$$SX = \{ \textstyle\sum_{i=1}^n s_i x_i,\ s_i \in S,\ x_i \in M \}$$

<u>Prop</u> 1.2 Let $\phi \neq X \subseteq M$ be a subset of R module M, then RX is submodule of M, called submodule generated by X and denoted as (X).

<u>Prop</u> 1.3 For $X \subseteq M$, $RX = (X) = \bigcap_{X \subseteq N,\, N \text{ submodule}} N$

Proof. Step 1 $RX \subseteq \bigcap_{X \subseteq N,\, N \text{ submodule}} N$, since $X \subseteq N$, $RX \subseteq N$.

Step 2 $\bigcap_{X \subseteq N,\, N \text{ submodule}} N \subseteq RX$, since RX is a submodule and $X \subseteq RX$.

<u>Def</u> 1.5 If $M = RX$, X is called the set of generators of M. If X is finite, and $RX = M$, M is called finitely generated. If $M = (x)$, M is called a cyclic module.

<u>Def</u> 1.6 Let $\{N_i \mid i \in I\}$ be a family of submodules of M, we define

$$\sum_{i \in I} N_i := (\bigcup_{i \in I} N_i) = \{ y_{i_1} + \cdots + y_{i_k} \mid y_{i_j} \in N_{i_j},\ i_j \in I \}.$$

<u>Def</u> 1.7 Let K be a submodule of M, consider the coset

$$M/K = \{ x + K \mid x \in M \}.$$

The addition is defined as $(x+K) + (y+K) := x+y+K$, the scalar product is defined as $a(x+K) = ax + K$. Then M/K is a module called quotient module.