

# Weak Hopf symmetries behind $2d$ topological phases

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# Overview

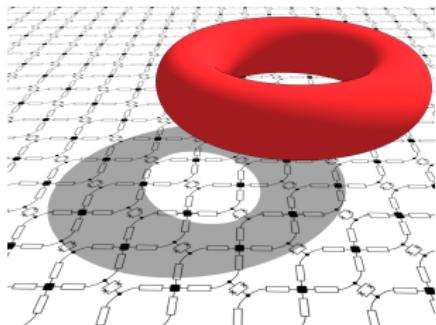
- 1 Topological phase and unitary modular tensor category
  - Physical perspective
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- 2 Lattice model of 2d topological order
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  - (Weak) Hopf quantum double model
  - Multifusion string-net model
- 3 Weak Hopf tube algebra
  - Bulk tube algebra
  - Boundary tube algebra
  - Morita theory
- 4 Open problems

# Topologically ordered phases

Anyon model: physical perspective

Anyon model:

- Topological charges;
- Fusion/splitting rule;
- Mutual statistics;
- Topological spin.



ENTANGLEMENT PATTERN!

Physics of topologically ordered phase (TOP):

- Topological order (compared with local order), phase transition beyond the Ginzburg–Landau spontaneous symmetry breaking (SSB) theory.
- Topologically protected ground state degeneracy;
- Topological entanglement entropy;
- Fractional charges/statistics;
- Boundary physics (edge states) and boundary-bulk duality;
- No TOP in 1d! Rich in 2d; higher dimensional case (not fully understood!).

# Topologically ordered phases

Anyon model: physical perspective

Symmetry enriched topological (SET) phase and symmetry protected topological (SPT) phase.

	No TO	TO
No Sym	Trivial	TO
Sym	SPT	SET

SET are richer and need a more complicated mathematical characterization

- SPT: Haldane chain, topological insulator.
- SET: FQHE.
- Exist even in  $1d$ .
- Lattice models are hard to construct.

# Topologically ordered phases

Anyon model: mathematical perspective

## The mathematical theory of anyons (2d)

- Topological order (TO) is characterized by a unitary modular tensor category (UMTC).
- SPT and SET are characterized by (i) modular extension of UMTC or (ii) G-crossed modular tensor category.
- Anyon condensation theory.
- Boundary-bulk duality.
- Some notions: gapped/gapless, chiral/non-chiral, anomalous/anomaly-free.

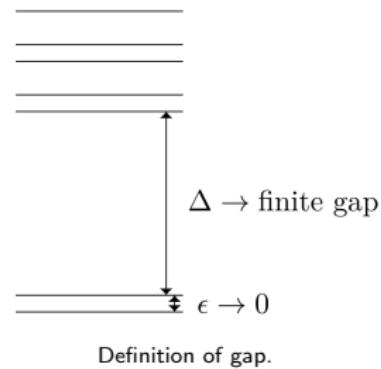
OUR MAIN FOCUS OF THIS TALK IS THE GAPPED ANOMALY-FREE  
NON-CHIRAL TOPOLOGICAL ORDER (WITHOUT SYMMETRY)!

# Lattice model of 2d topological order

## Big picture

A rigorous definition of topological order at the Hamiltonian level?

- Gapless case: far from reaching!
- Gapped case: adiabatic path  $H(\lambda)$  without closing the energy gap,  
 $H(0) \simeq H(1)$ .



Definition of gap.

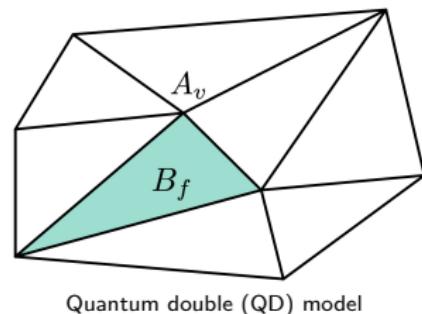
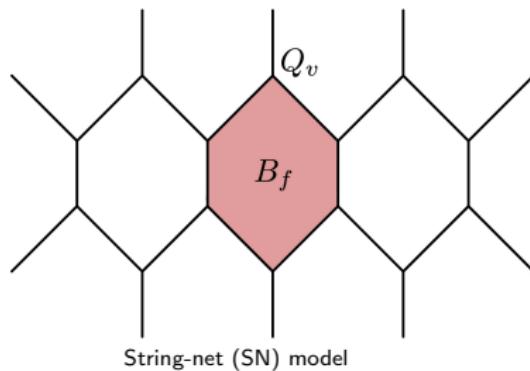
## Hamiltonian theory of topological phase

- Topologically ordered state: Long-range entangled states.
- Topological order: a class of (local) gapped Hamiltonians  $(H, \mathcal{H})$  which realize a TQFT.

# Lattice model of 2d topological order

## Big picture

Relation between Kitaev quantum double model and Levin-Wen string-net model :



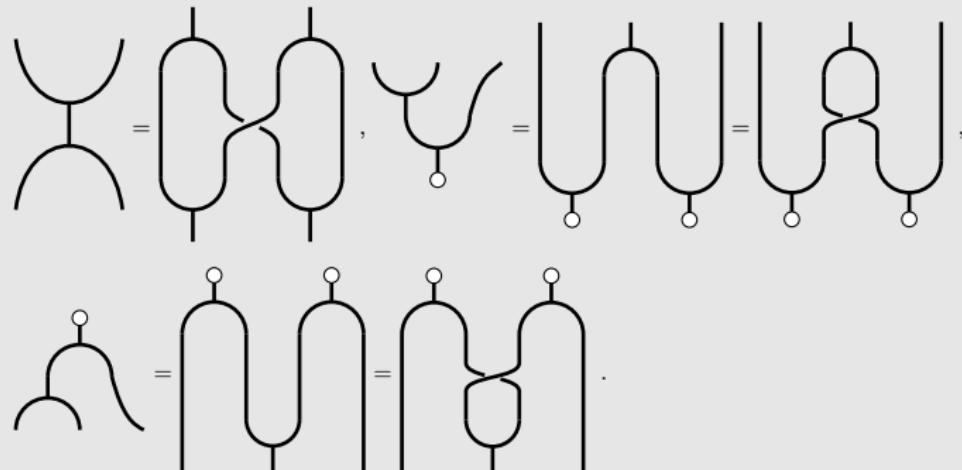
- SN with input UFC  $\Leftrightarrow$  QD with input a connected WHA.
- Multifusion SN  $\Leftrightarrow$  QD with input a general WHA.
- Morita equivalence.
- Duality between different models. (LU equivalence!)
- EM duality.

# Lattice model of 2d topological order

Weak Hopf quantum double model

## Weak bialgebra (WBA)

In a braided fusion category  $\mathcal{C}$ , a WBA is an object  $W \in \mathcal{C}$  that equipped with the following structure (i) Algebra  $(W, \mu, \eta)$  (ii) Coalgebra  $(W, \mu, \varepsilon)$ , such that



# Lattice model of 2d topological order

Weak Hopf quantum double model

## Weak Hopf algebra (WHA)

A WHA in  $\mathcal{C}$  is a WBA equipped with an antipode  $S : W \rightarrow W$  such that

$$\text{Diagram: } S \circlearrowleft = \text{Diagram: } S \circlearrowright , \quad S \circlearrowright = \text{Diagram: } S \circlearrowleft , \quad S \circlearrowuparrow = \text{Diagram: } S \circlearrowleft \text{ and } S \circlearrowright .$$

The diagrams consist of vertical lines with horizontal loops attached. In the first diagram, a box labeled  $S$  is on the left, followed by a crossing where the top line goes over the bottom line. In the second diagram, a box labeled  $S$  is on the right, preceded by a crossing where the top line goes under the bottom line. In the third diagram, a box labeled  $S$  is on the left, followed by a crossing, then another box labeled  $S$ .

Take  $\mathcal{C} = \text{Vect}_{\mathbb{C}}$ , we obtain the complex WHA.

- $\varepsilon_L(h) = (\varepsilon \otimes \text{id})(\Delta(1_W)(h \otimes 1_W)) = \sum_{(1_W)} \varepsilon(1_W^{(1)} h) 1_W^{(2)}$  and we denote  $W_L = \varepsilon_L(W)$ ;
- $\varepsilon_R(h) = (\text{id} \otimes \varepsilon)((1_W \otimes h)\Delta(1_W)) = \sum_{(1_W)} 1_W^{(1)} \varepsilon(h 1_W^{(2)})$  and we denote  $W_R = \varepsilon_R(W)$ .

# Lattice model of 2d topological order

## Weak Hopf quantum double model

- The linear span  $J$  of the elements

$$\begin{aligned}\varphi \otimes xh - \varphi(x \rightharpoonup \varepsilon) \otimes h, & \quad x \in W_L, \\ \varphi \otimes yh - \varphi(\varepsilon \leftharpoonup y) \otimes h, & \quad y \in W_R,\end{aligned}$$

is a two-sided ideal of  $\hat{W}^{\text{COP}} \otimes W$ . We denote the quotient algebra  $(\hat{W}^{\text{COP}} \otimes W)/J$  as  $D(W)$  and equivalent classes in  $D(W)$  as  $[\varphi \otimes h]$  for  $\varphi \otimes h \in \hat{W}^{\text{COP}} \otimes W$ .

- $J$  is used to ensure the operations below are well-defined and satisfy the axioms of WHA.

### Quantum double of WHA

For a WHA  $W \in \text{Vect}_{\mathbb{C}}$ , its quantum double is defined as the space  $D(W) = (\hat{W} \otimes W)/J$  equipped with the following weak Hopf algebra structure:

- (1) The multiplication  $[\varphi \otimes h][\psi \otimes g] = \sum_{(\psi), (h)} [\varphi \psi^{(2)} \otimes h^{(2)} g] \langle \psi^{(1)}, S^{-1}(h^{(3)}) \rangle \langle \psi^{(3)}, h^{(1)} \rangle$ .
- (2) The unit  $[\varepsilon \otimes 1_W]$ .
- (3) The comultiplication  $\Delta([\varphi \otimes h]) = \sum_{(\varphi), (h)} [\varphi^{(2)} \otimes h^{(1)}] \otimes [\varphi^{(1)} \otimes h^{(2)}]$ .
- (4) The counit  $\varepsilon([\varphi \otimes h]) = \langle \varphi, \varepsilon_R(S^{-1}(h)) \rangle$ .
- (5) The antipode  $S([\varphi \otimes h]) = \sum_{(\varphi), (h)} [\hat{S}^{-1}(\varphi^{(2)}) \otimes S(h^{(2)})] \langle \varphi^{(1)}, h^{(3)} \rangle \langle \varphi^{(3)}, S^{-1}(h^{(1)}) \rangle$ .

The quantum double has a canonical quasitriangular structure, ensuring that the representation category of  $D(W)$  is braided.

# Lattice model of 2d topological order

## Weak Hopf quantum double model

Quantum double model is defined for finite group  $G$ :

$$A_v |x_3 \xrightarrow{x_2} x_1\rangle = \frac{1}{|G|} \sum_{g \in G} |gx_3 \xrightarrow{gx_2 g^{-1}} x_1 g^{-1}\rangle.$$

$$B_f |x_3 \boxed{x_2 \\ x_4} x_1\rangle = \delta_{x_1^{-1}x_2x_3x_4^{-1}, e} |x_3 \boxed{x_2 \\ x_4} x_1\rangle.$$

### Haar integral

A left (resp. right) integral of  $W$  is an element  $l$  (resp.  $r$ ) satisfying  $xl = \varepsilon_L(x)l$  (resp.  $rx = r\varepsilon_R(x)$ ). A left (resp. right) integral  $l$  (resp.  $r$ ) is called left (resp. right) normalized if  $\varepsilon_L(l) = 1_W$  (resp.  $\varepsilon_R(r) = 1_W$ ). If  $h$  is both a left and right integral, it is called a two-side integral. A Haar integral in  $W$  (or Haar measure on  $\hat{W}$ ) is a two-side normalized two-side integral.

For weak Hopf quantum double model  $D(W)$ :

- $W$ -action:  $L_+^g |z\rangle = |gz\rangle$ ,  $L_-^g |z\rangle = |zS^{-1}(g)\rangle$ .
- $\hat{W}$ -action:
 
$$T_+^\varphi |x\rangle = |\varphi \rightharpoonup x\rangle = |\sum_{(x)} \langle \varphi, x^{(2)} \rangle x^{(1)}\rangle,$$

$$T_-^\varphi |x\rangle = |x \leftharpoonup \hat{S}(\varphi)\rangle = |\sum_{(x)} \langle \hat{S}(\varphi), x^{(1)} \rangle x^{(2)}\rangle.$$
- Vertex operator  $A_v^h = L^{h(1)} \otimes \cdots \otimes L^{h(n)}$ .
- Face operator  $B_f^\varphi = T_+^{\varphi(1)} \otimes \cdots \otimes T_-^{\varphi(n)}$ .

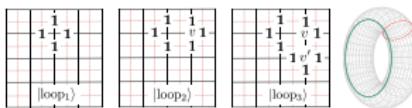
Haar integral  $h \in W$  and  $\varphi \in \hat{W}$ .

$$A^h(s) |x_3 \xrightarrow{\substack{v \\ f}} x_1\rangle = \sum_{(h)} |L_+^{h(3)} x_3 \xrightarrow{\substack{v \\ f}} L_-^{h(1)} x_1\rangle$$

$$B^\varphi(s) |x_3 \boxed{x_2 \\ x_4} x_1\rangle = \sum_{(\varphi)} |T_+^{\varphi(3)} x_3 \boxed{x_2 \\ x_4} T_-^{\varphi(1)} x_1\rangle$$

# Lattice model of 2d topological order

Weak Hopf quantum double model



- The bulk topological phase is characterized by the UMTC  $\text{Rep}(D(W))$ .
- $W$  is bulk gauge symmetry,  $D(W)$  is bulk charge symmetry.
- Ribbon operators create topological excitations.
- Boundary gauge symmetry is  $W$ -comodule algebra, boundary charge symmetry is a WHA.
- The boundary phase is a UFC  $\mathcal{B}$ , the boundary-bulk duality is given by  $\text{Rep}(D(W)) \simeq \mathcal{Z}(\mathcal{B})$ .
- Connection with string-net model?

# Lattice model of 2d topological order

## Multifusion string-net model

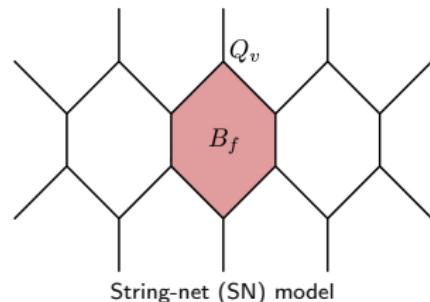
### String-net

A connected oriented trivalent lattice  $\Sigma$  is called a string-net. For a string-net model with an input UMFC  $\mathcal{D}$ , its topological excitation is given by UMTC  $\mathcal{Z}(\mathcal{D})$ .

Vertex space  $\mathcal{H}_v$ : (gauge choice  $Y_c^{ab} = (d_a d_b / d_c)^{1/2}$ )

$$(Y_c^{ab})^{-1/2} \begin{array}{c} a \\ \nearrow \\ \text{---} \\ \downarrow \\ b \\ \alpha \\ \downarrow \\ c \end{array} = |c \rightarrow a, b; \alpha\rangle,$$

$$(Y_c^{ab})^{-1/2} \begin{array}{c} c \\ \uparrow \\ a \\ \nearrow \\ b \\ \beta \end{array} = \langle a, b \rightarrow c; \beta|.$$



Total space  $\mathcal{H}_{\text{tot}} = \otimes_v \mathcal{H}_v$ .

# Lattice model of 2d topological order

Multifusion string-net model: Topological local move

- loop move:

$$\text{Diagram: A circular loop with boundary labels } a \text{ (bottom), } b \text{ (right), } c \text{ (left), and } c' \text{ (top). Arrows indicate a clockwise direction. Boundary labels } \alpha \text{ and } \beta \text{ are placed near the bottom-left and top-right respectively.}$$
$$= \delta_{c,c'} \delta_{\alpha,\beta} Y_c^{ab} .$$

- parallel move:

$$\text{Diagram: Two vertical strands labeled } a \text{ and } b \text{ meeting at a junction. Arrows point upwards along the strands. Labels } \alpha \text{ are placed at the junction.}$$
$$= \sum_{c,\alpha} \frac{1}{Y_c^{ab}} \text{Diagram: Three strands labeled } a, b, \text{ and } c \text{ meeting at a junction. Arrows point upwards along strands } a \text{ and } b, \text{ and downwards along strand } c. \text{ Labels } \alpha \text{ are placed at the junction.}$$

# Lattice model of 2d topological order

Multifusion string-net model: Topological local move

- F-move

$$\begin{array}{ccc} \text{Diagram 1: } & = \sum_{n,\mu,\nu} [F_I^{ijk}]_{m\alpha\beta}^{\eta\mu\nu} & \text{Diagram 2: } \\ \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \searrow \\ \beta \quad m \quad \alpha \\ \downarrow \\ l \end{array} & , & \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \searrow \\ n \quad \nu \\ \downarrow \\ l \end{array} \quad \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \searrow \\ n \quad \nu \\ \downarrow \\ l \end{array} \\ & = \sum_{m,\alpha,\beta} [(F_I^{ijk})^{-1}]_{n\mu\nu}^{m\alpha\beta} & \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \searrow \\ \beta \quad m \quad \alpha \\ \downarrow \\ l \end{array} \end{array}$$

$$\begin{array}{ccc} \text{Diagram 1: } & = \sum_{n,\mu,\nu} [F_I^{ijk}]_{m\alpha\beta}^{\eta\mu\nu} & \text{Diagram 2: } \\ \begin{array}{c} l \\ \uparrow \\ m \\ \swarrow \quad \searrow \\ \beta \quad i \quad j \quad k \end{array} & , & \begin{array}{c} l \\ \uparrow \\ \mu \\ \swarrow \quad \searrow \\ i \quad j \quad k \end{array} \quad \begin{array}{c} l \\ \uparrow \\ \mu \\ \swarrow \quad \searrow \\ j \quad k \end{array} \\ & = \sum_{m,\alpha,\beta} [(F_I^{ijk})^{-1}]_{n\mu\nu}^{m\alpha\beta} & \begin{array}{c} l \\ \uparrow \\ m \\ \swarrow \quad \searrow \\ \beta \quad i \quad j \quad k \end{array} \end{array}$$

## Topological local move

The loop move, parallel move, and F-move, collectively known as topological local moves, are equivalent to the Pachner moves.

# Lattice model of 2d topological order

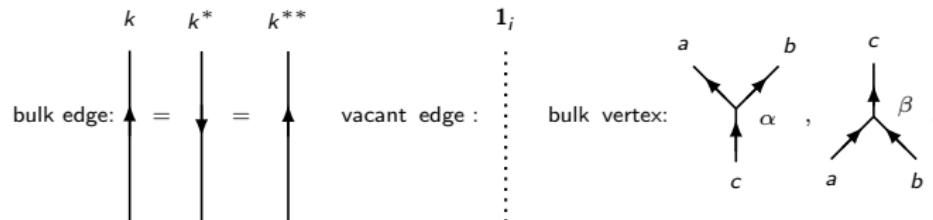
## Multifusion string-net model

### Input data of multifusion string-net

The input data for the generalized multifusion string-net are:

- 1 String type:  $\text{Irr}(\mathcal{D})$ ;
- 2 Fusion rule:  $N_{a,b}^c$  (Recall that quantum dimensions  $d_a$ 's are determined by the fusion rule);
- 3 Local normalization factor:  $Y_c^{ab}$ .
- 4 F-symbols:  $F_I^{ijk}$ ,  $F_{ijk}^I$ .

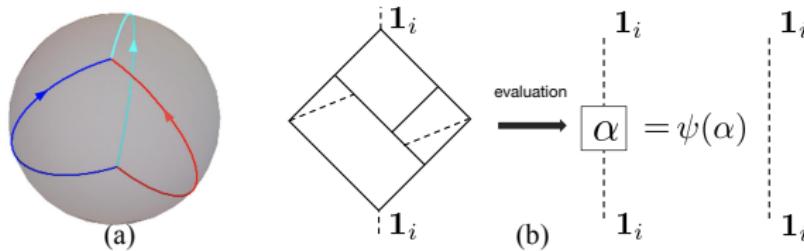
### ■ A fully labeled string-net



# Lattice model of 2d topological order

## Multifusion string-net model

- The multifusion category  $\mathcal{D} = \bigoplus_{i,j \in I} \mathcal{D}_{i,j}$ ,  $\mathbf{1} = \bigoplus_{i \in I} \mathbf{1}_i$ .  
 $X_{i,j} \otimes Y_{j,k} \in \mathcal{D}_{i,k}$ .
- For weak Hopf algebra  $W$ , its representation category  $\text{Rep}(W)$  is a multifusion category.
- String-net ground state



Ground state  $|\psi\rangle = \sum_{\alpha} \psi(\alpha)|\alpha\rangle$ . The coefficient is calculated by evaluation.

# Lattice model of 2d topological order

Multifusion string-net model

String-net lattice model (vertex and face operators)

$$Q_v | \begin{array}{c} a \\ b \\ c \\ \alpha \end{array} \rangle = \delta_{c \rightarrow a,b} | \begin{array}{c} a \\ b \\ c \\ \alpha \end{array} \rangle .$$

$$B_f^k | \begin{array}{c} i_2 \\ j_3 \\ i_3 \\ j_4 \\ j_5 \\ i_4 \\ j_6 \\ i_5 \\ j_1 \\ i_1 \\ j_2 \\ i_6 \\ j_6 \\ i_6 \end{array} \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \end{array} \rangle = | \begin{array}{c} i_2 \\ j_3 \\ i_3 \\ j_4 \\ j_5 \\ i_4 \\ j_6 \\ i_5 \\ j_1 \\ i_1 \\ j_2 \\ i_6 \\ j_6 \\ i_6 \\ k \end{array} \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \end{array} \rangle .$$

# Lattice model of 2d topological order

## Multifusion string-net model

### String-net lattice model

The local stabilizers  $Q_v$  and  $B_f$  functions are projectors and mutually commute. Consequently, the Hamiltonian of the generalized multifusion string-net ( $J_v, J_f > 0$ ,  
 $B_f = \sum_{k \in \text{Irr}(\mathcal{D})} w_k B_f^k$ ,  $w_k = Y_1^{k^* k} / \sum_{l \in \text{Irr}(\mathcal{D})} d_l^2$ ):

$$H = -J_v \sum_v Q_v - J_f \sum_f B_f$$

being a local commutative projector (LCP) Hamiltonian, exhibits a gap in the thermodynamic limit.

# Lattice model of 2d topological order

## Multifusion string-net model

### Multifusion string-net model

- For input UMFC  $\mathcal{D}$ , the topological excitation is given by UMTC  $\mathcal{Z}(\mathcal{D})$  (Drinfeld center).
- The bulk gauge symmetry and charge symmetry are weak Hopf algebras.
- The boundary gauge symmetry is a  $W$ -comodule algebra, the boundary charge symmetry is a weak Hopf algebra.
- The domain wall gauge symmetry is a  $W_1|W_2$ -comodule algebra, the domain wall charge symmetry is a weak Hopf algebra.

DEFECTIVE LEVIN-WEN STRING-NET CAN BE REGARDED AS A MULTIFUSION STRING-NET!

# Weak Hopf tube algebra

## Bulk tube algebra

Definition (Bulk tube algebra  $\mathbf{Tube}(\mathcal{D}\mathcal{D}_{\mathcal{D}})$ )

The bulk tube algebra  $\mathbf{Tube}(\mathcal{D}\mathcal{D}_{\mathcal{D}})$  is spanned by the following basis (up to planar isotopy):

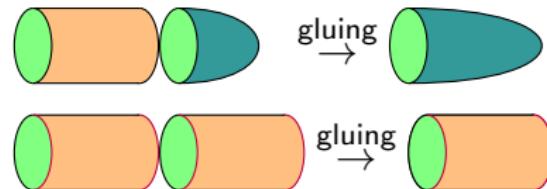
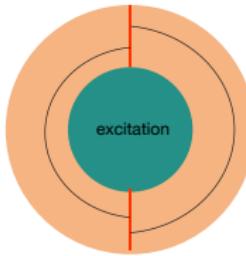
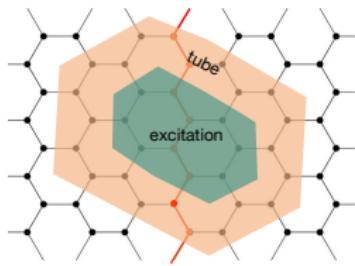
$$\left\{ \text{Diagram } a \text{ (a circle with boundary edges labeled } h, g, \zeta, f, e, d, \nu, \mu, c) \text{ and } b : a, \dots, h \in \mathrm{Irr}(\mathcal{D}), \mu, \nu, \gamma, \zeta \in \mathrm{Hom}_{\mathcal{D}} \right\}.$$

The diagram consists of a circle with a red boundary. Inside the circle, there are two vertical blue lines labeled  $h$  at the top and  $c$  at the bottom. Between these lines, there are four horizontal segments:  $\gamma$  (top),  $f$  (middle), and  $e$  (bottom). Outside the circle, there are two vertical blue lines labeled  $g$  at the top and  $d$  at the bottom. Between these lines, there are four horizontal segments:  $\zeta$  (top),  $\mu$  (middle), and  $\nu$  (bottom).

Note that the arrows have been omitted to avoid clutter in the equation; all edges are assumed to be directed upwards.

# Weak Hopf tube algebra

## Bulk tube algebra: Algebra structure



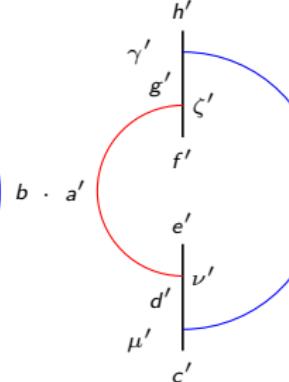
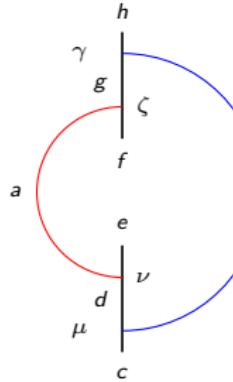
- The unit is given by

$$1 = \sum_{a,b} |a\rangle\langle b|$$

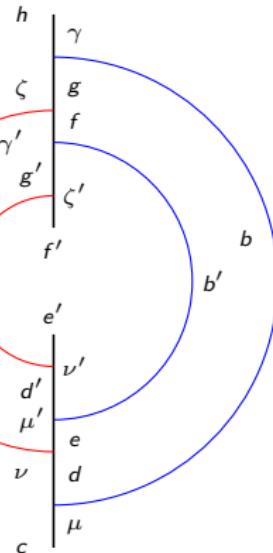
# Weak Hopf tube algebra

Bulk tube algebra: Algebra structure

- The multiplication is of the form



$$b' = \delta_{f,h'} \delta_{e,c'}$$



# Weak Hopf tube algebra

Bulk tube algebra: Coalgebra structure

- The counit is of the form:

$$\varepsilon \left( \begin{array}{c} h \\ \gamma \\ g \\ f \\ \zeta \\ \hline a \\ \text{red circle} \\ e \\ \nu \\ d \\ \mu \\ c \end{array} \right) b = \frac{\delta_{e,f}\delta_{c,h}}{d_h} \begin{array}{c} \gamma \\ g \\ \zeta \\ \hline a \\ \text{red circle} \\ f \\ b \\ \nu \\ d \\ \mu \end{array} h = \delta_{e,f}\delta_{c,h}\delta_{d,g}\delta_{\nu,\zeta}\delta_{\mu,\gamma} \sqrt{\frac{d_a d_f d_b}{d_h}}$$

# Weak Hopf tube algebra

Bulk tube algebra: Coalgebra structure

- The comultiplication is given by

$$\Delta \left( \begin{array}{c} h \\ \gamma \\ g \\ f \\ \zeta \\ \hline a \\ \text{red circle} \\ e \\ d \\ \mu \\ \nu \\ c \end{array} \right) = \sum_{i,j,k,\rho,\sigma} \sqrt{\frac{d_k}{d_a d_i d_b}} a \left( \begin{array}{c} h \\ \gamma \\ g \\ f \\ \zeta \\ \hline i \\ \text{red circle} \\ j \\ \rho \\ \sigma \\ k \end{array} \right) b \otimes a \left( \begin{array}{c} k \\ \rho \\ j \\ \sigma \\ i \\ \hline e \\ d \\ \mu \\ \nu \\ c \end{array} \right) b$$

# Weak Hopf tube algebra

Bulk tube algebra: Antipode morphism

- Antipode operation  $S$ :

$$S \left( \begin{array}{c} h \\ \gamma g | \zeta \\ f \\ e \\ \nu d | \mu \\ \mu d | c \\ c \end{array} \right) = \frac{d_f}{d_h} \left( \begin{array}{c} e \\ \nu d | \mu \\ c \\ h \\ \gamma g | \zeta \\ f \\ \bar{b} \end{array} \right) \bar{a} \bar{b}$$

The diagram illustrates the antipode operation  $S$  on a bulk tube algebra. On the left, a red circle represents the original state with components  $a, b, c, d, e, f, g, h$  labeled clockwise from bottom-left. A blue circle represents the transformed state after applying  $S$ , with components  $\bar{a}, \bar{b}, f, g, h, c, \mu, \nu, d, \zeta, \gamma, e$  labeled clockwise. The transformation is given by the formula  $S(a, b, c, d, e, f, g, h) = \frac{d_f}{d_h}(\bar{a}, \bar{b}, f, g, h, c, \mu, \nu, d, \zeta, \gamma, e)$ .

# Weak Hopf tube algebra

Bulk tube algebra:  $C^*$  structure

- The  $*$ -operation is given by

$$\left( \begin{array}{c} \text{Diagram A: A circle with red and blue boundary arcs. Vertices are labeled clockwise from top-left: } h, g, \zeta, f, e, \nu, d, \mu, c. \\ \text{Diagram B: A circle with red and blue boundary arcs. Vertices are labeled clockwise from top-left: } f, g, \gamma, h, c, \mu, d, \nu, e. \end{array} \right)^* = \frac{d_e}{d_c} \begin{array}{c} \text{Diagram C: A circle with red and blue boundary arcs. Vertices are labeled clockwise from top-left: } \bar{a}, \bar{b}, \bar{c}, \bar{\mu}, \bar{d}, \bar{\nu}, \bar{e}. \end{array}$$

# Weak Hopf tube algebra

Bulk tube algebra: weak Hopf symmetry

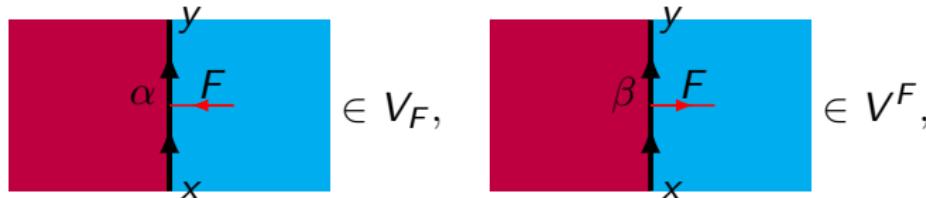
## Weak Hopf tube algebra

For any given UMFC  $\mathcal{D}$ , the tube algebra  $\mathbf{Tube}(\mathcal{D}\mathcal{D}_{\mathcal{D}})$  is a  $C^*$  weak Hopf algebra.

For any  $F \in \text{Fun}_{\mathcal{D}|\mathcal{D}}(\mathcal{D}, \mathcal{D})$ , we can construct a module  $V_F$  over the tube algebra.

$$V_F := \bigoplus_{x,y \in \text{Irr}(\mathcal{D}\mathcal{D}_{\mathcal{D}})} \text{Hom}_{\mathcal{D}\mathcal{D}_{\mathcal{D}}}(F(x), y),$$

$$V^F := \bigoplus_{x,y \in \text{Irr}(\mathcal{D}\mathcal{D}_{\mathcal{D}})} \text{Hom}_{\mathcal{D}\mathcal{D}_{\mathcal{D}}}(x, F(y)).$$



- Topological excitation are modules over tube algebra.
- Weak Hopf tube algebra is the charge symmetry for multifusion string-net.

# Weak Hopf tube algebra

## Boundary tube algebra

$$\mathbf{L}(\mathcal{D}) = \text{span} \left\{ \begin{array}{c} \text{Diagram showing a red circle } a \text{ with points } e, f, g \text{ on its boundary. A vertical line } \zeta \text{ passes through } f. \\ \text{A blue dotted circle } b \text{ passes through } g, \nu, d. \end{array} \right\} \text{ : } a, \dots, g \in \text{Irr}(\mathcal{D}), \nu, \zeta \in \text{Hom}_{\mathcal{D}}$$

$$\mathbf{R}(\mathcal{D}) = \text{span} \left\{ \begin{array}{c} \text{Diagram showing a blue circle } b \text{ with points } h, \gamma, f, e \text{ on its boundary. A vertical line } \mu \text{ passes through } c. \\ \text{A red dotted circle } a \text{ passes through } \mu, \nu, d. \end{array} \right\} \text{ : } b, \dots, h \in \text{Irr}(\mathcal{D}), \mu, \gamma, \nu \in \text{Hom}_{\mathcal{D}}$$

# Weak Hopf tube algebra

## Boundary tube algebra

Bulk tube algebra as crossed product of boundary tube algebra:

$$\boxtimes \left( \begin{array}{c} h \\ \gamma \\ g \\ d \\ \mu \\ c \end{array} \right) \otimes \left( \begin{array}{c} g' \\ \zeta \\ f \\ e \\ \nu \\ d' \end{array} \right) = \delta_{g,g'} \delta_{d,d'} \left( \begin{array}{c} h \\ \gamma \\ g \\ f \\ \zeta \\ e \\ d \\ \nu \\ \mu \\ c \end{array} \right)$$

### Boundary tube algebra

- The boundary tube algebra is a weak Hopf algebra.
- The boundary tube algebra can be regarded as the gauge symmetry of the bulk.

# Weak Hopf tube algebra

## Morita theory

General tube space  $\mathbf{T}^{m_0, m_1; n_0, n_1}$  spanned by the tube string-net configurations :

$$\mathbf{T}^{m_0, m_1; n_0, n_1} = \text{span} \left\{ \begin{array}{c} \text{Diagram showing a circular boundary with internal vertices and edges. Red edges form a loop on the left labeled } m_1 \text{ and } m_0. \text{ Blue edges form a loop on the right labeled } n_0 \text{ and } n_1. \\ \text{The diagram is enclosed in curly braces with the condition: edge } \in \text{Irr, vertex } \in \text{Hom} \end{array} \right\}$$

### Morita equivalence

The tube space  $\mathbf{T}^{m, m; n, n}$  are algebras for all  $m, n \in \mathbb{N}$ . The tube space  $\mathbf{T}^{m, s; n, t}$  forms a right- $\mathbf{T}^{m, m; n, n}$  and left- $\mathbf{T}^{s, s; t, t}$  bimodule. These structures form a Morita context in the sense that

$$\mathbf{T}^{m, s; n, t} \otimes_{\mathbf{T}^{m, m; n, n}} \mathbf{T}^{s, m; t, n} \cong \mathbf{T}^{s, s; t, t}, \quad \mathbf{T}^{s, m; t, n} \otimes_{\mathbf{T}^{s, s; t, t}} \mathbf{T}^{m, s; n, t} \cong \mathbf{T}^{m, m; n, n}.$$

Thus  $\mathbf{T}^{m, m; n, n}$ 's are Morita equivalent for all  $m, n \in \mathbb{N}$ .

# Weak Hopf tube algebra

## Summary

Weak Hopf symmetry behind  $2d$  non-chiral topological order

- There are two type of weak Hopf symmetries for non-chiral topological phase: weak Hopf gauge symmetry and weak Hopf charge symmetry
- The weak Hopf is not unique, they are related by categorical Morita equivalence.
- For (weak Hopf) quantum double model, the weak Hopf gauge symmetry is the input weak Hopf algebra  $W$ , the weak Hopf charge symmetry is its quantum double  $D(W)$ .
- For (multifusion) string-net model, the weak Hopf gauge symmetry is given by boundary tube algebra, the weak Hopf charge symmetry is given by the bulk tube algebra.

# Open problems

- Higher dimensional model and higher category structure.
- Entanglement property, entangle entropy is sensitive to defect and boundary.
- Weak Hopf quantum double  $\Leftrightarrow$  extended string-net model.
- SET/SPT generalization of quantum double model and string-net model.
- Operator algebra perspective: stability, Haag duality, infinite-volume sector, etc.

**THANK YOU FOR YOUR ATTENTIONS**