Chapter 1 Module

\$1.6 Injective Modules

- · Duality
- Hom(•, M) and injective modules

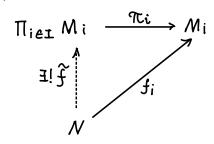
(I) Duality

Duality: reversing the direction of arrows.

1. monomorphism
$$f$$

$$\begin{array}{cccc}
 & f \\
 & K & \xrightarrow{g} & M & \xrightarrow{f} & N \\
 & & K & \xrightarrow{g} & M & \xrightarrow{f} & N
 \end{array}$$

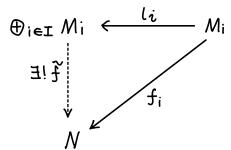
2. product



1' epimorphism
$$f$$

$$K \stackrel{g}{\longleftarrow} M \stackrel{f}{\longleftarrow} N$$

2' coproduct



(II) Injective module

For R module I, Hom (•, I) is a contravariount function:

$$(M \xrightarrow{f} N) \longmapsto Hom(M, I)$$

$$(M \xrightarrow{f} N) \longmapsto Hom(f, I) = f^* \text{ (or denoted as } \widehat{f})$$

$$M \xrightarrow{f} N$$

$$\downarrow h$$

$$\downarrow h$$

$$\downarrow f^*(h) = h \circ f$$

$$(f \cdot g)^* = g^* \cdot f^*$$
, $(f \cdot g)^* (h) = h \cdot (f \cdot g) = (h \cdot f) \cdot g = g^* \cdot f^* (h)$.

Det 6.1 Let I be an R module, if for any monomorphism $f: A \rightarrow B$, and module map $h:A \rightarrow I$, there is module map $\overline{h}:B \rightarrow I$ such that $h=\overline{h}f$, then I is called an injective module.

7hm 6.1 The following stutements are equivalent:

- (1) I is injective module.
- 2) If $f \in Hom(A, B)$ is monic, then $Hom(f, I) = \hat{f} = f^*$ is epic.
- (3) If $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is short exact sequence, then the following reguence is exact

$$o \longrightarrow Hom(C, I) \xrightarrow{g^*} Hom(B, I) \xrightarrow{f^*} Hom(A, I) \longrightarrow o.$$

(4) Exact sequence $0 \rightarrow I \xrightarrow{f} B \xrightarrow{g} c \rightarrow 0$ is split.

Proof. (1) (2): Obvious.

(2) \Rightarrow (3): Prop 5.2 guarantees the exact new of $0 \longrightarrow Hom(C, I) \xrightarrow{g^*} Hom(B, I) \xrightarrow{5^*} Hom(A, I).$

The only left part is to show f* is epic, this is guaranteed by assumption (2).

 $(3) \Rightarrow (4)$: Since $0 \rightarrow I \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ exact, by assumption,

 $0 \rightarrow Hom(C, I) \xrightarrow{g^*} Hom(B, I) \xrightarrow{f^*} Hom(I, I) \rightarrow 0$ exact.

Since f^* is epic, there is h s.t. $f^*(h) = id_{\perp}$. Thus $hf = id_{\perp}$. f is split monic. From splitting lemma, the sequence is split exact.

 $(4)\Rightarrow (1)$. To prove this, we need some subsequent result: 0 any R module is a submodule of an injective module; 0 $M\oplus N$ is injective iff M and N are injective From 0, there is an injective E such that $I \Longrightarrow E$, we have the following exact sequence

 $0 \longrightarrow I \stackrel{i}{\longleftrightarrow} E \stackrel{\ell}{\to} E/I \longrightarrow 0$.

By assumption, it is split exact, thus $E \cong I \oplus E/I$. Now using Q, we see I is injective.

Prop 6.2' For a finite family of R modules $\{M_j\}_{j\in J}$, $|J| < +\infty$, $\bigoplus_{j\in J} M_j$ is injective iff M_j are injective for all $j\in J$. Note that for infinite J, this does not hold.

Proof. Mod is additive actegory, then finite direct sum is isomorphic to finite direct product, we will prove a more general version in the next prop.

Prop 6.2. For index sex J cnot necessarily finite), direct product $\Pi_{j\in J}$ M_j is injective iff M_j is injective for all $j\in J$.

Proof. " \Leftarrow " Suppose $M_{\tilde{j}}$'s are injective, we need to show $\Pi_{\tilde{j}\in J}$ $M_{\tilde{j}}$ is injective. Let $\Pi_{\tilde{j}}: \Pi_{\tilde{j}\in J}$ $M_{\tilde{j}} \to M_{\tilde{j}}$ be canonical projections.

Consider monomorphism $f: A \longrightarrow B$, $Hom(f, M_j)$ is epic for all $j \in J$. For $g: A \to \Pi_j \in_J M_j$, define $g_j = \pi_j \circ g: A \to M_j$, the must exist $g_j: B \to M_j$ sub that

$$0 \longrightarrow A \xrightarrow{f} B$$

$$g_{5} \qquad \Re g = \widehat{g_{j}} f$$

Then define $\hat{g}: B \longrightarrow \Pi_{j \in J} M_j$ as $b \mapsto (\hat{g}_j(b))_{j \in J}$, which is a R-module map and $\hat{g} f(a) = (\hat{g}_j(f(a)))_{j \in J} = (\pi_j g(a))_{j \in J} = g(a)$.

" => " We need to show

$$0 \longrightarrow A \xrightarrow{f} B$$

$$|f| \downarrow \qquad |f| \downarrow$$

Define $h: A \to \Pi_{j \in J} M_j$, $\alpha \mapsto (h_j(\alpha))_{j \in J}$, there exists \hat{h} et. $h = \hat{\kappa} \circ f$

Pefine $\hat{h}_j = \mathcal{H}_j \hat{h}$, we are done.

(III) Bour criterion and its application.

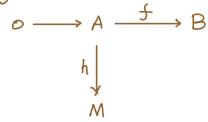
Every R module is a quotrent module of a projective module.

Every R module is a submodule of an injective module.

Lemmu 6.3 (Boer criterion). A left R module M is injective iff every R module map $f\colon L\to M$, where $J\in R$ is an ideal, can be excended to R.

proof. "⇒": This is obvious, since inclusion L: I → R is monic.

" \Leftarrow ": To show M is injective, consider



We can simply regard Imf the same as A, i.e., using $a \in B$ to represent $f(a) \in B$. We need to extend h to B.

Let X be the set of pairs (A',h') when $A \subseteq A' \subseteq B$ and $h':A' \to M$ extends h, meaning $h'|_A = h$. Define a partial order $(A',h') \preccurlyeq (A'',h'')$ iff $A' \subseteq A''$, $h''|_{A'} = h'$.

Using Zorn's lemma, there must exists a maximal element (A., h.) in X. If $A_0 = B$, we are done. Now assume $A_0 \neq B$.

There is be B and b & A.

Define I = {reR: rbeAo}, I is an ideal of R.

Define $f: I \to M$ as $f(r) = h_0(r \cdot b)$, by assumption, this can extend to R as $\widetilde{f}: R \to M$. Now, define $A_1 = A_0 + \langle b \rangle$, and $h_1: A_1 \to M$ as $h_1(a_0 + rb) := h_0(a_0) + r\widetilde{f}(1)$.

This means $(A_1, h_1) \in X$ and $(A_1, h_1) \not = (A_0, h_0)$, which leads a contradiction of maximality of (A_0, h_0) .

Def G.2 Let R be an integral domain, D is R module. If for any $y \in D$, and $o \neq r \in R$, there is $X \in D$ such that rx = y, the D is called divisible.

remark. This means $0 \neq r \in \mathbb{R}$, $r_{\bullet}(\bullet) : D \rightarrow D$ defines a surjective map-

Example 6.1. Q is divisible as a Z module.

Example 6.2. Q/Z is divisible as Z module.

Example 6.3. Z is not divisible as Z module.

Prop 6.4 Let R be an integral domain, then quotient of divisible module is divisible.

Prop 6.5 Let R be PID, then D is injective iff D is divisible.

Prop 6.6 Every abelian group can embed into a divisible abelian group cinjective z module).

For commutative ring R and abelian group A, $Hom_Z(R, A)$ is abelian group. Define $R \times Hom_Z(R, A) \longrightarrow Hom_Z(R, A)$

 $(r, f) \longmapsto rf: a \mapsto f(a\cdot r)$

then Hom Z (R, A) is a R module.

<u>Lemma</u> 6.7. If D is divisible abelian group, then $Hom_{\geq}(R,D)$ is injective R module. Theorem 6.8. Every R module can embed into an injective R module.

Proof. • Give $M \in \mathbb{R} \text{Mod}$, it is an abelian group, thus $M \in \mathbb{R} \text{Mod}$.

From prop 6.6, M can embed into a divisible abelian group D, $M \stackrel{f}{\Longrightarrow} D$. Consider $O \longrightarrow M \stackrel{f}{\Longrightarrow} D$ in Z Mod, $R \in Z$ Mod, we see Hom (R, f) is monic. $O \longrightarrow Hom_Z(R, M) \stackrel{f}{\Longrightarrow} Hom_Z(R, D)$.

• Since $Hom_{\mathbb{Z}}(R,1M)$, $Hom_{\mathbb{Z}}(R,D)$ are both R modules, we need to show f* is an R module map, in this way, we obtain a R module embedding.

For r, aeR, ae Homz (R, M),

 $[f_*(rd)](a) = (f \circ rd)(a) = f(r \cdot d(a)) = f(d(an)) = (f_*(d))(an)$ $= (r \cdot f_*)(a).$

Thus fx is R module map.

• Notice $Hom_R(R, M) \subseteq Hom_Z(R, M)$. As R modules $Hom_R(R, M) \subseteq Hom_Z(R, M)$.

But we know $M \cong Hom_R(R, M)$, thus we have (in RMod) $M \xrightarrow{\cong} Hom_R(R, M) \xrightarrow{\hookrightarrow} Hom_Z(R, M) \xrightarrow{f*} Hom_Z(R, D).$ Since D is divisible, prop 6.7 guarantees $Hom_Z(R, D)$ is injective.