Chapter 1 Module

1.1 Exercise

1.4 Let N, K be submodules of R module M, we define

$$(N:K) = \{a \in R \mid aK \subseteq N\}.$$

(1) Prove that $(N:K)$ is an ideal of R.

Proof. Step 1. Show $(N:K)$ is a subgroup

Suppose $a, b \in (N:K)$, then $aK \subseteq N$, $bK \subseteq N$. For any $k \in K$, $ak, bk \in N$, thus $ak - bk = (a-b)\cdot k \in N$. $\Rightarrow$ $a - b \in (N:K)$.

Step 2. Show $R \cdot (N:K) \subseteq (N:K)$

For $r \in R$ and $a \in (N:K)$, we have $r \cdot a \cdot K \subseteq r \cdot N \subseteq N$.

(2) A special case is $\mathrm{Ann}(M) = (0:M) = \{b \in R \mid bx = 0,\ x \in M\}$, called annihilator of M. For an ideal $\mathfrak{c}$ of R, if $\mathfrak{c} \subseteq \mathrm{Ann}(M)$, show that $(a+\mathfrak{c})\cdot x := ax$ makes M an $R/\mathfrak{c}$ module.

Proof. Step 1. Show that $(a+\mathfrak{c})\cdot x$ is well-defined.

If $a, b \in a+\mathfrak{c}$, $a-b \in \mathfrak{c} \subseteq \mathrm{Ann}(M)$, $(a-b)\cdot x = 0$. Thus $a\cdot x = b\cdot x$.

Step 2. The axioms of module is easy to check.

1.5 Prove the following

(1) $\mathrm{Ann}(N+K) = \mathrm{Ann}(N) \cap \mathrm{Ann}(K)$

(2) $(N:K) = \mathrm{Ann}((K+N)/N)$

proof. (1) "$\mathrm{Ann}(N+K) \subseteq \mathrm{Ann}(N) \cap \mathrm{Ann}(K)$"

For $a \in \mathrm{Ann}(N+K)$ $\quad a\cdot(N+K) = 0$. Since $N, K \subseteq N+K$, $a\cdot N = a\cdot K = 0$.

Thus $a \in \mathrm{Ann}(N) \cap \mathrm{Ann}(K)$

"$\mathrm{Ann}(N) \cap \mathrm{Ann}(K) \subseteq \mathrm{Ann}(N+K)$

For a such that $a\cdot N = a\cdot K = 0$, we have $a\cdot(x+y) = 0$ for all $x \in N$ and $y \in K$.

(2) Notice $\mathrm{Ann}((K+N)/N) = \{a \in R \mid a\cdot\bar{x} = \bar{0} \quad \forall\, \bar{x} \in (K+N)/N\}$, where $\bar{x} = x + N$ and $a\cdot\bar{x} = \overline{a\cdot x} = ax + N$.

"$(N:K) \subseteq \mathrm{Ann}((N+K)/N)$": For $a \in (N:K)$, $aK \subseteq N$. We have $a\cdot\bar{x} = \overline{a\cdot x} = a\cdot x + N$, $\quad x = y_1 + y_2$ with $y_1 \in K$ $\ y_2 \in N$. $a\cdot x \in N$.

Thus $a\cdot\bar{x} = \bar{0} = 0 + N$.

"$Ann((N+K)/N) \subseteq N:K$": For $a \in Ann((N+K)/N)$, $a \cdot \bar{x} = \bar{0}$ for all $\bar{x} \in (N+K)/N$. Let $x = y_1 + y_2$ with $y_1 \in K$ and $y_2 \in N$, we have $a \cdot y_1 + a \cdot y_2 \in N \Rightarrow a \cdot y_1 \in N$ for all $y_1 \in K$.

1.7 When R is not an integral domain, give an example of module M whose $T(M)$ is not a submodule.

Solution. Consider $R = M = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. $T(M) = \{0, 2, 3, 4\}$. $2+3 = 5 \notin T(M)$. Thus $T(M)$ is not a submodule.

1.10 Prove that nonzero module is simple iff M is generated by arbitrary nonzero element x, viz, $M = (x)$.

Proof. "$\Rightarrow$": Take an $0 \neq m \in M$. $R\{m\} = (m)$ is a submodule of M, since M is simple, $(m) \neq 0$, we have $(m) = M$.

"$\Leftarrow$" Suppose $m \neq 0$ generate M, i.e. $M = (m)$. If $0 \neq N$ is a submodule of $M = (m)$, there is $n \in N$ and $n \neq 0$. Since $n \in M$, we also have $(n) = M$ by assumption. Thus $(n) \subseteq N \subseteq M = (n)$. $N = M$.