

Chapter 2 Categories

§2.2 Functor and Natural Transformation

- Functor
- Natural transformation

(I) Functor

$$\text{category } \mathcal{C} \xrightarrow{\text{functor } F} \text{category } \mathcal{D}$$

$$\text{object } A, B \longmapsto \text{objects } F(A), F(B)$$

$$\text{map } A \xrightarrow{f} B \longmapsto \text{map } F(A) \xrightarrow{F(f)} F(B)$$

Functors are morphisms in the category of all categories.

Def 2.1 For two categories \mathcal{C} and \mathcal{D} , a **covariant functor** $F: \mathcal{C} \longrightarrow \mathcal{D}$ consists of

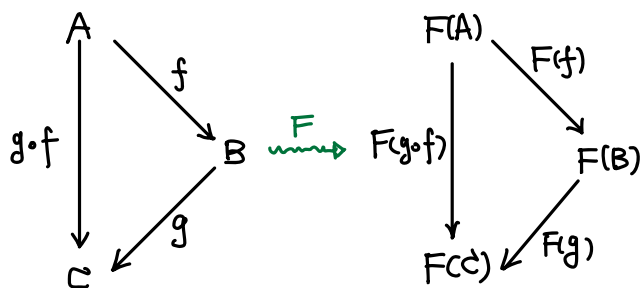
(i) Maps on objects: $A \longrightarrow F(A)$

(ii) Maps on morphisms: $f \in \text{Hom}_{\mathcal{C}}(A, B) \longmapsto F(f) \in \text{Hom}_{\mathcal{D}}(F(A), F(B))$.

They satisfy:

$$\textcircled{1} F(\text{id}_A) = \text{id}_{F(A)}$$

$$\textcircled{2} F(g \circ f) = F(g) \circ F(f).$$

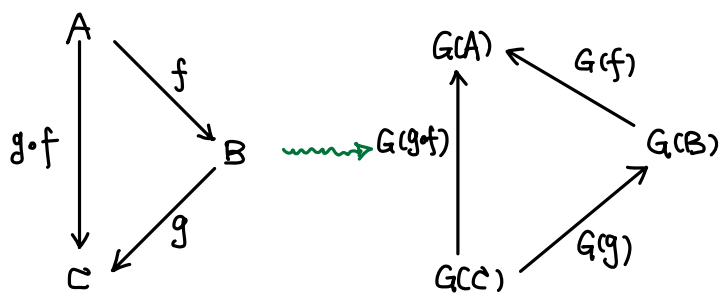


Def. 2.2 $G: \mathcal{C} \longrightarrow \mathcal{D}$ is called a **contravariant functor** if it maps $f \in \text{Hom}_{\mathcal{C}}(A, B)$ to

$G(f) \in \text{Hom}_{\mathcal{D}}(G(B), G(A))$ such that

$$\textcircled{1} G(\text{id}_A) = \text{id}_{G(A)}.$$

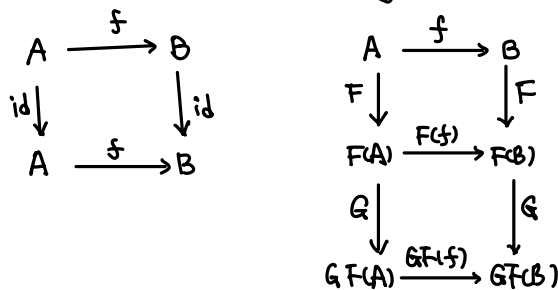
$$\textcircled{2} G(g \circ f) = G(f) \circ G(g).$$



Remark. (Duality). Dual category \mathcal{C}^{op} : ① $\text{ob } \mathcal{C}^{\text{op}} = \text{ob } \mathcal{C}$ ② $\text{Hom}_{\mathcal{C}^{\text{op}}}(A, B) = \text{Hom}_{\mathcal{C}}(B, A)$.
 $(f \circ g)^{\text{op}} = g^{\text{op}} \circ f^{\text{op}}$.

- A contravariant functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is a covariant functor $\tilde{F}: \mathcal{C}^{\text{op}} \longrightarrow \mathcal{D}$.
- A contravariant functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is a covariant functor $\tilde{F}: \mathcal{C} \longrightarrow \mathcal{D}^{\text{op}}$.

For two functors $F: \mathcal{C} \rightarrow \mathcal{D}$, $G: \mathcal{D} \rightarrow \mathcal{E}$, we can define their composition $G \circ F$. The identity functor can also be defined naturally.



Def 2.3. For functor $F: \mathcal{C} \rightarrow \mathcal{D}$, it is called:

(i) full, iff F is surjective on Hom set $F: \text{Hom}_{\mathcal{C}}(A, B) \longrightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B))$.

(ii) faithful, iff F is injective on Hom set $F: \text{Hom}_{\mathcal{C}}(A, B) \xrightarrow{\quad} \text{Hom}_{\mathcal{D}}(F(A), F(B))$.

Exp 2.1. Let \mathcal{D} be a subcategory of \mathcal{C} , we can define embedding by embedding of objects class and Hom set.

Exp 2.2 For group G , its commutator subgroup is defined as

$$[G, G] = \langle xyx^{-1}y^{-1} = [x, y] \mid x, y \in G \rangle.$$

The quotient group $G/[G, G]$ is called Abelianization of G .

This induces a functor $F: \text{Grp} \longrightarrow \text{Ab}$. (Check this !!)

This functor is left adjoint of inclusion functor $I: \text{Ab} \hookrightarrow \text{Grp}$.

Exp 2.3 For any set $X \in \text{Set}$. Define a free R module $\langle X \rangle$ with basis X . This gives a functor $F: \text{Set} \longrightarrow R\text{Mod}$.

Similarly, we have $F: \text{Set} \longrightarrow \text{Vect}_{\mathbb{C}}$. (Free functor)

Exp 2.4 Forgetful functor

$$\omega: {}_R\text{Mod} \longrightarrow \text{Set}$$

$$\omega: {}_R\text{Mod} \longrightarrow \text{Ab}$$

(II) Hom functor and tensor functor

- $\text{Hom}(\cdot, X) : \mathcal{C} \longrightarrow \text{Set}$ contravariant functor injective module
- $\text{Hom}(X, \cdot) : \mathcal{C} \longrightarrow \text{Set}$ covariant functor projective module
- Tensor functor $A \otimes_R \cdot : {}_R\text{Mod}_R \longrightarrow {}_R\text{Mod}_R$ covariant functor flat module

(III) Natural transformation

Natural transformation is maps between functors.

Def 2.4. For two functors $F, G: \mathcal{C} \longrightarrow \mathcal{D}$, a natural transformation is a set of morphisms

$$\tau: F \longrightarrow G = \{ \tau_A \in \text{Hom}_{\mathcal{D}}(F(A), G(A)) \mid A \in \mathcal{C} \}$$

such that for any $f \in \text{Hom}_{\mathcal{C}}(A, B)$, the following diagram commutes

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \tau_A \downarrow & & \downarrow \tau_B \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

If all τ_A are isomorphisms, τ is called natural isomorphism.

Exp 2.5. For $\text{Vect}_{\mathbb{F}}$, define double-dual functor

$$F: \text{Vect}_{\mathbb{F}} \longrightarrow \text{Vect}_{\mathbb{F}}$$

$$V \longmapsto V^{**}$$

Then $\{ \theta_V: V \longrightarrow V^{**} \}$ is a natural transformation

$$\theta: \text{id}_{\text{Vect}_{\mathbb{F}}} \longrightarrow F.$$

For $\text{Vect}_{\mathbb{F}}^{\text{FD}}$, θ is natural isomorphism.