Boundary and domain wall theories of 2d generalized quantum double model

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Abstract:

The generalized quantum double lattice realization of $2d$ topological orders based on Hopf algebras is discussed in this work. Both left-module and right-module constructions are investigated. The ribbon operators and the classification of topological excitations based on the representations of the quantum double of Hopf algebras are discussed. To generalize the model to a $2d$ surface with boundary and surface defect, we present a systematical construction of the boundary Hamiltonian and domain wall Hamiltonian using Hopf algebra pairing and generalized quantum double construction. Via the Hopf tensor network representation of the quantum many-body states, we solve the ground state of the model in the presence of the boundary and domain wall.

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1 Introduction

Topologically ordered phases extended our understanding of the notion of phases of matter beyond the Landau-Ginzburg symmetry-breaking picture [1, 2]. Besides their foundational importance, these exotic quantum phases of matter also found their applications in quantum information processing, such as robust topological quantum error-correction code (QECC) [3, 4] and topological quantum computation (TQC) [5–7]. The mathematical structure behind the usual symmetry-breaking phases is symmetry groups (G_H, G_Ψ) with G_H the symmetry group of Hamiltonian and G_Ψ the symmetry group of the wavefunction, while the mathematical frameworks behind topological orders are tensor categories. More precisely, a gapped $2d$ topological order is characterized by a unitary modular tensor category (UMTC) $\mathcal{D}$; for the gapless system, another data, central charge c, is needed, namely, $(\mathcal{D}, c)$ fully characterizes the topologically ordered phase.

A gapped topological phase is an equivalence class of gapped Hamiltonians together with their corresponding ground state spaces $\{(H, \mathcal{H})\}$, which realizes some topological quantum field theory (TQFT) at low energy. The excitations are characterized by a quantum group constructed from the gauge group of the theory, the quasi-particle types are given by irreducible representations of the quantum group. A crucial class of such kind of $(2+1)D$ model is the twisted quantum double model based on a finite group algebra $\mathbb{C}[G]$ [8–10], which is a Hamiltonian realization of Dijkgraaf-Witten TQFT [11]. Their topological excitations are characterized by the representation category of the twisted quantum double $D^{\alpha}(G)$ with α a 3-cocycle over G. When the 3-cocycle is trivial, the model becomes Kitaev's quantum double model [5], which corresponds to BF theory. The Levin-Wen's string-net model [12] has a more general setting, it realizes the Turaev-Viro-Barrett-Westbury TQFT [13, 14]. For arbitrary unitary fusion category (UFC) $\mathcal{C}$, the topological excitations of the Levin-Wen model are given by the Drinfeld center $\mathcal{Z}(\mathcal{C})$.

Various generalizations of the quantum double model have been studied (e.g. [10, 15, 16]), among which, Hopf algebraic quantum double model turned out to be a crucial class and attracts extensive studies from both lattice model perspective [15, 17, 18] and Hopf algebraic gauge theory perspective [19–21]. A quantum double model can be mapped into a string-net model [22–24], the reverse direction has also been studied [16, 25]. The physical properties (including topological excitation, ribbon operator, electric-magnetic duality, etc.) of the quantum double model for the finite group algebra case has be extensively studied [5, 25–28]. However, for the general Hopf algebra case, only some special examples are discussed [15]. In this work, we will present a systematic investigation of this generalized quantum double model based on C^* semisimple Hopf algebras.

On the other hand, although the topological phases on the closed surface seem to be natural from a mathematical perspective, real samples of topologically ordered material usually have boundaries and the boundary modes are easier to measure experimentally. Thus the topological phases on the surface with boundaries are of more practical and theoretical importance. Another crucial reason to investigate the boundary theory and domain wall theory is that there is a boundary-bulk duality, with which the boundary phase is obtained from the bulk phase using anyon condensation, and the bulk phase is recovered from the boundary phase by taking the Drinfeld center [29].

Not all topological phases allow the gapped boundaries (with a lattice realization). One of the crucial observations of the existence of a gapped boundary is that the chiral central charge $c_- = c_L - c_R$ must vanish [30–32]. Even for the $c_- = 0$ case, there exist some ungappable boundaries [32]. Therefore, a deep and comprehensive understanding of the boundary theory is of great importance. For the quantum double phase, the boundary is gappable, and the gapped boundary theories for finite group cases have been extensively explored [24, 25, 27, 28, 31, 33, 34]. However, for the general Hopf algebra case, the gapped boundary theory has not been systematically investigated yet. This is due, to some extent, to the mathematical difficulties when dealing with general Hopf algebras.

In this work, we will investigate the generalized quantum double model in detail and present the Hamiltonian construction of gapped boundaries and domain walls. To this end, we first review the generalized quantum double model on a closed surface and stress the problem of constructing the ribbon operators for this model and classifying the topological excitations. For the generalized quantum double model, there also exist electric charges, magnetic charges, and dyons, all these charges can be created with proper ribbon operators. Our construction of gapped boundary is parameterized by a triple of Hopf algebras (K, J, W) with some pairings among them. Each boundary site supports a representation of a generalized quantum double induced by the pairing. The domain wall between two quantum double phases is characterized by a quadruple of Hopf algebras (K, J_1, J_2, W) with some pairings among them. The left and right boundary sites support different representations of quantum doubles induced by different pairings. By utilizing the Hopf tensor network representation of quantum many-body states, we solve the ground state of the model with boundaries and domain walls and obtain the explicit ground states. Using this explicit exact ground state of the model, we can investigate various properties of the phase in presence of boundaries and domain walls. Especially, the entanglement entropy can be calculated directly. This also paves the way for applications of generalized quantum double phase in QECC and TQC.

The paper is organized as follows. In Sec. 2, we review the generalized quantum double model and present a detailed study of the ribbon operators and topological excitations. Sec. 3 establishes the boundary theory of the generalized quantum double model. Sec. 4 establishes the domain wall theory of the generalized quantum double model. In Sec. 5, we solve the extended generalized quantum double model with boundaries and domain walls, the explicit ground states are expressed as Hopf tensor network states. The appendices collect some detailed discussions and calculations.

2 Generalized quantum double model

Let's start with a brief review of the Hopf algebraic quantum double model [15, 18, 35] on a closed $2d$ surface. Kitaev's original construction of the quantum double model is based on finite group algebra $\mathbb{C}[G]$, and he also pointed out that the model can be generalized to the finite-dimensional Hopf algebra equipped with a Hermitian inner product with certain properties [5]. The first explicit construction is given in [15], and the corresponding ribbon operators are discussed in detail in [18, 35]. In this section, we will discuss the construction and stress the chirality of the construction. From the model, we can obtain Turaev-Viro type topological invariant [20, 36].

2.1 Generalized quantum double model

For a given $2d$ closed manifold Σ, consider a lattice[1] on it $C(\Sigma) = V(\Sigma) \cup E(\Sigma) \cup F(\Sigma)$, where $V(\Sigma)$, $E(\Sigma)$ and $F(\Sigma)$ are sets of vertices, edges and faces respectively. The dual lattice of $C(\Sigma)$ is a lattice $\tilde{C}(\Sigma)$ for which the vertices and faces of the original lattice are switched while the edge set remains unchanged. We assign a direction for each edge $e \in E(\Sigma)$, and the direction of corresponding dual edge $\tilde{e} \in \tilde{E}(\Sigma)$ is obtained by rotating the direction of e counterclockwise by $\pi/2$. As shown in Fig. 1, the original lattice is drawn

[1]Also known as a cellulation of Σ or a ribbon graph on Σ. We assume that the graph corresponding to the lattice is a simple graph, namely, a graph that no edge starts and ends at the same vertex.

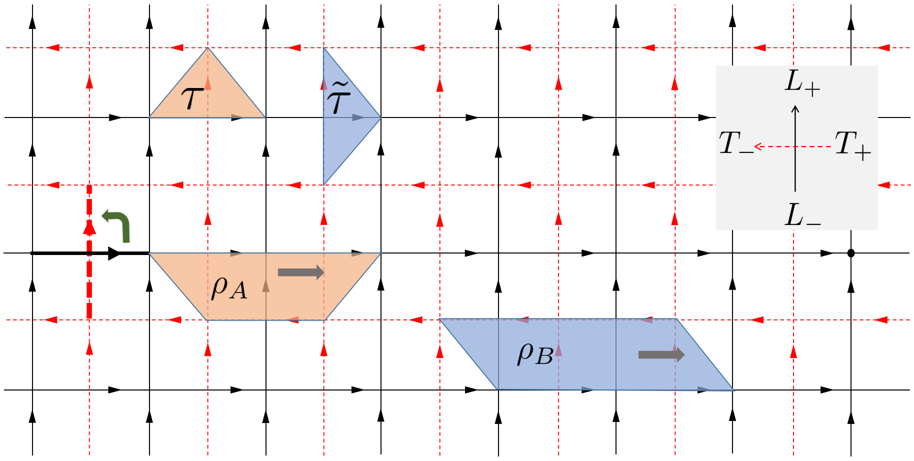


Figure 1. The depiction of geometric objects appeared in the definition of the generalized Kitaev model. The solid black lines represent direct lattice, the dashed red lines represent dual lattice.

with a black solid line, and the dual lattice is drawn with a red dashed line. A site $s = (v, f)$ is a pair of a vertex v and an adjacent face f (which is a dual vertex of the dual lattice). The site is drawn as a solid line connecting v and f. Two sites s, s' are called adjacent if they share a common vertex or face.

Hereinafter, we assume that H is a semisimple finite-dimensional C^* Hopf algebra (H has a Hilbert space structure given by Eq. (A.13)), its dual Hopf algebra will be denoted as $\hat{H}$ or $H^\vee$. For general facts of Hopf algebra, see Appendix A. To each edge of the lattice, we attach a copy of H, i.e., $\mathcal{H}_e = H$. The total Hilbert space is $\mathcal{H} = \otimes_{e\in E(\Sigma)}\mathcal{H}_e$. To define the model, we need to introduce four types of edge operators $L^h_+, L^h_-, T^\varphi_+, T^\varphi_-$ with $h \in H$ and $\varphi \in \hat{H}$, as follows:

$$L^h_+|x\rangle = |h \triangleright x\rangle = |hx\rangle, \tag{2.1}$$

$$L^h_-|x\rangle = |x \triangleleft S(h)\rangle = |xS(h)\rangle, \tag{2.2}$$

$$T^\varphi_+|x\rangle = |\varphi \rightharpoonup x\rangle = |\sum_{(x)}\langle\varphi, x^{(2)}\rangle x^{(1)}\rangle, \tag{2.3}$$

$$T^\varphi_-|x\rangle = |x \leftharpoonup \hat{S}(\varphi)\rangle = |\sum_{(x)}\langle\hat{S}(\varphi), x^{(1)}\rangle x^{(2)}\rangle = |\sum_{(x)}\langle\varphi, S(x^{(1)})\rangle x^{(2)}\rangle, \tag{2.4}$$

where we have adopted the Sweedler arrow notations. Notice that H can be regarded as a left H-modules with actions $h \triangleright x$ and $x \triangleleft S(h)$ (recall that $S(hg) = S(g)S(h)$ and $S(1_H) = 1_H$), so that L^h_+ and L^h_- are corresponding operator representations. H can also be regarded as left $\hat{H}$-modules with the actions $\varphi \rightharpoonup x$ and $x \leftharpoonup \hat{S}(\varphi)$ (recall that $\hat{S}(\varphi\psi) = \hat{S}(\psi)\hat{S}(\varphi)$ and $\hat{S}(\hat{1}) = \hat{1}$), so that T^φ_+ and T^φ_- are the corresponding operator representations.

Since the antipode is involutive $S^2 = \mathrm{id}$, the reverse of the edge direction is given by $x_e \mapsto \bar{x}_e = S(x_e)$. This is compatible with four edge actions, e.g., $S(L^h_-|x\rangle) = |S(xS(h))\rangle = |hS(x)\rangle = L^h_+|S(x)\rangle$ and $S(T^\varphi_+|x\rangle) = \sum_{(x)}\langle\varphi, x^{(2)}\rangle|S(x^{(1)})\rangle = T^\varphi_-|S(x)\rangle$. This means that all patterns of the edge directions are equivalent, hence the quantum double model can be constructed from arbitrary given pattern.

Let j be a directed edge and v one of its endpoints, we can define $L^h(j,v)$ as follows: if v is the origin of j, set $L^h(j,v) = L^h_-(j)$, otherwise, set $L^h(j,v) = L^h_+(j)$. Similarly, let j be a directed dual edge and f one of its endpoints, if f is the origin of j, set $T^\varphi(j,f) = T^\varphi_+(j)$, otherwise, set $T^\varphi(j,f) = T^\varphi_-(j)$. See Fig. 1 for an illustration of these choices. For a given site $s = (v,f)$, we order the edges around the vertex v and around the face f counterclockwise with the origin s. Using these conventions, the vertex operators and face operators on a site $s = (v,p)$ are defined as

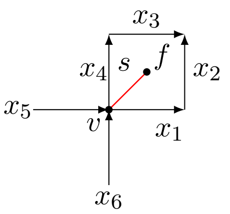
$$A^h(s) = \sum_{(h)} L^{h^{(1)}}(j_1,v) \otimes \cdots \otimes L^{h^{(n)}}(j_n,v), \tag{2.5}$$

$$B^\varphi(s) = \sum_{(\varphi)} T^{\varphi^{(1)}}(j_1,f) \otimes \cdots \otimes T^{\varphi^{(n)}}(j_n,f). \tag{2.6}$$

Notice that hereinafter we assume that comultiplication $\Delta(\varphi)$ is taken in $\hat{H}$, thus the order around the face is counterclockwise; if we use comultiplication Δ of $\hat{H}^{\mathrm{cop}}$, the orientation around the face must be clockwise.

Since $A^h(s)$ and $A^{h'}(s')$ can only share at most one edge (with opposite directions for $s \neq s'$), from the fact that $[L^x_+, L^y_-] = 0$ for all $x, y \in H$, we see $[A^h(s), A^{h'}(s')] = 0$ for all sites $s \neq s'$. Similarly, consider the dual lattice, from the fact that $[T^\psi_+, T^\zeta_-] = 0$ for all $\psi, \zeta \in \hat{H}$, we obtain $[B^\varphi(s), B^{\varphi'}(s)] = 0$ for all sites $s \neq s'$. For the non-adjacent sites s, s', $[A^h(s), B^\varphi(s')] = 0$, but for the adjacent sites, $A^h(s)$ and $B^\varphi(s')$ are in general not commutative. It can be checked that, when φ and h are cocommutative elements, we have $[A^h(s), B^\varphi(s')] = 0$ for $s \neq s'$.

Now consider a fixed site $s = (v,f)$, the corresponding vertex operator and face operator can only share two edges. For example



$$\begin{aligned}
A^h(s) &= \sum_{(h)} L^{h^{(1)}}_-(j_4) \otimes L^{h^{(2)}}_+(j_5) \otimes L^{h^{(3)}}_+(j_6) \otimes L^{h^{(4)}}_-(j_1), \\
B^\varphi(s) &= \sum_{(\varphi)} T^{\varphi^{(1)}}_-(j_1) \otimes T^{\varphi^{(2)}}_-(j_2) \otimes T^{\varphi^{(3)}}_+(j_3) \otimes T^{\varphi^{(4)}}_+(j_4), \\
A^h(s)B^\varphi(s) &= \sum_{(h)} B^{\varphi(S^{-1}(h^{(3)})\bullet h^{(1)})}(s) A^{h^{(2)}}(s).
\end{aligned} \tag{2.7}$$

In fact, this establishes an algebra homomorphism for arbitrary given site s:

$$\Phi : D(H) \to \mathrm{End}(\mathcal{H}_{tot}(s)), \quad \varphi \otimes h \mapsto D^{\varphi\otimes h}(s) := B^\varphi(s)A^h(s), \tag{2.8}$$

where $D(H) = \hat{H}^{\mathrm{cop}} \bowtie H$ is the quantum double of H (see Appendix A), and $\mathcal{H}_{tot}(s) = \otimes_{j\in\partial v, j\in\partial f}\mathcal{H}_j$. When h and φ are cocommutative elements, we have $[A^h(s), B^\varphi(s)] = 0$. We denote $\mathcal{D}(s) = \Phi(D(H))$. Then the mapping $h \mapsto \hat{1} \otimes h \mapsto B^{\hat{1}}(s)A^h$ provides a representation of H, and the mapping $\varphi \mapsto \varphi \otimes 1 \mapsto B^\varphi(s)A^1(s)$ provides a representation of $\hat{H}$. Therefore $[A^h(s), A^g(s)] = A^{[h,g]}(s)$, and similarly, $[B^\varphi(s), B^\psi(s)] = B^{[\varphi,\psi]}(s)$. We see that if $h \in Z(H)$ and $\varphi \in Z(\hat{H})$ (by $Z(H)$ we mean the center of Hopf algebra H), the commutators vanish.

Using the C^* structure, for involution invariant elements $h^* = h$ and $\varphi^* = \varphi$, the corresponding operators are Hermitian

$$(A^h(s))^\dagger = A^{h^*}(s) = A^h(s), \quad (B^\varphi(s'))^\dagger = B^{\varphi^*}(s') = B^\varphi(s'). \tag{2.9}$$

If we further require that h, φ are idempotents $h^2 = h$ and $\varphi^2 = \varphi$, then $A^h(s)$ and $B^\varphi(s')$ become projectors.

Now we are in a position to give the Hamiltonian construction for the Hopf algebraic quantum double model. The input data will be a finite-dimensional semisimple C^* Hopf algebra H. The Haar integrals $h_H \in H$ and $\varphi_{\hat{H}} \in \hat{H}$ exist and are unique, involutive, idempotent, cocommutative, and in the center of H and $\hat{H}$, respectively. The corresponding operator $A^{h_H}(s)$ only depends on vertex v and is thus denoted as A_v^H, and similarly $B^{\varphi_{\hat{H}}}(s)$ only depends on face f and is thus denoted as B_f^H. The local operators are projectors and they are commutative with each other. The frustration-free Hamiltonian is

$$H_H(\Sigma) = \sum_{v \in V(\Sigma)} (I - A_v^H) - \sum_{f \in F(\Sigma)} (I - B_f^H). \tag{2.10}$$

This model will be called a generalized quantum double model.

Remark 2.1. Notice that in the above construction, both A_v and B_f are defined from the left-module structures of H. We can also introduce a right-module construction. To define the model, we need to introduce four types of right-module edge operators $\tilde{L}_-^h, \tilde{L}_+^h, \tilde{T}_-^\varphi, \tilde{T}_+^\varphi$, as follows:

$$\tilde{L}_-^h|x\rangle = |x \triangleleft h\rangle = |xh\rangle, \tag{2.11}$$

$$\tilde{L}_+^h|x\rangle = |S(h) \triangleright x\rangle = |S(h)x\rangle, \tag{2.12}$$

$$\tilde{T}_-^\varphi|x\rangle = |x \leftharpoonup \varphi\rangle = |\sum_{(x)} \langle \varphi, x^{(1)}\rangle x^{(2)}\rangle, \tag{2.13}$$

$$\tilde{T}_+^\varphi|x\rangle = |\hat{S}(\varphi) \rightharpoonup x\rangle = |\sum_{(x)} \langle \hat{S}(\varphi), x^{(2)}\rangle x^{(1)}\rangle = |\sum_{(x)} \langle \varphi, S(x^{(2)})\rangle x^{(1)}\rangle, \tag{2.14}$$

where $h \in H$ and $\varphi \in \hat{H}$. Here H can be regarded as right H-modules via actions $x \triangleleft h$ and $S(h) \triangleright x$, $\tilde{L}_-^h$ and $\tilde{L}_+^h$ are the corresponding operator representations. H can also be regarded as right $\hat{H}$-modules via the actions $x \leftharpoonup \varphi$ and $\hat{S}(\varphi) \rightharpoonup x$, with $\tilde{T}_-^\varphi$ and $\tilde{T}_+^\varphi$ being the corresponding operator representations. For site $s = (v, f)$, we can order the edges around vertex v and around face f clockwise. The convention for choosing "+" or "−" remains unchanged. In this way, for Haar integrals $h_H \in H$ and $\varphi_{\hat{H}} \in \hat{H}$, the vertex operator $\tilde{A}_v^H$ and $\tilde{B}_f^H$ can be constructed. A right-module Hamiltonian $\tilde{H}_H(\Sigma) = -\sum_{v \in V(\Sigma)} \tilde{A}_v^H - \sum_{f \in F(\Sigma)} \tilde{B}_f^H$ is obtained.

The ground state space of the model (2.10) is given by

$$\mathcal{H}_{GS} = (\prod_{v \in V(\Sigma)} A_v^H \prod_{f \in F(\Sigma)} B_f^H)\mathcal{H}_{tot}. \tag{2.15}$$

The ground state degeneracy depends on the topology of surface Σ, and it is regardless of the choice of cellulation $C(\Sigma)$ (since different cellulations can be related with Pachner moves),

$$\mathrm{GSD} = \mathrm{Tr}(\prod_{v\in V(\Sigma)} A_v^H \prod_{f\in F(\Sigma)} B_f^H). \tag{2.16}$$

On a sphere (or infinite plane), GSD = 1, i.e., there is a unique ground state $|\Psi_{GS}\rangle$.

2.2 Ribbon operators

The ribbon operators are crucial for us to study the topological excitations. In this subsection, following the work of [18], we will construct ribbon operators and determine the corresponding ribbon operator algebra over a given ribbon. There are two kinds of ribbons, called type-A and type-B here, classified by the chirality of the triangles composing them. Their corresponding ribbon operator algebras are slightly different. A detailed presentation of ribbon operator algebra and the properties of ribbon operators will be given in Appendices B and C.

To begin with, we introduce several geometric objects [5, 18, 26, 28] (see Fig. 1 for illustrations and see Appendix B for a more comprehensive discussion):

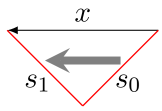
- A direct triangle $\tau = (s_0, s_1, e)$ consists of two adjacent sites s_0 and s_1 connected via the directed edge e. Similarly, the dual triangle $\tilde{\tau} = (s_0, s_1, \tilde{e})$ consists of two sites s_0 and s_1 connected via dual edge $\tilde{e}$. The direction of the triangle is defined as the direction from s_0 to s_1. Notice that the direction of the triangle may or may not match the direction of the edge.

- For a given (direct or dual) triangle, the chirality (which is called local orientation in [18]) of the triangle is defined as follows: it's called a left-handed (right-handed) triangle if the edge of the triangle is on the left-hand side (right-hand side) when we pass through the triangle along its positive direction. Notice that the chirality is fixed when the direction of the triangle is fixed. We will denote left-handed (right-handed) direct triangle and dual triangle as τ_L and $\tilde{\tau}_L$ (τ_R and $\tilde{\tau}_R$) respectively.

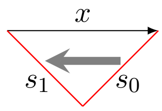
- A ribbon ρ is a collection of triangles $\tau_1, \cdots, \tau_n$ with a given direction such that $\partial_1\tau_j = \partial_0\tau_{j+1}$ and there is no self-intersection. A closed ribbon is the one for which there is no open ends, *viz.*, $\partial_1\tau_n = \partial_0\tau_1$. For a given directed ribbon, the direct triangle and dual triangle in it must have different chirality. A ribbon consisting of left-handed direct triangles and the right-handed dual triangle is called a type-A ribbon and will be denoted as ρ_A; similarly, a type-B ribbon consists of right-handed direct triangles and left-handed dual triangles and will be denoted as ρ_B.

The ribbon operator can be defined recursively. First, we define the triangle operator, then by introducing recursive relation, the ribbon operator is determined. To define triangle operators, we need to consider the different cases separately (the convention we use here is following Ref. [5], which is slightly different from the one in Ref. [18]).

For right-handed direct triangles, we have

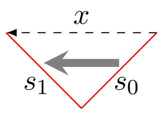


$$F^{h,\varphi}(\tau_R)|x\rangle = \varepsilon(h)T_{-}^{\varphi}|x\rangle = \varepsilon(h)|x \leftharpoonup \hat{S}(\varphi)\rangle, \tag{2.17}$$

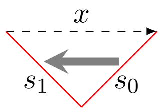


$$F^{h,\varphi}(\tau_R)|x\rangle = \varepsilon(h)T_{+}^{\varphi}|x\rangle = \varepsilon(h)|\varphi \rightharpoonup x\rangle. \tag{2.18}$$

For right-handed dual triangles, we have

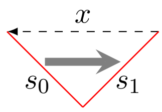


$$F^{h,\varphi}(\tilde{\tau}_R)|x\rangle = \hat{\varepsilon}(\varphi)L_{-}^{h}|x\rangle = \hat{\varepsilon}(\varphi)|x \triangleleft S(h)\rangle, \tag{2.19}$$

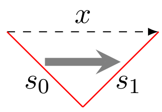


$$F^{h,\varphi}(\tilde{\tau}_R)|x\rangle = \hat{\varepsilon}(\varphi)L_{+}^{h}|x\rangle = \hat{\varepsilon}(\varphi)|h \triangleright x\rangle. \tag{2.20}$$

For left-handed dual triangles, we have

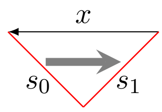


$$F^{h,\varphi}(\tilde{\tau}_L)|x\rangle = \hat{\varepsilon}(\varphi)\tilde{L}_{-}^{h}|x\rangle = \hat{\varepsilon}(\varphi)|x \triangleleft h\rangle. \tag{2.21}$$

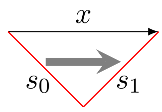


$$F^{h,\varphi}(\tilde{\tau}_L)|x\rangle = \hat{\varepsilon}(\varphi)\tilde{L}_{+}|x\rangle = \hat{\varepsilon}(\varphi)|S(h) \triangleright x\rangle. \tag{2.22}$$

For left-handed direct triangles, we have



$$F^{h,\varphi}(\tau_L)|x\rangle = \varepsilon(h)\tilde{T}_{-}^{\varphi}|x\rangle = \varepsilon(h)|x \leftharpoonup \varphi\rangle, \tag{2.23}$$



$$F^{h,\varphi}(\tau_L)|x\rangle = \varepsilon(h)\tilde{T}_{+}^{\varphi}|x\rangle = \varepsilon(h)|\hat{S}(\varphi) \rightharpoonup x\rangle. \tag{2.24}$$

The reason for the above choice of convention we have made is to let vertex and face operators be a special case of ribbon operators.

For general ribbon ρ, the ribbon operator on it can be defined recursively. We begin with the definition of type-B ribbon operators. They are built from the dual Hopf algebra $D(H)^\vee = D_B(H)^\vee = H^{\mathrm{op}} \otimes \hat{H}$, with

$$(h \otimes \varphi)(g \otimes \psi) = gh \otimes \varphi\psi, \tag{2.25}$$

$$\Delta_{D(H)^\vee}(h \otimes \alpha) = \sum_{k,(k),(h)} (h^{(1)} \otimes \hat{k}) \otimes (S(k^{(3)})h^{(2)}k^{(1)} \otimes \alpha(k^{(2)}\bullet)), \tag{2.26}$$

$$1_{D(H)^\vee} = 1_H \otimes \varepsilon_H, \quad \varepsilon_{D(H)^\vee}(h\otimes\alpha) = \varepsilon(h)\alpha(1), \tag{2.27}$$

where $\{k\}$ is an orthogonal basis of H and $\{\hat{k}\}$ its dual basis. For a ribbon ρ_B and an element $h\otimes\varphi \in D(H)^\vee$, we will define the ribbon operator $F^{h,\varphi}(\rho) = F^{h\otimes\varphi}(\rho_B)$. The operator acts nontrivially only on edges contained in ρ_B, and it commutes with all vertex and face operators except ones that act on the ending sites of the ribbon. These operators form an algebra $\mathcal{A}_{\rho_B} = \{F^{h,\varphi}(\rho_B) \mid h \in H^{\rm op}, \varphi \in \hat{H}\}$ called ribbon operator algebra. Consider the decomposition $\rho = \rho_B = \tau_1 \cup \tau_2$ where both τ_1 and τ_2 have the same direction with ρ and $\partial_1\tau_1 = \partial_0\tau_2$. For $h\otimes\varphi \in H^{\rm op}\otimes\hat{H}$, the ribbon operator on this composite ribbon can be defined as

$$\begin{aligned} F^{h,\varphi}(\rho) &= \sum_{(h\otimes\varphi)} F^{(h\otimes\varphi)^{(1)}}(\tau_1)F^{(h\otimes\varphi)^{(2)}}(\tau_2) \\ &= \sum_k \sum_{(k),(h)} F^{h^{(1)},\hat{k}}(\tau_1)F^{S(k^{(3)})h^{(2)}k^{(1)},\varphi(k^{(2)}\bullet)}(\tau_2). \end{aligned} \tag{2.28}$$

From the co-associativity of Hopf algebra $D(H)^\vee$, we see that this definition is independent of the decomposition of $\rho = \tau_1\cup\tau_2$. The construction for type-A ribbon is similar, but the ribbon operator is now built from $D(H)^{\vee,\rm op} = D_A(H)^\vee = H\otimes\hat{H}^{\rm op}$.

For closed ribbon, there is only one end $\partial\rho = \partial_0\rho = \partial_1\rho$. The vertex and face operators can be regarded as special cases of closed ribbon operators. $A_v = F^{h_H,\hat{1}}_{\sigma_A}$ is a type-A dual closed ribbon operator, and $B_f = F^{1,\varphi_{\hat{H}}}_{\sigma_B}$ is a type-B direct closed ribbon operator. $\tilde{A}_v = F^{h_H,\hat{1}}_{\sigma_B}$ is a type-B dual closed ribbon operator, and $\tilde{B}_f = F^{1,\varphi_{\hat{H}}}_{\sigma_A}$ is a type-A direct closed ribbon operator. A more comprehensive discussion of ribbon operators is given in Appendices B and C.

2.3 Topological excitations

The topological excitations of the model are point-like quasi-particles, the corresponding state $|\Psi\rangle$ violates some of stabilizer conditions $A_v^{h_H}|\Psi\rangle = |\Psi\rangle$ and $B_f^{\varphi_{\hat{H}}}|\Psi\rangle = |\Psi\rangle$ for some local vertex and face operators. For a ribbon ρ connecting sites s_0 and s_1, ribbon operator $F^{g,\psi}(\rho)$ commutes with all vertex and face operator in Hamiltonian of Eq. (2.10) except ones at sites $s_0 = \partial_0\rho$ and $\partial_1\rho = s_1$. Thus the ribbon operator creates excitations only at the ends of the ribbon.

Before we discuss the topological excitations of Hopf algebraic quantum double model $D(H)$, let's first recall the case that $H = \mathbb{C}[G]$ with G a finite group [5]. In this case, topological excitations are classified by $([g],\pi)$ where $[g]$ is conjugacy class of group G, and π is an irreducible representation of centralizer $C_G(g)$. Notice that there is a $\mathbb{C}C_G(g)$-module M_π corresponding to π, thus a topological charge can also be expressed as

$$a_{[g],\pi} = \mathbb{C}[G]\otimes_{\mathbb{C}C_G(g)} M_\pi. \tag{2.29}$$

The vacuum charge corresponds to $g = e_G$ and $\pi = \mathbb{1}$ (trivial representation). The antiparticle of the one in Eq. (2.29) is given by (note that $C_G(g) = C_G(g^{-1})$)

$$a_{[g^{-1}],\pi^\dagger} = \mathbb{C}[G]\otimes_{\mathbb{C}C_G(g^{-1})} M_{\pi^\dagger}. \tag{2.30}$$

The conjugacy class $[g]$ of a topological charge is called magnetic charge and the irrep π is called electric charge. When $g = e_G$, $a_{[e],\pi}$ is characterized by a representation of G and is called a chargeon; when $\pi = \mathbb{1}$, $a_{[g],\mathbb{1}}$ is called a fluxion; and when both $g \neq e_G$ and $\pi \neq \mathbb{1}$, $a_{[g],\pi}$ is called a dyon. The quantum dimension of the topological excitation is given by

$$\mathrm{FPdim}\, a_{[g],\pi} = |[g]| \dim \pi. \tag{2.31}$$

Here FPdim denotes the Frobenius-Peron dimension (see, e.g., [37]). These topological excitations form a UMTC $\mathbf{Rep}(D(G))$, the representation category of quantum double of a finite group G.

For general Hopf algebra H, a similar picture for classifying topological excitations exists but it is much more complicated (to our knowledge, this has not been discussed in physical literature). To introduce such a classification, let's first present several crucial mathematical notions. A fusion category $\mathcal{C}$ is called G-graded if there exists a decomposition

$$\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g, \tag{2.32}$$

where $\mathcal{C}_g$'s are some full abelian subcategories, and the tensor product of $\mathcal{C}$ maps $\mathcal{C}_g \times \mathcal{C}_h$ to $\mathcal{C}_{gh}$ for all g, h in the finite group G. G is called a grading group of $\mathcal{C}$; when G is maximal in the sense that any other grading is obtained by a quotient group of G, it is called a universal grading group and we denote it as $G = U(\mathcal{C})$. It follows from [37, Theorem 3.8] that there is a universal grading group $G = U(\mathbf{Rep}(H))$ for any semisimple Hopf algebra H.

Consider the largest central Hopf subalgebra $K(\hat{H}^{\mathrm{cop}})$ of $\hat{H}^{\mathrm{cop}}$, we have $K(\hat{H}^{\mathrm{cop}}) = \mathbb{C}G^{\vee}$, which is commutative, and $\mathbf{Rep}(\hat{H}^{\mathrm{cop}}) = \oplus_{g\in G}\mathbf{Rep}(\hat{H}^{\mathrm{cop}})_g$. Suppose that H_g is a Hopf subalgebra of H such that $\mathbf{Rep}(\hat{H}_g) = \oplus_{x\in C_G(g)}\mathbf{Rep}(\hat{H}^{\mathrm{cop}})_x$. It is proved that $K(\hat{H}^{\mathrm{cop}}) = \mathbb{C}G^{\vee}$ is a normal Hopf subalgebra of $D(H)$ [38], and $g \in G$ becomes an irreducible character of $K(\hat{H}^{\mathrm{cop}})$. We denote $\mathcal{I}_g = \{M_g\}$ the set of all irreducible representations of $\hat{H}^{\mathrm{cop}} \bowtie H_g$ (here "$\bowtie$" denotes bicrossed product) such that the character χ_{M_g}, when restricted on $K(\hat{H}^{\mathrm{cop}})$, satisfies $\chi_{M_g}|_{K(\hat{H}^{\mathrm{cop}})} = g \dim M$. With the above preparation, we are now at a position to present the classification of topological excitations, namely

$$a_{g,M_g} = H \otimes_{H_g} M_g, \tag{2.33}$$

where $g \in G$ and $M_g \in \mathcal{I}_g$. This completely classifies the irreducible representations of the quantum double $D(H)$, see [38] for a rigorous proof. The $g \in G$ can be regarded as magnetic charge and M electric charge. When $g = e_G$, $a_{e_G,M_{e_G}}$ is called a chargeon; when $M_g = \mathbb{1}$, $a_{g,\mathbb{1}}$ is called a fluxion; and when both $g \neq e$ and $M_g \neq \mathbb{1}$, a_{g,M_g} is called a dyon. The quantum dimension of the topological excitation is

$$\mathrm{FPdim}\, a_{g,M_g} = \frac{|G|}{\dim H_g} \dim M_g. \tag{2.34}$$

These topological excitations form a UMTC $\mathbf{Rep}(D(H))$, the representation category of Hopf algebra H.

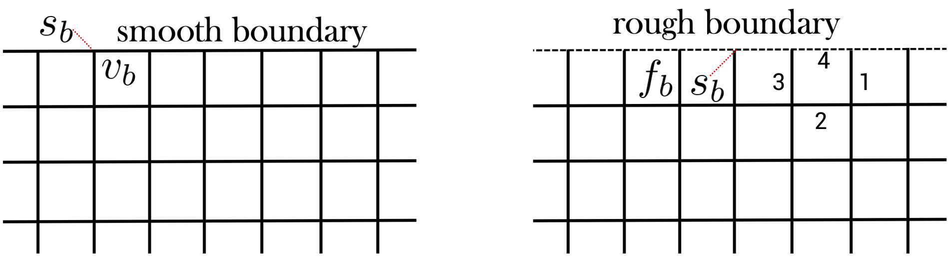


Figure 2. The depiction of smooth boundary and rough boundary.

3 Gapped boundary theory

In this section, we will establish the theory of gapped boundaries for the generalized quantum double model. For special case that $H = \mathbb{C}[G]$, the boundary theories have been investigated from aspects in Refs. [24, 25, 27, 28, 31, 33, 34]. But for the general Hopf algebra case, the problem is not yet systematically studied. Here, we will give the lattice construction of the gapped boundary.

3.1 Gapped boundary theory I

Let's first consider a simple construction, then, in the next subsection, a more general construction will be discussed. Consider the surface Σ with a single boundary $\partial\Sigma$. For a given cellulation $C(\Sigma) = C(\Sigma \setminus \partial\Sigma) \cup C(\partial\Sigma)$, we need to investigate the vertices, edges and faces in the vicinity of the boundary. Without loss of generality, we define the orientation of the boundary such that the bulk is always on the left-hand side when traversing the boundary.

Our first construction of the gapped boundary is based on a Hopf subalgebra $K \subseteq H$. The boundary edges are projected to the subspace

$$\mathcal{H}_{e_b} = \Pi_K H = K. \tag{3.1}$$

The boundary vertex operator is chosen as

$$A^K_{v_b} = A^{h_K}_{v_b}, \tag{3.2}$$

where h_K is the Haar integral of K. It is obvious that $[A^K_{v_b}, A^K_{v'_b}] = 0 = [A^K_{v_b}, A^H_v]$ for all v_b, v'_b, v. Since h_K and $\varphi_{\hat{H}}$ are Haar integral, they are cocommutative, which further implies that $[A^K_{v_b}, B^H_f] = 0$ for all v_b and f.

This kind of construction is a natural generalization of the constructions for the group algebra case [26, 27]. If we choose $H = \mathbb{C}[G]$ and $K = \mathbb{C}[N]$ with N as a subgroup of G, our model reduces to the group-algebra boundary model.

Let's take a closer look at two typical examples of this kind of construction.

Example 1 (Smooth boundary)**.** For the smooth boundary, the corresponding Hopf subalgebra is $K = H$. On a boundary site s_b, we have the local operator algebra H generated by

$$A^k(s_b)A^l(s_b) = A^{kl}(s_b). \tag{3.3}$$

The boundary excitations are characterized by the UFC $\mathsf{Rep}(H)$.

Example 2 (Rough boundary)**.** For the rough boundary, the corresponding Hopf subalgebra is the trivial one $K = \mathbb{C}1_H$ and the Haar integral is $h_K = 1_H$. In this case, boundary edges are fixed with label 1_H and $A^K_{v_b} = \mathrm{id}$. Consider the boundary face operator as shown in Fig. 2 acting on $x_1, x_2, x_3, x_4 = 1_H$ as

$$B^\varphi(s_b)|x_1x_2x_3x_4\rangle = (B^\varphi(s_b)|x_1x_2x_3\rangle) \otimes |1_H\rangle. \tag{3.4}$$

Thus the rough boundary can be obtained equivalently by removing all boundary edges and boundary vertices, the boundary local operator algebra is generated by

$$B^\varphi(s_b)B^\psi(s_b) = B^{\varphi\psi}(s_b). \tag{3.5}$$

In this way, the boundary excitations are characterized by the UFC $\mathsf{Rep}(\hat{H})$.

The boundary topological phases for smooth and rough boundaries are Morita equivalent. To see this, let's recall the Kitaev-Kong construction of boundary theory of the Levin-Wen model for $2d$ topological phase [25], for which the bulk phase is determined by a UFC $\mathcal{C}$, and the boundary is characterized by an indecomposable $\mathcal{C}$-module category $\mathcal{M}$. The boundary excitation is given by UFC $\mathsf{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ of all $\mathcal{C}$-module functors. By transforming the basis of the Hopf algebraic quantum double model to the fusion basis, the quantum double model can be mapped to a Levin-Wen model [17, 22, 36]. In this case, the input UFC for the bulk is $\mathcal{C} = \mathsf{Rep}(H)$. For the smooth boundary, the module category is $\mathcal{M}_s = \mathsf{Rep}(H)$, and for the rough boundary, the module category is $\mathcal{M}_r = \mathsf{Vect}$. It can be proved that there is a monoidal equivalence

$$\mathsf{Fun}_{\mathsf{Rep}(H)}(\mathsf{Vect}, \mathsf{Vect}) \simeq \mathsf{Rep}(\hat{H}). \tag{3.6}$$

This implies that the topological excitations of smooth and rough boundaries are Morita equivalent.

3.2 Gapped boundary theory II

Let's now introduce a more general construction based on a generalized quantum double. The cellulation of the surface for this construction is slightly different from the one given in the previous subsection. As shown in Fig. 3, the boundary is drawn as a bold solid line. The boundary vertices are vertices on the line, which we denote as v_b. The boundary face is the right end of the dual boundary edges. There is only one boundary face for each boundary and we denote it as f_b. There are two kinds of boundary sites: the one $s_b = (v_b, f_b)$ consists of boundary vertex and boundary face, which we call outer boundary site; the one $s_b = (v_b, f)$ consists of boundary vertex and inner face, which we call inner boundary site.

Definition 3. *A pairing $\lambda = \langle \bullet, \bullet \rangle : J \otimes K \to \mathbb{C}$ between two Hopf algebras J, K is a bilinear map satisfying*

$$\langle hg, a\rangle = \sum_{(a)} \langle h, a^{(1)}\rangle \langle g, a^{(2)}\rangle, \tag{3.7}$$

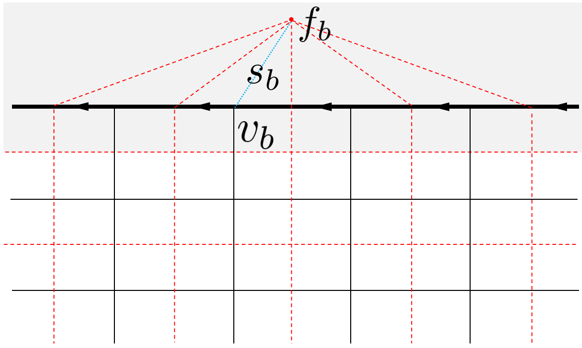


Figure 3. The depiction of the boundary, boundary face, boundary vertex and boundary site.

$$\langle h, ab\rangle = \sum_{(h)} \langle h^{(1)}, a\rangle \langle h^{(2)}, b\rangle, \tag{3.8}$$

$$\langle 1_J, a\rangle = \varepsilon_K(a), \quad \langle h, 1_K\rangle = \varepsilon_J(h). \tag{3.9}$$

There is a convolution algebra structure over $C = \mathrm{Hom}(J \otimes K, \mathbb{C})$, with the convolution defined as

$$\begin{aligned}(\varphi * \psi)(h \otimes a) :&= (\mu_{\mathbb{C}} \circ (\varphi \otimes \psi) \circ \Delta_{J\otimes K})(h \otimes a) \\ &= \sum_{(h),(a)} \varphi(h^{(1)} \otimes a^{(1)})\psi(h^{(2)} \otimes a^{(2)}).\end{aligned} \tag{3.10}$$

The multiplication and unit are defined as

$$\mu_C(\varphi \otimes \psi) = \varphi * \psi, \quad \eta_C = \eta_{\mathbb{C}} \circ \varepsilon_{J\otimes K} = \varepsilon_J \varepsilon_K. \tag{3.11}$$

It is easy to verify that (for semisimple Hopf algebras), the convolutional inverse of λ is

$$\lambda^{-1}(h \otimes a) = \langle S(h), a\rangle = \langle h, S(a)\rangle = \langle S^{-1}(h), a\rangle = \langle h, S^{-1}(a)\rangle. \tag{3.12}$$

Hereinafter, we assume that the pairing is also consistent with the C^* structure $\lambda(\varphi^* \otimes h) = \overline{\lambda(\varphi \otimes S(h)^*)}$.

Definition 4 (Generalized quantum double [39]). *For a given pairing $\lambda : J \otimes K \to \mathbb{C}$, the general quantum double $D_\lambda(J^{\mathrm{cop}}, K) := J^{\mathrm{cop}} \bowtie_\lambda K$ is a Hopf algebra built on $J^{\mathrm{cop}} \otimes K$ with Hopf algebra multiplication given by*

$$(h \otimes a)(g \otimes b) = \sum_{(a),(g)} hg^{(2)} \otimes a^{(2)} b \lambda(g^{(1)} \otimes a^{(1)})\lambda^{-1}(g^{(3)} \otimes a^{(3)}), \tag{3.13}$$

where the comultiplication of g is taken in J^{cop}.

When taking $J = H^\vee$, $K = H$ and $\lambda(\varphi \otimes x) = \varphi(x)$, one recovers the Drinfeld quantum double $D(H) = (H^\vee)^{\mathrm{cop}} \bowtie H$ as shown in Appendix A.

Now we are in a position to present the construction of boundary Hamiltonian. The boundary is characterized by the following collection of data:

- Two finite-dimensional C^* semisimple Hopf algebras K and J for which there exists a pairing $\lambda : J \otimes K \to \mathbb{C}$. And there is pairing between $\hat{H}$ and K, $\gamma : \hat{H} \otimes K \to \mathbb{C}$.

- A finite-dimensional C^* semisimple Hopf algebra W, which is a K-bimodule and the algebra morphisms μ_W, η_W are K-bimodule homomorphisms [2]. The left and right K-actions are denoted as $h \triangleright w$ and $w \triangleleft h$. The right action induces a left K-module structure $w \triangleleft S(h)$. We denote the operators corresponding to these two actions as

$$L_+^{K,h} w = h \triangleright w = hw, \quad L_-^{K,h} w = w \triangleleft S(h) = wS(h). \tag{3.14}$$

  We also assume that the antipode law holds: $S(h \triangleright w) = S(w) \triangleleft S(h)$ and $S(w \triangleleft h) = S(h) \triangleright S(w)$; and for C^* structure, $(h \triangleright w)^* = w^* \triangleleft h^*$ and $(w \triangleleft h)^* = h^* \triangleright w^*$; for Haar integral $h_W \in W$ and $k \in K$, we assume $kh_W = \varepsilon(k)h_W = h_W k$.

- H is a K-bimodule and H is a W-bimodule, and they both satisfy the antipode law.

- There is a pairing $\zeta : J \otimes W \to \mathbb{C}$ between J and W. The pairing induces the J-bimodule structure of W: left module $\varphi \rightharpoonup w = \sum_{(w)} w^{(1)} \zeta(\varphi \otimes w^{(2)})$; and right module $w \leftharpoonup \varphi = \sum_{(w)} w^{(2)} \zeta(\varphi \otimes w^{(1)})$. The right module structure induces a left module structure by $w \leftharpoonup S(\varphi)$. The corresponding two left module structure operators are

$$\begin{aligned} T_+^{J,\varphi} w &= w \rightharpoonup \varphi = \sum_{(w)} w^{(1)} \zeta(\varphi \otimes w^{(2)}), \\ T_-^{J,\varphi} w &= w \leftharpoonup S(\varphi) = \sum_{(w)} w^{(2)} \zeta^{-1}(\varphi \otimes w^{(1)}). \end{aligned} \tag{3.15}$$

  For the Haar integral of $\varphi_{\hat{W}}$, we assume that $\sum_{(x)} \langle \varphi_{\hat{W}}, x^{(1)} \rangle \zeta(\phi, x^{(2)}) = \varepsilon_J(\phi) \langle \varphi_{\hat{W}}, x \rangle$ for all $\phi \in J$.

- There is a pairing $\alpha : \hat{H} \otimes W \to \mathbb{C}$ which also induces $\hat{H}$-bimodule structure over W in a similar way as in Eq. (3.15). For the Haar integral of $\varphi_{\hat{W}}$, we assume that $\sum_{(x)} \langle \varphi_{\hat{W}}, x^{(1)} \rangle \zeta(\phi, x^{(2)}) = \varepsilon_J(\phi) \langle \varphi_{\hat{W}}, x \rangle$ for all $\phi \in \hat{H}$.

- Different pairings are consistent when acting on two Hopf algebras for which there is a module structure. For example, $\lambda : J \otimes K \to \mathbb{C}$ and $\zeta : J \otimes W \to \mathbb{C}$ are two pairings and W has left K-module structures $h \triangleright w$ and $w \triangleleft S(h)$, then the consistency conditions read

$$\sum_{(\varphi)} \lambda(\varphi^{(1)} \otimes h) \zeta(\varphi^{(2)} \otimes w) = \zeta(\varphi \otimes (h \triangleright w)), \tag{3.16}$$

$$\sum_{(\varphi)} \lambda(\varphi^{(1)} \otimes h) \zeta^{-1}(\varphi^{(2)} \otimes w) = \zeta(\varphi \otimes (w \triangleleft S(h))). \tag{3.17}$$

[2]This means that W is a K-module algebra, which implies that $x(h \triangleright y) = (x \triangleleft h)y$. Notice that this is different from the notion of Hopf module algebra.

Remark 3.1. As for the bulk, we can also construct the boundary using the right-module structure, the generalization is straightforward.

For each boundary edge e_b, we assign a Hopf algebra $\mathcal{H}_{e_b} = W$. The total boundary space becomes $\mathcal{H}(\partial\Sigma) = \otimes_{e_b \in E(\partial\Sigma)} \mathcal{H}_{e_b}$, thus, the total Hilbert space becomes $\mathcal{H}_{tot} = \mathcal{H}(\Sigma \setminus \partial\Sigma) \otimes \mathcal{H}(\partial\Sigma)$. As shown in Fig. 3, the boundary corresponds to the shaded region, and there is only one boundary face outside a given boundary. For boundary site $s_b = (v_b, f_b)$ and $h \in K$, we can define (in counterclockwise order)

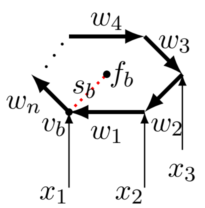
$$A^h(s_b) = \sum_{(h)} L^{K,h^{(1)}}(j_1, v_b) \otimes \cdots \otimes L^{K,h^{(n)}}(j_n, v_b), \tag{3.18}$$

where, for bulk edge, $L^{K,h^{(l)}}(j_l, v_b)$ act on H; and for boundary edge they act on W. The convention to choose L_+, L_- is the same as for the bulk.

For the outer boundary site s_b and $\varphi \in J$, we introduce an operator which acts non-trivially only on all boundary edges (in counterclockwise order),

$$B^\varphi(s_b) = \sum_{(\varphi)} T^{J,\varphi^{(1)}}(j_1, f_b) \otimes T^{J,\varphi^{(2)}}(j_2, f_b) \otimes \cdots \otimes T^{J,\varphi^{(n)}}(j_n, f_b). \tag{3.19}$$

The convention to choose T_+, T_- is also the same as for the bulk. For example,



$$\begin{aligned} & B^\varphi(s_b)|w_1, \cdots, w_n\rangle \\ = & \sum_{(w_1),\cdots,(w_n),(\varphi)} \zeta(\varphi^{(1)} \otimes w_1^{(2)}) \cdots \zeta(\varphi^{(n)} \otimes w_n^{(2)})|w_1^{(1)}, \cdots, w_n^{(1)}\rangle \\ = & \sum_{(w_1),\cdots,(w_n)} \zeta(\varphi \otimes w_1^{(2)} \cdots w_n^{(2)})|w_1^{(1)}, \cdots, w_n^{(1)}\rangle, \end{aligned} \tag{3.20}$$

where comultiplication component $\varphi^{(i)}$ are taken in J.

Proposition 1. *For the outer boundary site s_b, the operators $A^h(s_b)$ and $B^\varphi(s_b)$ satisfy*

$$A^h(s_b)B^\varphi(s_b) = \sum_{(h),(\varphi)} \lambda(\varphi^{(3)} \otimes h^{(1)})\lambda^{-1}(\varphi^{(1)} \otimes h^{(3)})B^{\varphi^{(2)}}(s_b)A^{h^{(2)}}(s_b). \tag{3.21}$$

Therefore, by mapping $\varphi \otimes h \in J^{\mathrm{cop}} \bowtie_\lambda K$ to $B^\varphi(s_b)A^h(s_b)$, we obtain a representation of $J^{\mathrm{cop}} \bowtie_\lambda K$ over $\mathcal{H}(s_b) = \otimes_{e \in \partial s_b} \mathcal{H}_e$.

Proof. Consider the setting as in Eq. (3.20), we see that

$$\begin{aligned} & A^h(s_b)B^\varphi(s_b)|w_1, \cdots, w_n\rangle|x_1\rangle \\ = & \sum_{(w_i),(h)} \zeta(\varphi \otimes w_1^{(2)} \cdots w_n^{(2)})|h^{(3)}w_1^{(1)}, \cdots, w_n^{(1)}S(h^{(1)})\rangle|h^{(2)}x_1\rangle. \end{aligned} \tag{3.22}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{(h),(\varphi)} \lambda(\varphi^{(3)} \otimes h^{(1)})\lambda^{-1}(\varphi^{(1)} \otimes h^{(3)})B^{\varphi^{(2)}}(s_b)A^{h^{(2)}}(s_b)|w_1, \cdots, w_n\rangle|x_1\rangle \\
=& \sum_{(w_i),(h),(\varphi)} \lambda(\varphi^{(3)} \otimes h^{(1)})\lambda^{-1}(\varphi^{(1)} \otimes h^{(5)})B^{\varphi^{(2)}}(s_b)|h^{(4)}w_1, \cdots, w_n S(h^{(2)})\rangle|h^{(3)}x_1\rangle \\
=& \sum_{(w_i),(h),(\varphi)} \lambda(\varphi^{(3)} \otimes h^{(1)})\lambda^{-1}(\varphi^{(1)} \otimes h^{(5)})\zeta(\varphi^{(2)} \otimes (h^{(4)}w_1)^{(2)} w_2^{(2)} \cdots (w_n S(h^{(2)}))^{(2)}) \\
& |(h^{(4)}w_1)^{(1)}, w_2^{(1)}, \cdots, (w_n S(h^{(2)}))^{(1)}\rangle|h^{(3)}x_1\rangle \\
=& \sum_{(w_i),(h)} \zeta(\varphi \otimes S^{-1}(h^{(7)})h^{(6)}w_1^{(2)}w_2^{(2)} \cdots w_n^{(2)} S(h^{(2)})h^{(1)}) \\
& |h^{(5)}w_1^{(1)}, w_2^{(1)}, \cdots, w_n^{(1)} S(h^{(3)})\rangle|h^{(4)}x_1\rangle \\
=& \sum_{(w_i),(h)} \zeta(\varphi \otimes w_1^{(2)} \cdots w_n^{(2)})|h^{(3)}w_1^{(1)}, \cdots, w_n^{(1)} S(h^{(1)})\rangle|h^{(2)}x_1\rangle.
\end{aligned} \tag{3.23}$$

This completes the proof of Eq. (3.21). Using Eq. (3.21), it's easy to check that $B^\varphi(s_b)A^h(s_b)$ is a representation of $J^{\mathrm{cop}} \bowtie K$. ■

Since K and J are both C^* semisimple Hopf algebras, they both have unique Haar integrals h_K and φ_J. Then we can define the boundary stabilizer operators as

$$A^K_{v_b} = A^{h_K}(s_b), \quad B^J_{f_b} = B^{\varphi_J}(s_b). \tag{3.24}$$

Since both of h_K and φ_J are cocommutative, the vertex and face operators only depend on their respective vertices and faces.

Proposition 2. *All boundary stabilizer operators are projectors, and they satisfy*

$$(A^{h_K}_{v_b})^\dagger = A^{h_K^*}_{v_b} = A^{h_K}_{v_b}, \quad (A^{h_K}_{v_b})^2 = A^{h_K^2} = A^{h_K}_{v_b}, \tag{3.25}$$

$$(B^{\varphi_J}_{f_b})^\dagger = B^{\varphi_J^*}_{f_b} = B^{\varphi_J}_{f_b}, \quad (B^{\varphi_J}_{f_b})^2 = B^{\varphi^2}_{f_b} = B^{\varphi_J}_{f_b}. \tag{3.26}$$

All boundary stabilizer operators commute with each other and they commute with all bulk stabilizer operators.

Proof. The inner product of W is determined by the Haar integral $\varphi_{\hat{W}}$, i.e., $\langle x, y\rangle = \varphi_{\hat{W}}(x^* y)$. We see that $\langle x, L_+^h y\rangle = \varphi_{\hat{W}}(x^*(h \triangleright y)) = \varphi_{\hat{W}}((h^* \triangleright x)^* y) = \langle L_+^{h^*} x, y\rangle$, thus $(L_+^h)^\dagger = L_+^{h^*}$. Similarly, $(L_-^h)^\dagger = L_-^{h^*}$. This implies that $(A^{h_K}_{v_b})^\dagger = \sum_{(h_K)} \otimes_j (L^{h_K^{(j)}}(j, v_b))^\dagger = \sum_{(h_K)} \otimes_j L^{h_K^{(j)*}}(j, v_b) = A^{h_K^*}_{v_b} = A^{h_K}_{v_b}$.

Similarly, it follows that

$$\begin{aligned}
\langle x, T_+^{\varphi} y\rangle &= \sum_{(y)} \varphi_{\hat{W}}(x^* y^{(1)} \zeta(\varphi \otimes y^{(2)})) \\
&= \sum_{(y),(x^*),(\varphi)} \varphi_{\hat{W}}(x^{*(1)} \zeta(1_J \otimes x^{*(2)}) \varepsilon_J(\varphi^{(2)}) \zeta(\varphi^{(1)} \otimes y^{(2)}) y^{(1)}) \\
&= \sum_{(y),(x^*),(\varphi)} \varphi_{\hat{W}}(x^{*(1)} \zeta(\varphi^{(2)} S(\varphi^{(3)}) \otimes x^{*(2)}) \zeta(\varphi^{(1)} \otimes y^{(2)}) y^{(1)}) \\
&= \sum_{(y),(x^*),(\varphi)} \varphi_{\hat{W}}(x^{*(1)} y^{(1)} \zeta(\varphi^{(2)} \otimes x^{*(2)}) \zeta(S(\varphi^{(3)}) \otimes x^{*(3)}) \zeta(\varphi^{(1)} \otimes y^{(2)})) \\
&= \sum_{(y),(x^*),(\varphi)} \varphi_{\hat{W}}(y^{(1)} x^{*(1)} \zeta(\varphi^{(2)} \otimes y^{(2)} x^{*(2)}) \zeta(S(\varphi^{(1)}) \otimes x^{*(3)})) \\
&= \sum_{(x^*),(y),(\varphi)} \varphi_{\hat{W}}((y x^{*(1)})^{(1)} \zeta(\varphi^{(2)} \otimes (y x^{*(1)})^{(2)}) \zeta(S(\varphi^{(1)}) \otimes x^{*(3)})) \\
&= \sum_{(x^*)} \varphi_{\hat{W}}(x^{*(1)} y \zeta(S(\varphi) \otimes x^{*(2)})) = \langle T_+^{\varphi^*} x, y\rangle.
\end{aligned} \tag{3.27}$$

This means that $(T_+^{\varphi})^\dagger = T_+^{\varphi^*}$. We can similarly show that $(T_-^{\varphi})^\dagger = T_-^{\varphi^*}$. Then we see that $(B_{f_b}^{\varphi_J})^\dagger = B_{f_b}^{\varphi_J^*} = B_{f_b}^{\varphi_J}$.

Using the proposition 1 and fact that $h_K^2 = h_K$ and $\varphi_J^2 = \varphi_J$, we see that $A_{v_b}^{h_K}$ and $B_{f_b}^{\varphi_J}$ are idempotent. Further using the fact that h_K and φ_J are cocommutative, we can show that all stabilizers are commutative. ■

The boundary Hamiltonian is given by the local commuting projector Hamiltonian

$$H_{K,J}(\partial\Sigma) = \sum_{v_b \in V(\partial\Sigma)} (I - A_{v_b}^K) + \sum_{f_b \in F(\partial\Sigma)} (I - B_{f_b}^J). \tag{3.28}$$

The total Hamiltonian is thus

$$H_H^{K,J} = H_H(\Sigma \setminus \partial\Sigma) + H_{K,J}(\partial\Sigma). \tag{3.29}$$

This model will be called the extended Hopf algebraic quantum double model.

The ground state degeneracy now depends on both the topology of $2d$ surface Σ and its boundaries. It's still regardless of the choice of cellulation $C(\Sigma)$.

$$\mathrm{GSD} = \mathrm{Tr}\Big(\prod_{v \in V(\Sigma \setminus \partial\Sigma)} A_v^H \prod_{f \in F(\Sigma \setminus \partial\Sigma)} B_f^{\hat{H}} \prod_{v_b \in V(\partial\Sigma)} A_{v_b}^K \prod_{f_b \in F(\partial\Sigma)} B_{f_b}^J \Big) \tag{3.30}$$

When Σ is a sphere, by cutting boundaries, the ground state space will become degenerate.

To shed further light on this construction of boundaries, let's consider two examples.

Example 5. Consider the case that $W = H$, $J \leq \hat{H}$ and $K \leq H$ are Hopf subalgebras, we see that they satisfy all the requirements for constructing the boundary. In this kind of model, it's more convenient to discuss boundary condensation and confinement using the ribbon operators.

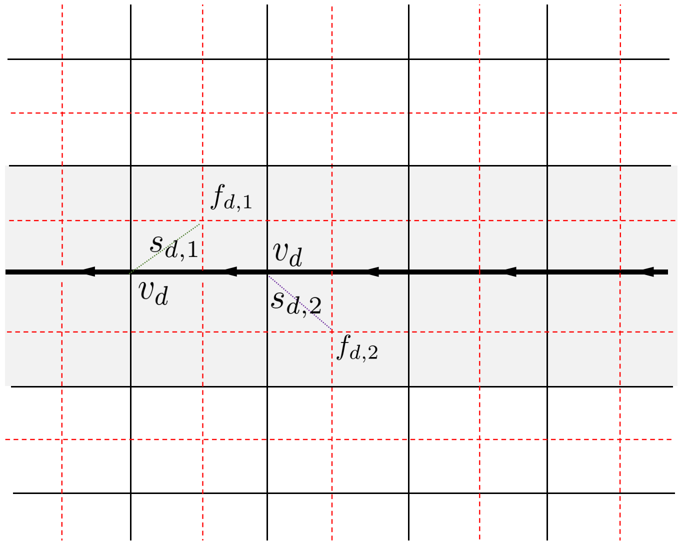


Figure 4. The depiction of the domain wall, domain wall face, domain wall vertex and domain wall site.

Example 6. For finite group algebra $H = \mathbb{C}[G]$, take $W = \mathbb{C}[G]$, $K = \mathbb{C}[M]$, $J = \mathbb{C}^{G/N}$ with M a subgroup and N a normal subgroup of G. We can also obtain a corresponding generalized model. This model is similar as the one given in [26].

4 Gapped domain wall theory

Let's now consider the gapped domain wall, which is closely related to the gapped boundary: the gapped boundary can be regarded as a domain wall that separates the generalized quantum double phase from the vacuum phase; conversely, using the folding trick, a gapped domain wall can be transformed into a gapped boundary. In this section, let's generalize the construction of gapped boundaries to the case of general domain walls.

Suppose that two $2d$ quantum double phases, which are characterized by Hopf algebras H_1 and H_2, are separated by a $1d$ domain wall. To construct the domain wall Hamiltonian, we need the following input data (all Hopf algebras here are finite-dimensional C^* semisimple Hopf algebras):

- Two domain wall face Hopf algebras J_1, J_2 and one domain wall vertex Hopf algebra K, and there exist pairings, $\lambda_1 : J_1 \otimes K \to \mathbb{C}$ and $\lambda_2 : J_2 \otimes K \to \mathbb{C}$.
- A domain Hopf algebra W which is a K-bimodule. The left action of K on W induced by $h \triangleright w$ and $w \triangleleft S(h)$ are denoted as $T_+^{K,h}$ and $T_-^{K,h}$ respectively.
- Two bulk Hopf algebras H_1, H_2 are two K-bimodules. The bimodule structures also induce two inequivalent left module structures on H_i.
- There are pairings $\zeta_1 : J_1 \otimes W \to \mathbb{C}$ and $\zeta_2 : J_2 \otimes W \to \mathbb{C}$. These two pairings induce the J_1 and J_2 bimodule structures on W: $\phi \rightharpoonup w = \sum_{(w)} w^{(1)} \zeta_i(\phi \otimes w^{(2)})$ and

$w \leftharpoonup \phi = \sum_{(w)} w^{(2)} \zeta_i(\phi \otimes w^{(1)})$. This further induce two left J_i module structure operators

$$T_+^{J_i,\phi} w = \phi \rightharpoonup w, \quad T_-^{J_i,\phi} w = w \leftharpoonup S(\phi). \tag{4.1}$$

- There are pairings $\xi_1 : J_1 \otimes H_1 \to \mathbb{C}$ and $\xi_2 : J_2 \otimes H_2 \to \mathbb{C}$, which also induce two inequivalent left J_i-module structures on H_i.

Similar to the gapped boundary construction, we will also assume some consistency conditions for these pairings and actions.

For each edge in the first and second bulks $C(\Sigma_1)$ and $C(\Sigma_2)$, we assign the Hopf algebras H_1 and H_2 respectively. Two bulk total Hilbert spaces are thus $\mathcal{H}(\Sigma_i) = \otimes_{e\in C(\Sigma_i)} H_i$ with $i = 1, 2$. For each domain wall edge $e_d \in C(\Sigma_d)$ (where $\Sigma_d = \partial\Sigma_1 = \partial\Sigma_2$), we attach the Hopf algebra W to it. The total domain wall Hilbert space is $\mathcal{H}(\Sigma_d) = \otimes_{e_d \in C(\Sigma_d)} W$. We define the domain wall vertex as the vertex on the wall and denote them as v_d. The domain wall face on the first bulk is the face near the wall, in the first bulk, and they are denoted as $f_{d,1}$. Similarly, we have $f_{d,2}$.

For a domain wall site $s_d = (v_d, f_d)$ and $h \in K$, we defined (in counterclockwise order)

$$A^h(s_d) = \sum_{(h)} L^{K,h^{(1)}}(j_1, v_d) \otimes \cdots \otimes L^{K,h^{(n)}}(j_n, v_d), \tag{4.2}$$

where, for first and second bulk edges, $L^{K,h^{(1)}}(j_l, v_d)$ act on H_i; for domain wall edges, they act on W. The convention for choosing L_+, L_- is the same as for the bulk.

For the domain wall site $s_{d,i}$ on the i-th bulk and $\varphi \in J_i^{\mathrm{cop}}$, we defined the face operator (in counterclockwise order)

$$B^\varphi(s_{d,i}) = \sum_{(\varphi)} L^{J_i,\varphi^{(1)}}(j_1, f_{d,i}) \otimes L^{J_i,\varphi^{(2)}}(j_2, f_{d,i}) \otimes \cdots \otimes L^{J_i,\varphi^{(n)}}(j_n, f_{d,i}), \tag{4.3}$$

where the comultiplication is in taken J_i.

Proposition 3. *For the domain wall site $s_{d,i}$, the operator $A^h(s_{d,i})$ and $B^\varphi(s_{d,i})$ satisfy*

$$A^h(s_{d,i}) B^\varphi(s_{d,i}) = \sum_{(h),(\varphi)} \lambda_i(\varphi^{(3)} \otimes h^{(1)}) \lambda_i^{-1}(\varphi^{(1)} \otimes h^{(3)}) B^{\varphi^{(2)}}(s_{d,i}) A^{h^{(2)}}(s_{d,i}). \tag{4.4}$$

Therefore, by mapping $\varphi \otimes h \in J_i^{\mathrm{cop}} \bowtie_{\lambda_i} K$ to $B^\varphi(s_{d,i}) A^h(s_{d,i})$, we obtain a representation of $J_i^{\mathrm{cop}} \bowtie_{\lambda_i} K$ over $\mathcal{H}_i(s_{d,i}) = \otimes_{e\in\partial s_{d,i}} \mathcal{H}_e$.

Proof. The proof is similar to the one in Proposition 1, we omit it here. ■

Now by choosing the Haar integrals $h_K \in K$ and $\varphi_{J_i} \in J_i$, we defined domain wall stabilizers as

$$A^K_{v_d} = A^{h_K}(s_d), \quad B^{J_i}_{f_{d,i}} = B^{\varphi_{J_i}}(s_{d,i}). \tag{4.5}$$

All domain wall stabilizer operators are projectors, and they satisfy

$$(A^{h_K}_{v_b})^\dagger = A^{h_K^*}_{v_b} = A^{h_K}_{v_b}, \quad (A^{h_K}_{v_b})^2 = A^{h_K^2} = A^{h_K}_{v_b}, \tag{4.6}$$

$$(B_{f_{d,i}}^{\varphi_{J_i}})^\dagger = B_{f_{d,i}}^{\varphi_{J_i}^*} = B_{f_{d,i}}^{\varphi_{J_i}}, \quad (B_{f_{d,i}}^{\varphi_{J_i}})^2 = B_{f_{d,i}}^{\varphi_{J_i}^2} = B_{f_{d,i}}^{\varphi_{J_i}}. \tag{4.7}$$

All boundary stabilizer operators commute with each other and they commute with all bulk stabilizer operators. The domain wall Hamiltonian can be constructed as

$$H^{K,J_1,J_2}(\Sigma_d) = \sum_{v_d}(I - A_{v_d}^K) + \sum_{f_{d,1}}(I - B_{f_{d,1}}^{J_1}) + \sum_{f_{d,2}}(I - B_{f_{d,2}}^{J_2}). \tag{4.8}$$

The total Hamiltonian is thus

$$H_{H_1,H_2}^{K,J_1,J_2} = H_{H_1}(\Sigma_1) + H_{H_2}(\Sigma_2) + H^{K,J_1,J_2}(\Sigma_d). \tag{4.9}$$

Example 7. To take a closer look at the construction, let's consider a simple example. Choose $J_1 \subseteq \hat{H}_1, J_2 \subseteq \hat{H}_2$ and $W = H_1 \otimes H_2$. The pairing is set as

$$\langle \varphi_1, h_1 \otimes h_2 \rangle := \langle \varphi_1, h_1 \rangle \langle \varepsilon, h_2 \rangle = \varphi_1(h_1)\varepsilon(h_2), \tag{4.10}$$

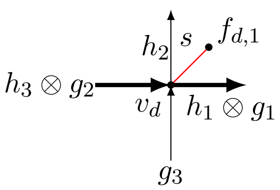
$$\langle \varphi_2, h_1 \otimes h_2 \rangle := \langle \varepsilon, h_1 \rangle \langle \varphi_2, h_2 \rangle = \varepsilon(h_1)\varphi(h_2), \tag{4.11}$$

for $\varphi_i \in J_i$. It's easy to verify that these give the well-defined pairings. For the edge operators induced by these pairings, we have

$$T_\pm^{J_1,\varphi} h_1 \otimes h_2 = (T_\pm^{J_1,\varphi} h_1) \otimes h_2, \tag{4.12}$$

$$T_\pm^{J_2,\varphi} h_1 \otimes h_2 = h_1 \otimes (T_\pm^{J_2,\varphi} h_2). \tag{4.13}$$

The action decouples from the tensor product of two phases. We could introduce a domain wall Hopf algebra K to sew them together. The action of K on H_i naturally induce an action of K on $W = H_1 \otimes H_2$ as $k \triangleright (h_1 \otimes h_2) = \sum_{(k)} k^{(1)} h_1 \otimes k^{(2)} h_2$. To simplify the notation, we will denote the action of K on H_i as kh and $h_i S(k)$. For example



$$A^k|h_1 \otimes g_1, h_2, h_3 \otimes g_2, g_3\rangle = \sum_{(k)} |h_2 S(k^{(1)}), k^{(2)}(h_3 \otimes g_2), k^{(3)} g_3, (h_1 \otimes g_1) S(k^{(4)})\rangle. \tag{4.14}$$

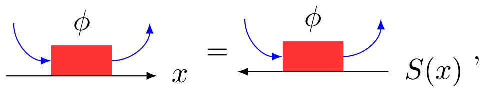
We see that the gapped domain wall are determined by K, J_1, J_2.

5 Ground states in the presence of boundaries and domain walls

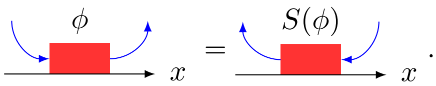
In this section, we will solve the extended generalized quantum double model with gapped boundaries and domain walls and obtain the corresponding ground states. To this end, we firstly review the Hopf tensor network representation of the states proposed in [15], which is convenient for investigating the entanglement properties of the states.

5.1 Hopf tensor network states

Consider two Hopf algebras J, W with a pairing $\langle \bullet, \bullet \rangle : J \otimes W \to \mathbb{C}$ and $\phi \in J, x \in W$. The basic ingredient of the Hopf tensor network is the rank-2 Hopf tensor $\Psi(x, \phi) = \langle \phi, x \rangle$,

$$\langle \phi, x \rangle =$$ 

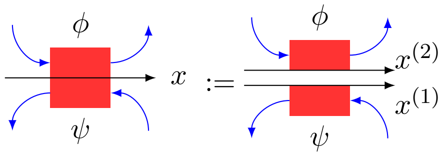
(5.1)

$$\langle \phi, x \rangle =$$ 

(5.2)

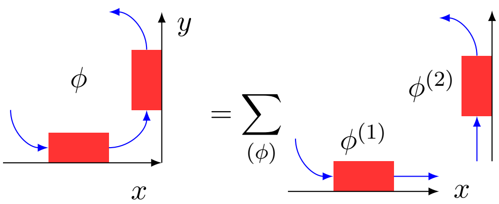
Notice that here the blue edges (corresponding to faces of the lattice) are labeled with $\phi \in J$ and the black edges (corresponding to edges of the lattice) are labeled with $x \in W$. For fixed ϕ, x, the evaluation is determined by the pairing, and the process will be called Hopf trace. Therefore, the Hopf tensor network representation is a diagrammatic representation of the pairing.

For two Hopf algebras J and I which both have their pairings with Hopf algebra W, we can contract the rank-2 tensors to obtain rank-3 tensors. There are two types of contractions. The first one is parallel gluing, which is determined by the comultiplication of W. For example, fix $\phi \in J$, $\psi \in I$ and $x \in W$, we define



$$= \sum_{(x)} \langle \phi, x^{(2)} \rangle \langle \psi, S(x^{(1)}) \rangle. \tag{5.3}$$

The second one is vertical gluing, which is determined by the comultiplication of J. For example, fix $\phi \in J$ and $x, y \in W$, we have



$$= \sum_{(\phi)} \langle \phi^{(1)}, x \rangle \langle \phi^{(2)}, y \rangle. \tag{5.4}$$

Notice that when dealing with a face, the starting site determined the starting point of ordered pairing according to Eq. (5.4).

With the above rules, for arbitrary Hopf tensor network Γ with no free indices (which is a lattice for which all edges and faces have been labelled, and each face has its fixed starting site), the above Hopf trace give us a complex number. We denote the Hopf trace as

$$\Psi_\Gamma(\{x_e\}_{e \in E_\Gamma}, \{\phi_f\}_{f \in F_\Gamma}) = \mathrm{ttr}_\Gamma(\{x_e\}_{e \in E_\Gamma}, \{\phi_f\}_{f \in F_\Gamma}), \tag{5.5}$$

whose evaluation rules are given by Eqs. (5.3) and (5.4).

Now consider a lattice $C(\Sigma)$ of a $2d$ surface Σ with face set F and edge set E. We assign each face a Hopf algebra J_f and each edge a Hopf algebra W_e. Assuming that for each face, J_f has the corresponding pairings with the edge Hopf algebras for edges $e \in \partial f$. If we set the edge with values $x_e \in W_e$ and face with values $\phi_f \in J_f$, then we obtain a Hopf tensor network

$$\otimes_{e\in E} x_e \otimes_{f\in F} \phi_f \mapsto \Psi_{C(\Sigma)}(\{x_e\},\{\phi_e\}) = \operatorname{ttr}_{C(\Sigma)}(\{x_e\},\{\phi_f\}). \tag{5.6}$$

The corresponding Hopf tensor network states are defined as

$$|\Psi_{C(\Sigma)}(\{x_e\},\{\phi_f\})\rangle = \sum_{(x_e)} \operatorname{ttr}_{C(\Sigma)}(\{x_e^{(2)}\},\{\phi_e\}) \otimes_{e\in E} |x_e^{(1)}\rangle. \tag{5.7}$$

5.2 Ground state of the boundary model

From an anyon-condensation point of view, the bulk phase of the Hopf algebraic quantum double model is characterized by the UMTC $\mathcal{B} = \mathsf{Rep}(D(H)) \simeq \mathcal{Z}(\mathsf{Rep}(H))$. A gapped boundary is characterized by a Lagrangian algebra L in $\mathcal{B}$. For a sphere Σ with n boundaries $\partial^i\Sigma$ and each boundary is characterized by its Lagrangian algebra L_i, the ground state degeneracy is given by

$$\mathrm{GSD} = \dim \operatorname{Hom}(\mathbb{1}, L_1 \otimes \cdots \otimes L_n), \tag{5.8}$$

where $\mathbb{1}$ is the bulk vacuum charge. For the disk, there is only one boundary. From the fact that, for arbitrary Lagrangian algebra, $\dim \operatorname{Hom}(\mathbb{1}, L) = 1$, this implies that the ground state on a disk is unique. Notice that Eq. (5.8) is universal, namely, it's independent of the lattice realizations.

From our construction of the extended quantum double model with boundary, it's clear that the boundary can be regarded as a big face (punch a hole in the sphere). For the construction in subsection 3.1, we choose each the Hopf algebra for each internal edge as H, and internal face $\hat{H}$. The boundary edge Hopf algebra is K, the boundary faces Hopf algebra is trivial Hopf algebra. For the construction in subsection 3.2, all internal edges are set as Hopf algebra H, internal faces Hopf algebra $\hat{H}$. The boundary edge is set as W and boundary face is set as J. In this way, we obtain the corresponding Hopf tensor network states of the model,

$$\begin{aligned} &|\Psi_{C(\Sigma\setminus\partial\Sigma);C(\partial\Sigma)}(\{x_e\},\{\phi_f\};\{y_{e_b}\},\varphi_{f_b})\rangle \\ =& \sum_{(x_e),(y_{e_b})} \operatorname{ttr}_{C(\Sigma\setminus\partial\Sigma);C(\partial\Sigma)}(\{x_e^{(2)}\},\{\phi_f\};\{y_{e_b}^{(2)}\},\varphi_{f_b}) \otimes_{e\in E(\Sigma\setminus\partial\Sigma)} |x_e^{(1)}\rangle \otimes_{e_b\in E(\partial\Sigma)} |y_{e_b}^{(1)}\rangle. \end{aligned} \tag{5.9}$$

The ground state of the model is the one for which we choose all elements as Haar integrals.

Proposition 4 (Ground state on a disk)**.** *Suppose that Σ is a disk, then the ground state is unique. The ground state of the extended generalized quantum double model is the Hopf tensor network state with all edge and face elements are chosen as Haar integrals*

$$|\Psi_{GS}\rangle = |\Psi_{C(\Sigma\setminus\partial\Sigma);C(\partial\Sigma)}(\{h_{H,e}\},\{\varphi_{\hat{H},f}\};\{h_{K,e_b}\},\{\varphi_{J,f_b}\})\rangle \tag{5.10}$$

For each bulk site s, we have

$$B^{\phi}(s)A^{g}(s)|\Psi_{GS}\rangle = \varepsilon(g)\phi(1_H)|\Psi_{GS}\rangle. \tag{5.11}$$

For the boundary site s_b, we have

$$B^{\phi}(s_b)A^{g}(s_b)|\Psi_{GS}\rangle = \varepsilon(g)\phi(1_H)|\Psi_{GS}\rangle. \tag{5.12}$$

Proof. Since the ground state is unique, we only need to check that the state we construct here satisfies the stabilizer conditions. For the bulk part, the proof is the same as given in Ref [15]. We only need to check the boundary stabilizers. Since in our construction, the boundary can be treated as a big face. Using the fact that $\varphi_J\varphi_J = \varphi_J$ and $h_W \in \mathrm{Cocom}(W)$, it's easy to check $B^{\varphi_J}|\Psi_{GS}\rangle = |\Psi_{GS}\rangle$. For vertex stabilizers, using the fact that $h_K \in \mathrm{Comm}(K)$, $h_W \in \mathrm{Cocom}(W)$ and $kh_W = \varepsilon(k)h_W = h_W k$, we have $A^{h_K}|\Psi_{GS}\rangle = |\Psi_{GS}\rangle$. Eqs. (5.11) and (5.12) are direct result of the definition of Haar integrals. ■

5.3 Ground state of the domain wall model

Using the Hopf tensor network representation, we can also solve the model with a domain wall. The spirit is the same as the construction of the boundary, we put the corresponding Haar integrals over each edge, and all the faces also take the Haar integrals. The ground state is

$$|\Psi_{GS}\rangle = |\Psi_{C(\Sigma\backslash\partial\Sigma);C(\partial\Sigma)}(\{h_{H_1,e_1}\},\{\varphi_{\hat{H}_1,f_1}\};\{\varphi_{J_1,f_{d,1}}\};\{h_{W,e_d}\};\{\varphi_{J_2,f_{d,2}}\};\{h_{H_2,e_2}\},\{\varphi_{\hat{H}_2,f_2}\})\rangle, \tag{5.13}$$

where e_i and f_i is the edges and faces in region $i = 1, 2$; e_d are edges on the wall; f_{d_i} are wall faces in region $i = 1, 2$.

Notice that these states are local quasi-product states [40], thus they satisfy the entanglement area law. Recently, it's discovered (in group algebra case) that the entanglement entropy of the quantum double phase is sensitive to the existence of the boundary and boundary types [41–44] (and domain walls [45, 46]). Using this explicit ground state, we can also analyze the entanglement feature of the extended generalized quantum double model in more detail, this will be done in our future work.

6 Conclusion and discussion

In this work, we discussed the boundary and domain wall theories of the generalized quantum double model. A gapped boundary is characterized by a triple of three semisimple Hopf algebras with some pairings. A domain wall is characterized by a quadruple of semisimple Hopf algebras equipped with some pairings. The ground states of these extended quantum double model are solved by using the Hopf tensor network representations of the quantum states.

However, despite the progress made, much work remains to be done in many different aspects toward a deep understanding of the boundary and domain wall theory of the generalized quantum double model. The Hamiltonian construction of the gapped boundary and domain wall is only the first step. As for the universal properties of the model, there are several problems to be investigated: (i) The anyon condensation and confinement of the model. We have presented a Hamiltonian theory of the gapped boundary and gapped domain wall, however, the connection between this Hamiltonian theory and the algebraic theory of anyon condensation and confinement still needs to be investigated. (ii) We have mentioned that the entanglement features of the generalized quantum double model are sensitive to the existence of types of gapped boundaries and domain walls, this part is also left for our future study. (iii) Another interesting aspect is the symmetry enriched case, some simple cases (cyclic group, Abelian group, etc.) are considered in Refs. [47–53]. However, the general Hopf algebra case is still largely left open. These topics will be covered in our future studies.

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A C^* semisimple Hopf algebra

In this appendix, we will introduce the input data of our construction of lattice realization of $2d$ topological ordered phase, that is, a finite-dimensional C^* semisimple Hopf algebra. And we will collect some properties of them that will be used in this paper.

A Hopf algebra is a complex vector space H equipped with several structure morphisms: multiplication $\mu : H \otimes H \to H$, unit $\eta : \mathbb{C} \to H$, comultiplication $\Delta : H \to H \otimes H$, counit $\varepsilon : H \to \mathbb{C}$ and antipode $S : H \to H$, for which some consistency conditions are satisfied:

1. (H, μ, η) is an algebra: $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$, and $\mu \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = \mu \circ (\mathrm{id} \otimes \eta)$.

2. (H, Δ, ε) is a coalgebra: $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, and $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$.

3. $(H, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra: Δ and ε are algebra homomorphisms (equivalently μ and η are coalgebra homomorphisms).

4. The antipode S satisfies: $\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = \mu \circ (\mathrm{id} \otimes S) \circ \Delta$.

We will use the notation $\mu(x \otimes y) = xy$ and $\eta(1) = 1_H$ for multiplication and unit. We will also adopt the Sweedler's notation $\Delta(u) = \sum_{(u)} u^{(1)} \otimes u^{(2)} := \sum_i u_i^{(1)} \otimes u_i^{(2)}$. The comultiplication law ensures that $(\Delta \otimes \mathrm{id}) \circ \Delta(u) = (\mathrm{id} \otimes \Delta) \circ \Delta(u) = \sum_{(u)} u^{(1)} \otimes u^{(2)} \otimes u^{(3)}$.

In general, we define $\Delta_1 = \Delta$ and $\Delta_n = (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \Delta) \circ \Delta_{n-1}$, then $\Delta_n(u) = \sum_{(u)} u^{(1)} \otimes \cdots \otimes u^{(n+1)}$. From the definition above, we have the following useful identity:

$$\sum_{(x)} \varepsilon(x^{(1)})x^{(2)} = x = \sum_{(x)} x^{(1)}\varepsilon(x^{(2)}). \tag{A.1}$$

The twist operation is defined as $\tau(x \otimes y) = y \otimes x$, by which one denotes $\mu^{\mathrm{op}} = \mu \circ \tau$ and $\Delta^{\mathrm{op}} = \tau \circ \Delta$. The opposite Hopf algebra H^{op} is defined as $(H, \mu^{\mathrm{op}}, \eta, \Delta, \varepsilon, S^{-1})$, and the co-opposite weak Hopf algebra H^{cop} is defined as $(H, \mu, \eta, \Delta^{\mathrm{op}}, \varepsilon, S^{-1})$.

A Hopf algebra H is called simple (or indecomposable) if its algebra (H, μ, η) does not have nontrivial subalgebra, and it is called semisimple if its algebra can be written as a direct sum of simple algebras. From the Larson-Radford theorem, H is semisimple if and only if $S^2 = \mathrm{id}$ (also equivalent to the simplicity of the dual Hopf algebra $\hat{H}$). A $*$-Hopf algebra $(H, *)$ is a Hopf algebra H equipped with a star structure $* : H \to H$ such that Δ is a $*$-homomorphism. That is

$$(x^*)^* = x, (x+y)^* = x^* + y^*, (cx)^* = \bar{c}x^*, (xy)^* = y^*x^*, \Delta(x)^* = \Delta(x^*), \tag{A.2}$$

for all $x, y \in H$ and $c \in \mathbb{C}$. $(H, *)$ is called a C^* Hopf algebra if there exists a faithful $*$-representation $\rho : H \to \mathbf{B}(\mathcal{H})$ for some operator space over Hilbert space $\mathcal{H}$. It can be proved that

$$S(S(x^*)^*) = x. \tag{A.3}$$

In particular, for semisimple Hopf algebra, this implies that $S(x^*) = S(x)^*$.

For a finite-dimensional C^* semisimple Hopf algebra, the antipode satisfies the following properties:

$$S(xy) = S(y)S(x), \quad S(1_H) = 1_H, \quad S^2 = \mathrm{id}, \quad \varepsilon \circ S = \varepsilon, \tag{A.4}$$

$$\sum_{(S(x))} S(x)^{(1)} \otimes S(x)^{(2)} = \sum_{(x)} S(x^{(2)}) \otimes S(x^{(1)}), \tag{A.5}$$

$$\sum_{(x)} x^{(1)} S(x^{(2)}) = \varepsilon(x)1_H = \sum_{(x)} S(x^{(1)})x^{(2)}. \tag{A.6}$$

These properties are useful for our discussion.

Another notion we will use is the Haar integral $h \in H$, which is defined as a normalized two-side integral. A left (resp. right) integral of H is an element ℓ (resp. r) satisfying $x\ell = \varepsilon(x)\ell$ (resp. $rx = r\varepsilon(x)$) for all $x \in H$. If h is simultaneously left and right integral, it's called a two-side integral. A left (resp. right) integral ℓ (resp. r) is called normalized if $\varepsilon(\ell) = 1$ (resp. $\varepsilon(r) = 1$). We see that normalized integral is idempotent $\ell^2 = \varepsilon(\ell)\ell = \ell$ (resp. $r^2 = r\varepsilon(r) = r$). A Haar integral of H is a normalized two-side integral. Notice that Haar integral, if exists, must be unique. To see this, suppose that h, h' are two Haar integrals, then $h' = \varepsilon(h)h' = hh' = h\varepsilon(h') = h$. It's easy to see that $S(h)$ is a Haar integral if h is, thus from uniqueness we see that Haar integral is S-invariant, i.e. $S(h) = h$. A C^* semisimple Hopf algebra always have a unique Haar integral h which satisfies $h^* = h$, $h^2 = h$ and $S(h) = h$. An element $x \in H$ is called cocommutative if $\Delta^{\mathrm{op}}(x) = \Delta(x)$; the set of all cocommutative elements in H is denoted as $\mathrm{Cocom}(H)$. It can be proved that the Haar integral h is always cocommutative $\Delta(h) = \Delta^{\mathrm{op}}(h)$.

For a given finite-dimensional C^* semisimple Hopf algebra H, its dual space $\hat{H} := \mathrm{Hom}(H, \mathbb{C}) = H^\vee$ has a canonical finite-dimensional C^* semisimple Hopf algebra structure induced by canonical pairing $\langle \bullet, \bullet \rangle : \hat{H} \times H \to \mathbb{C}$, $\langle \varphi, h \rangle := \varphi(h)$. More precisely,

$$\langle \hat{\mu}(\varphi \otimes \psi), x \rangle = (\varphi \otimes \psi)(\Delta(x)), \tag{A.7}$$

$$\langle \hat{\eta}(1), x \rangle = \varepsilon(x), \hat{1} = \varepsilon, \tag{A.8}$$

$$\hat{\Delta}(\varphi)(x \otimes y) = \langle \varphi, \mu(x \otimes y) \rangle, \tag{A.9}$$

$$\hat{\varepsilon}(\varphi) = \langle \varphi, \eta(1) \rangle = \varphi(1_H), \tag{A.10}$$

$$\langle \hat{S}(\varphi), x \rangle = \langle \varphi, S(x) \rangle. \tag{A.11}$$

The star operation on $\hat{H}$ is defined as

$$\langle \varphi^*, x \rangle = \overline{\langle \varphi, S(x)^* \rangle}. \tag{A.12}$$

It is easily checked that $(H^{\mathrm{op}})^\vee \simeq (H^\vee)^{\mathrm{cop}}$ and $(H^{\mathrm{cop}})^\vee \simeq (H^\vee)^{\mathrm{op}}$ as Hopf algebras. If H is a C^* semisimple Hopf algebra, then there exists a unique Haar integral h. The pairing $\langle \varphi, \psi \rangle := \langle \varphi^* \psi, h \rangle =: \int_h \varphi^* \psi$ for $\varphi, \psi \in \hat{H}$, is an inner product making $\hat{H}$ a Hilbert space. Since $\hat{H}$ is also a C^* Hopf algebra, there also exists a unique Haar integral φ which induces a Hilbert space structure on H,

$$\langle x, y \rangle_H := \langle \varphi, x^* y \rangle \tag{A.13}$$

The Haar integral of $\hat{H}$ is also called the Haar measure on H.

The quantum double of H is the vector space $(H^\vee)^{\mathrm{cop}} \otimes H$ equipped with a Hopf algebra structure[3]; we denote it as $D(H) = (H^\vee)^{\mathrm{cop}} \bowtie H$. The multiplication is given by

$$(\varphi \otimes x)(\psi \otimes y) := \sum_{(x)} \varphi \psi (S^{-1}(x^{(3)}) \bullet x^{(1)}) \otimes x^{(2)} y. \tag{A.14}$$

where "$\bullet$" denotes the argument of the function. The other data are given by

$$1_{D(H)} = \hat{1} \otimes 1, \tag{A.15}$$

$$\Delta_{D(H)}(\varphi \otimes x) = \sum_{(\varphi)} \sum_{(x)} (\varphi^{(1)} \otimes x^{(1)}) \otimes (\varphi^{(2)} \otimes x^{(2)}), \tag{A.16}$$

$$\varepsilon_{D(H)}(\varphi \otimes x) = \varepsilon(x) \varphi(1_H), \tag{A.17}$$

$$S_{D(H)}(\varphi \otimes x) = \sum_{(\varphi)} \sum_{(x)} \langle \varphi^{(1)} \otimes \varphi^{(3)}, h^{(3)} \otimes S^{-1}(h^{(1)}) \rangle \hat{S}^{-1}(\varphi^{(2)}) \otimes S(h^{(2)}). \tag{A.18}$$

From expression of antipode, we see that the antipode of $D(H)$ is involutive if and only if the antipode of H is involutive. Then using the Larson-Radford theorem, $D(H)$ is semisimple if and only if H is semisimple. Both $(H^\vee)^{\mathrm{cop}}, H$ can be embedded in $D(H)$ as

[3] There are several different but equivalent constructions of quantum double, see [39]. Here we choose the one built from $(H^\vee)^{\mathrm{cop}} \otimes H$. See also Sec. 3.2 for a more general construction.

Hopf subalgebras by $\varphi \mapsto \varphi \otimes 1$ and $x \mapsto \hat{1} \otimes x$, respectively. For more details, the reader can consult Refs. [39, 54–56]. In particular, we have the following useful straightening formula

$$x\varphi = \sum_{(x)} \varphi(S^{-1}(x^{(3)}) \bullet x^{(1)}) x^{(2)}. \tag{A.19}$$

B Geometric objects on $2d$ lattice

In this appendix, following Ref. [26], we provide a detailed discussion of geometric objects that appeared in the quantum double model.

Definition 8 (Lattice on a surface)**.** *Let Σ be a 2d surface, a lattice $C(\Sigma)$ on it is a simple graph embedded in Σ, where, by simple graph we mean a graph for which no edge starts and ends at the same vertex.*

It can be proved that, for an arbitrary $2d$ surface, this kind of lattice embedding always exists. Another way to understand the lattice is from the perspective of cellulation, and the result coincides with the graph embedding. A typical example is a triangulation, which always exists for topological manifolds with dimensions less than or equal to three [57]. A slightly more complicated case is to consider the lattice as a ribbon graph embedded in the surface [20, 21], but there is no big difference. We can always break the ribbon graph into a simple graph by breaking edges and adding some new edges. Notice that for a given surface, different lattices on it have the same topological invariant (e.g., the same Euler characteristics), thus they are topologically equivalent. For the surface with boundaries, similar results hold. We first make a graph embedding of the boundary, which is nothing but an m-vertex cycle. The bulk graph shares these vertices and edges on the boundary.

Definition 9. *For a lattice $C(\Sigma)$ on the surface Σ, we denote its vertex set, edge set, and face set as $V(\Sigma)$, $E(\Sigma)$ and $F(\Sigma)$ respectively. The dual lattice $\tilde{C}(\Sigma)$ is the Poincaré dual of $C(\Sigma)$, for which the role of face set and vertex set interchange. We also introduce a direction of each edge $e \in E(\Sigma)$, and the direction of the dual edge is obtained by rotating the direct edge counterclockwise with $\pi/2$. The inverse edge $\bar{e}$ is obtained by reversing the direction of edge e. Two ends of edge e are denoted as $\partial_i e$ with $i = 0, 1$ indicating the starting and terminal ends. A direct path p is a list $p = (v_0, e_1, \cdots, e_n, v_n)$ for which $\partial_0 e_k = v_{k-1}$ and $\partial e_k = v_k$. The dual path is thus a list $\tilde{p} = (f_0, \tilde{e}_1, \cdots, \tilde{e}_n, f_n)$ such that $\partial_0 e_k = f_{k-1}$ and $\partial e_k = f_k$.*

Definition 10. *A site is a pair $s = (v, f)$ with adjacent $v \in V(\Sigma)$ and $f \in F(\Sigma)$. A direct triangle $\tau = (s_0, s_1, e)$ consists of two sites s_0, s_1 and a direct edge e, for which s_0 and s_1 share a common face. A dual triangle $\tilde{\tau} = (s_0, s_1, \tilde{e})$ consists of two sites s_0, s_1 and a dual edge $\tilde{e}$, for which $_0$ and s_1 shares a common vertex. The two sites of the triangle are denoted as $\partial_i \tau$ with $i = 0, 1$ indicating the starting and terminal ends. Notice that each edge has its direction, which may or may not match the direction of the triangle. A left-handed (right-handed) triangle is the one for which the edge is on the left-hand (right-hand) side when we pass through the triangle along its positive direction. Two triangles overlap if there* *share some common area. Two direct (dual) triangles overlap only if they are the same triangle. A direct triangle overlaps with a dual triangle when their direct edge and dual edge coincide.*

Definition 11. *A strip is an alternating sequence of triangles $\rho = (\tau_1, \cdots, \tau_n)$ such that $\partial_1\tau_k = \partial_0\tau_{k+1}$. The starting and terminal site are denoted as $\partial_0\rho = \partial\tau_1$ and $\partial_1\rho = \partial_1\tau_n$. Notice that a strip may have self-overlapping. A strip without self-overlapping is called a ribbon.*

- *A strip (ribbon) is called empty, if its length is zero (no triangle).*
- *A strip (ribbon) is called direct (dual) if it consists only of direct (dual) triangles.*
- *A strip (ribbon) is called proper if it contains both direct and dual triangles. For a proper strip (ribbon), there is an associated direct path p_ρ and a dual path $\tilde{p}_\rho$.*
- *A strip (ribbon) is called closed if $\partial_0\rho = \partial_1\rho$,* viz., *its starting site and terminal site are the same site. The only end site of closed strip (ribbon) is denoted as $\partial\rho$.*
- *A strip (ribbon) is called open if $\partial_0\rho$ and $\partial_1\rho$ have no overlap ($v_{\partial_0\rho} \neq v_{\partial_1\rho}$ and $v_{\partial_0\rho} \neq v_{\partial_1\rho}$).*

Definition 12. *A directed strip (ribbon) is called type-A (resp. type-B) if all the direct triangles in it are left-handed (resp. right-handed).*

Definition 13. *A direct closed ribbon is a closed ribbon only consisting of direct triangles. A dual closed ribbon is a closed ribbon only consisting of dual triangles.*

C Properties of ribbon operators

In this appendix, we provide some detailed discussion and calculations of ribbon operators.

Triangle operators

The building blocks for the ribbon operators are eight triangle operators given in Eqs. (2.17)-(2.20). We see that they are determined by the left-handed edge operators $T_\pm$, $L_\pm$, and right-handed edge operators $\tilde{T}_\pm$ and $\tilde{L}_\pm$. Recall that the inner product on the Hopf algebra H is given by $\langle x, y\rangle = \varphi_{\hat{H}}(x^* y)$, where $\varphi_{\hat{H}}$ the Haar integral of $\hat{H}$.

The left-handed operator and right-handed operator are independent, and they won't appear in a given ribbon simultaneously. Only when considering the overlap of two ribbon operators with different chiralities, need we be concerned with the relation between them.

It's easy to verify that the left-handed operators satisfy

$$L_-^h = S \circ L_+^h \circ S^{-1}, \quad T_-^\phi = S \circ T_+^\phi \circ S^{-1}. \tag{C.1}$$

These edge operators are also compatible with $*$-structures of H and $\hat{H}$:

$$S^\dagger = S, \quad (L_\pm^h)^\dagger = L_\pm^{h^*}, \quad (T_\pm^\varphi)^\dagger = T_\pm^{\varphi^*}. \tag{C.2}$$

For completeness, here we give proof of Eq. (C.2). Consider $\langle S^\dagger(x), y\rangle = \varphi_{\hat{H}}(x^* S(y)) = S(\varphi_{\hat{H}})(S(x^* S(y)) = \varphi_{\hat{H}}(S(x^*)y) = \langle S(x), y\rangle$, thus $S^\dagger = S$. The proof of $(L_\pm^h)^\dagger = L_\pm^{h^*}$ is almost straightforward (see Sec. III B of [15] for a detailed proof). We will prove the $T_\pm^\phi$ part, which is relatively technical. Namely, we have

$$
\begin{aligned}
\langle x, T_+^\phi y\rangle &= \sum_{(y)} \varphi_{\hat{H}}(x^* y^{(1)} \phi(y^{(2)})) \\
&= \sum_{(y),(\phi),(x^*)} \varphi_{\hat{H}}(x^{*(1)} \varepsilon(x^{*(2)}) \phi^{(2)}(1_H) \phi^{(1)}(y^{(2)}) y^{(1)}) \\
&= \sum_{(y),(\phi),(x^*)} \varphi_{\hat{H}}(x^{*(1)} y^{(1)} 1_{\hat{H}}(x^{*(2)}) \hat{\varepsilon}(\phi^{(2)}) \phi^{(1)}(y^{(2)})) \\
&= \sum_{(y),(\phi),(x^*)} \varphi_{\hat{H}}(x^{*(1)} y^{(1)} [\phi^{(2)} S(\phi^{(3)})](x^{*(2)}) \phi^{(1)}(y^{(2)})) \\
&= \sum_{(y),(\phi),(x^*)} \varphi_{\hat{H}}(x^{*(1)} y^{(1)} \phi^{(2)}(x^{*(2)}) S(\phi^{(3)})(x^{*(3)}) \phi^{(1)}(y^{(2)})) \qquad \text{(C.3)} \\
&= \sum_{(y),(\phi),(x^*)} \varphi_{\hat{H}}((y x^{*(1)})^{(1)} \phi^{(1)}((y x^{*(1)})^{(2)}) S(\phi^{(2)})(x^{*(2)})) \\
&= \sum_{(y),(\phi),(x^*)} \varphi_{\hat{H}}((y x^{*(1)})^{(1)} \phi^{(1)}((y x^{*(1)})^{(2)}) S(\phi^{(2)})(x^{*(2)})) \\
&= \sum_{(\phi),(x^*)} \varphi_{\hat{H}}(x^{*(1)} y \hat{\varepsilon}(\phi^{(1)}) S(\phi^{(2)})(x^{*(2)})) \\
&= \langle T_+^{\phi^*} x, y\rangle,
\end{aligned}
$$

where we have used $\varphi_{\hat{H}} \phi^{(1)} = \hat{\varepsilon}(\phi^{(1)}) \varphi_{\hat{H}}$ and Eq. (A.12). Using Eq. (C.1), we have $(T_-^\phi)^\dagger = S^\dagger \circ T_+^{\phi^*} \circ S^\dagger = T_-^{\phi^*}$.

Using the definition of these operators, it's not difficult to work out the commutation relations among these operators,

$$
T_+^\varphi L_+^h = \sum_{(h)} L_+^{h^{(1)}} T_+^{\varphi(h^{(2)}\bullet)}, \quad T_+^\varphi L_-^h = \sum_{(S(h))} L_-^{S(h)^{(1)}} T_+^{\varphi(\bullet S(h)^{(2)})}, \tag{C.4}
$$

$$
T_-^\varphi L_+^h = \sum_{(h)} L_+^{h^{(2)}} T_-^{\varphi(\bullet S(h^{(1)}))}, \quad T_-^\varphi L_-^h = \sum_{(h)} L_-^{h^{(2)}} T_-^{\varphi(h^{(2)}\bullet)}, \tag{C.5}
$$

where "$\bullet$" denotes the argument of the function.

The right-handed edge operators and left-handed edge operator are related via antipode by

$$
\tilde{L}_\pm^h = L_\mp^{S(h)}, \quad \tilde{T}_\pm^\phi = T_\mp^{\hat{S}(\phi)}. \tag{C.6}
$$

Thus from properties of left-handed edge operators, we similarly have:

$$
\tilde{L}_+^h = S \circ \tilde{L}_-^h \circ S^{-1}, \quad \tilde{T}_+^\phi = S \circ \tilde{T}_-^\phi \circ S^{-1}, \tag{C.7}
$$

and

$$
(\tilde{L}_\pm^h)^\dagger = \tilde{L}_\pm^{h^*}, \quad (\tilde{T}_\pm^\varphi)^\dagger = \tilde{T}_\pm^{\varphi^*}. \tag{C.8}
$$

The commutation relation for right-handed edge operators (and the commutation relation between left-handed and right-handed edge operators) can also be obtained from Eqs. (C.2).

It's easy to prove that

$$L_\pm^h L_\pm^g = L_\pm^{hg}, \quad T_\pm^\phi T_\pm^\psi = T_\pm^{\phi\psi}, \tag{C.9}$$

$$\tilde{L}_\pm^h \tilde{L}_\pm^g = L_\pm^{gh}, \quad \tilde{T}_\pm^\phi T_\pm^\psi = \tilde{T}_\pm^{\psi\phi}. \tag{C.10}$$

From the above equalities and the definition of triangle operators, it's clear that

$$F^{h,\phi}(\tau_L)F^{g,\psi}(\tau_L) = F^{hg,\psi\phi}(\tau_L), \quad F^{h,\phi}(\tilde{\tau}_R)F^{g,\psi}(\tilde{\tau}_R) = F^{hg,\psi\phi}(\tilde{\tau}_R), \tag{C.11}$$

$$F^{h,\phi}(\tau_R)F^{g,\psi}(\tau_R) = F^{gh,\phi\psi}(\tau_R), \quad F^{h,\phi}(\tilde{\tau}_L)F^{g,\psi}(\tilde{\tau}_L) = F^{gh,\phi\psi}(\tilde{\tau}_L), \tag{C.12}$$

This means that, as algebras, the triangle operator algebras satisfy $\mathcal{A}_{\tau_L} \simeq \mathcal{A}_{\tilde{\tau}_R} \simeq H \otimes \hat{H}^{\mathrm{op}}$, and $\mathcal{A}_{\tau_R} \simeq \mathcal{A}_{\tilde{\tau}_L} \simeq H^{\mathrm{op}} \otimes \hat{H}$.

Vertex and face operators

The type-B construction of the quantum double model based on the left-module structure of H. Vertex operators $A^h(s)$ are built from $\tilde{L}_\pm$ in a clockwise order. Face operators $B^\varphi(s)$ are built from $\tilde{T}_\pm$ in a clockwise order. It's easy to verify that

$$A^h(s)B^\varphi(s) = \sum_{(h)} B^{\varphi(S^{-1}(h^{(3)})\bullet h^{(1)})}(s)A^{h^{(2)}}(s). \tag{C.13}$$

See Sec III A in Ref. [15]. They form a representation of $D_B(H) = \hat{H}^{\mathrm{cop}} \bowtie H$.

The type-A construction of quantum double model based on the right-module structure of H. Vertex operators $\tilde{A}^h(s)$ are built from $\tilde{L}_\pm$ in a clockwise order. Face operators $\tilde{B}^\varphi(s)$ are built from $\tilde{T}_\pm$ in a clockwise order. It's easy to verify that

$$\tilde{A}^h(s)\tilde{B}^\varphi(s) = \sum_{(h)} \tilde{B}^{\varphi(h^{(3)}\bullet S^{-1}(h^{(1)}))}(s)\tilde{A}^{h^{(2)}}(s). \tag{C.14}$$

They form a representation of $D_A(H) = \hat{H} \bowtie H^{\mathrm{cop}} \simeq D_B(H)^{\mathrm{cop}}$.

Properties of ribbon operators

There are several crucial properties for open ribbon operators that will be used for constructing topologically excited states of the model.

As we have mentioned before, the topological excitations are given at two ends of the ribbon operators, thus the commutation relation between vertex and face operator and ribbon operators is crucial:

1. At the starting points of ribbons ρ_A and ρ_B, we have

$$A^g(s_0)F^{h,\varphi}(\rho_A) = \sum_{(g)} F^{g^{(1)}hS(g^{(3)}),\varphi(S(g^{(2)})\bullet)}(\rho_A)A^{g^{(4)}}(s_0), \tag{C.15}$$

$$A^g(s_0)F^{h,\varphi}(\rho_B) = \sum_{(g)} F^{g^{(2)}hS(g^{(4)}),\varphi(S(g^{(3)})\bullet)}(\rho_B)A^{g^{(1)}}(s_0), \tag{C.16}$$

$$B^{\psi}(s_0)F^{h,\varphi}(\rho_A)=\sum_{(h)}F^{h^{(2)},\varphi}(\rho_A)B^{\psi(\bullet S(h^{(1)}))}(s_0), \tag{C.17}$$

$$B^{\psi}(s_0)F^{h,\varphi}(\rho_B)=\sum_{(h)}F^{h^{(2)},\varphi}(\rho_B)B^{\psi(S(h^{(1)})\bullet)}(s_0). \tag{C.18}$$

2. Similarly, at the ending points, we have

$$A^{g}(s_1)F^{h,\varphi}(\rho_A)=\sum_{(g)}F^{h,\varphi(\bullet g^{(2)})}(\rho_A)A^{g^{(1)}}(s_1), \tag{C.19}$$

$$A^{g}(s_1)F^{h,\varphi}(\rho_B)=\sum_{(g)}F^{h,\varphi(\bullet g^{(1)})}(\rho_B)A^{g^{(2)}}(s_1), \tag{C.20}$$

$$B^{\psi}(s_1)F^{h,\varphi}(\rho_A)=\sum_{k,(k),(h)}\varphi(k^{(2)})F^{h^{(1)},\hat{k}}(\rho_A)B^{\psi(S(k^{(3)})h^{(2)}k^{(1)}\bullet)}(s_1), \tag{C.21}$$

$$B^{\psi}(s_1)F^{h,\varphi}(\rho_B)=\sum_{k,(k),(h)}\varphi(k^{(2)})F^{h^{(1)},\hat{k}}(\rho_B)B^{\psi(\bullet S(k^{(3)})h^{(2)}k^{(1)})}(s_1). \tag{C.22}$$

See [18] for detailed proofs (although the convention we use here is different from the one in [18], but the same result can be obtained).

Proposition 5. *Let ρ be a closed ribbon with end site $s=\partial\rho$.*

1. *If $\rho=\rho_A$ is of type-A, we have*

$$A^{h}(s)F^{g,\psi}(\rho_A)=\sum_{(h)}F^{h^{(1)}gS(h^{(3)}),\psi(S(h^{(2)})\bullet h^{(5)})}(\rho_A)A^{h^{(4)}}(s), \tag{C.23}$$

$$B^{\varphi}(s)F^{g,\psi}(\rho_A)=\sum_{(g)}F^{g^{(2)},\psi}(\rho_A)B^{\varphi(g^{(3)}\bullet S(g^{(1)}))}(s). \tag{C.24}$$

2. *If $\rho=\rho_B$ is of type-B, we have*

$$A^{h}(s)F^{g,\psi}(\rho_B)=\sum_{(h)}F^{h^{(3)}gS(h^{(5)}),\psi(S(h^{(4)})\bullet h^{(1)})}(\rho_B)A^{h^{(2)}}(s), \tag{C.25}$$

$$B^{\varphi}(s)F^{g,\psi}(\rho_B)=\sum_{(g)}F^{g^{(2)},\psi}(\rho_B)B^{\varphi(S(g^{(1)})\bullet g^{(3)})}(s). \tag{C.26}$$

Proof. Decompose ρ into $\rho=\rho_1\cup\rho_2$ with $\partial_1\rho_1=\partial_0\rho_2$ and $\partial_0\rho_1=\partial_1\rho_2=s$. Assume ρ is of type-B first. Using Eqs. (C.16) and (C.20), we have

$$\begin{aligned}
&A^{h}(s)F^{g,\psi}(\rho_B)\\
&=\sum_{k}\sum_{(k),(g)}A^{h}(s)F^{g^{(1)},\hat{k}}(\rho_1)F^{S(k^{(3)})g^{(2)}k^{(1)},\psi(k^{(2)}\bullet)}(\rho_2)\\
&=\sum_{k}\sum_{(k),(g)}\sum_{(h)}F^{h^{(2)}g^{(1)}S(h^{(4)}),\hat{k}(S(h^{(3)})\bullet)}(\rho_1)A^{h^{(1)}}(s)F^{S(k^{(3)})g^{(2)}k^{(1)},\psi(k^{(2)}\bullet)}(\rho_2)\\
&=\sum_{k}\sum_{(k),(g)}\sum_{(h)}F^{h^{(3)}g^{(1)}S(h^{(5)}),\hat{k}(S(h^{(4)})\bullet)}(\rho_1)F^{S(k^{(3)})g^{(2)}k^{(1)},\psi(k^{(2)}\bullet h^{(1)})}(\rho_2)A^{h^{(2)}}(s)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_k \sum_{(k),(g)} \sum_{(h)} \sum_j F^{h^{(3)}g^{(1)}S(h^{(5)}),\hat{k}(S(h^{(4)})j)\hat{j}}(\rho_1) F^{S(k^{(3)})g^{(2)}k^{(1)},\psi(k^{(2)}\bullet h^{(1)})}(\rho_2) A^{h^{(2)}}(s) \\
&= \sum_{(h)} \sum_j \sum_{(j),(g)} F^{h^{(3)}g^{(1)}S(h^{(7)}),\hat{j}}(\rho_1) F^{S(j^{(3)})h^{(4)}g^{(2)}S(h^{(6)})j^{(1)},\psi(S(h^{(5)})j^{(2)}\bullet h^{(1)})}(\rho_2) A^{h^{(2)}}(s) \\
&= \sum_{(h)} F^{h^{(3)}gS(h^{(5)}),\psi(S(h^{(4)})\bullet h^{(1)})}(\rho_B) A^{h^{(2)}}(s).
\end{aligned} \tag{C.27}
$$

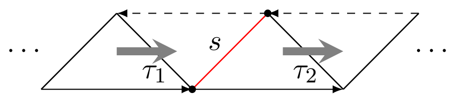
Similarly, using Eqs. (C.18) and (C.22), we have

$$
\begin{aligned}
&B^{\varphi}(s)F^{g,\psi}(\rho_B) \\
&= \sum_k \sum_{(k),(g)} B^{\varphi}(s) F^{g^{(1)},\hat{k}}(\rho_1) F^{S(k^{(3)})g^{(2)}k^{(1)},\psi(k^{(2)}\bullet)}(\rho_2) \\
&= \sum_k \sum_{(k),(g)} F^{g^{(2)},\hat{k}}(\rho_1) B^{\varphi(S(g^{(1)})\bullet)}(s) F^{S(k^{(3)})g^{(3)}k^{(1)},\psi(k^{(2)}\bullet)}(\rho_2) \\
&= \sum_k \sum_{(k),(g)} F^{g^{(2)},\hat{k}}(\rho_1) \sum_{j,(j)} F^{S(k^{(5)})g^{(3)}k^{(1)},\hat{j}\psi(k^{(3)}j^{(2)})}(\rho_2) B^{\varphi(S(g^{(1)})\bullet S(j^{(3)})S(k^{(4)})g^{(4)}k^{(2)}j^{(1)})}(s) \\
&= \sum_k \sum_{(k),(g)} F^{g^{(2)},\hat{k}}(\rho_1) \sum_{j} F^{S(k^{(5)})g^{(3)}k^{(1)},\hat{j}\psi(k^{(3)}j)}(\rho_2) B^{\varphi(S(g^{(1)})\bullet S(k^{(4)})g^{(4)}k^{(2)})}(s) \\
&= \sum_k \sum_{(k),(g)} F^{g^{(2)},\hat{k}}(\rho_1) F^{S(k^{(5)})g^{(3)}k^{(1)},\psi(k^{(3)}\bullet)}(\rho_2) B^{\varphi(S(g^{(1)})\bullet S(k^{(4)})g^{(4)}k^{(2)})}(s) \\
&= \sum_{(g)} \sum_k \sum_{(k),(g^{(2)})} F^{g^{(2)},\hat{k}}(\rho_1) F^{S(k^{(3)})g^{(3)}k^{(1)},\psi(k^{(2)}\bullet)}(\rho_2) B^{\varphi(S(g^{(1)})\bullet g^{(4)})}(s) \\
&= \sum_{(g)} F^{g^{(2)},\psi}(\rho) B^{\varphi(S(g^{(1)})\bullet g^{(3)})}(s).
\end{aligned} \tag{C.28}
$$

Here, in the fourth and sixth equality, we apply the straightening formula; in the fifth equality, we use the fact that $\psi = \sum_j \psi(j)\hat{j}$ for an orthogonal basis $\{j\}$. This completes the proof for the assertion of closed ribbons of type-B. The proof for the assertion of closed ribbons of type-A is similar. ■

Proposition 6. *The ribbon operator $F^{g,\psi}(\rho)$ for an open ribbon ρ commutes with all operators $A_v^H = A^{h_H}(s)$ and $B_f^{\hat{H}} = B^{\varphi_{\hat{H}}}(s)$ for $s \neq \partial_0\rho, \partial_1\rho$, where h_H and $\varphi_{\hat{H}}$ are Haar integrals.*

Proof. We prove the assertion for type-B ribbons. Denote $h = h_H$ and $\varphi = \varphi_{\hat{H}}$ for simplicity. Decompose ρ into $\rho = \tau_1 \cup \tau_2$, with $\partial_1\tau_1 = \partial_0\tau_2 = s$:



Using Eqs. (C.16) and (C.20), one gets

$$
A^h(s)F^{g,\psi}(\rho)
$$

$$
\begin{aligned}
&= \sum_{(g\otimes\psi)} A^h(s) F^{g^{(1)},\psi^{(1)}}(\tau_1) F^{g^{(2)},\psi^{(2)}}(\tau_2) \\
&= \sum_{(g),(\psi),(h)} F^{g^{(1)},\psi^{(1)}(\bullet h^{(2)})}(\tau_1) A^{h^{(1)}}(s) F^{g^{(2)},\psi^{(2)}}(\tau_2) \\
&= \sum_{(g),(\psi),(h)} F^{g^{(1)},\psi^{(1)}(\bullet h^{(5)})}(\tau_1) F^{h^{(2)}g^{(2)}S(h^{(4)}),\psi^{(2)}(S(h^{(3)})\bullet)}(\tau_2) A^{h^{(1)}}(s) \\
&= \sum_{(g),(\psi),(h)} F^{g^{(1)},\psi^{(1)}(\bullet h^{(5)})}(\tau_1) F^{g^{(2)}h^{(2)}S(h^{(3)}),\psi^{(2)}(S(h^{(4)})\bullet)}(\tau_2) A^{h^{(1)}}(s) \qquad \text{(C.29)} \\
&= \sum_{(g),(\psi),(h)} F^{g^{(1)},\psi^{(1)}(\bullet h^{(3)})}(\tau_1) F^{g^{(2)},\psi^{(2)}(S(h^{(2)})\bullet)}(\tau_2) A^{h^{(1)}}(s) \\
&= \sum_{(g),(\psi),(h)} F^{g^{(1)},\psi^{(1)}}(\tau_1) F^{g^{(2)},\psi^{(2)}}(\tau_2) A^{h^{(1)}\varepsilon(h^{(2)})}(s) \\
&= F^{g,\psi}(\rho) A^h(s).
\end{aligned}
$$

Here, in the fourth equality, we use that facts that $gh = \varepsilon(g)h = hg$ and h is cocommutative by which we can switch $h^{(3)}$ and $h^{(4)}$; in the sixth equality we use the linearity of comultiplication and the following (taking sum over $(h^{(2)})$:

$$
\begin{aligned}
\psi^{(1)}(\bullet h^{(3)}) \otimes \psi^{(2)}(S(h^{(2)})\bullet) &= \langle \Delta\psi, \bullet h^{(3)} \otimes S(h^{(2)})\bullet \rangle = \langle \psi, \bullet h^{(3)} S(h^{(2)})\bullet \rangle \\
&= \langle \psi, \bullet \varepsilon(h^{(2)})\bullet \rangle = \psi^{(1)}(\bullet) \otimes \psi^{(2)}(\varepsilon(h^{(2)})\bullet).
\end{aligned} \qquad \text{(C.30)}
$$

For the second statement, using Eqs. (C.18) and (C.22), one has

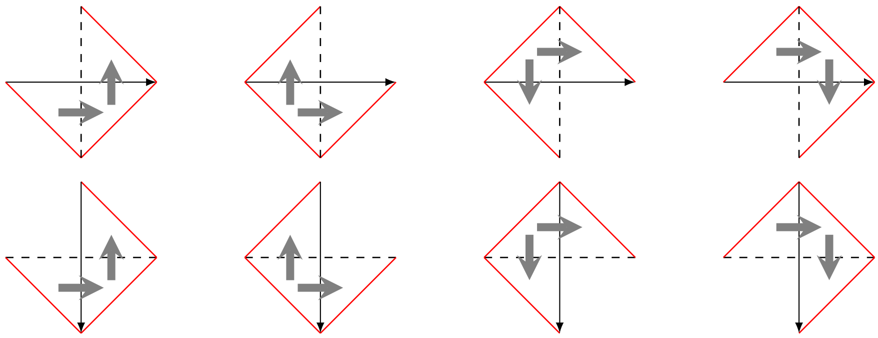
$$
\begin{aligned}
& B^\varphi(s) F^{g,\psi}(\rho) \\
&= \sum_{(g\otimes\psi)} B^\varphi(s) F^{g^{(1)},\psi^{(1)}}(\tau_1) F^{g^{(2)},\psi^{(2)}}(\tau_2) \\
&= \sum_{(g),(\psi)} \sum_{k,(k)} F^{g^{(1)},\psi^{(1)}(k^{(2)})\hat{k}}(\tau_1) B^{\varphi(\bullet S(k^{(3)})g^{(2)}k^{(1)})}(s) F^{g^{(3)},\psi^{(2)}}(\tau_2) \\
&= \sum_{(g),(\psi)} \sum_{k,(k)} F^{g^{(1)},\psi^{(1)}(k^{(2)})\hat{k}}(\tau_1) F^{g^{(4)},\psi^{(2)}}(\tau_2) B^{\varphi(S(g^{(3)})\bullet S(k^{(3)})g^{(2)}k^{(1)})}(s) \\
&= \sum_{(g),(\psi)} \sum_{k} F^{g^{(1)},\psi^{(1)}(k)\hat{k}}(\tau_1) F^{g^{(4)},\psi^{(2)}}(\tau_2) B^{\varphi(S(g^{(3)})\bullet g^{(2)})}(s) \qquad \text{(C.31)} \\
&= \sum_{(g),(\psi)} F^{g^{(1)},\psi^{(1)}}(\tau_1) F^{g^{(4)},\psi^{(2)}}(\tau_2) B^{\varphi(S(g^{(3)})g^{(2)}\bullet)}(s) \\
&= \sum_{(g),(\psi)} F^{g^{(1)},\psi^{(1)}}(\tau_1) F^{\varepsilon(g^{(2)})g^{(3)},\psi^{(2)}}(\tau_2) B^\varphi(s) \\
&= \sum_{(g\otimes\psi)} F^{g^{(1)},\psi^{(1)}}(\tau_1) F^{g^{(2)},\psi^{(2)}}(\tau_2) B^\varphi(s) \\
&= F^{g,\psi}(\rho) B^\varphi(s).
\end{aligned}
$$

Here, in the fourth equality, we apply the straightening formula; in the fifth equality we use the fact that φ is cocommutative and $\psi^{(1)} = \sum_k \psi^{(1)}(k)\hat{k}$. This completes the proof of the assertion for type-B ribbons. The assertion for type-A ribbons can be proved similarly. ■

Proposition 7. *The ribbon operator $F^{h,\phi}(\rho)$ of a closed ribbon ρ commutes with all stabilizers $A_v^H = A^{h_H}(s)$ and $B_f^{\hat{H}} = B^{\varphi_H}(s)$ with $s \neq \partial\rho$.*

Proof. Denote $s_0 = \partial\rho$ and $s = (v, f)$. We only need to consider the case that s lies on ρ. Note that we can always find an s_1 in ρ such that $\rho = \rho_1\rho_2$ with $\partial_0\rho_1 = \partial_1\rho_2 = s_0, \partial_1\rho_1 = \partial_0\rho_2 = s_1$ and $s \neq s_1$. Then ρ_1 and ρ_2 are open ribbons and s is not end site of them. It follows from Proposition 6 that A_v^H and $B_f^{\hat{H}}$ commute with $F^{h^{(1)},\varphi^{(1)}}(\rho_1)$ and $F^{h^{(2)},\varphi^{(2)}}(\rho_2)$, and hence with $F^{h,\varphi}(\rho)$ by the recursive definition of ribbon operator. ∎

Next we characterize the behavior of ribbon operators on overlapped triangles. There are eight possibilities:



(C.32)

In the upper row, from left to right, (τ_1, τ_2) is equal to $(\tau_L, \tilde{\tau}_L), (\tau_L, \tilde{\tau}_R), (\tau_R, \tilde{\tau}_L)$ and $(\tau_R, \tilde{\tau}_R)$, respectively; in the lower row, (τ_1, τ_2) is equal to $(\tilde{\tau}_L, \tau_L), (\tilde{\tau}_L, \tau_R), (\tilde{\tau}_R, \tau_L)$ and $(\tilde{\tau}_R, \tau_R)$, respectively. The following lemma provides the commutation of $F^{h,\varphi}(\tau_1)$ and $F^{g,\psi}(\tau_2)$ in different cases.

Proposition 8. *We have the following basic identities:*

$$F^{h,\varphi}(\tau_L)F^{g,\psi}(\tilde{\tau}_L) = \sum_{(g)} F^{g^{(2)},\psi}(\tilde{\tau}_L)F^{h,\varphi(\bullet g^{(1)})}(\tau_L), \tag{C.33}$$

$$F^{h,\varphi}(\tau_L)F^{g,\psi}(\tilde{\tau}_R) = \sum_{(g)} F^{g^{(1)},\psi}(\tilde{\tau}_R)F^{h,\varphi(\bullet S(g^{(2)}))}(\tau_L), \tag{C.34}$$

$$F^{h,\varphi}(\tau_R)F^{g,\psi}(\tilde{\tau}_L) = \sum_{(g)} F^{g^{(1)},\psi}(\tilde{\tau}_L)F^{h,\varphi(\bullet g^{(2)})}(\tau_R), \tag{C.35}$$

$$F^{h,\varphi}(\tau_R)F^{g,\psi}(\tilde{\tau}_R) = \sum_{(g)} F^{g^{(2)},\psi}(\tilde{\tau}_R)F^{h,\varphi(\bullet S(g^{(1)}))}(\tau_R), \tag{C.36}$$

$$F^{h,\varphi}(\tilde{\tau}_L)F^{g,\psi}(\tau_L) = \sum_{(h)} F^{g,\psi(h^{(2)}\bullet)}(\tau_L)F^{h^{(1)},\varphi}(\tilde{\tau}_L), \tag{C.37}$$

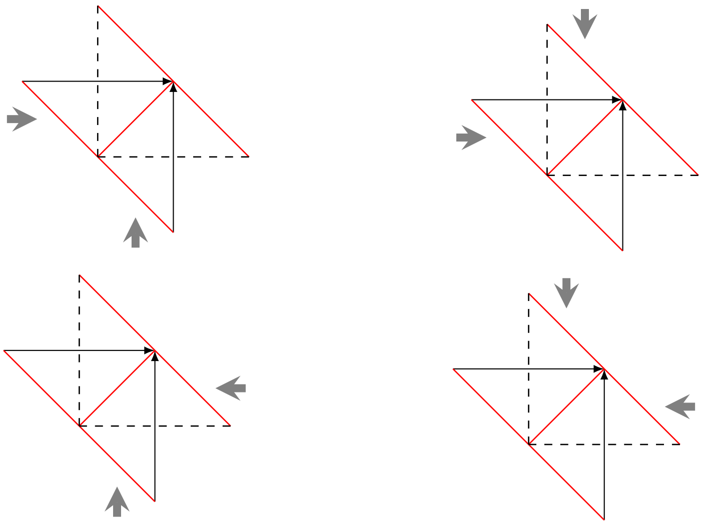
$$F^{h,\varphi}(\tilde{\tau}_L)F^{g,\psi}(\tau_R) = \sum_{(h)} F^{g,\psi(\bullet S(h^{(2)}))}(\tau_R)F^{h^{(1)},\varphi}(\tilde{\tau}_L), \tag{C.38}$$

$$F^{h,\varphi}(\tilde{\tau}_R)F^{g,\psi}(\tau_L) = \sum_{(h)} F^{g,\psi(\bullet h^{(2)})}(\tau_L)F^{h^{(1)},\varphi}(\tilde{\tau}_R), \tag{C.39}$$

$$F^{h,\varphi}(\tilde{\tau}_R)F^{g,\psi}(\tau_R) = \sum_{(h)} F^{g,\psi(S(h^{(2)})\bullet)}(\tau_R)F^{h^{(1)},\varphi}(\tilde{\tau}_R). \tag{C.40}$$

Proof. These follow from straightforward computation using the definition of ribbon operator on triangles. One can also prove these by Eqs. (C.4), (C.5) and similar formulas for $\tilde{L}$ and $\tilde{T}$. ∎

Now we are in the position to characterize the behavior of ribbon operators for which the ribbons intersect in segments of length two. All possible cases can be obtained by complementing eight cases in (C.32). But there are only four non-equivalent cases:



(C.41)

In what follows, we denote the horizontal ribbon by ρ_1 and the vertical by ρ_2.

Proposition 9. *For the four cases listed above, we have the corresponding commutation relations:*

$$F^{h,\varphi}(\rho_1)F^{g,\psi}(\rho_2) = \sum_{(g)}\sum_{(h)} F^{g^{(2)},\psi(\bullet h^{(1)})}(\rho_2)F^{h^{(2)},\varphi(\bullet g^{(1)})}(\rho_1), \tag{C.42}$$

$$F^{h,\varphi}(\rho_1)F^{g,\psi}(\rho_2) = \sum_{(g)}\sum_{(h)} F^{g^{(1)},\psi(S(h^{(1)})\bullet)}(\rho_2)F^{h^{(2)},\varphi(S(g^{(2)})\bullet)}(\rho_1), \tag{C.43}$$

$$F^{h,\varphi}(\rho_1)F^{g,\psi}(\rho_2) = \sum_{(g)}\sum_{(h)} F^{g^{(1)},\psi(\bullet h^{(2)})}(\rho_2)F^{h^{(1)},\varphi(S(g^{(2)})\bullet)}(\rho_1), \tag{C.44}$$

$$F^{h,\varphi}(\rho_1)F^{g,\psi}(\rho_2) = \sum_{(g)}\sum_{(h)} F^{g^{(1)},\psi(h^{(2)}\bullet)}(\rho_2)F^{h^{(1)},\varphi(g^{(2)}\bullet)}(\rho_1). \tag{C.45}$$

Proof. Label the horizontal edge by x and the vertical by y. In the first case, $\rho_1 = \tau_L\tilde{\tau}_R$ and $\rho_2 = \tau_R\tilde{\tau}_L$. The corresponding triangle operators are as follows:

$$\begin{aligned} F^{h,\varphi}(\tau_L)|x\rangle &= \varepsilon(h)\tilde{T}_-^{\varphi}|x\rangle, \quad F^{h,\varphi}(\tilde{\tau}_R)|y\rangle = \hat{\varepsilon}(\varphi)L_+^{h}|y\rangle, \\ F^{g,\psi}(\tau_R)|y\rangle &= \varepsilon(g)T_-^{\psi}|y\rangle, \quad F^{g,\psi}(\tilde{\tau}_L)|x\rangle = \hat{\varepsilon}(\psi)\tilde{L}_-^{g}|x\rangle. \end{aligned} \tag{C.46}$$

Using definition of ribbon operator and commutation relation of L and T ($\tilde{L}$ and $\tilde{T}$), one has

$$F^{h,\varphi}(\rho_1)F^{g,\psi}(\rho_2)|x\rangle \otimes |y\rangle$$

$$
\begin{aligned}
&= \sum_{(h),(\varphi)} F^{h^{(1)},\varphi^{(1)}}(\tau_L) F^{h^{(2)},\varphi^{(2)}}(\tilde{\tau}_R) \sum_{(g),(\psi)} F^{g^{(1)},\psi^{(1)}}(\tau_R) F^{g^{(2)},\psi^{(2)}}(\tilde{\tau}_L) |x\rangle \otimes |y\rangle \\
&= \sum_{(h),(\varphi)} \sum_{(g),(\psi)} \varepsilon(h^{(1)}) \hat{\varepsilon}(\psi^{(2)}) \tilde{T}_-^{\varphi^{(1)}} \tilde{L}_-^{g^{(2)}} |x\rangle \otimes \varepsilon(g^{(1)}) \hat{\varepsilon}(\varphi^{(2)}) L_+^{h^{(2)}} T_-^{\psi^{(1)}} |y\rangle \\
&= \tilde{T}_-^{\varphi} \tilde{L}_-^{g} |x\rangle \otimes L_+^{h} T_-^{\psi} |y\rangle \\
&= \sum_{(g)} \tilde{L}_-^{g^{(2)}} \tilde{T}_-^{\varphi(\bullet g^{(1)})} |x\rangle \otimes \sum_{(h)} T_-^{\psi(\bullet h^{(1)})} L_+^{h^{(2)}} |y\rangle \\
&= \sum_{(g)} \sum_{(\varphi)} \varepsilon(g^{(2)}) \hat{\varepsilon}(\varphi^{(2)}(\bullet g^{(1)})) \tilde{L}_-^{g^{(3)}} \tilde{T}_-^{\varphi^{(1)}} |x\rangle \otimes \sum_{(h)} \sum_{(\psi)} \varepsilon(h^{(2)}) \hat{\varepsilon}(\psi^{(2)}(\bullet h^{(1)})) T_-^{\psi^{(1)}} L_+^{h^{(3)}} |y\rangle \\
&= \sum_{(g),(\psi)} F^{g^{(2)},\psi^{(1)}}(\tau_R) F^{g^{(3)},\psi^{(2)}(\bullet h^{(1)})}(\tilde{\tau}_L) \sum_{(h),(\varphi)} F^{h^{(2)},\varphi^{(1)}}(\tau_L) F^{h^{(3)},\varphi^{(2)}(\bullet g^{(1)})}(\tilde{\tau}_R) |x\rangle \otimes y\rangle \\
&= \sum_{(g)} \sum_{(h)} F^{g^{(2)},\psi(\bullet h^{(1)})}(\rho_2) F^{h^{(2)},\varphi(\bullet g^{(1)})}(\rho_1) |x\rangle \otimes y\rangle. \qquad \text{(C.47)}
\end{aligned}
$$

The other cases can be proved similarly. Hence we complete the proof. ■

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