Chapter 1 Module

§1.4 Free Modules

- Definition and Examples.
- Properties and applications.

(I) Definition and examples

Def. 4.1 Let X be a subset of R module M, if for any finite $\{x_i\}_{i=1}^n \subseteq X$ and $\alpha_i \in R$, we have $\sum_{i=1}^n \alpha_i \alpha_i = 0 \implies \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

Then X is called linearly independent.

If X can generate M and X is linearly independent, X is called a basis of M.

If M has a basis, then M is called a free module.

Example. Every vector space (finite or inifinite dimensional) has a basis, thus they are all free modules.

(Try to prove it, you will need to use Zom's lemma).

Example. Free Z modules are free Abelian group.

Prop 4.1 V is free R module iff V is inner direct sums of its cyclic submodules: $V = \bigoplus_{i \in I} V_i$, each Vi is cyclic and Vi $\cong R$. (Cyclic module $M \cong R/Amn(x)$ is free iff Ann(x) = 0, $M = \langle x \rangle \cong R$.)

Aroof. " \Rightarrow " Since V is free module, suppose x is a basis of V, for each $x \in X$, define $V_x := \langle x \rangle = Rx$.

Since X is linearly independent, Ann(x) = 0. Thus $V_{\infty} \cong \mathbb{R}/Ann(x) = \mathbb{R}$ (Prop 2.7).

 $\mathbb{D} \Sigma_{x \in X} \ V_x = V \quad \textcircled{a} \quad V_x \cap (\Sigma_{y \neq x} \ V_y) = 0$

Thus V= @xex Vx.

"=" Suppose $Vi = \langle \alpha_i \rangle = \Re \alpha_i$. Let $X = \{\alpha_i\}$, we need to show X is a basis. $OV = \langle X \rangle$ is clear.

- ② $\Sigma_i r_i x_i = 0$ \Rightarrow $r_i x_i = 0$ for all i (zero has unique decomposition in direct sum). But $Vi \cong \mathbb{R}/Ann(x_i) = \mathbb{R}$. We see $Ann(x_i) = 0$. Thus $r_i = 0$.
- Prop 4.2. Let V be a free module, there exists a set X and set map $L: X \longrightarrow V$ such that for any module M and set map $f: X \longrightarrow M$, there exists unique module map $f: V \longrightarrow M$ such that $f = \overline{f} L$.

Diagramatically:

$$\times \xrightarrow{f} V \text{ (free)}$$

Proof. Choose X as basis of V, define $\overline{f}(\Sigma r_i x_i) = r_i f(x_i)$.

Prop. 4.3 Any R module is a quotient module of a free module. If M is finitely generated, then M is quotient module of finitely generated free module.

Proof. Let X be a generating set of M (always exists, since we can choose X=M), when M is finitely generated, then X can be chosen as a finite set.

Set free module $V=\mathcal{D}_{x\in X}\ R_x$ with cyclic module $R_x\cong R$ and g_x is the generating element. Define $L(x)=g_x$, then we obtain a map $L\colon X\to V$.

Let $f: X \longrightarrow M$ be embedding map. Then by prop 4.2, we have module map $\bar{f}: V \to M$.

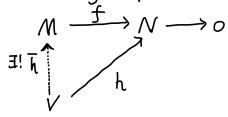
$$\times \xrightarrow{\downarrow} V = \bigoplus_{x \in X} R_x$$

$$\exists ! \overline{f}$$

$$M$$

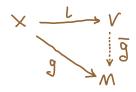
Since $X = f(X) \subseteq \overline{f}(V)$ and M = RX. We see $M \subseteq \overline{f}(V)$. \overline{f} is surjective.

Prop 4.4. Let $f: M \to N$ be module epimorphism, V be a free module, $h: V \to N$ be a module map. Then there exists a not necessarily unique module map $h: V \to M$ such that h = fh.



Proof. Choose a basis X of V. $L: X \hookrightarrow V$ be embedding. Since $f: M \to N$ is surjective, For any $X \in X$ we could find a $m_X \in M$ such that $f(m_X) = h(X)$. Define $g: X \to M$, $x \mapsto m_X$.

Then by prop 4.2, we have commutative diagram:



Notice \bar{g} is a module map and fg(x) = h(x), we have $f\bar{g}l = hl$. Since x is basis, $\bar{g}l$ can

be extended into a module map h.

- Thm 4.5. Let V be α finitely generated free module, then all bases of V have the same number cotclements.
- Proof. This is the same as that for vector space.
- Def. 4.2 Let R be a commutative ring, V is a finitely generated free module, then we call the number of elements in basis X as the rank of V, denotes rk(V).
- Remark. 1. Any free module V with rk(V) = n is isomorphic to $R_{\infty} = 0$.
 - 2. $Hom_R(R^m, R^n) \cong M_{n,m}(R) = \{n \times m \text{ matrices}\}$ as additive group.
 - 3. Home (R", R") \(Mn(R) as ring.
- Prop 4.6 When R is a PID, V is a free R module with rkcV) = n. Then any submodule N are free module and $rk(N) \leq n$. (Also holds for infinite rank modules).