

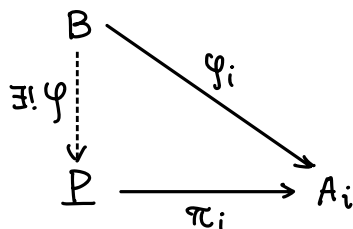
Chapter 2 Categories

§2.2 Products, Coproducts, and Universal Constructions

- Product
- Coproduct
- Free objects, initial and terminal objects, pull-back, push-out.

(I) Product

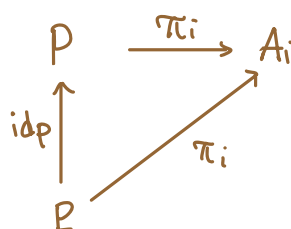
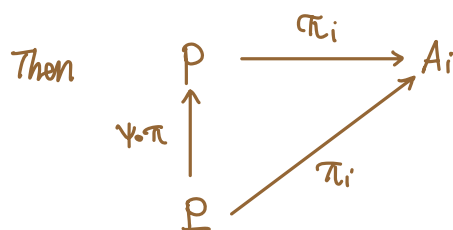
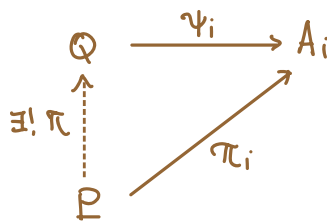
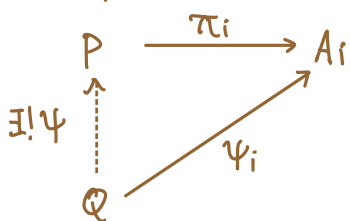
Def 3.1 For $\{A_i\}_{i \in I}$ in a given category \mathcal{C} , the product is defined as $(P, \{\pi_i\})$ with $P \in \text{Ob } \mathcal{C}$ and $\pi_i: P \rightarrow A_i$ a family of maps such that for any $(B, \{\varphi_i\})$ with $\varphi_i: B \rightarrow A_i$, there exists a **unique** $\varphi: B \rightarrow P$ s.t. $\pi_i \circ \varphi = \varphi_i, \forall i \in I$



We denote the product as $\prod_{i \in I} A_i$.

Prop 3.1 If $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both product of $\{A_i\}_{i \in I}$, then they are isomorphic.

Proof. For each product, we have



Uniqueness implies $\psi \circ \pi = \text{id}$. Similarly $\pi \circ \psi = \text{id}$.

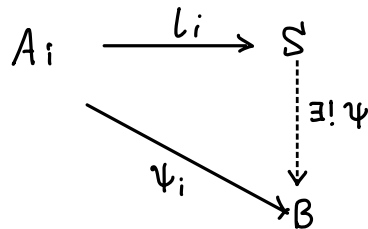
Remark. Generally, for $\{A_i\}_{i \in I} \subseteq \text{Ob } \mathcal{C}$, there may not exist a product.

But for Abelian category, the product and coproduct always exist.

(II) Coproduct

Coproduct is dual concept to product.

Def 3.2 For $\{A_i\}_{i \in I} \subseteq \text{Ob } \mathcal{C}$, its coproduct is defined as $(S, \{l_i: A_i \rightarrow S\})$ which satisfies: for any $(B, \{\psi_i: A_i \rightarrow B\})$, there exists a unique ψ s.t. $\psi l_i = \psi_i$:



We denote the coproduct as $\coprod_{i \in I} A_i$.

Prop 3.2. If $(S, \{l_i\})$, $(S', \{l'_i\})$ are both coproducts of $\{A_i\}_{i \in I}$, then they are isomorphic.

Proof. Similar to Prop 3.1

Example 3.1. In module category Mod_R , product and coproduct are direct product and direct sum.

Example 3.2. In Set : (1) product = Cartesian product

(2) coproduct = disjoint union

(III) Some special objects.

1. Free object.

Def 3.3 A concrete category \mathcal{C} is a category that is equipped with a (faithful) functor to the Set category. More precisely:

(1) Every object A is assigned with a set $G(A)$.

(2) Every map $A \xrightarrow{f} B$ is assigned with a set map $G(A) \rightarrow G(B)$.

$$(3) \text{ id}_A = \text{id}_{\sigma(A)}$$

$$(4) \sigma(f \circ g) = \sigma(f) \circ \sigma(g)$$

$$A \xrightarrow{g} B \xrightarrow{f} \rightsquigarrow \sigma(A) \xrightarrow{\sigma(f)} \sigma(B) \xrightarrow{\sigma(g)} \sigma(C)$$

Remark. In concrete category, we can regard object as a set equipped with some additional structure.

Example: Grp , Ring , Vect , Mod_R are all concrete categories.

Recall that for free module M and its basis X , consider the inclusion $L: X \hookrightarrow M$, then for any module N and set map $f: X \rightarrow N$, there exists unique \tilde{f} s.t. $f = \tilde{f} \circ L$.

$$\begin{array}{ccc} X & \xrightarrow{L} & M \\ & \searrow f & \downarrow \exists! \tilde{f} \\ & & N \end{array}$$

This inspire the following definition of free objects:

Def 3.4 Let V be an object of some concrete category \mathcal{C} , $i: X \rightarrow V$ is a set map. If for any object $A \in \text{Ob } \mathcal{C}$ and set map $f: X \rightarrow A$, there is a unique $\tilde{f}: V \rightarrow A$ s.t. $f = \tilde{f} \circ i$, then V is called a free object over X .

$$\begin{array}{ccc} X & \xrightarrow{i} & V \\ & \searrow f & \downarrow \exists! \tilde{f} \\ & & A \end{array}$$

Example. free module is free object in Mod_R .

Prop 3.3 Let \mathcal{C} be a concrete category and V be a free object over X . V' be a free object over X' , if $|X| = |X'|$, then $V \cong V'$.

Proof. Exercise.

2. Initial and terminal object.

Def 3.5 For a category \mathcal{C}

means $A \xrightarrow{f} B$ exists & unique

(1) A is called an initial object if for any B , $\# \text{Hom}(A, B) = 1$.

(2) A is called a terminal object if for any B , $\# \text{Hom}(B, A) = 1$.

(3) A is called a zero or null object if it is initial and terminal.

Prop Initial, terminal and zero object, if exist, must be unique up to isomorphisms.

Proof. Exercise.

Example. In Set , \emptyset is initial and $\{*\}$ is terminal.

Example. In Grp , $\{1\}$ is zero object.

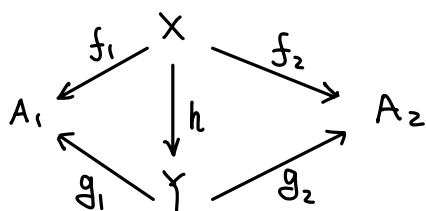
Example. In Mod_R , $\{0\}$ is zero object.

3. Product and coproduct as terminal and initial object.

• Let \mathcal{C} be a category, $A_1, A_2 \in \text{Ob } \mathcal{C}$. Define

$$\text{Ob}(\mathcal{C}/\{A_1, A_2\}) = \{(X, f_1, f_2) \mid X \in \text{Ob } \mathcal{C}, f_i \in \text{Hom}_{\mathcal{C}}(X, A_i), i=1, 2\}.$$

$$\text{Hom}_{\mathcal{C}/\{A_1, A_2\}}((X, f_1, f_2), (Y, g_1, g_2)) = \{h \in \text{Hom}_{\mathcal{C}}(X, Y) \mid g_i h = f_i, i=1, 2\}.$$

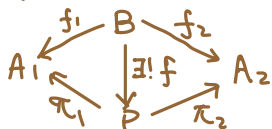


$\mathcal{C}/\{A_1, A_2\}$ is a category.

Prop. Terminal object in $\mathcal{C}/\{A_1, A_2\}$ is product of A_1, A_2 in \mathcal{C} .

Proof. Terminal (P, π_1, π_2)

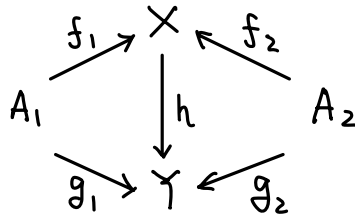
For any (B, f_1, f_2) , there exists unique f such that



- Similarly, we can define $\mathcal{C}/\{A_1, A_2\}$ for coproduct

$$\text{Ob } \mathcal{C}/\{A_1, A_2\} = \{ (X, f_1, f_2) \mid X \in \text{Ob } \mathcal{C}, f_i \in \text{Hom}(A_i, X), i=1,2 \}.$$

$$\text{Hom}_{\mathcal{C}/\{A_1, A_2\}}((X, f_1, f_2), (Y, g_1, g_2)) = \{ h \in \text{Hom}(X, Y) \mid h f_i = g_i, i=1,2 \}.$$



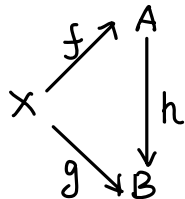
Prop In above $\mathcal{C}/\{A_1, A_2\}$, the initial object is coproduct of A_1 and A_2 in the category \mathcal{C} .

Proof. Exercise.

4. Free object as initial object.

For concrete category \mathcal{C} and a set X , we could define a category $\text{Hom}(X, \mathcal{C})$.

- $\text{Ob } \text{Hom}(X, \mathcal{C}) = \bigcup_{A \in \text{Ob } \mathcal{C}} \text{Hom}_{\mathcal{C}}(X, A)$
- Map between $x \xrightarrow{f} A$ and $x \xrightarrow{g} B$ is

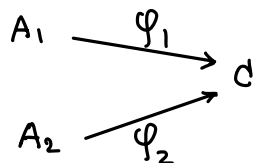


Prop In $\text{Hom}(X, \mathcal{C})$, an initial object is a free object over X in \mathcal{C} .

Proof. Exercise.

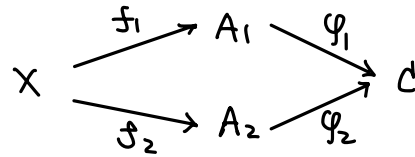
5. Pull-back

Consider a category \mathcal{C} and two maps with the same codomain

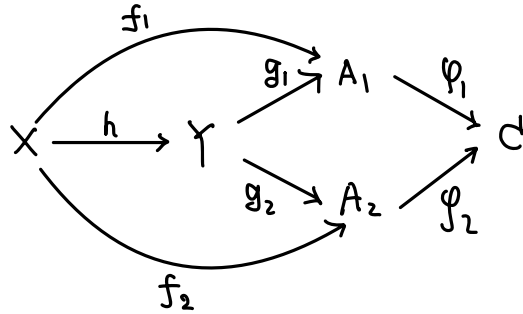


we define a category \mathcal{D} as follows:

- $\text{Ob } \mathcal{D} = \{ (X, f_1, f_2) \mid X \in \text{Ob } \mathcal{C}, f_i \in \text{Hom}(X, A_i), i=1,2, \varphi_1 f_1 = \varphi_2 f_2 \}$

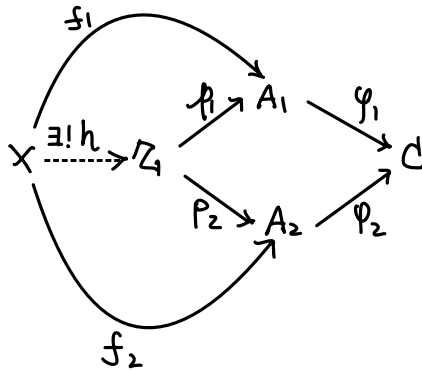


- $\text{Hom}_{\mathcal{D}}((X, f_1, f_2), (Y, g_1, g_2)) = \{ h \in \text{Hom}_{\mathcal{C}}(X, Y) \mid g_i h = f_i, i=1,2 \}$.



In this category, a terminal object (if exist) is called a pull-back of (φ_1, φ_2) .

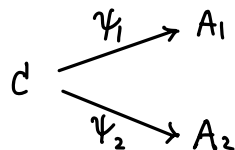
In other words, a pull-back is (Z, p_1, p_2) such that for any (X, f_1, f_2) satisfying $\varphi_1 f_1 = \varphi_2 f_2$, there exists unique $h: X \rightarrow Z$ s.t. $p_i h = f_i, i=1,2$



This is also called fiber product $A_1 \times_C A_2$.

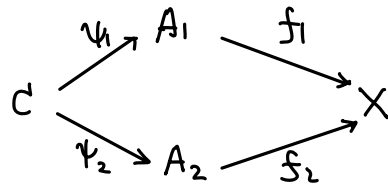
6. Push-out

Consider a category \mathcal{C} and two maps with the same domain

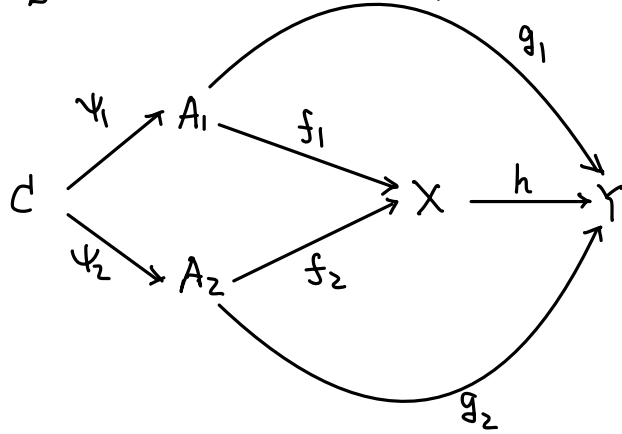


We define a category \mathcal{D} as follows

- $\text{Ob } \mathcal{D} = \{ (X, f_1, f_2) \mid X \in \text{Ob } \mathcal{C}, f_i \in \text{Hom}_{\mathcal{C}}(A_i, X), i=1,2, f_1 \psi_1 = f_2 \psi_2 \}$

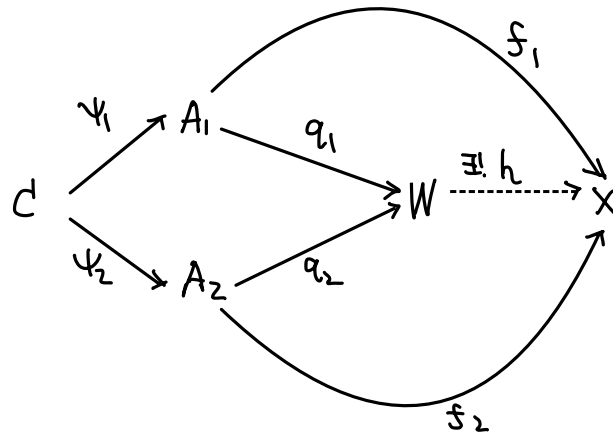


- $\text{Hom}_{\mathcal{D}}((X, f_1, f_2), (Y, g_1, g_2)) = \{ h \in \text{Hom}_{\mathcal{C}}(X, Y) \mid g_i = h f_i, i=1,2 \}$



The initial object (W, q_1, q_2) in \mathcal{D} is called push-out of ψ_1 and ψ_2 .

In other words, a push-out (W, q_1, q_2) satisfies that: for any (X, f_1, f_2) with $f_1 \psi_1 = f_2 \psi_2$, there is a unique $h: W \rightarrow X$ such that $h q_i = f_i$.



7. Tensor product as initial objects

Consider module category Mod_R , fix $A, B \in \text{Mod}_R$, we define a category

$\mathcal{B}(A, B)$ as follows:

- $\text{Ob } \mathcal{B}(A, B) = \{ \text{bilinear } f: A \times B \rightarrow C, C \in \text{Mod}_R \}$
- $\text{Hom}_{\mathcal{B}(A, B)}(f, g) = \{ h \in \text{Hom}_{\text{Mod}_R}(C, D) \mid g = hf \}$.

$$\begin{array}{ccc}
 A \times B & \xrightarrow{f} & C \\
 & \searrow g & \downarrow h \\
 & & D
 \end{array}$$

Then tensor product $A \otimes_R B$ is an initial object in $\mathcal{B}(A, B)$.

$$\begin{array}{ccc}
 A \times B & \xrightarrow{\otimes} & A \otimes_R B \\
 & \searrow f & \downarrow \exists! \tilde{f} \\
 & & X
 \end{array}$$