HKUST MATH 1014 L1 assignment 3 submission

1

1.a

Let m and n be non-negative integers. Evaluate the following integrals, distinguishing all possible cases for m and n.

1.a.i

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, \mathrm{d}x$$

$$\begin{aligned}
&\int_{-\pi}^{\pi} \cos mx \cos nx \, \mathrm{d}x \\
&= \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) + \cos((m-n)x)) \, \mathrm{d}x \\
&= \begin{cases} \frac{1}{2} \int_{-\pi}^{\pi} (1+1) \, \mathrm{d}x & m+n=0, m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (1+\cos((m-n)x)) \, \mathrm{d}x & m+n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x)+1) \, \mathrm{d}x & m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x)+\cos((m-n)x)) \, \mathrm{d}x & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{2} [2x]_{x=-\pi}^{x=\pi} & m+n=0, m-n=0 \\ \frac{1}{2} \left[x + \frac{1}{m-n} \sin((m-n)x)\right]_{x=-\pi}^{x=\pi} & m+n=0 \\ \frac{1}{2} \left[\frac{1}{m+n} \sin((m+n)x) + x\right]_{x=-\pi}^{x=\pi} & m-n=0 \\ \frac{1}{2} \left[\frac{1}{m+n} \sin((m+n)x) + \frac{1}{m-n} \sin((m-n)x)\right]_{x=-\pi}^{x=\pi} & \text{otherwise} \end{cases} \\
&= \begin{cases} 2\pi & m+n=0, m-n=0 \\ \pi & m+n=0 \\ \pi & m-n=0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

1.a.ii

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x$$

$$\begin{aligned}
&\int_{-\pi}^{\pi} \sin mx \sin nx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) - \cos((m-n)x)) \, \mathrm{d}x \\
&= \begin{cases} \frac{1}{2} \int_{-\pi}^{\pi} (1-1) \, \mathrm{d}x & m+n=0, m-n=0 \\ \frac{1}{2} \int > & \\ {}^{\pi}_{-\pi}(1-\cos((m-n)x)) \, \mathrm{d}x & m+n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x)-1) \, \mathrm{d}x & m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x)-\cos((m-n)x)) \, \mathrm{d}x & \text{otherwise} \end{cases} \\
&= \begin{cases} 0 & m+n=0, m-n=0 \\ \frac{1}{2} \left[x - \frac{1}{m-n} \sin((m-n)x)\right]_{x=-\pi}^{x=\pi} & m+n=0 \\ \frac{1}{2} \left[\frac{1}{m+n} \sin((m+n)x) - x\right]_{x=-\pi}^{x=\pi} & m-n=0 \\ \frac{1}{2} \left[\frac{1}{m+n} \sin((m+n)x) - \frac{1}{m-n} \sin((m-n)x)\right]_{x=-\pi}^{x=\pi} & \text{otherwise} \end{cases} \\
&= \begin{cases} 0 & m+n=0, m-n=0 \\ \pi & m+n=0 \\ -\pi & m-n=0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

1.a.iii

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, \mathrm{d}x$$

$$\begin{aligned}
&\int_{-\pi}^{\pi} \cos mx \sin nx \, \mathrm{d}x \\
&= \int_{0}^{\pi} \cos mx \sin nx \, \mathrm{d}x + \int_{-\pi}^{0} \cos mx \sin nx \, \mathrm{d}x \\
&= \int_{0}^{\pi} \cos mx \sin nx \, \mathrm{d}x - \int_{\pi}^{0} \cos(-mu) \sin(-nu) \, \mathrm{d}u \qquad (u := -x) \\
&= \int_{0}^{\pi} \cos mx \sin nx \, \mathrm{d}x - \int_{0}^{\pi} \cos mu \sin nu \, \mathrm{d}u \\
&= 0
\end{aligned}$$

2

Evaluate the following antiderivatives.

2.a

$$\int x^2 \arctan x \, \mathrm{d}x$$

$$\begin{aligned}
&\int x^2 \arctan x \, \mathrm{d}x \\
&= \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int \frac{x^3}{1+x^2} \, \mathrm{d}x \\
&= \frac{1}{3}x^3 \arctan x - \frac{1}{6}\int \frac{x^2}{1+x^2} \, \mathrm{d}x^2 \\
&= \frac{1}{3}x^3 \arctan x - \frac{1}{6}\left(\int \mathrm{d}x^2 - \int \frac{1}{1+x^2} \, \mathrm{d}x^2\right) \\
&= \frac{1}{3}x^3 \arctan x - \frac{1}{6}\left(x^2 - \ln\left(1+x^2\right)\right) + C \qquad (1+x^2>0) \\
&= \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln\left(1+x^2\right) + C
\end{aligned}$$

2.d

$$\int (2x^2+1)e^{x^2} \, \mathrm{d}x$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}e^{x^2} &= 2xe^{x^2} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(2xe^{x^2}\right) &= 2e^{x^2} + 4x^2e^{x^2} \\
&= 2(2x^2+1)e^{x^2} \\
\int (2x^2+1)e^{x^2} \, \mathrm{d}x &= xe^{x^2} + C
\end{aligned}$$

7

Let $f : [0, +\infty) \to \mathbb{R}$ be the function defined by $f(x) = xe^x$.

7.a

Show that f is strictly increasing.

$$\begin{aligned}
f(x) &= xe^x \\
f'(x) &= e^x + xe^x \\
&= (1+x)e^x
\end{aligned}$$

$$\begin{aligned}
&\forall x \in [0, +\infty) \\
&\qquad 1+x \geq 1 \\
&\qquad e^x \geq 1 \\
\\
&(1+x)e^x \geq 1 \\
&f'(x) \geq 1 \\
&\qquad > 0 \qquad \implies f(x) \text{ is strictly increasing}
\end{aligned}$$

7.b

Now f is one-to-one according to (a), so we let g be the **inverse** of f, i.e. $g = f^{-1}$.

7.b.i

Write down the domain of g. Show that

$$g'(x) = \frac{1}{x + e^{g(x)}}$$

for every x in the interior of the domain of g.

The domain of g is $[0,+\infty)$.

$$\begin{aligned}
&f'(x) = (1+x)e^x = e^x + f(x) \in C^0([0,+\infty),[0,+\infty)) \\
\implies\ & f(x) \in C^1((0,+\infty),(0,+\infty)) \\
\implies\ & g'(x) = \frac{1}{f'(g(x))} \qquad \forall x \in (0,+\infty) \qquad\qquad \text{(inverse function theorem)}
\end{aligned}$$

$$\begin{aligned}
&\forall x \in (0,+\infty) \\
&\ g'(x) \\
&= \frac{1}{f'(g(x))} \\
&= \frac{1}{f(g(x)) + e^{g(x)}} \\
&= \frac{1}{x + e^{g(x)}}
\end{aligned}$$

7.b.ii

Using the result from 7.b.i or otherwise, evaluate the antiderivative $\int g(x)\,\mathrm{d}x$, expressing your answer in terms of g and other elementary functions only.

$$\begin{aligned}
&\forall x \in (0,+\infty) \\
&\ \int g(x)\,\mathrm{d}x \\
&= xg(x) - \int x\,\mathrm{d}g(x) \\
&= xg(x) - \int \frac{x}{x+e^{g(x)}}\,\mathrm{d}x \\
&= xg(x) - \left(\int \mathrm{d}x - \int \frac{e^{g(x)}}{x+e^{g(x)}}\,\mathrm{d}x\right) \\
&= xg(x) - \left(x - \int e^{g(x)} g'(x)\,\mathrm{d}x\right) \\
&= xg(x) - x + \int e^{g(x)}\,\mathrm{d}g(x) \\
&= xg(x) - x + e^{g(x)} + C
\end{aligned}$$

7.b.iii

Hence, or otherwise, evaluate the integral $\int_0^e g(x)\,\mathrm{d}x$.

$$\begin{aligned}
&f(1) = 1e^1 = e \\
&g(e) = 1 \\
&f(0) = 0e^0 = 0 \\
&g(0) = 0
\end{aligned}$$

$$\begin{aligned}
&\ \int_0^e g(x)\,\mathrm{d}x \\
&= \left[xg(x) - x + e^{g(x)}\right]_0^e \\
&= eg(e) - e + e^{g(e)} - e^{g(0)} \\
&= e - e + e^1 - e^0 \\
&= e - 1
\end{aligned}$$

10

For each non-negative integer n, let

$$I_n = \int_0^1 t^n e^t\,\mathrm{d}t$$

.

10.a

Show that $\frac{1}{n+1} \le I_n \le \frac{e}{n+1}$ for every non-negative integer n.

$$\begin{aligned}
\int_0^1 t^n e^0\,\mathrm{d}t \le \int_0^1 t^n e^t\,\mathrm{d}t \le \int_0^1 t^n e^1\,\mathrm{d}t \qquad & \left((\forall t \in [0,1])\left(e^0 \le e^t \le e^1\right)\right) \\
\int_0^1 t^n\,\mathrm{d}t \le I_n \le e\int_0^1 t^n\,\mathrm{d}t & \\
\left[\frac{t^{n+1}}{n+1}\right]_{t=0}^{t=1} \le I_n \le e\left[\frac{t^{n+1}}{n+1}\right]_{t=0}^{t=1} & \\
\frac{1}{n+1} \le I_n \le \frac{e}{n+1} &
\end{aligned}$$

10.b

Express I_n in terms of I_{n-1} for each $n \geq 1$. Hence show that

$$I_n = (-1)^{n+1} n! + e \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!}$$

.

$$\begin{aligned}
& I_n \\
&= \int_0^1 t^n e^t \, \mathrm{d}t \\
&= \left[t^n e^t\right]_{t=0}^{t=1} - n \int_0^1 t^{n-1} e^t \, \mathrm{d}t \\
&= e - n I_{n-1}
\end{aligned}$$

$$I_0 = \int_0^1 t^0 e^t \, \mathrm{d}t = \left[e^t\right]_0^1 = e - 1 = (-1)^{0+1} 0! + e \sum_{k=0}^{0} (-1)^k \frac{0!}{(0-k)!}$$

Assume $I_m = (-1)^{m+1} m! + e \sum_{k=0}^{m} (-1)^k \frac{m!}{(m-k)!}$.

$$\begin{aligned}
& I_{m+1} \\
&= e - (m+1) I_m && (m+1 \geq 1) \\
&= e - (m+1) \left((-1)^{m+1} m! + e \sum_{k=0}^{m} (-1)^k \frac{m!}{(m-k)!} \right) \\
&= e \left(1 - (m+1) \sum_{k=0}^{m} (-1)^k \frac{m!}{(m-k)!} \right) + (-1)^{m+2} (m+1)! \\
&= e \left(1 + \sum_{k=0}^{m} (-1)^{k+1} \frac{(m+1)!}{(m-k)!} \right) + (-1)^{m+2} (m+1)! \\
&= e \left((-1)^0 \frac{(m+1)!}{(m+1-0)!} + \sum_{k=1}^{m+1} (-1)^k \frac{(m+1)!}{(m+1-k)!} \right) + (-1)^{m+2} (m+1)! \\
&= e \sum_{k=0}^{m+1} (-1)^k \frac{(m+1)!}{(m+1-k)!} + (-1)^{m+2} (m+1)! \\
&= (-1)^{(m+1)+1} (m+1)! + e \sum_{k=0}^{m+1} (-1)^k \frac{(m+1)!}{((m+1)-k)!}
\end{aligned}$$

By induction, $\forall n \in \mathbb{Z}_{\geq 0}$

$$I_n = (-1)^{n+1} n! + e \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!}$$

10.c

Using 10.a and 10.b, deduce that e is an irrational number.

Hint: There is no integer in the open interval $(0, 1)$.

Assume e is rational, then $e = \dfrac{a}{b}$ where $a, b \in \mathbb{Z}, b \neq 0$.

$$
\begin{aligned}
&\forall n \in \mathbb{Z}_{\geq 0} \\
I_n &= (-1)^{n+1} n! + e \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!} \\
e &= \frac{I_n - (-1)^{n+1} n!}{\sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!}} \\
\frac{a}{b} &= \frac{I_n - (-1)^{n+1} n!}{\sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!}} \\
a \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!} &= b\left(I_n - (-1)^{n+1} n!\right) \\
&= bI_n - b(-1)^{n+1} n! \\
c &= bI_n - d && (c, d \in \mathbb{Z}) \\
bI_n &= c + d \\
bI_n &\in \mathbb{Z} && (\mathbb{Z} \text{ is closed under addition})
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{n+1} &\leq I_n \leq \frac{e}{n+1} \\
\frac{1}{n+1} &\leq I_n < \frac{3}{n+1} && (e < 3) \\
\frac{|b|}{n+1} &\leq |b| I_n < \frac{3|b|}{n+1} \\
\frac{|b|}{3|b|+1} &\leq |bI_{3|b|}| < \frac{3|b|}{3|b|+1} && (\text{set } n = 3|b|) \\
|bI_{3|b|}| &\notin \mathbb{Z} && (|b| < |3b+1|, |3b| < |3b+1|) \\
bI_{3|b|} &\notin \mathbb{Z}
\end{aligned}
$$

$$
\begin{aligned}
&(\forall n \in \mathbb{Z}_{\geq 0})(bI_n \notin \mathbb{Z}) \Rightarrow\Leftarrow bI_{3|b|} \notin \mathbb{Z} \\
\implies &\ e \text{ is irrational.}
\end{aligned}
$$

11

11.a

For every pair of non-negative integers m and n, let

$$B(m, n) = \int_0^1 x^m (1-x)^n \, \mathrm{d}x$$

.

Show that $B(m, n) = \frac{n}{m+1} B(m+1, n-1)$ for every pair of integers $m \geq 0$ and $n \geq 1$.

Hence or otherwise, deduce that $B(m, n) = \frac{m!n!}{(m+n+1)!}$.

$$
\begin{aligned}
&\forall (m,n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1} \\
&\quad B(m,n) \\
&= \int_0^1 x^m (1-x)^n \,\mathrm{d}x \\
&= \left[\frac{1}{m+1} x^{m+1} (1-x)^n \right]_{x=0}^{x=1} + \frac{n}{m+1} \int_0^1 x^{(m+1)} (1-x)^{n-1} \,\mathrm{d}x \\
&= 0 + \frac{n}{m+1} B(m+1, n-1) \\
&= \frac{n}{m+1} B(m+1, n-1)
\end{aligned}
$$

$$
\begin{aligned}
&\quad B(m,n) \\
&= \frac{n}{m+1} B(m+1, n-1) \\
&= \frac{n}{m+1} \frac{n-1}{m+2} B(m+2, n-2) \\
&\vdots \\
&= \frac{n!}{\frac{(m+n)!}{m!}} B(m+n, 0) \\
&= \frac{m!n!}{(m+n)!} B(m+n, 0) \\
&= \frac{m!n!}{(m+n)!} \int_0^1 x^{m+n} (1-x)^0 \,\mathrm{d}x \\
&= \frac{m!n!}{(m+n)!} \left[\frac{1}{m+n+1} x^{m+n+1} \right]_{x=0}^{x=1} \\
&= \frac{m!n!}{(m+n)!} \frac{1}{m+n+1} \\
&= \frac{m!n!}{(m+n+1)!}
\end{aligned}
$$

11.b

Show that

$$\int_0^1 \frac{x^4 (x-1)^4}{x^2+1} \,\mathrm{d}x = \frac{22}{7} - \pi$$

.

Using this together with the result from 11.a, deduce that

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}$$

.

$$
\begin{aligned}
&\int_0^1 \frac{x^4(x-1)^4}{x^2+1}\,\mathrm{d}x \\
&= \int_0^1 \frac{x^4\left(x^4-4x^3+6x^2-4x+1\right)}{x^2+1}\,\mathrm{d}x \\
&= \int_0^1 \frac{x^8-4x^7+6x^6-4x^5+x^4}{x^2+1}\,\mathrm{d}x \\
&= \int_0^1 \left(x^6-4x^5+5x^4-4x^2+4-\frac{4}{x^2+1}\right)\mathrm{d}x \\
&= \left[\frac{1}{7}x^7-\frac{2}{3}x^6+x^5-\frac{4}{3}x^3+4x-4\arctan x\right]_0^1 \\
&= \frac{1}{7}-\frac{2}{3}+1-\frac{4}{3}+4-\pi \\
&= \frac{22}{7}-\pi
\end{aligned}
$$

$$
\begin{aligned}
\frac{x^4(x-1)^4}{1^2+1} < \frac{x^4(x-1)^4}{x^2+1} &< \frac{x^4(x-1)^4}{0^2+1} && (x\in(0,1)) \\
\int_0^1 \frac{x^4(x-1)^4}{1^2+1}\,\mathrm{d}x < \int_0^1 \frac{x^4(x-1)^4}{x^2+1}\,\mathrm{d}x &< \int_0^1 \frac{x^4(x-1)^4}{0^2+1}\,\mathrm{d}x \\
\frac{1}{2}\int_0^1 x^4(x-1)^4\,\mathrm{d}x < \frac{22}{7}-\pi &< \int_0^1 x^4(x-1)^4\,\mathrm{d}x \\
\frac{1}{2}B(4,4) < \frac{22}{7}-\pi &< B(4,4) \\
\frac{1}{2}\frac{4!4!}{9!} < \frac{22}{7}-\pi &< \frac{4!4!}{9!} \\
\frac{1}{1260} < \frac{22}{7}-\pi &< \frac{1}{630} \\
-\frac{1}{1260} > \pi-\frac{22}{7} &> -\frac{1}{630} \\
\frac{22}{7}-\frac{1}{1260} > \pi &> \frac{22}{7}-\frac{1}{630} \\
\frac{22}{7}-\frac{1}{630} < \pi &< \frac{22}{7}-\frac{1}{1260}
\end{aligned}
$$

13

Evaluate the following antiderivatives, using trigonometric substitutions when appropriate.

13.a

$$\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}\,\mathrm{d}x$$

$$
\begin{aligned}
&\forall x\in(-1,1) \\
&\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}\,\mathrm{d}x \\
&= \int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1+x^2}}\,\mathrm{d}x \\
&= \int \left(\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1+x^2}}\right)\mathrm{d}x \\
&= \arcsin x + \int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x \\
&= \arcsin x + \int \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\,\mathrm{d}\theta && \left(x:=\tan\theta, \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right) \\
&= \arcsin x + \int \sec\theta\,\mathrm{d}\theta \\
&= \arcsin x + \int \frac{(\sec\theta)(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}\,\mathrm{d}\theta \\
&= \arcsin x + \int \frac{\sec^2\theta+\tan\theta\sec\theta}{\sec\theta+\tan\theta}\,\mathrm{d}\theta \\
&= \arcsin x + \ln|\tan\theta+\sec\theta| + C \\
&= \arcsin x + \ln\left|x+\sqrt{1+x^2}\right| + C && \left(\sec\theta>0 \quad \forall\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right) \\
&= \arcsin x + \ln\left(x+\sqrt{1+x^2}\right) + C && \left(x+\sqrt{1+x^2}>0 \quad \forall x\in(-1,1)\right)
\end{aligned}
$$

13.b

$$\int \frac{x+1}{(x^2+x+1)\sqrt{x^2+x+1}}\,\mathrm{d}x$$

$$
\begin{aligned}
&\int \frac{x+1}{(x^2+x+1)\sqrt{x^2+x+1}}\,\mathrm{d}x \\
&= \frac{1}{2}\int \frac{2x+1}{(x^2+x+1)\sqrt{x^2+x+1}}\,\mathrm{d}x + \frac{1}{2}\int \frac{1}{(x^2+x+1)\sqrt{x^2+x+1}}\,\mathrm{d}x \\
&= \frac{1}{2}\int \frac{1}{(x^2+x+1)^{1.5}}\,\mathrm{d}\left(x^2+x+1\right)\mathrm{d}x + \frac{1}{2}\int \frac{1}{((x+0.5)^2+0.75)\sqrt{(x+0.5)^2+0.75}}\,\mathrm{d}x \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{1}{2}\sqrt{0.75}\int \frac{\sec^2\theta}{\left(0.75\tan^2\theta+0.75\right)\sqrt{0.75\tan^2\theta+0.75}}\,\mathrm{d}\theta && \left(x := \sqrt{0.75}\tan\theta - 0.5, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right) \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{1}{2}\sqrt{0.75}\int \frac{1}{0.75\sqrt{0.75}\sec\theta}\,\mathrm{d}\theta \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3}\int \cos\theta\,\mathrm{d}\theta \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3}\sin\theta + C \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3}\operatorname{sgn}(x+0.5)\frac{1}{\sqrt{\frac{1}{\frac{4}{3}(x+0.5)^2}+1}} + C \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3}\operatorname{sgn}(x+0.5)\frac{1}{\sqrt{\frac{3+4(x+0.5)^2}{4(x+0.5)^2}}} + C \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3}\operatorname{sgn}(x+0.5)\sqrt{\frac{4(x+0.5)^2}{4x^2+4x+4}} + C \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2(x+0.5)}{3\sqrt{x^2+x+1}} + C \\
&= \frac{2x+1-3}{3\sqrt{x^2+x+1}} + C \\
&= \frac{2x-2}{3\sqrt{x^2+x+1}} + C
\end{aligned}
$$

14

14.a

For each non-negative integer n, let

$$I_n(x) = \int \frac{x^n}{\sqrt{x^2+1}}\,\mathrm{d}x$$

which is defined up to addition by a constant function. Find a reduction formula that connects I_n and I_{n-2} for $n \geq 2$.

$$
\begin{aligned}
&\forall n \geq 2 \\
I_n(x) &= \int \frac{x^n}{\sqrt{x^2+1}}\,\mathrm{d}x \\
&= x^{n-1}\sqrt{x^2+1} - (n-1)\int x^{n-2}\sqrt{x^2+1}\,\mathrm{d}x \\
&= x^{n-1}\sqrt{x^2+1} - (n-1)\left(\int x^{n-2}\left(\sqrt{x^2+1} - \frac{1}{\sqrt{x^2+1}}\right)\mathrm{d}x + I_{n-2}(x)\right) \\
&= x^{n-1}\sqrt{x^2+1} - (n-1)\left(\int x^{n-2}\frac{x^2}{\sqrt{x^2+1}}\,\mathrm{d}x + I_{n-2}(x)\right) \\
&= x^{n-1}\sqrt{x^2+1} - (n-1)\left(\int \frac{x^n}{\sqrt{x^2+1}}\,\mathrm{d}x + I_{n-2}(x)\right) \\
&= x^{n-1}\sqrt{x^2+1} - (n-1)\left(I_n(x) + I_{n-2}(x)\right) \\
&= x^{n-1}\sqrt{x^2+1} - (n-1)I_n(x) - (n-1)I_{n-2}(x) \\
nI_n(x) &= x^{n-1}\sqrt{x^2+1} - (n-1)I_{n-2}(x) \\
I_n(x) &= \frac{1}{n}x^{n-1}\sqrt{x^2+1} - \frac{n-1}{n}I_{n-2}(x)
\end{aligned}
$$

14.b

Hence evaluate

$$\int_0^1 \frac{x^5}{\sqrt{x^2+1}}\,\mathrm{d}x$$

.

$$
\begin{aligned}
&I_1(x)\\
&=\int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x\\
&=\frac{1}{2}\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x^2\\
&=\frac{1}{2}\left(2\sqrt{x^2+1}\right)+C\\
&=\sqrt{x^2+1}+C
\end{aligned}
$$

$$
\begin{aligned}
&I_5(x)\\
&=\frac{1}{5}x^4\sqrt{x^2+1}-\frac{4}{5}I_3(x)\\
&=\frac{1}{5}x^4\sqrt{x^2+1}-\frac{4}{5}\frac{1}{3}x^2\sqrt{x^2+1}+\frac{4}{5}\frac{2}{3}I_1(x)\\
&=\frac{1}{5}x^4\sqrt{x^2+1}-\frac{4}{15}x^2\sqrt{x^2+1}+\frac{8}{15}\sqrt{x^2+1}+C
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \frac{x^5}{\sqrt{x^2+1}}\,\mathrm{d}x\\
&=I_5(1)-I_5(0)\\
&=\left[\frac{1}{5}x^4\sqrt{x^2+1}-\frac{4}{15}x^2\sqrt{x^2+1}+\frac{8}{15}\sqrt{x^2+1}\right]_0^1\\
&=\frac{1}{5}\sqrt{2}-\frac{4}{15}\sqrt{2}+\frac{8}{15}\sqrt{2}-\frac{8}{15}\\
&=\frac{7}{15}\sqrt{2}-\frac{8}{15}
\end{aligned}
$$