

HKUST MATH 1014 L1 assignment 7 submission

MATH1014 Calculus II Problem Set 7
L01 (Spring 2024)

Problem Set 7

Note: The problem sets serve as additional exercise problems for your own practice.
Problem Set 7 covers materials from §8.2 - §8.4.

Q3

Let (a_n) be a sequence of positive real numbers.

Q3.a

Show that if $\sum_{k=1}^{+\infty} a_k$ converges, then $\sum_{k=1}^{+\infty} \frac{1}{a_k}$ diverges.

$$\begin{aligned} & \sum_{k=1}^{+\infty} a_k \text{ converges} \\ \implies & \lim_{k \rightarrow +\infty} a_k = 0 \\ \implies & \lim_{k \rightarrow +\infty} \frac{1}{a_k} = +\infty \quad (a_k > 0, \text{ algebraic limit theorem}) \\ \implies & \sum_{k=1}^{+\infty} \frac{1}{a_k} \text{ diverges} \end{aligned}$$

Q3.b

Show that if $\lim_{n \rightarrow +\infty} na_n = L > 0$, then $\sum_{k=1}^{+\infty} a_k$ diverges.

$$\begin{aligned} & \sum_{k=1}^{+\infty} \frac{1}{k} \text{ diverges by the } p\text{-test} \\ & \lim_{n \rightarrow +\infty} na_n = L > 0 \\ \implies & \lim_{n \rightarrow +\infty} \frac{a_n}{\frac{1}{n}} = L > 0 \\ \implies & \sum_{k=1}^{+\infty} a_k \text{ diverges by the limit comparison test} \quad (a_k > 0) \end{aligned}$$

Q3.c

Show that if $\sum_{k=1}^{+\infty} a_k$ converges, then $\sum_{k=1}^{+\infty} a_k^2$ converges. Is the converse true?

$$\sum_{k=1}^{+\infty} a_k \text{ converges}$$

$$\implies \lim_{k \rightarrow +\infty} a_k = 0$$

$$\lim_{n \rightarrow +\infty} \frac{a_k^2}{a_k} = \lim_{n \rightarrow +\infty} a_k = 0$$

$$\implies \sum_{k=1}^{+\infty} a_k^2 \text{ converges by the limit comparison test} \quad (a_k > 0)$$

The converse is not true. Let $a_k = \frac{1}{k}$.

Then $\sum_{k=1}^{+\infty} a_k^2$ converges by the p -test,

But $\sum_{k=1}^{+\infty} a_k$ diverges by the p -test.

Q3.d

Show that if $\sum_{k=1}^{+\infty} a_k^2$ converges, then $\sum_{k=1}^{+\infty} \frac{a_k}{k}$ converges.

Hint: AM-GM inequality.

$$\sum_{k=1}^{+\infty} \frac{1}{k} \text{ diverges by the } p\text{-test and } \sum_{k=1}^{+\infty} a_k^2 \text{ converges}$$

$$\implies \lim_{k \rightarrow +\infty} \frac{a_k^2}{\frac{1}{k}} = 0 \quad (a_k^2 > 0, k > 0, \text{contrapositive of the limit comparison test})$$

$$\implies \lim_{k \rightarrow +\infty} \frac{a_k}{\frac{1}{\sqrt{k}}} = 0 \quad (a_k > 0, k > 0, \text{algebraic limit theorem})$$

$$\implies \lim_{k \rightarrow +\infty} \frac{\frac{1}{k}}{\frac{1}{\sqrt{k}}} \cdot \frac{a_k}{\frac{1}{\sqrt{k}}} = \lim_{k \rightarrow +\infty} \frac{\frac{a_k}{k}}{\frac{1}{k^{\frac{3}{2}}}} = 0$$

$$\sum_{k=1}^{+\infty} \frac{1}{k^{\frac{3}{2}}} \text{ converges by the } p\text{-test}$$

$$\implies \sum_{k=1}^{+\infty} \frac{a_k}{k} \text{ converges by the limit comparison test} \quad (a_k > 0)$$

Q7

For each of the following series, find all the values of $p \in \mathbb{R}$ such that the series converges.

Q7.a

$$\sum_{k=1}^{+\infty} k^2 \sin^p \frac{1}{k}$$

$$\begin{aligned}\frac{1}{k} &\in (0, 1] \\ \implies \sin^p \frac{1}{k} &> 0 \\ \implies k^2 \sin^p \frac{1}{k} &> 0\end{aligned}$$

$$\begin{aligned}k &> 0 \\ \implies k^{2-p} &> 0\end{aligned}$$

$$\begin{aligned}\lim_{k \rightarrow +\infty} \frac{k^2 \sin^p \frac{1}{k}}{k^{2-p}} \\ &= \lim_{k \rightarrow +\infty} \frac{\sin^p \frac{1}{k}}{k^{-p}} \\ &= \lim_{k \rightarrow +\infty} \left(\frac{\sin \frac{1}{k}}{\frac{1}{k}} \right)^p \\ &= 1^p \\ &= 1\end{aligned}$$

$$\sum_{k=1}^{+\infty} k^{2-p} \text{ converges by the } p\text{-test iff } p \in (3, +\infty)$$

$$\sum_{k=1}^{+\infty} k^2 \sin^p \frac{1}{k} \text{ converges by the limit comparison test iff } p \in (3, +\infty)$$

Q7.b

$$\sum_{k=2}^{+\infty} \frac{1}{(\ln \ln k)^{p \ln k}}$$

$$\sum_{k=2}^{+\infty} \frac{1}{(\ln \ln k)^{p \ln k}} \quad \dots (1)$$

$$\begin{aligned}\int_2^{+\infty} \frac{1}{(\ln \ln k)^{p \ln k}} dk &\quad \dots (2) \\ = \int_{\ln 2}^{+\infty} \frac{e^k}{(\ln k)^{pk}} dk &\quad \dots (3) \quad (\text{change of variable: } \ln k \mapsto k)\end{aligned}$$

$$\sum_{k=1}^{+\infty} \frac{e^k}{(\ln k)^{pk}} \quad \dots (4)$$

$$= \sum_{k=1}^{+\infty} \left(\frac{e}{(\ln k)^p} \right)^k \quad \dots (5)$$

$$\lim_{k \rightarrow +\infty} \sqrt[k]{\left| \left(\frac{e}{(\ln k)^p} \right)^k \right|} \quad \dots (6)$$

$$= \lim_{k \rightarrow +\infty} \sqrt[k]{\left(\frac{e}{(\ln k)^p} \right)^k} \quad \left(k > 1 \implies \frac{e}{\ln k} > 0 \right)$$

$$\begin{aligned}&= \lim_{k \rightarrow +\infty} \frac{e}{(\ln k)^p} \\ &= \begin{cases} 0, & p > 0 \\ e, & p = 0 \\ +\infty, & p < 0 \end{cases}\end{aligned}$$

$$(6) < 1 \text{ iff } p \in (0, +\infty)$$

$$\implies (5) \text{ is convergent iff } p \in (0, +\infty) \text{ by the root test}$$

$$\implies (4) \text{ is convergent iff } p \in (0, +\infty)$$

$$\implies (3) \text{ is convergent iff } p \in (0, +\infty) \text{ by the integral test}$$

$$\implies (2) \text{ is convergent iff } p \in (0, +\infty)$$

$$\implies (1) \text{ is convergent iff } p \in (0, +\infty) \text{ by the integral test}$$

Q7.c

$$\sum_{k=3}^{+\infty} \frac{1}{k(\ln k)(\ln \ln k)^p}$$

$$\sum_{k=3}^{+\infty} \frac{1}{k(\ln k)(\ln \ln k)^p} \quad \dots (1)$$

$$\begin{aligned} & \int_3^{+\infty} \frac{1}{k(\ln k)(\ln \ln k)^p} dk \quad \dots (2) \\ &= \int_{\ln 3}^{+\infty} \frac{1}{k(\ln k)^p} dk \quad (\text{change of variable: } \ln k \mapsto k) \\ &= \int_{\ln \ln 3}^{+\infty} \frac{1}{k^p} dk \quad \dots (3) \quad (\text{change of variable: } \ln k \mapsto k) \end{aligned}$$

(3) is convergent iff $p \in (1, +\infty)$ by the p -test
 \implies (2) is convergent iff $p \in (1, +\infty)$
 \implies (1) is convergent iff $p \in (1, +\infty)$ by the integral test

Q8

Let (a_n) be a sequence of real numbers, and define

$$a_n^+ := \max\{a_n, 0\} \quad \text{and} \quad a_n^- := \max\{-a_n, 0\}$$

for every n . Show that

Q8.a

If $\sum_{k=1}^{+\infty} a_k$ **converges absolutely**, then both $\sum_{k=1}^{+\infty} a_k^+$ and $\sum_{k=1}^{+\infty} a_k^-$ converge.

$$\begin{aligned} & \sum_{k=1}^{+\infty} a_k \text{ converges absolutely} \\ \implies & \sum_{k=1}^{+\infty} |a_k| \text{ converges} \\ & a_n \geq 0 \\ \implies & a_n^+ = a_n = |a_n| \\ & a_n < 0 \\ \implies & a_n^+ = 0 < |a_n| \\ & a_n^+ \leq |a_n| \\ \implies & \sum_{k=1}^{+\infty} a_k^+ \text{ converges by the direct comparison test} \\ & a_n \leq 0 \\ \implies & a_n^- = -a_n = |a_n| \\ & a_n > 0 \\ \implies & a_n^- = 0 < |a_n| \\ & a_n^- \leq |a_n| \\ \implies & \sum_{k=1}^{+\infty} a_k^- \text{ converges by the direct comparison test} \end{aligned}$$

Q8.b

If $\sum_{k=1}^{+\infty} a_k$ **converges conditionally**, then both $\sum_{k=1}^{+\infty} a_k^+$ and $\sum_{k=1}^{+\infty} a_k^-$ diverge.

$$\sum_{k=1}^{+\infty} a_k \text{ converges conditionally}$$

$$\implies \sum_{k=1}^{+\infty} |a_k| \text{ diverges}$$

$$a_n \geq 0$$

$$\implies a_n^+ = a_n$$

$$a_n < 0$$

$$\implies a_n^- = -a_n \implies a_n = -a_n^-$$

$\therefore (a_n)$ can be rewritten as a sequence in terms of a_n^+ and $-a_n^-$ only.

$(|a_n|)$ can be rewritten as a sequence in terms of a_n^+ and a_n^- only.

$$a_k^+ \geq 0 \implies a_k^+ = |a_k^+|$$

$$a_k^- \geq 0 \implies a_k^- = |a_k^-|$$

assume both $\sum_{k=1}^{+\infty} a_k^+ = L^+$ and $\sum_{k=1}^{+\infty} a_k^- = L^-$ converge

$$\implies \text{both } \sum_{k=1}^{+\infty} a_k^+ \text{ and } \sum_{k=1}^{+\infty} a_k^- \text{ converge absolutely}$$

$$\implies \sum_{k=1}^{+\infty} |a_n| = L^+ + L^- \text{ converges}$$

...since absolutely converging sequences can be rearranged without changing their sums.

the above conclusion contradicts that $\sum_{k=1}^{+\infty} |a_k|$ diverges

$$\implies \text{both } \sum_{k=1}^{+\infty} a_k^+ \text{ and } \sum_{k=1}^{+\infty} a_k^- \text{ cannot converge simultaneously} \quad \dots (1)$$

Without loss of generality,
assume one of the sum converges while the other diverges:

Assume $\sum_{k=1}^{+\infty} a_k^+ = L^+$ converges and $\sum_{k=1}^{+\infty} a_k^-$ diverges.

Then, $\sum_{k=1}^{n \in \mathbb{Z}_{>1}} a_k^+ \leq L^+$ by the monotone convergence theorem.

Consider $\sum_{k=1}^{n \in \mathbb{Z}_{>1}} a_k = \sum_{k=1}^n a_k^+ - \sum_{k=1}^n a_k^-$.

$$\sum_{k=1}^{n \in \mathbb{Z}_{>1}} a_k = \sum_{k=1}^n a_k^+ - \sum_{k=1}^n a_k^-$$

$$\leq L^+ - \sum_{k=1}^n a_k^-$$

$$a_k^- \geq 0 \text{ and } \sum_{k=1}^{+\infty} a_k^- \text{ diverges}$$

$$\implies \sum_{k=1}^{+\infty} a_k^- = +\infty$$

$$\sum_{k=1}^{+\infty} a_k^- = +\infty \text{ diverges}$$

$$\implies L^+ - \sum_{k=1}^{+\infty} a_k^- = -\infty \text{ diverges}$$

$$\implies \sum_{k=1}^{+\infty} a_k \leq -\infty \text{ diverges by the direct comparison test}$$

the above conclusion contradicts that $\sum_{k=1}^{+\infty} a_k$ converges

$\sum_{k=1}^{+\infty}$

$\sum_{k=1}^{+\infty}$

$$\Rightarrow \sum_{k=1}^{\infty} a_k^+ \text{ cannot converge and } \sum_{k=1}^{\infty} a_k^- \text{ cannot diverge simultaneously} \quad \dots (2)$$

$$\text{Similarly, } \sum_{k=1}^{\infty} a_k^- \text{ cannot converge and } \sum_{k=1}^{\infty} a_k^+ \text{ cannot diverge simultaneously} \quad \dots (3)$$

(1), (2), (3) combined implies that both integrals must diverge simultaneously.

Q9

For each of the following series, determine whether it diverges, converges absolutely or converges conditionally.

Q9.b

$$\sum_{k=0}^{+\infty} (-1)^{k+1} (\sqrt{k+1} - \sqrt{k})$$

$$\sum_{k=0}^{+\infty} (-1)^{k+1} (\sqrt{k+1} - \sqrt{k}) \quad \dots (1)$$

$$a_k := \sqrt{k+1} - \sqrt{k} \quad k \geq 0$$

$$a_{k+1} = \sqrt{k+2} - \sqrt{k+1}$$

$$a_k > 0 \quad (\sqrt{*} \text{ is increasing})$$

$$\begin{aligned} & |a_k| - |a_{k+1}| \\ &= |\sqrt{k+1} - \sqrt{k}| - |\sqrt{k+2} - \sqrt{k+1}| \\ &= \sqrt{k+1} - \sqrt{k} + \sqrt{k+2} - \sqrt{k+1} \\ &= \sqrt{k+2} - \sqrt{k} \\ &> 0 \\ &\therefore |a_k| \geq |a_{k+1}| \quad \dots (2) \end{aligned}$$

$$\begin{aligned} & \lim_{k \rightarrow +\infty} a_k \\ &= \lim_{k \rightarrow +\infty} (\sqrt{k+1} - \sqrt{k}) \\ &= \lim_{k \rightarrow +\infty} \frac{(k+1) - k}{\sqrt{k+1} + \sqrt{k}} \\ &= \lim_{k \rightarrow +\infty} \frac{1}{\sqrt{k+1} + \sqrt{k}} \\ &= 0 \quad \dots (3) \end{aligned}$$

(1) is an alternating series in terms of a_k .

(1) converges by the alternating series test due to (2), (3).

$$\begin{aligned} & \sum_{k=0}^{+\infty} |(-1)^{k+1} (\sqrt{k+1} - \sqrt{k})| \quad \dots (4) \\ &= \sum_{k=0}^{+\infty} (\sqrt{k+1} - \sqrt{k}) \quad (|(-1)^{k+1}| = 1, \sqrt{k+1} - \sqrt{k} > 0) \\ &= \lim_{n \rightarrow +\infty} \sum_{k=0}^n (\sqrt{k+1} - \sqrt{k}) \\ &= \lim_{n \rightarrow +\infty} \sqrt{n+1} \quad (\text{telescope}) \\ &= +\infty \\ &\therefore (4) \text{ diverges} \end{aligned}$$

\therefore (1) converges but (4) diverges

\therefore (1) converges conditionally

Q9.d

$$\sum_{k=2}^{+\infty} \frac{(-1)^k}{\sqrt{k} + (-1)^k}$$

$$\sum_{k=2}^{+\infty} \frac{(-1)^k}{\sqrt{k} + (-1)^k} \quad \dots (1)$$

$$= \sum_{k=2}^{+\infty} \frac{(-1)^k (\sqrt{k} - (-1)^k)}{k - 1}$$

$$= \sum_{k=2}^{+\infty} \frac{(-1)^k \sqrt{k} - 1}{k - 1}$$

$$= \sum_{k=2}^{+\infty} \left(\frac{(-1)^k \sqrt{k}}{k - 1} - \frac{1}{k - 1} \right) \quad \dots (2)$$

Consider the sum $\sum_{k=2}^{+\infty} \frac{(-1)^k \sqrt{k}}{k - 1}$.

It is alternating series with $a_k = \frac{\sqrt{k}}{k - 1}$.

$$\text{Also, } \frac{da_k}{dk} = \frac{0.5k^{-0.5}(k - 1) - \sqrt{k}}{(k - 1)^2} = \frac{-0.5\sqrt{k} - 0.5k^{-0.5}}{(k - 1)^2} < 0.$$

Thus a_k is strictly decreasing when $k \geq 2$.

$$\text{Furthermore, } \lim_{k \rightarrow +\infty} \frac{\sqrt{k}}{k - 1} = \lim_{k \rightarrow +\infty} \frac{\sqrt{\frac{k}{k^2}}}{1 - \frac{1}{k}} = 0.$$

By the alternating series test,
the sum being considered converges.

$$\text{Consider the sum } \sum_{k=2}^{+\infty} \frac{1}{k - 1} = \sum_{k=1}^{+\infty} \frac{1}{k}.$$

The sum diverges by the p -test.

Finally, by above and the algebraic limit theorem,

$$(2) \text{ equals } \sum_{k=2}^{+\infty} \frac{(-1)^k \sqrt{k}}{k - 1} - \sum_{k=2}^{+\infty} \frac{1}{k - 1} \text{ and diverges.}$$

Thus (1) diverges.

Q11

Find the radius and interval of convergence for each of the following power series.

Q11.b

$$\sum_{k=1}^{+\infty} \frac{x^k}{2^k k^2}$$

The center of the power series is $x = 0$.

The coefficient of the power series is $c_k = \frac{1}{2^k k^2}$.

$$\begin{aligned} & \text{radius of convergence} \\ &= \lim_{k \rightarrow +\infty} \left| \frac{c_k}{c_{k+1}} \right| \\ &= \lim_{k \rightarrow +\infty} \left| \frac{2^{k+1}(k+1)^2}{2^k k^2} \right| \\ &= 2 \lim_{k \rightarrow +\infty} \left| \frac{k^2 + 2k + 1}{k^2} \right| \\ &= 2 \lim_{k \rightarrow +\infty} \left| \frac{1 + 2k^{-1} + k^{-2}}{1} \right| \\ &= 2 \end{aligned}$$

When $x = 2$,

$$\begin{aligned} & \sum_{k=1}^{+\infty} \frac{2^k}{2^k k^2} \\ &= \sum_{k=1}^{+\infty} \frac{1}{k^2} \end{aligned}$$

which converges by the p -test.

As $k \geq 1 \implies \frac{1}{k^2} > 0$,

The integral converges absolutely.

Then its alternating series counterpart,

$$\begin{aligned} & \sum_{k=1}^{+\infty} \frac{(-1)^k 2^k}{2^k k^2} \\ &= \sum_{k=1}^{+\infty} \frac{(-2)^k}{2^k k^2} \end{aligned}$$

also converges by the direct comparison test,

and is the expression when $x = -2$.

interval of convergence = $[-2, 2]$

Q14

For each of the following power series, evaluate its sum whenever it converges. What happens at the end-points of its interval of convergence?

Hint: In each part, apply term-wise differentiation or integration on some power series whose sum is well-known.

Q14.b

$$\sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (x-1)^k$$

The center of the power series is

$$x = 1.$$

The coefficients of the power series are

$$c_k = \frac{1}{k(k-1)}.$$

radius of convergence

$$\begin{aligned} &= \lim_{k \rightarrow +\infty} \left| \frac{c_k}{c_{k+1}} \right| \\ &= \lim_{k \rightarrow +\infty} \left| \frac{\frac{1}{k(k-1)}}{\frac{1}{(k+1)k}} \right| \\ &= \lim_{k \rightarrow +\infty} \left| \frac{k+1}{k-1} \right| \\ &= \lim_{k \rightarrow +\infty} \frac{k+1}{k-1} && (k \rightarrow +\infty) \\ &= \lim_{k \rightarrow +\infty} \frac{1 + \frac{1}{k}}{1 - \frac{1}{k}} \\ &= 1 \end{aligned}$$

When $x = 2$,

$$\begin{aligned} &\sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (2-1)^k \\ &= \sum_{k=2}^{+\infty} \frac{1}{k(k-1)} \\ &= \sum_{k=2}^{+\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\ &= \lim_{k \rightarrow +\infty} \left(1 - \frac{1}{k} \right) && (\text{telescope}) \\ &= 1 \end{aligned}$$

The sum converges.

When $x = 0$,

$$\begin{aligned} &\sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (0-1)^k \\ &= \sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (-1)^k \\ &\leq \sum_{k=2}^{+\infty} \left| \frac{1}{k(k-1)} (-1)^k \right| \\ &= \sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (2-1)^k \\ &= 1 \end{aligned}$$

The sum converges by the direct comparison test.

interval of convergence = $[0, 2]$

$$S(x) := \sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (x-1)^k \quad x \in [0, 2]$$

$$S(2) = 1$$

$$S'(x) = \frac{d}{dx} \sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (x-1)^k \quad x \in (0, 2)$$

$$= \sum_{k=2}^{+\infty} \frac{d}{dx} \frac{1}{k(k-1)} (x-1)^k \quad (\text{power series can be differentiated term-wise})$$

$$= \sum_{k=2}^{+\infty} \frac{1}{k-1} (x-1)^{k-1}$$

$$S'(1) = \sum_{k=2}^{+\infty} \frac{1}{k-1} (1-1)^{k-1}$$

$$= \sum_{k=2}^{+\infty} 0 \quad (k-1 > 0)$$

$$\begin{aligned}
&= 0 \\
S''(x) &= \frac{d^2}{dx^2} \sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (x-1)^k \quad x \in (0, 2) \\
&= \sum_{k=2}^{+\infty} \frac{d^2}{dx^2} \frac{1}{k(k-1)} (x-1)^k && \text{(power series can be differentiated term-wise)} \\
&= \sum_{k=2}^{+\infty} (x-1)^{k-2} \\
&= \lim_{k \rightarrow +\infty} \frac{(x-1)^{k-1} - (x-1)^0}{(x-1) - 1} \\
&= \lim_{k \rightarrow +\infty} \frac{(x-1)^{k-1} - 1}{x-2} \\
&= \frac{-1}{x-2} && (x \in (0, 2) \implies |x-1| < 1) \\
&= \frac{1}{2-x} \\
S'(x) &= \int S''(x) dx \\
&= \int \frac{1}{2-x} dx \\
&= -\ln|2-x| + C \\
S'(1) &= -\ln|2-1| + C \\
0 &= C \\
C &= 0 \\
S'(x) &= -\ln|2-x| \quad x \in (0, 2) \\
S(x) &= \int S'(x) dx \\
&= -\int \ln|2-x| dx \\
&= \int \ln|u| du && (u := 2-x) \\
&= u \ln|u| - \int \frac{u}{u} du \\
&= u \ln|u| - u + C \\
&= (2-x) \ln|2-x| - 2 + x + C \\
&\lim_{x \rightarrow 2^-} (2-x) \ln|2-x| \\
&= \lim_{x \rightarrow 2^-} \frac{\ln|2-x|}{\frac{1}{2-x}} \\
&= \lim_{x \rightarrow 2^-} \frac{-\frac{1}{2-x}}{\frac{1}{(2-x)^2}} && \text{(L'Hopital rule)} \\
&= \lim_{x \rightarrow 2^-} (x-2) \\
&= 0 \\
S(2) &= \lim_{x \rightarrow 2^-} ((2-x) \ln|2-x| - 2 + x + C) && \text{(make } S(x) \text{ continuous)} \\
1 &= C && \text{(algebraic limit theorem)} \\
C &= 1 \\
S(x) &= \begin{cases} (2-x) \ln|2-x| + x - 1, & x \in [0, 2) \\ 1, & x = 2 \end{cases} \\
&= \begin{cases} (2-x) \ln(2-x) + x - 1, & x \in [0, 2) \\ 1, & x = 2 \end{cases} && (2-x > 0)
\end{aligned}$$

The sum converges on the endpoints of the interval of convergence.

Q14.e

$$\sum_{k=1}^{+\infty} \frac{k}{k+1} x^k$$

The center of the power series is

$$x = 0.$$

The coefficients of the power series are

$$c_k = \frac{k}{k+1}.$$

radius of convergence

$$\begin{aligned} &= \lim_{k \rightarrow +\infty} \left| \frac{c_k}{c_{k+1}} \right| \\ &= \lim_{k \rightarrow +\infty} \left| \frac{\frac{k}{k+1}}{\frac{k+1}{k+2}} \right| \\ &= \lim_{k \rightarrow +\infty} \left| \frac{k(k+2)}{(k+1)^2} \right| \\ &= \lim_{k \rightarrow +\infty} \left| \frac{(k+1)^2 - 1}{(k+1)^2} \right| \\ &= \lim_{k \rightarrow +\infty} \left| 1 - \frac{1}{(k+1)^2} \right| \\ &= 1 \end{aligned}$$

When $x = -1$,

$$\sum_{k=1}^{+\infty} \frac{k}{k+1} (-1)^k$$

$$\text{Consider } a_k = \frac{k}{k+1} (-1)^k.$$

$$\text{Consider the subsequence given by } b_k = a_{2k} = \frac{2k}{2k+1} (-1)^{2k}.$$

$$\lim_{k \rightarrow +\infty} \frac{2k}{2k+1} (-1)^{2k} = \lim_{k \rightarrow +\infty} \frac{2}{2 + \frac{1}{k}} = 1$$

$$\text{Consider the subsequence given by } c_k = a_{2k-1} = \frac{2k-1}{2k} (-1)^{2k-1}.$$

$$\lim_{k \rightarrow +\infty} \frac{2k-1}{2k} (-1)^{2k-1} = \lim_{k \rightarrow +\infty} \left(-\frac{2 - \frac{1}{k}}{2} \right) = -1$$

As the two subsequences of $(a_k)_{k \in \mathbb{N}}$ approaches different values as $k \rightarrow +\infty$,

$$\lim_{k \rightarrow +\infty} \frac{k}{k+1} (-1)^k \text{ does not exist.}$$

$$\text{Thus, } \sum_{k=1}^{+\infty} \frac{k}{k+1} (-1)^k \text{ diverges by the tail test.}$$

When $x = 1$,

$$\begin{aligned} &\sum_{k=1}^{+\infty} \frac{k}{k+1} 1^k \\ &\geq \sum_{k=1}^{+\infty} \frac{k}{k+1} (-1)^k \end{aligned}$$

$$\text{Thus, } \sum_{k=1}^{+\infty} \frac{k}{k+1} 1^k \text{ diverges by the direct comparison test.}$$

$$\text{interval of convergence} = (-1, 1)$$

$$\begin{aligned} &\sum_{k=1}^{+\infty} x^k \quad x \in (-1, 1) \\ &= \lim_{k \rightarrow +\infty} \frac{x^{k+1} - x}{x - 1} \\ &= \frac{x}{1 - x} \end{aligned} \quad (|x| < 1)$$

$$S(x) := \sum_{k=1}^{+\infty} \frac{k}{k+1} x^k \quad x \in (-1, 1)$$

$$S(0) = \sum_{k=1}^{+\infty} \frac{k}{k+1} 0^k$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} 0 \\
&= 0 \\
S(x) &= \sum_{k=1}^{\infty} \frac{k}{k+1} x^k \\
&= \sum_{k=1}^{\infty} \left(1 - \frac{1}{k+1}\right) x^k \\
&= \sum_{k=1}^{\infty} x^k + \sum_{k=1}^{\infty} \left(1 - \frac{1}{k+1}\right) x^k - \sum_{k=1}^{\infty} x^k \\
&= \sum_{k=1}^{\infty} x^k - \sum_{k=1}^{\infty} \frac{1}{k+1} x^k \quad (\text{algebraic limit theorem}) \\
&= \frac{x}{1-x} - \sum_{k=1}^{\infty} \frac{1}{k+1} x^k
\end{aligned}$$

Not using differential equations:

$$\begin{aligned}
 T(x) &:= \sum_{k=1}^{+\infty} \frac{1}{k+1} x^k & x \in (-1, 1) \\
 &= \begin{cases} \frac{1}{x} \sum_{k=1}^{+\infty} \frac{1}{k+1} x^{k+1}, & x \in (-1, 0) \cup (0, 1) \\ 0, & x = 0 \end{cases}
 \end{aligned}$$

$$U(x) := \sum_{k=1}^{+\infty} \frac{1}{k+1} x^{k+1} \quad x \in (-1, 1)$$

$$\begin{aligned}
 U'(x) &= \frac{d}{dx} \sum_{k=1}^{+\infty} \frac{1}{k+1} x^{k+1} \\
 &= \sum_{k=1}^{+\infty} \frac{d}{dx} \frac{1}{k+1} x^{k+1} & (\text{power series can be differentiated term-wise}) \\
 &= \sum_{k=1}^{+\infty} x^k \\
 &= \frac{x}{1-x}
 \end{aligned}$$

$$\begin{aligned}
 U(0) &= \sum_{k=1}^{+\infty} \frac{1}{k+1} 0^{k+1} \\
 &= \sum_{k=1}^{+\infty} 0 & (k+1 > 0) \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 U(x) &= \int U'(x) dx \\
 &= \int \frac{x}{1-x} dx \\
 &= \int \frac{u-1}{u} du & u := 1-x \\
 &= u - \ln|u| + C \\
 &= 1-x - \ln|1-x| + C \\
 &= -x - \ln|1-x| + C & (C \text{ is arbitrary})
 \end{aligned}$$

$$\begin{aligned}
 U(0) &= -0 - \ln|1-0| + C \\
 C &= 0
 \end{aligned}$$

$$U(x) = -x - \ln|1-x| \quad x \in (-1, 1)$$

$$\begin{aligned}
 T(x) &= \begin{cases} \frac{1}{x} \sum_{k=1}^{+\infty} \frac{1}{k+1} x^{k+1}, & x \in (-1, 0) \cup (0, 1) \\ 0, & x = 0 \end{cases} \\
 &= \begin{cases} \frac{1}{x} U(x), & x \in (-1, 0) \cup (0, 1) \\ 0, & x = 0 \end{cases} \\
 &= \begin{cases} -1 - \frac{\ln|1-x|}{x}, & x \in (-1, 0) \cup (0, 1) \\ 0, & x = 0 \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 S(x) &= \frac{x}{1-x} - \sum_{k=1}^{+\infty} \frac{1}{k+1} x^k \\
 &= \frac{x}{1-x} - T(x) \\
 &= \begin{cases} \frac{x}{1-x} + 1 + \frac{\ln|1-x|}{x}, & x \in (-1, 0) \cup (0, 1) \\ \frac{0}{1-0}, & x = 0 \end{cases} \\
 &= \begin{cases} \frac{x+1-x}{1-x} + \frac{\ln|1-x|}{x}, & x \in (-1, 0) \cup (0, 1) \\ 0, & x = 0 \end{cases} \\
 &= \begin{cases} \frac{1}{1-x} + \frac{\ln|1-x|}{x}, & x \in (-1, 0) \cup (0, 1) \\ 0, & x = 0 \end{cases}
 \end{aligned}$$

The sum diverges on the endpoints of the interval of convergence.

Alternatively, using differential equations:

$$\begin{aligned}
S'(x) &= \frac{d}{dx} \left(\frac{x}{1-x} - \sum_{k=1}^{+\infty} \frac{1}{k+1} x^k \right) \\
&= \frac{(1-x) + x}{(1-x)^2} - \sum_{k=1}^{+\infty} \frac{d}{dx} \frac{1}{k+1} x^k && \text{(power series can be differentiated term-wise)} \\
&= \begin{cases} \frac{1}{(1-x)^2} - \sum_{k=1}^{+\infty} \frac{k}{k+1} x^{k-1}, & x \in (-1, 0) \cup (0, 1) \\ 1 - \frac{1}{2}, & x = 0 \end{cases} \\
&= \begin{cases} \frac{1}{(1-x)^2} - \frac{1}{x} \sum_{k=1}^{+\infty} \frac{k}{k+1} x^k, & x \in (-1, 0) \cup (0, 1) \\ \frac{1}{2}, & x = 0 \end{cases} \\
&= \begin{cases} \frac{1}{(1-x)^2} - \frac{S(x)}{x}, & x \in (-1, 0) \cup (0, 1) \\ \frac{1}{2}, & x = 0 \end{cases}
\end{aligned}$$

$$y := S(x) \quad x \in (-1, 0) \cup (0, 1)$$

$$y' = \frac{1}{(1-x)^2} - \frac{y}{x}$$

The above is a linear ordinary differential equation.

An unique solution exists on each of $(-1, 0)$ and $(0, 1)$.

Solve the homogeneous equation.

$$y' = -\frac{y}{x} \quad x \in (-1, 0) \cup (0, 1)$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\frac{1}{y} dy = -\frac{1}{x} dx$$

$$\int \frac{1}{y} dy = - \int \frac{1}{x} dx$$

$$\ln|y| = -\ln|x| + C$$

$$e^{\ln|y|} = e^{-\ln|x|+C}$$

$$|y| = \frac{e^C}{|x|}$$

$$y = \pm \frac{e^C}{|x|}$$

$$y = \frac{c_1}{x} \quad c_1 \in \mathbb{R}_{\neq 0}$$

When $c_1 = 0$,

$$y = \frac{0}{x} \quad x \in (-1, 0) \cup (0, 1)$$

$$y = 0$$

$$y' = 0$$

... which satisfies the homogeneous equation.

$$y = \frac{c_1}{x} \quad c_1 \in \mathbb{R}$$

Different c_1 can be chosen on each of $(-1, 0)$ and $(0, 1)$.

Solve the inhomogeneous equation.

$$y = \frac{1}{x} \int \frac{\det \left[\frac{1}{(1-x)^2} \right]}{\det \left[\frac{1}{x} \right]} dx \quad x \in (-1, 0) \cup (0, 1) \quad \text{(variation of parameters)}$$

$$= \frac{1}{x} \int \frac{x}{(1-x)^2} dx$$

$$= -\frac{1}{x} \int \frac{-u+1}{u^2} du \quad u := 1-x$$

$$= -\frac{1}{x} \left(-\ln|u| - \frac{1}{u} + c_1 \right) \quad c_1 \in \mathbb{R}$$

$$= \frac{\ln|1-x|}{x} + \frac{1}{x(1-x)} - \frac{c_1}{x}$$

Different c_1 can be chosen on each of $(-1, 0)$ and $(0, 1)$,

which will be denoted c_1^- and c_1^+ respectively below.

$$\lim_{x \rightarrow 0^+} \frac{\ln|1-x|}{x} = \frac{1}{\ln(1-x)}$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{\ln|1-x|}{x} + \frac{1}{x(1-x)} - \frac{c_1^+}{x}}{1} \quad (1-x \geq 0)$$

$$= \lim_{x \rightarrow 0^+} \frac{-\frac{1}{1-x}}{1} \quad (\text{L'Hopital rule})$$

$$= -1$$

$$\lim_{x \rightarrow 0^-} \frac{\ln|1-x|}{x} \quad (1-x \leq 0)$$

$$= \lim_{x \rightarrow 0^-} \frac{\ln(x-1)}{x}$$

$$= \lim_{x \rightarrow 0^-} \frac{\frac{1}{x-1}}{1} \quad (\text{L'Hopital rule})$$

$$= -1$$

$$\lim_{x \rightarrow 0} \frac{\ln|1-x|}{x} = -1$$

$$S(0) = \lim_{x \rightarrow 0^+} \left(\frac{\ln|1-x|}{x} + \frac{1}{x(1-x)} - \frac{c_1^+}{x} \right) \quad (\text{make } S(x) \text{ continuous})$$

$$0 = -1 + \lim_{x \rightarrow 0^+} \frac{1 - c_1^+ + c_1^+ x}{x(1-x)} \quad (\text{algebraic limit theorem})$$

$$1 = \lim_{x \rightarrow 0^+} \frac{c_1^+}{(1-x) - x} \quad (\text{L'Hopital rule})$$

$$1 = c_1^+$$

$$c_1^+ = 1$$

$$S(0) = \lim_{x \rightarrow 0^-} \left(\frac{\ln|1-x|}{x} + \frac{1}{x(1-x)} - \frac{c_1^-}{x} \right) \quad (\text{make } S(x) \text{ continuous})$$

$$0 = -1 + \lim_{x \rightarrow 0^-} \frac{1 - c_1^- + c_1^- x}{x(1-x)} \quad (\text{algebraic limit theorem})$$

$$1 = \lim_{x \rightarrow 0^-} \frac{c_1^-}{(1-x) - x} \quad (\text{L'Hopital rule})$$

$$1 = c_1^-$$

$$c_1^- = 1$$

Therefore,

$$S(x) = \begin{cases} \frac{\ln|1-x|}{x} + \frac{1}{x(1-x)} - \frac{1}{x}, & x \in (-1, 0) \cup (0, 1) \\ 0, & x = 0 \end{cases}$$

$$= \begin{cases} \frac{\ln|1-x|}{x} + \frac{1-1+x}{x(1-x)}, & x \in (-1, 0) \cup (0, 1) \\ 0, & x = 0 \end{cases}$$

$$= \begin{cases} \frac{\ln|1-x|}{x} + \frac{1}{1-x}, & x \in (-1, 0) \cup (0, 1) \\ 0, & x = 0 \end{cases}$$

The sum diverges on the endpoints of the interval of convergence.