HKUST MATH 1014 L1 assignment 6 submission

MATH1014 Calculus II Problem Set 6
L01 (Spring 2024)

Problem Set 6

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 6 covers materials from §7.7 – §8.2.

Q4

Compute the area of the region in $\mathbb{R}^2$ that is inside the curve with polar equation

$$r = 1 + \cos\theta$$

but outside the curve with polar equation

$$r = 3\cos\theta$$

.

Hint: Sketch the required region first.

The 1st curve $[0, 2\pi)$ is a cardioid, with the cusp facing the left.

The 2nd curve on $\left[0, \frac{\pi}{2}\right)$ is the upper half of the circle, with the leftmost point of the circle touching the origin.

The 2nd curve on $\left[\frac{3\pi}{2}, 2\pi\right)$ is the lower half of the circle, with the leftmost point of the circle touching the origin.

The 2nd curve on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$ contains negative r and is ignored.

For $\left[0, \frac{\pi}{2}\right)$, there is 1 and only 1 intersection point excluding the origin.

$$\begin{aligned} 1+\cos\theta &= 3\cos\theta \\ \cos\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{3} \qquad \theta \in \left[0, \frac{\pi}{2}\right) \end{aligned}$$

For $\left[\frac{3\pi}{2}, 2\pi\right)$, there is 1 and only 1 intersection point excluding the origin.

$$\begin{aligned} 1+\cos\theta &= 3\cos\theta \\ \cos\theta &= \frac{1}{2} \\ \theta &= \frac{5\pi}{3} \qquad \theta \in \left[\frac{3\pi}{2}, 2\pi\right) \end{aligned}$$

For intersections at the origin,

$$\begin{aligned} 1+\cos\theta &= 0 \\ \cos\theta &= -1 \\ \theta &= \pi \qquad \theta \in [0, 2\pi) \\ 3\cos\theta &= 0 \\ \cos\theta &= 0 \\ \theta &\in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\} \qquad \theta \in [0, 2\pi) \end{aligned}$$

Therefore, the area is...

$$\begin{aligned} &\int (1+\cos\theta)^2 \, \mathrm{d}\theta \\ =& \int \left(1 + 2\cos\theta + \cos^2\theta\right) \mathrm{d}\theta \\ =& \int \left(1 + 2\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{2}\right) \mathrm{d}\theta \\ =& \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta + C \end{aligned}$$

$$\begin{aligned} &\int (3\cos\theta)^2 \, \mathrm{d}\theta \\ =& \int 9\cos^2\theta \, \mathrm{d}\theta \\ =& \int \left(\frac{9}{2}\cos 2\theta + \frac{9}{2}\right) \mathrm{d}\theta \\ =& \frac{9}{2}\theta + \frac{9}{4}\sin 2\theta + C \end{aligned}$$

$$\begin{aligned} &\int \left((1+\cos\theta)^2 - (3\cos\theta)^2\right) \mathrm{d}\theta \\ =& \int (1+\cos\theta)^2 \, \mathrm{d}\theta - \int (3\cos\theta)^2 \, \mathrm{d}\theta \\ =& \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta - \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + C \\ =& -3\theta + 2\sin\theta - 2\sin 2\theta + C \end{aligned}$$

$$\begin{aligned} &\text{area} \\ =& \frac{1}{2}\left(\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left((1+\cos\theta)^2 - (3\cos\theta)^2\right) \mathrm{d}\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1+\cos\theta)^2 \, \mathrm{d}\theta + \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \left((1+\cos\theta)^2 - (3\cos\theta)^2\right) \mathrm{d}\theta\right) \\ =& \frac{1}{2}\left(\left[-3\theta + 2\sin\theta - 2\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} + \left[-3\theta + 2\sin\theta - 2\sin 2\theta\right]_{\frac{3\pi}{2}}^{\frac{5\pi}{3}} + \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right) \\ =& \frac{1}{2}\left(-3\left(\frac{\pi}{2} - \frac{\pi}{3} + \frac{5\pi}{3} - \frac{3\pi}{2}\right) + 2\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{3} + \sin\frac{5\pi}{3} - \sin\frac{3\pi}{2}\right) - 2\left(\sin\pi - \sin\frac{2\pi}{3} + \sin\frac{10\pi}{3} - \sin 3\pi\right)\right. \\ &\left. + \frac{3}{2}\left(\frac{3\pi}{2} - \frac{\pi}{2}\right) + 2\left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right) + \frac{1}{4}(\sin 3\pi - \sin\pi)\right) \\ =& \frac{1}{2}\left(-3\cdot\frac{\pi}{3} + 2\left(1 - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + 1\right) - 2\left(0 - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} - 0\right) + \frac{3\pi}{2} + 2(-1-1) + \frac{1}{4}(0-0)\right) \\ =& \frac{1}{2}\left(-\pi + 4 - 2\sqrt{3} + 2\sqrt{3} + \frac{3\pi}{2} - 4\right) \\ =& \frac{1}{2}\cdot\frac{\pi}{2} \end{aligned}$$

$$= \frac{\pi}{4}$$

Q5

Let $f:[0,\pi]\to[0,+\infty)$ be a continuously differentiable function, and consider the curve in $\mathbb{R}^2$ defined by the polar equation

$$r = f(\theta)$$

.

Such a curve can be viewed as a polar curve on the one hand, and as a parameterized curve on the other hand. Show that the area bounded between the curve and the x-axis evaluated using Theorem 7.98 (as a polar curve) is the same as that evaluated using Theorem 7.96 (as a parameterized curve).

$$\begin{aligned}
& f(\theta) \text{ is a nonnegative continuous function.} \\
A_1 &:= \frac{1}{2}\int_0^\pi f(\theta)^2 \,\mathrm{d}\theta && \text{(theorem 7.98)} \\
f(\theta)\in C^1([0,\pi]) &\implies \text{the curve is } C^1 \text{ smooth} \\
A_2 &:= -\int_0^\pi (f(\theta)\sin\theta)(f(\theta)\cos\theta)' \,\mathrm{d}\theta && (\text{theorem 7.96, } f(\theta)\in C^1([0,\pi]) \\
&= \int_0^\pi (f(\theta)\sin\theta)(f(\theta)\sin\theta - f'(\theta)\cos\theta)\,\mathrm{d}\theta && \big(f(\theta)\in C^1([0,\pi])\big) \\
&= \int_0^\pi \big(f(\theta)^2\sin^2\theta - f(\theta)f'(\theta)\sin\theta\cos\theta\big)\,\mathrm{d}\theta \\
&= \int_0^\pi f(\theta)^2\,\mathrm{d}\theta - \int_0^\pi \big(f(\theta)^2\cos^2\theta + f(\theta)f'(\theta)\sin\theta\cos\theta\big)\,\mathrm{d}\theta \\
&= \int_0^\pi f(\theta)^2\,\mathrm{d}\theta - \int_0^\pi (f(\theta)\cos\theta)\,\big(f(\theta)\cos\theta + f'(\theta)\sin\theta\big)\,\mathrm{d}\theta \\
&= \int_0^\pi f(\theta)^2\,\mathrm{d}\theta - \int_0^\pi (f(\theta)\cos\theta)(f(\theta)\sin\theta)'\,\mathrm{d}\theta \\
&= \int_0^\pi f(\theta)^2\,\mathrm{d}\theta - \big[f(\theta)^2\sin\theta\cos\theta\big]_0^\pi - \int_0^\pi (f(\theta)\sin\theta)(f(\theta)\cos\theta)'\,\mathrm{d}\theta \\
2A_2 &= \int_0^\pi f(\theta)^2\,\mathrm{d}\theta - \big[f(\theta)^2\sin\theta\cos\theta\big]_0^\pi \\
&= \int_0^\pi f(\theta)^2\,\mathrm{d}\theta - (0-0) \\
A_2 &= \frac{1}{2}\int_0^\pi f(\theta)^2\,\mathrm{d}\theta \\
&= A_1
\end{aligned}$$

Q8

Consider the region in the coordinate plane bounded by the curve

$$y = e^{-x}$$

, the x- and y-axes, and the line $x = \ln 2$. Find the volume of the solid obtained by revolving this region about the line $x = \ln 2$.

$$y = e^{-0}$$
$$= 1$$

$$y = e^{-\ln 2}$$
$$= \frac{1}{2}$$

$$y = e^{-x}$$
$$\ln y = -x$$
$$x = -\ln y$$

$$\begin{aligned}
&\int \ln y \, \mathrm{d}y \\
&= y \ln y - \int \frac{y}{y} \, \mathrm{d}y \\
&= y(\ln y - 1) + C
\end{aligned}$$

$$\begin{aligned}
&\int (\ln y)^2 \, \mathrm{d}y \\
&= y(\ln y)^2 - 2 \int \frac{y \ln y}{y} \, \mathrm{d}y \\
&= y(\ln y)^2 - 2 \int \ln y \, \mathrm{d}y \\
&= y(\ln y)^2 - 2y(\ln y - 1) + C \\
&= y\left((\ln y)^2 - 2\ln y + 2\right) + C
\end{aligned}$$

$$\begin{aligned}
&\text{volume} \\
&= \pi \left(\int_0^{\frac{1}{2}} (\ln 2 - 0)^2 \, \mathrm{d}y + \int_{\frac{1}{2}}^1 \left((\ln 2 - 0)^2 - (\ln 2 + \ln y)^2\right) \mathrm{d}y \right) \\
&= \pi \left(\int_0^{\frac{1}{2}} (\ln 2)^2 \, \mathrm{d}y + \int_{\frac{1}{2}}^1 \left((\ln 2)^2 - (\ln 2)^2 - 2\ln 2 \ln y - (\ln y)^2\right) \mathrm{d}y \right) \\
&= \pi \left(\frac{1}{2}(\ln 2)^2 - 2\ln 2 [y(\ln y - 1)]_{\frac{1}{2}}^1 - \left[y\left((\ln y)^2 - 2\ln y + 2\right)\right]_{\frac{1}{2}}^1 \right) \\
&= \pi \left(0.5(\ln 2)^2 - 2\ln 2(1(0-1) - 0.5(-\ln 2 - 1)) - \left(1(0 - 0 + 2) - 0.5\left((\ln 2)^2 + 2\ln 2 + 2\right)\right)\right) \\
&= \pi \left(0.5(\ln 2)^2 + 2\ln 2 - \ln 2(\ln 2 + 1) - 2 + 0.5(\ln 2)^2 + \ln 2 + 1\right) \\
&= \pi \left((\ln 2)^2 + 3\ln 2 - 1 - (\ln 2)^2 - \ln 2\right) \\
&= \pi (2\ln 2 - 1)
\end{aligned}$$

Q14

Let $f : [1, +\infty) \to [0, +\infty)$ be the function

$$f(x) = \frac{1}{x}$$

and consider its graph in the plane.

Q14.a

Consider the (unbounded) region under the graph of f and above the x-axis. Show that the solid obtained by revolving this region about the x-axis has a finite volume.

$$\begin{aligned}
&\text{volume} \\
&= \pi \int_1^{+\infty} \left(\frac{1}{x}\right)^2 \mathrm{d}x \\
&= \pi \int_1^{+\infty} \frac{1}{x^2} \, \mathrm{d}x
\end{aligned}$$

By the p-test, the integral is convergent.
So the volume is finite.

Q14.b

Show that the surface obtained revolving the graph of f about the x-axis has an infinite surface area.

$$
\begin{aligned}
&\text{surface area}\\
&= 2\pi \int_{1}^{+\infty} \frac{1}{x} \sqrt{1 + \left(\left(\frac{1}{x}\right)'\right)^2} \, \mathrm{d}x\\
&= 2\pi \int_{1}^{+\infty} \frac{1}{x} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2} \, \mathrm{d}x\\
&= 2\pi \int_{1}^{+\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, \mathrm{d}x\\
&> 2\pi \int_{1}^{+\infty} \frac{1}{x} \, \mathrm{d}x && \left(x \in [1, +\infty) \implies \sqrt{1 + \frac{1}{x^4}} > 1\right)
\end{aligned}
$$

By the p-test, the integral one line above this line is divergent.
By the comparison test, the original integral is divergent.
So the surface area is infinite.

Q15

Let $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^2$ be the curve defined by

$$\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$$

.

Q15.a

Find the arc-length of this curve.

$$
\begin{aligned}
&\mathbf{r}'(t)\\
&= \left\langle -3\cos^2 t \sin t, 3\sin^2 t \cos t \right\rangle\\
&\text{arc-length}\\
&= \int_0^{2\pi} \|\mathbf{r}'(t)\| \, \mathrm{d}t\\
&= \int_0^{2\pi} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} \, \mathrm{d}t\\
&= 3 \int_0^{2\pi} |\sin t \cos t| \sqrt{\cos^2 + \sin^2 t} \, \mathrm{d}t\\
&= 3 \int_0^{2\pi} |\sin t \cos t| \, \mathrm{d}t\\
&= \frac{3}{2} \int_0^{2\pi} |\sin 2t| \, \mathrm{d}t\\
&= \frac{3}{4} \int_0^{4\pi} |\sin t| \, \mathrm{d}t && (\text{change of variable } 2t \mapsto t)\\
&= 3 \int_0^{\pi} \sin t \, \mathrm{d}t && (\text{symmetry}, (\forall t \in [0, \pi])(\sin t \geq 0))\\
&= 3[-\cos t]_0^{\pi}\\
&= 6
\end{aligned}
$$

Q15.b

Find the area of the region in $\mathbb{R}^2$ bounded by this curve.

The curve is a smooth curve.

$$\begin{aligned}
&\text{area}\\
&= \int_0^{2\pi} \cos^3 t \left(\sin^3 t\right)' \,\mathrm{d}t\\
&= 3\int_0^{2\pi} \cos^4 t \sin^2 t \,\mathrm{d}t\\
&= \frac{3}{8}\int_0^{2\pi} (\cos 2t + 1)\sin^2 2t \,\mathrm{d}t\\
&= \frac{3}{8}\int_0^{2\pi} (\cos 2t + 1)\sin^2 2t \,\mathrm{d}t\\
&= \frac{3}{16}\int_0^{4\pi} (\cos t + 1)\sin^2 t \,\mathrm{d}t && \text{(change of variable } 2t \mapsto t\text{)}\\
&= \frac{3}{16}\left[\frac{1}{3}\sin^3 t\right]_0^{4\pi} + \frac{3}{16}\int_0^{4\pi} \sin^2 t \,\mathrm{d}t\\
&= \frac{3}{32}\int_0^{4\pi} (1 - \cos 2t) \,\mathrm{d}t\\
&= \frac{3}{32}\left[\theta - \frac{1}{2}\sin 2t\right]_0^{4\pi}\\
&= \frac{3\pi}{8} - \frac{3}{64}(\sin 8\pi - \sin 0)\\
&= \frac{3\pi}{8}
\end{aligned}$$

Q15.c

Find the volume of the solid obtained by revolving this curve about the x-axis.

The curve is a 4-folded symmetric star.
By symmetry, we only need to find the volume by considering one quadrant of the star.
Considering quadrant I,

$$\begin{aligned}
x &= \cos^3 t\\
y &= \sin^3 t\\
&= \left(\sin^2 t\right)^{\frac{3}{2}} && (y \geq 0 \implies \sin t \geq 0)\\
&= \left(1 - \cos^2 t\right)^{\frac{3}{2}}\\
&= \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} && (x \geq 0 \implies \cos t \geq 0)
\end{aligned}$$

$$\begin{aligned}
&\text{volume}\\
&= 2\pi \int_0^1 y^2 \,\mathrm{d}x\\
&= 2\pi \int_0^1 \left(1 - x^{\frac{2}{3}}\right)^3 \mathrm{d}x\\
&= 2\pi \int_0^1 \left(1 - 3x^{\frac{2}{3}} + 3x^{\frac{4}{3}} - x^2\right) \mathrm{d}x\\
&= 2\pi \left[x - \frac{9}{5}x^{\frac{5}{3}} + \frac{9}{7}x^{\frac{7}{3}} - \frac{1}{3}x^3\right]_0^1\\
&= 2\pi \left(1 - \frac{9}{5} + \frac{9}{7} - \frac{1}{3}\right)\\
&= 2\pi \cdot \frac{105 - 189 + 135 - 35}{105}\\
&= 2\pi \cdot \frac{16}{105}\\
&= \frac{32\pi}{105}
\end{aligned}$$

Q15.d

Find the area of the surface obtained by revolving this curve about the x-axis.

The curve is a 4-folded symmetric star.
By symmetry, we only need to the surface area using one quadrant of the star.
Considering quadrant I,

$$
\begin{aligned}
x &= \cos^3 t \\
y &= \sin^3 t \\
&= \left(\sin^2 t\right)^{\frac{3}{2}} && (y \geq 0 \implies \sin t \geq 0) \\
&= \left(1 - \cos^2 t\right)^{\frac{3}{2}} \\
&= \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} && (x \geq 0 \implies \cos t \geq 0) \\
y' &= \frac{3}{2}\left(1 - x^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(-\frac{2}{3}x^{-\frac{1}{3}}\right) \\
&= -\left(1 - x^{\frac{2}{3}}\right)^{\frac{1}{2}} x^{-\frac{1}{3}} \\
&\text{surface area} \\
&= 4\pi \int_0^1 y\sqrt{1 + (y')^2}\,\mathrm{d}x \\
&= 4\pi \int_0^1 \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} \sqrt{1 + \left(1 - x^{\frac{2}{3}}\right)x^{-\frac{2}{3}}}\,\mathrm{d}x \\
&= 4\pi \int_0^1 \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} x^{-\frac{1}{3}}\,\mathrm{d}x \\
&= 6\pi \int_0^1 (1 - x)^{\frac{3}{2}}\,\mathrm{d}x && \left(\text{change of variable } x^{\frac{2}{3}} \to x\right) \\
&= 6\pi \left[-\frac{2}{5}(1 - x)^{\frac{5}{2}}\right]_0^1 \\
&= \frac{12\pi}{5}
\end{aligned}
$$

Q16

Consider the cardioid in $\mathbb{R}^2$ defined by the polar equation

$$r = 1 + \sin\theta$$

.

Hint: The given cardioid is symmetric about the y-axis. To generate a solid or a surface by revolving about the y-axis, we only need the **right-half** of the cardioid.

Q16.a

Find the volume of the solid obtained by revolving this curve about the y-axis.

r is always nonnegative because $\sin\theta \geq -1$.
As the cardioid is symmetric about the y-axis,
only considering the right-half...

$$\begin{aligned}
&\text{volume}\\
&= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \mathrm{d}y\\
&= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 y' \mathrm{d}\theta\\
&= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \cos^2\theta (r' \sin\theta + r\cos\theta)\, \mathrm{d}\theta\\
&= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\sin\theta)^2 \cos^2\theta (2\cos\theta\sin\theta + \cos\theta)\, \mathrm{d}\theta\\
&= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\sin\theta)^2 \left(1-\sin^2\theta\right)(2\sin\theta+1)\cos\theta\, \mathrm{d}\theta\\
&= \pi \int_{-1}^{1} (1+u)^3 (1-u)(2u+1)\, \mathrm{d}u && (\text{substitute } u := \sin\theta)\\
&= \pi \int_{0}^{2} u^3 (2-u)(2u-1)\, \mathrm{d}u && (\text{change of variable } 1+u \mapsto u)\\
&= \pi \int_{0}^{2} u^3 \left(4u - 2 - 2u^2 + u\right) \mathrm{d}u\\
&= \pi \int_{0}^{2} u^3 \left(-2u^2 + 5u - 2\right) \mathrm{d}u\\
&= \pi \int_{0}^{2} \left(-2u^5 + 5u^4 - 2u^3\right) \mathrm{d}u\\
&= \pi \left[-\frac{1}{3}u^6 + u^5 - \frac{1}{2}u^4\right]_0^2\\
&= \pi \left(-\frac{1}{3}(64) + 32 - \frac{1}{2}(16)\right)\\
&= \pi \left(-\frac{64}{3} + 32 - 8\right)\\
&= \frac{8\pi}{3}
\end{aligned}$$

Q16.b

Find the area of the surface obtained by revolving this curve about the y-axis.

r is always nonnegative because $\sin\theta \geq -1$.
As the cardioid is symmetric about the y-axis,
only considering the right-half...

$$\begin{aligned}
&\text{surface area} \\
&= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sqrt{r^2 + r'^2}\,\mathrm{d}\theta \\
&= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - y^2}\sqrt{r^2 + r'^2}\,\mathrm{d}\theta && (x \geq 0) \\
&= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2\theta}\sqrt{r^2 + r'^2}\,\mathrm{d}\theta \\
&= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |r\cos\theta|\sqrt{r^2 + r'^2}\,\mathrm{d}\theta \\
&= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin\theta)\cos\theta\sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\,\mathrm{d}\theta && \left(r \geq 0, \left(\forall\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)(\cos\theta \geq 0)\right) \\
&= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin\theta)\cos\theta\sqrt{2 + 2\sin\theta}\,\mathrm{d}\theta \\
&= 2\pi\sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin\theta)\cos\theta\sqrt{1 + \sin\theta}\,\mathrm{d}\theta \\
&= 2\pi\sqrt{2} \int_0^2 u^{\frac{3}{2}}\,\mathrm{d}u && (\text{substitute } u := \sin\theta + 1) \\
&= 2\pi\sqrt{2}\left[\frac{2}{5}u^{\frac{5}{2}}\right]_0^2 \\
&= 2\pi\sqrt{2} \cdot \frac{2}{5} \cdot 4\sqrt{2} \\
&= \frac{32\pi}{5}
\end{aligned}$$

Q17

Determine whether each of the following series of real numbers converges or diverges. Also compute its limit (i.e. the sum) if it converges.

Q17.b

$$\sum_{k=1}^{+\infty} \frac{2}{k(k+1)(k+2)}$$

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{2}{k(k+1)(k+2)} && (n \in \mathbb{Z}_{\geq 1}) \\
&= \sum_{k=1}^{n} \left(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right) \\
&= \frac{1}{1 \cdot 2} - \frac{1}{(n+1)(n+2)} && (\text{telescope}) \\
&= \frac{1}{2} - \frac{1}{(n+1)(n+2)} \\
&\sum_{k=1}^{+\infty} \frac{2}{k(k+1)(k+2)} \\
&= \lim_{n \to +\infty} \sum_{k=1}^{n} \frac{2}{k(k+1)(k+2)} \\
&= \lim_{n \to +\infty} \left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right) \\
&= \frac{1}{2} - 0 \\
&= \frac{1}{2}
\end{aligned}$$

$\therefore$ The limit converges.

Q17.c

$$\sum_{k=1}^{+\infty} \ln\left(1 + \frac{1}{k}\right)$$

$$
\begin{aligned}
&\sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right) \qquad\qquad (n \in \mathbb{Z}_{\geq 1})\\
&= \ln\left(\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)\right)\\
&= \ln\left(\prod_{k=1}^{n}\frac{1}{k}(k+1)\right)\\
&= \ln\left(\frac{(n+1)!}{n!}\right)\\
&= \ln(n+1)\\
&\sum_{k=1}^{+\infty} \ln\left(1+\frac{1}{k}\right)\\
&= \lim_{n\to+\infty}\sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right)\\
&= \lim_{n\to+\infty}\ln(n+1)\\
&= +\infty\\
&\therefore \text{The limit diverges.}
\end{aligned}
$$

Q17.e

$$\sum_{k=2}^{+\infty}\frac{k}{2^{k-1}}$$

Hint: $\frac{\mathrm{d}}{\mathrm{d}x}x^k = kx^{k-1}$.

$$f_k(x) := x^k$$
$$f_k'(x) = kx^{k-1}$$

$$\begin{aligned}
&\sum_{k=2}^{n} \frac{k}{2^{k-1}} && (n \in \mathbb{Z}_{\geq 2}) \\
&= \sum_{k=2}^{n} k \left(\frac{1}{2}\right)^{k-1} \\
&= \sum_{k=2}^{n} f_k' \left(\frac{1}{2}\right) \\
&= \left(\sum_{k=2}^{n} f_k(x)\right)' \Bigg|_{x=\frac{1}{2}} && \text{(linearity)} \\
&= \left(\sum_{k=2}^{n} x^k\right)' \Bigg|_{x=\frac{1}{2}} \\
&= \left(\frac{x^{n+1} - x^2}{x-1}\right)' \Bigg|_{x=\frac{1}{2}} \\
&= \left(x^2 \left(\frac{x^{n-1} - 1}{x-1}\right)\right)' \Bigg|_{x=\frac{1}{2}} \\
&= \left((2x)\frac{x^{n-1} - 1}{x-1} + x^2 \left(\frac{(n-1)x^{n-2}(x-1) - \left(x^{n-1} - 1\right)}{(x-1)^2}\right)\right) \Bigg|_{x=\frac{1}{2}} \\
&= 2(0.5)\frac{0.5^{n-1} - 1}{0.5 - 1} + (0.5)^2 \left(\frac{(n-1)(0.5)^{n-2}(0.5-1) - \left(0.5^{n-1} - 1\right)}{(0.5-1)^2}\right) \\
&= 2\left(1 - 0.5^{n-1}\right) + \left(-(n-1)(0.5)^{n-1} + \left(1 - 0.5^{n-1}\right)\right) \\
&= 3\left(1 - 0.5^{n-1}\right) - (n-1)(0.5)^{n-1} \\
&= 3 - 3(0.5)^{n-1} - n(0.5)^{n-1} + 0.5^{n-1} \\
&= 3 - 2^{1-n}(n+2)
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=2}^{+\infty} \frac{k}{2^{k-1}} \\
&= \lim_{n \to +\infty} \sum_{k=2}^{n} \frac{k}{2^{k-1}} \\
&= \lim_{n \to +\infty} \left(3 - 2^{1-n}(n+2)\right) \\
&= 3 - 0 && \text{(exponential is faster growing than linear)} \\
&= 3
\end{aligned}$$

$\therefore$ The limit converges.