1.

1.

(a) $F\{e^{-a|t|}\} = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt$

$= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$

$= \left[\frac{1}{a-j\omega} e^{(a-j\omega)t}\right]_{-\infty}^{0} + \left[-\frac{1}{a+j\omega} e^{-(a+j\omega)t}\right]_{0}^{\infty}$

$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$

Sub $a=2$ and timeshift to the right $t_0 = 1$

$X_1(j\omega) = \frac{4}{4+\omega^2} e^{-j\omega}$

(b)

$F\left\{\frac{\sin(\pi t)}{\pi t}\right\} = F\{\text{sinc}(\pi t)\} = \begin{cases} 1 & |\omega| < \pi \\ 0 & |\omega| > \pi \end{cases}$

$F\{\cos(4\pi t)\} = \pi[\delta(\omega - 4\pi) + \delta(\omega + 4\pi)]$

$x_2(t) = \text{sinc}(\pi t)\cos(4\pi t)$

$\Rightarrow X_2(j\omega) = \frac{1}{2\pi}\left(\begin{cases} 1 & |\omega| < \pi \\ 0 & |\omega| > \pi \end{cases}\right) * \left(\pi[\delta(\omega - 4\pi) + \delta(\omega + 4\pi)]\right)$

$= \begin{cases} \frac{1}{2} & \omega \in [-5\pi, -3\pi] \cup [3\pi, 5\pi] \\ 0 & \text{otherwise} \end{cases}$

2. (a) $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + x(t)$

$\Rightarrow (j\omega)^2 Y(j\omega) + 4j\omega Y(j\omega) + 4Y(j\omega) = j\omega X(j\omega) + X(j\omega)$

~~(omit jw for brevity)~~ $\cancel{\Rightarrow (j\omega)^2 Y + 4j\omega Y + Y = j\omega X + X}$

(continue below)

(continue from above)

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 1}{(j\omega)^2 + 4j\omega + 4}$$

(b)

$$H(j\omega) = \frac{j\omega + 1}{(j\omega + 2)^2}$$

$$= \frac{1}{j\omega + 2} - \frac{1}{(j\omega + 2)^2}$$

$$\Rightarrow h(t) = e^{-2t}u(t) - te^{-2t}u(t)$$

(c)

$$x(t) = e^{-2t}u(t)$$

$$\Rightarrow X(j\omega) = \frac{1}{j\omega + 2}$$

$$x(t) * h(t)$$

$$\Rightarrow X(j\omega)H(j\omega)$$

$$= \frac{j\omega + 1}{(j\omega + 2)^3}$$

$$= \frac{1}{(j\omega + 2)^2} - \frac{1}{(j\omega + 2)^3}$$

$$\Rightarrow y(t) = te^{-2t}u(t) - \frac{t^2}{2}e^{-2t}u(t)$$

2.2 (a) ~~For it to~~ For $f_{X,Y}$ to be pdf

$$\int_0^1 dx \left(\int_x^\infty dy\, K x e^{-y} \right) = 1$$

$$K \int_0^1 dx \left(x \int_x^\infty dy\, e^{-y} \right) = 1$$

$$K \int_0^1 dx \left(x \left[-e^{-y} \right]_x^\infty \right) = 1$$

$$K \int_0^1 dx\, x e^{-x} = 1$$

$$K \left(\left[-x e^{-x} \right]_0^1 + \int_0^1 dx\, e^{-x} \right) = 1$$

$$K \left(-e^{-1} + \left[-e^{-x} \right]_0^1 \right) = 1$$

$$K \left(-e^{-1} - e^{-1} + 1 \right) = 1$$

$$K = \cancel{\frac{1}{1+e^{-2}}}\ \cancel{\frac{1}{1+2e^{-1}}}\ \frac{1}{1-2e^{-1}}$$

$$K \approx 3.7844$$

(b)

$$f_X(x) = \int_x^\infty dy \left(K x e^{-y} \right) \qquad x \in [0, 1]$$

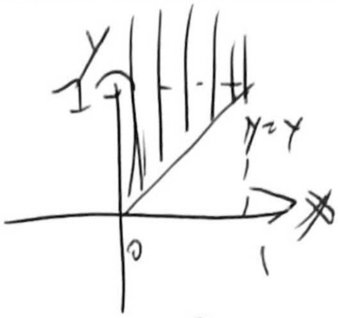
$$= Kx \left[-e^{-y} \right]_x^\infty$$

$$= K x e^{-x} \qquad \cancel{x \in [0, 1)}$$

$$\cancel{f_Y(y) = \int} = \begin{cases} \dfrac{x e^{-x}}{1-2e^{-1}} & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

(continue)

$\cancel{f_Y(y)} =$



$$f_Y(y) = \begin{cases} \int_0^{y} dx \; Kxe^{-y} & 0 \le y < 1 \\ \int_0^{1} dx \; Kxe^{-y} & y \ge 1 \end{cases}$$

$\cancel{\int dx \; Kxe^{-y}}$

$=$

$\int dx \; Kxe^{-y}$

$= Ke^{-y} \int dx \; x$

$= \frac{Ke^{-y}}{2} x^2 + C$

$$= \begin{cases} \frac{Ke^{-y}}{2} [x^2]_0^y & y \in [0, 1) \\ \frac{Ke^{-y}}{2} [x^2]_0^1 & y \ge 1 \end{cases}$$

$$= \begin{cases} \frac{e^{-y} y^2}{2-4e^{-1}} & y \in [0, 1) \\ \frac{e^{-y}}{2-4e^{-1}} & y \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

~~2. 2. (~~

2. 2. (c) $P(X > 0.5 \mid Y = 2)$

$$= \cancel{\int_{0.5}^{1} f_X(x)\, d}$$

$$= \cancel{\int}$$

$$= \frac{P(X > 0.5 \text{ and } Y = 2)}{P(Y = 2)}$$

$P(X > 0.5 \text{ and } Y = 2)$

$$= \int_{0.5}^{1} K x \cdot e^{-2}\, dx$$

$$= \frac{e^{-2}}{2-4e^{-1}} \left[x^2 \right]_{0.5}^{1}$$

$$= \cancel{\frac{0.75 e^{-2}}{2-2}} \frac{3e^{-2}}{8-16e^{-1}}$$

$$= \frac{\left(\frac{3e^{-2}}{8-16e^{-1}} \right)}{\left(\frac{e^{-2}}{2-4e^{-1}} \right)} = \frac{3}{4}$$

$$\cancel{= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) = \frac{3}{16}}$$

$P(Y=2) = f_Y(2)$

$$= \frac{e^{-2}}{2-4e^{-1}}$$

2. 3. (a) $m_Z(t) = E[Z(t)]$

$$= E[X \cos(\omega_0 t) + Y \sin(\omega_0 t)]$$

$$= E[X] \cos(\omega_0 t) + E[Y] \sin(\omega_0 t)$$

$$= 0 + 0$$

$$= 0$$

(b) $R_Z(t_1, t_2) = E[Z(t_1) Z(t_2)]$

$$\cancel{= E[X(\cos(\omega_0 t_1) + \cos(\omega_0 t_2)) + Y(\sin(\omega}$$

(continue)

(continue from above)

$= E\left[\left(X\cos(w_0 t_1)+Y\sin(w_0 t_1)\right)\left(X\cos(w_0 t_2)+Y\sin(w_0 t_2)\right)\right]$

$\cancel{(X_2 \cos}$

$= E[X^2\cos(w_0 t_1)\cos(w_0 t_2)$

$+ XY\cos(w_0 t_1)\sin(w_0 t_2)$

$+ YX\sin(w_0 t_1)\cos(w_0 t_2)$

$+ Y^2\sin(w_0 t_1)\sin(w_0 t_2)]$

$= E[X^2]\cos(w_0 t_1)\cos(w_0 t_2)$

$+ E[XY]\cos(w_0 t_1)\sin(w_0 t_2)$

$+ E[YX]\sin(w_0 t_1)\cos(w_0 t_2)$

$+ E[Y^2]\sin(w_0 t_1)\sin(w_0 t_2)$

$= \sigma^2\left(\cos(w_0 t_1)\cos(w_0 t_2)+\sin(w_0 t_1)\sin(w_0 t_2)\right)$

$= \sigma^2\cos(w_0(t_1 - t_2))$

$E[X]=0$

$E[X^2]$

$= E[(X-E[X])^2]$

$= Var(X)$

$= \sigma^2$

~~Same for $E[Y^2]$~~

Same for $E[Y^2]$

$E[XY]$

$= E[X]E[Y]$ independent

$= 0$

(c) ~~$z(t)$~~ has

$\therefore$ ~~Mean~~ $m_Z(t)$ ~~$=0$~~ is the same for all t

and $R_Z(t_1, t_2)$ only depends on $t_1 - t_2$ (difference in time)

$\therefore$ Yes, it is WSS

3.

4. (a) ~~Assume receiver & integrates from~~ received signal during $[0, T]$

Assume receiver is the one talked about in lessons.

Because noise is additive, and symmetric, optimal threshold is $\frac{-A+A}{2} = 0$

Then, before determining the bit, obtain

$$\text{value} = \int_0^T A + n(t)\, dt$$

$$= AT + \int_0^T n(t)\, dt$$

$\int_0^T n(t)\, dt$

$$\text{mean} = E\left[\int_0^T n(t)\,dt\right]$$

$$= \int_0^T E[n(t)]\, dt$$

$$= \int_0^T 0\, dt$$

$$= 0$$

(mean is 0)

~~variance~~ $= E\left[\left(\int_0^T n(t)\,dt\right)^2\right]$

$$\text{variance} = E\left[\left(\int_0^T n(t)\,dt - 0\right)^2\right]$$

$$= E\left[\int_0^T\int_0^T n(t_0)n(t_1)\,dt_0\,dt_1\right]$$

$$= \int_0^T\int_0^T E[n(t_0)n(t_1)]\,dt_0\,dt_1$$

$$= \frac{N_0}{2}\int_0^T\int_0^T \delta(t_0 - t_1)\,dt_0\,dt_1$$

$$= \frac{N_0}{2}\int_0^T dt_0$$

$$= \frac{N_0 T}{2}$$

$$= AT + N$$

where $N \sim \text{Normal}\left(0, \frac{N_0 T}{2}\right)$

When bit 1 is sent

error occurs when

$$-AT + N > 0$$

$$N > AT$$

$$P_{e0} = Q\left(\frac{AT - 0}{\sqrt{N_0 T/2}}\right)$$

$$= Q\left(\sqrt{\frac{2A^2 T}{N_0}}\right)$$

$P_{e1} = P_{e0}$ (noise is additive and symmetric)

$$P_e = \frac{1}{2}P_{e1} + \frac{1}{2}P_{e0} = P_{e0} = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

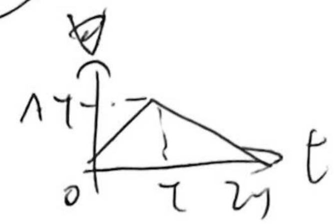
(both bits equally likely)

4. (b) $g(t) = s_1(t) - s_0(t) = \begin{cases} -2A & t\in[0,T] \\ 0 & \text{otherwise} \end{cases}$

$h(t) = g(T-t) = \begin{cases} -2A & t\in[0,T] \\ 0 & \text{otherwise} \end{cases}$

(c)

Because noise is additive and symmetric, optimal decision threshold is $\frac{-A+A}{2} = 0$ the middle, i.e.

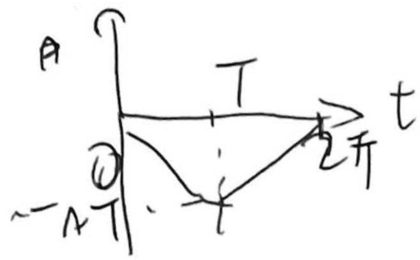


For bit 0, $s_0(t) * h(t) = \begin{cases} AT & t\in[0,T) \\ A(2T-t) & t\in[T,2T) \\ 0 & \text{otherwise} \end{cases}$

For bit 1, $s_1(t) * h(t) = \begin{cases} -AT & t\in[0,T) \\ -A(2T-t) & t\in[T,2T) \\ 0 & \text{otherwise} \end{cases}$

They have max difference when $t=T$,

so optimal sampling time should be T.



~~寿~~.

3.5. (a) $\vec{\varphi}_1 = \frac{\vec{s}_1}{\|\vec{s}_1\|} = \frac{[1\ 1\ 1]}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}[1, 1, 1]$

$\vec{v}_2 = \vec{s}_2 - \langle \vec{s}_2, \vec{\varphi}_2 \rangle \vec{\varphi}_2$

$= [1, -1, -1] - \frac{(-1)}{3}[1, 1, 1]$

$= [1, -1, -1] + \frac{1}{3}[1, 1, 1]$

$= \cancel{\left[\frac{4}{3}, \frac{2}{3}, \frac{2}{3}\right]}\ \frac{1}{3}[4, -2, -2]$

$\vec{\varphi}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{(\frac{1}{3})\sqrt{24}}\left(\frac{1}{3}\right)[4, -2, -2]$

$= \frac{1}{\sqrt{6}}[2, -1, -1]$

$\vec{v}_3 = \vec{s}_3 - \langle \vec{s}_3, \vec{\varphi}_1 \rangle \vec{\varphi}_1 - \langle \vec{s}_3, \vec{\varphi}_2 \rangle \vec{\varphi}_2$

$= [1\ 1\ -1] - \frac{1}{3}[1, 1, 1] - \frac{2}{6}[2, -1, -1]$

$= [1\ 1\ -1] - \frac{1}{3}[1\ 1\ 1] - \frac{1}{3}[2\ -1\ -1]$

$= [0, 1, -1]$

$\vec{\varphi}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$

$= \frac{1}{\sqrt{2}}[0\ 1\ -1]$

$= \frac{1}{\sqrt{3}}[1, 1, 1] + \frac{2}{\sqrt{6}}[2, -1, -1]$

$- \sqrt{2}[0, 1, -1]$

(b) $x(t) = \langle \vec{x}, \vec{\varphi}_1 \rangle \vec{\varphi}_1 + \langle \vec{x}, \vec{\varphi}_2 \rangle \vec{\varphi}_2 + \langle \vec{x}, \vec{\varphi}_3 \rangle \vec{\varphi}_3$

$= \frac{1}{\sqrt{3}}[1, 1, 1] + \frac{2}{\sqrt{6}}[2, -1, -1] - \frac{2}{\sqrt{2}}[0, 1, -2]$