# Problem Set 8

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 8 covers materials from chapter 9.

1. Let $f(x) = e^x$ and let $n$ be a non-negative integer.
   (a) Show **from definition** that the polynomial
   $$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$
   is the $n^{\text{th}}$ order approximation of $f$ at $0$.
   (b) Using the result from (a) (but without using Taylor's Theorem), find the $n^{\text{th}}$ order approximation of $f$ at $1$.
2. Find the $6^{\text{th}}$ order approximation of each of the following functions at $0$.
   (a) $f(x) = e^x \cos x$
   (b) $g(x) = e^{\cos x}$
   (c) $h(x) = \sec x$
3. Evaluate each of the following limits using polynomial approximations.
   (a) $\displaystyle\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x^3}$
   (b) $\displaystyle\lim_{x\to 0} \frac{\sin^2 x - \sin(x^2) + \frac{1}{3}x^4}{x^6}$
   (c) $\displaystyle\lim_{x\to +\infty} x^2 \left(e - \frac{e}{2x} - \left(1 + \frac{1}{x}\right)^x\right)$
4. Let $a$ be real number and let $x > a$, let $n$ be a non-negative integer and let $f$ be a function such that $f^{(n+1)}$ is continuous on $[a, x]$.
   (a) Prove the "**integral remainder formula**"
   $$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt.$$
   *Hint*: Integration by parts for $n$ times.
   (b) Using (a), give another proof of Lagrange's remainder formula; i.e. show that there exists a number $c \in (a, x)$ such that
   $$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$
   *Hint*: Generalized Mean Value Theorem for integrals (Example 5.47 (a)).

5. In Examples 9.31 and 9.32, we have seen that in some cases, Lagrange's remainder formula is not strong enough to show that $\lim_{n\to+\infty} R_n(x) = 0$. Let's develop another remainder formula.

(a) Let $a$ be real number and let $x > a$, let $n$ be a non-negative integer and let $f$ be a function such that $f^{(n)}$ is continuous on $[a, x]$ and differentiable on $(a, x)$.

(i) Let $g: [a, x] \to \mathbb{R}$ be the function

$$g(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k.$$

Compute $g'(t)$ for $t \in (a, x)$.

(ii) **(Cauchy's remainder formula)** By applying Mean Value Theorem to the function $g$, show that there exists a number $c \in (a, x)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a).$$

*Remark*: If we assume further that $f^{(n+1)}$ is integrable on $[a, x]$, then another way of obtaining (a) (ii) is to use Q4(a) and then the MVT for integrals.

(b) Using the result from (a) (ii) (which obviously still holds if $x < a$), show that for each of the following functions, the remainder term at $0$ satisfies

$$\lim_{n\to+\infty} R_n(x) = 0 \qquad \text{for each fixed } x \in (-1, 1).$$

(i) **(Example 9.31)** $f(x) = \ln(1 + x)$

(ii) **(Example 9.32)** $f(x) = (1 + x)^p$, where $p$ is a real number.

*Hint*: In (b) (ii), it is useful to note that if $\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\lim_{n\to+\infty} a_n = 0$.

6. Let $p$ be a real number which is not a non-negative integer. Consider the power series

$$\sum_{k=0}^{+\infty} \frac{p(p-1)\cdots(p-k+1)}{k!} x^k,$$

which is the Maclaurin series of the function $f(x) = (1 + x)^p$. We investigate the behavior of this power series at the end-points $1$ and $-1$ of the interval of convergence.

(a) At $1$, show that the series converges if $p > -1$ and diverges if $p \le -1$.

(b) At $-1$, show that the series converges if $p > 0$ and diverges if $p < 0$.

7. For each of the following, compute its Maclaurin series and find its radius of convergence.

(a) $f(x) = \sin^2 x$

(b) $f(x) = \int_0^x \frac{\sin t}{t}\, dt$

(c) $f(x) = \arcsin x$

(d) $f(x) = \ln\left(x + \sqrt{1 + x^2}\right)$

*Hint*: In (d), first consider $f'$.

8. For each of the following power series, evaluate its sum whenever it converges. What happens at the end-points of its interval of convergence?

   (a) $\displaystyle\sum_{k=0}^{+\infty} \frac{(-1)^k}{4k^2-1} x^{2k+1}$

   *Hint*: Differentiate the series.

   (b) $\displaystyle\sum_{k=1}^{+\infty} a_k x^k$, where $a_k = \begin{cases} -\dfrac{2}{n} & \text{if } n = 3k \text{ for some } k \in \mathbb{N} \\ \dfrac{1}{n} & \text{if } n \neq 3k \text{ for any } k \in \mathbb{N} \end{cases}$

   *Hint*: Rewrite it as the difference of two power series whose sums are known.

9. Let $f(x) = x^3 e^x$. Using the Taylor series of $f$, compute
   (a) $f^{(n)}(0)$ and
   (b) $f^{(n)}(1)$

   for every positive integer $n$. (Do not try to really differentiate for $n$ times!)

10. Let

$$f(x) = \frac{x}{1-x-x^2}$$

and suppose that the Maclaurin series of $f$ is

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

whose radius of convergence is positive.

   (a) By considering the function $(1-x-x^2)f(x)$, show that $(a_n)$ is the ***Fibonacci sequence*** (see Example 2.4 for its definition).

   (b) Write the partial fraction decomposition of $f$ as

$$f(x) = \frac{x}{1-x-x^2} = \frac{A}{x-p} + \frac{B}{x-q}.$$

   What are the numbers $p$ and $q$? By considering this partial fraction decomposition of $f$, express $a_n$ in terms of $p$, $q$ and $n$.

   *Hint*: The answer is already given in Example 2.4.

11. Let $a_0, a_1, \dots, a_n, b_1, \dots, b_n$ be real numbers and let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

be a trigonometric polynomial. (Note that $f$ is a finite sum.) Show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = \frac{{a_0}^2}{2} + \sum_{k=1}^{n} \left({a_k}^2 + {b_k}^2\right).$$

12. Let $f(x) = |x|$ be defined on $[-\pi, \pi]$ and extended periodically on $\mathbb{R}$ to become a function with period $2\pi$.
    (a) Compute the Fourier series of $f$.
    (b) (i) By setting $x = 0$ in (a), show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

    (ii) Using (b) (i), give another proof of the equality

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

13. Let $a$ be a real number which is not an integer. Let $f(x) = \cos ax$ be defined on $[-\pi, \pi]$ and extended periodically to become a function with period $2\pi$.
    (a) Compute the Fourier series of $f$.
    (b) Using (a), prove that

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k^2 - a^2} = -\frac{1}{2a^2} + \frac{\pi}{2a\sin(a\pi)},$$

    and in a similar way also compute

$$\sum_{k=1}^{+\infty} \frac{1}{k^2 - a^2}.$$

14. Consider the function $f(x) = e^x$ defined on $(0, 2\pi)$ and extended periodically to become a function with period $2\pi$. Let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series of $f$.
    (a) Compute the Fourier series of $f$.
    (b) (i) By considering the sum of the Fourier series of $f$ at $x = 0$, show that

$$\sum_{k=0}^{+\infty} \frac{1}{1 + k^2} = \frac{\pi}{2} \frac{e^{2\pi} + 1}{e^{2\pi} - 1} + \frac{1}{2}.$$

    (ii) By considering the sum of the Fourier series of $f$ at $x = \pi$, evaluate the sum

$$\sum_{k=0}^{+\infty} \frac{(-1)^k}{1 + k^2}.$$