Solution to Problem Set 6

1. (a) The given curve is the graph of $f:[1,4]\to\mathbb{R}$ defined by $f(x)=x^{\frac{1}{2}}-(1/3)x^{\frac{3}{2}}$. Since

$$f'(x)=\frac{1}{2\sqrt{x}}-\frac{1}{2}\sqrt{x}$$

for every $x\in(1,4)$, the given curve has arc-length

$$l=\int_1^4\sqrt{1+[f'(x)]^2}\,dx=\int_1^4\sqrt{1+\left(\frac{1}{2\sqrt{x}}-\frac{1}{2}\sqrt{x}\right)^2}\,dx=\int_1^4\sqrt{\left(\frac{1}{2\sqrt{x}}+\frac{1}{2}\sqrt{x}\right)^2}\,dx$$

$$=\int_1^4\left(\frac{1}{2\sqrt{x}}+\frac{1}{2}\sqrt{x}\right)dx=\left[x^{\frac{1}{2}}+\frac{1}{3}x^{\frac{3}{2}}\right]_1^4=\frac{10}{3}.$$

(b) Since $\frac{dx}{dy}=y^{-\frac{1}{3}}$ for every $y\in(1,8)$, the given curve has arc-length

$$l=\int_1^8\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_1^8\sqrt{1+\left(y^{-\frac{1}{3}}\right)^2}\,dy=\int_1^8\sqrt{1+y^{-\frac{2}{3}}}\,dy.$$

With the substitution $u=y^{\frac{2}{3}}$, we have $y=u^{\frac{3}{2}}$ and $dy=(3/2)u^{\frac{1}{2}}\,du$. Therefore

$$l=\int_1^4\sqrt{1+u^{-1}}\cdot\frac{3}{2}u^{\frac{1}{2}}\,du=\frac{3}{2}\int_1^4\sqrt{u+1}\,du=\left[(u+1)^{\frac{3}{2}}\right]_1^4=5\sqrt{5}-2\sqrt{2}.$$

(c) The given curve is the graph of $f:\left[1,\sqrt{2}\right]\to\mathbb{R}$ defined by $f(x)=\ln\left(x+\sqrt{x^2-1}\right)$. Since

$$f'(x)=\frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{1}{2\sqrt{x^2-1}}\cdot 2x\right)=\frac{1}{\sqrt{x^2-1}}$$

for every $x\in\left(1,\sqrt{2}\right)$, the given curve has arc-length

$$l=\int_1^{\sqrt{2}}\sqrt{1+[f'(x)]^2}\,dx=\int_1^{\sqrt{2}}\sqrt{1+\left(\frac{1}{\sqrt{x^2-1}}\right)^2}\,dx=\int_1^{\sqrt{2}}\frac{x}{\sqrt{x^2-1}}\,dx=\left[\sqrt{x^2-1}\right]_1^{\sqrt{2}}=1.$$

<u>Alternative solution</u>: The equation $y=\ln\left(x+\sqrt{x^2-1}\right)$ of the curve for $x\in\left[1,\sqrt{2}\right]$ can be rewritten as

$$x=\frac{e^y+e^{-y}}{2}$$

for $y\in\left[0,\ln\left(1+\sqrt{2}\right)\right]$. Since

$$\frac{dx}{dy}=\frac{e^y-e^{-y}}{2}$$

for every $y\in\left(0,\ln\left(1+\sqrt{2}\right)\right)$, the given curve has arc-length

$$l=\int_0^{\ln(1+\sqrt{2})}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_0^{\ln(1+\sqrt{2})}\sqrt{1+\left(\frac{e^y-e^{-y}}{2}\right)^2}\,dy=\int_0^{\ln(1+\sqrt{2})}\sqrt{\left(\frac{e^y+e^{-y}}{2}\right)^2}\,dy$$

$$=\int_0^{\ln(1+\sqrt{2})}\left|\frac{e^y+e^{-y}}{2}\right|dy=\int_0^{\ln(1+\sqrt{2})}\frac{e^y+e^{-y}}{2}\,dy=\left[\frac{e^y-e^{-y}}{2}\right]_0^{\ln(1+\sqrt{2})}=1.$$

(d) The given curve is the graph of $f:[e,e^3]\to\mathbb{R}$ defined by $f(x)=\int_e^x\sqrt{(\ln t)^2-1}\,dt$. By the first version of the Fundamental Theorem of Calculus we have

$$f'(x)=\sqrt{(\ln x)^2-1}$$

for every $x\in(e,e^3)$, so the given curve has arc-length

$$l=\int_e^{e^3}\sqrt{1+[f'(x)]^2}\,dx=\int_e^{e^3}\sqrt{1+\left(\sqrt{(\ln x)^2-1}\right)^2}\,dx=\int_e^{e^3}\sqrt{(\ln x)^2}\,dx$$

$$=\int_e^{e^3}|\ln x|dx=\int_e^{e^3}\ln x\,dx=[x\ln x-x]_e^{e^3}=2e^3.$$

2. Let f be the function $f(\theta)=1+\sin\theta$. The required arc-length is given by

$$l=\int_0^{\frac{3\pi}{2}}\sqrt{[f(\theta)]^2+[f'(\theta)]^2}\,d\theta=\int_0^{\frac{3\pi}{2}}\sqrt{(1+\sin\theta)^2+(\cos\theta)^2}\,d\theta$$

$$=\int_0^{\frac{3\pi}{2}}\sqrt{2+2\sin\theta}\,d\theta=\int_0^{\frac{3\pi}{2}}\sqrt{2+2\cos\left(\frac{\pi}{2}-\theta\right)}\,d\theta\,.$$

With a substitution $u=\frac{\pi}{2}-\theta$, we obtain

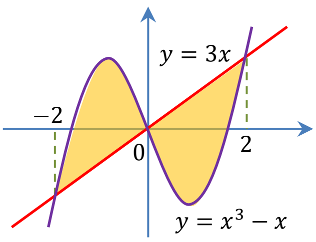
$$l=\int_{-\pi}^{\frac{\pi}{2}}\sqrt{2+2\cos u}\,du=\int_{-\pi}^{\frac{\pi}{2}}\sqrt{4\cos^2\frac{u}{2}}du=\int_{-\pi}^{\frac{\pi}{2}}\left|2\cos\frac{u}{2}\right|du$$

$$=\int_{-\pi}^{\frac{\pi}{2}}2\cos\frac{u}{2}du=\left[4\sin\frac{u}{2}\right]_{-\pi}^{\frac{\pi}{2}}=4+2\sqrt{2}.$$

3. (a) We first find the x-coordinates of the points of intersection of the two graphs by solving

$$\begin{cases}y=x^3-x\\ y=3x\end{cases}.$$

The solutions are $x=-2$, $x=0$ and $x=2$. So the required area is

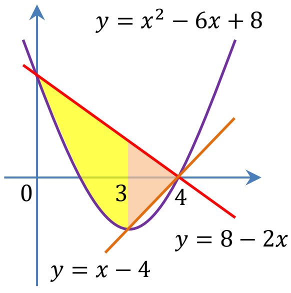


$$A=\int_{-2}^{0}[(x^3-x)-(3x)]dx+\int_0^2[(3x)-(x^3-x)]dx$$

$$=\left[\frac{1}{4}x^4-2x^2\right]_{-2}^{0}+\left[2x^2-\frac{1}{4}x^4\right]_0^2$$

$$=8.$$

(b) The x-coordinates of the points of intersection of the graphs are 0, 3 and 4. The area of the required region is given by



$$A=\int_0^3[(8-2x)-(x^2-6x+8)]dx+\int_3^4[(8-2x)-(x-4)]dx$$

$$=\left[-\frac{1}{3}x^3+2x^2\right]_0^3+\left[-\frac{3}{2}x^2+12x\right]_3^4$$

$$=\frac{21}{2}.$$

4. On solving the inequality $1+\cos\theta \geq 3\cos\theta$, we find that $\theta \in \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$. But we note that $3\cos\theta \geq 0$ only when θ is in Quadrant I or IV, so the required region can be written as

$$\left\{(r,\theta): \theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, \frac{5\pi}{3}\right] \text{ and } 3\cos\theta \leq r \leq 1+\cos\theta\right\} \cup \left\{(r,\theta): \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \text{ and } 0 \leq r \leq 1+\cos\theta\right\}.$$

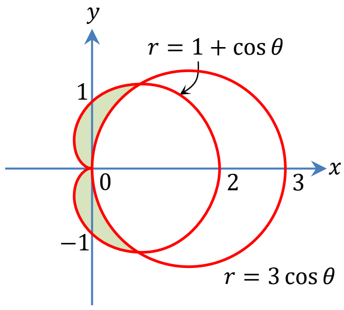
Its area is given by

$$A = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}[(1+\cos\theta)^2 - (3\cos\theta)^2]\,d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2}(1+\cos\theta)^2 d\theta + \int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}} \frac{1}{2}[(1+\cos\theta)^2 - (3\cos\theta)^2]\,d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \frac{1}{2}[(1+\cos\theta)^2 - (3\cos\theta)^2]\,d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2}(3\cos\theta)^2 d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \left(-\frac{3}{2} + \cos\theta - 2\cos 2\theta\right) d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{9}{4} + \frac{9}{4}\cos 2\theta\right) d\theta$$

$$= \left[-\frac{3}{2}\theta + \sin\theta - \sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}} + \left[\frac{9}{4}\theta + \frac{9}{8}\sin 2\theta\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \frac{\pi}{4}.$$



5. The polar curve $r = f(\theta)$ can be parametrized by the vector-valued function $\mathbf{r}: [0,\pi] \to \mathbb{R}^2$ defined by

$$\mathbf{r}(t) = \langle f(t)\cos t, f(t)\sin t\rangle.$$

According to Theorem 7.96, the area between the x-axis and this parametrized curve is

$$A = -\int_{t=0}^{t=\pi} y\,dx = -\int_0^{\pi} (f(t)\sin t)(f'(t)\cos t - f(t)\sin t)dt$$

$$= \int_0^{\pi} f(t)^2 \sin^2 t\,dt - \int_0^{\pi} f(t)f'(t)\sin t\cos t\,dt.$$

Our goal is to show that this area equals to $\int_0^{\pi} \frac{1}{2}[f(t)]^2 dt$ as in Theorem 7.98. Now in the second integral on the right-hand side, we integrate by parts to obtain

$$A = \int_0^{\pi} f(t)^2 \sin^2 t\,dt - \frac{1}{2}\int_0^{\pi} \sin t\cos t\,d[f(t)]^2$$

$$= \int_0^{\pi} f(t)^2 \sin^2 t\,dt - \frac{1}{2}\underbrace{[f(t)^2 \sin t\cos t]_0^{\pi}}_{=0} + \frac{1}{2}\int_0^{\pi} f(t)^2(\cos^2 t - \sin^2 t)dt$$

$$= \int_0^{\pi} f(t)^2 \left(\sin^2 t + \frac{1}{2}(\cos^2 t - \sin^2 t)\right) dt$$

$$= \int_0^{\pi} f(t)^2 \left(\frac{1}{2}(\cos^2 t + \sin^2 t)\right) dt = \int_0^{\pi} \frac{1}{2}[f(t)]^2 dt$$

as desired. Therefore the area evaluated using the two approaches are the same. ■

6. (a) At each $x \in [0,1]$, the cross-section of the solid perpendicular to the x-axis is a disk of diameter $\sqrt{x} - x^2$, whose area is $A(x) = \pi\left(\frac{\sqrt{x}-x^2}{2}\right)^2$. So the given solid has volume

$$V = \int_0^1 \pi\left(\frac{\sqrt{x}-x^2}{2}\right)^2 dx = \pi\int_0^1 \left(\frac{1}{4}x - \frac{1}{2}x^{\frac{5}{2}} + \frac{1}{4}x^4\right) dx = \pi\left[\frac{1}{8}x^2 - \frac{1}{7}x^{\frac{7}{2}} + \frac{1}{20}x^5\right]_0^1 = \frac{9\pi}{280}.$$

(b) The y-coordinates of the points of intersection of the line $y = x$ and the parabola $y = x^2/4$ are 0 and 4, so the given solid is bounded between the planes $y = 0$ and $y = 4$. At each $y \in [0, 4]$, the cross-section of the solid perpendicular to the y-axis is an equilateral triangle having an edge with length $2\sqrt{y} - y$, so its area is $A(y) = (\sqrt{3}/4)(2\sqrt{y} - y)^2$. Therefore the given solid has volume

$$V = \int_0^4 \frac{\sqrt{3}}{4}\left(2\sqrt{y} - y\right)^2 dy = \sqrt{3}\int_0^4 \left(y - y^{\frac{3}{2}} + \frac{1}{4}y^2\right) dy = \sqrt{3}\left[\frac{1}{2}y^2 - \frac{2}{5}y^{\frac{5}{2}} + \frac{1}{12}y^3\right]_0^4 = \frac{8\sqrt{3}}{15}.$$

(c) At each $x \in [0, 6]$, the cross-section of the solid perpendicular to the x-axis is a square having an edge with length $\left(\sqrt{6} - \sqrt{x}\right)^2$, so its area is $A(x) = \left(\sqrt{6} - \sqrt{x}\right)^4$. Therefore the given solid has volume

$$V = \int_0^6 \left(\sqrt{6} - \sqrt{x}\right)^4 dx = \int_0^6 \left(36 - 24\sqrt{6}x^{\frac{1}{2}} + 36x - 4\sqrt{6}x^{\frac{3}{2}} + x^2\right) dx$$

$$= \left[36x - 16\sqrt{6}x^{\frac{3}{2}} + 18x^2 - \frac{8}{5}\sqrt{6}x^{\frac{5}{2}} + \frac{1}{3}x^3\right]_0^6 = \frac{72}{5}.$$

7. The curve $y = \frac{4}{x^3}$ and the lines $x = 1$ and $y = \frac{1}{2}$ intersect at the three points

$$(1, 4), \qquad (1, 1/2) \qquad \text{and} \qquad (2, 1/2).$$

(a) The solid obtained by revolving the region about the x-axis has volume

$$V = \int_1^2 \pi\left[\left(\frac{4}{x^3}\right)^2 - \left(\frac{1}{2}\right)^2\right] dx = \pi\int_1^2 \left(\frac{16}{x^6} - \frac{1}{4}\right) dx = \pi\left[-\frac{16}{5x^5} - \frac{1}{4}x\right]_1^2 = \frac{57\pi}{20}.$$

(b) (Shell method) The solid obtained by revolving the region about the y-axis has volume

$$V = \int_1^2 2\pi x\left(\frac{4}{x^3} - \frac{1}{2}\right) dx = \pi\int_1^2 \left(\frac{8}{x^2} - x\right) dx = \pi\left[-\frac{8}{x} - \frac{1}{2}x^2\right]_1^2 = \frac{5\pi}{2}.$$

(Slice method) The equation of the curve can be written as $x = 4^{\frac{1}{3}}y^{-\frac{1}{3}}$, so the solid obtained by revolving the region about the y-axis has volume

$$V = \int_{\frac{1}{2}}^4 \pi\left[\left(4^{\frac{1}{3}}y^{-\frac{1}{3}}\right)^2 - 1^2\right] dy = \pi\int_{\frac{1}{2}}^4 \left(4^{\frac{2}{3}}y^{-\frac{2}{3}} - 1\right) dy = \pi\left[4^{\frac{2}{3}} \cdot 3y^{\frac{1}{3}} - y\right]_{\frac{1}{2}}^4 = \frac{5\pi}{2}.$$

(c) (Shell method) The solid obtained by revolving the region about the line $x = 2$ has volume

$$V = \int_1^2 2\pi(2 - x)\left(\frac{4}{x^3} - \frac{1}{2}\right) dx = \pi\int_1^2 \left(\frac{16}{x^3} - \frac{8}{x^2} - 2 + x\right) dx = \pi\left[-\frac{8}{x^2} + \frac{8}{x} - 2x + \frac{1}{2}x^2\right]_1^2 = \frac{3\pi}{2}.$$

(Slice method) The equation of the curve can be written as $x = 4^{\frac{1}{3}}y^{-\frac{1}{3}}$, so the solid obtained by revolving the region about the line $x = 2$ has volume

$$V = \int_{\frac{1}{2}}^4 \pi\left[(2 - 1)^2 - \left(2 - 4^{\frac{1}{3}}y^{-\frac{1}{3}}\right)^2\right] dy = \pi\int_{\frac{1}{2}}^4 \left(-3 + 4^{\frac{4}{3}}y^{-\frac{1}{3}} - 4^{\frac{2}{3}}y^{-\frac{2}{3}}\right) dy$$

$$= \pi\left[-3y + 4^{\frac{4}{3}} \cdot \frac{3}{2}y^{\frac{2}{3}} - 4^{\frac{2}{3}} \cdot 3y^{\frac{1}{3}}\right]_{\frac{1}{2}}^4 = \frac{3\pi}{2}.$$

(d) The solid obtained by revolving the region about the line $y = 4$ has volume

$$V = \int_1^2 \pi\left[\left(4 - \frac{1}{2}\right)^2 - \left(4 - \frac{4}{x^3}\right)^2\right]dx = \pi\int_1^2 \left(-\frac{15}{4} + \frac{32}{x^3} - \frac{16}{x^6}\right)dx$$

$$= \pi\left[-\frac{15}{4}x - \frac{16}{x^2} + \frac{16}{5x^5}\right]_1^2 = \frac{103\pi}{20}.$$

8. We divide the solid obtained by revolving the region about the line $x = \ln 2$ into cylindrical shells. Now for each $x \in [0, \ln 2]$, the shell of radius $\ln 2 - x$ has height e^{-x}, so the whole solid has volume

$$V = \int_0^{\ln 2} 2\pi(\ln 2 - x)e^{-x}dx = 2\pi\int_0^{\ln 2} (x - \ln 2)(-e^{-x})dx\,.$$

To handle this integral, we let $f(x) = x - \ln 2$ and $g'(x) = -e^{-x}$ (so that we may take $g(x) = e^{-x}$). Then integrating by parts we get

$$V = 2\pi[(x - \ln 2)e^{-x}]_0^{\ln 2} - 2\pi\int_0^{\ln 2} e^{-x}dx$$

$$= 2\pi[(x - \ln 2)e^{-x} + e^{-x}]_0^{\ln 2} = 2\pi\ln 2 - \pi.$$

Alternative solution: We may also use the usual slice method. The point of intersection of the curve $y = e^{-x}$ and the line $x = \ln 2$ has y-coordinate $\frac{1}{2}$. For each $y \in \left[0, \frac{1}{2}\right]$, the cross-section of the required solid of revolution is a disk of radius $\ln 2$, while for each $y \in \left[\frac{1}{2}, 1\right]$, the cross section is the difference of two concentric disks with outer radius $\ln 2$ and inner radius $\ln 2 + \ln y$ (note that $\ln y$ is negative). So the whole solid has volume

$$V = \int_0^{\frac{1}{2}} \pi(\ln 2)^2 dy + \int_{\frac{1}{2}}^1 \pi((\ln 2)^2 - (\ln 2 + \ln y)^2)dy$$

$$= \int_0^1 \pi(\ln 2)^2 dy - \int_{\frac{1}{2}}^1 \pi(\ln(2y))^2 dy = \pi(\ln 2)^2 - \frac{\pi}{2}\int_1^2 (\ln u)^2 du$$

$$= \pi(\ln 2)^2 - \frac{\pi}{2}[u(\ln u)^2]_1^2 + \frac{\pi}{2}\int_1^2 u \cdot \frac{2\ln u}{u} du$$

$$= \pi(\ln 2)^2 - \frac{\pi}{2}[u(\ln u)^2 - 2u\ln u + 2u]_1^2 = 2\pi\ln 2 - \pi.$$

9. We first compute the remaining volume. Since the curves $y = \sqrt{2^2 - x^2}$ and $y = \sqrt{3}$ intersect at the points $(-1, \sqrt{3})$ and $(1, \sqrt{3})$, the remaining volume is given by

$$V_0 = \int_{-1}^1 \pi\left[(2^2 - x^2) - \left(\sqrt{3}\right)^2\right]dx = \pi\int_{-1}^1 (1 - x^2)dx = \pi\left[x - \frac{1}{3}x^3\right]_{-1}^1 = \frac{4\pi}{3}.$$

Therefore the volume of material removed from the ball is

$$V = (\text{Volume of a ball of radius 2}) - V_0$$

$$= \frac{4}{3}\pi(2)^3 - \frac{4\pi}{3} = \frac{28\pi}{3}.$$

10. The volume of the given solid is $\int_a^t \pi[f(x)]^2 dx$. So now we are given that

$$\int_a^t \pi[f(x)]^2 dx = t^2 - at \qquad \text{for every } t > a.$$

Differentiating both sides with respect to t, we obtain

$$\pi[f(t)]^2 = 2t - a$$

by the first version of the Fundamental Theorem of Calculus. Noting that f is non-negative, we have

$$f(x) = \sqrt{\frac{2x - a}{\pi}}$$

for every $x \in [a, +\infty)$.

11. (a) The ellipse can be parametrized by the vector-valued function $\mathbf{r}: [0, 2\pi] \to \mathbb{R}$ defined by

$$\mathbf{r}(t) = \langle a\cos t, b\sin t\rangle,$$

whose derivative is $\mathbf{r}'(t) = \langle -a\sin t, b\cos t\rangle$. So the arc-length of the ellipse is

$$l = \int_0^{2\pi} \|\mathbf{r}'(t)\| dt = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (b\cos t)^2}\, dt = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt.$$

(b) (i) The upper-half of the ellipse can be regarded as the graph of the function $f: [-a, a] \to [0, +\infty)$ given by

$$f(x) = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2 - x^2}.$$

The derivative of this function is

$$f'(x) = \frac{b}{a} \cdot \frac{1}{2\sqrt{a^2 - x^2}} \cdot (-2x) = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}$$

for every $x \in (-a, a)$. The required surface is obtained by revolving the graph of f about the x-axis, so its surface area is

$$S = \int_{-a}^{a} 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx = \int_{-a}^{a} 2\pi \frac{b}{a}\sqrt{a^2 - x^2}\sqrt{1 + \left(-\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}\right)^2}\, dx$$

$$= \frac{2\pi b}{a^2}\int_{-a}^{a} \sqrt{a^2(a^2 - x^2) + b^2x^2}\, dx.$$

(ii) The right-half of the ellipse has equation

$$x = \frac{a}{b}\sqrt{b^2 - y^2} \qquad \text{for } y \in [-b, b].$$

Now

$$\frac{dx}{dy} = \frac{a}{b} \cdot \frac{1}{2\sqrt{b^2 - y^2}} \cdot (-2y) = -\frac{a}{b}\frac{y}{\sqrt{b^2 - y^2}}$$

for every $y \in (-b, b)$. With a similar computation as in (b)(i), the required surface area is

$$S = \int_{-b}^{b} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \frac{2\pi a}{b^2}\int_{-b}^{b} \sqrt{b^2(b^2 - y^2) + a^2y^2}\, dy.$$

12. (a) The given curve is the graph of $f\colon [0,1] \to \mathbb{R}$ defined by $f(x) = \frac{x^3}{3}$. Since

$$f'(x) = x^2$$

for every $x \in (0,1)$, the area of the given surface is

$$S = \int_0^1 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx = \int_0^1 2\pi \frac{x^3}{3}\sqrt{1 + (x^2)^2}\, dx = \frac{\pi}{6}\int_0^1 \sqrt{1 + x^4}\, dx^4$$

$$= \frac{\pi}{6}\left[\frac{2}{3}(1 + x^4)^{\frac{3}{2}}\right]_0^1 = \frac{\pi}{9}\left(2\sqrt{2} - 1\right).$$

(b) Since

$$\frac{dx}{dy} = \frac{1}{2\sqrt{4y - y^2}} \cdot (4 - 2y) = \frac{2 - y}{\sqrt{4y - y^2}}$$

for every $y \in (1,2)$, the area of the given surface is

$$S = \int_1^2 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^2 2\pi\sqrt{4y - y^2}\sqrt{1 + \left(\frac{2 - y}{\sqrt{4y - y^2}}\right)^2}\, dy = \int_1^2 4\pi\, dy = 4\pi.$$

(c) The equation $y = \frac{1}{2}\ln\left(2x + \sqrt{4x^2 - 1}\right)$ of the curve for $x \in \left[\frac{1}{2}, \frac{17}{16}\right]$ can be rewritten as

$$x = \frac{e^{2y} + e^{-2y}}{4}$$

for $y \in [0, \ln 2]$. Since

$$\frac{dx}{dy} = \frac{e^{2y} - e^{-2y}}{2}$$

for every $y \in (0, \ln 2)$, the area of the given surface is

$$S = \int_0^{\ln 2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^{\ln 2} 2\pi \frac{e^{2y} + e^{-2y}}{4}\sqrt{1 + \left(\frac{e^{2y} - e^{-2y}}{2}\right)^2}\, dy$$

$$= \int_0^{\ln 2} 2\pi \frac{e^{2y} + e^{-2y}}{4}\sqrt{\left(\frac{e^{2y} + e^{-2y}}{2}\right)^2}\, dy = \frac{\pi}{4}\int_0^{\ln 2} (e^{2y} + e^{-2y})^2 dy$$

$$= \frac{\pi}{4}\int_0^{\ln 2} (e^{4y} + 2 + e^{-4y}) dy = \frac{\pi}{4}\left[\frac{1}{4}e^{4y} + 2y - \frac{1}{4}e^{-4y}\right]_0^{\ln 2} = \left(\frac{255}{256} + \frac{1}{2}\ln 2\right)\pi.$$

13. (a) The polar curve $r = f(\theta)$ can be parametrized by the vector-valued function $\mathbf{r}\colon [a,b] \to \mathbb{R}^2$ defined by

$$\mathbf{r}(t) = \langle f(t)\cos t, f(t)\sin t\rangle.$$

The parametric equations of this curve are

$$x = f(t)\cos t \qquad \text{and} \qquad y = f(t)\sin t,$$

so

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)\sin t + f(t)\cos t}{f'(t)\cos t - f(t)\sin t}$$

Revolving this curve about the x-axis, we obtain a surface whose area is

$$S=\int_{t=a}^{t=b}2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_a^b 2\pi f(t)\sin t\sqrt{1+\left(\frac{f'(t)\sin t+f(t)\cos t}{f'(t)\cos t-f(t)\sin t}\right)^2}\,(f'(t)\cos t-f(t)\sin t)dt$$

$$=\int_a^b 2\pi f(t)\sin t\sqrt{(f'(t)\cos t-f(t)\sin t)^2+(f'(t)\sin t+f(t)\cos t)^2}dt$$

$$=\int_a^b 2\pi f(t)\sin t\sqrt{\left(f(t)\right)^2+\left(f'(t)\right)^2}dt.$$

(b) Let $f(\theta)=\sqrt{\cos 2\theta}$. Note that $\cos 2\theta\geq 0$ for $\theta\in\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$, and by symmetry only $\theta\in\left[0,\frac{\pi}{4}\right]$ is needed to generate the required surface. Thus the surface area is given by

$$S=\int_0^{\frac{\pi}{4}}2\pi f(\theta)\sin\theta\sqrt{\left(f(\theta)\right)^2+\left(f'(\theta)\right)^2}d\theta=\int_0^{\frac{\pi}{4}}2\pi\sqrt{\cos 2\theta}\sin\theta\sqrt{\cos 2\theta+\left(\frac{-2\sin 2\theta}{2\sqrt{\cos 2\theta}}\right)^2}\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}2\pi\sin\theta\,d\theta=2\pi[-\cos\theta]_0^{\frac{\pi}{4}}=\left(2-\sqrt{2}\right)\pi.$$

(c) By a similar computation as in (a), the area of the surface obtained by revolving the polar curve $r=f(\theta)$ about the y-axis is given by

$$S=\int_{t=a}^{t=b}2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_a^b 2\pi f(t)\cos t\sqrt{\left(f(t)\right)^2+\left(f'(t)\right)^2}dt\,.$$

Now with $f(\theta)=\sqrt{\cos 2\theta}$, this time we need all of $\theta\in\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$ to generate the required surface. The surface area is given by

$$S=\int_{-\pi/4}^{\pi/4}2\pi f(\theta)\cos\theta\sqrt{\left(f(\theta)\right)^2+\left(f'(\theta)\right)^2}d\theta=\int_{-\pi/4}^{\pi/4}2\pi\sqrt{\cos 2\theta}\cos\theta\sqrt{\cos 2\theta+\left(\frac{-2\sin 2\theta}{2\sqrt{\cos 2\theta}}\right)^2}\,d\theta$$

$$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}2\pi\cos\theta\,d\theta=2\pi[\sin\theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=2\sqrt{2}\pi.$$

14. (a) The required volume is given by the improper integral

$$V=\int_1^{+\infty}\pi[f(x)]^2dx=\pi\int_1^{+\infty}\frac{1}{x^2}dx=\pi\left[-\frac{1}{x}\right]_1^{+\infty}=\pi$$

which is finite.

(b) The required surface area is given by the improper integral

$$S=\int_1^{+\infty}2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx=2\pi\int_1^{+\infty}\frac{1}{x}\sqrt{1+(-1/x^2)^2}dx=2\pi\int_1^{+\infty}\frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx\,.$$

Since $\frac{1}{x}\sqrt{1+\frac{1}{x^4}}>\frac{1}{x}>0$ for every $x\in[1,+\infty)$ and since the improper integral $\int_1^{+\infty}\frac{1}{x}dx$ diverges by p-test, the improper integral $\int_1^{+\infty}\frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx$ diverges by comparison test. Hence the surface area is infinite.

15. Given $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t\rangle$, we have $\mathbf{r}'(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t\rangle$ for every $t \in (0, 2\pi)$.

(a) The arc-length of the curve is

$$\begin{aligned} l &= \int_0^{2\pi} \|\mathbf{r}'(t)\| dt = \int_0^{2\pi} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{9\sin^2 t \cos^2 t\,(\cos^2 t + \sin^2 t)}\, dt = \int_0^{2\pi} |3\sin t \cos t| dt \\ &= 4\int_0^{\frac{\pi}{2}} 3\sin t \cos t\, dt = [6\sin^2 t]_0^{\frac{\pi}{2}} = 6. \end{aligned}$$

(b) The required area is

$$A = \int_{t=0}^{t=2\pi} x\, dy = \int_0^{2\pi} \cos^3 t\,(3\sin^2 t \cos t) dt = 3\int_0^{2\pi} \sin^2 t \cos^4 t\, dt\,.$$

Recall from Example 6.18 (c) we have

$$\int \sin^2 t \cos^4 t\, dt = \frac{1}{16}t + \frac{1}{64}\sin 2t - \frac{1}{64}\sin 4t - \frac{1}{192}\sin 6t + C;$$

so

$$A = 3\left[\frac{1}{16}t + \frac{1}{64}\sin 2t - \frac{1}{64}\sin 4t - \frac{1}{192}\sin 6t\right]_0^{2\pi} = \frac{3\pi}{8}.$$

(c) The required volume is

$$V = \int_{x=-1}^{x=1} \pi y^2 dx = \int_{t=\pi}^{t=0} \pi y^2 dx = \int_\pi^0 \pi(\sin^3 t)^2(-3\cos^2 t \sin t) dt = 3\pi\int_\pi^0 \cos^2 t \sin^6 t\, d\cos t\,.$$

With a substitution $u = \cos t$, we obtain

$$V = 3\pi\int_{-1}^1 u^2(1-u^2)^3 du = 3\pi\int_{-1}^1 (u^2 - 3u^4 + 3u^6 - u^8) du = 3\pi\left[\frac{1}{3}u^3 - \frac{3}{5}u^5 + \frac{3}{7}u^7 - \frac{1}{9}u^9\right]_{-1}^1 = \frac{32\pi}{105}.$$

(d) The required surface area is

$$\begin{aligned} S &= \int_{x=-1}^{x=1} 2\pi y \underbrace{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}_{=-\|\mathbf{r}'(t)\|dt} = \int_{t=0}^{t=\pi} 2\pi(\sin^3 t)\|\mathbf{r}'(t)\| dt \\ &= \int_0^\pi 2\pi(\sin^3 t)\sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\, dt \\ &= \int_0^\pi 2\pi(\sin^3 t)|3\sin t \cos t| dt = 2\int_0^{\frac{\pi}{2}} 2\pi(\sin^3 t)(3\sin t \cos t) dt \\ &= 12\pi\int_0^{\frac{\pi}{2}} \sin^4 t\, d\sin t = 12\pi\left[\frac{1}{5}\sin^5 t\right]_0^{\frac{\pi}{2}} = \frac{12\pi}{5}. \end{aligned}$$

16. (a) Note that the right-half of the cardioid can be parametrized by the vector-valued function

$$\mathbf{r}(t) = \langle (1+\sin t)\cos t\,, (1+\sin t)\sin t\rangle$$

for $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The required volume is given by

$$V = \int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} \pi x^2 dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi\big((1+\sin t)\cos t\big)^2 \big((1+2\sin t)\cos t\big)dt\,.$$

With a substitution $u = \sin t$, we obtain

$$V = \pi\int_{-1}^{1}(1+u)^2(1-u^2)(1+2u)du = \pi\int_{-1}^{1}\left(1+\underbrace{4u}_{\text{odd}}+4u^2-\underbrace{2u^3}_{\text{odd}}-5u^4-\underbrace{2u^5}_{\text{odd}}\right)du$$

$$= \pi\int_{-1}^{1}(1+4u^2-5u^4)du = \pi\left[u+\frac{4}{3}u^3-u^5\right]_{-1}^{1} = \frac{8\pi}{3}.$$

(b) First note that

$$\mathbf{r}'(t) = \langle \cos^2 t - (1+\sin t)\sin t\,, (1+2\sin t)\cos t\rangle = \langle -\sin t + \cos 2t\,, \cos t + \sin 2t\rangle$$

for every $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and so

$$\|\mathbf{r}'(t)\| = \sqrt{(-\sin t + \cos 2t)^2 + (\cos t + \sin 2t)^2} = \sqrt{2(1+\sin t)}$$

for every $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The required surface area is

$$S = \int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} 2\pi x \underbrace{\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy}_{=\|\mathbf{r}'(t)\|dt} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi(1+\sin t)\cos t\,\|\mathbf{r}'(t)\|dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi(1+\sin t)\cos t\sqrt{2(1+\sin t)}\,dt\,.$$

With a substitution $u = \sin t$, we obtain

$$S = \int_{-1}^{1} 2\pi(1+u)\sqrt{2(1+u)}du = 2\sqrt{2}\pi\int_{-1}^{1}(1+u)^{\frac{3}{2}}du$$

$$= 2\sqrt{2}\pi\left[\frac{2}{5}(1+u)^{\frac{5}{2}}\right]_{-1}^{1} = \frac{32\pi}{5}.$$

17. (a) For each $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n}\left(3^{\frac{1}{k}} - 3^{\frac{1}{k+1}}\right) = \left(3 - 3^{\frac{1}{2}}\right) + \left(3^{\frac{1}{2}} - 3^{\frac{1}{3}}\right) + \left(3^{\frac{1}{3}} - 3^{\frac{1}{4}}\right) + \cdots + \left(3^{\frac{1}{n}} - 3^{\frac{1}{n+1}}\right) = 3 - 3^{\frac{1}{n+1}}.$$

This implies that

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\left(3^{\frac{1}{k}} - 3^{\frac{1}{k+1}}\right) = \lim_{n\to+\infty}\left(3 - 3^{\frac{1}{n+1}}\right) = 3 - 3^{\lim_{n\to+\infty}\frac{1}{n+1}} = 3 - 3^0 = 2$$

(because the function $f(x) = 3^x$ is continuous). Therefore $\sum_{k=1}^{+\infty}\left(3^{\frac{1}{k}} - 3^{\frac{1}{k+1}}\right)$ converges to 2.

(b) For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{2}{k(k+1)(k+2)} \\
&= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}\right) \\
&= \left(\frac{1}{1} - \frac{2}{2} + \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) + \cdots + \left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) \\
&= \frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n+1} - \frac{2}{n+1} + \frac{1}{n+2} \\
&= \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}.
\end{aligned}$$

Thus

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{2}{k(k+1)(k+2)} = \lim_{n \to +\infty} \left(\frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}\right) = \frac{1}{2}.$$

Therefore $\sum_{k=1}^{+\infty} \frac{2}{k(k+1)(k+2)}$ converges to $\frac{1}{2}$.

(c) For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{k=1}^{n} \ln\left(1 + \frac{1}{k}\right) &= \ln\left(1 + \frac{1}{1}\right) + \ln\left(1 + \frac{1}{2}\right) + \ln\left(1 + \frac{1}{3}\right) + \cdots + \ln\left(1 + \frac{1}{n}\right) \\
&= \ln\frac{2}{1} + \ln\frac{3}{2} + \ln\frac{4}{3} + \cdots + \ln\frac{n+1}{n} = \ln\left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \cdots \cdot \frac{n+1}{n}\right) \\
&= \ln(n+1).
\end{aligned}$$

Thus

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \ln\left(1 + \frac{1}{k}\right) = \lim_{n \to +\infty} \ln(n+1) = +\infty.$$

Therefore $\sum_{k=1}^{+\infty} \ln\left(1 + \frac{1}{k}\right)$ diverges to $+\infty$.

(d) For each $n \in \mathbb{N}$, let s_n be the n^{th} partial sum. Then

- The sequence (s_n) is increasing because each term is positive.
- Moreover, for each $m \in \mathbb{N}$ we have

$$\begin{aligned}
s_{2^m - 1} &= 1 + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) + \cdots + \underbrace{\left(\frac{1}{2^{m-1}} + \cdots + \frac{1}{2^{m-1}}\right)}_{2^{m-1}\text{ terms}} \\
&= \underbrace{1 + 1 + 1 + \cdots + 1}_{m\text{ terms}} = m,
\end{aligned}$$

  which shows that (s_n) is unbounded.

Therefore (s_n) diverges (to $+\infty$) by the Monotone Sequence Theorem.

(e) For each integer $n \geq 2$, we have

$$\sum_{k=2}^{n} \frac{k}{2^{k-1}} = \sum_{k=2}^{n} \left(\frac{d}{dx} x^k \bigg|_{x=\frac{1}{2}} \right) = \left(\frac{d}{dx} \sum_{k=2}^{n} x^k \right) \bigg|_{x=\frac{1}{2}}$$

$$= \left(\frac{d}{dx} \frac{x^2 - x^{n+1}}{1-x} \right) \bigg|_{x=\frac{1}{2}} = \frac{(2x - (n+1)x^n)(1-x) - (x^2 - x^{n+1})(-1)}{(1-x)^2} \bigg|_{x=\frac{1}{2}}$$

$$= 3 - \frac{n+2}{2^{n-1}}.$$

Thus

$$\lim_{n\to+\infty} \sum_{k=2}^{n} \frac{k}{2^{k-1}} = \lim_{n\to+\infty} \left(3 - \frac{n+2}{2^{n-1}} \right) = 3.$$

Therefore $\sum_{k=2}^{+\infty} \frac{k}{2^{k-1}}$ converges to 3.

Remark: (For those who are taking / will take MATH2411) The probability that it takes exactly k births to get babies of both sex is $\frac{1}{2^{k-1}}$. The sum of this series in (e) therefore represents the **expected number of births** to get babies of both sex.

18. (a) For every $x \in \mathbb{R}$, we have

$$\begin{aligned} \sin 3x = \sin(2x + x) &= \sin 2x \cos x + \cos 2x \sin x \\ &= (2 \sin x \cos x)(\cos x) + (1 - 2\sin^2 x)(\sin x) \\ &= 2 \sin x\,(1 - \sin^2 x) + (1 - 2\sin^2 x)(\sin x) \\ &= 3 \sin x - 4 \sin^3 x\,. \end{aligned}$$

Therefore $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$. ■

(b) When $n = 1$, we have

$$\sum_{k=1}^{1} 3^{k-1} \sin^3 \frac{1}{3^k} = \sin^3 \frac{1}{3} = \frac{3}{4} \sin \frac{1}{3} - \frac{1}{4} \sin 1\,,$$

according to (a) with $x = \frac{1}{3}$. Assume that for some positive integer m we have

$$\sum_{k=1}^{m} 3^{k-1} \sin^3 \frac{1}{3^k} = \frac{3^m}{4} \sin \frac{1}{3^m} - \frac{1}{4} \sin 1\,.$$

Then when $n = m + 1$, we have

$$\sum_{k=1}^{m+1} 3^{k-1} \sin^3 \frac{1}{3^k} = 3^m \sin^3 \frac{1}{3^{m+1}} + \sum_{k=1}^{m} 3^{k-1} \sin^3 \frac{1}{3^k} = 3^m \sin^3 \frac{1}{3^{m+1}} + \frac{3^m}{4} \sin \frac{1}{3^m} - \frac{1}{4} \sin 1$$

$$= 3^m \left(\sin^3 \frac{1}{3^{m+1}} + \frac{1}{4} \sin \frac{1}{3^m} \right) - \frac{1}{4} \sin 1 = 3^m \left(\frac{3}{4} \sin \frac{1}{3^{m+1}} \right) - \frac{1}{4} \sin 1$$

$$= \frac{3^{m+1}}{4} \sin \frac{1}{3^{m+1}} - \frac{1}{4} \sin 1\,,$$

according to (a) with $x = \frac{1}{3^{m+1}}$. So by induction, the equality is true for every positive integer n. ■

(c) According to the result from (b), we have

$$\begin{aligned}\lim_{n\to+\infty}\sum_{k=1}^{n} 3^{k-1}\sin^3\frac{1}{3^k} &= \lim_{n\to+\infty}\left(\frac{3^n}{4}\sin\frac{1}{3^n}-\frac{1}{4}\sin 1\right)\\ &= \frac{1}{4}\left(\lim_{n\to+\infty}\frac{\sin\frac{1}{3^n}}{\frac{1}{3^n}}\right)-\frac{1}{4}\sin 1\\ &= \frac{1}{4}(1-\sin 1).\end{aligned}$$

Therefore the series $\sum_{k=1}^{+\infty} 3^{k-1}\sin^3\frac{1}{3^k}$ converges to $\frac{1}{4}(1-\sin 1)$.

19. (a) Suppose that $\sum_{k=1}^{+\infty} a_k$ converges to a number L. Then for each $\varepsilon > 0$, there exists $N > 0$ such that

if p is an integer with $p \geq N$, then $\left|\left(a_1+a_2+\cdots+a_p\right)-L\right| < \frac{\varepsilon}{2}$.

Now if m and n are integers with $n > m \geq N$, then

$$|(a_1+a_2+\cdots+a_m)-L| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |(a_1+a_2+\cdots+a_n)-L| < \frac{\varepsilon}{2}.$$

Thus by triangle inequality,

$$\begin{aligned}|a_{m+1}+a_{m+2}+\cdots+a_n| &= |(a_1+a_2+\cdots+a_n)-L+L-(a_1+a_2+\cdots+a_m)|\\ &\leq |(a_1+a_2+\cdots+a_n)-L|+|L-(a_1+a_2+\cdots+a_m)|\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

This shows that $\sum_{k=1}^{+\infty} a_k$ satisfies Cauchy's criterion. ■

(b) Suppose that $\sum_{k=1}^{+\infty} a_k$ satisfies Cauchy's criterion. Then (for $\varepsilon = 1$) there exists $N > 0$ such that

if m and p are integers with $p > m \geq N$, then $\left|a_{m+1}+a_{m+2}+\cdots+a_p\right| < 1$.

Now let

$$M = 1+\max\underbrace{\{|a_1|, |a_1+a_2|, |a_1+a_2+a_3|, \ldots, |a_1+a_2+\cdots+a_N|\}}_{\text{a set of } N \text{ elements}}.$$

Then for each positive integer n,

- If $n \leq N$, then $|a_1+a_2+\cdots+a_n| < M$ because $|a_1+a_2+\cdots+a_n|$ is a member of the set above.
- If $n > N$, then by triangle inequality we also have

$$\begin{aligned}|a_1+a_2+\cdots+a_n| &= |a_1+a_2+\cdots+a_N+a_{N+1}+a_{N+2}+\cdots+a_n|\\ &\leq \underbrace{|a_1+a_2+\cdots+a_N|}_{\leq\max\{|a_1|,\ldots,|a_1+\cdots+a_N|\}}+\underbrace{|a_{N+1}+a_{N+2}+\cdots+a_n|}_{<1}\\ &< \max\{|a_1|, |a_1+a_2|, |a_1+a_2+a_3|, \ldots, |a_1+a_2+\cdots+a_N|\}+1\\ &= M.\end{aligned}$$

Therefore the sequence of partial sums $\left(\sum_{k=1}^{n} a_k\right)_{n\in\mathbb{N}}$ is bounded. ■