HKUST MATH 1014 L1 assignment 3 submission

1

1.a

Let m and n be non-negative integers. Evaluate the following integrals, distinguishing all possible cases for m and n.

1.a.i

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) + \cos((m-n)x)) \, dx$$

$$= \begin{cases} \frac{1}{2} \int_{-\pi}^{\pi} (1+1) \, dx & m+n=0, m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos((m-n)x)) \, dx & m+n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) + 1) \, dx & m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) + \cos((m-n)x)) \, dx & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{2} [2x]_{x=-\pi}^{x=\pi} & m+n=0, m-n=0 \\ \frac{1}{2} \left[x + \frac{1}{m-n}\sin((m-n)x)\right]_{x=-\pi}^{x=\pi} & m+n=0 \\ \frac{1}{2} \left[\frac{1}{m+n}\sin((m+n)x) + x\right]_{x=-\pi}^{x=\pi} & m-n=0 \\ \frac{1}{2} \left[\frac{1}{m+n}\sin((m+n)x) + \frac{1}{m-n}\sin((m-n)x)\right]_{x=-\pi}^{x=\pi} & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2\pi & m+n=0, m-n=0 \\ \pi & m+n=0 \\ \pi & m-n=0 \\ 0 & \text{otherwise} \end{cases}$$

1.a.ii

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) - \cos((m-n)x)) \, dx$$

$$= \begin{cases} \frac{1}{2} \int_{-\pi}^{\pi} (1-1) \, dx & m+n=0, m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos((m-n)x)) \, dx & m+n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) - 1) \, dx & m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) - \cos((m-n)x)) \, dx & \text{otherwise} \end{cases}$$

$$= \begin{cases} 0 & m+n=0, m-n=0 \\ \frac{1}{2} \left[x - \frac{1}{m-n}\sin((m-n)x)\right]_{x=-\pi}^{x=\pi} & m+n=0 \\ \frac{1}{2} \left[\frac{1}{m+n}\sin((m+n)x) - x\right]_{x=-\pi}^{x=\pi} & m-n=0 \\ \frac{1}{2} \left[\frac{1}{m+n}\sin((m+n)x) - \frac{1}{m-n}\sin((m-n)x)\right]_{x=-\pi}^{x=\pi} & \text{otherwise} \end{cases}$$

$$= \begin{cases} 0 & m+n=0, m-n=0 \\ \pi & m+n=0 \\ -\pi & m-n=0 \\ 0 & \text{otherwise} \end{cases}$$

1.a.iii

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx$$

$$= \int_{0}^{\pi} \cos mx \sin nx \, dx + \int_{-\pi}^{0} \cos mx \sin nx \, dx$$

$$= \int_{0}^{\pi} \cos mx \sin nx \, dx - \int_{\pi}^{0} \cos(-mu) \sin(-nu) \, du \qquad (u := -x)$$

$$= \int_{0}^{\pi} \cos mx \sin nx \, dx - \int_{0}^{\pi} \cos mu \sin nu \, du$$

$$= 0$$

2

Evaluate the following antiderivatives.

2.a

$$\int x^2 \arctan x \, dx$$

$$\int x^2 \arctan x \, dx$$

$$= \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int \frac{x^3}{1+x^2} \, dx$$

$$= \frac{1}{3}x^3 \arctan x - \frac{1}{6}\int \frac{x^2}{1+x^2} \, dx^2$$

$$= \frac{1}{3}x^3 \arctan x - \frac{1}{6}\left(\int dx^2 - \int \frac{1}{1+x^2} \, dx^2 \right)$$

$$= \frac{1}{3}x^3 \arctan x - \frac{1}{6}\left(x^2 - \ln\left(1+x^2\right) \right) + C \qquad \left(1+x^2 > 0\right)$$

$$= \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln\left(1+x^2\right) + C$$

2.d

$$\int \left(2x^2 + 1\right)e^{x^2} \, dx$$

$$\frac{d}{dx}e^{x^2} = 2xe^{x^2}$$

$$\frac{d}{dx}\left(2xe^{x^2}\right) = 2e^{x^2} + 4x^2 e^{x^2}$$

$$= 2(2x^2 + 1)e^{x^2}$$

$$\int \left(2x^2 + 1\right)e^{x^2} \, dx = xe^{x^2} + C$$

7

Let $f : [0, +\infty) \to \mathbb{R}$ be the function defined by $f(x) = xe^x$.

7.a

Show that f is strictly increasing.

$$f(x) = xe^x$$
$$f'(x) = e^x + xe^x$$
$$= (1+x)e^x$$

$$\forall x \in [0, +\infty)$$
$$1 + x \geq 1$$
$$e^x \geq 1$$

$$(1+x)e^x \geq 1$$
$$f'(x) \geq 1$$
$$> 0 \qquad\qquad \implies f(x) \text{ is strictly increasing}$$

7.b

Now f is one-to-one according to (a), so we let g be the **inverse** of f, i.e. $g = f^{-1}$.

7.b.i

Write down the domain of g. Show that

$$g'(x) = \frac{1}{x + e^{g(x)}}$$

for every x in the interior of the domain of g.

The domain of g is $[0, +\infty)$.

$$f'(x) = (1+x)e^x = e^x + f(x) \in C^0([0,+\infty), [0,+\infty))$$
$$\implies f(x) \in C^1((0,+\infty), (0,+\infty))$$
$$\implies g'(x) = \frac{1}{f'(g(x))} \qquad \forall x \in (0,+\infty) \qquad\qquad \text{(inverse function theorem)}$$

$$\forall x \in (0,+\infty)$$
$$g'(x)$$
$$= \frac{1}{f'(g(x))}$$
$$= \frac{1}{f(g(x)) + e^{g(x)}}$$
$$= \frac{1}{x + e^{g(x)}}$$

7.b.ii

Using the result from 7.b.i or otherwise, evaluate the antiderivative $\int g(x)\, dx$, expressing your answer in terms of g and other elementary functions only.

$$\forall x \in (0,+\infty)$$
$$\int g(x)\, dx$$
$$= xg(x) - \int x\, dg(x)$$
$$= xg(x) - \int \frac{x}{x + e^{g(x)}}\, dx$$
$$= xg(x) - \left(\int dx - \int \frac{e^{g(x)}}{x + e^{g(x)}}\, dx \right)$$
$$= xg(x) - \left(x - \int e^{g(x)} g'(x)\, dx \right)$$
$$= xg(x) - x + \int e^{g(x)}\, dg(x)$$
$$= xg(x) - x + e^{g(x)} + C$$

7.b.iii

Hence, or otherwise, evaluate the integral $\int_0^e g(x)\, dx$.

$$f(1) = 1e^1 = e$$
$$g(e) = 1$$
$$f(0) = 0e^0 = 0$$
$$g(0) = 0$$

$$\int_0^e g(x)\, dx$$
$$= \left[xg(x) - x + e^{g(x)} \right]_0^e$$
$$= eg(e) - e + e^{g(e)} - e^{g(0)}$$
$$= e - e + e^1 - e^0$$
$$= e - 1$$

10

For each non-negative integer n, let

$$I_n = \int_0^1 t^n e^t\, dt$$

.

10.a

Show that $\frac{1}{n+1} \le I_n \le \frac{e}{n+1}$ for every non-negative integer n.

$$\int_0^1 t^n e^0\, dt \le \int_0^1 t^n e^t\, dt \le \int_0^1 t^n e^1\, dt \qquad \left((\forall t \in [0,1])\, (e^0 \le e^t \le e^1) \right)$$
$$\int_0^1 t^n\, dt \le I_n \le e \int_0^1 t^n\, dt$$
$$\left[\frac{t^{n+1}}{n+1} \right]_{t=0}^{t=1} \le I_n \le e \left[\frac{t^{n+1}}{n+1} \right]_{t=0}^{t=1}$$
$$\frac{1}{n+1} \le I_n \le \frac{e}{n+1}$$

10.b

Express I_n in terms of I_{n-1} for each $n \geq 1$. Hence show that

$$I_n = (-1)^{n+1}n! + e\sum_{k=0}^{n}(-1)^k\frac{n!}{(n-k)!}$$

.

$$I_n$$
$$= \int_0^1 t^n e^t \, \mathrm{d}t$$
$$= \left[t^n e^t\right]_{t=0}^{t=1} - n\int_0^1 t^{n-1}e^t \, \mathrm{d}t$$
$$= e - nI_{n-1}$$

$$I_0 = \int_0^1 t^0 e^t \, \mathrm{d}t = \left[e^t\right]_0^1 = e - 1 = (-1)^{0+1}0! + e\sum_{k=0}^{0}(-1)^k\frac{0!}{(0-k)!}$$

Assume $I_m = (-1)^{m+1}m! + e\sum_{k=0}^{m}(-1)^k\frac{m!}{(m-k)!}$.

$$I_{m+1}$$
$$= e - (m+1)I_m \hspace{4cm} (m+1 \geq 1)$$
$$= e - (m+1)\left((-1)^{m+1}m! + e\sum_{k=0}^{m}(-1)^k\frac{m!}{(m-k)!}\right)$$
$$= e\left(1 - (m+1)\sum_{k=0}^{m}(-1)^k\frac{m!}{(m-k)!}\right) + (-1)^{m+2}(m+1)!$$
$$= e\left(1 + \sum_{k=0}^{m}(-1)^{k+1}\frac{(m+1)!}{(m-k)!}\right) + (-1)^{m+2}(m+1)!$$
$$= e\left((-1)^0\frac{(m+1)!}{(m+1-0)!} + \sum_{k=1}^{m+1}(-1)^k\frac{(m+1)!}{(m+1-k)!}\right) + (-1)^{m+2}(m+1)!$$
$$= e\sum_{k=0}^{m+1}(-1)^k\frac{(m+1)!}{(m+1-k)!} + (-1)^{m+2}(m+1)!$$
$$= (-1)^{(m+1)+1}(m+1)! + e\sum_{k=0}^{m+1}(-1)^k\frac{(m+1)!}{((m+1)-k)!}$$

By induction, $\forall n \in \mathbb{Z}_{\geq 0}$

$$I_n = (-1)^{n+1}n! + e\sum_{k=0}^{n}(-1)^k\frac{n!}{(n-k)!}$$

10.c

Using 10.a and 10.b, deduce that e is an irrational number.

Hint: There is no integer in the open interval $(0,1)$.

Assume e is rational, then $e = \dfrac{a}{b}$ where $a, b \in \mathbb{Z}, b \neq 0$.

$\forall n \in \mathbb{Z}_{\geq 0}$

$$I_n = (-1)^{n+1} n! + e \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!}$$

$$e = \frac{I_n - (-1)^{n+1} n!}{\sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!}}$$

$$\frac{a}{b} = \frac{I_n - (-1)^{n+1} n!}{\sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!}}$$

$$a \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!} = b\left(I_n - (-1)^{n+1} n!\right)$$

$$= bI_n - b(-1)^{n+1} n!$$

$$c = bI_n - d \qquad\qquad (c, d \in \mathbb{Z})$$

$$bI_n = c + d$$

$$bI_n \in \mathbb{Z} \qquad\qquad (\mathbb{Z} \text{ is closed under addition})$$

$$\frac{1}{n+1} \leq I_n \leq \frac{e}{n+1}$$

$$\frac{1}{n+1} \leq I_n < \frac{3}{n+1} \qquad\qquad (e < 3)$$

$$\frac{|b|}{n+1} \leq |b| I_n < \frac{3|b|}{n+1}$$

$$\frac{|b|}{3|b|+1} \leq |bI_{3|b|}| < \frac{3|b|}{3|b|+1} \qquad\qquad (\text{set } n = 3|b|)$$

$$|bI_{3|b|}| \notin \mathbb{Z} \qquad\qquad (|b| < |3b+1|, |3b| < |3b+1|)$$

$$bI_{3|b|} \notin \mathbb{Z}$$

$$(\forall n \in \mathbb{Z}_{\geq 0})(bI_n \notin \mathbb{Z}) \Rightarrow\Leftarrow bI_{3|b|} \notin \mathbb{Z}$$
$$\implies e \text{ is irrational.}$$

11

11.a

For every pair of non-negative integers m and n, let

$$B(m, n) = \int_0^1 x^m (1-x)^n \, dx$$

.

Show that $B(m, n) = \frac{n}{m+1} B(m+1, n-1)$ for every pair of integers $m \geq 0$ and $n \geq 1$.

Hence or otherwise, deduce that $B(m, n) = \frac{m! n!}{(m+n+1)!}$.

$$\forall (m,n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$$

$$B(m,n)$$

$$= \int_0^1 x^m (1-x)^n \, \mathrm{d}x$$

$$= \left[\frac{1}{m+1} x^{m+1} (1-x)^n \right]_{x=0}^{x=1} + \frac{n}{m+1} \int_0^1 x^{(m+1)} (1-x)^{n-1} \, \mathrm{d}x$$

$$= 0 + \frac{n}{m+1} B(m+1, n-1)$$

$$= \frac{n}{m+1} B(m+1, n-1)$$

$$B(m,n)$$

$$= \frac{n}{m+1} B(m+1, n-1)$$

$$= \frac{n}{m+1} \frac{n-1}{m+2} B(m+2, n-2)$$

$$\vdots$$

$$= \frac{n!}{\frac{(m+n)!}{m!}} B(m+n, 0)$$

$$= \frac{m! n!}{(m+n)!} B(m+n, 0)$$

$$= \frac{m! n!}{(m+n)!} \int_0^1 x^{m+n} (1-x)^0 \, \mathrm{d}x$$

$$= \frac{m! n!}{(m+n)!} \left[\frac{1}{m+n+1} x^{m+n+1} \right]_{x=0}^{x=1}$$

$$= \frac{m! n!}{(m+n)!} \frac{1}{m+n+1}$$

$$= \frac{m! n!}{(m+n+1)!}$$

11.b

Show that

$$\int_0^1 \frac{x^4 (x-1)^4}{x^2 + 1} \, \mathrm{d}x = \frac{22}{7} - \pi$$

.

Using this together with the result from , deduce that

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}$$

.

$$\int_0^1 \frac{x^4(x-1)^4}{x^2+1}\,dx$$

$$= \int_0^1 \frac{x^4\left(x^4 - 4x^3 + 6x^2 - 4x + 1\right)}{x^2+1}\,dx$$

$$= \int_0^1 \frac{x^8 - 4x^7 + 6x^6 - 4x^5 + x^4}{x^2+1}\,dx$$

$$= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}\right)dx$$

$$= \left[\frac{1}{7}x^7 - \frac{2}{3}x^6 + x^5 - \frac{4}{3}x^3 + 4x - 4\arctan x\right]_0^1$$

$$= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - \pi$$

$$= \frac{22}{7} - \pi$$

$$\frac{x^4(x-1)^4}{1^2+1} < \frac{x^4(x-1)^4}{x^2+1} < \frac{x^4(x-1)^4}{0^2+1} \qquad (x \in (0,1))$$

$$\int_0^1 \frac{x^4(x-1)^4}{1^2+1}\,dx < \int_0^1 \frac{x^4(x-1)^4}{x^2+1}\,dx < \int_0^1 \frac{x^4(x-1)^4}{0^2+1}\,dx$$

$$\frac{1}{2}\int_0^1 x^4(x-1)^4\,dx < \frac{22}{7} - \pi < \int_0^1 x^4(x-1)^4\,dx$$

$$\frac{1}{2}B(4,4) < \frac{22}{7} - \pi < B(4,4)$$

$$\frac{1}{2}\frac{4!4!}{9!} < \frac{22}{7} - \pi < \frac{4!4!}{9!}$$

$$\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630}$$

$$-\frac{1}{1260} > \pi - \frac{22}{7} > -\frac{1}{630}$$

$$\frac{22}{7} - \frac{1}{1260} > \pi > \frac{22}{7} - \frac{1}{630}$$

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}$$

13

Evaluate the following antiderivatives, using trigonometric substitutions when appropriate.

13.a

$$\int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^4}}\,dx$$

$\forall x \in (-1,1)$

$$\int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^4}}\,dx$$

$$= \int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1+x^2}}\,dx$$

$$= \int \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1+x^2}}\right)dx$$

$$= \arcsin x + \int \frac{1}{\sqrt{1+x^2}}\,dx$$

$$= \arcsin x + \int \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\,d\theta \qquad \left(x := \tan\theta, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$$

$$= \arcsin x + \int \sec\theta\,d\theta$$

$$= \arcsin x + \int \frac{(\sec\theta)(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}\,d\theta$$

$$= \arcsin x + \int \frac{\sec^2\theta + \tan\theta\sec\theta}{\sec\theta + \tan\theta}\,d\theta$$

$$= \arcsin x + \ln|\tan\theta + \sec\theta| + C$$

$$= \arcsin x + \ln\left|x + \sqrt{1+x^2}\right| + C \qquad \left(\sec\theta > 0 \quad \forall \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$$

$$= \arcsin x + \ln\left(x + \sqrt{1+x^2}\right) + C \qquad \left(x + \sqrt{1+x^2} > 0 \quad \forall x \in (-1,1)\right)$$

13.b

$$\int \frac{x+1}{(x^2+x+1)\sqrt{x^2+x+1}}\,dx$$

$$\int \frac{x+1}{(x^2+x+1)\sqrt{x^2+x+1}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+1}{(x^2+x+1)\sqrt{x^2+x+1}}\,dx + \frac{1}{2}\int \frac{1}{(x^2+x+1)\sqrt{x^2+x+1}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{(x^2+x+1)^{1.5}}\,d\left(x^2+x+1\right)\,dx + \frac{1}{2}\int \frac{1}{((x+0.5)^2+0.75)\sqrt{(x+0.5)^2+0.75}}\,dx$$

$$= -\frac{1}{\sqrt{x^2+x+1}} + \frac{1}{2}\sqrt{0.75}\int \frac{\sec^2\theta}{\left(0.75\tan^2\theta+0.75\right)\sqrt{0.75\tan^2\theta+0.75}}\,d\theta \qquad \left(x := \sqrt{0.75}\tan\theta - 0.5, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$$

$$= -\frac{1}{\sqrt{x^2+x+1}} + \frac{1}{2}\sqrt{0.75}\int \frac{1}{0.75\sqrt{0.75}\sec\theta}\,d\theta$$

$$= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3}\int \cos\theta\,d\theta$$

$$= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3}\sin\theta + C$$

$$= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3}\operatorname{sgn}(x+0.5)\frac{1}{\sqrt{\frac{1}{\frac{4}{3}(x+0.5)^2}+1}} + C$$

$$= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3}\operatorname{sgn}(x+0.5)\frac{1}{\sqrt{\frac{3+4(x+0.5)^2}{4(x+0.5)^2}}} + C$$

$$= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3}\operatorname{sgn}(x+0.5)\sqrt{\frac{4(x+0.5)^2}{4x^2+4x+4}} + C$$

$$= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2(x+0.5)}{3\sqrt{x^2+x+1}} + C$$

$$= \frac{2x+1-3}{3\sqrt{x^2+x+1}} + C$$

$$= \frac{2x-2}{3\sqrt{x^2+x+1}} + C$$

14

14.a

For each non-negative integer n, let

$$I_n(x) = \int \frac{x^n}{\sqrt{x^2+1}}\,dx$$

which is defined up to addition by a constant function. Find a reduction formula that connects I_n and I_{n-2} for $n \geq 2$.

$$\forall n \geq 2$$

$$I_n(x) = \int \frac{x^n}{\sqrt{x^2+1}}\,dx$$

$$= x^{n-1}\sqrt{x^2+1} - (n-1)\int x^{n-2}\sqrt{x^2+1}\,dx$$

$$= x^{n-1}\sqrt{x^2+1} - (n-1)\left(\int x^{n-2}\left(\sqrt{x^2+1} - \frac{1}{\sqrt{x^2+1}}\right)dx + I_{n-2}(x)\right)$$

$$= x^{n-1}\sqrt{x^2+1} - (n-1)\left(\int x^{n-2}\frac{x^2}{\sqrt{x^2+1}}\,dx + I_{n-2}(x)\right)$$

$$= x^{n-1}\sqrt{x^2+1} - (n-1)\left(\int \frac{x^n}{\sqrt{x^2+1}}\,dx + I_{n-2}(x)\right)$$

$$= x^{n-1}\sqrt{x^2+1} - (n-1)\left(I_n(x) + I_{n-2}(x)\right)$$

$$= x^{n-1}\sqrt{x^2+1} - (n-1)I_n(x) - (n-1)I_{n-2}(x)$$

$$nI_n(x) = x^{n-1}\sqrt{x^2+1} - (n-1)I_{n-2}(x)$$

$$I_n(x) = \frac{1}{n}x^{n-1}\sqrt{x^2+1} - \frac{n-1}{n}I_{n-2}(x)$$

14.b

Hence evaluate

$$\int_0^1 \frac{x^5}{\sqrt{x^2+1}}\,dx$$

.

$$I_1(x)$$
$$= \int \frac{x}{\sqrt{x^2+1}} \, dx$$
$$= \frac{1}{2} \int \frac{1}{\sqrt{x^2+1}} \, dx^2$$
$$= \frac{1}{2} \left(2\sqrt{x^2+1} \right) + C$$
$$= \sqrt{x^2+1} + C$$

$$I_5(x)$$
$$= \frac{1}{5} x^4 \sqrt{x^2+1} - \frac{4}{5} I_3(x)$$
$$= \frac{1}{5} x^4 \sqrt{x^2+1} - \frac{4}{5} \frac{1}{3} x^2 \sqrt{x^2+1} + \frac{4}{5} \frac{2}{3} I_1(x)$$
$$= \frac{1}{5} x^4 \sqrt{x^2+1} - \frac{4}{15} x^2 \sqrt{x^2+1} + \frac{8}{15} \sqrt{x^2+1} + C$$

$$\int_0^1 \frac{x^5}{\sqrt{x^2+1}} \, dx$$
$$= I_5(1) - I_5(0)$$
$$= \left[\frac{1}{5} x^4 \sqrt{x^2+1} - \frac{4}{15} x^2 \sqrt{x^2+1} + \frac{8}{15} \sqrt{x^2+1} \right]_0^1$$
$$= \frac{1}{5} \sqrt{2} - \frac{4}{15} \sqrt{2} + \frac{8}{15} \sqrt{2} - \frac{8}{15}$$
$$= \frac{7}{15} \sqrt{2} - \frac{8}{15}$$