Solution to Problem Set 5

1. (a) The coordinates (x, y) of P satisfy

$$\frac{\sqrt{(x-4)^2+(y-0)^2}}{|x|}=\frac{1}{3},$$

i.e. $3\sqrt{(x-4)^2+(y-0)^2}=|x|$. Squaring both sides we get

$$9[(x-4)^2+y^2]=x^2,$$

i.e.

$$8x^2+9y^2-72x+144=0,$$

which is an equation of an ellipse since the coefficients of x^2 and y^2 have the same sign.

(b) The coordinates (x, y) of P satisfy

$$\frac{\sqrt{(x-4)^2+(y-0)^2}}{|x|}=3,$$

i.e. $\sqrt{(x-4)^2+(y-0)^2}=3|x|$. Squaring both sides we get

$$(x-4)^2+y^2=9x^2,$$

i.e.

$$8x^2-y^2+8x-16=0,$$

which is an equation of a hyperbola since the coefficients of x^2 and y^2 have the same sign.

2. Differentiating both sides of the equation of the ellipse with respect to x, we have

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

The slope of tangent lines to the ellipse at the point (x, y) is given by

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}.$$

Now $\frac{dy}{dx}=m$ only when $-\frac{b^2x}{a^2y}=m$, i.e. when $x=-\frac{ma^2y}{b^2}$. Putting this into the equation of the ellipse, we obtain $\frac{y^2}{b^2}(1+m^2a^2)=1$, i.e. the points of tangency are

$$(x,y)=\left(-\frac{ma^2b}{\sqrt{1+m^2a^2}},\frac{b}{\sqrt{1+m^2a^2}}\right) \quad \text{and} \quad (x,y)=\left(\frac{ma^2b}{\sqrt{1+m^2a^2}},-\frac{b}{\sqrt{1+m^2a^2}}\right).$$

Therefore the equations of tangents are given by

$$y-\frac{b}{\sqrt{1+m^2a^2}}=m\left(x+\frac{ma^2b}{\sqrt{1+m^2a^2}}\right) \quad \text{and} \quad y+\frac{b}{\sqrt{1+m^2a^2}}=m\left(x-\frac{ma^2b}{\sqrt{1+m^2a^2}}\right)$$

i.e.

$$y=mx+\sqrt{1+m^2a^2}b \quad \text{and} \quad y=mx-\sqrt{1+m^2a^2}b.$$

3. Since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2,$$

so $r = \sqrt{x^2 + y^2}$. On the other hand, we have

$$\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x},$$

so

$$\theta = \begin{cases} \arctan\dfrac{y}{x} - \pi & \text{if } x < 0 \text{ and } y < 0 \text{ (i.e. Quadrant III)} \\ -\dfrac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \text{ (i.e. negative } y\text{-axis)} \\ \arctan\dfrac{y}{x} & \text{if } x > 0 \text{ (i.e. Quadrants I or IV or positive } x\text{-axis)} \\ \dfrac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \text{ (i.e. positive } y\text{-axis)} \\ \arctan\dfrac{y}{x} + \pi & \text{if } x < 0 \text{ and } y \geq 0 \text{ (i.e. Quadrant II or negative } x\text{-axis)} \end{cases}.$$

4. The given polar equation $r = a\sin\theta + b\cos\theta$ can be rewritten as

$$r^2 = ar\sin\theta + br\cos\theta,$$

which becomes

$$x^2 + y^2 = ax + by$$

in rectangular coordinates. Further rewriting this equation into the form

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4},$$

we conclude that it represents a circle in $\mathbb{R}^2$ centered at the point $(x, y) = \left(\frac{a}{2}, \frac{b}{2}\right)$, with radius $\frac{\sqrt{a^2+b^2}}{2}$.

5. (a) $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\frac{\pi}{3} = (4)(3)\left(\frac{1}{2}\right) = 6.$

(b) To require that $\mathbf{u} + k\mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$ are orthogonal, we need to have

$$\begin{aligned} (\mathbf{u} + k\mathbf{v}) \cdot (\mathbf{u} - 2\mathbf{v}) &= 0 \\ \mathbf{u} \cdot \mathbf{u} + (k-2)\mathbf{u} \cdot \mathbf{v} - 2k\mathbf{v} \cdot \mathbf{v} &= 0 \\ \|\mathbf{u}\|^2 + (k-2)\mathbf{u} \cdot \mathbf{v} - 2k\|\mathbf{v}\|^2 &= 0. \end{aligned}$$

This means $4^2 + (k-2)(6) - 2k(3^2) = 0$, and so $k = \frac{1}{3}$.

(c) First we have

$$\mathbf{a} \times \mathbf{b} = (3\mathbf{u} + 4\mathbf{v}) \times (-2\mathbf{u} - \mathbf{v}) = -6\underbrace{\mathbf{u} \times \mathbf{u}}_{=\mathbf{0}} - 3\mathbf{u} \times \mathbf{v} - 8\mathbf{v} \times \mathbf{u} - 4\underbrace{\mathbf{v} \times \mathbf{v}}_{=\mathbf{0}} = 5\mathbf{u} \times \mathbf{v}.$$

The area of the required parallelogram is given by

$$\|\mathbf{a} \times \mathbf{b}\| = 5\|\mathbf{u} \times \mathbf{v}\| = 5\|\mathbf{u}\|\|\mathbf{v}\| \sin\frac{\pi}{3} = 5(4)(3)\left(\frac{\sqrt{3}}{2}\right) = 30\sqrt{3}.$$

Alternative solution: [This approach is valid even if $\mathbf{u}$ and $\mathbf{v}$ are vectors in dimensions other than 3.]

First we have $\mathbf{a}\cdot\mathbf{a} = 9\mathbf{u}\cdot\mathbf{u} + 24\mathbf{u}\cdot\mathbf{v} + 16\mathbf{v}\cdot\mathbf{v} = 432$, $\mathbf{a}\cdot\mathbf{b} = -6\mathbf{u}\cdot\mathbf{u} - 11\mathbf{u}\cdot\mathbf{v} - 4\mathbf{v}\cdot\mathbf{v} = -198$ and $\mathbf{b}\cdot\mathbf{b} = 4\mathbf{u}\cdot\mathbf{u} + 4\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} = 97$. Note that $\mathbf{a} - \operatorname{proj}_{\mathbf{b}}\mathbf{a} = \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$ and $\mathbf{b}$ are orthogonal vectors, so the area of the required parallelogram is given by

$$\begin{aligned} A &= \left\|\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}\right\| \|\mathbf{b}\| = \sqrt{\left[\left(\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}\right)\cdot\left(\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}\right)\right](\mathbf{b}\cdot\mathbf{b})} \\ &= \sqrt{(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2} = \sqrt{(432)(97) - (-198)^2} = 30\sqrt{3}. \end{aligned}$$

6. For vectors $\mathbf{u}$ and $\mathbf{v}$ of the same dimension,

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= (\mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}) \\ &= 2(\mathbf{u}\cdot\mathbf{u}) + 2(\mathbf{v}\cdot\mathbf{v}) = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2). \end{aligned}$$

■

7. (a) The statement is false.

Elements of $\mathbb{R}^2$ are ordered pairs (x,y) of real numbers or two-dimensional vectors $\langle x,y\rangle$, while elements of $\mathbb{R}^3$ are ordered triples of real numbers or three-dimensional vectors. We may say that the plane $\{(x,y,0): x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$ is a subset of $\mathbb{R}^3$, but $\mathbb{R}^2 = \{(x,y): x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$ is not a subset of $\mathbb{R}^3$.

(b) The statement is true.

If $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u}+\mathbf{v}\|^2$, then we have

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}).$$

This implies that $\mathbf{u}\cdot\mathbf{v} = 0$, so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(c) The statement is true.

If $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2$, then we have

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u}\cdot\mathbf{v}).$$

This also implies that $\mathbf{u}\cdot\mathbf{v} = 0$, so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal again.

(d) The statement is false.

In $\mathbb{R}^2$, let $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$ and $\mathbf{w} = 2\mathbf{j}$. Then $\mathbf{u}\cdot\mathbf{v} = 0 = \mathbf{u}\cdot\mathbf{w}$, but $\mathbf{v} \neq \mathbf{w}$.

(e) The statement is false.

In $\mathbb{R}^2$, let $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$ and $\mathbf{w} = \mathbf{i}$. Then $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\mathbf{v}$ is orthogonal to $\mathbf{w}$, but $\mathbf{u}$ is not orthogonal to $\mathbf{w}$.

(f) The statement is true.

If $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{u}$ is parallel to $\mathbf{v}$ and $\mathbf{v}$ is parallel to $\mathbf{w}$, then $\mathbf{u} = t\mathbf{v}$ and $\mathbf{w} = s\mathbf{v}$ for some non-zero scalars t and s. Then $\mathbf{u} = t\mathbf{v} = t\left(\frac{1}{s}\mathbf{w}\right) = \frac{t}{s}\mathbf{w}$, i.e. $\mathbf{u}$ is also a scalar multiple of $\mathbf{w}$, and so $\mathbf{u}$ is also parallel to $\mathbf{w}$.

8. (a) Written in component form, the vector equation $(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b})=0$ becomes

$$(\langle x,y\rangle-\langle a_1,a_2\rangle)\cdot(\langle x,y\rangle-\langle b_1,b_2\rangle)=0,$$

i.e. $(x-a_1)(x-b_1)+(y-a_2)(y-b_2)=0$, which simplifies to

$$x^2+y^2-(a_1+b_1)x-(a_2+b_2)y+a_1b_1+a_2b_2=0.$$

Completing squares and rearranging, we get

$$\left(x-\frac{a_1+b_1}{2}\right)^2+\left(y-\frac{a_2+b_2}{2}\right)^2=\frac{(a_1-b_1)^2+(a_2-b_2)^2}{4},$$

which is the equation of a circle in $\mathbb{R}^3$ with radius $\frac{1}{2}\sqrt{(a_1-b_1)^2+(a_2-b_2)^2}$ centered at $\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right)$.

(b) Written in component form, the vector equation $(\mathbf{r}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})=(\mathbf{r}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$ becomes

$$(\langle x,y\rangle-\langle a_1,a_2\rangle)\cdot(\langle b_1,b_2\rangle-\langle a_1,a_2\rangle)=(\langle x,y\rangle-\langle b_1,b_2\rangle)\cdot(\langle a_1,a_2\rangle-\langle b_1,b_2\rangle),$$

i.e.

$$(x-a_1)(b_1-a_1)+(y-a_2)(b_2-a_2)=(x-b_1)(a_1-b_1)+(y-b_2)(a_2-b_2).$$

This simplifies to

$$2(a_1-b_1)x+2(a_2-b_2)y=a_1^2-b_1^2+a_2^2-b_2^2,$$

which is the equation of a line in $\mathbb{R}^3$ passing through $\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right)$ and perpendicular to $\mathbf{a}-\mathbf{b}$.

9. Let $\theta,\alpha,\beta\in[0,\pi]$ be the angles between $\mathbf{u}$ and $\mathbf{v}$, between $\mathbf{u}$ and $\mathbf{w}$ and between $\mathbf{v}$ and $\mathbf{w}$ respectively. Then since

$$\mathbf{u}\cdot\mathbf{w}=\mathbf{u}\cdot(\|\mathbf{u}\|\mathbf{v}+\|\mathbf{v}\|\mathbf{u})=\|\mathbf{u}\|(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|(\mathbf{u}\cdot\mathbf{u})=\|\mathbf{u}\|(\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|\|\mathbf{v}\|),$$

$$\mathbf{v}\cdot\mathbf{w}=\mathbf{v}\cdot(\|\mathbf{u}\|\mathbf{v}+\|\mathbf{v}\|\mathbf{u})=\|\mathbf{u}\|(\mathbf{v}\cdot\mathbf{v})+\|\mathbf{v}\|(\mathbf{u}\cdot\mathbf{v})=\|\mathbf{v}\|(\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|\|\mathbf{v}\|),$$

$$\mathbf{w}\cdot\mathbf{w}=(\|\mathbf{u}\|\mathbf{v}+\|\mathbf{v}\|\mathbf{u})\cdot(\|\mathbf{u}\|\mathbf{v}+\|\mathbf{v}\|\mathbf{u})=2\|\mathbf{u}\|\|\mathbf{v}\|(\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|\|\mathbf{v}\|);$$

we have

$$\cos 2\alpha=2\cos^2\alpha-1=2\left(\frac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{u}\|\|\mathbf{w}\|}\right)^2-1=\frac{2(\mathbf{u}\cdot\mathbf{w})^2}{\|\mathbf{u}\|^2\|\mathbf{w}\|^2}-1=\frac{2(\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|\|\mathbf{v}\|)^2}{2\|\mathbf{u}\|\|\mathbf{v}\|(\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|\|\mathbf{v}\|)}-1$$

$$=\frac{\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|\|\mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|}-1=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}=\cos\theta$$

and similarly $\cos 2\beta=\cos\theta$. Now observe that $\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|\|\mathbf{v}\|\geq 0$ by Cauchy-Schwarz inequality, so $\mathbf{u}\cdot\mathbf{w}$ and $\mathbf{v}\cdot\mathbf{w}$ are both non-negative, i.e. $\cos\alpha=\frac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{u}\|\|\mathbf{w}\|}\geq 0$ and $\cos\beta=\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\geq 0$. Thus $\alpha,\beta\in\left[0,\frac{\pi}{2}\right]$ and so we must have $\alpha=\frac{\theta}{2}$ and $\beta=\frac{\theta}{2}$. Therefore $\mathbf{w}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$. ■

10. We have

$$\mathrm{proj}_{\mathbf{u}}\mathbf{v}\cdot\mathrm{proj}_{\mathbf{v}}\mathbf{u}=\left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right)\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\right)=\frac{(\mathbf{u}\cdot\mathbf{v})^3}{(\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v})}$$

$$=(\mathbf{u}\cdot\mathbf{v})\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)^2=(\mathbf{u}\cdot\mathbf{v})\cos^2\theta.$$

11. (a) For each $\mathbf{u} \in S$, there exists $x, y \in \mathbb{R}$ such that $\mathbf{u} = x\mathbf{a} + y\mathbf{b}$. Thus we have

$$\mathbf{u} \cdot \mathbf{a} = (x\mathbf{a} + y\mathbf{b}) \cdot \mathbf{a} = x\,\underbrace{(\mathbf{a} \cdot \mathbf{a})}_{=1} + y\,\underbrace{(\mathbf{a} \cdot \mathbf{b})}_{=0} = x,$$

$$\mathbf{u} \cdot \mathbf{b} = (x\mathbf{a} + y\mathbf{b}) \cdot \mathbf{b} = x\,\underbrace{(\mathbf{a} \cdot \mathbf{b})}_{=0} + y\,\underbrace{(\mathbf{b} \cdot \mathbf{b})}_{=1} = y,$$

and so $\mathbf{u} = (\mathbf{u} \cdot \mathbf{a})\mathbf{a} + (\mathbf{u} \cdot \mathbf{b})\mathbf{b}$. ■

(b) For every $\mathbf{u} \in S$, we have $\mathbf{u} = (\mathbf{u} \cdot \mathbf{a})\mathbf{a} + (\mathbf{u} \cdot \mathbf{b})\mathbf{b}$ according to (a). Thus

$$\begin{aligned}(\mathbf{v} - \mathbf{w}) \cdot \mathbf{u} &= \mathbf{v} \cdot \mathbf{u} - \mathbf{w} \cdot \mathbf{u} \\ &= \mathbf{v} \cdot \big((\mathbf{u} \cdot \mathbf{a})\mathbf{a} + (\mathbf{u} \cdot \mathbf{b})\mathbf{b}\big) - \big((\mathbf{v} \cdot \mathbf{a})\mathbf{a} + (\mathbf{v} \cdot \mathbf{b})\mathbf{b}\big) \cdot \mathbf{u} \\ &= (\mathbf{u} \cdot \mathbf{a})(\mathbf{v} \cdot \mathbf{a}) + (\mathbf{u} \cdot \mathbf{b})(\mathbf{v} \cdot \mathbf{b}) - (\mathbf{v} \cdot \mathbf{a})(\mathbf{a} \cdot \mathbf{u}) - (\mathbf{v} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{u}) = 0,\end{aligned}$$

which implies that $\mathbf{v} - \mathbf{w}$ is orthogonal to $\mathbf{u}$. ■

12. (a) If $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} = \mathbf{0}$, then we have

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot (-\mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot \underbrace{\big((\mathbf{b} - \mathbf{c}) \times \mathbf{a}\big)}_{\perp \mathbf{a}} = 0,$$

so $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar. ■

(b) If $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} = \mathbf{0}$, then we have

$$\overrightarrow{AB} \times \overrightarrow{AC} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} + \underbrace{\mathbf{a} \times \mathbf{a}}_{=\mathbf{0}} = \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} = \mathbf{0},$$

so A, B and C are collinear. ■

13. (a) For every three-dimensional vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, we have

$$\begin{aligned}\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \langle u_1, u_2, u_3 \rangle \times \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle \\ &= [u_2(v_1 w_2 - v_2 w_1) - u_3(v_3 w_1 - v_1 w_3)]\mathbf{i} + [u_3(v_2 w_3 - v_3 w_2) - u_1(v_1 w_2 - v_2 w_1)]\mathbf{j} \\ &\quad + [u_1(v_3 w_1 - v_1 w_3) - u_2(v_2 w_3 - v_3 w_2)]\mathbf{k}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \\ &= (u_1 w_1 + u_2 w_2 + u_3 w_3)\langle v_1, v_2, v_3 \rangle - (u_1 v_1 + u_2 v_2 + u_3 v_3)\langle w_1, w_2, w_3 \rangle \\ &= [(u_1 w_1 + u_2 w_2 + u_3 w_3)v_1 - (u_1 v_1 + u_2 v_2 + u_3 v_3)w_1]\mathbf{i} \\ &\qquad + [(u_1 w_1 + u_2 w_2 + u_3 w_3)v_2 - (u_1 v_1 + u_2 v_2 + u_3 v_3)w_2]\mathbf{j} \\ &\qquad + [(u_1 w_1 + u_2 w_2 + u_3 w_3)v_3 - (u_1 v_1 + u_2 v_2 + u_3 v_3)w_3]\mathbf{k} \\ &= [(u_2 w_2 + u_3 w_3)v_1 - (u_2 v_2 + u_3 v_3)w_1]\mathbf{i} + [(u_1 w_1 + u_3 w_3)v_2 - (u_1 v_1 + u_3 v_3)w_2]\mathbf{j} \\ &\qquad + [(u_1 w_1 + u_2 w_2)v_3 - (u_1 v_1 + u_2 v_2)w_3]\mathbf{k} \\ &= [u_2(v_1 w_2 - v_2 w_1) - u_3(v_3 w_1 - v_1 w_3)]\mathbf{i} + [u_3(v_2 w_3 - v_3 w_2) - u_1(v_1 w_2 - v_2 w_1)]\mathbf{j} \\ &\qquad + [u_1(v_3 w_1 - v_1 w_3) - u_2(v_2 w_3 - v_3 w_2)]\mathbf{k}\end{aligned}$$

also. Therefore $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$. ■

(b) According to the result from (a), we have

$$\begin{aligned}
&\mathbf{u}\times(\mathbf{v}\times\mathbf{w})+\mathbf{v}\times(\mathbf{w}\times\mathbf{u})+\mathbf{w}\times(\mathbf{u}\times\mathbf{v})\\
&=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}+(\mathbf{v}\cdot\mathbf{u})\mathbf{w}-(\mathbf{v}\cdot\mathbf{w})\mathbf{u}+(\mathbf{w}\cdot\mathbf{v})\mathbf{u}-(\mathbf{w}\cdot\mathbf{u})\mathbf{v}\\
&=(\mathbf{w}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{w})\mathbf{u}+(\mathbf{u}\cdot\mathbf{w}-\mathbf{w}\cdot\mathbf{u})\mathbf{v}+(\mathbf{v}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v})\mathbf{w}=\mathbf{0}.
\end{aligned}$$

14. We can regard the points A, B and C in $\mathbb{R}^2$ as points lying in the xy-plane in $\mathbb{R}^3$, so that their coordinates become $(-3,0,0)$, $(-1,3,0)$ and $(5,2,0)$ respectively. Now

$$\overrightarrow{AB}\times\overrightarrow{AC}=\langle -1-(-3),3-0,0-0\rangle\times\langle 5-(-3),2-0,0-0\rangle=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&0\\8&2&0\end{vmatrix}=\langle 0,0,-20\rangle,$$

so the area of ΔABC is half of the area of a parallelogram with A, B and C to be three of its vertices, i.e.

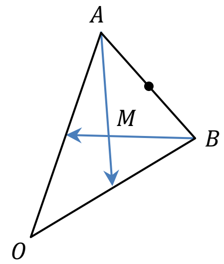
$$\text{Area of }\Delta ABC=\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\frac{1}{2}|-20\mathbf{k}|=10.$$

15. The volume of the parallelepiped in $\mathbb{R}^3$ determined by the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ is

$$V=\left|\overrightarrow{OA}\cdot\left(\overrightarrow{OB}\times\overrightarrow{OC}\right)\right|=\left\|\begin{matrix}3&-5&1\\0&2&-2\\3&1&1\end{matrix}\right\|=\left|3\begin{vmatrix}2&-2\\1&1\end{vmatrix}-(-5)\begin{vmatrix}0&-2\\3&1\end{vmatrix}+1\begin{vmatrix}0&2\\3&1\end{vmatrix}\right|=36$$

which is non-zero, so the points A, B, C and O are not coplanar.

16. (a) Without loss of generality, we let the origin O be one of the vertices of the triangle, and let A and B be the other two vertices, whose position vectors are $\mathbf{a}=\overrightarrow{OA}$ and $\mathbf{b}=\overrightarrow{OB}$ which are not parallel. The mid-point of OA has position vector $\frac{1}{2}\mathbf{a}$, and the mid-point of OB has position vector $\frac{1}{2}\mathbf{b}$. So the median from the vertex A is parallel to the vector $\frac{1}{2}\mathbf{b}-\mathbf{a}$, and the median from the vertex B is parallel to the vector $\frac{1}{2}\mathbf{a}-\mathbf{b}$. Now let M be the point of intersection of the medians from A and from B. Then $\overrightarrow{AM}=t\left(\frac{1}{2}\mathbf{b}-\mathbf{a}\right)$ for some $t\in[0,1]$ and $\overrightarrow{BM}=s\left(\frac{1}{2}\mathbf{a}-\mathbf{b}\right)$ for some $s\in[0,1]$. The position vector of M is therefore simultaneously given by



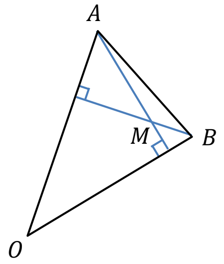
$$\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AM}=\mathbf{a}+t\left(\frac{1}{2}\mathbf{b}-\mathbf{a}\right)\quad\text{and}\quad\overrightarrow{OM}=\overrightarrow{OB}+\overrightarrow{BM}=\mathbf{b}+s\left(\frac{1}{2}\mathbf{a}-\mathbf{b}\right)$$

for some $s,t\in[0,1]$. Equating the above we get $\left(1-t-\frac{s}{2}\right)\mathbf{a}=\left(1-s-\frac{t}{2}\right)\mathbf{b}$. Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, we must have

$$1-t-\frac{s}{2}=0\qquad\text{and}\qquad 1-s-\frac{t}{2}=0,$$

and on solving this system we get $s=t=\frac{2}{3}$. Therefore $\overrightarrow{OM}=\mathbf{a}+\frac{2}{3}\left(\frac{1}{2}\mathbf{b}-\mathbf{a}\right)=\frac{1}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}$. Finally, the mid-point of AB has position vector $\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}=\frac{3}{2}\overrightarrow{OM}$, which is a scalar multiple of $\overrightarrow{OM}$. Therefore the median from O also passes through the point M, and in other words, the three medians are concurrent. ∎

(b) Without loss of generality, we let the origin be one of the vertices of the triangle, and let A and B be the other two vertices, whose position vectors are $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$ which are not parallel.



Let M be the point of intersection of the altitudes from A and from B, and let $\mathbf{m} = \overrightarrow{OM}$. Then we have $\overrightarrow{AM} \cdot \overrightarrow{OB} = 0$ and $\overrightarrow{BM} \cdot \overrightarrow{OA} = 0$, i.e.

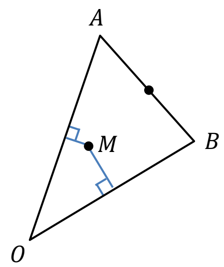
$$(\mathbf{m} - \mathbf{a}) \cdot \mathbf{b} = 0 \qquad \text{and} \qquad (\mathbf{m} - \mathbf{b}) \cdot \mathbf{a} = 0.$$

This implies that $\mathbf{m} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ and $\mathbf{m} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a}$, and so

$$\overrightarrow{OM} \cdot \overrightarrow{AB} = \mathbf{m} \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{m} \cdot \mathbf{b} - \mathbf{m} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} = 0,$$

which shows that OM and AB are perpendicular. Therefore the altitude from O also passes through the point M, and in other words, the three altitudes are concurrent. ■

(c) Without loss of generality, we let the origin be one of the vertices of the triangle, and let A and B be the other two vertices, whose position vectors are $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$ which are not parallel. The mid-point of OA has position vector $\frac{1}{2}\mathbf{a}$, and the mid-point of OB has position vector $\frac{1}{2}\mathbf{b}$.



Now let M be the point of intersection of the perpendicular bisectors of the edge OA and of the edge OB, and let $\mathbf{m} = \overrightarrow{OM}$. Then we have

$$\left(\mathbf{m} - \frac{1}{2}\mathbf{a}\right) \cdot \mathbf{a} = 0 \qquad \text{and} \qquad \left(\mathbf{m} - \frac{1}{2}\mathbf{b}\right) \cdot \mathbf{b} = 0.$$

Now the mid-point of AB has position vector $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$, and we have

$$\left(\mathbf{m} - \left(\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}\right)\right) \cdot (\mathbf{b} - \mathbf{a}) = \left(\mathbf{m} - \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b}\right) \cdot \mathbf{b} - \left(\mathbf{m} - \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b}\right) \cdot \mathbf{a} = -\frac{1}{2}\mathbf{a} \cdot \mathbf{b} + \frac{1}{2}\mathbf{b} \cdot \mathbf{a} = 0,$$

which shows that the line joining M and the mid-point of AB is perpendicular to the edge AB. Therefore the perpendicular bisector of the edge AB also passes through the point M, and in other words, the three perpendicular bisectors are concurrent. ■

17. Let $OABC$ be a rhombus, so that the position vectors $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{c} = \overrightarrow{OC}$ are not parallel and has the same length $\|\mathbf{a}\| = \|\mathbf{c}\|$. Now the two diagonals of the rhombus are given by $\overrightarrow{OB} = \mathbf{a} + \mathbf{c}$ and $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$. Since

$$\overrightarrow{OB} \cdot \overrightarrow{AC} = (\mathbf{a} + \mathbf{c}) \cdot (\mathbf{c} - \mathbf{a}) = \|\mathbf{c}\|^2 - \|\mathbf{a}\|^2 = 0,$$

it follows that the diagonals of a rhombus are perpendicular to each other. ■

18. Let $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{AD}$. Then $\mathbf{u}$ and $\mathbf{v}$ are not parallel, and we have

$$\overrightarrow{AX} = \overrightarrow{AB} + \overrightarrow{BX} = \mathbf{u} + \frac{1}{2}\mathbf{v} \qquad \text{and} \qquad \overrightarrow{AY} = \overrightarrow{AD} + \overrightarrow{DY} = \mathbf{v} + \frac{1}{2}\mathbf{u}.$$

Now let M be the point of intersection of AX and the diagonal BD, and let $\overrightarrow{BM} = t\overrightarrow{BD}$. Then there exists k such that

$$\overrightarrow{AM} = k\overrightarrow{AX} = k\left(\mathbf{u} + \frac{1}{2}\mathbf{v}\right) \qquad \text{and} \qquad \overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} = \mathbf{u} + t(\mathbf{v} - \mathbf{u}).$$

This implies that $k\left(\mathbf{u} + \frac{1}{2}\mathbf{v}\right) = \mathbf{u} + t(\mathbf{v} - \mathbf{u})$, i.e. $(k - 1 + t)\mathbf{u} = \left(t - \frac{1}{2}k\right)\mathbf{v}$. Since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we have $k - 1 + t = 0$ and $t - \frac{1}{2}k = 0$, and on solving this system we get $t = \frac{1}{3}$ (and $k = \frac{2}{3}$).

In a similar way, let N be the point of intersection of AY and the diagonal BD, and let $\overrightarrow{DN} = s\overrightarrow{DB}$. Then there exists m such that

$$\overrightarrow{AN} = m\overrightarrow{AY} = m\left(\mathbf{v} + \frac{1}{2}\mathbf{u}\right) \qquad \text{and} \qquad \overrightarrow{AN} = \overrightarrow{AD} + \overrightarrow{DN} = \mathbf{v} + s(\mathbf{u} - \mathbf{v}).$$

This implies that $m\left(\mathbf{v} + \frac{1}{2}\mathbf{u}\right) = \mathbf{v} + s(\mathbf{u} - \mathbf{v})$, i.e. $(m - 1 + s)\mathbf{v} = \left(s - \frac{1}{2}m\right)\mathbf{u}$. Since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we have $m - 1 + s = 0$ and $s - \frac{1}{2}m = 0$, and on solving this system we get $s = \frac{1}{3}$ (and $m = \frac{2}{3}$).
Therefore the line segments AX and AY divide the diagonal BD into three portions of equal length. ■

19. (a) Given the point $(6, -5, 2)$ on the line and the direction vector $\langle 3, 9, -2\rangle$, a vector equation of the line is

$$\mathbf{r} = \langle 6, -5, 2\rangle + t\langle 3, 9, -2\rangle$$

where $t \in (-\infty, +\infty)$, and the parametric equations of the line are

$$x = 6 + 3t \qquad \text{and} \qquad y = -5 + 9t \qquad \text{and} \qquad z = 2 - 2t,$$

where $t \in (-\infty, +\infty)$.

(b) A direction vector of the required line segment is the vector from $(4, -6, 6)$ to $(2, 3, 1)$, which is $\langle -2, 9, -5\rangle$. Together with the end-point $(4, -6, 6)$ of the line segment, a vector equation of the line segment is

$$\mathbf{r} = \langle 4, -6, 6\rangle + t\langle -2, 9, -5\rangle,$$

where $t \in [0, 1]$, and the parametric equations of the line segment are

$$x = 4 - 2t \qquad \text{and} \qquad y = -6 + 9t \qquad \text{and} \qquad z = 6 - 5t,$$

where $t \in [0, 1]$.

(c) A direction vector of the required line is $(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k}) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Together with the point $(2, 1, 0)$ on the line, a vector equation of the line is

$$\mathbf{r} = \langle 2, 1, 0\rangle + t\langle 1, -1, 1\rangle,$$

where $t \in (-\infty, +\infty)$, and the parametric equations of the line are

$$x = 2 + t \qquad \text{and} \qquad y = 1 - t \qquad \text{and} \qquad z = t,$$

where $t \in (-\infty, +\infty)$.

(d) We require that the vector from $(0, 1, 2)$ to the point of intersection $(1 + t, 1 - t, 2t)$ is perpendicular to the direction vector $\langle 1, -1, 2\rangle$ of the given line. So $\langle 1 + t - 0, 1 - t - 1, 2t - 2\rangle \cdot \langle 1, -1, 2\rangle = 0$, which gives $t = \frac{1}{2}$. Now a direction vector of the required line is $\langle 1 + t - 0, 1 - t - 1, 2t - 2\rangle = \langle \frac{3}{2}, -\frac{1}{2}, -1\rangle$, so a vector equation of the line is

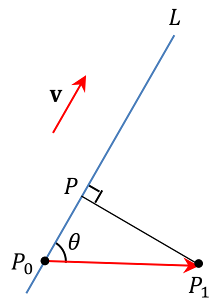
$$\mathbf{r} = \langle 0, 1, 2\rangle + t\langle 3, -1, -2\rangle,$$

where $t \in (-\infty, +\infty)$, and the parametric equations of the line are

$$x = 3t \qquad \text{and} \qquad y = 1 - t \qquad \text{and} \qquad z = 2 - 2t,$$

where $t \in (-\infty, +\infty)$.

20. Let P be the point on the line L that is the closest to the given point P_1. Then the line joining P and P_1 is perpendicular to L, i.e. the vector $\overrightarrow{PP_1}$ is perpendicular to $\mathbf{v}$. Now the distance between P_1 and L is given by the length of the line segment PP_1, which is



$$d(P_1, L) = \|\overrightarrow{P_0P_1}\| \sin\theta = \frac{\|\overrightarrow{P_0P_1} \times \mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\|(\mathbf{r}_1 - \mathbf{r}_0) \times \mathbf{v}\|}{\|\mathbf{v}\|}.$$

■

21. At a point of intersection of the two given polar curves we have

$$a\sin\theta = a\cos\theta\,,$$

so $\tan\theta = 1$, i.e. $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. Now the curve $r = a\sin\theta$ can be parametrized by $\mathbf{r}_1\colon [0, 2\pi] \to \mathbb{R}^2$,

$$\mathbf{r}_1(t) = \langle a\sin t\cos t\,, a\sin t\sin t\rangle = \langle \frac{a}{2}\sin 2t\,, \frac{a}{2}(1 - \cos 2t)\rangle\,,$$

whose tangent vector is $\mathbf{r}_1{}'(t) = \langle a\cos 2t\,, a\sin 2t\rangle$. On the other hand, the curve $r = a\cos\theta$ can be parametrized by $\mathbf{r}_2\colon [0, 2\pi] \to \mathbb{R}^2$,

$$\mathbf{r}_2(t) = \langle a\cos t\cos t\,, a\cos t\sin t\rangle = \langle \frac{a}{2}(1 + \cos 2t), \frac{a}{2}\sin 2t\rangle\,,$$

whose tangent vector is $\mathbf{r}_2{}'(t) = \langle -a\sin 2t\,, a\cos 2t\rangle$. Now for $t = \frac{\pi}{4}$ and for $t = \frac{3\pi}{4}$, we have both

$$\mathbf{r}_1{}'\left(\frac{\pi}{4}\right) \cdot \mathbf{r}_2{}'\left(\frac{\pi}{4}\right) = \langle 0, a\rangle \cdot \langle -a, 0\rangle = 0 \qquad \text{and} \qquad \mathbf{r}_1{}'\left(\frac{3\pi}{4}\right) \cdot \mathbf{r}_2{}'\left(\frac{3\pi}{4}\right) = \langle 0, -a\rangle \cdot \langle a, 0\rangle = 0,$$

which shows that the tangent vectors are perpendicular to each other at each point of intersection. In other words, the two curves intersect at right angles. ■

22. (a) The given curve $r = f(\theta)$ can be parametrized by a vector-valued function $\mathbf{r}\colon I \to \mathbb{R}^2$,

$$\mathbf{r}(t) = \langle f(t)\cos t\,, f(t)\sin t\rangle.$$

At each point P on the curve with position vector $\overrightarrow{OP} = \mathbf{r}(t) = \langle f(t)\cos t\,, f(t)\sin t\rangle$, the tangent vector is given by

$$\mathbf{r}'(t) = \langle f'(t)\cos t - f(t)\sin t\,, f'(t)\sin t + f(t)\cos t\rangle.$$

Now α is the acute angle between $\mathbf{r}(t)$ and $\mathbf{r}'(t)$, so

$$\cos\alpha = \frac{|\mathbf{r}(t) \cdot \mathbf{r}'(t)|}{\|\mathbf{r}(t)\|\|\mathbf{r}'(t)\|} = \frac{|f(t)f'(t)\cos^2 t - f(t)^2\sin t\cos t + f(t)f'(t)\sin^2 t + f(t)^2\sin t\cos t|}{|f(t)|\sqrt{f(t)^2 + f'(t)^2}}$$

$$= \frac{|f(t)f'(t)|}{|f(t)|\sqrt{f(t)^2 + f'(t)^2}} = \frac{|f'(t)|}{\sqrt{f(t)^2 + f'(t)^2}} = \frac{|f'(\theta)|}{\sqrt{f(\theta)^2 + f'(\theta)^2}}$$

as the parameter t represents the angle θ in the polar coordinates. ■

(b) Now with the function $f(\theta) = e^{\theta}$, we have $f'(\theta) = e^{\theta}$; so by (a),

$$\cos\alpha = \frac{|f'(\theta)|}{\sqrt{f(\theta)^2 + f'(\theta)^2}} = \frac{e^{\theta}}{\sqrt{e^{2\theta} + e^{2\theta}}} = \frac{1}{\sqrt{2}}.$$

Therefore $\alpha = \frac{\pi}{4}$ for every θ, i.e. the angle is always $\frac{\pi}{4}$ at every point on the curve.

(c) If the angle α is always constant for all θ, then $\cos\alpha$ is also a constant, so by (a), there exists a constant c such that

$$\frac{|f'(\theta)|}{\sqrt{f(\theta)^2 + f'(\theta)^2}} = c \qquad \text{for all } \theta \in I.$$

Note that $c \in (0,1)$ (why?). From this we have $\sqrt{1 + \frac{f(\theta)^2}{f'(\theta)^2}} = \frac{1}{c}$, and so $1 - c^2 = \frac{f(\theta)^2}{f(\theta)^2 + f'(\theta)^2}$, i.e.

$$\frac{f'(\theta)}{f(\theta)} = \frac{\pm c}{\sqrt{1 - c^2}},$$

which is always a constant. (The $\pm$ sign on the right-hand side is independent of θ because the function on the left-hand side is continuous.) Now let k be this constant on the right-hand side, i.e. $\frac{f'(\theta)}{f(\theta)} = k$. Since $\frac{f'(\theta)}{f(\theta)} = \frac{d}{d\theta}\ln f(\theta)$, we have $\ln f(\theta) = k\theta + C_0$ for some constant C_0. Thus if we relabel $e^{C_0} = C$, then

$$f(\theta) = e^{k\theta + C_0} = Ce^{k\theta}$$

for all θ. ■