HKUST MATH 1014 L1 assignment 7 submission

MATH1014 Calculus II Problem Set 7
L01 (Spring 2024)

Problem Set 7

Note: The problem sets serve as additional exercise problems for your own practice.
Problem Set 7 covers materials from §8.2 – §8.4.

Q3

Let (a_n) be a sequence of positive real numbers.

Q3.a

Show that if $\sum_{k=1}^{+\infty} a_k$ converges, then $\sum_{k=1}^{+\infty} \frac{1}{a_k}$ diverges.

$$
\begin{aligned}
& \sum_{k=1}^{+\infty} a_k \text{ converges} \\
\implies & \lim_{k\to+\infty} a_k = 0 && \text{(tail test)} \\
\implies & \lim_{k\to+\infty} \frac{1}{a_k} = +\infty && (a_k > 0, \text{algebraic limit theorem}) \\
\implies & \sum_{k=1}^{+\infty} \frac{1}{a_k} \text{ diverges by the tail test}
\end{aligned}
$$

Q3.b

Show that if $\lim_{n\to+\infty} na_n = L > 0$, then $\sum_{k=1}^{+\infty} a_k$ diverges.

$$
\sum_{k=1}^{+\infty} \frac{1}{k} \text{ diverges by the } p\text{-test}
$$

$$
\begin{aligned}
& \lim_{n\to+\infty} na_n = L > 0 \\
\implies & \lim_{n\to+\infty} \frac{a_n}{\frac{1}{n}} = L > 0 \\
\implies & \sum_{k=1}^{+\infty} a_k \text{ diverges by the limit comparison test} && \left(a_n > 0, \frac{1}{n} > 0\right)
\end{aligned}
$$

Q3.c

Show that if $\sum_{k=1}^{+\infty} a_k$ converges, then $\sum_{k=1}^{+\infty} a_k^2$ converges. Is the converse true?

$$
\begin{aligned}
& \sum_{k=1}^{+\infty} a_k \text{ converges} \\
\implies & \lim_{k\to+\infty} a_k = 0
\end{aligned}
$$

$$
\begin{aligned}
& \lim_{n\to+\infty} \frac{a_k^2}{a_k} = \lim_{n\to+\infty} a_k = 0 \\
\implies & \sum_{k=1}^{+\infty} a_k^2 \text{ converges by the limit comparison test} && (a_k^2 > 0, a_k > 0)
\end{aligned}
$$

The converse is not true. Let $a_k = \dfrac{1}{k}$.

Then $\displaystyle\sum_{k=1}^{+\infty} a_k^2 = \sum_{k=1}^{+\infty} \frac{1}{k^2}$ converges by the p-test,

But $\displaystyle\sum_{k=1}^{+\infty} a_k = \sum_{k=1}^{+\infty} \frac{1}{k}$ diverges by the p-test.

Q3.d

Show that if $\sum_{k=1}^{+\infty} a_k^2$ converges, then $\sum_{k=1}^{+\infty} \frac{a_k}{k}$ converges.

Hint: AM-GM inequality.

$$
\begin{aligned}
& \sum_{k=1}^{+\infty} \frac{1}{k} \text{ diverges by the } p\text{-test and } \sum_{k=1}^{+\infty} a_k^2 \text{ converges} \\
\implies & \lim_{k\to+\infty} \frac{a_k^2}{\frac{1}{k}} = 0 && \left(a_k^2 > 0, \frac{1}{k} > 0, \text{contrapositive of the limit comparison test}\right) \\
\implies & \lim_{k\to+\infty} \frac{a_k}{\frac{1}{\sqrt{k}}} = \sqrt{0} && \left(a_k^2 > 0, \frac{1}{k} > 0, \text{algebraic limit theorem}\right) \\
\implies & \lim_{k\to+\infty} \frac{\frac{1}{k}}{\frac{1}{k}} \cdot \frac{a_k}{\frac{1}{\sqrt{k}}} = \lim_{k\to+\infty} \frac{\frac{a_k}{k}}{\frac{1}{k^{\frac{3}{2}}}} = 0
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{k=1}^{+\infty} \frac{1}{k^{\frac{3}{2}}} \text{ converges by the } p\text{-test} \\
\implies & \sum_{k=1}^{+\infty} \frac{a_k}{k} \text{ converges by the limit comparison test} && \left(\frac{1}{k^{\frac{3}{2}}} > 0, \frac{a_k}{k} > 0\right)
\end{aligned}
$$

Q7

For each of the following series, find all the values of $p \in \mathbb{R}$ such that the series converges.

Q7.a

$$
\sum_{k=1}^{+\infty} k^2 \sin^p \frac{1}{k}
$$

$$
\begin{aligned}
&\frac{1}{k} \in (0,1] \\
\implies &\sin^p \frac{1}{k} > 0 \\
\implies &k^2 \sin^p \frac{1}{k} > 0
\end{aligned}
$$

$$
\begin{aligned}
&k > 0 \\
\implies &k^{2-p} > 0
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{k\to+\infty} \frac{k^2 \sin^p \frac{1}{k}}{k^{2-p}} \\
= &\lim_{k\to+\infty} \frac{\sin^p \frac{1}{k}}{k^{-p}} \\
= &\lim_{k\to+\infty} \left(\frac{\sin \frac{1}{k}}{\frac{1}{k}}\right)^p \\
= &1^p \\
= &1
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k=1}^{+\infty} k^{2-p} \text{ converges by the } p\text{-test iff } p \in (3, +\infty) \\
&\sum_{k=1}^{+\infty} k^2 \sin^p \frac{1}{k} \text{ converges by the limit comparison test iff } p \in (3,+\infty) \qquad \left(k^{2-p} > 0, \frac{1}{k} \in (0,1] \implies k^2 \sin^p \frac{1}{k} > 0\right)
\end{aligned}
$$

Q7.b

$$
\sum_{k=2}^{+\infty} \frac{1}{(\ln \ln k)^{p \ln k}}
$$

$$
\sum_{k=2}^{+\infty} \frac{1}{(\ln \ln k)^{p \ln k}} \qquad \cdots (1)
$$

$$
\begin{aligned}
&\int_2^{+\infty} \frac{1}{(\ln \ln k)^{p \ln k}} \, \mathrm{d}k && \cdots (2) \\
= &\int_{\ln 2}^{+\infty} \frac{e^k}{(\ln k)^{pk}} \, \mathrm{d}k && \cdots (3) \quad (\text{change of variable: } \ln k \mapsto k)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k=1}^{+\infty} \frac{e^k}{(\ln k)^{pk}} && \cdots (4) \\
= &\sum_{k=1}^{+\infty} \left(\frac{e}{(\ln k)^p}\right)^k && \cdots (5)
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{k\to+\infty} \sqrt[k]{\left|\left(\frac{e}{(\ln k)^p}\right)^k\right|} && \cdots (6) \\
= &\lim_{k\to+\infty} \sqrt[k]{\left(\frac{e}{(\ln k)^p}\right)^k} && \left(k > 1 \implies \frac{e}{\ln k} > 0\right) \\
= &\lim_{k\to+\infty} \frac{e}{(\ln k)^p} \\
= &\begin{cases} 0, & p > 0 \\ e, & p = 0 \\ +\infty, & p < 0 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&(6) < 1 \text{ iff } p \in (0, +\infty) \\
\implies &(5) \text{ is convergent iff } p \in (0,+\infty) \text{ by the root test} \\
\implies &(4) \text{ is convergent iff } p \in (0,+\infty) \\
\implies &(3) \text{ is convergent iff } p \in (0,+\infty) \text{ by the integral test} \\
\implies &(2) \text{ is convergent iff } p \in (0,+\infty) \\
\implies &(1) \text{ is convergent iff } p \in (0,+\infty) \text{ by the integral test}
\end{aligned}
$$

Q7.c

$$\sum_{k=3}^{+\infty} \frac{1}{k(\ln k)(\ln\ln k)^p}$$

$$\sum_{k=3}^{+\infty} \frac{1}{k(\ln k)(\ln\ln k)^p} \qquad \cdots(1)$$

$$\begin{aligned}
&\int_3^{+\infty} \frac{1}{k(\ln k)(\ln\ln k)^p}\,\mathrm{d}k && \cdots(2) \\
&= \int_{\ln 3}^{+\infty} \frac{1}{k(\ln k)^p}\,\mathrm{d}k && (\text{change of variable: } \ln k \mapsto k) \\
&= \int_{\ln\ln 3}^{+\infty} \frac{1}{k^p}\,\mathrm{d}k && \cdots(3) \quad (\text{change of variable: } \ln k \mapsto k)
\end{aligned}$$

$$\begin{aligned}
&(3) \text{ is convergent iff } p \in (1, +\infty) \text{ by the } p\text{-test} \\
\implies &(2) \text{ is convergent iff } p \in (1, +\infty) \\
\implies &(1) \text{ is convergent iff } p \in (1, +\infty) \text{ by the integral test}
\end{aligned}$$

Q8

Let (a_n) be a sequence of real numbers, and define

$$a_n^+ := \max\{a_n, 0\} \qquad \text{and} \qquad a_n^- := \max\{-a_n, 0\}$$

for every n. Show that

Q8.a

If $\sum_{k=1}^{+\infty} a_k$ **converges absolutely**, then both $\sum_{k=1}^{+\infty} a_k^+$ and $\sum_{k=1}^{+\infty} a_k^-$ converge.

$$\begin{aligned}
&\sum_{k=1}^{+\infty} a_k \text{ converges absolutely} \\
\implies &\sum_{k=1}^{+\infty} |a_k| \text{ converges}
\end{aligned}$$

$$\begin{aligned}
&a_n \geq 0 \\
\implies &a_n^+ = a_n = |a_n| \\
&a_n < 0 \\
\implies &a_n^+ = 0 < |a_n|
\end{aligned}$$

$$\begin{aligned}
&a_n^+ \leq |a_n| \\
\implies &\sum_{k=1}^{+\infty} a_n^+ \text{ converges by the direct comparison test} \qquad \left(a_n^+ \geq 0, |a_n| \geq 0\right)
\end{aligned}$$

$$\begin{aligned}
&a_n \leq 0 \\
\implies &a_n^- = -a_n = |a_n| \\
&a_n > 0 \\
\implies &a_n^- = 0 < |a_n|
\end{aligned}$$

$$\begin{aligned}
&a_n^- \leq |a_n| \\
\implies &\sum_{k=1}^{+\infty} a_n^- \text{ converges by the direct comparison test} \qquad \left(a_n^- \geq 0, |a_n| \geq 0\right)
\end{aligned}$$

Q8.b

If $\sum_{k=1}^{+\infty} a_k$ **converges conditionally**, then both $\sum_{k=1}^{+\infty} a_k^+$ and $\sum_{k=1}^{+\infty} a_k^-$ diverge.

$$\sum_{k=1}^{+\infty} a_k \text{ converges conditionally}$$
$$\implies \sum_{k=1}^{+\infty} |a_k| \text{ diverges}$$

$$a_n \geq 0$$
$$\implies a_n^+ = a_n$$
$$a_n < 0$$
$$\implies a_n^- = -a_n \implies a_n = -a_n^-$$
$\therefore$ (a_n) can be rewritten as a sequence in terms of a_n^+ and $-a_n^-$ only.
$(|a_n|)$ can be rewritten as a sequence in terms of a_n^+ and a_n^- only.

$$a_k^+ \geq 0 \implies a_k^+ = |a_k^+|$$
$$a_k^- \geq 0 \implies a_k^- = |a_k^-|$$

assume both $\sum_{k=1}^{+\infty} a_k^+ = L^+$ and $\sum_{k=1}^{+\infty} a_k^- = L^-$ converge

$\implies$ both $\sum_{k=1}^{+\infty} a_k^+$ and $\sum_{k=1}^{+\infty} a_k^-$ converge absolutely

$$\implies \sum_{k=1}^{+\infty} |a_n| = L^+ + L^- \text{ converges}$$

...since absolutely converging sequences can be rearranged without changing their sums.

the above conclusion contradicts that $\sum_{k=1}^{+\infty} |a_k|$ diverges

$\implies$ both $\sum_{k=1}^{+\infty} a_k^+$ and $\sum_{k=1}^{+\infty} a_k^-$ cannot converge simultaneously $\cdots(1)$

Without loss of generality,
assume one of the sum converges while the other diverges:

Assume $\sum_{k=1}^{+\infty} a_k^+ = L^+$ converges and $\sum_{k=1}^{+\infty} a_k^-$ diverges.

Then, $\sum_{k=1}^{n\in\mathbb{Z}_{\geq 1}} a_k^+ \leq L^+$ by the monotone convergence theorem.

Consider $\sum_{k=1}^{n\in\mathbb{Z}_{\geq 1}} a_k = \sum_{k=1}^{n} a_k^+ - \sum_{k=1}^{n} a_k^-$.

$$\begin{aligned}\sum_{k=1}^{n\in\mathbb{Z}_{\geq 1}} a_k &= \sum_{k=1}^{n} a_k^+ - \sum_{k=1}^{n} a_k^-. \\ &\leq L^+ - \sum_{k=1}^{n} a_k^-\end{aligned} \qquad \cdots(2)$$

$a_k^- \geq 0$ and $\sum_{k=1}^{+\infty} a_k^-$ diverges

$\implies \sum_{k=1}^{+\infty} a_k^- = +\infty$ as the partial sums are increasing

$\sum_{k=1}^{+\infty} a_k^- = +\infty$ diverges

$\implies L^+ - \sum_{k=1}^{+\infty} a_k^- = -\infty$ diverges by the algebraic limit theorem

$\implies \sum_{k=1}^{+\infty} a_k \leq -\infty$ diverges by (2)

the above conclusion contradicts that $\sum_{k=1}^{+\infty} a_k$ converges

$$\implies \sum_{k=1} a_k^+ \text{ cannot converge and } \sum_{k=1} a_k^- \text{ cannot diverge simultaneously} \qquad \cdots(3)$$

$$\text{Similarly, } \sum_{k=1}^{+\infty} a_k^- \text{ cannot converge and } \sum_{k=1}^{+\infty} a_k^+ \text{ cannot diverge simultaneously} \qquad \cdots(4)$$

$(1), (3), (4)$ combined implies that both integrals must diverge simultaneously.

Q9

For each of the following series, determine whether it diverges, converges absolutely or converges conditionally.

Q9.b

$$\sum_{k=0}^{+\infty}(-1)^{k+1}\left(\sqrt{k+1}-\sqrt{k}\right)$$

$$\sum_{k=0}^{+\infty}(-1)^{k+1}\left(\sqrt{k+1}-\sqrt{k}\right) \qquad \cdots(1)$$

$$\begin{aligned} a_k &:= \sqrt{k+1}-\sqrt{k} \qquad k \geq 0 \\ a_{k+1} &= \sqrt{k+2}-\sqrt{k+1} \end{aligned}$$

$$a_k > 0 \qquad (\sqrt{*} \text{ is increasing})$$

$$\begin{aligned} & |a_k| - |a_{k+1}| \\ &= \left|\sqrt{k+1}-\sqrt{k}\right| - \left|\sqrt{k+2}-\sqrt{k+1}\right| \\ &= \sqrt{k+1}-\sqrt{k}+\sqrt{k+2}-\sqrt{k+1} \\ &= \sqrt{k+2}-\sqrt{k} \\ &> 0 \\ &\therefore |a_k| \geq |a_{k+1}| \qquad \cdots(2) \end{aligned}$$

$$\begin{aligned} & \lim_{k\to+\infty} a_k \\ &= \lim_{k\to+\infty}\left(\sqrt{k+1}-\sqrt{k}\right) \\ &= \lim_{k\to+\infty}\frac{(k+1)-k}{\sqrt{k+1}+\sqrt{k}} \\ &= \lim_{k\to+\infty}\frac{1}{\sqrt{k+1}+\sqrt{k}} \\ &= 0 \qquad \cdots(3) \end{aligned}$$

(1) is an alternating series in terms of a_k.
(1) converges by the alternating series test due to $(2), (3)$.

$$\begin{aligned} & \sum_{k=0}^{+\infty}\left|(-1)^{k+1}\left(\sqrt{k+1}-\sqrt{k}\right)\right| \qquad \cdots(4) \\ &= \sum_{k=0}^{+\infty}\left(\sqrt{k+1}-\sqrt{k}\right) \qquad \left(\left|(-1)^{k+1}\right|=1, \sqrt{k+1}-\sqrt{k}>0\right) \\ &= \lim_{n\to+\infty}\sum_{k=0}^{n}\left(\sqrt{k+1}-\sqrt{k}\right) \\ &= \lim_{n\to+\infty}\sqrt{n+1} \qquad \text{(telescope)} \\ &= +\infty \\ &\therefore (4) \text{ diverges} \end{aligned}$$

$\because$ (1) converges but (4) diverges
$\therefore$ (1) converges conditionally

Q9.d

$$\sum_{k=2}^{+\infty} \frac{(-1)^k}{\sqrt{k}+(-1)^k}$$

$$\begin{aligned}
&\sum_{k=2}^{+\infty} \frac{(-1)^k}{\sqrt{k}+(-1)^k} && \cdots(1) \\
=&\sum_{k=2}^{+\infty} \frac{(-1)^k\left(\sqrt{k}-(-1)^k\right)}{k-1} \\
=&\sum_{k=2}^{+\infty} \frac{(-1)^k\sqrt{k}-1}{k-1} \\
=&\sum_{k=2}^{+\infty} \left(\frac{(-1)^k\sqrt{k}}{k-1}-\frac{1}{k-1}\right) && \cdots(2)
\end{aligned}$$

Consider the sum $\sum_{k=2}^{+\infty} \frac{(-1)^k\sqrt{k}}{k-1}$.

It is alternating series with $a_k = \frac{\sqrt{k}}{k-1}$.

Also, $\frac{\mathrm{d}a_k}{\mathrm{d}k} = \frac{0.5k^{-0.5}(k-1)-\sqrt{k}}{(k-1)^2} = \frac{-0.5\sqrt{k}-0.5k^{-0.5}}{(k-1)^2} < 0$.

Thus a_k is strictly decreasing when $k \geq 2$.

Furthermore, $\lim_{k\to+\infty} \frac{\sqrt{k}}{k-1} = \lim_{k\to+\infty} \frac{\sqrt{\frac{k}{k^2}}}{1-\frac{1}{k}} = 0$.

By the alternating series test,
the sum being considered converges.

Consider the sum $\sum_{k=2}^{+\infty} \frac{1}{k-1} = \sum_{k=1}^{+\infty} \frac{1}{k}$.

The sum diverges by the p-test.

Finally, by above and the algebraic limit theorem,
(2) equals $\sum_{k=2}^{+\infty} \frac{(-1)^k\sqrt{k}}{k-1} - \sum_{k=2}^{+\infty} \frac{1}{k-1}$ and diverges.

Thus (1) diverges.

Q11

Find the radius and interval of convergence for each of the following power series.

Q11.b

$$\sum_{k=1}^{+\infty} \frac{x^k}{2^k k^2}$$

The center of the power series is $x = 0$.

The coefficient of the power series is $c_k = \dfrac{1}{2^k k^2}$.

$$\begin{aligned}
&\text{radius of convergence} \\
&= \lim_{k \to +\infty} \left| \frac{c_k}{c_{k+1}} \right| \\
&= \lim_{k \to +\infty} \left| \frac{2^{k+1}(k+1)^2}{2^k k^2} \right| \\
&= 2 \lim_{k \to +\infty} \left| \frac{k^2 + 2k + 1}{k^2} \right| \\
&= 2 \lim_{k \to +\infty} \left| \frac{1 + 2k^{-1} + k^{-2}}{1} \right| \\
&= 2
\end{aligned}$$

When $x = 2$,

$$\begin{aligned}
&\sum_{k=1}^{+\infty} \frac{2^k}{2^k k^2} \\
&= \sum_{k=1}^{+\infty} \frac{1}{k^2}
\end{aligned}$$

which converges by the p-test.

As $k \geq 1 \implies \dfrac{1}{k^2} > 0$,

The integral converges absolutely.

Then its alternating series counterpart,

$$\begin{aligned}
&\sum_{k=1}^{+\infty} \frac{(-1)^k 2^k}{2^k k^2} \\
&= \sum_{k=1}^{+\infty} \frac{(-2)^k}{2^k k^2}
\end{aligned}$$

also converges by the absolute convergence test,

and is the expression when $x = -2$.

interval of convergence $= [-2, 2]$

Q14

For each of the following power series, evaluate its sum whenever it converges. What happens at the end-points of its interval of convergence?

Hint: In each part, apply term-wise differentiation or integration on some power series whose sum is well-known.

Q14.b

$$\sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (x-1)^k$$

The center of the power series is
$x = 1.$
The coefficients of the power series are
$c_k = \frac{1}{k(k-1)}.$

$$
\begin{aligned}
&\text{radius of convergence} \\
&= \lim_{k\to+\infty} \left| \frac{c_k}{c_{k+1}} \right| \\
&= \lim_{k\to+\infty} \left| \frac{\frac{1}{k(k-1)}}{\frac{1}{(k+1)k}} \right| \\
&= \lim_{k\to+\infty} \left| \frac{k+1}{k-1} \right| \\
&= \lim_{k\to+\infty} \frac{k+1}{k-1} && (k \to +\infty) \\
&= \lim_{k\to+\infty} \frac{1+\frac{1}{k}}{1-\frac{1}{k}} \\
&= 1
\end{aligned}
$$

When $x = 2$,

$$
\begin{aligned}
&\sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (2-1)^k \\
&= \sum_{k=2}^{+\infty} \frac{1}{k(k-1)} \\
&= \sum_{k=2}^{+\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\
&= \lim_{k\to+\infty} \left(1 - \frac{1}{k} \right) && \text{(telescope)} \\
&= 1
\end{aligned}
$$

The sum converges.
When $x = 0$,

$$
\begin{aligned}
&\sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (0-1)^k \\
&= \sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (-1)^k \\
&= \sum_{k=2}^{+\infty} \frac{(-1)^k}{k(k-1)} (2-1)^k
\end{aligned}
$$

... which is the alternating counterpart of the series
when $x = 2$.
Thus, the sum converges by the absolute convergence test.
interval of convergence $= [0, 2]$

$$
\begin{aligned}
S(x) &:= \sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (x-1)^k \qquad x \in [0,2] \\
S(2) &= 1 \\
S'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (x-1)^k \qquad x \in (0,2) \\
&= \sum_{k=2}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{k(k-1)} (x-1)^k && \text{(power series can be differentiated term-wise)} \\
&= \sum_{k=2}^{+\infty} \frac{1}{k-1} (x-1)^{k-1} \\
S'(1) &= \sum_{k=2}^{+\infty} \frac{1}{k-1} (1-1)^{k-1} \\
&= \sum_{k=2}^{+\infty} 0 && (k-1 > 0) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
S''(x) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sum_{k=2}^{+\infty}\frac{1}{k(k-1)}(x-1)^k \qquad x\in(0,2) \\
&= \sum_{k=2}^{+\infty}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{1}{k(k-1)}(x-1)^k && \text{(power series can be differentiated term-wise)} \\
&= \sum_{k=2}^{+\infty}(x-1)^{k-2} \\
&= \lim_{k\to+\infty}\frac{(x-1)^{k-1}-(x-1)^0}{(x-1)-1} \\
&= \lim_{k\to+\infty}\frac{(x-1)^{k-1}-1}{x-2} \\
&= \frac{-1}{x-2} && (x\in(0,2)\implies |x-1|<1) \\
&= \frac{1}{2-x} \\
S'(x) &= \int S''(x)\,\mathrm{d}x \\
&= \int\frac{1}{2-x}\,\mathrm{d}x \\
&= -\ln|2-x|+C \\
S'(1) &= -\ln|2-1|+C \\
0 &= C \\
C &= 0 \\
S'(x) &= -\ln|2-x| \qquad x\in(0,2) \\
S(x) &= \int S'(x)\,\mathrm{d}x \\
&= -\int\ln|2-x|\,\mathrm{d}x \\
&= \int\ln|u|\,\mathrm{d}u && (u:=2-x) \\
&= u\ln|u|-\int\frac{u}{u}\,\mathrm{d}u \\
&= u\ln|u|-u+C \\
&= (2-x)\ln|2-x|-2+x+C \\
\\
&\lim_{x\to2^-}(2-x)\ln|2-x| \\
&= \lim_{x\to2^-}\frac{\ln|2-x|}{\frac{1}{2-x}} \\
&= \lim_{x\to2^-}\frac{-\frac{1}{2-x}}{\frac{1}{(2-x)^2}} && \text{(L'Hopital rule)} \\
&= \lim_{x\to2^-}(x-2) \\
&= 0 \\
\\
S(2) &= \lim_{x\to2^-}((2-x)\ln|2-x|-2+x+C) && \text{(make } S(x) \text{ continuous)} \\
1 &= C && \text{(algebraic limit theorem)} \\
C &= 1 \\
S(x) &= \begin{cases}(2-x)\ln|2-x|+x-1, & x\in[0,2) \\ 1, & x=2\end{cases} \\
&= \begin{cases}(2-x)\ln(2-x)+x-1, & x\in[0,2) \\ 1, & x=2\end{cases} && (2-x>0)
\end{aligned}
$$

The sum converges on the endpoints of the interval of convergence.

Q14.e

$$\sum_{k=1}^{+\infty}\frac{k}{k+1}x^k$$

The center of the power series is
$x = 0$.
The coefficients of the power series are
$c_k = \dfrac{k}{k+1}$.

radius of convergence

$$= \lim_{k\to+\infty}\left|\frac{c_k}{c_{k+1}}\right|$$
$$= \lim_{k\to+\infty}\left|\frac{\frac{k}{k+1}}{\frac{k+1}{k+2}}\right|$$
$$= \lim_{k\to+\infty}\left|\frac{k(k+2)}{(k+1)^2}\right|$$
$$= \lim_{k\to+\infty}\left|\frac{(k+1)^2-1}{(k+1)^2}\right|$$
$$= \lim_{k\to+\infty}\left|1-\frac{1}{(k+1)^2}\right|$$
$$= 1$$

When $x = -1$,
$$\sum_{k=1}^{+\infty}\frac{k}{k+1}(-1)^k$$
Consider $a_k = \dfrac{k}{k+1}(-1)^k$.
Consider the subsequence given by $b_k = a_{2k} = \dfrac{2k}{2k+1}(-1)^{2k}$.
$$\lim_{k\to+\infty}\frac{2k}{2k+1}(-1)^{2k} = \lim_{k\to+\infty}\frac{2}{2+\frac{1}{k}} = 1$$
Consider the subsequence given by $c_k = a_{2k-1} = \dfrac{2k-1}{2k}(-1)^{2k-1}$.
$$\lim_{k\to+\infty}\frac{2k-1}{2k}(-1)^{2k-1} = \lim_{k\to+\infty}\left(-\frac{2-\frac{1}{k}}{2}\right) = -1$$
As the two subsequences of $(a_k)_{k\in\mathbb{N}}$ approaches different values as $k \to +\infty$,
$\lim_{k\to+\infty}\dfrac{k}{k+1}(-1)^k$ does not exist.
Thus, $\sum_{k=1}^{+\infty}\dfrac{k}{k+1}(-1)^k$ diverges by the tail test.
When $x = 1$,
$$\sum_{k=1}^{+\infty}\frac{k}{k+1}1^k$$
$$\lim_{k\to+\infty}\frac{k}{k+1}1^k$$
$$= \lim_{k\to+\infty}\frac{1}{1+\frac{1}{k}}$$
$$= 1$$
Thus, $\sum_{k=1}^{+\infty}\dfrac{k}{k+1}1^k$ diverges by the tail test.
interval of convergence $= (-1, 1)$

$$\sum_{k=1}^{+\infty}x^k \qquad x \in (-1,1)$$
$$= \lim_{k\to+\infty}\frac{x^{k+1}-x}{x-1}$$
$$= \frac{x}{1-x} \qquad (|x|<1)$$

$$S(x) := \sum_{k=1}^{+\infty}\frac{k}{k+1}x^k \qquad x \in (-1,1)$$
$+\infty$

$$
\begin{aligned}
S(0) &= \sum_{k=1}^{+\infty} \frac{k}{k+1} 0^k \\
&= \sum_{k=1}^{+\infty} 0 \\
&= 0 \\
S(x) &= \sum_{k=1}^{+\infty} \frac{k}{k+1} x^k \\
&= \sum_{k=1}^{+\infty} \left(1 - \frac{1}{k+1}\right) x^k \\
&= \sum_{k=1}^{+\infty} x^k + \sum_{k=1}^{+\infty} \left(1 - \frac{1}{k+1}\right) x^k - \sum_{k=1}^{+\infty} x^k \\
&= \sum_{k=1}^{+\infty} x^k - \sum_{k=1}^{+\infty} \frac{1}{k+1} x^k && \text{(algebraic limit theorem)} \\
&= \frac{x}{1-x} - \sum_{k=1}^{+\infty} \frac{1}{k+1} x^k
\end{aligned}
$$

```
Not using differential equations:
```

$$\begin{aligned}
T(x) &:= \sum_{k=1}^{+\infty} \frac{1}{k+1} x^k \qquad x \in (-1,1) \\
&= \begin{cases} \frac{1}{x} \sum_{k=1}^{+\infty} \frac{1}{k+1} x^{k+1}, & x \in (-1,0) \cup (0,1) \\ 0, & x = 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
U(x) &:= \sum_{k=1}^{+\infty} \frac{1}{k+1} x^{k+1} \qquad x \in (-1,1) \\
U'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=1}^{+\infty} \frac{1}{k+1} x^{k+1} \\
&= \sum_{k=1}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{k+1} x^{k+1} && \text{(power series can be differentiated term-wise)} \\
&= \sum_{k=1}^{+\infty} x^k \\
&= \frac{x}{1-x} \\
U(0) &= \sum_{k=1}^{+\infty} \frac{1}{k+1} 0^{k+1} \\
&= \sum_{k=1}^{+\infty} 0 && (k+1>0) \\
&= 0 \\
U(x) &= \int U'(x)\,\mathrm{d}x \\
&= \int \frac{x}{1-x}\,\mathrm{d}x \\
&= \int \frac{u-1}{u}\,\mathrm{d}u \qquad u := 1-x \\
&= u - \ln|u| + C \\
&= 1 - x - \ln|1-x| + C \\
&= -x - \ln|1-x| + C && (C \text{ is arbitrary}) \\
U(0) &= -0 - \ln|1-0| + C \\
C &= 0 \\
U(x) &= -x - \ln|1-x| \qquad x \in (-1,1)
\end{aligned}$$

$$\begin{aligned}
T(x) &= \begin{cases} \frac{1}{x} \sum_{k=1}^{+\infty} \frac{1}{k+1} x^{k+1}, & x \in (-1,0) \cup (0,1) \\ 0, & x = 0 \end{cases} \\
&= \begin{cases} \frac{1}{x} U(x), & x \in (-1,0) \cup (0,1) \\ 0, & x = 0 \end{cases} \\
&= \begin{cases} -1 - \frac{\ln|1-x|}{x}, & x \in (-1,0) \cup (0,1) \\ 0, & x = 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
S(x) &= \frac{x}{1-x} - \sum_{k=1}^{+\infty} \frac{1}{k+1} x^k \\
&= \frac{x}{1-x} - T(x) \\
&= \begin{cases} \frac{x}{1-x} + 1 + \frac{\ln|1-x|}{x}, & x \in (-1,0) \cup (0,1) \\ \frac{0}{1-0}, & x = 0 \end{cases} \\
&= \begin{cases} \frac{x+1-x}{1-x} + \frac{\ln|1-x|}{x}, & x \in (-1,0) \cup (0,1) \\ 0, & x = 0 \end{cases} \\
&= \begin{cases} \frac{1}{1-x} + \frac{\ln|1-x|}{x}, & x \in (-1,0) \cup (0,1) \\ 0, & x = 0 \end{cases}
\end{aligned}$$

The sum diverges on the endpoints of the interval of convergence.

Alternatively, using differential equations:

$$
\begin{aligned}
S'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{1-x} - \sum_{k=1}^{+\infty}\frac{1}{k+1}x^k\right) \\
&= \frac{(1-x)+x}{(1-x)^2} - \sum_{k=1}^{+\infty}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{k+1}x^k && \text{(power series can be differentiated term-wise)} \\
&= \begin{cases} \frac{1}{(1-x)^2} - \sum_{k=1}^{+\infty}\frac{k}{k+1}x^{k-1}, & x \in (-1,0)\cup(0,1) \\ 1-\frac{1}{2}, & x = 0 \end{cases} \\
&= \begin{cases} \frac{1}{(1-x)^2} - \frac{1}{x}\sum_{k=1}^{+\infty}\frac{k}{k+1}x^{k}, & x \in (-1,0)\cup(0,1) \\ \frac{1}{2}, & x = 0 \end{cases} \\
&= \begin{cases} \frac{1}{(1-x)^2} - \frac{S(x)}{x}, & x \in (-1,0)\cup(0,1) \\ \frac{1}{2}, & x = 0 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
y &:= S(x) \qquad x \in (-1,0)\cup(0,1) \\
y' &= \frac{1}{(1-x)^2} - \frac{y}{x}
\end{aligned}
$$

The above is a linear ordinary differential equation.
An unique solution exists on each of $(-1,0)$ and $(0,1)$.

Solve the homogeneous equation.

$$
\begin{aligned}
y' &= -\frac{y}{x} \qquad x \in (-1,0)\cup(0,1) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{y}{x} \\
\frac{1}{y}\,\mathrm{d}y &= -\frac{1}{x}\,\mathrm{d}x \\
\int\frac{1}{y}\,\mathrm{d}y &= -\int\frac{1}{x}\,\mathrm{d}x \\
\ln|y| &= -\ln|x| + C \\
e^{\ln|y|} &= e^{-\ln|x|+C} \\
|y| &= \frac{e^C}{|x|} \\
y &= \pm\frac{e^C}{|x|} \\
y &= \frac{c_1}{x} \qquad c_1 \in \mathbb{R}_{\neq 0}
\end{aligned}
$$

When $c_1 = 0$,

$$
\begin{aligned}
y &= \frac{0}{x} \qquad x \in (-1,0)\cup(0,1) \\
y &= 0 \\
y' &= 0
\end{aligned}
$$

... which satisfies the homogeneous equation.

$$
y = \frac{c_1}{x} \qquad c_1 \in \mathbb{R}
$$

Different c_1 can be chosen on each of $(-1,0)$ and $(0,1)$.

Solve the inhomogeneous equation.

$$
\begin{aligned}
y &= \frac{1}{x}\int\frac{\det\left[\frac{1}{(1-x)^2}\right]}{\det\left[\frac{1}{x}\right]}\,\mathrm{d}x \qquad x \in (-1,0)\cup(0,1) && \text{(variation of parameters)} \\
&= \frac{1}{x}\int\frac{x}{(1-x)^2}\,\mathrm{d}x \\
&= -\frac{1}{x}\int\frac{-u+1}{u^2}\,\mathrm{d}u \qquad u := 1-x \\
&= -\frac{1}{x}\left(-\ln|u| - \frac{1}{u} + c_1\right) \qquad c_1 \in \mathbb{R} \\
&= \frac{\ln|1-x|}{x} + \frac{1}{x(1-x)} - \frac{c_1}{x}
\end{aligned}
$$

Different c_1 can be chosen on each of $(-1,0)$ and $(0,1)$, which will be denoted c_1^- and c_1^+ respectively below.

$$
\lim_{x\to 0^+}\frac{\ln|1-x|}{x}
$$

ln(1 − x)

$$
\begin{aligned}
&= \lim_{x\to 0^+} \frac{\ln(1-x)}{x} && (1-x \geq 0)\\
&= \lim_{x\to 0^+} \frac{-\frac{1}{1-x}}{1} && \text{(L'Hopital rule)}\\
&= -1\\
&\lim_{x\to 0^-} \frac{\ln|1-x|}{x}\\
&= \lim_{x\to 0^-} \frac{\ln(x-1)}{x} && (1-x \leq 0)\\
&= \lim_{x\to 0^-} \frac{\frac{1}{x-1}}{1} && \text{(L'Hopital rule)}\\
&= -1\\
&\lim_{x\to 0} \frac{\ln|1-x|}{x} = -1
\end{aligned}
$$

$$
\begin{aligned}
S(0) &= \lim_{x\to 0^+} \left(\frac{\ln|1-x|}{x} + \frac{1}{x(1-x)} - \frac{c_1^+}{x}\right) && \text{(make } S(x) \text{ continuous)}\\
0 &= -1 + \lim_{x\to 0^+} \frac{1 - c_1^+ + c_1^+ x}{x(1-x)} && \text{(algebraic limit theorem)}\\
1 &= \lim_{x\to 0^+} \frac{c_1^+}{(1-x)-x} && \text{(L'Hoptial rule)}\\
1 &= c_1^+\\
c_1^+ &= 1\\
S(0) &= \lim_{x\to 0^-} \left(\frac{\ln|1-x|}{x} + \frac{1}{x(1-x)} - \frac{c_1^-}{x}\right) && \text{(make } S(x) \text{ continuous)}\\
0 &= -1 + \lim_{x\to 0^-} \frac{1 - c_1^- + c_1^- x}{x(1-x)} && \text{(algebraic limit theorem)}\\
1 &= \lim_{x\to 0^-} \frac{c_1^-}{(1-x)-x} && \text{(L'Hopital rule)}\\
1 &= c_1^-\\
c_1^- &= 1
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
S(x) &= \begin{cases} \frac{\ln|1-x|}{x} + \frac{1}{x(1-x)} - \frac{1}{x}, & x \in (-1,0)\cup(0,1)\\ 0, & x = 0\end{cases}\\
&= \begin{cases} \frac{\ln|1-x|}{x} + \frac{1-1+x}{x(1-x)}, & x \in (-1,0)\cup(0,1)\\ 0, & x = 0\end{cases}\\
&= \begin{cases} \frac{\ln|1-x|}{x} + \frac{1}{1-x}, & x \in (-1,0)\cup(0,1)\\ 0, & x = 0\end{cases}
\end{aligned}
$$

The sum diverges on the endpoints of the interval of convergence.