# HKUST MATH 1014 L1 assignment 2 submission

## 1

Let $n$ be a positive integer. Evaluate each of the following limits.

### 1.a

$$\lim_{x \to 0} \frac{1}{x^n} \int_0^{x^n} \cos\left(t^2\right) dt$$

$$
\begin{aligned}
\lim_{x \to 0} \frac{1}{x^n} \int_0^{x^n} \cos\left(t^2\right) dt &= \lim_{x \to 0} \int_0^1 \cos\left(x^{2n} t^2\right) dt && \left(\text{change of variable } \frac{t}{x^n} \mapsto t\right) \\
&= \int_0^1 \lim_{x \to 0} \cos\left(x^{2n} t^2\right) dt && (\text{dominated convergence theorem; dominated by } f(x) = 1 \text{ on } [0,1]) \\
&= \int_0^1 \cos\left(\left(\lim_{x \to 0} x\right)^{2n} t\right) && \left(f(x) = \cos\left(x^{2n} t\right) \text{ is continuous at } 0\right) \\
&= \int_0^1 \cos 0 \, dt && (n > 0) \\
&= \int_0^1 dt \\
&= 1
\end{aligned}
$$

### 1.b

$$\lim_{x \to 0} \frac{1}{x^n} \int_0^{x^n} \cos\left(x^2 t\right) dt$$

$$
\begin{aligned}
\lim_{x \to 0} \frac{1}{x^n} \int_0^{x^n} \cos\left(x^2 t\right) d &= \lim_{x \to 0} \int_0^1 \cos\left(x^{n+2} t\right) dt && \left(\text{change of variable } \frac{t}{x^n} \mapsto t\right) \\
&= \int_0^1 \lim_{x \to 0} \cos\left(x^{n+2} t\right) dt && (\text{dominated convergence theorem; dominated by } f(x) = 1 \text{ on } [0,1]) \\
&= \int_0^1 \cos\left(\left(\lim_{x \to 0} x\right)^{n+2} t\right) dt && \left(f(x) = \cos\left(x^{n+2} t\right) \text{ is continuous at } 0\right) \\
&= \int_0^1 \cos 0 \, dt && (n + 2 > 0) \\
&= \int_0^1 dt \\
&= 1
\end{aligned}
$$

## 2

Let $f, g : \mathbb{R} \to \mathbb{R}$ be **increasing** continuous functions, and let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$F(x) = x \int_0^x f(t)g(t) \, dt - \left(\int_0^x f(t) \, dt\right)\left(\int_0^x g(t) \, dt\right)$$

.

### 2.a

Show that $F$ is differentiable on $\mathbb{R}$ and

$$F'(x) = \int_0^x (f(x) - f(t))(g(x) - g(t)) \, dt$$

.

$$f(t) \in C(\mathbb{R}, \mathbb{R}) \implies X(x) := \int_0^x f(t)\,\mathrm{d}t \in C^1(\mathbb{R}, \mathbb{R}) \qquad \text{(first fundamental theorem of calculus)}$$

$$g(t) \in C(\mathbb{R}, \mathbb{R}) \implies Y(x) := \int_0^x g(t)\,\mathrm{d}t \in C^1(\mathbb{R}, \mathbb{R}) \qquad \text{(first fundamental theorem of calculus)}$$

$$f(t), g(t) \in C(\mathbb{R}, \mathbb{R}) \implies f(t)g(t) \in C(\mathbb{R}, \mathbb{R})$$

$$\implies Z(x) := \int_0^x f(t)g(t)\,\mathrm{d}t \in C^1(\mathbb{R}, \mathbb{R}) \qquad \text{(first fundamental theorem of calculus)}$$

$$x, Z(x) \in C^1(\mathbb{R}, \mathbb{R}) \implies xZ(x) \in C^1(\mathbb{R}, \mathbb{R}) \qquad \text{(product rule)}$$

$$X(x), Y(x) \in C^1(\mathbb{R}, \mathbb{R}) \implies X(x)Y(x) \in C^1(\mathbb{R}, \mathbb{R}) \qquad \text{(product rule)}$$

$$xZ(x), X(x)Y(x) \in C^1(\mathbb{R}, \mathbb{R}) \implies F(x) = xZ(x) - X(x)Y(x) \in C^1(\mathbb{R}, \mathbb{R}) \qquad \text{(linearity of differentiation)}$$

$$\therefore F(x) \text{ is differentiable on } \mathbb{R}.$$

$$F(x) = x\int_0^x f(t)g(t)\,\mathrm{d}t - \left(\int_0^x f(t)\,\mathrm{d}t\right)\left(\int_0^x g(t)\,\mathrm{d}t\right)$$

$$= xZ(x) - X(x)Y(x)$$

$$F'(x) = Z(x) + xZ'(x) - X'(x)Y(x) - X(x)Y'(x)$$

$$= \int_0^x f(t)g(t)\,\mathrm{d}t + xf(x)g(x) - f(x)\int_0^x g(t)\,\mathrm{d}t - g(x)\int_0^x f(t)\,\mathrm{d}t \qquad \text{(first fundamental theorem of calculus)}$$

$$= \int_0^x f(t)g(t)\,\mathrm{d}t + \int_0^x f(x)g(x)\,\mathrm{d}t - \int_0^x f(x)g(t)\,\mathrm{d}t - \int_0^x f(t)g(x)\,\mathrm{d}t$$

$$= \int_0^x (f(t)g(t) + f(x)g(x) - f(x)g(t) - f(t)g(x))\,\mathrm{d}t \qquad \text{(linearity of integral)}$$

$$= \int_0^x (f(x)(g(x) - g(t)) + f(t)(g(t) - g(x)))\,\mathrm{d}t$$

$$= \int_0^x (f(x) - f(t))(g(x) - g(t))\,\mathrm{d}t$$

## 2.b

Using the result from 2.a, find the global minimum of $F$ on $\mathbb{R}$.

Solve for $F'(x) = 0$.

When $x = 0$, obviously $F'(x) = 0$.

When $x > 0$, $t \in [0, x]$, then

$$x \geq t \implies f(x) - f(t) \geq 0 \qquad (f \text{ is increasing})$$

$$x \geq t \implies g(x) - g(t) \geq 0 \qquad (g \text{ is increasing})$$

$$f(x) - f(t) \geq 0 \wedge g(x) - g(t) \geq 0 \implies (f(x) - f(t))(g(x) - g(t)) \geq 0$$

$$\implies \int_0^x (f(x) - f(t))(g(x) - g(t))\,\mathrm{d}t \geq 0 \qquad \text{(integrand is nonnegative; } x > 0\text{)}$$

$$\implies F'(x) \geq 0$$

When $x < 0$, $t \in [x, 0]$, then

$$x \leq t \implies f(x) - f(t) \leq 0 \qquad (f \text{ is increasing})$$

$$x \leq t \implies g(x) - g(t) \leq 0 \qquad (g \text{ is increasing})$$

$$f(x) - f(t) \leq 0 \wedge g(x) - g(t) \leq 0 \implies (f(x) - f(t))(g(x) - g(t)) \geq 0$$

$$\implies \int_x^0 (f(x) - f(t))(g(x) - g(t))\,\mathrm{d}t \geq 0 \qquad \text{(integrand is nonnegative; } 0 > x\text{)}$$

$$\implies \int_0^x (f(x) - f(t))(g(x) - g(t))\,\mathrm{d}t \leq 0$$

$$\implies F'(x) \leq 0$$

The above shows that $F(x)$ is decreasing on $(-\infty, 0)$, stationary on $\{0\}$, and increasing on $(0, +\infty)$.

Then $x = 0$ is one of the global minima of $F(x)$.

So the global minimum value is $F(0) = 0$.

# 6

## 6.a

By considering the function $f(x) = x - \sin x$, show that

$$\sin x \leq x$$

for every $x \geq 0$.

$$f(x) = x - \sin x$$
$$f(x) \in C^\infty(\mathbb{R}, \mathbb{R})$$
$$f(0) = 0 - \sin 0 = 0$$

$$f'(x) = 1 - \cos x$$
$$\geq 0 \qquad\qquad (\cos x \leq 1)$$
$$f'(x) \geq 0 \implies (x \geq y \implies f(x) \geq f(y)) \qquad \text{(increasing)}$$

$$\forall x \geq 0$$
$$x \geq 0 \implies f(x) \geq f(0)$$
$$\implies f(x) \geq 0$$
$$f(x) \geq 0$$
$$x - \sin x \geq 0$$
$$x \geq \sin x$$
$$\sin x \leq x$$

## 6.b

Using the result of 6.a and integration, show that each of the following inequalities holds for every $x \geq 0$.

### 6.b.i

$$\cos x \geq 1 - \frac{x^2}{2}$$

$$\forall x \geq 0$$
$$\sin x \leq x$$
$$\int_0^x \sin \xi \, d\xi \leq \int_0^x \xi \, d\xi \qquad \text{(the integrands are continuous and thus integrable by FTC I; } x \geq 0)$$
$$[-\cos \xi]_0^x \leq \left[\frac{\xi^2}{2}\right]_0^x$$
$$\cos 0 - \cos x \leq \frac{x^2}{2} - \frac{0^2}{2}$$
$$1 - \cos x \leq \frac{x^2}{2}$$
$$-\cos x \leq -1 + \frac{x^2}{2}$$
$$\cos x \geq 1 - \frac{x^2}{2}$$

### 6.b.ii

$$\sin x \geq x - \frac{x^3}{6}$$

$$\forall x \geq 0$$
$$\cos x \geq 1 - \frac{x^2}{2}$$
$$\int_0^x \cos \xi \, d\xi \geq \int_0^x \left(1 - \frac{\xi^2}{2}\right) d\xi \qquad \text{(the integrands are continuous and thus integrable by FTC I; } x \geq 0)$$
$$[\sin \xi]_0^x \geq \left[\xi - \frac{\xi^3}{6}\right]_0^x$$
$$\sin x - \sin 0 \geq x - \frac{x^3}{6} - 0 + \frac{0^3}{6}$$
$$\sin x \geq x - \frac{x^3}{6}$$

### 6.b.iii

$$\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{2}4$$

$$\forall x \geq 0$$

$$\sin x \geq x - \frac{x^3}{6}$$

$$\int_0^x \sin \xi \, \mathrm{d}\xi \geq \int_0^x \left( \xi - \frac{\xi^3}{6} \right) \mathrm{d}\xi \qquad \text{(the integrands are continuous and thus integrable by FTC I; } x \geq 0 \text{)}$$

$$[-\cos \xi]_0^x \geq \left[ \frac{\xi^2}{2} - \frac{\xi^4}{24} \right]_0^x$$

$$\cos 0 - \cos x \geq \frac{x^2}{2} - \frac{x^4}{24} - \frac{0^2}{2} + \frac{0^4}{24}$$

$$1 - \cos x \geq \frac{x^2}{2} - \frac{x^4}{24}$$

$$-\cos x \geq -1 + \frac{x^2}{2} - \frac{x^4}{24}$$

$$\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

# 7

## 7.a

Let $a < b$ be real numbers and let $f : [a, b] \to (0, +\infty)$ be a **positive** continuous function. Using Cauchy-Schwarz inequality, show that

$$\left( \int_a^b f(x) \, \mathrm{d}x \right) \left( \int_a^b \frac{1}{f(x)} \, \mathrm{d}x \right) \geq (b - a)^2$$

.

$$g(x) := \frac{1}{f(x)}$$

$$F(x) := \sqrt{f(x)}$$

$$G(x) := \sqrt{g(x)} = \frac{1}{\sqrt{f(x)}}$$

$f(x)$ is integrable on $[a, b]$ because it is bounded and continuous.

$g(x)$ is integrable on $[a, b]$ because it is bounded and continuous. $\qquad (f(x) > 0$ and is bounded $\implies g(x) = \frac{1}{f(x)} > 0$ and is bounded$)$

$F(x)$ is integrable on $[a, b]$ because it is bounded and continuous. $\qquad (f(x) > 0$ and is bounded $\implies F(x) = \sqrt{f(x)} > 0$ and is bounded$)$

$G(x)$ is integrable on $[a, b]$ because it is bounded and continuous. $\qquad (g(x) > 0$ and is bounded $\implies G(x) = \sqrt{g(x)} > 0$ and is bounded$)$

By the Cauchy-Schwarz inequality,

$$\left( \int_a^b (F(x))^2 \, \mathrm{d}x \right) \left( \int_a^b (G(x))^2 \, \mathrm{d}x \right) \geq \left( \int_a^b F(x) G(x) \, \mathrm{d}x \right)^2$$

$$\left( \int_a^b \left( \sqrt{f(x)} \right)^2 \mathrm{d}x \right) \left( \int_a^b \left( \frac{1}{\sqrt{f(x)}} \right)^2 \mathrm{d}x \right) \geq \left( \int_a^b \sqrt{f(x)} \frac{1}{\sqrt{f(x)}} \, \mathrm{d}x \right)^2$$

$$\left( \int_a^b |f(x)| \, \mathrm{d}x \right) \left( \int_a^b \frac{1}{|f(x)|} \, \mathrm{d}x \right) \geq \left( \int_a^b \mathrm{d}x \right)^2$$

$$\geq (b - a)^2$$

$$\left( \int_a^b f(x) \, \mathrm{d}x \right) \left( \int_a^b \frac{1}{f(x)} \, \mathrm{d}x \right) \geq (b - a)^2 \qquad\qquad (f(x) > 0)$$

## 7.b

Using the result from and Cauchy-Schwarz inequality again, show that

$$\int_0^{2\pi} \frac{1}{\sqrt{1 - \frac{1}{2}\cos x}} \, \mathrm{d}x \geq 2\pi$$

.

$$f(x) := \sqrt{1 - \frac{1}{2}\cos x} \qquad\qquad x \in \mathbb{R}$$

$f(x)$ is continuous and bounded.

$$\cos x \in [-1, 1]$$

$$1 - \frac{1}{2}\cos x \in [0.5, 1.5]$$

$$\sqrt{1 - \frac{1}{2}\cos x} > 0$$

$$f(x) > 0$$

$$\left(\int_0^{2\pi} \mathrm{d}x\right)\left(\int_0^{2\pi} (f(x))^2 \,\mathrm{d}x\right) \geq \left(\int_0^{2\pi} f(x)\,\mathrm{d}x\right)^2 \qquad\qquad \text{(Cauchy-Schwarz inequality)}$$

$$2\pi \left(\int_0^{2\pi} \left(1 - \frac{1}{2}\cos x\right)\mathrm{d}x\right) \geq \left(\int_0^{2\pi} f(x)\,\mathrm{d}x\right)^2$$

$$2\pi \left[x - \frac{1}{2}\sin x\right]_0^{2\pi} \geq \left(\int_0^{2\pi} f(x)\,\mathrm{d}x\right)^2$$

$$2\pi \left(2\pi - \frac{1}{2}\sin(2\pi) - 0 + \frac{1}{2}\sin 0\right) \geq \left(\int_0^{2\pi} f(x)\,\mathrm{d}x\right)^2$$

$$4\pi^2 \geq \left(\int_0^{2\pi} f(x)\,\mathrm{d}x\right)^2$$

$$2\pi \geq \left|\int_0^{2\pi} f(x)\,\mathrm{d}x\right|$$

$$2\pi \geq \int_0^{2\pi} f(x)\,\mathrm{d}x \qquad\qquad (f(x) > 0, 2\pi > 0)$$

$$\left(\int_0^{2\pi} f(x)\,\mathrm{d}x\right)\left(\int_0^{2\pi} \frac{1}{f(x)}\,\mathrm{d}x\right) \geq (2\pi - 0)^2 \qquad\qquad \text{(7.a)}$$

$$\geq 4\pi^2$$

$$\int_0^{2\pi} \frac{1}{f(x)}\,\mathrm{d}x \geq \frac{4\pi^2}{\int_0^{2\pi} f(x)\,\mathrm{d}x}$$

$$\geq \frac{4\pi^2}{2\pi} \qquad\qquad \left(2\pi \geq \int_0^{2\pi} f(x)\,\mathrm{d}x\right)$$

$$\int_0^{2\pi} \frac{1}{\sqrt{1 - \frac{1}{2}\cos x}}\,\mathrm{d}x \geq 2\pi$$

# 8

Let $m \in (0,1)$ be a fixed number, and let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \int_0^x \frac{1}{\sqrt{1 - m\sin^2 t}}\,\mathrm{d}t$$

.

## 8.a

Show that $f$ is strictly increasing on $\mathbb{R}$.

$$\forall t \in \mathbb{R}$$

$$\sin t \in [-1, 1]$$

$$\sin^2 t \in [0, 1]$$

$$m\sin^2 t \in [0, 1) \qquad\qquad (m \in (0,1))$$

$$1 - m\sin^2 t \in (0, 1]$$

$$\sqrt{1 - m\sin^2 t} \in (0, 1]$$

$$\frac{1}{\sqrt{1 - m\sin^2 t}} \in [1, +\infty)$$

$$> 0$$

$$f(x) = \int_0^x \frac{1}{\sqrt{1 - m\sin^2 t}}\,\mathrm{d}t$$

$$f'(x) = \frac{1}{\sqrt{1 - m\sin^2 x}} \qquad\qquad \text{(first fundamental theorem of calculus)}$$

$$> 0$$

$$f'(x) > 0 \implies f \text{ is strictly increasing on } \mathbb{R}.$$

## 8.b

Show that $f(x) \geq x$ for every $x > 0$. Hence deduce that

$$\lim_{x \to +\infty} f(x) = +\infty$$

and

$$\lim_{x \to -\infty} f(x) = -\infty$$

.

$$\forall x > 0$$

$$\frac{1}{\sqrt{1 - m \sin^2 x}} \in [1, +\infty)$$

$$\geq 1$$

$$\int_0^x \frac{1}{\sqrt{1 - m \sin^2 \xi}} \, d\xi \geq \int_0^x d\xi \qquad \text{(the integrands are continuous and thus integrable by FTC I; } x > 0)$$

$$f(x) \geq [\xi]_0^x$$

$$\geq x$$

When $x > 0$,

$$f(x) \geq x$$

$$\lim_{x \to +\infty} f(x) \geq \lim_{x \to +\infty} x \qquad (x > 0)$$

$$\geq +\infty$$

$$\lim_{x \to +\infty} f(x) = +\infty$$

$$f(-x) = \int_0^{-x} \frac{1}{\sqrt{1 - m \sin^2 x}} \, dx$$

$$= -\int_0^x \frac{1}{\sqrt{1 - m \sin^2(-x)}} \, dx \qquad \text{(change of variable } -x \mapsto x)$$

$$= -\int_0^x \frac{1}{\sqrt{1 - m \sin^2 x}} \, dx$$

$$= -f(x)$$

When $x < 0$,

$$f(-x) \geq -x$$

$$\lim_{x \to -\infty} f(-x) \geq \lim_{x \to -\infty} -x \qquad (x < 0)$$

$$-\lim_{x \to -\infty} f(x) \geq +\infty$$

$$\lim_{x \to -\infty} f(x) \leq -\infty$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

## 8.c

Using the results from (a) and (b), deduce that $f$ has an inverse which is defined on $\mathbb{R}$.

The range of $f$ is $(-\infty, +\infty)$, i.e. $\mathbb{R}$. (8.b)

$f$ is invertible as it is strictly increasing on all of its domain. (8.a)

The domain of the inverse of $f$ is the range of $f$, which is $\mathbb{R}$.

## 8.d

For each $y \in \mathbb{R}$, let's write $x := f^{-1}(y)$ and define three functions $p, q, r : \mathbb{R} \to \mathbb{R}$ by

$$\begin{cases} p(y) = \sin x \\ q(y) = \cos x \\ r(y) = \sqrt{1 - m \sin^2 x} \end{cases}$$

.

Show that

$$p'(y) = q(y) r(y)$$

for every $y \in \mathbb{R}$.

In a similar way, also compute $q'(y)$ and $r'(y)$ in terms of $p(y)$, $q(y)$, and $r(y)$.

$$f(x) = \int_0^x \frac{1}{\sqrt{1 - m \sin^2 t}} \, dt$$

$$f'(x) = \frac{1}{\sqrt{1 - m \sin^2 x}} \qquad \text{(first fundamental theorem of calculus)}$$

$$> 0$$

$f(x)$ is continuously differentiable with nonzero derivative everywhere on $\mathbb{R}$, thus $f^{-1}(y)$ is continuously differentiable on $\mathbb{R}$ by the inverse function theorem.

$$p(y) = \sin\left(f^{-1}(y)\right)$$

$$p'(y) = (\cos x)(f^{-1}(y))' \qquad \text{(chain rule)}$$

$$= \frac{\cos x}{f'(x)} \qquad \text{(inverse function theorem)}$$

$$= (\cos x)\left(\sqrt{1 - m \sin^2 x}\right)$$

$$= q(y)r(y)$$

$$q(y) = \cos\left(f^{-1}(y)\right)$$

$$q'(y) = (-\sin x)(f^{-1}(y))' \qquad \text{(chain rule)}$$

$$= -\frac{\sin x}{f'(x)} \qquad \text{(inverse function theorem)}$$

$$= -(\sin x)\left(\sqrt{1 - m \sin^2 x}\right)$$

$$= -p(y)r(y)$$

$$r(y) = \sqrt{1 - m \sin^2\left(f^{-1}(y)\right)}$$

$$r'(y) = \frac{-2m \sin x \cos x \left(f^{-1}(y)\right)'}{2\sqrt{1 - m \sin^2 x}} \qquad \text{(chain rule)}$$

$$r'(y) = \frac{-2m \sin x \cos x}{2\sqrt{1 - m \sin^2 x} f'(x)} \qquad \text{(inverse function theorem)}$$

$$r'(y) = -m \sin x \cos x$$

$$= -m p(y) q(y)$$

# 12

## 12.a

Let $x$ be a fixed non-negative number with $x \neq 1$. Evaluate the integral

$$\int_0^\pi \frac{\sin t}{\sqrt{1 - 2x \cos t + x^2}} \, dt$$

in terms of $x$.

$$x \neq 0$$

$$\int_0^\pi \frac{\sin t}{\sqrt{1 - 2x \cos t + x^2}} \, dt = -\int_1^{-1} \frac{1}{\sqrt{1 - 2xt + x^2}} \, dt \qquad \text{(change of variable } \cos t \mapsto t)$$

$$= \int_{-1}^1 \frac{1}{\sqrt{1 + x^2 - 2xt}} \, dt$$

$$= -\frac{1}{2x} \int_{1+2x+x^2}^{1-2x+x^2} \frac{1}{\sqrt{t}} \, dt \qquad \text{(change of variable } 1 + x^2 - 2xt \mapsto t, x \neq 0)$$

$$= \frac{1}{x}\left[\sqrt{t}\right]_{1-2x+x^2}^{1+2x+x^2}$$

$$= \frac{1}{x}\left[\sqrt{t}\right]_{(x-1)^2}^{(x+1)^2}$$

$$= \frac{|x+1| - |x-1|}{x}$$

$$= \begin{cases} \frac{(x+1)-(x-1)}{x} & x > 1 \\ \frac{(x+1)-(1-x)}{x} & 0 < x < 1 \end{cases}$$

$$= \begin{cases} \frac{2}{x} & x > 1 \\ 2 & 0 < x < 1 \end{cases}$$

$$x = 0$$

$$\int_0^\pi \frac{\sin t}{\sqrt{1 - 2x \cos t + x^2}} \, dt = \int_0^\pi \sin t \, dt$$

$$= [-\cos t]_0^\pi$$

$$= 2$$

$$\int_0^\pi \frac{\sin t}{\sqrt{1 - 2x \cos t + x^2}} \, dt = \begin{cases} \frac{2}{x} & x > 1 \\ 2 & 0 \leq x < 1 \end{cases}$$

## 12.b

Let $f: [0, +\infty) \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \int_0^\pi \frac{\sin t}{\sqrt{1 - 2x\cos t + x^2}}\, dt & \text{if } x \neq 1 \\ a & \text{if } x = 1 \end{cases}$$

.

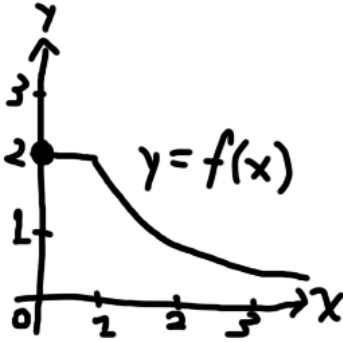Using the result from , find the value of $a$ so that $f$ is a continuous function. Hence sketch the graph of $f$.

Using the definition in 12.a,
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2 = 2$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{2}{x} = 2$$
$$\therefore \lim_{x \to 1} f(x) = 2$$
$\therefore$ To make $f$ continuous, $a = \lim_{x \to 1} f(x) = 2$.



## 13

## 13.a

Using the substitution $u = \frac{1}{x}$, show that

$$\int_{\frac{1}{2}}^{2} \frac{\ln x}{1 + x^2}\, dx = 0$$

.

$$\int_{\frac{1}{2}}^{2} \frac{\ln x}{1 + x^2}\, dx = -\int_{u=2}^{u=\frac{1}{2}} \frac{\ln x}{1 + x^{-2}}\, du \qquad \text{(change of variable)}$$
$$= \int_{2}^{\frac{1}{2}} \frac{\ln u}{1 + u^2}\, du$$
$$= \int_{2}^{\frac{1}{2}} \frac{\ln x}{1 + x^2}\, dx \qquad \text{(rename dummy variable)}$$
$$= -\int_{\frac{1}{2}}^{2} \frac{\ln x}{1 + x^2}\, dx$$
$$2\int_{\frac{1}{2}}^{2} \frac{\ln x}{1 + x^2}\, dx = 0$$
$$\int_{\frac{1}{2}}^{2} \frac{\ln x}{1 + x^2}\, dx = 0$$

## 13.b

Using or otherwise, evaluate the limit

$$\lim_{n \to +\infty} \sum_{k=1}^{3n} \frac{1}{2n} \frac{\ln\left(2\left(\frac{1}{2} + \frac{k}{2n}\right)\right)}{1 + \left(\frac{1}{2} + \frac{k}{2n}\right)^2}$$

.

$$\lim_{n \to +\infty} \sum_{k=1}^{3n} \frac{1}{2n} \frac{\ln\left(2\left(\frac{1}{2} + \frac{k}{2n}\right)\right)}{1 + \left(\frac{1}{2} + \frac{k}{2n}\right)^2} = \lim_{n \to +\infty} \sum_{k=1}^{3n} \frac{\frac{3}{2}}{3n} \frac{\ln\left(2\left(\frac{1}{2} + \frac{k}{2n}\right)\right)}{1 + \left(\frac{1}{2} + \frac{k}{2n}\right)^2}$$

$$= \int_{\frac{1}{2}}^{2} \frac{\ln 2x}{1 + x^2}\, dx \qquad \left(\text{the integrand is bounded and continuous on } \left[\frac{1}{2}, 2\right], \text{ so it is integrable}\right)$$

$$= \int_{\frac{1}{2}}^{2} \frac{\ln 2 + \ln x}{1 + x^2}\, dx$$

$$= (\ln 2) \int_{\frac{1}{2}}^{2} \frac{1}{1 + x^2}\, dx + \int_{\frac{1}{2}}^{2} \frac{\ln x}{1 + x^2}\, dx$$

$$= (\ln 2)[\arctan x]_{\frac{1}{2}}^{2} + 0$$

$$= (\ln 2)\left(\arctan 2 - \arctan \frac{1}{2}\right)$$

$$= (\ln 2)\left(\arctan 2 - \left(\frac{\pi}{2} - \arctan 2\right)\right)$$

$$= (\ln 2)\left(2\arctan 2 - \frac{\pi}{2}\right)$$

# 16

## 16.a

Let $f : [0, \pi] \to \mathbb{R}$ be a continuous function such that

$$f(\pi - x) = -f(x)$$

for every $x \in [0, \pi]$.

Using the substitution $u = \pi - x$, show that

$$\int_0^{\pi} f(x) \ln\left(1 + e^{\cos x}\right) dx = \frac{1}{2} \int_0^{\pi} f(x) \cos x\, dx$$

.

$$\int_0^{\pi} f(x) \ln\left(1 + e^{\cos x}\right) dx = -\int_{\pi}^{0} f(\pi - u) \ln\left(1 + e^{\cos(\pi - u)}\right) du \qquad \text{(change of variable)}$$

$$= -\int_0^{\pi} f(u) \ln\left(1 + e^{-\cos u}\right) du$$

$$= -\int_0^{\pi} f(x) \ln\left(1 + e^{-\cos x}\right) dx \qquad \text{(rename dummy variable)}$$

$$2\int_0^{\pi} f(x) \ln\left(1 + e^{\cos x}\right) dx = \int_0^{\pi} f(x) \ln\left(1 + e^{\cos x}\right) dx - \int_0^{\pi} f(x) \ln\left(1 + e^{-\cos x}\right) dx$$

$$= \int_0^{\pi} f(x) \left(\ln\left(1 + e^{-\cos x}\right) - \ln\left(1 + e^{-\cos x}\right)\right) dx$$

$$= \int_0^{\pi} f(x) \ln\left(\frac{1 + e^{\cos x}}{e^{-\cos x} + 1}\right) dx$$

$$= \int_0^{\pi} f(x) \ln\left(e^{\cos x}\right) dx$$

$$= \int_0^{\pi} f(x) \cos x\, dx$$

$$\int_0^{\pi} f(x) \ln\left(1 + e^{\cos x}\right) dx = \frac{1}{2} \int_0^{\pi} f(x) \cos x\, dx$$

## 16.b

Compute the derivative of the function $g : [0, \pi] \to \mathbb{R}$ defined by

$$g(x) = \frac{\cos x}{1 + \sin x}$$

.

Using this together with the result from , evaluate the integral

$$\int_0^{\pi} \frac{(\cos x) \ln\left(1 + e^{\cos x}\right)}{(1 + \sin x)^2}\, dx$$

.

$$g(x) = \frac{\cos x}{1 + \sin x}$$

$$g'(x) = \frac{-(\sin x)(1 + \sin x) - \cos^2 x}{(1 + \sin x)^2} \qquad \text{(quotient rule)}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$$

$$= -\frac{1 + \sin x}{(1 + \sin x)^2}$$

$$= -\frac{1}{1 + \sin x}$$

$$g''(x) = -\frac{-\cos x}{(1 + \sin x)^2} \qquad \text{(quotient rule)}$$

$$= \frac{\cos x}{(1 + \sin x)^2}$$

$$f : [0, \pi] \to \mathbb{R}$$

$$f(x) := \frac{\cos x}{(1 + \sin x)^2}$$

$f(x)$ is continuous on $[0, \pi]$.

$$f(\pi - x) = \frac{\cos(\pi - x)}{(1 + \sin(\pi - x))^2}$$

$$= -\frac{\cos x}{(1 + \sin x)^2}$$

$$= -f(x)$$

$$\int_0^\pi \frac{(\cos x) \ln\left(1 + e^{\cos x}\right)}{(1 + \sin x)^2} \, dx = \int_0^\pi f(x) \ln\left(1 + e^{\cos x}\right) dx$$

$$= \frac{1}{2} \int_0^\pi f(x) \cos x \, dx \qquad \text{(16.a)}$$

$$= \frac{1}{2} \int_0^\pi \frac{\cos^2 x}{(1 + \sin x)^2} \, dx$$

$$= \frac{1}{2} \int_0^\pi g''(x) \cos x \, dx$$

$$= \frac{1}{2} \int_{x=0}^{x=\pi} \cos x \, dg'(x)$$

$$= \frac{1}{2} \left( [g'(x) \cos x]_0^\pi - \int_{x=0}^{x=\pi} g'(x) \, d(\cos x) \right) \qquad \text{(integration by parts)}$$

$$= \frac{1}{2} \left( -g'(\pi) - g'(0) + \int_0^\pi \frac{\sin x}{1 + \sin x} \, dx \right)$$

$$= \frac{1}{2} \left( 2 + \int_0^\pi g'(x) \sin x \, dx \right)$$

$$= \frac{1}{2} \left( 2 + \int_{x=0}^{x=\pi} \sin x \, dg(x) \right)$$

$$= \frac{1}{2} \left( 2 + [g'(x) \sin x]_0^\pi - \int_{x=0}^{x=\pi} g(x) \, d(\sin x) \right) \qquad \text{(integration by parts)}$$

$$= \frac{1}{2} \left( 2 - \int_0^\pi g(x) \cos x \, dx \right)$$

$$= \frac{1}{2} \left( 2 - \int_0^\pi \frac{\cos^2 x}{1 + \sin x} \, dx \right)$$

$$= \frac{1}{2} \left( 2 - \int_0^\pi \frac{(1 - \sin x)(1 + \sin x)}{1 + \sin x} \, dx \right)$$

$$= \frac{1}{2} \left( 2 - \int_0^\pi (1 - \sin x) \, dx \right)$$

$$= \frac{1}{2} \left( 2 - [x + \cos x]_0^\pi \right)$$

$$= \frac{1}{2} (2 - \pi + 1 + 0 + 1)$$

$$= 2 - \frac{\pi}{2}$$