HKUST MATH 1014 L1 assignment 4 submission

MATH1014 Calculus II Problem Set 4
L01 (Spring 2024)

Problem Set 4

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 4 covers materials from §6.4 - §6.6.

Q2

Evaluate the following antiderivatives.

Q2.b

$$\int \ln\left(x^3+1\right) \mathrm{d}x$$

$$\begin{aligned}
&\int \ln\left(x^3+1\right) \mathrm{d}x \\
=&\int \ln\left((x+1)\left(x^2-x+1\right)\right) \mathrm{d}x \\
=&\int \left(\ln(x+1)+\ln\left(x^2-x+1\right)\right) \mathrm{d}x \\
=&\int \ln(x+1)\,\mathrm{d}x+\int \ln\left(x^2-x+1\right) \mathrm{d}x \\
=&\int \ln(x+1)\,\mathrm{d}(x+1)+\int \ln\left(x^2-x+1\right) \mathrm{d}x \\
=&(x+1)\ln(x+1)-\int (x+1)\,\mathrm{d}\ln(x+1)+x\ln\left(x^2-x+1\right)-\int x\,\mathrm{d}\ln\left(x^2-x+1\right) \\
=&(x+1)\ln(x+1)-\int \mathrm{d}x+x\ln\left(x^2-x+1\right)-\int \frac{2x^2-x}{x^2-x+1}\,\mathrm{d}x \\
=&(x+1)\ln(x+1)-x+x\ln\left(x^2-x+1\right)-\int\left(2+\frac{1}{2}\frac{2x-1}{x^2-x+1}-\frac{3}{2}\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\right) \\
=&(x+1)\ln(x+1)-x+x\ln\left(x^2-x+1\right)-2x-\frac{1}{2}\ln\left(x^2-x+1\right)+\frac{3}{2}\frac{\sqrt{3}}{2}\frac{4}{3}\arctan\left(\left(x-\frac{1}{2}\right)\frac{2}{\sqrt{3}}\right)+C \\
=&(x+1)\ln(x+1)-3x+\left(x-\frac{1}{2}\right)\ln\left(x^2-x+1\right)+\sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+C \qquad (x>-1)
\end{aligned}$$

Q3

Q3.a

Using the factorization $x^4+1=\left(x^2-\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)$, evaluate

$$\int \frac{x^2}{x^4+1}\,\mathrm{d}x$$

.

$$
\begin{aligned}
&\int \frac{x^2}{x^4+1}\,\mathrm{d}x \\
&= \int \frac{x^2}{\left(x^2-\sqrt{2}x+1\right)\left(x^2+\sqrt{2}x+1\right)}\,\mathrm{d}x \\
&= \frac{1}{2\sqrt{2}}\int\left(\frac{x}{x^2-\sqrt{2}x+1}-\frac{x}{x^2+\sqrt{2}x+1}\right)\mathrm{d}x \\
&= \frac{1}{2\sqrt{2}}\int\left(\frac{1}{2}\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}+\frac{\sqrt{2}}{2}\frac{1}{x^2-\sqrt{2}x+1}-\frac{1}{2}\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}+\frac{\sqrt{2}}{2}\frac{1}{x^2+\sqrt{2}x+1}\right)\mathrm{d}x \\
&= \frac{1}{4\sqrt{2}}\left(\ln\left|x^2-\sqrt{2}x+1\right|-\ln\left|x^2+\sqrt{2}x+1\right|\right)+\frac{1}{4}\int\left(\frac{1}{x^2-\sqrt{2}x+1}+\frac{1}{x^2+\sqrt{2}x+1}\right)\mathrm{d}x \\
&= \frac{1}{4\sqrt{2}}\left(\ln\left|x^2-\sqrt{2}x+1\right|-\ln\left|x^2+\sqrt{2}x+1\right|\right)+\frac{1}{4}\int\left(\frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}+\frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right)\mathrm{d}x \\
&= \frac{1}{4\sqrt{2}}\left(\ln\left|x^2-\sqrt{2}x+1\right|-\ln\left|x^2+\sqrt{2}x+1\right|\right)+\frac{1}{4}\left(\frac{1}{\sqrt{2}}(2)\arctan\left(\left(x-\frac{1}{\sqrt{2}}\right)\sqrt{2}\right)+\frac{1}{\sqrt{2}}(2)\arctan\left(\left(x+\frac{1}{\sqrt{2}}\right)\sqrt{2}\right)\right)+C \\
&= \frac{1}{4\sqrt{2}}\left(\ln\left|x^2-\sqrt{2}x+1\right|-\ln\left|x^2+\sqrt{2}x+1\right|\right)+\frac{1}{4}\left(\sqrt{2}\arctan\left(\sqrt{2}x-1\right)+\sqrt{2}\arctan\left(\sqrt{2}x+1\right)\right)+C \\
&= \frac{1}{4\sqrt{2}}\ln\left|x^2-\sqrt{2}x+1\right|-\frac{1}{4\sqrt{2}}\ln\left|x^2+\sqrt{2}x+1\right|+\frac{1}{2\sqrt{2}}\arctan\left(\sqrt{2}x-1\right)+\frac{1}{2\sqrt{2}}\arctan\left(\sqrt{2}x+1\right)+C
\end{aligned}
$$

Q3.b

Using (a) and the substitution $u=\sqrt{\tan x}$, evaluate

$$\int_0^{\frac{\pi}{4}}\sqrt{\tan x}\,\mathrm{d}x$$

.

$$
\begin{aligned}
&\int_0^{\frac{\pi}{4}}\sqrt{\tan x}\,\mathrm{d}x \\
&= 2\int_{\sqrt{\tan x}=0}^{\sqrt{\tan x}=1}\frac{\tan x}{\sec^2 x}\,\mathrm{d}\sqrt{\tan x} \\
&= 2\int_{\sqrt{\tan x}=0}^{\sqrt{\tan x}=1}\frac{\tan x}{\tan^2 x+1}\,\mathrm{d}\sqrt{\tan x} \\
&= 2\int_0^1\frac{u^2}{u^4+1}\,\mathrm{d}u \\
&= 2\left[\frac{1}{4\sqrt{2}}\ln\left|u^2-\sqrt{2}u+1\right|-\frac{1}{4\sqrt{2}}\ln\left|u^2+\sqrt{2}u+1\right|+\frac{1}{2\sqrt{2}}\arctan\left(\sqrt{2}u-1\right)+\frac{1}{2\sqrt{2}}\arctan\left(\sqrt{2}u+1\right)\right]_0^1 \\
&= 2\left(\frac{1}{4\sqrt{2}}\ln\left|1-\sqrt{2}+1\right|-\frac{1}{4\sqrt{2}}\ln\left|1+\sqrt{2}+1\right|+\frac{1}{2\sqrt{2}}\arctan\left(\sqrt{2}-1\right)+\frac{1}{2\sqrt{2}}\arctan\left(\sqrt{2}+1\right)\right) \\
&= \frac{1}{2\sqrt{2}}\ln\left|2-\sqrt{2}\right|-\frac{1}{2\sqrt{2}}\ln\left|2+\sqrt{2}\right|+\frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}-1\right)+\frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}-1}\right) \\
&= \frac{1}{2\sqrt{2}}\ln\left|2-\sqrt{2}\right|-\frac{1}{2\sqrt{2}}\ln\left|2+\sqrt{2}\right|+\frac{\pi}{2\sqrt{2}}
\end{aligned}
$$

Q7

Let a be a positive real number. Evaluate

$$\int\frac{1}{1-a\sin x}\,\mathrm{d}x$$

for each of the following cases:

Q7.a

$0<a<1$ (for $x\in(-\pi,\pi)$),

$$
\begin{aligned}
&\int \frac{1}{1-a\sin x}\,\mathrm{d}x \\
&= \int \frac{1}{1-a\frac{2t}{1+t^2}}\frac{2\,\mathrm{d}t}{1+t^2} && \left(t := \tan\frac{x}{2}, x\in(-\pi,\pi)\right) \\
&= \int \frac{1}{\frac{1+t^2-2at}{1+t^2}}\frac{2\,\mathrm{d}t}{1+t^2} \\
&= 2\int \frac{1}{1+t^2-2at}\,\mathrm{d}t \\
&= 2\int \frac{1}{(t-a)^2+(1-a^2)}\,\mathrm{d}t \\
&= 2\int \frac{1}{(t-a)^2+(1-a^2)}\,\mathrm{d}t && \left(0<a<1 \implies 1-a^2>0\right) \\
&= 2\sqrt{1-a^2}\frac{1}{1-a^2}\arctan\left((t-a)\frac{1}{\sqrt{1-a^2}}\right)+C \\
&= \frac{2}{\sqrt{1-a^2}}\arctan\left(\frac{t-a}{\sqrt{1-a^2}}\right)+C \\
&= \frac{2}{\sqrt{1-a^2}}\arctan\left(\frac{\tan\frac{x}{2}-a}{\sqrt{1-a^2}}\right)+C, x\in(-\pi,\pi)
\end{aligned}
$$

Q7.b

$a=1$,

$$
\begin{aligned}
&\int \frac{1}{1-a\sin x}\,\mathrm{d}x \\
&= \int \frac{1}{1-\sin x}\,\mathrm{d}x && (a=1) \\
&= \int \frac{1}{1-\frac{2t}{1+t^2}}\frac{2\,\mathrm{d}t}{1+t^2} && \left(t := \tan\frac{x}{2}, x\in(-\pi,\pi)\right) \\
&= \int \frac{1}{\frac{1+t^2-2t}{1+t^2}}\frac{2\,\mathrm{d}t}{1+t^2} \\
&= 2\int \frac{1}{1+t^2-2t}\,\mathrm{d}t \\
&= 2\int \frac{1}{(t-1)^2}\,\mathrm{d}t \\
&= -\frac{2}{t-1}+C \\
&= -\frac{2}{\tan\frac{x}{2}-1}+C && (x\in(-\pi,\pi))
\end{aligned}
$$

Notice that the integrand $\dfrac{1}{1-\sin x}$ is periodic with a period of 2π.

Further, the integrand has vertical asymptotes when $x=\dfrac{\pi}{2}+2n\pi, n\in\mathbb{Z}$,

which separates each cycle, so we can extend the domain:

$$
\begin{cases}
-\frac{2}{\tan\frac{x}{2}-1}+C & x\neq \pi+2n\pi \\
\lim_{t\to\pm\infty} -\frac{2}{t-1}=0 & x=\pi+2n\pi
\end{cases}, x\neq\frac{\pi}{2}+2n\pi, n\in\mathbb{Z}
$$

Q7.c

$a>1$.

$$
\begin{aligned}
&\int \frac{1}{1-a\sin x}\,\mathrm{d}x \\
&= \int \frac{1}{1-a\frac{2t}{1+t^2}}\frac{2\,\mathrm{d}t}{1+t^2} && \left(t := \tan\frac{x}{2}, x\in(-\pi,\pi)\right) \\
&= \int \frac{1}{\frac{1+t^2-2at}{1+t^2}}\frac{2\,\mathrm{d}t}{1+t^2} \\
&= 2\int \frac{1}{1+t^2-2at}\,\mathrm{d}t \\
&= 2\int \frac{1}{(t-a)^2+(1-a^2)}\,\mathrm{d}t \\
&= 2\int \frac{1}{(t-a)^2-\sqrt{a^2-1}^2}\,\mathrm{d}t && \left(a>1 \implies a^2-1>0\right) \\
&= 2\int \frac{1}{\left(t-a-\sqrt{a^2-1}\right)\left(t-a+\sqrt{a^2-1}\right)}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{a^2-1}}\int\left(\frac{1}{t-a-\sqrt{a^2-1}}-\frac{1}{t-a+\sqrt{a^2-1}}\right)\mathrm{d}t \\
&= \frac{1}{\sqrt{a^2-1}}\left(\ln\left|t-a-\sqrt{a^2-1}\right|+\ln\left|t-a+\sqrt{a^2-1}\right|\right)+C \\
&= \frac{1}{\sqrt{a^2-1}}\ln\left|\tan\frac{x}{2}-a-\sqrt{a^2-1}\right|+\frac{1}{\sqrt{a^2-1}}\ln\left|\tan\frac{x}{2}-a+\sqrt{a^2-1}\right|+C, \\
&x\neq\pi+2n\pi, \tan\frac{x}{2}\neq a+\sqrt{a^2-1}, \tan\frac{x}{2}\neq a-\sqrt{a^2-1}
\end{aligned}
$$

Q10

Let $f:[0,+\infty)\to\mathbb{R}$ be the function

$$f(x)=\int_1^{\sqrt{x}} e^{-t^2}\,\mathrm{d}t$$

.

Q10.a

Find $f'(x)$ for every $x\in(0,+\infty)$.

$$\begin{aligned}
&f'(x)\\
&=\frac{\mathrm{d}}{\mathrm{d}x}\int_1^{\sqrt{x}} e^{-t^2}\,\mathrm{d}t\\
&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\int_1^{x}\frac{e^{-u}}{\sqrt{u}}\,\mathrm{d}u && (u:=t^2,u>0)\\
&=\frac{e^{-u}}{2\sqrt{u}} && \text{(FTC)}\\
&=\frac{e^{-t^2}}{2t} && (u>0)
\end{aligned}$$

Q10.b

Evaluate the improper integral

$$\int_0^1\frac{f(x)}{\sqrt{x}}\,\mathrm{d}x$$

.

$$\begin{aligned}
&\int_0^1\frac{f(x)}{\sqrt{x}}\,\mathrm{d}x\\
&=2\int_0^1 f(x)\,\mathrm{d}\sqrt{x}\\
&=2\big[f(x)\sqrt{x}\big]_0^1-2\int_0^1\sqrt{x}\,\mathrm{d}f(x)\\
&=2\,(0-0)-2\int_0^1\sqrt{x}f'(x)\,\mathrm{d}x\\
&=-\int_0^1\frac{e^{-t^2}}{\sqrt{x}}\,\mathrm{d}x\\
&=-2\int_0^1 e^{-u}\,\mathrm{d}u && (u:=\sqrt{x},u>0)\\
&=-2\big[e^{-u}\big]_0^1\\
&=-2\left(e^{-1}-1\right)\\
&=2-\frac{2}{e}
\end{aligned}$$

Q15

For each of the following improper integrals, determine whether it converges or not.

Q15.a

$$\int_1^{+\infty}\frac{2+\cos x}{\sqrt{x+5}}\,\mathrm{d}x$$

$$\begin{aligned}
&\int_1^{+\infty}\frac{2+\cos x}{\sqrt{x+5}}\,\mathrm{d}x\\
&\geq\int_1^{+\infty}\frac{1}{\sqrt{x+5}}\,\mathrm{d}x && (2+\cos x\geq 1)\\
&=\int_6^{+\infty}\frac{1}{\sqrt{x}}\,\mathrm{d}x && (\text{change of variables: } x+5\mapsto x)
\end{aligned}$$

By p-test, the last integral is divergent.
By comparison test, the original integral is divergent.

Q15.b

$$\int_0^1\frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}}\,\mathrm{d}x$$

$$
\begin{aligned}
&\lim_{x\to0^+}\frac{\frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}}}{\frac{1}{\sqrt{x}}}\\
&=\lim_{x\to0^+}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}\\
&=\lim_{x\to0^+}\frac{\sqrt{\frac{x}{x^{\frac{1}{4}}}}}{\sqrt{\frac{x}{x^{\frac{1}{4}}}+\sqrt{\frac{x}{\sqrt{x}}+\sqrt{\frac{x}{x}}}}}\\
&=\lim_{x\to0^+}\frac{x^{\frac{3}{8}}}{\sqrt{x^{\frac{3}{4}}+\sqrt{\sqrt{x}+1}}}\\
&=\frac{0}{\sqrt{0+\sqrt{0+1}}}\\
&=0
\end{aligned}
$$

By p-test, $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ is convergent.

By limit comparison test, the original integral is convergent.

Q15.c

$$\int_0^{+\infty}\frac{x}{1+x^2\sin^2 x}\,\mathrm{d}x$$

$$
\begin{aligned}
&\int_0^{+\infty}\frac{x}{1+x^2\sin^2 x}\,\mathrm{d}x\\
&\geq\int_0^{+\infty}\frac{x}{1+x^2}\,\mathrm{d}x && (\sin^2 x\leq 1)\\
&=\frac{1}{2}\int_1^{+\infty}\frac{1}{x}\,\mathrm{d}x && (\text{change of variables: } 1+x^2\mapsto x)
\end{aligned}
$$

By p-test, the last integral is divergent.

By comparison test, the original integral is divergent.

Q15.d

$$\int_0^{+\infty}\frac{\sin x}{x}\,\mathrm{d}x$$

$$
\begin{aligned}
&\int_0^{+\infty}\frac{\sin x}{x}\,\mathrm{d}x\\
&\leq\int_0^{+\infty}\left|\frac{\sin x}{x}\right|\mathrm{d}x && (\text{triangle inequality})\\
&\leq\int_0^{+\infty}\left|\frac{1}{x}\right|\mathrm{d}x\\
&=\int_0^{+\infty}\frac{1}{x}\,\mathrm{d}x && (x\geq 0)
\end{aligned}
$$

By p-test, the last integral is divergent.

By absolute convergence test, the second last integral is convergent.

By comparison test, the original integral is convergent.

Q17

Let $f:[0,+\infty)\to[0,+\infty)$ be a decreasing continuous function.

Q17.a

Show that if $\int_0^{+\infty} f(x)\,\mathrm{d}x$ converges, then $\lim_{x\to+\infty} xf(x)=0$.

Hint: Show that $0\leq xf(x)\leq 2\int_{\frac{x}{2}}^{x} f(t)\,\mathrm{d}t$ and apply Squeeze Theorem.

Assume that if $\int_0^{+\infty} f(x)\,\mathrm{d}x$ is convergent, then $\lim_{x\to+\infty} xf(x) \neq 0$.

$\int_1^{+\infty} \frac{1}{x}\,\mathrm{d}x$ is divergent by p-test.

$$\begin{aligned}
&\lim_{x\to+\infty} \frac{f(x)}{\frac{1}{x}} \\
&= \lim_{x\to+\infty} xf(x) \\
&> 0 && \text{(assumption, } f(x) \geq 0, x > 0)
\end{aligned}$$

By limit comparison test, $\int_1^{+\infty} f(x)\,\mathrm{d}x$ diverges. $\left(f(x) \geq 0, \frac{1}{x} > 0\right)$

By comparison test, $\int_0^{+\infty} f(x)\,\mathrm{d}x \geq \int_1^{+\infty} f(x)\,\mathrm{d}x$ diverges. $(f(x) \geq 0)$

This contradicts our initial assumption that $\int_0^{+\infty} f(x)\,\mathrm{d}x$ is convergent.

By contradiction, if $\int_0^{+\infty} f(x)\,\mathrm{d}x$ is convergent, then $\lim_{x\to+\infty} xf(x) = 0$.

Q17.b

Show that the converse of the result from (a) is not true, i.e. give an example of $f(x)$ so that $\lim_{x\to+\infty} xf(x) = 0$ but $\int_0^{+\infty} f(x)\,\mathrm{d}x$ diverges.

$$f(x) := \begin{cases} \frac{1}{x\ln x} & x \geq e \\ \frac{1}{e} & x \in [0, e] \end{cases}$$

Because $x\ln x$ is increasing and nonnegative,
$f(x)$ is decreasing and nonnegative.

$$\begin{aligned}
\lim_{x\to e^+} \frac{1}{x\ln x} &= \frac{1}{e\ln e} \\
&= \frac{1}{e}
\end{aligned}$$

$f(x)$ is continuous.

$$\begin{aligned}
\int_0^{+\infty} f(x)\,\mathrm{d}x &= \int_0^e \frac{1}{e}\,\mathrm{d}x + \int_e^{+\infty} \frac{1}{x\ln x}\,\mathrm{d}x \\
&= 1 + \int_e^{+\infty} \frac{1}{x\ln x}\,\mathrm{d}x \\
&= 1 + \int_e^{+\infty} \frac{1}{\ln x}\,\mathrm{d}(\ln x) \\
&= 1 + [\ln(\ln x)]_e^{+\infty} \\
&= 1 + \lim_{x\to+\infty} \ln(\ln x) \\
&= +\infty
\end{aligned}$$

So $\int_0^{+\infty} f(x)\,\mathrm{d}x$ diverges.

However,

$$\begin{aligned}
\lim_{x\to+\infty} xf(x) &= \lim_{x\to+\infty} \frac{x}{x\ln x} \\
&= \lim_{x\to+\infty} \frac{1}{\ln x} \\
&= 0
\end{aligned}$$

Thus it is proved.

Now let $g: [1, +\infty) \to [e, +\infty)$ be an increasing continuous function.

Q17.c

If $\lim_{x\to+\infty} \frac{x}{\ln(g(e^x))} = 0$, show that for every sufficiently large $x > 0$ we have $\frac{e^x}{g(e^x)} < e^{-x}$.

Hint: In the definition of limit (Definition 2.91), take $\epsilon = \frac{1}{2}$.

$$
\begin{aligned}
\lim_{x\to+\infty} \frac{x}{\ln(g(e^x))} &= 0 \\
\lim_{x\to+\infty} (e^x - g(e^x)) &= e^0 \\
&= 1
\end{aligned}
$$

By limit definition, for large sufficiently $x > 0$,

$$
\begin{aligned}
\frac{x}{\ln(g(e^x))} &< \frac{1}{2} \\
x &< \frac{1}{2}\ln(g(e^x)) \\
x - \ln(g(e^x)) &< -\frac{1}{2}\ln(g(e^x)) \\
x - \ln(g(e^x)) &< -x && \left(\frac{x}{\ln(g(e^x))} < \frac{1}{2} \implies -x > -\frac{1}{2}\ln(g(e^x))\right) \\
\frac{e^x}{g(e^x)} &< e^{-x} && (e^* \text{ is strictly increasing})
\end{aligned}
$$

Q17.d

Using the results from (a) and (c), show that if $\int_1^{+\infty} \frac{1}{g(x)}\,\mathrm{d}x$ diverges, then $\int_1^{+\infty} \frac{1}{x\ln g(x)}\,\mathrm{d}x$ also diverges.

$$
\int_1^{+\infty} \frac{1}{g(x)}\,\mathrm{d}x = \int_0^{+\infty} \frac{e^x}{g(e^x)}\,\mathrm{d}x \qquad (\text{change of variables: } \ln x \mapsto x)
$$

$$
\begin{aligned}
\int_0^{+\infty} e^{-x}\,\mathrm{d}x &= \left[-e^{-x}\right]_0^{+\infty} \\
&= 1
\end{aligned}
$$

$\int_0^{+\infty} e^{-x}\,\mathrm{d}x$ is convergent but

$\int_0^{+\infty} \frac{e^x}{g(e^x)}\,\mathrm{d}x$ is divergent,

By contraposition of comparison test,
for large enough $x \geq X > 0$,

$$
\frac{e^x}{g(e^x)} > e^{-x}
$$

$$
\lim_{x\to+\infty} \frac{x}{\ln(g(e^x))} > 0 \qquad (\text{contraposition of (c)}, x > 0, \ln * > 0)
$$

$\int_0^{+\infty} \frac{1}{\ln(g(e^x))}\,\mathrm{d}x$ diverges by contraposition of (a).

$$
\int_0^{+\infty} \frac{1}{\ln(g(e^x))}\,\mathrm{d}x = \int_1^{+\infty} \frac{1}{x\ln g(x)}\,\mathrm{d}x \qquad (\text{change of variables: } e^x \mapsto x)
$$

$\int_1^{+\infty} \frac{1}{x\ln g(x)}\,\mathrm{d}x$ diverges.