Solution to Problem Set 7

1. (a) Suppose on the contrary that $(\cos n)$ converges to some real number L.   Since

$$\cos(n+1) = \cos n \cos 1 - \sin n \sin 1 \qquad \text{for every } n \in \mathbb{N},$$

it follows that

$$\sin n = \frac{\cos n \cos 1 - \cos(n+1)}{\sin 1} \qquad \text{for every } n \in \mathbb{N}$$

and so $\lim_{n \to +\infty} \sin n = \frac{L \cos 1 - L}{\sin 1}$ also exists as a real number; let's call this limit M.   Now since

$$\cos^2 n + \sin^2 n = 1 \qquad \text{for every } n \in \mathbb{N},$$

we have $L^2 + M^2 = 1$.   But on the other hand since

$$\cos 1 = \cos\big((n+1) - n\big) = \cos(n+1)\cos n - \sin(n+1)\sin n \qquad \text{for every } n \in \mathbb{N},$$

we have $L^2 + M^2 = \cos 1$.   These together imply that $\cos 1 = 1$, which is a contradiction.   ∎

  (b) Since $(\cos n)$ diverges according to (a), the series $\sum_{k=0}^{+\infty} \cos n$ diverges by term test.

2. (a) Since the exponential function is continuous, we have

$$\lim_{n \to +\infty} e^{\frac{1}{n^2}} = e^{\lim_{n \to +\infty} \frac{1}{n^2}} = e^0 = 1 \neq 0.$$

Therefore the series $\sum_{k=1}^{+\infty} e^{\frac{1}{k^2}}$ diverges by term test.

  (b) Since

$$\lim_{n \to +\infty} \sqrt[n]{|a_n|} = \lim_{n \to +\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n \to +\infty} \frac{1}{\ln n} = 0,$$

the series $\sum_{k=2}^{+\infty} \frac{1}{(\ln k)^k}$ converges by root test.   (Other possible solutions include ratio test, or comparison test with a geometric series.)

  (c) Since the cosine and sine functions are continuous, we have

$$\lim_{n \to +\infty} \cos\left(\sin \frac{1}{n}\right) = \cos\left(\sin \lim_{n \to +\infty} \frac{1}{n}\right) = \cos(\sin 0) = 1 \neq 0.$$

Therefore the series $\sum_{k=1}^{+\infty} \cos\left(\sin \frac{1}{n}\right)$ diverges by term test.

  (d) The terms of the series are all positive.   Since

$$\lim_{n \to +\infty} \frac{1 - \cos \frac{1}{n}}{\frac{1}{n^2}} = \lim_{n \to +\infty} \frac{2 \sin^2 \frac{1}{n}}{\frac{1}{n^2}} = 2\left(\lim_{n \to +\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}}\right)^2 = 2,$$

which exists and is finite, and since $\sum_{k=1}^{+\infty} \frac{1}{k^2}$ converges (by p-test), the series $\sum_{k=1}^{+\infty} \left(1 - \cos \frac{1}{k}\right)$ converges by limit comparison test.

(e)　Let $f:[1,+\infty) \to [0,+\infty)$ be the non-negative function $f(x) = xe^{-x^2}$.　Since

$$f'(x) = (1 - 2x^2)e^{-x^2} < 0 \qquad \text{for every } x > 1,$$

f is decreasing on $[1,+\infty)$.　Since the improper integral

$$\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty} xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_1^{+\infty} = \frac{1}{2e}$$

converges, the series $\sum_{k=1}^{+\infty} ke^{-k^2}$ also converges by integral test.　(Other possible solutions include ratio test, root test and comparison test with a geometric series.)

(f)　The terms of the series are all positive.　Since

$$\lim_{n\to+\infty} \frac{\dfrac{\ln n}{n(n-1)}}{\dfrac{1}{n^{3/2}}} = \lim_{n\to+\infty} \frac{\ln n}{\sqrt{n} - \dfrac{1}{\sqrt{n}}} = 0,$$

which exists and is finite, and since $\sum_{k=2}^{+\infty} \frac{1}{k^{3/2}}$ converges (by p-test), the series $\sum_{k=2}^{+\infty} \frac{\ln k}{k(k-1)}$ converges by limit comparison test.

(g)　The terms of the series are all positive.　Since

$$\lim_{n\to+\infty} \frac{\dfrac{1}{n^{1+\frac{1}{n}}}}{\dfrac{1}{n}} = \lim_{n\to+\infty} \frac{1}{n^{1/n}} = 1,$$

(cf. Example (b) on page 2 of chapter 8) which exists and is positive, and since $\sum_{k=1}^{+\infty} \frac{1}{k}$ diverges (by p-test), the series $\sum_{k=1}^{+\infty} \frac{1}{k^{1+1/k}}$ diverges by limit comparison test.　(Another possible solution is to apply comparison test with $\sum_{k=1}^{+\infty} \frac{1}{2k}$.)

(h)　Since

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = \lim_{n\to+\infty} \sqrt[n]{\left(\frac{1}{2} + \frac{1}{n}\right)^n} = \lim_{n\to+\infty} \left(\frac{1}{2} + \frac{1}{n}\right) = \frac{1}{2} < 1,$$

the series $\sum_{k=1}^{+\infty} \left(\frac{1}{2} + \frac{1}{k}\right)^k$ converges by root test.　(Other possible solutions include ratio test, or comparison test with a geometric series.)

(i)　Since

$$\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \frac{\dfrac{(2(n+1))!}{(n+2)!\,n!}}{\dfrac{2n!}{(n+1)!\,(n-1)!}} = \lim_{n\to+\infty} \frac{(2n+1)(2n+2)}{(n+2)n} = \lim_{n\to+\infty} \frac{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}{\left(1+\frac{2}{n}\right)} = 4 > 1,$$

the series $\sum_{k=1}^{+\infty} \frac{(2k)!}{(k+1)!(k-1)!}$ diverges by ratio test.　(Other possible solutions include root test, or comparison test with a geometric series.)

(j) Since

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\frac{\dfrac{3^{n+1}+4^{n+1}}{2^{n+1}+5^{n+1}}}{\dfrac{3^n+4^n}{2^n+5^n}} = \lim_{n\to+\infty}\frac{3^{n+1}+4^{n+1}}{3^n+4^n}\frac{2^n+5^n}{2^{n+1}+5^{n+1}}$$

$$= \lim_{n\to+\infty}\frac{3\cdot\left(\frac{3}{4}\right)^n+4}{\left(\frac{3}{4}\right)^n+1}\frac{\left(\frac{2}{5}\right)^n+1}{2\cdot\left(\frac{2}{5}\right)^n+5} = \frac{4}{5} < 1,$$

the series $\sum_{k=1}^{+\infty}\frac{3^k+4^k}{2^k+5^k}$ converges by ratio test.   (Other possible solutions include root test, or comparison test

with a geometric series.)

3. (a) Suppose that $\sum_{k=1}^{+\infty}a_k$ converges.   Then by term test, the sequence (a_n) must converge to 0.   Since (a_n)

is a sequence of positive numbers, $\left(\frac{1}{a_n}\right)$ must diverge to $+\infty$.   In particular, $\left(\frac{1}{a_n}\right)$ does not converge to 0

and so $\sum_{k=1}^{+\infty}\frac{1}{a_k}$ diverges by term test.   ■

(b) If $\lim_{n\to+\infty}na_n = L > 0$, then (for $\varepsilon = \frac{L}{2} > 0$) there exists $N > 0$ such that for every integer $n \geq N$, we have

$$|na_n - L| < \frac{L}{2},$$

which implies that $na_n > \frac{L}{2}$.   Therefore $a_n > \frac{L}{2n} > 0$ for every $n \geq N$.   Now the series $\sum_{k=1}^{+\infty}\frac{L}{2k} = \frac{L}{2}\sum_{k=1}^{+\infty}\frac{1}{k}$
diverges by p-test, so $\sum_{k=1}^{+\infty}a_k$ diverges by comparison test.   ■

 Alternative solution: Since $\lim_{n\to+\infty}\frac{a_n}{\frac{1}{n}} = \lim_{n\to+\infty}na_n = L > 0$ and the harmonic series $\sum_{k=1}^{+\infty}\frac{1}{k}$ diverges, the

series $\sum_{k=1}^{+\infty}a_k$ also diverges by limit comparison test.   ■

(c) Suppose that $\sum_{k=1}^{+\infty}a_k$ converges.   Then by term test, the sequence (a_n) must converge to 0.   In particular,
there exists $N > 0$ such that for every integer $n \geq N$, we have $0 < a_n < 1$.   This implies that $0 < a_n^2 < a_n$
for every $n \geq N$.   Recall again that $\sum_{k=1}^{+\infty}a_k$ converges, so $\sum_{k=1}^{+\infty}a_k^2$ converges by comparison test.   ■

The converse of this result from (c) is not true.   To see this, let $a_n = \frac{1}{n}$.   Then $\sum_{k=1}^{+\infty}a_k^2 = \sum_{k=1}^{+\infty}\frac{1}{k^2}$ converges
but $\sum_{k=1}^{+\infty}a_k = \sum_{k=1}^{+\infty}\frac{1}{k}$ diverges.

(d) For each $n \in \mathbb{N}$, since $a_n > 0$, we have

$$\frac{1}{2}\left(a_n^2 + \frac{1}{n^2}\right) \geq \sqrt{a_n^2\cdot\frac{1}{n^2}} = \left|\frac{a_n}{n}\right| = \frac{a_n}{n} > 0$$

by the AM-GM inequality.   Now we know that $\sum_{k=1}^{+\infty}\frac{1}{k^2}$ converges.   If $\sum_{k=1}^{+\infty}a_k^2$ also converges, then
$\sum_{k=1}^{+\infty}\frac{1}{2}\left(a_k^2 + \frac{1}{k^2}\right)$ converges, and so $\sum_{k=1}^{+\infty}\frac{a_k}{k}$ converges by comparison test.   ■

(e) For each $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (ka_k)\left(\frac{1}{k}\right) \leq \sqrt{\sum_{k=1}^{n} k^2 a_k^2} \sqrt{\sum_{k=1}^{n} \frac{1}{k^2}}$$

by Cauchy-Schwarz inequality.   (Let $\mathbf{u} = \langle 1a_1, 2a_2, \ldots, na_n \rangle$ and $\mathbf{v} = \langle 1, \frac{1}{2}, \ldots, \frac{1}{n} \rangle$ be vectors in $\mathbb{R}^n$ and apply Cauchy-Schwarz inequality $\mathbf{u} \cdot \mathbf{v} \leq \|\mathbf{u}\|\|\mathbf{v}\|$ as in Theorem 7.33.)   Now we know that $\sum_{k=1}^{+\infty} \frac{1}{k^2}$ converges.   If $\sum_{k=1}^{+\infty} k^2 a_k^2$ also converges, then the above inequality shows that $\sum_{k=1}^{+\infty} a_k$ also converges.   ∎

4. (a) Note that for each $j \in \mathbb{N}$, the sum $\left(\sum_{k=N+1}^{+\infty} \frac{1}{p_k}\right)^j$, after expanding, consists of **all possible terms** of the form

$$\frac{1}{p_{N+m_1} p_{N+m_2} \cdots p_{N+m_j}},$$

in which there are exactly j factors in the denominator which may or may not repeat, and all these factors come from the set $\{p_{N+1}, p_{N+2}, \ldots\}$.   Now each number $1 + M$, $1 + 2M$, $1 + 3M$, ... is not divisible by any of the primes $p_1, p_2, \ldots, p_N$; so their prime factorizations take exactly the form of the above denominator.   Therefore

$$\frac{1}{1+M} + \frac{1}{1+2M} + \cdots + \frac{1}{1+nM} \leq \sum_{j=1}^{+\infty} \left(\sum_{k=N+1}^{+\infty} \frac{1}{p_k} \right)^j \qquad \text{for all } n \in \mathbb{N}.$$

(b) Suppose that $\sum_{k=1}^{+\infty} \frac{1}{p_k}$ converges.   Then by definition, (for $\varepsilon = \frac{1}{2} > 0$) there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{+\infty} \frac{1}{p_k} < \frac{1}{2}.$$

According to (a), we have

$$\frac{1}{1+M} + \frac{1}{1+2M} + \cdots + \frac{1}{1+nM} \leq \sum_{j=1}^{+\infty} \frac{1}{2^j} = 1 \qquad \text{for all } n \in \mathbb{N}.$$

This implies that $\sum_{k=1}^{+\infty} \frac{1}{1+kM}$ converges, which is impossible as it would imply that $\sum_{k=1}^{+\infty} \frac{1}{k}$ converges by limit comparison test (or by comparison test).   ∎

5. (a) For every $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

$$= h_{2n} - h_n.$$

(b) According to (a), we have

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = h_{2n} - h_n = (h_{2n} - \ln 2n) - (h_n - \ln n) + \ln 2,$$

so with γ denoting the Euler-Mascheroni constant, we have

$$\lim_{n \to +\infty} \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \lim_{n \to +\infty} (h_{2n} - \ln 2n) - \lim_{n \to +\infty} (h_n - \ln n) + \ln 2 = \gamma - \gamma + \ln 2 = \ln 2,$$

i.e. the $(2n)^{\text{th}}$ partial sum converges to $\ln 2$.   On the other hand, we have

$$\lim_{n \to +\infty} \sum_{k=1}^{2n+1} \frac{(-1)^{k+1}}{k} = \lim_{n \to +\infty} \left(\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} - \frac{1}{2n+1} \right) = \left(\lim_{n \to +\infty} \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} \right) - \left(\lim_{n \to +\infty} \frac{1}{2n+1} \right)$$

$$= \ln 2 - 0 = \ln 2$$

the $(2n+1)^{\text{st}}$ partial sum also converges to $\ln 2$.   Combining the odd and even terms in the sequence of

partial sums, we conclude that $\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = \ln 2$; i.e. $\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k}$ converges to $\ln 2$.   ∎

Alternative solution: For each $n \in \mathbb{N}$, $h_{2n} - h_n$ can be treated as the right Riemann-sum of $f(x) = \frac{1}{x}$ with

respect to the regular partition of $[1, 2]$ into n subintervals.   Therefore

$$\lim_{n \to +\infty} (h_{2n} - h_n) = \lim_{n \to +\infty} \sum_{k=n+1}^{2n} \frac{1}{k} = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=n+1}^{2n} \frac{1}{k/n} = \int_1^2 \frac{1}{x} dx = \ln 2.$$

Thus according to (a), we have $\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = \ln 2$; i.e. $\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k}$ converges to $\ln 2$.   ∎

6.   For each $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} \left(\frac{1}{k} - \ln \left(1 + \frac{1}{k} \right) \right) = (1 - \ln 2) + \left(\frac{1}{2} - \ln \frac{3}{2} \right) + \left(\frac{1}{3} - \ln \frac{4}{3} \right) + \cdots + \left(\frac{1}{n} - \ln \frac{n+1}{n} \right)$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) - \left(\ln 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \cdots \cdot \frac{n+1}{n} \right) = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) - \ln(n+1)$$

$$= \left(\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) - \ln n \right) + \ln \frac{n}{n+1}.$$

So

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \ln \left(1 + \frac{1}{k} \right) \right) = \lim_{n \to +\infty} \left(\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) - \ln n \right) + \lim_{n \to +\infty} \ln \frac{n}{n+1} = \gamma + \ln 1 = \gamma.$$

In other words, the series $\sum_{k=1}^{+\infty} \left(\frac{1}{k} - \ln \left(1 + \frac{1}{k} \right) \right)$ converges to the Euler-Mascheroni constant γ.

7.   (a)   The terms of the series are all positive.   Since

$$\lim_{n \to +\infty} \frac{n^2 \sin^p(1/n)}{1/n^{p-2}} = \left(\lim_{n \to +\infty} \frac{\sin(1/n)}{1/n} \right)^p = 1,$$

which exists and is in $(0, +\infty)$, according to limit comparison test, the series $\sum_{k=1}^{+\infty} k^2 \sin^p \frac{1}{k}$ converges if and

only if $\sum_{k=1}^{+\infty} \frac{1}{k^{p-2}}$ converges.   Now $\sum_{k=1}^{+\infty} \frac{1}{k^{p-2}}$ converges if and only if $p - 2 > 1$ according to p-test, so

$\sum_{k=1}^{+\infty} k^2 \sin^p \frac{1}{k}$ converges if and only if $p > 3$.

(b) Let $f:[3,+\infty) \to [0,+\infty)$ be the non-negative function $f(x) = \dfrac{1}{x(\ln x)(\ln \ln x)^p}$.   Then f is decreasing on $[3,+\infty)$ regardless of the value of p.   Since the improper integral

$$\int_3^{+\infty} f(x)\,dx = \int_3^{+\infty} \frac{1}{x(\ln x)(\ln \ln x)^p}\,dx = \int_3^{+\infty} \frac{1}{(\ln \ln x)^p}\,d\ln\ln x = \begin{cases} \left[\dfrac{1}{1-p}(\ln \ln x)^{1-p}\right]_3^{+\infty} & \text{if } p \neq 1 \\[2mm] [\ln \ln \ln x]_3^{+\infty} & \text{if } p = 1 \end{cases}$$

converges if and only if $1-p < 0$, according to integral test, the series $\sum_{k=3}^{+\infty} \dfrac{1}{k(\ln k)(\ln \ln k)^p}$ converges if and only if $1-p < 0$, i.e. $p > 1$.

(c) For each $n \in \mathbb{N}$, we have

$$\frac{1}{(\ln \ln n)^{p \ln n}} = \frac{1}{e^{p(\ln n)(\ln \ln \ln n)}} = \frac{1}{n^{p \ln \ln \ln n}}.$$

⊙ If $p = 0$, then $\lim\limits_{n \to +\infty} \dfrac{1}{n^{p \ln \ln \ln n}} = \lim\limits_{n \to +\infty} 1 = 1$, so the series diverges by term test.

⊙ If $p < 0$, then $\lim\limits_{n \to +\infty} \dfrac{1}{n^{p \ln \ln \ln n}} = +\infty$, so the series also diverges by term test.

⊙ If $p > 0$, then since $\lim\limits_{n \to +\infty} p \ln \ln \ln n = +\infty$, there exists $N > 0$ such that for every integer $n \geq N$ we have $p \ln \ln \ln n \geq 2$, i.e. $0 < \dfrac{1}{n^{p \ln \ln \ln n}} \leq \dfrac{1}{n^2}$.   Since $\sum_{k=3}^{+\infty} \dfrac{1}{k^2}$ converges by p-test, $\sum_{k=3}^{+\infty} \dfrac{1}{(\ln \ln k)^{p \ln k}}$ also converges by comparison test.

Therefore the series $\sum_{k=3}^{+\infty} \dfrac{1}{(\ln \ln k)^{p \ln k}}$ converges if and only if $p > 0$.

(d) For every $p \in \mathbb{R}$, since

$$\lim_{n \to +\infty} \sqrt[n]{|a_n|} = \lim_{n \to +\infty} \sqrt[n]{\frac{n^p}{(\ln n)^n}} = \lim_{n \to +\infty} \frac{(\sqrt[n]{n})^p}{\ln n} = 0 < 1,$$

the series $\sum_{k=2}^{+\infty} \dfrac{k^p}{(\ln k)^k}$ converges by root test.   Therefore the series converges for every $p \in \mathbb{R}$.

8. With $a_n^+ := \max\{a_n, 0\}$ and $a_n^- := \max\{-a_n, 0\}$, we have
$$a_n = a_n^+ - a_n^- \qquad \text{and} \qquad |a_n| = a_n^+ + a_n^- \qquad \text{for every } n.$$

(a) Suppose that $\sum_{k=1}^{+\infty} a_k$ converges absolutely.   Then $\sum_{k=1}^{+\infty} |a_k|$ converges.   Since
$$0 \leq a_n^+ \leq |a_n| \qquad \text{and} \qquad 0 \leq a_n^- \leq |a_n| \qquad \text{for every } n,$$
we must have both $\sum_{k=1}^{+\infty} a_k^+$ and $\sum_{k=1}^{+\infty} a_k^-$ converge, by comparison test.   ∎

(b) Suppose that $\sum_{k=1}^{+\infty} a_k$ converges conditionally.   Then $\sum_{k=1}^{+\infty} a_k$ converges but $\sum_{k=1}^{+\infty} |a_k|$ diverges.

⊙ If $\sum_{k=1}^{+\infty} a_k^+$ converges, then $\sum_{k=1}^{+\infty} a_k^- = \sum_{k=1}^{+\infty}(a_k^+ - a_k)$ also converges.

⊙ If $\sum_{k=1}^{+\infty} a_k^-$ converges, then $\sum_{k=1}^{+\infty} a_k^+ = \sum_{k=1}^{+\infty}(a_k^- + a_k)$ also converges.

In any case, the series

$$\sum_{k=1}^{+\infty} |a_k| = \sum_{k=1}^{+\infty}(a_k^+ + a_k^-)$$

converges, which is a contradiction.   Therefore $\sum_{k=1}^{+\infty} a_k^+$ and $\sum_{k=1}^{+\infty} a_k^-$ must both diverge.   ∎

9. (a) Consider the series of absolute values $\sum_{k=1}^{+\infty} \left| \frac{\cos k}{k^3} \right|$. For each $n \in \mathbb{N}$, we have

$$\left| \frac{\cos n}{n^3} \right| \le \frac{1}{n^3}.$$

Since $\sum_{k=0}^{+\infty} \frac{1}{k^3}$ converges by p-test, it follows that $\sum_{k=1}^{+\infty} \left| \frac{\cos k}{k^3} \right|$ converges by comparison test. Therefore we conclude that $\sum_{k=1}^{+\infty} \frac{\cos k}{k^3}$ **converges absolutely**.

(b) ⊙ Consider the series of absolute values $\sum_{k=0}^{+\infty} \left| (-1)^{k+1} (\sqrt{k+1} - \sqrt{k}) \right| = \sum_{k=0}^{+\infty} (\sqrt{k+1} - \sqrt{k})$. Now for each $n \in \mathbb{N}$, we have the partial sum

$$\sum_{k=0}^{n} (\sqrt{k+1} - \sqrt{k}) = (\sqrt{1} - \sqrt{0}) + (\sqrt{2} - \sqrt{1}) + \cdots + (\sqrt{n+1} - \sqrt{n}) = \sqrt{n+1},$$

so $\sum_{k=0}^{+\infty} (\sqrt{k+1} - \sqrt{k})$ diverges to $+\infty$, i.e. $\sum_{k=0}^{+\infty} (-1)^{k+1} (\sqrt{k+1} - \sqrt{k})$ does not converge absolutely.

⊙ On the other hand, we have

$$\lim_{n \to +\infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n \to +\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0.$$

Moreover, for each $n \in \mathbb{N}$ we have

$$(\sqrt{n+2} - \sqrt{n+1}) - (\sqrt{n+1} - \sqrt{n}) = \frac{1}{\sqrt{n+2} + \sqrt{n+1}} - \frac{1}{\sqrt{n+1} + \sqrt{n}} < 0,$$

so the sequence $(\sqrt{n+1} - \sqrt{n})$ is decreasing. Thus the series $\sum_{k=0}^{+\infty} (-1)^{k+1} (\sqrt{k+1} - \sqrt{k})$ converges by alternating series test.

Therefore we conclude that $\sum_{k=0}^{+\infty} (-1)^{k+1} (\sqrt{k+1} - \sqrt{k})$ **converges conditionally**.

(c) Note that $\cos k\pi = (-1)^k$ for every positive integer k.

⊙ Consider the series of absolute values $\sum_{k=1}^{+\infty} \left| \cos k\pi \sin \frac{1}{k\pi} \right| = \sum_{k=1}^{+\infty} \left| (-1)^k \sin \frac{1}{k\pi} \right| = \sum_{k=1}^{+\infty} \sin \frac{1}{k\pi}$. Since $\sum_{k=1}^{+\infty} \frac{1}{k}$ diverges (by p-test) and since

$$\lim_{n \to +\infty} \frac{\sin(1/n\pi)}{1/n} = \frac{1}{\pi}$$

which exists and is positive, it follows that $\sum_{k=1}^{+\infty} \sin \frac{1}{k\pi}$ also diverges by limit comparison test, i.e. $\sum_{k=1}^{+\infty} \cos k\pi \sin \frac{1}{k\pi}$ does not converge absolutely.

⊙ On the other hand, since $\sin$ is continuous at 0 we have

$$\lim_{n \to +\infty} \sin \frac{1}{n\pi} = \sin \lim_{n \to +\infty} \frac{1}{n\pi} = 0.$$

Moreover, for each $n \in \mathbb{N}$ we have $\frac{1}{(n+1)\pi} < \frac{1}{n\pi}$; since $\sin$ is (strictly) increasing on $\left[0, \frac{\pi}{2} \right]$, we have

$$\sin \frac{1}{(n+1)\pi} < \sin \frac{1}{n\pi},$$

and so the sequence $\left(\sin \frac{1}{n\pi} \right)$ is decreasing. Thus the series $\sum_{k=1}^{+\infty} \cos k\pi \sin \frac{1}{k\pi} = \sum_{k=1}^{+\infty} (-1)^k \sin \frac{1}{k\pi}$ converges by alternating series test.

Therefore we conclude that $\sum_{k=1}^{+\infty} \cos k\pi \sin \frac{1}{k\pi}$ **converges conditionally**.

(d) For each integer $n \geq 2$, we have the partial sum

$$\sum_{k=2}^{n} \frac{(-1)^k}{\sqrt{k}+(-1)^k} = \sum_{k=2}^{n} \frac{(-1)^k\left(\sqrt{k}-(-1)^k\right)}{k-1} = \sum_{k=2}^{n}(-1)^k \frac{1}{\sqrt{k}-1/\sqrt{k}} - \sum_{k=2}^{n} \frac{1}{k-1}.$$

Now consider the series $\sum_{k=2}^{+\infty}(-1)^k \frac{1}{\sqrt{k}-\frac{1}{\sqrt{k}}}$.   We have

$$\lim_{n\to+\infty} \frac{1}{\sqrt{n}-\dfrac{1}{\sqrt{n}}} = 0.$$

Moreover, for each integer $n \geq 2$ we have $\sqrt{n+1} > \sqrt{n}$ and $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$, so $\left(\frac{1}{\sqrt{n}-\frac{1}{\sqrt{n}}}\right)$ is obviously a

decreasing sequence.   Thus the series $\sum_{k=2}^{+\infty}(-1)^k \frac{1}{\sqrt{k}-\frac{1}{\sqrt{k}}}$ converges by alternating series test.   Finally we see

that the given series $\sum_{k=2}^{+\infty} \frac{(-1)^k}{\sqrt{k}+(-1)^k}$ **diverges**, because otherwise if it converges, then the above would imply

that the harmonic series

$$\sum_{k=2}^{+\infty} \frac{1}{k-1} = \sum_{k=2}^{+\infty}(-1)^k \frac{1}{\sqrt{k}-1/\sqrt{k}} - \sum_{k=2}^{+\infty} \frac{(-1)^k}{\sqrt{k}+(-1)^k}$$

converges, which is a contradiction.

Remark: Note that the **alternating series test does not apply** on the given series $\sum_{k=2}^{+\infty} \frac{(-1)^k}{\sqrt{k}+(-1)^k}$ in (d),

because $\left(\frac{1}{\sqrt{n}+(-1)^n}\right)$ is not a decreasing sequence.

10. (a) (i)   Note that $b_1 = B_1$, so for $n = 1$ we have

$$a_2 B_1 - \sum_{k=1}^{1} B_k(a_{k+1} - a_k) = a_2 B_1 - B_1(a_2 - a_1) = a_1 B_1 = a_1 b_1 = \sum_{k=1}^{1} a_k b_k.$$

Assume that for some positive integer m we have

$$\sum_{k=1}^{m} a_k b_k = a_{m+1} B_m - \sum_{k=1}^{m} B_k(a_{k+1} - a_k).$$

Then for $n = m + 1$ we have

$$\sum_{k=1}^{m+1} a_k b_k = a_{m+1}b_{m+1} + \sum_{k=1}^{m} a_k b_k = a_{m+1}b_{m+1} + a_{m+1}B_m - \sum_{k=1}^{m} B_k(a_{k+1} - a_k)$$

$$= a_{m+1}(B_m + b_{m+1}) - \sum_{k=1}^{m} B_k(a_{k+1} - a_k) = a_{m+1}B_{m+1} - \sum_{k=1}^{m} B_k(a_{k+1} - a_k)$$

$$= a_{m+2}B_{m+1} - B_{m+1}(a_{m+2} - a_{m+1}) - \sum_{k=1}^{m} B_k(a_{k+1} - a_k) = a_{m+2}B_{m+1} - \sum_{k=1}^{m+1} B_k(a_{k+1} - a_k).$$

So by mathematical induction, the given formula is true for every positive integer n.   ■

(ii) Since (B_n) is bounded, there exists $M \geq 0$ such that $|B_n| \leq M$ for every $n \in \mathbb{N}$. Now for each $n \in \mathbb{N}$, summation by parts gives the nth partial sum

$$\sum_{k=1}^{n} a_k b_k = a_{n+1} B_n + \sum_{k=1}^{n} B_k(a_k - a_{k+1}).$$

⊙ Since $0 \leq |a_{n+1} B_n| \leq M|a_{n+1}|$ for every $n \in \mathbb{N}$ and since $\lim_{n \to +\infty} (M|a_{n+1}|) = M\left(\lim_{n \to +\infty} |a_{n+1}|\right) = 0$,

we have $\lim_{n \to +\infty} |a_{n+1} B_n| = 0$ by squeeze theorem and therefore $\lim_{n \to +\infty} a_{n+1} B_n = 0$.

⊙ Since (a_n) is decreasing, we have

$$|B_n(a_n - a_{n+1})| = |B_n|(a_n - a_{n+1}) \leq M(a_n - a_{n+1}) \qquad \text{for every } n \in \mathbb{N}.$$

But since the telescoping series $\sum_{k=1}^{+\infty} M(a_k - a_{k+1}) = Ma_1$ converges, so $\sum_{k=1}^{+\infty} |B_k(a_k - a_{k+1})|$ also converges by comparison test. Therefore $\sum_{k=1}^{+\infty} B_k(a_k - a_{k+1})$ converges (absolutely).

Combining the two paragraphs above, it follows that $\sum_{k=1}^{+\infty} a_k b_k$ converges.   ∎

(b) We consider the following two cases.

⊙ If t is an integer multiple of π, then every term in $\sum_{k=1}^{+\infty} \frac{\sin kt}{k}$ is just 0, so the series converges to 0.

⊙ If t is not an integer multiple of π, then we let $a_n = \frac{1}{n}$ and $b_n = \sin nt$ for each $n \in \mathbb{N}$. Now (a_n) is obviously decreasing and $\lim_{n \to +\infty} a_n = 0$. Since $\sin \frac{t}{2} \neq 0$, for every $n \in \mathbb{N}$ we have

$$|B_n| = \left| \sum_{k=1}^{n} \sin kt \right| = \left| \frac{\cos \frac{t}{2} - \cos\left(n + \frac{1}{2}\right)t}{2\sin \frac{t}{2}} \right| \leq \frac{\left|\cos \frac{t}{2}\right| + \left|\cos\left(n + \frac{1}{2}\right)t\right|}{\left|2\sin \frac{t}{2}\right|} \leq \frac{1+1}{2\left|\sin \frac{t}{2}\right|} \leq \frac{1}{\left|\sin \frac{t}{2}\right|},$$

so (B_n) is a bounded sequence (Note that t is fixed, the upper bound works <u>for all n</u>). Therefore applying the result from (a)(ii), it follows that the series $\sum_{k=1}^{+\infty} \frac{\sin kt}{k} = \sum_{k=1}^{+\infty} a_k b_k$ converges also.

Remark: The convergence test proved in (a)(ii) is called ***Dirichlet's test***. It is a generalization of the alternating series test, which is the case when $b_n = (-1)^n$.

11. (a) The coefficients of the given power series are given by $c_n = n^{\sqrt{n}}$. Now the limit

$$\lim_{n \to +\infty} \sqrt[n]{|c_n|} = \lim_{n \to +\infty} n^{\frac{\sqrt{n}}{n}} = \lim_{n \to +\infty} e^{\frac{\ln n}{\sqrt{n}}} = e^{\left(2 \lim_{n \to +\infty} \frac{\ln \sqrt{n}}{\sqrt{n}}\right)} = e^0 = 1$$

exists, so the radius of convergence of the given power series is

$$R = \frac{1}{\lim_{n \to +\infty} \sqrt[n]{|c_n|}} = 1.$$

The center of the power series is 0, so its interval of convergence has end-points -1 and 1.

⊙ At 1, the power series becomes $\sum_{k=1}^{+\infty} k^{\sqrt{k}}$ which diverges by term test.

⊙ At -1, the power series becomes $\sum_{k=1}^{+\infty} k^{\sqrt{k}}(-1)^k$ which diverges by term test also.

Therefore the interval of convergence of the given power series is $(-1, 1)$.

(b) The coefficients of the given power series are given by $c_n = \frac{1}{2^n n^2}$.   Now the limit

$$\lim_{n \to +\infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n \to +\infty} \frac{2^{n+1}(n+1)^2}{2^n n^2} = \lim_{n \to +\infty} 2\left(1 + \frac{1}{n}\right)^2 = 2(1+0)^2 = 2$$

exists, so the radius of convergence of the given power series is

$$R = \lim_{n \to +\infty} \frac{|c_n|}{|c_{n+1}|} = 2.$$

The center of the power series is 0, so its interval of convergence has end-points -2 and 2.

⊙   At 2, the power series becomes $\sum_{k=0}^{+\infty} \frac{2^k}{2^k k^2} = \sum_{k=0}^{+\infty} \frac{1}{k^2}$ which converges by p-test.

⊙   At -2, the power series becomes $\sum_{k=0}^{+\infty} \frac{(-2)^k}{2^k k^2} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k^2}$ which converges by either alternating series

test or by absolute convergence test.

Therefore the interval of convergence of the given power series is $[-2, 2]$.

(c) The coefficients of the given power series $\sum_{k=1}^{+\infty} \frac{(1-2x)^k}{k} = \sum_{k=1}^{+\infty} \frac{(-2)^k}{k}\left(x - \frac{1}{2}\right)^k$ are given by $c_n = \frac{(-2)^n}{n}$ (but not

just $\frac{1}{n}$).   Now the limit

$$\lim_{n \to +\infty} \sqrt[n]{|c_n|} = \lim_{n \to +\infty} \sqrt[n]{\frac{2^n}{n}} = \lim_{n \to +\infty} \frac{2}{\sqrt[n]{n}} = \frac{2}{1} = 2$$

exists, so the radius of convergence of the given power series is

$$R = \frac{1}{\lim_{n \to +\infty} \sqrt[n]{|c_n|}} = \frac{1}{2}.$$

The center of the power series is $\frac{1}{2}$, so its interval of convergence has end-points 0 and 1.

⊙   At 0, the power series becomes $\sum_{k=1}^{+\infty} \frac{1}{k}$ which diverges by p-test.

⊙   At 1, the power series becomes $\sum_{k=1}^{+\infty} \frac{(-1)^k}{k}$ which converges by alternating series test.

Therefore the interval of convergence of the given power series is $(0, 1]$.

(d) The coefficients of the given power series are given by $c_n = \frac{(-1)^{n+1}}{\sqrt{n!}}$.   Now the limit

$$\lim_{n \to +\infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n \to +\infty} \frac{\sqrt{(n+1)!}}{\sqrt{n!}} = \lim_{n \to +\infty} \sqrt{n+1} = +\infty$$

exists, so the radius of convergence of the given power series is also

$$R = +\infty.$$

The interval of convergence of the given power series is $\mathbb{R}$.

12. The coefficients of the given power series are given by $c_n = a^n + b^n$. Now since $a > b > 0$, we have

$$a^n \leq a^n + b^n \leq a^n + a^n \qquad \text{for every } n \in \mathbb{N},$$

so taking n^{th} root of each component we obtain

$$a \leq \sqrt[n]{a^n + b^n} \leq \sqrt[n]{2}\, a \qquad \text{for every } n \in \mathbb{N}.$$

Since $\displaystyle\lim_{n \to +\infty} \sqrt[n]{2} = 2^{\lim_{n \to +\infty} \frac{1}{n}} = 2^0 = 1$, the limit

$$\lim_{n \to +\infty} \sqrt[n]{|c_n|} = \lim_{n \to +\infty} \sqrt[n]{a^n + b^n} = a$$

exists by squeeze theorem.   Therefore the radius of convergence of the given power series is

$$R = \frac{1}{\displaystyle\lim_{n \to +\infty} \sqrt[n]{|c_n|}} = \frac{1}{a}.$$

13. (a) Suppose that the power series $\sum_{k=0}^{+\infty} a_k x^k$ has radius of convergence R.   Then the series $\sum_{k=0}^{+\infty} a_k t^k$ converges for every $t \in (-R, R)$.

   ⊙ Now if $x \in \left(-R^{\frac{1}{m}}, R^{\frac{1}{m}}\right)$, then $x^m \in (-R, R)$, so the series $\sum_{k=0}^{+\infty} a_k x^{mk} = \sum_{k=0}^{+\infty} a_k (x^m)^k$ converges.

   This shows that the radius of convergence of $\sum_{k=0}^{+\infty} a_k x^{mk}$ is at least $R^{\frac{1}{m}}$.

   ⊙ If the radius of convergence of $\sum_{k=0}^{+\infty} a_k x^{mk}$, call it ρ, is strictly greater than $R^{\frac{1}{m}}$, then we take a number

   $t \in \left(R^{\frac{1}{m}}, \rho\right)$.   The series $\sum_{k=0}^{+\infty} a_k t^{mk}$ converges.   But this implies that the series $\sum_{k=0}^{+\infty} a_k x^k$ converges

   at the number $x = t^m > R$, which contradicts with the fact that $\sum_{k=0}^{+\infty} a_k x^k$ has radius of convergence R.

   Therefore the radius of convergence of $\sum_{k=0}^{+\infty} a_k x^{mk}$ is exactly $R^{\frac{1}{m}}$.   ∎

   (b) (i) We consider the power series $\sum_{k=0}^{+\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^k$ whose coefficients are given by $c_n = \frac{(-1)^n}{2^{2n}(n!)^2}$.   Since

   $$\lim_{n \to +\infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n \to +\infty} \frac{2^{2(n+1)}\big((n+1)!\big)^2}{2^{2n}(n!)^2} = \lim_{n \to +\infty} 4(n+1)^2 = +\infty$$

   exists, so the radius of convergence of the given power series is also $R = +\infty$.   According to (a), the radius

   of convergence of $J_0(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k}$ is also $+\infty$.

   (ii) We consider the power series $1 + \frac{1}{2 \cdot 3} x + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^2 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^3 + \cdots$ whose coefficients are given by

   $c_n = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdots (3n)}$.   Since

   $$\lim_{n \to +\infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n \to +\infty} \frac{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdots (3n)(3n+2)(3n+3)}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdots (3n)} = \lim_{n \to +\infty} (3n+2)(3n+3) = +\infty$$

   exists, so the radius of convergence of the given power series is also $R = +\infty$.   According to (a), the radius

   of convergence of $A(x) = 1 + \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \cdots$ is also $+\infty$.

(c) (i) For every $x \in \mathbb{R}$, we differentiate the power series $J_0(x)$ termwise to obtain

$$J_0{}'(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{2^{2k}(k!)^2} 2kx^{2k-1} = \sum_{k=1}^{+\infty} \frac{(-1)^k}{2^{2k-1}(k!)(k-1)!} x^{2k-1} = \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k+1}(k+1)!\, k!} x^{2k+1}.$$

Differentiating once more, we have

$$\begin{aligned}
J_0{}''(x) &= \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k+1}(k+1)!\, k!}(2k+1)x^{2k} \\
&= \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k+1}(k+1)!\, k!}(2k+2)x^{2k} - \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k+1}(k+1)!\, k!} x^{2k} \\
&= \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k}(k!)^2} x^{2k} - \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k+1}(k+1)!\, k!} x^{2k}.
\end{aligned}$$

Thus for every $x \in \mathbb{R}$ we have

$$\begin{aligned}
xJ_0{}''(x) + J_0{}'(x) &= \left(\sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k}(k!)^2} x^{2k+1} - \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k+1}(k+1)!\, k!} x^{2k+1} \right) + \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k+1}(k+1)!\, k!} x^{2k+1} \\
&= \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k}(k!)^2} x^{2k+1} = -x \sum_{k=0}^{+\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k} = -xJ_0(x),
\end{aligned}$$

i.e. $xJ_0{}''(x) + J_0{}'(x) + xJ_0(x) = 0$.   ∎

(ii) For every $x \in \mathbb{R}$, we differentiate the power series $A(x)$ termwise to obtain

$$A'(x) = \frac{1}{2 \cdot 3} 3x^2 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} 6x^5 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} 9x^8 + \cdots$$

$$A''(x) = \frac{1}{2 \cdot 3} 3 \cdot 2x + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} 6 \cdot 5x^4 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} 9 \cdot 8x^7 + \cdots$$

$$= x + \frac{1}{2 \cdot 3} x^4 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^7 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^{10} + \cdots$$

$$= x \left(1 + \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \cdots \right) = xA(x),$$

i.e. $A''(x) - xA(x) = 0$.   ∎

14. (a) Consider the geometric series

$$\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x} \qquad \text{for every } x \in (-1,1).$$

Differentiating both sides, we have $\sum_{k=1}^{+\infty} kx^{k-1} = \frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}$ for every $x \in (-1,1)$.   Multiplying both

sides by x, we have

$$\sum_{k=1}^{+\infty} kx^k = \frac{x}{(1-x)^2} \qquad \text{for every } x \in (-1,1).$$

Differentiating both sides again, we have $\sum_{k=1}^{+\infty} k^2 x^{k-1} = \frac{d}{dx} \frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^3}$ for every $x \in (-1,1)$.

Multiplying both sides by x, we have

$$\sum_{k=1}^{+\infty} k^2 x^k = \frac{x + x^2}{(1-x)^3} \qquad \text{for every } x \in (-1, 1).$$

At both the end-points -1 and 1 of the interval of convergence, the series diverges by term test.

(b) Consider the geometric series

$$\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x} \qquad \text{for every } x \in (-1, 1).$$

Shifting the indices of the series, we also have $\sum_{k=2}^{+\infty} x^{k-2} = \frac{1}{1-x}$ for every $x \in (-1, 1)$.　Now relabeling x by t and then integrating termwise from 0 to x, we obtain

$$\sum_{k=2}^{+\infty} \frac{1}{k-1} x^{k-1} = \int_0^x \frac{1}{1-t} dt = -\ln(1-x) \qquad \text{for every } x \in (-1, 1).$$

Integrating once more, we obtain

$$\sum_{k=2}^{+\infty} \frac{1}{k(k-1)} x^k = \int_0^x -\ln(1-t)\, dt = (1-x)\ln(1-x) + x \qquad \text{for every } x \in (-1, 1).$$

Finally replacing x by $x - 1$, we have

$$\sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (x-1)^k = (2-x)\ln(2-x) + x - 1 \qquad \text{for every } x \in (0, 2).$$

⊙ At the end-point 2 of the interval of convergence, the series becomes $\sum_{k=2}^{+\infty} \frac{1}{k(k-1)} = \sum_{k=2}^{+\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right)$

which is a telescoping series which converges to 1.

⊙ At the end-point 0 of the interval of convergence, the series becomes $\sum_{k=2}^{+\infty} \frac{(-1)^k}{k(k-1)}$ which converges by

alternating series test.　According to Abel's limit theorem, we have

$$\sum_{k=2}^{+\infty} \frac{(-1)^k}{k(k-1)} = \lim_{x \to 0^+} [(2-x)\ln(2-x) + x - 1] = 2\ln 2 - 1.$$

(c) Consider the geometric series

$$\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x} \qquad \text{for every } x \in (-1, 1).$$

Shifting the indices of the series, we also have $\sum_{k=1}^{+\infty} x^{k-1} = \frac{1}{1-x}$ for every $x \in (-1, 1)$.　Now relabeling x by t and then integrating termwise from 0 to x, we obtain

$$\sum_{k=1}^{+\infty} \frac{1}{k} x^k = \int_0^x \frac{1}{1-t} dt = -\ln(1-x) \qquad \text{for every } x \in (-1, 1).$$

Integrating once more, we obtain

$$\sum_{k=1}^{+\infty} \frac{1}{k(k+1)} x^{k+1} = \int_0^x -\ln(1-t)\, dt = (1-x)\ln(1-x) + x \qquad \text{for every } x \in (-1, 1).$$

Integrating once more, we have

$$\sum_{k=1}^{+\infty} \frac{1}{k(k+1)(k+2)} x^{k+2} = \int_0^x [(1-t)\ln(1-t) + t]dt = \frac{3}{4}x^2 - \frac{1}{2}x - \frac{1}{2}(1-x)^2 \ln(1-x)$$

for every $x \in (-1, 1)$.   Dividing by x^2, we get

$$\sum_{k=1}^{+\infty} \frac{1}{k(k+1)(k+2)} x^k = \frac{3}{4} - \frac{1}{2x} - \frac{1}{2}\left(\frac{1}{x} - 1\right)^2 \ln(1-x)$$

for every $x \in (-1, 1) \setminus \{0\}$.   For $x = 0$, the series becomes 0 as all terms are 0.

⊙   At the end-point 1 of the interval of convergence, the series becomes $\sum_{k=1}^{+\infty} \frac{1}{k(k+1)(k+2)}$ which converges

by comparing to the p-series $\sum_{k=1}^{+\infty} \frac{1}{k^3}$.   According to Abel's limit theorem, we have

$$\sum_{k=1}^{+\infty} \frac{1}{k(k+1)(k+2)} = \lim_{x \to 1^-}\left[\frac{3}{4} - \frac{1}{2x} - \frac{1}{2}\left(\frac{1}{x} - 1\right)^2 \ln(1-x)\right] = \frac{1}{4}.$$

⊙   At the end-point -1 of the interval of convergence, the series becomes $\sum_{k=1}^{+\infty} \frac{(-1)^k}{k(k+1)(k+2)}$ which converges

by alternating series test.   According to Abel's limit theorem, we have

$$\sum_{k=1}^{+\infty} \frac{(-1)^k}{k(k+1)(k+2)} = \lim_{x \to -1^+}\left[\frac{3}{4} - \frac{1}{2x} - \frac{1}{2}\left(\frac{1}{x} - 1\right)^2 \ln(1-x)\right] = \frac{5}{4} - 2\ln 2.$$

(d) Consider the geometric series

$$\sum_{k=0}^{+\infty} x^{2k} = \frac{1}{1-x^2} \qquad \text{for every } x \in (-1, 1).$$

Relabeling x by t and then integrating termwise from 0 to x, we obtain

$$\sum_{k=0}^{+\infty} \frac{1}{2k+1} x^{2k+1} = \int_0^x \frac{1}{1-t^2} dt = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x) \qquad \text{for every } x \in (-1, 1).$$

⊙   At the end-point 1 of the interval of convergence, the series becomes $\sum_{k=0}^{+\infty} \frac{1}{2k+1}$ which diverges by

comparing to the harmonic series $\sum_{k=0}^{+\infty} \frac{1}{2k+2} = \frac{1}{2}\sum_{k=1}^{+\infty} \frac{1}{k}$.

⊙   At the end-point -1 of the interval of convergence, the series becomes $\sum_{k=0}^{+\infty} \frac{-1}{2k+1} = -\sum_{k=0}^{+\infty} \frac{1}{2k+1}$ which

also diverges.

(e) Consider the geometric series

$$\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x} \qquad \text{for every } x \in (-1, 1).$$

Differentiating both sides, we have $\sum_{k=1}^{+\infty} kx^{k-1} = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$ for every $x \in (-1, 1)$.   Multiplying both

sides by x, we have

$$\sum_{k=1}^{+\infty} kx^k = \frac{x}{(1-x)^2} \qquad \text{for every } x \in (-1, 1).$$

Now relabeling x by t and then integrating termwise from 0 to x, we obtain

$$\sum_{k=1}^{+\infty} \frac{k}{k+1} x^{k+1} = \int_0^x \frac{t}{(1-t)^2}\, dt = \ln(1-x) + \frac{x}{1-x} \qquad \text{for every } x \in (-1,1).$$

Dividing by x, we get

$$\sum_{k=1}^{+\infty} \frac{k}{k+1} x^{k} = \frac{\ln(1-x)}{x} + \frac{1}{1-x} \qquad \text{for every } x \in (-1,1) \setminus \{0\};$$

while for $x = 0$ the series converges to 0 since all terms become 0.

At both the end-points -1 and 1 of the interval of convergence, the series diverges by term test.

15. (a) Consider the power series

$$\sum_{k=1}^{+\infty} (x^{3k-1} - x^{3k+1}) = x^2 - x^4 + x^5 - x^7 + x^8 - x^{10} + \cdots,$$

which converges absolutely for $x \in (-1,1)$.   By the sum of geometric series we have

$$\sum_{k=1}^{+\infty} (x^{3k-1} - x^{3k+1}) = \frac{x^2}{1-x^3} - \frac{x^4}{1-x^3} = \frac{x^2(1-x^2)}{1-x^3} = \frac{x^2+x^3}{1+x+x^2} \qquad \text{for every } x \in (-1,1).$$

Relabeling x by t and then integrating termwise from 0 to x, we obtain

$$f(x) = \sum_{k=1}^{+\infty} \left(\frac{1}{3k} x^{3k} - \frac{1}{3k+2} x^{3k+2} \right) = \int_0^x \frac{t^2+t^3}{1+t+t^2}\, dt$$

$$= \int_0^x \left(t - \frac{1}{2}\frac{2t+1}{1+t+t^2} + \frac{1}{2}\frac{1}{\left(t+\frac{1}{2}\right)^2 + \frac{3}{4}} \right) dt$$

$$= \frac{1}{2}x^2 - \frac{1}{2}\ln(1+x+x^2) + \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} - \frac{\pi}{6\sqrt{3}}$$

for every $x \in (-1,1)$.

(b) Since

$$0 \le \frac{1}{3n^2+2n} \le \frac{1}{3n^2}$$

for every $n \in \mathbb{N}$ and since $\sum_{k=1}^{+\infty} \frac{1}{3k^2} = \frac{1}{3}\sum_{k=1}^{+\infty} \frac{1}{k^2}$ converges by p-test, the given series $\sum_{k=1}^{+\infty} \frac{1}{3k^2+2k}$ converges by comparison test.   Now according to (a) and Abel's limit theorem, we have

$$\sum_{k=1}^{+\infty} \frac{1}{3k^2+2k} = \frac{3}{2}\sum_{k=1}^{+\infty} \left(\frac{1}{3k} - \frac{1}{3k+2} \right) = \frac{3}{2}f(1) = \frac{3}{2}\lim_{x\to 1^-} f(x)$$

$$= \frac{3}{2}\lim_{x\to 1^-} \left[\frac{1}{2}x^2 - \frac{1}{2}\ln(1+x+x^2) + \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} - \frac{\pi}{6\sqrt{3}} \right]$$

$$= \frac{3}{4} - \frac{3}{4}\ln 3 + \frac{\pi}{4\sqrt{3}}.$$