

# HKUST MATH 1014 L1 assignment 6 submission

MATH1014 Calculus II Problem Set 6  
L01 (Spring 2024)

Problem Set 6

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 6 covers materials from §7.7 – §8.2.

## Q4

Compute the area of the region in  $\mathbb{R}^2$  that is inside the curve with polar equation

$$r = 1 + \cos \theta$$

but outside the curve with polar equation

$$r = 3 \cos \theta$$

.

*Hint:* Sketch the required region first.

The 1st curve  $[0, 2\pi)$  is a cardioid, with the cusp facing the left.

The 2nd curve on  $\left[0, \frac{\pi}{2}\right)$  is the upper half of the circle, with the leftmost point of the circle touching the origin.

The 2nd curve on  $\left[\frac{3\pi}{2}, 2\pi\right)$  is the lower half of the circle, with the leftmost point of the circle touching the origin.

The 2nd curve on  $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$  contains negative  $r$  and is ignored.

For  $\left[0, \frac{\pi}{2}\right)$ , there is 1 and only 1 intersection point excluding the origin.

$$1 + \cos \theta = 3 \cos \theta$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3} \quad \theta \in \left[0, \frac{\pi}{2}\right)$$

For  $\left[\frac{3\pi}{2}, 2\pi\right)$ , there is 1 and only 1 intersection point excluding the origin.

$$1 + \cos \theta = 3 \cos \theta$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{5\pi}{3} \quad \theta \in \left[\frac{3\pi}{2}, 2\pi\right)$$

For intersections at the origin,

$$1 + \cos \theta = 0$$

$$\cos \theta = -1$$

$$\theta = \pi \quad \theta \in [0, 2\pi)$$

$$3 \cos \theta = 0$$

$$\cos \theta = 0$$

$$\theta \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\} \quad \theta \in [0, 2\pi)$$

Therefore, the area is...

$$\begin{aligned} & \int (1 + \cos \theta)^2 d\theta \\ &= \int (1 + 2 \cos \theta + \cos^2 \theta) d\theta \\ &= \int \left(1 + 2 \cos \theta + \frac{1}{2} \cos 2\theta + \frac{1}{2}\right) d\theta \\ &= \frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta + C \\ & \int (3 \cos \theta)^2 d\theta \\ &= \int 9 \cos^2 \theta d\theta \\ &= \int \left(\frac{9}{2} \cos 2\theta + \frac{9}{2}\right) d\theta \\ &= \frac{9}{2} \theta + \frac{9}{4} \sin 2\theta + C \\ & \int ((1 + \cos \theta)^2 - (3 \cos \theta)^2) d\theta \\ &= \int (1 + \cos \theta)^2 d\theta - \int (3 \cos \theta)^2 d\theta \\ &= \frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta - \frac{9}{2} \theta - \frac{9}{4} \sin 2\theta + C \\ &= -3\theta + 2 \sin \theta - 2 \sin 2\theta + C \\ & \text{area} \\ &= \frac{1}{2} \left( \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} ((1 + \cos \theta)^2 - (3 \cos \theta)^2) d\theta + \int_{\frac{\pi}{2}}^{\frac{5\pi}{3}} (1 + \cos \theta)^2 d\theta + \int_{\frac{5\pi}{3}}^{2\pi} ((1 + \cos \theta)^2 - (3 \cos \theta)^2) d\theta \right) \\ &= \frac{1}{2} \left( [-3\theta + 2 \sin \theta - 2 \sin 2\theta]_{\frac{\pi}{3}}^{\frac{\pi}{2}} + [-3\theta + 2 \sin \theta - 2 \sin 2\theta]_{\frac{\pi}{2}}^{\frac{5\pi}{3}} + \left[ \frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \right) \\ &= \frac{1}{2} \left( -3 \left( \frac{\pi}{2} - \frac{\pi}{3} + \frac{5\pi}{3} - \frac{3\pi}{2} \right) + 2 \left( \sin \frac{\pi}{2} - \sin \frac{\pi}{3} + \sin \frac{5\pi}{3} - \sin \frac{3\pi}{2} \right) - 2 \left( \sin \pi - \sin \frac{2\pi}{3} + \sin \frac{10\pi}{3} - \sin 3\pi \right) \right. \\ & \quad \left. + \frac{3}{2} \left( \frac{3\pi}{2} - \frac{\pi}{2} \right) + 2 \left( \sin \frac{3\pi}{2} - \sin \frac{\pi}{2} \right) + \frac{1}{4} (\sin 3\pi - \sin \pi) \right) \\ &= \frac{1}{2} \left( -3 \cdot \frac{\pi}{3} + 2 \left( 1 - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + 1 \right) - 2 \left( 0 - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} - 0 \right) + \frac{3\pi}{2} + 2(-1 - 1) + \frac{1}{4}(0 - 0) \right) \\ &= \frac{1}{2} \left( -\pi + 4 - 2\sqrt{3} + 2\sqrt{3} + \frac{3\pi}{2} - 4 \right) \\ &= \frac{1}{2} \cdot \frac{\pi}{2} \end{aligned}$$

$$= \frac{\pi}{4}$$

## Q5

Let  $f: [0, \pi] \rightarrow [0, +\infty)$  be a continuously differentiable function, and consider the curve in  $\mathbb{R}^2$  defined by the polar equation

$$r = f(\theta)$$

.

Such a curve can be viewed as a polar curve on the one hand, and as a parameterized curve on the other hand. Show that the area bounded between the curve and the  $x$ -axis evaluated using Theorem 7.98 (as a polar curve) is the same as that evaluated using Theorem 7.96 (as a parameterized curve).

$f(\theta)$  is a nonnegative continuous function.

$$A_1 := \frac{1}{2} \int_0^\pi f(\theta)^2 d\theta \quad (\text{theorem 7.98})$$

$f(\theta) \in C^1([0, \pi]) \implies$  the curve is  $C^1$  smooth

$$A_2 := - \int_0^\pi (f(\theta) \sin \theta)(f(\theta) \cos \theta)' d\theta \quad (\text{theorem 7.96, } f(\theta) \in C^1([0, \pi]))$$

$$= \int_0^\pi (f(\theta) \sin \theta)(f(\theta) \sin \theta - f'(\theta) \cos \theta) d\theta \quad (f(\theta) \in C^1([0, \pi]))$$

$$= \int_0^\pi (f(\theta)^2 \sin^2 \theta - f(\theta) f'(\theta) \sin \theta \cos \theta) d\theta$$

$$= \int_0^\pi f(\theta)^2 d\theta - \int_0^\pi (f(\theta)^2 \cos^2 \theta + f(\theta) f'(\theta) \sin \theta \cos \theta) d\theta$$

$$= \int_0^\pi f(\theta)^2 d\theta - \int_0^\pi (f(\theta) \cos \theta) (f(\theta) \cos \theta + f'(\theta) \sin \theta) d\theta$$

$$= \int_0^\pi f(\theta)^2 d\theta - \int_0^\pi (f(\theta) \cos \theta)(f(\theta) \sin \theta)' d\theta$$

$$= \int_0^\pi f(\theta)^2 d\theta - [f(\theta)^2 \sin \theta \cos \theta]_0^\pi + \int_0^\pi (f(\theta) \sin \theta)(f(\theta) \cos \theta)' d\theta$$

$$= \int_0^\pi f(\theta)^2 d\theta - [f(\theta)^2 \sin \theta \cos \theta]_0^\pi - A_2$$

$$2A_2 = \int_0^\pi f(\theta)^2 d\theta - [f(\theta)^2 \sin \theta \cos \theta]_0^\pi$$

$$= \int_0^\pi f(\theta)^2 d\theta - (0 - 0)$$

$$A_2 = \frac{1}{2} \int_0^\pi f(\theta)^2 d\theta$$

$$= A_1$$

## Q8

Consider the region in the coordinate plane bounded by the curve

$$y = e^{-x}$$

, the  $x$ - and  $y$ -axes, and the line  $x = \ln 2$ . Find the volume of the solid obtained by revolving this region about the line  $x = \ln 2$ .

$$y = e^{-0}$$

$$= 1$$

$$y = e^{-\ln 2}$$

$$= \frac{1}{2}$$

$$y = e^{-x}$$

$$\ln y = -x$$

$$x = -\ln y$$

$$\int \ln y \, dy$$

$$= y \ln y - \int \frac{y}{y} \, dy$$

$$= y(\ln y - 1) + C$$

$$\int (\ln y)^2 \, dy$$

$$= y(\ln y)^2 - 2 \int \frac{y \ln y}{y} \, dy$$

$$= y(\ln y)^2 - 2 \int \ln y \, dy$$

$$= y(\ln y)^2 - 2y(\ln y - 1) + C$$

$$= y((\ln y)^2 - 2 \ln y + 2) + C$$

$$\text{volume}$$

$$= \pi \left( \int_0^{\frac{1}{2}} (\ln 2 - 0)^2 \, dy + \int_{\frac{1}{2}}^1 ((\ln 2 - 0)^2 - (\ln 2 + \ln y)^2) \, dy \right)$$

$$= \pi \left( \int_0^{\frac{1}{2}} (\ln 2)^2 \, dy + \int_{\frac{1}{2}}^1 ((\ln 2)^2 - (\ln 2)^2 - 2 \ln 2 \ln y - (\ln y)^2) \, dy \right)$$

$$= \pi \left( \frac{1}{2} (\ln 2)^2 - 2 \ln 2 [y(\ln y - 1)]_{\frac{1}{2}}^1 - [y((\ln y)^2 - 2 \ln y + 2)]_{\frac{1}{2}}^1 \right)$$

$$= \pi (0.5(\ln 2)^2 - 2 \ln 2(1(0 - 1) - 0.5(-\ln 2 - 1)) - (1(0 - 0 + 2) - 0.5((\ln 2)^2 + 2 \ln 2 + 2)))$$

$$= \pi (0.5(\ln 2)^2 + 2 \ln 2 - \ln 2(\ln 2 + 1) - 2 + 0.5(\ln 2)^2 + \ln 2 + 1)$$

$$= \pi ((\ln 2)^2 + 3 \ln 2 - 1 - (\ln 2)^2 - \ln 2)$$

$$= \pi (2 \ln 2 - 1)$$

## Q14

Let  $f: [1, +\infty) \rightarrow [0, +\infty)$  be the function

$$f(x) = \frac{1}{x}$$

and consider its graph in the plane.

### Q14.a

Consider the (unbounded) region under the graph of  $f$  and above the  $x$ -axis. Show that the solid obtained by revolving this region about the  $x$ -axis has a finite volume.

$$\text{volume}$$

$$= \pi \int_1^{+\infty} \left( \frac{1}{x} \right)^2 \, dx$$

$$= \pi \int_1^{+\infty} \frac{1}{x^2} \, dx$$

By the  $p$ -test, the integral is convergent.  
So the volume is finite.

### Q14.b

Show that the surface obtained revolving the graph of  $f$  about the  $x$ -axis has an infinite surface area.

surface area

$$\begin{aligned}
 &= 2\pi \int_1^{+\infty} \frac{1}{x} \sqrt{1 + \left(\left(\frac{1}{x}\right)'\right)^2} dx \\
 &= 2\pi \int_1^{+\infty} \frac{1}{x} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2} dx \\
 &= 2\pi \int_1^{+\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx \\
 &> 2\pi \int_1^{+\infty} \frac{1}{x} dx
 \end{aligned}$$

$$\left(x \in [1, +\infty) \implies \sqrt{1 + \frac{1}{x^4}} > 1\right)$$

By the  $p$ -test, the integral one line above this line is divergent.

By the comparison test, the original integral is divergent.

So the surface area is infinite.

## Q15

Let  $\mathbf{r}: [0, 2\pi] \rightarrow \mathbb{R}^2$  be the curve defined by

$$\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$$

.

### Q15.a

Find the arc-length of this curve.

$$\begin{aligned}
 &\mathbf{r}'(t) \\
 &= \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t \rangle \\
 &\text{arc-length} \\
 &= \int_0^{2\pi} \|\mathbf{r}'(t)\| dt \\
 &= \int_0^{2\pi} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} dt \\
 &= 3 \int_0^{2\pi} |\sin t \cos t| \sqrt{\cos^2 t + \sin^2 t} dt \\
 &= 3 \int_0^{2\pi} |\sin t \cos t| dt \\
 &= \frac{3}{2} \int_0^{2\pi} |\sin 2t| dt \\
 &= \frac{3}{4} \int_0^{4\pi} |\sin t| dt && (\text{change of variable } 2t \mapsto t) \\
 &= 3 \int_0^{\pi} \sin t dt && (\text{symmetry, } (\forall t \in [0, \pi])(\sin t \geq 0)) \\
 &= 3[-\cos t]_0^{\pi} \\
 &= 6
 \end{aligned}$$

### Q15.b

Find the area of the region in  $\mathbb{R}^2$  bounded by this curve.

The curve is a smooth curve.

$$\begin{aligned}
 & \text{area} \\
 &= \int_0^{2\pi} \cos^3 t (\sin^3 t)' dt \\
 &= 3 \int_0^{2\pi} \cos^4 t \sin^2 t dt \\
 &= \frac{3}{8} \int_0^{2\pi} (\cos 2t + 1) \sin^2 2t dt \\
 &= \frac{3}{8} \int_0^{2\pi} (\cos 2t + 1) \sin^2 2t dt \\
 &= \frac{3}{16} \int_0^{4\pi} (\cos t + 1) \sin^2 t dt && \text{(change of variable } 2t \mapsto t) \\
 &= \frac{3}{16} \left[ \frac{1}{3} \sin^3 t \right]_0^{4\pi} + \frac{3}{16} \int_0^{4\pi} \sin^2 t dt \\
 &= \frac{3}{32} \int_0^{4\pi} (1 - \cos 2t) dt \\
 &= \frac{3}{32} \left[ t - \frac{1}{2} \sin 2t \right]_0^{4\pi} \\
 &= \frac{3\pi}{8} - \frac{3}{64} (\sin 8\pi - \sin 0) \\
 &= \frac{3\pi}{8}
 \end{aligned}$$

### Q15.c

Find the volume of the solid obtained by revolving this curve about the  $x$ -axis.

The curve is a 4-folded symmetric star.

By symmetry, we only need to find the volume by considering one quadrant of the star.

Considering quadrant I,

$$x = \cos^3 t$$

$$y = \sin^3 t$$

$$= (\sin^2 t)^{\frac{3}{2}} \quad (y \geq 0 \implies \sin t \geq 0)$$

$$= (1 - \cos^2 t)^{\frac{3}{2}}$$

$$= \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} \quad (x \geq 0 \implies \cos t \geq 0)$$

volume

$$= 2\pi \int_0^1 y^2 dx$$

$$= 2\pi \int_0^1 \left(1 - x^{\frac{2}{3}}\right)^3 dx$$

$$= 2\pi \int_0^1 \left(1 - 3x^{\frac{2}{3}} + 3x^{\frac{4}{3}} - x^2\right) dx$$

$$= 2\pi \left[ x - \frac{9}{5} x^{\frac{5}{3}} + \frac{9}{7} x^{\frac{7}{3}} - \frac{1}{3} x^3 \right]_0^1$$

$$= 2\pi \left(1 - \frac{9}{5} + \frac{9}{7} - \frac{1}{3}\right)$$

$$= 2\pi \cdot \frac{105 - 189 + 135 - 35}{105}$$

$$= 2\pi \cdot \frac{16}{105}$$

$$= \frac{32\pi}{105}$$

### Q15.d

Find the area of the surface obtained by revolving this curve about the  $x$ -axis.

The curve is a 4-folded symmetric star.

By symmetry, we only need to the surface area using one quadrant of the star.

Considering quadrant I,

$$x = \cos^3 t$$

$$y = \sin^3 t$$

$$= (\sin^2 t)^{\frac{3}{2}} \quad (y \geq 0 \implies \sin t \geq 0)$$

$$= (1 - \cos^2 t)^{\frac{3}{2}}$$

$$= \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} \quad (x \geq 0 \implies \cos t \geq 0)$$

$$y' = \frac{3}{2} \left(1 - x^{\frac{2}{3}}\right)^{\frac{1}{2}} \left(-\frac{2}{3} x^{-\frac{1}{3}}\right)$$

$$= -\left(1 - x^{\frac{2}{3}}\right)^{\frac{1}{2}} x^{-\frac{1}{3}}$$

surface area

$$= 4\pi \int_0^1 y \sqrt{1 + (y')^2} dx$$

$$= 4\pi \int_0^1 \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} \sqrt{1 + \left(1 - x^{\frac{2}{3}}\right) x^{-\frac{2}{3}}} dx$$

$$= 4\pi \int_0^1 \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} x^{-\frac{1}{3}} dx$$

$$= 6\pi \int_0^1 (1 - x)^{\frac{3}{2}} dx \quad \left(\text{change of variable } x^{\frac{2}{3}} \rightarrow x\right)$$

$$= 6\pi \left[-\frac{2}{5} (1 - x)^{\frac{5}{2}}\right]_0^1$$

$$= \frac{12\pi}{5}$$

## Q16

Consider the cardioid in  $\mathbb{R}^2$  defined by the polar equation

$$r = 1 + \sin \theta$$

.

*Hint:* The given cardioid is symmetric about the  $y$ -axis. To generate a solid or a surface by revolving about the  $y$ -axis, we only need the **right-half** of the cardioid.

### Q16.a

Find the volume of the solid obtained by revolving this curve about the  $y$ -axis.

$r$  is always nonnegative because  $\sin \theta \geq -1$ .  
 As the cardioid is symmetric about the  $y$ -axis,  
 only considering the right-half...

volume

$$\begin{aligned}
 &= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 dy \\
 &= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 y' d\theta \\
 &= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \cos^2 \theta (r' \sin \theta + r \cos \theta) d\theta \\
 &= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin \theta)^2 \cos^2 \theta (2 \cos \theta \sin \theta + \cos \theta) d\theta \\
 &= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin \theta)^2 (1 - \sin^2 \theta) (2 \sin \theta + 1) \cos \theta d\theta \\
 &= \pi \int_{-1}^1 (1 + u)^3 (1 - u) (2u + 1) du && \text{(substitute } u := \sin \theta) \\
 &= \pi \int_0^2 u^3 (2 - u) (2u - 1) du && \text{(change of variable } 1 + u \mapsto u) \\
 &= \pi \int_0^2 u^3 (4u - 2 - 2u^2 + u) du \\
 &= \pi \int_0^2 u^3 (-2u^2 + 5u - 2) du \\
 &= \pi \int_0^2 (-2u^5 + 5u^4 - 2u^3) du \\
 &= \pi \left[ -\frac{1}{3}u^6 + u^5 - \frac{1}{2}u^4 \right]_0^2 \\
 &= \pi \left( -\frac{1}{3}(64) + 32 - \frac{1}{2}(16) \right) \\
 &= \pi \left( -\frac{64}{3} + 32 - 8 \right) \\
 &= \frac{8\pi}{3}
 \end{aligned}$$

## Q16.b

Find the area of the surface obtained by revolving this curve about the  $y$ -axis.



$r$  is always nonnegative because  $\sin \theta \geq -1$ .  
 As the cardioid is symmetric about the  $y$ -axis,  
 only considering the right-half...

surface area

$$\begin{aligned}
 &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sqrt{r^2 + r'^2} \, d\theta \\
 &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - y^2} \sqrt{r^2 + r'^2} \, d\theta & (x \geq 0) \\
 &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - r^2 \sin^2 \theta} \sqrt{r^2 + r'^2} \, d\theta \\
 &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |r \cos \theta| \sqrt{r^2 + r'^2} \, d\theta \\
 &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin \theta) \cos \theta \sqrt{1 + 2 \sin \theta + \sin^2 \theta + \cos^2 \theta} \, d\theta & \left( r \geq 0, \left( \forall \theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \right) (\cos \theta \geq 0) \right) \\
 &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin \theta) \cos \theta \sqrt{2 + 2 \sin \theta} \, d\theta \\
 &= 2\pi \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin \theta) \cos \theta \sqrt{1 + \sin \theta} \, d\theta \\
 &= 2\pi \sqrt{2} \int_0^2 u^{\frac{3}{2}} \, du & (\text{substitute } u := \sin \theta + 1) \\
 &= 2\pi \sqrt{2} \left[ \frac{2}{5} u^{\frac{5}{2}} \right]_0^2 \\
 &= 2\pi \sqrt{2} \cdot \frac{2}{5} \cdot 4\sqrt{2} \\
 &= \frac{32\pi}{5}
 \end{aligned}$$

## Q17

Determine whether each of the following series of real numbers converges or diverges. Also compute its limit (i.e. the sum) if it converges.

### Q17.b

$$\begin{aligned}
 &\sum_{k=1}^{+\infty} \frac{2}{k(k+1)(k+2)} \\
 &\sum_{k=1}^n \frac{2}{k(k+1)(k+2)} & (n \in \mathbb{Z}_{\geq 1}) \\
 &= \sum_{k=1}^n \left( \frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} \right) \\
 &= \frac{1}{1 \cdot 2} - \frac{1}{(n+1)(n+2)} & (\text{telescope}) \\
 &= \frac{1}{2} - \frac{1}{(n+1)(n+2)} \\
 &\sum_{k=1}^{+\infty} \frac{2}{k(k+1)(k+2)} \\
 &= \lim_{n \rightarrow +\infty} \sum_{k=1}^n \frac{2}{k(k+1)(k+2)} \\
 &= \lim_{n \rightarrow +\infty} \left( \frac{1}{2} - \frac{1}{(n+1)(n+2)} \right) \\
 &= \frac{1}{2} - 0 \\
 &= \frac{1}{2} \\
 &\therefore \text{The limit converges.}
 \end{aligned}$$

### Q17.c

$$\sum_{k=1}^{+\infty} \ln \left( 1 + \frac{1}{k} \right)$$

$$\begin{aligned}
& \sum_{k=1}^n \ln \left( 1 + \frac{1}{k} \right) && (n \in \mathbb{Z}_{\geq 1}) \\
&= \ln \left( \prod_{k=1}^n \left( 1 + \frac{1}{k} \right) \right) \\
&= \ln \left( \prod_{k=1}^n \frac{1}{k} (k+1) \right) \\
&= \ln \left( \frac{(n+1)!}{n!} \right) \\
&= \ln(n+1) \\
& \sum_{k=1}^{+\infty} \ln \left( 1 + \frac{1}{k} \right) \\
&= \lim_{n \rightarrow +\infty} \sum_{k=1}^n \ln \left( 1 + \frac{1}{k} \right) \\
&= \lim_{n \rightarrow +\infty} \ln(n+1) \\
&= +\infty \\
&\therefore \text{The limit diverges.}
\end{aligned}$$

**Q17.e**

$$\sum_{k=2}^{+\infty} \frac{k}{2^{k-1}}$$

*Hint:*  $\frac{d}{dx} x^k = kx^{k-1}$ .

$$f_k(x) := x^k$$

$$f'_k(x) = kx^{k-1}$$

$$\sum_{k=2}^n \frac{k}{2^{k-1}}$$

$$(n \in \mathbb{Z}_{\geq 2})$$

$$= \sum_{k=2}^n k \left( \frac{1}{2} \right)^{k-1}$$

$$= \sum_{k=2}^n f'_k \left( \frac{1}{2} \right)$$

$$= \left( \sum_{k=2}^n f_k(x) \right)' \Big|_{x=\frac{1}{2}}$$

(linearity)

$$= \left( \sum_{k=2}^n x^k \right)' \Big|_{x=\frac{1}{2}}$$

$$= \left( \frac{x^{n+1} - x^2}{x - 1} \right)' \Big|_{x=\frac{1}{2}}$$

$$= \left( x^2 \left( \frac{x^{n-1} - 1}{x - 1} \right) \right)' \Big|_{x=\frac{1}{2}}$$

$$= \left( (2x) \frac{x^{n-1} - 1}{x - 1} + x^2 \left( \frac{(n-1)x^{n-2}(x-1) - (x^{n-1} - 1)}{(x-1)^2} \right) \right)' \Big|_{x=\frac{1}{2}}$$

$$= 2(0.5) \frac{0.5^{n-1} - 1}{0.5 - 1} + (0.5)^2 \left( \frac{(n-1)(0.5)^{n-2}(0.5-1) - (0.5^{n-1} - 1)}{(0.5-1)^2} \right)$$

$$= 2(1 - 0.5^{n-1}) + (-(n-1)(0.5)^{n-1} + (1 - 0.5^{n-1}))$$

$$= 3(1 - 0.5^{n-1}) - (n-1)(0.5)^{n-1}$$

$$= 3 - 3(0.5)^{n-1} - n(0.5)^{n-1} + 0.5^{n-1}$$

$$= 3 - 2^{1-n}(n+2)$$

$$\sum_{k=2}^{+\infty} \frac{k}{2^{k-1}}$$

$$= \lim_{n \rightarrow +\infty} \sum_{k=2}^n \frac{k}{2^{k-1}}$$

$$= \lim_{n \rightarrow +\infty} (3 - 2^{1-n}(n+2))$$

$$= 3 - 0$$

$$= 3$$

(exponential is faster growing than linear)

$\therefore$  The limit converges.