MATH 1014 L1 assignment 1 submission

Please do not use the fundamental theorem of calculus in this assignment.

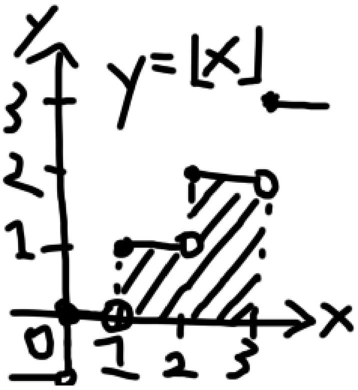
1

Evaluate each of the following integrals by considering some simple geometric shape.

Hint: First sketch the graph of each integrand.

1.a

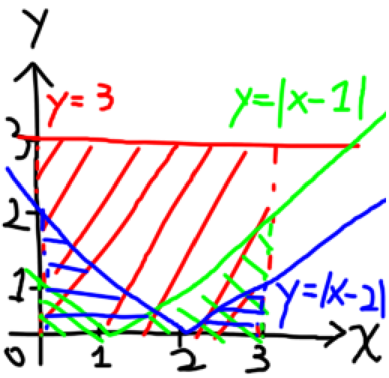
$\int_0^3 \lfloor x \rfloor \, \mathrm{d}x$



$$\begin{aligned}\int_0^3 \lfloor x \rfloor \, \mathrm{d}x &= 0(1-0) + 1(2-1) + 2(3-2) \\ &= 0 + 1 + 2 \\ &= 3\end{aligned}$$

1.b

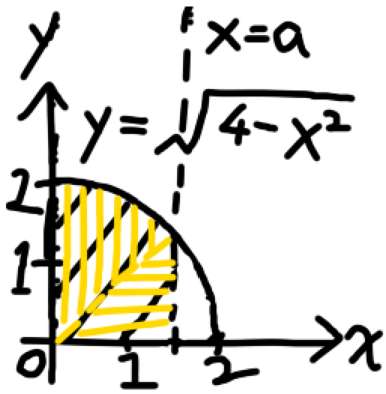
$\int_0^3 (3 - |x-1| - |x-2|) \, \mathrm{d}x$



$$\begin{aligned}\int_0^3 (3 - |x-1| - |x-2|) \, \mathrm{d}x &= \int_0^3 3 \, \mathrm{d}x - \int_0^3 |x-1| \, \mathrm{d}x - \int_0^3 |x-2| \, \mathrm{d}x \\ &= 3(3-0) - \left(\frac{1(1-0)}{2} + \frac{2(3-1)}{2}\right) - \left(\frac{2(2-0)}{2} + \frac{1(3-2)}{2}\right) \\ &= 9 - \left(\frac{1}{2} + 2\right) - \left(2 + \frac{1}{2}\right) \\ &= 4\end{aligned}$$

1.c

$\int_0^a \sqrt{4 - x^2} \, \mathrm{d}x$, where $0 \le a \le 2$



$$\begin{aligned}\int_0^a \sqrt{4-x^2}\,\mathrm{d}x &= \pi \cdot 2^2 \cdot \frac{\arcsin \frac{a}{2}}{2\pi} + \frac{a\sqrt{4-a^2}}{2} \\ &= 2\arcsin\frac{a}{2} + \frac{1}{2}a\sqrt{4-a^2}\end{aligned}$$

2

Find the pair of real numbers $a, b \in [0, 2\pi]$ with $a < b$ such that the integral

$$\int_a^b (\sin x - \cos x)\,\mathrm{d}x$$

...

Give justifications to your answers.

2.a

... attains its maximum possible value.

$$\begin{aligned}&\sin x - \cos x = 0 \\ &\qquad \sin x = \cos x \\ &\text{Case 1. } \cos x = 0 \\ &\sin^2 x + \cos^2 x = 1 \\ &\qquad \sin^2 x = 1 \\ &\qquad \sin x = \pm 1 \\ &\qquad\qquad \neq \cos x\end{aligned}$$

There are no solutions for x when $\cos x = 0$.

Case 2. $\cos x \neq 0$

$$\begin{aligned}&\sin x = \cos x \\ &\frac{\sin x}{\cos x} = 1 \qquad\qquad (\cos x \neq 0) \\ &\tan x = 1 \\ &\quad x = \frac{\pi}{4} + n\pi \qquad n \in \mathbb{Z}\end{aligned}$$

Within $[0, 2\pi]$, $x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$.

Using $\sin x - \cos x$ is continuous on $[0, 2\pi]$,

by the intermediate value theorem,

$\sin x - \cos x$ does not change its sign on $\left[0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{5\pi}{4}, 2\pi\right]$.

By evaluating $\sin x - \cos x$ in each of the above intervals, we get

$$\operatorname{sgn}(\sin x - \cos x) = \begin{cases} -1 & x \in \left[0, \frac{\pi}{4}\right) \\ 0 & x = \frac{\pi}{4} \\ 1 & x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right) \\ 0 & x = \frac{5\pi}{4} \\ -1 & x \in \left(\frac{5\pi}{4}, 2\pi\right] \end{cases}$$

To attain the maximum value for the integral,

select an interval such that the integrand

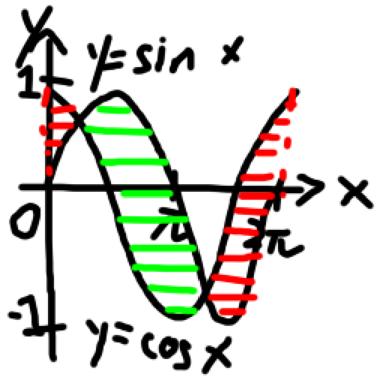
has the most area for positive values

and the least area in negative values.

In this case, $(a, b) = \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

2.b

... attains its minimum possible value.



Continuing from 2.a.
To attain the maximum value for the integral,
select an interval such that the integrand
has the least area for positive values
and the most area in negative values.

In this case, $(a, b) = \left(\frac{5\pi}{4}, 2\pi\right)$

We can confirm this by inspecting the sketch above.

3

Let $f(x) = \cos\left(x^2\right)$ and $g(x) = \cos\left(x^3\right)$. For each of the following pairs of integrals, explain which one has a greater value.

3.a

$\int_0^1 f(x)\,\mathrm{d}x$ and $\int_0^1 g(x)\,\mathrm{d}x$

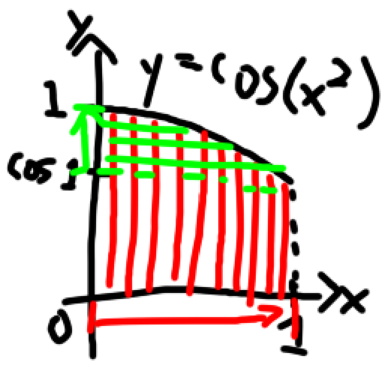
Considering $x \in (0, 1)$

$$\begin{aligned} x^2 &> x^3 \\ \cos\left(x^2\right) &< \cos\left(x^3\right) && (\cos * \text{ is strictly decreasing on } (0,1)) \\ f(x) &< g(x) \\ \int_0^1 f(x)\,\mathrm{d}x &< \int_0^1 g(x)\,\mathrm{d}x && (\text{the values of } f(0), f(1), g(0), g(1) \text{ do not affect the inequality as } \{0,1\} \text{ has zero Lebesgue measure}) \end{aligned}$$

3.b

$\int_0^1 f(x)\,\mathrm{d}x$ and $\int_{\cos 1}^1 f^{-1}(x)\,\mathrm{d}x$

(Note that f has an inverse defined on $[\cos 1, 1]$ because $f : [0, 1] \to [\cos 1, 1]$ is strictly decreasing.)



Refer to the sketch above.
The red area is the value of the first integral,
and the green area is the value of the second integral.
The colored arrows shows the integration orientation,
showing that both orientation are both positive
to check that the red and green area are indeed the values of the integral.
It can be seen that the red area is greater than the green area.
Hence, $\int_0^1 f(x)\,\mathrm{d}x > \int_{\cos 1}^1 f^{-1}(x)\,\mathrm{d}x$.

5

5.a

Let $a \geq 2$ be a real number. By considering appropriate trapeziums, show that

$$\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} \ln x\,\mathrm{d}x \leq \ln a \qquad \text{and} \qquad \int_{a-1}^{a} \ln x\,\mathrm{d}x \geq \frac{\ln(a-1)+\ln a}{2}$$

Considering $a \geq 2$, all operations below involving $\ln *$ produces nonnegative results.

Consider the rectangle under by $y = \ln a$ on $\left[a - \frac{1}{2}, a + \frac{1}{2}\right]$.
Its area is $\left(a + \frac{1}{2} - \left(a - \frac{1}{2}\right)\right) \ln a = \ln a$.
Rotate the graph of $y = \ln a$ with $(a, \ln a)$ as the pivot
such that the graph becomes tangent to $\ln x$ at $(a, \ln a)$,
i.e. $y = \left(\frac{\mathrm{d}}{\mathrm{d}u}\Big|_{u=a} \ln u\right)(x-a) + \ln a$.
The rectangle becomes a trapezium.
The area of the trapezium is the same as the rectangle because
the left side decreases by as much as the right side increases.
Since $\ln x$ is a concave function,
the tangent line is always higher than $\ln x$ except at a,
where they meet.
Then,

$$\begin{aligned}
\ln x &\leq \left(\frac{\mathrm{d}}{\mathrm{d}u}\Big|_{u=a} \ln u\right)(x-a) + \ln a && \left(x \in \left[a - \frac{1}{2}, a + \frac{1}{2}\right]\right) \\
\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} \ln x\,\mathrm{d}x &\leq \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} \left(\left(\frac{\mathrm{d}}{\mathrm{d}u}\Big|_{u=a} \ln u\right)(x-a) + \ln a\right) \mathrm{d}x \\
&\leq \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} \ln a\,\mathrm{d}x && \text{(the trapezium and the rectangle have the same area)} \\
&\leq \left(a + \frac{1}{2} - \left(a - \frac{1}{2}\right)\right) \ln a \\
&\leq \ln a
\end{aligned}$$

$\frac{\ln(a-1)+\ln a}{2}$ is the area of the trapezium enclosed by
the line segment joining $(a-1, \ln(a-1))$ and $(a, \ln a)$,
$x = a - 1$, $x = a$, and the x-axis.
The equation of the line segment extended to infinity is
$y = (\ln a - \ln(a-1))(x-a) + \ln a$.
Since $\ln x$ is a concave function,
all points in a line segment connecting two points on $\ln x$
are always lower than $\ln x$ except for the end points, where they meet.
Then,

$$\begin{aligned}
\ln x &\geq (\ln a - \ln(a-1))(x-a) + \ln a && (x \in [a-1, a]) \\
\int_{a-1}^{a} \ln x\,\mathrm{d}x &\geq \int_{a-1}^{a} ((\ln a - \ln(a-1))(x-a) + \ln a)\,\mathrm{d}x \\
&\geq \frac{\ln(a-1)+\ln a}{2} && \text{(area of the trapezium)}
\end{aligned}$$

5.b

Using the result from 5.a, deduce that

$$\int_{\frac{3}{2}}^{n} \ln x \, \mathrm{d}x \leq \ln(n!) - \frac{1}{2}\ln n \leq \int_{1}^{n} \ln x \, \mathrm{d}x$$

for every integer $n \geq 2$.

$$\begin{aligned}
\ln(n!) - \frac{1}{2}\ln n &= \sum_{i=1}^{n} \ln i - \frac{1}{2}\ln n \\
&= \sum_{i=2}^{n} \ln i - \frac{1}{2}\ln n \\
&\geq \sum_{i=2}^{n} \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \ln x \, \mathrm{d}x - \frac{1}{2}\ln n && \text{(5.a)} \\
&= \int_{\frac{3}{2}}^{n} \ln x \, \mathrm{d}x + \int_{n}^{n+\frac{1}{2}} \ln x \, \mathrm{d}x - \frac{1}{2}\ln n \\
&\geq \int_{\frac{3}{2}}^{n} \ln x \, \mathrm{d}x + \left(n + \frac{1}{2} - n\right) \ln x - \frac{1}{2}\ln x && \text{(}\ln * \text{ is strictly increasing)} \\
&= \int_{\frac{3}{2}}^{n} \ln x \, \mathrm{d}x \\
\ln(n!) - \frac{1}{2}\ln n &= \sum_{i=1}^{n} \ln i - \frac{1}{2}\ln n \\
&= \sum_{i=2}^{n} \ln i - \frac{1}{2}\ln n \\
&= \sum_{i=2}^{n} \frac{\ln(i-1) + \ln i}{2} - \frac{1}{2}\ln 1 + \frac{1}{2}\ln n - \frac{1}{2}\ln n \\
&= \sum_{i=2}^{n} \frac{\ln(i-1) + \ln i}{2} \\
&\leq \sum_{i=2}^{n} \int_{i-1}^{i} \ln x \, \mathrm{d}x && \text{(5.a)} \\
&= \int_{1}^{n} \ln x \, \mathrm{d}x \\
\int_{\frac{3}{2}}^{n} \ln x \, \mathrm{d}x \leq \ln(n!) &- \frac{1}{2}\ln n \leq \int_{1}^{n} \ln x \, \mathrm{d}x
\end{aligned}$$

8

Evaluate the following limits. Do not use the fundamental theorem of calculus when computing any integral.

8.a

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{\sqrt{n^2 - k^2}}{n^2}$$

$$\begin{aligned}
\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{\sqrt{n^2 - k^2}}{n^2} &= \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\sqrt{n^2 - k^2}}{n} \\
&= \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \sqrt{1 - \left(\frac{k}{n}\right)^2} \\
&= \int_{0}^{1} \sqrt{1 - x^2} \, \mathrm{d}x && \text{(the integrand is bounded and continuous, so it is integrable)} \\
&= \frac{\pi}{4} && \text{(area of quarter of circle of radius 1)}
\end{aligned}$$

8.b

$$\lim_{n \to +\infty} \frac{1}{n^2} \left(\sqrt{n-1} + \sqrt{2n-4} + \sqrt{3n-9} + \cdots + \sqrt{n^2 - n^2}\right)$$

$$
\begin{aligned}
\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \sqrt{\frac{k}{n} - \left(\frac{k}{n}\right)^2} &= \int_0^1 \sqrt{x - x^2}\,\mathrm{d}x && \text{(the integrand is bounded and continuous, so it is integrable)} \\
&= \int_0^1 \sqrt{x(1-x)}\,\mathrm{d}x \\
&= 2\int_0^{\frac{1}{2}} \sqrt{x(1-x)}\,\mathrm{d}x && \text{(symmetry)} \\
&= 2\int_0^{\frac{1}{2}} \sqrt{\left(\frac{1}{2}+x\right)\left(\frac{1}{2}-x\right)}\,\mathrm{d}x && \text{(integrate in reverse direction)} \\
&= 2\int_0^{\frac{1}{2}} \sqrt{\frac{1}{4} - x^2}\,\mathrm{d}x \\
&= 2 \cdot \frac{\pi}{4}\left(\frac{1}{2}\right)^2 && \left(\text{area of quarter of circle of radius } \frac{1}{2}\right) \\
&= \frac{\pi}{8}
\end{aligned}
$$

10

10.a

Let t be a real number such that $\sin\frac{t}{2} \neq 0$ and let n be a positive integer. Show that

$$
\sum_{k=1}^{n} \sin kt = \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}
$$

$$
\begin{aligned}
\sum_{k=1}^{n} \sin kt &= \frac{1}{\sin\frac{t}{2}} \sum_{k=1}^{n} \sin\frac{t}{2}\sin kt \\
&= \frac{1}{2\sin\frac{t}{2}} \sum_{k=1}^{n} \left(\cos\left(k-\frac{1}{2}\right)t - \cos\left(k+\frac{1}{2}\right)t\right) \\
&= \frac{1}{2\sin\frac{t}{2}} \left(\cos\left(1-\frac{1}{2}\right)t - \cos\left(n+\frac{1}{2}\right)t\right) \\
&= \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
\sum_{k=1}^{n} \sin kt &= \sum_{k=1}^{n} \operatorname{Im}\left\{e^{kti}\right\} \\
&= \operatorname{Im}\left\{\sum_{k=1}^{n} e^{kti}\right\} \\
&= \operatorname{Im}\left\{\frac{e^{(n+1)ti} - e^{ti}}{e^{ti}-1}\right\} \\
&= \operatorname{Im}\left\{\frac{e^{(n+\frac{1}{2})ti} - e^{\frac{t}{2}i}}{e^{\frac{t}{2}i} - e^{-\frac{t}{2}i}}\right\} \\
&= \operatorname{Im}\left\{\frac{e^{(n+\frac{1}{2})ti} - e^{\frac{t}{2}i}}{2i\sin\frac{t}{2}}\right\} \\
&= \operatorname{Im}\left\{\frac{\cos\left(n+\frac{1}{2}\right)t + i\sin\left(n+\frac{1}{2}\right)t - \cos\frac{t}{2} - i\sin\frac{t}{2}}{2i\sin\frac{t}{2}}\right\} \\
&= \operatorname{Im}\left\{\frac{-i\cos\left(n+\frac{1}{2}\right)t + \sin\left(n+\frac{1}{2}\right)t + i\cos\frac{t}{2} - \sin\frac{t}{2}}{2\sin\frac{t}{2}}\right\} \\
&= \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}
\end{aligned}
$$

10.b

Hence for each $a > 0$, evaluate

$$
\int_0^a \sin t\,\mathrm{d}t
$$

from the definition of a Riemann integral.

$$
\begin{aligned}
\int_0^a \sin t \,\mathrm{d}t &= \lim_{n\to+\infty} \frac{a}{n} \sum_{k=1}^{n} \sin k\frac{a}{n} \\
&= \lim_{n\to+\infty} \frac{a}{n} \frac{\cos\frac{a}{2n} - \cos\left(n+\frac{1}{2}\right)\frac{a}{n}}{2\sin\frac{a}{2n}} && \left((\forall a \in \mathbb{R}_{>0})(\exists N \in \mathbb{N}_{\neq 0})(\forall n \in \mathbb{N}_{\neq 0})\left(n \geq N \implies \sin\frac{a}{2n} \neq 0\right)\right) \\
&= \lim_{n\to+\infty} \frac{a}{n} \frac{2\sin\frac{n+1}{2}\frac{a}{n}\sin\frac{n}{2}\frac{a}{n}}{2\sin\frac{a}{2n}} \\
&= \lim_{n\to+\infty} \frac{a}{n} \frac{\sin\frac{na+a}{2n}\sin\frac{a}{2}}{\sin\frac{a}{2n}} \\
&= \lim_{n\to+\infty} \frac{a}{n} \left(\sin\frac{a}{2}\right) \frac{\sin\frac{a}{2}\cos\frac{a}{2n} + \sin\frac{a}{2n}\cos\frac{a}{2}}{\sin\frac{a}{2n}} \\
&= \lim_{n\to+\infty} \frac{a}{n} \left(\sin^2\frac{a}{2}\cot\frac{a}{2n} + \sin\frac{a}{2}\cos\frac{a}{2}\right) \\
&= \lim_{n\to+\infty} \left(2\sin^2\frac{a}{2}\frac{\frac{a}{2n}}{\tan\frac{a}{2n}} + \frac{a}{n}\sin\frac{a}{2}\cos\frac{a}{2}\right) \\
&= 2\sin^2\frac{a}{2} \\
&= 1 - \cos a
\end{aligned}
$$

13

Let m be a non-negative real number.

13.a

Let $g : [0,1] \to [0,+\infty)$ be a non-negative integrable function such that

$$
g(x) \leq mx \qquad \text{and} \qquad g(x) \leq m(1-x)
$$

for every $x \in [0,1]$. Show that

$$
\int_0^1 g(x)\,\mathrm{d}x \leq \frac{m}{4}
$$

$$
\begin{aligned}
\int_0^1 g(x)\,\mathrm{d}x &= \int_0^{\frac{1}{2}} g(x)\,\mathrm{d}x + \int_{\frac{1}{2}}^1 g(x)\,\mathrm{d}x \\
&\leq \int_0^{\frac{1}{2}} mx\,\mathrm{d}x + \int_{\frac{1}{2}}^1 m(1-x)\,\mathrm{d}x \\
&= 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{m}{2} && \left(\text{two triangles of base } \frac{1}{2} \text{ and height } \frac{m}{2}\right) \\
&= \frac{m}{4}
\end{aligned}
$$

13.b

Let $f : [0,1] \to \mathbb{R}$ be a function that is continuous on $[0,1]$ and differentiable on $(0,1)$. Suppose that $f(0) = f(1) = 0$ and $|f'(x)| \leq m$ for every $x \in (0,1)$. Using the result from 13.a, show that

$$
\int_0^1 |f(x)|\,\mathrm{d}x \leq \frac{m}{4}
$$

$$
\begin{aligned}
&\forall x \in (0,1] \\
\frac{f(x)-f(0)}{x-0} &= f'(a) \quad \exists a \in (0,x) && \text{(mean value theorem)} \\
\frac{f(x)}{x} &= f'(a) \\
f(x) &= xf'(a) && (x>0) \\
|f(x)| &= x|f'(a)| && (x>0) \\
&\le mx \\
f(x) &\le mx \quad \forall x \in [0,1] && (f(0)=0)
\end{aligned}
$$

$$
\begin{aligned}
&\forall x \in [0,1) \\
\frac{f(1)-f(x)}{1-x} &= f'(a) \quad \exists a \in (x,1) && \text{(mean value theorem)} \\
-\frac{f(x)}{1-x} &= f'(a) \\
-f(x) &= (1-x)f'(a) && (x<1) \\
|f(x)| &= (1-x)|f'(a)| && (x<1) \\
&\le m(1-x) \\
f(x) &\le m(1-x) \quad \forall x \in [0,1] && (f(1)=0)
\end{aligned}
$$

Obviously $|f(x)|$ is nonnegative.
$\because f(x)$ is bounded and continuous on $[0,1]$
$\therefore |f(x)|$ is bounded and continuous on $[0,1]$
$\therefore |f(x)|$ is (Riemann) integrable on $[0,1]$

$\because |f(x)|$ satisfies all conditions for $g(x)$ in 13.a,
$\therefore \int_0^1 |f(x)|\,\mathrm{d}x \le \frac{m}{4}$

13.c

Using the result from 13.b, show that

$$\int_0^1 |\sin(mx(x-1))|\,\mathrm{d}x \le \frac{m}{4}$$

$h(x) := \sin(mx(x-1))$
By inspection, $h(x)$ is continuous and differentiable on $\mathbb{R}$.

$$
\begin{aligned}
h(0) &= \sin(m(0)(0-1)) \\
&= 0 \\
h(1) &= \sin(m(1)(1-1)) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
&\forall x \in [0,1] \\
h'(x) &= \cos(mx(x-1))m(2x-1) \\
|h'(x)| &= m|\cos(mx(x-1))(2x-1)| && (m \ge 0) \\
&\le m|2x-1| && (|\cos *| \le 1) \\
&\le m && (|2x-1| \le 1)
\end{aligned}
$$

$\because$ The restriction of $h(x)$ to $[0,1]$ satisfies all conditions for $f(x)$ in 13.b,
$\therefore \int_0^1 |\sin(mx(x-1))|\,\mathrm{d}x \le \frac{m}{4}$

15

Let $a<b$ be real numbers, let $f:[a,b]\to(0,+\infty)$ be a **positive** continuous function, and let $g:[a,b]\to\mathbb{R}$ be the function

$$g(x) = \int_a^x f(t)\,\mathrm{d}t + \int_b^x \frac{1}{f(t)}\,\mathrm{d}t$$

15.a

Show from definition (Definition 5.49) that g is **strictly** increasing on $[a,b]$.

$$
\begin{aligned}
&\forall x \in [a,b] \\
&\because f(x) > 0 \\
&\therefore \frac{1}{f(x)} > 0
\end{aligned}
$$

$$
\begin{aligned}
&\forall x, y \in [a,b], \text{where } x > y \\
g(x) &= \int_a^x f(t)\,\mathrm{d}t + \int_b^x \frac{1}{f(t)}\,\mathrm{d}t \\
&= \int_a^y f(t)\,\mathrm{d}t + \int_b^y \frac{1}{f(t)}\,\mathrm{d}t + \int_y^x f(t)\,\mathrm{d}t + \int_y^x \frac{1}{f(t)}\,\mathrm{d}t \\
&= g(y) + \int_y^x f(t)\,\mathrm{d}t + \int_y^x \frac{1}{f(t)}\,\mathrm{d}t \\
&> g(y) + \int_y^x 0\,\mathrm{d}t + \int_y^x 0\,\mathrm{d}t && \left(x > y, f(t) > 0, \frac{1}{f(t)} > 0\right) \\
&> g(y)
\end{aligned}
$$

15.b

Suppose it is given that g is continuous on $[a,b]$ (This fact actually follows immediately from Lemma 5.50 in §5.4). Show that g has one and only one root in $[a,b]$.

$$
\begin{aligned}
g(a) &= \int_a^a f(t)\,\mathrm{d}t + \int_b^a \frac{1}{f(t)}\,\mathrm{d}t \\
&= \int_b^a \frac{1}{f(t)}\,\mathrm{d}t \\
&= \int_a^b -\frac{1}{f(t)}\,\mathrm{d}t \\
&< \int_a^b 0\,\mathrm{d}t && \left(a < b, -\frac{1}{f(t)} < 0\right) \\
g(b) &= \int_a^b f(t)\,\mathrm{d}t + \int_b^b \frac{1}{f(t)}\,\mathrm{d}t \\
&= \int_a^b f(t)\,\mathrm{d}t \\
&> \int_a^b 0\,\mathrm{d}t && (a < b, f(t) > 0)
\end{aligned}
$$

$\because$ g is continuous on $[a,b]$,
$\therefore$ $(\exists x \in [a,b])(g(x) = 0)$ by the intermediate value theorem.

$\because$ g is strictly increasing on $[a,b]$,
$\therefore$ g is injective on $[a,b]$.
$\therefore g(x) = g(y) \implies x = y$
Let x, y be roots of g that may or may not be distinct.
$\because g(x) = g(y) = 0$
$\therefore x = y$
$\therefore$ g has at most one root.

$\because$ g has at least one root and at most one root,
$\therefore$ g has one and only one root.