Problem Set 4

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 4 covers materials from §6.4 – §6.6.

1. For each of the following rational functions f, evaluate the antiderivative $\int f(x)dx$.

   (a) $f(x) = \frac{1}{x^2+1}$

   (b) $f(x) = \frac{1}{x^2+2x}$

   (c) $f(x) = \frac{1}{x^2+2x+1}$

   (d) $f(x) = \frac{1}{x^2+2x+2}$

   (e) $f(x) = \frac{1}{x^2+2x+3}$

   (f) $f(x) = \frac{x}{x^2+2x+1}$

   (g) $f(x) = \frac{x}{x^2+2x+2}$

   (h) $f(x) = \frac{x+2}{x^2+2x+2}$

   (i) $f(x) = \frac{1}{x^2(x+2)}$

   (j) $f(x) = \frac{1}{x(x+2)^2}$

2. Evaluate the following antiderivatives.

   (a) $\displaystyle\int \frac{1}{\sqrt{e^x+1}}\,dx$

   (b) $\displaystyle\int \ln(x^3+1)\,dx$

   (c) $\displaystyle\int \frac{1}{x^2\sqrt{x+1}}\,dx$

   (d) $\displaystyle\int (x+2)\sqrt{\frac{1+x}{1-x}}\,dx$

3. (a) Using the factorization $x^4+1 = \left(x^2-\sqrt{2}x+1\right)\left(x^2+\sqrt{2}x+1\right)$, evaluate
$$\int \frac{x^2}{x^4+1}\,dx.$$
   (b) Using (a) and the substitution $u = \sqrt{\tan x}$, evaluate
$$\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\,dx.$$

4. Evaluate the antiderivative
$$\int \frac{1}{(1+x^2)\sqrt{1-x^2}}\,dx$$
   using the substitution $u = \sqrt{\frac{1-x}{1+x}}$.

5. (a) Show that the polynomial x^3+3x+1 has exactly one real root.

   (b) Let r be the real root of x^3+3x+1. Evaluate the antiderivative
$$\int \frac{1}{x^3+3x+1}\,dx$$
   in terms of r.

6. Evaluate the antiderivatives of each of the following trigonometric rational functions f.

(a) $f(x) = \dfrac{1}{1 + 2\sin x \cos x + \cos^2 x}$ (for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

(b) $f(x) = \dfrac{1}{3\sin x + 4\cos x}$

(c) $f(x) = \dfrac{1 + \sin x - 3\cos x}{2 + 2\sin x - \cos x}$

7. Let a be a positive real number. Evaluate

$$\int \frac{1}{1 - a\sin x}\,dx$$

for each of the following cases:

(a) $0 < a < 1$ (for $x \in (-\pi, \pi)$),

(b) $a = 1$,

(c) $a > 1$.

8. Evaluate the antiderivative

$$\int e^x \frac{1 + \sin x}{1 + \cos x}\,dx$$

using the substitution $t = \tan(x/2)$.

9. Evaluate each of the following improper integrals if it converges.

(a) $\displaystyle\int_0^{+\infty} \frac{3 - 5x}{(1 + x)(1 - x + 2x^2)}\,dx$

(b) $\displaystyle\int_1^{+\infty} \frac{1}{x^2 - 1}\,dx$

(c) $\displaystyle\int_0^{\pi/2} \ln(\sin x)\,dx$

Hint: In (c), break the interval into two halves, and let $u = \frac{\pi}{2} - x$ in the second half.

10. Let $f\colon [0, +\infty) \to \mathbb{R}$ be the function

$$f(x) = \int_1^{\sqrt{x}} e^{-t^2}\,dt.$$

(a) Find $f'(x)$ for every $x \in (0, +\infty)$.

(b) Evaluate the improper integral

$$\int_0^1 \frac{f(x)}{\sqrt{x}}\,dx.$$

11. Find the value of the real number a such that the improper integral

$$\int_0^{+\infty} \left(\frac{1}{\sqrt{x^2 + 1}} + \frac{a}{x + 1}\right) dx$$

converges. Also evaluate the improper integral for this value of a.

12. Let n be a positive integer.
   (a) Show that the improper integral
$$\int_0^1 \ln x\, dx$$
converges and find its value. Using integration by parts, deduce that the improper integral
$$\int_0^1 (\ln x)^n dx$$
also converges and find its value in terms of n.
   (b) Let α be a positive real number. Show that
$$\int_t^1 x^{\alpha-1}(\ln x)^n dx = \frac{1}{\alpha^{n+1}}\int_{t^\alpha}^1 (\ln x)^n dx \qquad \text{for every } t \in (0,1).$$
Using the result from (a), show that the improper integral
$$\int_0^1 x^{\alpha-1}(\ln x)^n dx$$
also converges and find its value in terms of n and α.

13. Let $f\colon [0,+\infty) \to \mathbb{R}$ be a continuous function. The ***Laplace transform*** of f is a function F defined by the improper integral
$$F(s) = \int_0^{+\infty} e^{-st} f(t)dt\,,$$
and the domain of F consists of all the numbers s for which the integral converges.
   (a) Show that if there exist $a \in \mathbb{R}$ and $M > 0$ such that
$$|f(t)| \le Me^{at} \qquad \text{for every } t \ge 0,$$
then the Laplace transform of f exists for $s > a$.
   (b) Compute the Laplace transforms of the following functions:
      (i) $f(t) = 1$
      (ii) $f(t) = e^t$
      (iii) $f(t) = t^2$
      (iv) $f(t) = \cos t$
   (c) Suppose that f is continuously differentiable, and that there exist $a \in \mathbb{R}$ and $M > 0$ such that
$$|f(t)| \le Me^{at} \quad \text{and} \quad |f'(t)| \le Me^{at} \qquad \text{for every } t \ge 0.$$
If F and G denote the Laplace transforms of f and of f' respectively, show that
$$G(s) = sF(s) - f(0) \qquad \text{for every } s > a.$$

14. Let f be a function defined by

$$f(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

(a) Show that $f(n)$ is well-defined (i.e. the improper integral converges) for each $n \in \mathbb{N}$. Hence deduce that $f(x)$ is well-defined for each $x \geq 1$.

Hint: Compare with $\int_0^{+\infty} e^{-\frac{1}{2}t} dt$.

(b) Show that $f(x)$ is also well-defined for each $x \in (0, 1)$.

(c) Show that $f(x+1) = xf(x)$ for every $x > 0$, and $f(n) = (n-1)!$ for every $n \in \mathbb{N}$.

15. For each of the following improper integrals, determine whether it converges or not.

(a) $\displaystyle\int_1^{+\infty} \frac{2 + \cos x}{\sqrt{x+5}} dx$

(b) $\displaystyle\int_0^1 \frac{1}{\sqrt{x + \sqrt{x + \sqrt{x}}}} dx$

(c) $\displaystyle\int_0^{+\infty} \frac{x}{1 + x^2 \sin^2 x} dx$

(d) $\displaystyle\int_0^{+\infty} \frac{\sin x}{x} dx$

16. Let $f, g: [0, +\infty) \to \mathbb{R}$ be continuous functions such that the improper integrals

$$\int_0^{+\infty} f(x)^2 dx \quad \text{and} \quad \int_0^{+\infty} g(x)^2 dx$$

both converge. Show that the improper integrals

$$\int_0^{+\infty} f(x)g(x) dx \quad \text{and} \quad \int_0^{+\infty} \big(f(x) + g(x)\big)^2 dx$$

both converge too.

17. Let $f: [0, +\infty) \to [0, +\infty)$ be a decreasing continuous function.

(a) Show that if $\int_0^{+\infty} f(x) dx$ converges, then $\lim\limits_{x \to +\infty} xf(x) = 0$.

Hint: Show that $0 \leq xf(x) \leq 2\int_{x/2}^{x} f(t) dt$ and apply Squeeze Theorem.

(b) Show that the converse of the result from (a) is not true, i.e. give an example of $f(x)$ so that $\lim\limits_{x \to +\infty} xf(x) = 0$ but $\int_0^{+\infty} f(x) dx$ diverges.

Now let $g: [1, +\infty) \to [e, +\infty)$ be an increasing continuous function.

(c) If $\lim\limits_{x \to +\infty} \frac{x}{\ln(g(e^x))} = 0$, show that for every sufficiently large $x > 0$ we have $\frac{e^x}{g(e^x)} < e^{-x}$.

Hint: In the definition of limit (Definition 2.91), take $\varepsilon = \frac{1}{2}$.

(d) Using the results from (a) and (c), show that if $\int_1^{+\infty} \frac{1}{g(x)} dx$ diverges, then $\int_1^{+\infty} \frac{1}{x \ln g(x)} dx$ also diverges.