HKUST MATH 1014 L1 assignment 2 submission

1

Let n be a positive integer. Evaluate each of the following limits.

1.a

$$\lim_{x \to 0} \frac{1}{x^n} \int_0^{x^n} \cos\left(t^2\right) \mathrm{d}t$$

> $$\begin{aligned} \lim_{x \to 0} \frac{1}{x^n} \int_0^{x^n} \cos\left(t^2\right) \mathrm{d}t &= \lim_{x \to 0} \int_0^1 \cos\left(x^{2n} t^2\right) \mathrm{d}t && \left(\text{change of variable } \frac{t}{x^n} \mapsto t\right) \\ &= \int_0^1 \lim_{x \to 0} \cos\left(x^{2n} t^2\right) \mathrm{d}t && (\text{dominated convergence theorem; dominated by } f(x) = 1 \text{ on } [0, 1]) \\ &= \int_0^1 \cos\left(\left(\lim_{x \to 0} x\right)^{2n} t\right) && \left(f(x) = \cos\left(x^{2n} t\right) \text{ is continuous at } 0\right) \\ &= \int_0^1 \cos 0 \,\mathrm{d}t && (n > 0) \\ &= \int_0^1 \mathrm{d}t \\ &= 1 \end{aligned}$$

1.b

$$\lim_{x \to 0} \frac{1}{x^n} \int_0^{x^n} \cos\left(x^2 t\right) \mathrm{d}t$$

> $$\begin{aligned} \lim_{x \to 0} \frac{1}{x^n} \int_0^{x^n} \cos\left(x^2 t\right) \mathrm{d} &= \lim_{x \to 0} \int_0^1 \cos\left(x^{n+2} t\right) \mathrm{d}t && \left(\text{change of variable } \frac{t}{x^n} \mapsto t\right) \\ &= \int_0^1 \lim_{x \to 0} \cos\left(x^{n+2} t\right) \mathrm{d}t && (\text{dominated convergence theorem; dominated by } f(x) = 1 \text{ on } [0, 1]) \\ &= \int_0^1 \cos\left(\left(\lim_{x \to 0} x\right)^{n+2} t\right) \mathrm{d}t && \left(f(x) = \cos\left(x^{n+2} t\right) \text{ is continuous at } 0\right) \\ &= \int_0^1 \cos 0 \,\mathrm{d}t && (n + 2 > 0) \\ &= \int_0^1 \mathrm{d}t \\ &= 1 \end{aligned}$$

2

Let $f, g : \mathbb{R} \to \mathbb{R}$ be **increasing** continuous functions, and let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$F(x) = x \int_0^x f(t) g(t) \,\mathrm{d}t - \left(\int_0^x f(t) \,\mathrm{d}t\right) \left(\int_0^x g(t) \,\mathrm{d}t\right)$$

.

2.a

Show that F is differentiable on $\mathbb{R}$ and

$$F'(x) = \int_0^x (f(x) - f(t))(g(x) - g(t)) \,\mathrm{d}t$$

.

$$
\begin{aligned}
f(t) \in C(\mathbb{R},\mathbb{R}) &\implies X(x) := \int_0^x f(t)\,\mathrm{d}t \in C^1(\mathbb{R},\mathbb{R}) && \text{(first fundamental theorem of calculus)} \\
g(t) \in C(\mathbb{R},\mathbb{R}) &\implies Y(x) := \int_0^x g(t)\,\mathrm{d}t \in C^1(\mathbb{R},\mathbb{R}) && \text{(first fundamental theorem of calculus)} \\
f(t), g(t) \in C(\mathbb{R},\mathbb{R}) &\implies f(t)g(t) \in C(\mathbb{R},\mathbb{R}) \\
&\implies Z(x) := \int_0^x f(t)g(t)\,\mathrm{d}t \in C^1(\mathbb{R},\mathbb{R}) && \text{(first fundamental theorem of calculus)} \\
x, Z(x) \in C^1(\mathbb{R},\mathbb{R}) &\implies xZ(x) \in C^1(\mathbb{R},\mathbb{R}) && \text{(product rule)} \\
X(x), Y(x) \in C^1(\mathbb{R},\mathbb{R}) &\implies X(x)Y(x) \in C^1(\mathbb{R},\mathbb{R}) && \text{(product rule)} \\
xZ(x), X(x)Y(x) \in C^1(\mathbb{R},\mathbb{R}) &\implies F(x) = xZ(x) - X(x)Y(x) \in C^1(\mathbb{R},\mathbb{R}) && \text{(linearity of differentiation)}
\end{aligned}
$$

$\therefore F(x)$ is differentiable on $\mathbb{R}$.

$$
\begin{aligned}
F(x) &= x\int_0^x f(t)g(t)\,\mathrm{d}t - \left(\int_0^x f(t)\,\mathrm{d}t\right)\left(\int_0^x g(t)\,\mathrm{d}t\right) \\
&= xZ(x) - X(x)Y(x) \\
F'(x) &= Z(x) + xZ'(x) - X'(x)Y(x) - X(x)Y'(x) \\
&= \int_0^x f(t)g(t)\,\mathrm{d}t + xf(x)g(x) - f(x)\int_0^x g(t)\,\mathrm{d}t - g(x)\int_0^x f(t)\,\mathrm{d}t && \text{(first fundamental theorem of calculus)} \\
&= \int_0^x f(t)g(t)\,\mathrm{d}t + \int_0^x f(x)g(x)\,\mathrm{d}t - \int_0^x f(x)g(t)\,\mathrm{d}t - \int_0^x f(t)g(x)\,\mathrm{d}t \\
&= \int_0^x (f(t)g(t) + f(x)g(x) - f(x)g(t) - f(t)g(x))\,\mathrm{d}t && \text{(linearity of integral)} \\
&= \int_0^x (f(x)(g(x) - g(t)) + f(t)(g(t) - g(x)))\,\mathrm{d}t \\
&= \int_0^x (f(x) - f(t))(g(x) - g(t))\,\mathrm{d}t
\end{aligned}
$$

2.b

Using the result from 2.a, find the global minimum of F on $\mathbb{R}$.

Solve for $F'(x) = 0$.
When $x = 0$, obviously $F'(x) = 0$.
When $x > 0$, $t \in [0, x]$, then

$$
\begin{aligned}
x \geq t &\implies f(x) - f(t) \geq 0 && (f \text{ is increasing}) \\
x \geq t &\implies g(x) - g(t) \geq 0 && (g \text{ is increasing}) \\
f(x) - f(t) \geq 0 \land g(x) - g(t) \geq 0 &\implies (f(x) - f(t))(g(x) - g(t)) \geq 0 \\
&\implies \int_0^x (f(x) - f(t))(g(x) - g(t))\,\mathrm{d}t \geq 0 && (\text{integrand is nonnegative; } x > 0) \\
&\implies F'(x) \geq 0
\end{aligned}
$$

When $x < 0$, $t \in [x, 0]$, then

$$
\begin{aligned}
x \leq t &\implies f(x) - f(t) \leq 0 && (f \text{ is increasing}) \\
x \leq t &\implies g(x) - g(t) \leq 0 && (g \text{ is increasing}) \\
f(x) - f(t) \leq 0 \land g(x) - g(t) \leq 0 &\implies (f(x) - f(t))(g(x) - g(t)) \geq 0 \\
&\implies \int_x^0 (f(x) - f(t))(g(x) - g(t))\,\mathrm{d}t \geq 0 && (\text{integrand is nonnegative; } 0 > x) \\
&\implies \int_0^x (f(x) - f(t))(g(x) - g(t))\,\mathrm{d}t \leq 0 \\
&\implies F'(x) \leq 0
\end{aligned}
$$

The above shows that $F(x)$ is decreasing on $(-\infty, 0)$, stationary on $\{0\}$, and increasing on $(0, +\infty)$.
Then $x = 0$ is one of the global minima of $F(x)$.
So the global minimum value is $F(0) = 0$.

6

6.a

By considering the function $f(x) = x - \sin x$, show that

$$\sin x \leq x$$

for every $x \geq 0$.

$$
\begin{aligned}
f(x) &= x - \sin x \\
f(x) &\in C^\infty(\mathbb{R}, \mathbb{R}) \\
f(0) &= 0 - \sin 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
f'(x) &= 1 - \cos x \\
&\geq 0 && (\cos x \leq 1) \\
f'(x) \geq 0 &\implies (x \geq y \implies f(x) \geq f(y)) && \text{(increasing)}
\end{aligned}
$$

$$
\begin{aligned}
&\forall x \geq 0 \\
x \geq 0 &\implies f(x) \geq f(0) \\
&\implies f(x) \geq 0 \\
f(x) &\geq 0 \\
x - \sin x &\geq 0 \\
x &\geq \sin x \\
\sin x &\leq x
\end{aligned}
$$

6.b

Using the result of 6.a and integration, show that each of the following inequalities holds for every $x \geq 0$.

6.b.i

$$\cos x \geq 1 - \frac{x^2}{2}$$

$$
\begin{aligned}
&\forall x \geq 0 \\
\sin x &\leq x \\
\int_0^x \sin \xi \, \mathrm{d}\xi &\leq \int_0^x \xi \, \mathrm{d}\xi && \text{(the integrands are continuous and thus integrable by FTC I; } x \geq 0) \\
[-\cos \xi]_0^x &\leq \left[\frac{\xi^2}{2}\right]_0^x \\
\cos 0 - \cos x &\leq \frac{x^2}{2} - \frac{0^2}{2} \\
1 - \cos x &\leq \frac{x^2}{2} \\
-\cos x &\leq -1 + \frac{x^2}{2} \\
\cos x &\geq 1 - \frac{x^2}{2}
\end{aligned}
$$

6.b.ii

$$\sin x \geq x - \frac{x^3}{6}$$

$$
\begin{aligned}
&\forall x \geq 0 \\
\cos x &\geq 1 - \frac{x^2}{2} \\
\int_0^x \cos \xi \, \mathrm{d}\xi &\geq \int_0^x \left(1 - \frac{\xi^2}{2}\right) \mathrm{d}\xi && \text{(the integrands are continuous and thus integrable by FTC I; } x \geq 0) \\
[\sin \xi]_0^x &\geq \left[\xi - \frac{\xi^3}{6}\right]_0^x \\
\sin x - \sin 0 &\geq x - \frac{x^3}{6} - 0 + \frac{0^3}{6} \\
\sin x &\geq x - \frac{x^3}{6}
\end{aligned}
$$

6.b.iii

$$\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{2}4$$

$$
\begin{aligned}
&\forall x \geq 0 \\
\sin x &\geq x - \frac{x^3}{6} \\
\int_0^x \sin \xi \, \mathrm{d}\xi &\geq \int_0^x \left(\xi - \frac{\xi^3}{6}\right) \mathrm{d}\xi && \text{(the integrands are continuous and thus integrable by FTC I; } x \geq 0\text{)} \\
[-\cos \xi]_0^x &\geq \left[\frac{\xi^2}{2} - \frac{\xi^4}{24}\right]_0^x \\
\cos 0 - \cos x &\geq \frac{x^2}{2} - \frac{x^4}{24} - \frac{0^2}{2} + \frac{0^4}{24} \\
1 - \cos x &\geq \frac{x^2}{2} - \frac{x^4}{24} \\
-\cos x &\geq -1 + \frac{x^2}{2} - \frac{x^4}{24} \\
\cos x &\leq 1 - \frac{x^2}{2} + \frac{x^4}{24}
\end{aligned}
$$

7

7.a

Let $a < b$ be real numbers and let $f : [a, b] \to (0, +\infty)$ be a **positive** continuous function. Using Cauchy-Schwarz inequality, show that

$$\left(\int_a^b f(x) \, \mathrm{d}x\right) \left(\int_a^b \frac{1}{f(x)} \, \mathrm{d}x\right) \geq (b-a)^2$$

.

$$
\begin{aligned}
g(x) &:= \frac{1}{f(x)} \\
F(x) &:= \sqrt{f(x)} \\
G(x) &:= \sqrt{g(x)} = \frac{1}{\sqrt{f(x)}}
\end{aligned}
$$

$f(x)$ is integrable on $[a, b]$ because it is bounded and continuous.

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$F(x)$ is integrable on $[a, b]$ because it is bounded and continuous. $\quad \left(f(x) > 0 \text{ and is bounded} \implies F(x) = \sqrt{f(x)} > 0 \text{ and is bounded}\right)$

$G(x)$ is integrable on $[a, b]$ because it is bounded and continuous. $\quad \left(g(x) > 0 \text{ and is bounded} \implies G(x) = \sqrt{g(x)} > 0 \text{ and is bounded}\right)$

By the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\left(\int_a^b (F(x))^2 \, \mathrm{d}x\right) \left(\int_a^b (G(x))^2 \, \mathrm{d}x\right) &\geq \left(\int_a^b F(x)G(x) \, \mathrm{d}x\right)^2 \\
\left(\int_a^b \left(\sqrt{f(x)}\right)^2 \mathrm{d}x\right) \left(\int_a^b \left(\frac{1}{\sqrt{f(x)}}\right)^2 \mathrm{d}x\right) &\geq \left(\int_a^b \sqrt{f(x)} \frac{1}{\sqrt{f(x)}} \, \mathrm{d}x\right)^2 \\
\left(\int_a^b |f(x)| \, \mathrm{d}x\right) \left(\int_a^b \frac{1}{|f(x)|} \, \mathrm{d}x\right) &\geq \left(\int_a^b \mathrm{d}x\right)^2 \\
&\geq (b-a)^2 \\
\left(\int_a^b f(x) \, \mathrm{d}x\right) \left(\int_a^b \frac{1}{f(x)} \, \mathrm{d}x\right) &\geq (b-a)^2 && (f(x) > 0)
\end{aligned}
$$

7.b

Using the result from 7.a and Cauchy-Schwarz inequality again, show that

$$\int_0^{2\pi} \frac{1}{\sqrt{1 - \frac{1}{2}\cos x}} \, \mathrm{d}x \geq 2\pi$$

.

$$
\begin{aligned}
f(x) &:= \sqrt{1-\frac{1}{2}\cos x} && x \in \mathbb{R} \\
& f(x) \text{ is continuous and bounded.} \\
& \cos x \in [-1,1] \\
1-\frac{1}{2}\cos x &\in [0.5, 1.5] \\
\sqrt{1-\frac{1}{2}\cos x} &> 0 \\
f(x) &> 0
\end{aligned}
$$

$$
\begin{aligned}
\left(\int_0^{2\pi} \mathrm{d}x\right)\left(\int_0^{2\pi} (f(x))^2 \,\mathrm{d}x\right) &\geq \left(\int_0^{2\pi} f(x)\,\mathrm{d}x\right)^2 && \text{(Cauchy-Schwarz inequality)} \\
2\pi\left(\int_0^{2\pi}\left(1-\frac{1}{2}\cos x\right)\mathrm{d}x\right) &\geq \left(\int_0^{2\pi} f(x)\,\mathrm{d}x\right)^2 \\
2\pi\left[x-\frac{1}{2}\sin x\right]_0^{2\pi} &\geq \left(\int_0^{2\pi} f(x)\,\mathrm{d}x\right)^2 \\
2\pi\left(2\pi-\frac{1}{2}\sin(2\pi)-0+\frac{1}{2}\sin 0\right) &\geq \left(\int_0^{2\pi} f(x)\,\mathrm{d}x\right)^2 \\
4\pi^2 &\geq \left(\int_0^{2\pi} f(x)\,\mathrm{d}x\right)^2 \\
2\pi &\geq \left|\int_0^{2\pi} f(x)\,\mathrm{d}x\right| \\
2\pi &\geq \int_0^{2\pi} f(x)\,\mathrm{d}x && (f(x) > 0, 2\pi > 0)
\end{aligned}
$$

$$
\begin{aligned}
\left(\int_0^{2\pi} f(x)\,\mathrm{d}x\right)\left(\int_0^{2\pi} \frac{1}{f(x)}\,\mathrm{d}x\right) &\geq (2\pi-0)^2 && \text{(7.a)} \\
&\geq 4\pi^2 \\
\int_0^{2\pi} \frac{1}{f(x)}\,\mathrm{d}x &\geq \frac{4\pi^2}{\int_0^{2\pi} f(x)\,\mathrm{d}x} \\
&\geq \frac{4\pi^2}{2\pi} && \left(2\pi \geq \int_0^{2\pi} f(x)\,\mathrm{d}x\right) \\
\int_0^{2\pi} \frac{1}{\sqrt{1-\frac{1}{2}\cos x}}\,\mathrm{d}x &\geq 2\pi
\end{aligned}
$$

8

Let $m \in (0,1)$ be a fixed number, and let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$
f(x) = \int_0^x \frac{1}{\sqrt{1-m\sin^2 t}}\,\mathrm{d}t
$$

.

8.a

Show that f is strictly increasing on $\mathbb{R}$.

$$
\begin{aligned}
& \forall t \in \mathbb{R} \\
\sin t &\in [-1,1] \\
\sin^2 t &\in [0,1] \\
m\sin^2 t &\in [0,1) && (m \in (0,1)) \\
1-m\sin^2 t &\in (0,1] \\
\sqrt{1-m\sin^2 t} &\in (0,1] \\
\frac{1}{\sqrt{1-m\sin^2 t}} &\in [1,+\infty) \\
&> 0
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= \int_0^x \frac{1}{\sqrt{1-m\sin^2 t}}\,\mathrm{d}t \\
f'(x) &= \frac{1}{\sqrt{1-m\sin^2 x}} && \text{(first fundamental theorem of calculus)} \\
&> 0 \\
f'(x) > 0 &\implies f \text{ is strictly increasing on } \mathbb{R}.
\end{aligned}
$$

8.b

Show that $f(x) \geq x$ for every $x > 0$. Hence deduce that

$$
\lim_{x\to+\infty} f(x) = +\infty
$$

and

$$\lim_{x\to-\infty} f(x) = -\infty$$

.

$$\begin{aligned}
&\forall x > 0 \\
\frac{1}{\sqrt{1-m\sin^2 x}} &\in [1,+\infty) \\
&\geq 1 \\
\int_0^x \frac{1}{\sqrt{1-m\sin^2\xi}}\,\mathrm{d}\xi &\geq \int_0^x \mathrm{d}\xi && \text{(the integrands are continuous and thus integrable by FTC I; } x>0\text{)} \\
f(x) &\geq [\xi]_0^x \\
&\geq x
\end{aligned}$$

$$\begin{aligned}
&\text{When } x > 0, \\
f(x) &\geq x \\
\lim_{x\to+\infty} f(x) &\geq \lim_{x\to+\infty} x && (x>0) \\
&\geq +\infty \\
\lim_{x\to+\infty} f(x) &= +\infty
\end{aligned}$$

$$\begin{aligned}
f(-x) &= \int_0^{-x} \frac{1}{\sqrt{1-m\sin^2 x}}\,\mathrm{d}x \\
&= -\int_0^{x} \frac{1}{\sqrt{1-m\sin^2(-x)}}\,\mathrm{d}x && (\text{change of variable } -x\mapsto x) \\
&= -\int_0^{x} \frac{1}{\sqrt{1-m\sin^2 x}}\,\mathrm{d}x \\
&= -f(x)
\end{aligned}$$

$$\begin{aligned}
&\text{When } x < 0, \\
f(-x) &\geq -x \\
\lim_{x\to-\infty} f(-x) &\geq \lim_{x\to-\infty} -x && (x<0) \\
-\lim_{x\to-\infty} f(x) &\geq +\infty \\
\lim_{x\to-\infty} f(x) &\leq -\infty \\
\lim_{x\to-\infty} f(x) &= -\infty
\end{aligned}$$

8.c

Using the results from (a) and (b), deduce that f has an inverse which is defined on $\mathbb{R}$.

$$\begin{aligned}
&\text{The range of } f \text{ is } (-\infty,+\infty)\text{, i.e. } \mathbb{R}. && \text{(8.b)} \\
&f \text{ is invertible as it is strictly increasing on all of its domain.} && \text{(8.a)} \\
&\text{The domain of the inverse of } f \text{ is the range of } f\text{, which is } \mathbb{R}.
\end{aligned}$$

8.d

For each $y\in\mathbb{R}$, let's write $x := f^{-1}(y)$ and define three functions $p,q,r:\mathbb{R}\to\mathbb{R}$ by

$$\begin{cases} p(y) = \sin x \\ q(y) = \cos x \\ r(y) = \sqrt{1-m\sin^2 x} \end{cases}$$

.

Show that

$$p'(y) = q(y)r(y)$$

for every $y\in\mathbb{R}$.

In a similar way, also compute $q'(y)$ and $r'(y)$ in terms of $p(y)$, $q(y)$, and $r(y)$.

$$
\begin{aligned}
f(x) &= \int_0^x \frac{1}{\sqrt{1-m\sin^2 t}}\,\mathrm{d}t \\
f'(x) &= \frac{1}{\sqrt{1-m\sin^2 x}} && \text{(first fundamental theorem of calculus)} \\
&> 0
\end{aligned}
$$

$f(x)$ is continuously differentiable with nonzero derivative everywhere on $\mathbb{R}$, thus $f^{-1}(y)$ is continuously differentiable on $\mathbb{R}$ by the inverse function theorem.

$$
\begin{aligned}
p(y) &= \sin\left(f^{-1}(y)\right) \\
p'(y) &= (\cos x)(f^{-1}(y))' && \text{(chain rule)} \\
&= \frac{\cos x}{f'(x)} && \text{(inverse function theorem)} \\
&= (\cos x)\left(\sqrt{1-m\sin^2 x}\right) \\
&= q(y)r(y)
\end{aligned}
$$

$$
\begin{aligned}
q(y) &= \cos\left(f^{-1}(y)\right) \\
q'(y) &= (-\sin x)(f^{-1}(y))' && \text{(chain rule)} \\
&= -\frac{\sin x}{f'(x)} && \text{(inverse function theorem)} \\
&= -(\sin x)\left(\sqrt{1-m\sin^2 x}\right) \\
&= -p(y)r(y)
\end{aligned}
$$

$$
\begin{aligned}
r(y) &= \sqrt{1-m\sin^2\left(f^{-1}(y)\right)} \\
r'(y) &= \frac{-2m\sin x\cos x\left(f^{-1}(y)\right)'}{2\sqrt{1-m\sin^2 x}} && \text{(chain rule)} \\
r'(y) &= \frac{-2m\sin x\cos x}{2\sqrt{1-m\sin^2 x}\,f'(x)} && \text{(inverse function theorem)} \\
r'(y) &= -m\sin x\cos x \\
&= -mp(y)q(y)
\end{aligned}
$$

12

12.a

Let x be a fixed non-negative number with $x \neq 1$. Evaluate the integral

$$\int_0^\pi \frac{\sin t}{\sqrt{1-2x\cos t+x^2}}\,\mathrm{d}t$$

in terms of x.

$$x \neq 0$$

$$
\begin{aligned}
\int_0^\pi \frac{\sin t}{\sqrt{1-2x\cos t+x^2}}\,\mathrm{d}t &= -\int_1^{-1} \frac{1}{\sqrt{1-2xt+x^2}}\,\mathrm{d}t && \text{(change of variable } \cos t \mapsto t\text{)} \\
&= \int_{-1}^{1} \frac{1}{\sqrt{1+x^2-2xt}}\,\mathrm{d}t \\
&= -\frac{1}{2x}\int_{1+2x+x^2}^{1-2x+x^2} \frac{1}{\sqrt{t}}\,\mathrm{d}t && \text{(change of variable } 1+x^2-2xt \mapsto t, x \neq 0\text{)} \\
&= \frac{1}{x}\left[\sqrt{t}\right]_{1-2x+x^2}^{1+2x+x^2} \\
&= \frac{1}{x}\left[\sqrt{t}\right]_{(x-1)^2}^{(x+1)^2} \\
&= \frac{|x+1|-|x-1|}{x} \\
&= \begin{cases} \frac{(x+1)-(x-1)}{x} & x > 1 \\ \frac{(x+1)-(1-x)}{x} & 0 < x < 1 \end{cases} \\
&= \begin{cases} \frac{2}{x} & x > 1 \\ 2 & 0 < x < 1 \end{cases}
\end{aligned}
$$

$$x = 0$$

$$
\begin{aligned}
\int_0^\pi \frac{\sin t}{\sqrt{1-2x\cos t+x^2}}\,\mathrm{d}t &= \int_0^\pi \sin t\,\mathrm{d}t \\
&= [-\cos t]_0^\pi \\
&= 2
\end{aligned}
$$

$$
\int_0^\pi \frac{\sin t}{\sqrt{1-2x\cos t+x^2}}\,\mathrm{d}t = \begin{cases} \frac{2}{x} & x > 1 \\ 2 & 0 \leq x < 1 \end{cases}
$$

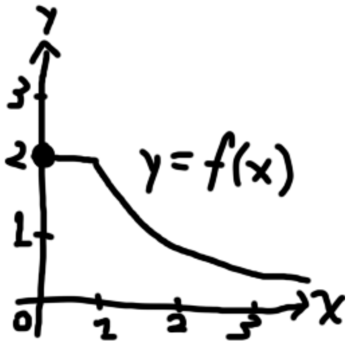
12.b

Let $f:[0,+\infty)\to\mathbb{R}$ be the function defined by

$$f(x)=\begin{cases}\int_0^{\pi}\frac{\sin t}{\sqrt{1-2x\cos t+x^2}}\,\mathrm{d}t & \text{if } x\neq 1\\ a & \text{if } x=1\end{cases}$$

.

Using the result from 12.a, find the value of a so that f is a continuous function. Hence sketch the graph of f.

> Using the definition in 12.a,
> $\lim\limits_{x\to1^-} f(x)=\lim\limits_{x\to1^-} 2=2$
> $\lim\limits_{x\to1^+} f(x)=\lim\limits_{x\to1^+} \dfrac{2}{x}=2$
> $\therefore \lim\limits_{x\to1} f(x)=2$
> $\therefore$ To make f continuous, $a=\lim\limits_{x\to1} f(x)=2$.
>
> 
>

13

13.a

Using the substitution $u=\frac{1}{x}$, show that

$$\int_{\frac{1}{2}}^{2}\frac{\ln x}{1+x^2}\,\mathrm{d}x=0$$

.

> $$\begin{aligned}\int_{\frac{1}{2}}^{2}\frac{\ln x}{1+x^2}\,\mathrm{d}x &= -\int_{u=2}^{u=\frac{1}{2}}\frac{\ln x}{1+x^{-2}}\,\mathrm{d}u && \text{(change of variable)}\\ &= \int_{2}^{\frac{1}{2}}\frac{\ln u}{1+u^2}\,\mathrm{d}u\\ &= \int_{2}^{\frac{1}{2}}\frac{\ln x}{1+x^2}\,\mathrm{d}x && \text{(rename dummy variable)}\\ &= -\int_{\frac{1}{2}}^{2}\frac{\ln x}{1+x^2}\,\mathrm{d}x\\ 2\int_{\frac{1}{2}}^{2}\frac{\ln x}{1+x^2}\,\mathrm{d}x &= 0\\ \int_{\frac{1}{2}}^{2}\frac{\ln x}{1+x^2}\,\mathrm{d}x &= 0\end{aligned}$$

13.b

Using 13.a or otherwise, evaluate the limit

$$\lim_{n\to+\infty}\sum_{k=1}^{3n}\frac{1}{2n}\,\frac{\ln\left(2\left(\frac{1}{2}+\frac{k}{2n}\right)\right)}{1+\left(\frac{1}{2}+\frac{k}{2n}\right)^2}$$

.

$$
\begin{aligned}
\lim_{n\to+\infty}\sum_{k=1}^{3n}\frac{1}{2n}\frac{\ln\left(2\left(\frac{1}{2}+\frac{k}{2n}\right)\right)}{1+\left(\frac{1}{2}+\frac{k}{2n}\right)^2} &= \lim_{n\to+\infty}\sum_{k=1}^{3n}\frac{\frac{3}{2}}{3n}\frac{\ln\left(2\left(\frac{1}{2}+\frac{k}{2n}\right)\right)}{1+\left(\frac{1}{2}+\frac{k}{2n}\right)^2} \\
&= \int_{\frac{1}{2}}^{2}\frac{\ln 2x}{1+x^2}\,\mathrm{d}x && \left(\text{the integrand is bounded and continuous on } \left[\frac{1}{2},2\right]\text{, so it is integrable}\right) \\
&= \int_{\frac{1}{2}}^{2}\frac{\ln 2+\ln x}{1+x^2}\,\mathrm{d}x \\
&= (\ln 2)\int_{\frac{1}{2}}^{2}\frac{1}{1+x^2}\,\mathrm{d}x + \int_{\frac{1}{2}}^{2}\frac{\ln x}{1+x^2}\,\mathrm{d}x \\
&= (\ln 2)[\arctan x]_{\frac{1}{2}}^{2} + 0 \\
&= (\ln 2)\left(\arctan 2 - \arctan\frac{1}{2}\right) \\
&= (\ln 2)\left(\arctan 2 - \left(\frac{\pi}{2} - \arctan 2\right)\right) \\
&= (\ln 2)\left(2\arctan 2 - \frac{\pi}{2}\right)
\end{aligned}
$$

16

16.a

Let $f : [0,\pi] \to \mathbb{R}$ be a continuous function such that

$$f(\pi - x) = -f(x)$$

for every $x \in [0,\pi]$.

Using the substitution $u = \pi - x$, show that

$$\int_0^{\pi} f(x)\ln\left(1+e^{\cos x}\right)\mathrm{d}x = \frac{1}{2}\int_0^{\pi} f(x)\cos x\,\mathrm{d}x$$

.

$$
\begin{aligned}
\int_0^{\pi} f(x)\ln\left(1+e^{\cos x}\right)\mathrm{d}x &= -\int_{\pi}^{0} f(\pi-u)\ln\left(1+e^{\cos(\pi-u)}\right)\mathrm{d}u && \text{(change of variable)} \\
&= -\int_0^{\pi} f(u)\ln\left(1+e^{-\cos u}\right)\mathrm{d}u \\
&= -\int_0^{\pi} f(x)\ln\left(1+e^{-\cos x}\right)\mathrm{d}x && \text{(rename dummy variable)} \\
2\int_0^{\pi} f(x)\ln\left(1+e^{\cos x}\right)\mathrm{d}x &= \int_0^{\pi} f(x)\ln\left(1+e^{\cos x}\right)\mathrm{d}x - \int_0^{\pi} f(x)\ln\left(1+e^{-\cos x}\right)\mathrm{d}x \\
&= \int_0^{\pi} f(x)\left(\ln\left(1+e^{-\cos x}\right) - \ln\left(1+e^{-\cos x}\right)\right)\mathrm{d}x \\
&= \int_0^{\pi} f(x)\ln\left(\frac{1+e^{\cos x}}{e^{-\cos x}+1}\right)\mathrm{d}x \\
&= \int_0^{\pi} f(x)\ln\left(e^{\cos x}\right)\mathrm{d}x \\
&= \int_0^{\pi} f(x)\cos x\,\mathrm{d}x \\
\int_0^{\pi} f(x)\ln\left(1+e^{\cos x}\right)\mathrm{d}x &= \frac{1}{2}\int_0^{\pi} f(x)\cos x\,\mathrm{d}x
\end{aligned}
$$

16.b

Compute the derivative of the function $g : [0,\pi] \to \mathbb{R}$ defined by

$$g(x) = \frac{\cos x}{1+\sin x}$$

.

Using this together with the result from 16.a, evaluate the integral

$$\int_0^{\pi}\frac{(\cos x)\ln\left(1+e^{\cos x}\right)}{(1+\sin x)^2}\,\mathrm{d}x$$

.

$$\begin{aligned}
g(x) &= \frac{\cos x}{1+\sin x} \\
g'(x) &= \frac{-(\sin x)(1+\sin x) - \cos^2 x}{(1+\sin x)^2} && \text{(quotient rule)} \\
&= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} \\
&= -\frac{1+\sin x}{(1+\sin x)^2} \\
&= -\frac{1}{1+\sin x} \\
g''(x) &= -\frac{-\cos x}{(1+\sin x)^2} && \text{(quotient rule)} \\
&= \frac{\cos x}{(1+\sin x)^2}
\end{aligned}$$

$$\begin{aligned}
f &: [0,\pi] \to \mathbb{R} \\
f(x) &:= \frac{\cos x}{(1+\sin x)^2} \\
f(x) & \text{ is continuous on } [0,\pi]. \\
f(\pi - x) &= \frac{\cos(\pi - x)}{(1+\sin(\pi - x))^2} \\
&= -\frac{\cos x}{(1+\sin x)^2} \\
&= -f(x)
\end{aligned}$$

$$\begin{aligned}
\int_0^\pi \frac{(\cos x)\ln(1+e^{\cos x})}{(1+\sin x)^2}\,\mathrm{d}x &= \int_0^\pi f(x)\ln(1+e^{\cos x})\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^\pi f(x)\cos x\,\mathrm{d}x && \text{(16.a)} \\
&= \frac{1}{2}\int_0^\pi \frac{\cos^2 x}{(1+\sin x)^2}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^\pi g''(x)\cos x\,\mathrm{d}x \\
&= \frac{1}{2}\int_{x=0}^{x=\pi} \cos x\,\mathrm{d}g'(x) \\
&= \frac{1}{2}\left([g'(x)\cos x]_0^\pi - \int_{x=0}^{x=\pi} g'(x)\,\mathrm{d}(\cos x)\right) && \text{(integration by parts)} \\
&= \frac{1}{2}\left(-g'(\pi) - g'(0) + \int_0^\pi \frac{\sin x}{1+\sin x}\,\mathrm{d}x\right) \\
&= \frac{1}{2}\left(2 + \int_0^\pi g'(x)\sin x\,\mathrm{d}x\right) \\
&= \frac{1}{2}\left(2 + \int_{x=0}^{x=\pi} \sin x\,\mathrm{d}g(x)\right) \\
&= \frac{1}{2}\left(2 + [g'(x)\sin x]_0^\pi - \int_{x=0}^{x=\pi} g(x)\,\mathrm{d}(\sin x)\right) && \text{(integration by parts)} \\
&= \frac{1}{2}\left(2 - \int_0^\pi g(x)\cos x\,\mathrm{d}x\right) \\
&= \frac{1}{2}\left(2 - \int_0^\pi \frac{\cos^2 x}{1+\sin x}\,\mathrm{d}x\right) \\
&= \frac{1}{2}\left(2 - \int_0^\pi \frac{(1-\sin x)(1+\sin x)}{1+\sin x}\,\mathrm{d}x\right) \\
&= \frac{1}{2}\left(2 - \int_0^\pi (1-\sin x)\,\mathrm{d}x\right) \\
&= \frac{1}{2}(2 - [x+\cos x]_0^\pi) \\
&= \frac{1}{2}(2 - \pi + 1 + 0 + 1) \\
&= 2 - \frac{\pi}{2}
\end{aligned}$$