ELEC 4110 Homework 2 (Fall 2025)

1. (20 pts) Let the unit-energy rectangular pulse be $p(t) = \frac{1}{\sqrt{T}} \operatorname{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$. Consider four equiprobable signals

$$s_1(t) = -3A\,p(t), \quad s_2(t) = -A\,p(t), \quad s_3(t) = A\,p(t), \quad s_4(t) = 3A\,p(t),$$

where $A > 0$ and $T > 0$. The channel noise is AWGN with double-sided PSD $N_0/2$.

$$\operatorname{rect}(x) \triangleq \begin{cases} 1, & |x| < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) (5 pts) Compute the average symbol energy E_s in terms of A and T.

   (b) (7 pts) Derive the *optimal* decision rule using one correlator with template $p(t)$, and depict the 1-D decision regions.

   (c) (8 pts) Derive the exact symbol error probability P_e of this 4-PAM over AWGN.

2. (20 pts) Let $\{\phi_1(t), \phi_2(t)\}$ be orthonormal over $[0, T]$. Consider *three* equiprobable signals defined by a fixed rotation angle $\alpha = \pi/6$:

$$\begin{aligned} s_1(t) &= A\,\phi_1(t), \\ s_2(t) &= A\big(\cos\alpha\,\phi_1(t) + \sin\alpha\,\phi_2(t)\big), \\ s_3(t) &= A\big(-\sin\alpha\,\phi_1(t) + \cos\alpha\,\phi_2(t)\big), \end{aligned}$$

with $A > 0$. These correspond to the coordinate points $[A, 0]$, $[A\cos\alpha, A\sin\alpha]$, $[-A\sin\alpha, A\cos\alpha]$. The channel noise is AWGN with double-sided PSD $N_0/2$.

   (a) (6 pts) Compute the pairwise Euclidean distances d_{12}, d_{13}, d_{23} in the signal space.

   (b) (8 pts) *Sketch* the three points and the *optimal* decision regions in the (y_1, y_2)-plane. Write the *exact equations* of the three decision boundaries.

   (c) (6 pts) Assuming ML detection, write the *union bound* expression of the symbol error probability P_e.

3. (20 pts) A set of M equiprobable orthogonal signals $\{s_m(t)\}_{m=1}^{M}$ satisfies $\langle s_i, s_j \rangle = 0$ if $i \neq j$ and $\|s_m\|^2 = E_s$. The channel noise is AWGN with double-sided PSD $N_0/2$.

   (a) (10 pts) Describe the *optimum* receiver structure (bank of matched filters or correlators) and its decision rule in terms of correlator outputs.

   (b) (10 pts) Using pairwise error probabilities, derive the *union bound* on the symbol error probability P_e in terms of M, E_s, and N_0.

4. (20 pts) Let $p(t) = \frac{1}{\sqrt{T}} \operatorname{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$ and choose two *distinct* integers $n \neq m$. Consider three equiprobable signals

$$\begin{aligned}
s_a(t) &= \sqrt{2E_s}\, \cos\bigl(2\pi n t/T\bigr)\, p(t), \\
s_b(t) &= \sqrt{2E_s}\, \cos\bigl(2\pi m t/T\bigr)\, p(t), \\
s_c(t) &= \sqrt{2E_s}\, \cos\bigl(2\pi n t/T + \theta\bigr)\, p(t),
\end{aligned}$$

with fixed phase offset $\theta \in (0, \pi)$, and $E_s > 0$. The channel noise is AWGN with double-sided PSD $N_0/2$.

(a) (6 pts) Show that $\langle s_a, s_b \rangle = 0$ and $\langle s_b, s_c \rangle = 0$, while $\langle s_a, s_c \rangle = E_s \cos\theta$. Hence obtain the three pairwise distances d_{ab}, d_{bc}, d_{ac}.

(b) (8 pts) Using pairwise distances, write the *union bound* on P_e for this 3-signal set. Express the bound in terms of E_s/N_0 and θ.

(c) (6 pts) For $\theta = \pi/3$ and $E_s/N_0 = 10$ (linear), evaluate the numeric value of your union bound.