HKUST MATH 1014 L1 assignment 5 submission

MATH1014 Calculus II Problem Set 5
L01 (Spring 2024)

Problem Set 5

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 5 covers materials from §7.1 - §7.6.

Q5

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^3$ such that $\|\mathbf{u}\| = 4$, $\|\mathbf{v}\| = 3$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{3}$.

Q5.a

Find $\mathbf{u} \cdot \mathbf{v}$.

> $$\begin{aligned} &\mathbf{u} \cdot \mathbf{v} \\ &= \|\mathbf{u}\| \|\mathbf{v}\| \cos \frac{\pi}{3} \\ &= 4 \cdot 3 \cdot \frac{1}{2} \\ &= 6 \end{aligned}$$

Q5.b

Find the real number k such that the vectors $\mathbf{u} + k\mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$ are orthogonal.

> $$\begin{aligned} \text{Set the dot product of the vectors to } 0. \\ (\mathbf{u} + k\mathbf{v}) \cdot (\mathbf{u} - 2\mathbf{v}) &= 0 \\ \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + k\mathbf{v} \cdot \mathbf{u} - 2k\mathbf{v} \cdot \mathbf{v} &= 0 \\ \|\mathbf{u}\|^2 + (k-2)\mathbf{u} \cdot \mathbf{v} - 2k\|\mathbf{v}\|^2 &= 0 \\ 4^2 + (k-2)6 - 2k \cdot 3^2 &= 0 \\ 16 + 6k - 12 - 18k &= 0 \\ -12k &= -4 \\ k &= \frac{1}{3} \end{aligned}$$

Q5.c

Let $\mathbf{a} = 3\mathbf{u} + 4\mathbf{v}$ and $\mathbf{b} = -2\mathbf{u} - \mathbf{v}$. Find the area of the parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as two adjacent edges.

$$
\begin{aligned}
&\|\mathbf{a}\|^2 \\
&= \mathbf{a}\cdot\mathbf{a} \\
&= (3\mathbf{u}+4\mathbf{v})\cdot(3\mathbf{u}+4\mathbf{v}) \\
&= 9\|\mathbf{u}\|^2 + 24\mathbf{u}\cdot\mathbf{v} + 16\|\mathbf{v}\|^2 \\
&= 9\cdot 4^2 + 24\cdot 6 + 16\cdot 3^2 \\
&= 144+144+144 \\
&= 432 \\
&\|\mathbf{b}\|^2 \\
&= \mathbf{b}\cdot\mathbf{b} \\
&= (-2\mathbf{u}-\mathbf{v})\cdot(-2\mathbf{u}-\mathbf{v}) \\
&= 4\|\mathbf{u}\|^2 + 4\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 \\
&= 4\cdot 4^2 + 4\cdot 6 + 3^2 \\
&= 64+24+9 \\
&= 97 \\
&\mathbf{a}\cdot\mathbf{b} \\
&= (3\mathbf{u}+4\mathbf{v})\cdot(-2\mathbf{u}-\mathbf{v}) \\
&= -6\mathbf{u}\cdot\mathbf{u} - 3\mathbf{u}\cdot\mathbf{v} - 8\mathbf{v}\cdot\mathbf{u} - 4\mathbf{v}\cdot\mathbf{v} \\
&= -6\|\mathbf{u}\|^2 - 11\mathbf{u}\cdot\mathbf{v} - 4\|\mathbf{v}\|^2 \\
&= -6\cdot 4^2 - 11\cdot 6 - 4\cdot 3^2 \\
&= -96-66-36 \\
&= -198
\end{aligned}
$$

Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$
\begin{aligned}
&\text{area} \\
&= \|\mathbf{a}\times\mathbf{b}\| \\
&= \sqrt{\|\mathbf{a}\times\mathbf{b}\|^2} \\
&= \sqrt{\|\mathbf{a}\|^2\|\mathbf{b}\|^2\sin^2\theta} \\
&= \sqrt{\|\mathbf{a}\|^2\|\mathbf{b}\|^2(1-\cos^2\theta)} \\
&= \sqrt{\|\mathbf{a}\|^2\|\mathbf{b}\|^2-(\mathbf{a}\cdot\mathbf{b})^2} \\
&= \sqrt{432\cdot 97 - 198^2} \\
&= \sqrt{41904-39204} \\
&= \sqrt{2700} \\
&= 30\sqrt{3}
\end{aligned}
$$

Q9

Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors of the same dimension. Show that the vector

$$\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$$

bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

Let $\theta \in [0, \pi]$ be the angle between $\mathbf{u}$ and $\mathbf{w}$.

$$\begin{aligned} \mathbf{u} \cdot \mathbf{w} &= \|\mathbf{u}\| \|\mathbf{w}\| \cos\theta \\ \|\mathbf{u}\| \mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\| \|\mathbf{u}\|^2 &= \|\mathbf{u}\| \|\mathbf{w}\| \cos\theta \\ \cos\theta &= \frac{\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\| \|\mathbf{v}\|}{\|\mathbf{w}\|} \end{aligned}$$

Let $\phi \in [0, \pi]$ be the angle between $\mathbf{v}$ and $\mathbf{w}$.

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= \|\mathbf{v}\| \|\mathbf{w}\| \cos\phi \\ \|\mathbf{u}\| \|\mathbf{v}\|^2 + \|\mathbf{v}\| \mathbf{u} \cdot \mathbf{v} &= \|\mathbf{v}\| \|\mathbf{w}\| \cos\phi \\ \cos\phi &= \frac{\|\mathbf{u}\| \|\mathbf{v}\| + \mathbf{u} \cdot \mathbf{v}}{\|\mathbf{w}\|} \\ &= \cos\theta \\ \phi &= \theta \qquad (\cos(*) \text{ is injective on } [0, \pi]) \end{aligned}$$

$\therefore$ $\mathbf{w}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

Q11

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^n$ suchh that

$$\mathbf{a} \cdot \mathbf{a} = 1, \mathbf{b} \cdot \mathbf{b} = 1 \text{ and } \mathbf{a} \cdot \mathbf{b} = 0$$

.

Let $S = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u} = x\mathbf{a} + y\mathbf{b} \text{ for some } x, y \in \mathbb{R}\}$.

Q11.a

Show that for every $\mathbf{u} \in S$, we have

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{a})\mathbf{a} + (\mathbf{u} \cdot \mathbf{b})\mathbf{b}$$

.

$$\begin{aligned} &(\mathbf{u} \cdot \mathbf{a})\mathbf{a} + (\mathbf{u} \cdot \mathbf{b})\mathbf{b} \\ &= ((x\mathbf{a} + y\mathbf{b}) \cdot \mathbf{a})\mathbf{a} + ((x\mathbf{a} + y\mathbf{b}) \cdot \mathbf{b})\mathbf{b} \\ &= (x\mathbf{a} \cdot \mathbf{a} + y\mathbf{b} \cdot \mathbf{a})\mathbf{a} + (x\mathbf{a} \cdot \mathbf{b} + y\mathbf{b} \cdot \mathbf{b})\mathbf{b} \\ &= (x + 0)\mathbf{a} + (0 + y)\mathbf{b} \\ &= x\mathbf{a} + y\mathbf{b} \\ &= \mathbf{u} \end{aligned}$$

Q11.b

For each $\mathbf{v} \in \mathbb{R}^n$, let $\mathbf{w} = (\mathbf{v} \cdot \mathbf{a})\mathbf{a} + (\mathbf{v} \cdot \mathbf{b})\mathbf{b}$. Show that $\mathbf{v} - \mathbf{w}$ is orthogonal to every $\mathbf{u} \in S$.

$$
\begin{aligned}
&(\mathbf{v}-\mathbf{w})\cdot\mathbf{u}\\
&=\mathbf{v}\cdot\mathbf{u}-\mathbf{w}\cdot\mathbf{u}\\
&=\mathbf{v}\cdot\mathbf{u}-((\mathbf{v}\cdot\mathbf{a})\mathbf{a}+(\mathbf{v}\cdot\mathbf{b})\mathbf{b})\cdot((\mathbf{u}\cdot\mathbf{a})\mathbf{a}+(\mathbf{u}\cdot\mathbf{b})\mathbf{b})\\
&=\mathbf{v}\cdot\mathbf{u}-(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{a})\mathbf{a}\cdot\mathbf{a}-(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{b})\mathbf{a}\cdot\mathbf{b}\\
&\quad-(\mathbf{v}\cdot\mathbf{b})(\mathbf{u}\cdot\mathbf{a})\mathbf{b}\cdot\mathbf{a}-(\mathbf{v}\cdot\mathbf{b})(\mathbf{u}\cdot\mathbf{b})\mathbf{b}\cdot\mathbf{b}\\
&=\mathbf{v}\cdot\mathbf{u}-(\mathbf{v}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{a})-(\mathbf{v}\cdot\mathbf{b})(\mathbf{u}\cdot\mathbf{b})\\
&=\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot((\mathbf{u}\cdot\mathbf{a})\mathbf{a}+(\mathbf{u}\cdot\mathbf{b})\mathbf{b})\\
&=\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{u}\\
&=0\\
&\qquad(\mathbf{v}-\mathbf{w})\cdot\mathbf{u}=0\\
&\implies \mathbf{v}-\mathbf{w}\perp\mathbf{u}
\end{aligned}
$$

Q16

Three lines are said to be ***concurrent*** if they pass through the same point.

Q16.a

A **median** of a triangle is a line that passes through both a vertex of the triangle and the mid-point of the edge opposite the vertex. Prove that the three medians of a triangle are concurrent.

$n \in \mathbb{Z}_{\geq 2}$

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ be the position vectors of vertices of a triangle.

$\mathbf{x} := \dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$.

$\mathbf{x_a} := \mathbf{x} - \mathbf{a} = \dfrac{-2\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$

$\mathbf{x_b} := \mathbf{x} - \mathbf{b} = \dfrac{\mathbf{a}-2\mathbf{b}+\mathbf{c}}{3}$

$\mathbf{x_c} := \mathbf{x} - \mathbf{c} = \dfrac{\mathbf{a}+\mathbf{b}-2\mathbf{c}}{3}$

$\dfrac{3}{2}\mathbf{x_a}$

$= \dfrac{-6\mathbf{a}+3\mathbf{b}+3\mathbf{c}}{6}$

$= -\mathbf{a} + \dfrac{1}{2}\mathbf{b} + \dfrac{1}{2}\mathbf{c}$

$= \dfrac{\mathbf{b}+\mathbf{c}}{2} - \mathbf{a}$

$\implies$ $\mathbf{x_a}$ is a direction vector of the median starting from $\mathbf{a}$.

$\dfrac{3}{2}\mathbf{x_b}$

$= \dfrac{3\mathbf{a}-6\mathbf{b}+3\mathbf{c}}{6}$

$= \dfrac{1}{2}\mathbf{a} - \mathbf{b} + \dfrac{1}{2}\mathbf{c}$

$= \dfrac{\mathbf{a}+\mathbf{c}}{2} - \mathbf{b}$

$\implies$ $\mathbf{x_b}$ is a direction vector of the median starting from $\mathbf{b}$.

$\dfrac{3}{2}\mathbf{x_c}$

$= \dfrac{3\mathbf{a}+3\mathbf{b}-6\mathbf{c}}{6}$

$= \dfrac{1}{2}\mathbf{a} + \dfrac{1}{2}\mathbf{b} - \mathbf{c}$

$= \dfrac{\mathbf{a}+\mathbf{b}}{2} - \mathbf{c}$

$\implies$ $\mathbf{x_c}$ is a direction vector of the median starting from $\mathbf{c}$.

$\because \mathbf{a} + \mathbf{x_a} = \mathbf{x}$

$\mathbf{b} + \mathbf{x_b} = \mathbf{x}$

$\mathbf{c} + \mathbf{x_c} = \mathbf{x}$

$\therefore$ The three medians are concurrent.

Q16.b

Prove that the three altitudes of a triangle are concurrent.

$n \in \mathbb{Z}_{\geq 2}$
Let the origin $\mathbf{o}$ be one of the vertex of the triangle.
Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ be the position vectors of the other two vertices of the triangle.
$\mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$ because otherwise two vertices are the same position,
making it no longer a triangle.
$\mathbf{a}$ and $\mathbf{b}$ are linearly independent because otherwise the three vertices are collinear,
making it no longer a triangle.

Consider the 2-dimensional linear span S generated by $\mathbf{a}$ and $\mathbf{b}$.
The triangle is on S as its three vertices are on S.
So its altitudes are on S.
Let $\mathbf{A}_*$ be the direction vector (not necessarily normalized)
of the altitude of the triangle starting from $*$.

$$\mathbf{A}_\mathbf{o} := \omega\mathbf{a} + \mathbf{b} \qquad \omega \in \mathbb{R}$$
$$\mathbf{A}_\mathbf{a} := \alpha\mathbf{a} + \mathbf{b} \qquad \alpha \in \mathbb{R}$$
$$\mathbf{A}_\mathbf{b} := \beta\mathbf{a} + \mathbf{b} \qquad \beta \in \mathbb{R}$$

$$\begin{aligned}
\mathbf{A}_\mathbf{o} \cdot (\mathbf{b} - \mathbf{a}) &= 0 \\
(\omega\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) &= 0 \\
(\omega - 1)\mathbf{a} \cdot \mathbf{b} - \omega\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 &= 0 \\
\omega &= \frac{\mathbf{a} \cdot \mathbf{b} - \|\mathbf{b}\|^2}{\mathbf{a} \cdot \mathbf{b} - \|\mathbf{a}\|^2}
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}_\mathbf{a} \cdot \mathbf{b} &= 0 \\
(\alpha\mathbf{a} + \mathbf{b}) \cdot \mathbf{b} &= 0 \\
\alpha\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2 &= 0 \\
\alpha &= -\frac{\|\mathbf{b}\|^2}{\mathbf{a} \cdot \mathbf{b}}
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}_\mathbf{b} \cdot \mathbf{a} &= 0 \\
(\beta\mathbf{a} + \mathbf{b}) \cdot \mathbf{a} &= 0 \\
\beta\|\mathbf{a}\|^2 + \mathbf{a} \cdot \mathbf{b} &= 0 \\
\beta &= -\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}
\end{aligned}$$

Find the intersection between the two medians starting from $\mathbf{a}$ and $\mathbf{b}$,
which is always possible on a plane, considering the medians are not parallel.

$$\mathbf{a} + x\mathbf{A}_\mathbf{a} = \mathbf{b} + y\mathbf{A}_\mathbf{b} \qquad (x, y \in \mathbb{R})$$
$$\mathbf{a} + x\alpha\mathbf{a} + x\mathbf{b} = \mathbf{b} + y\beta\mathbf{a} + y\mathbf{b}$$
$$\begin{cases} 1 + x\alpha = y\beta \\ x = 1 + y \end{cases}$$
$$\begin{aligned}
1 + \alpha + y\alpha &= y\beta \\
y &= \frac{1 + \alpha}{\beta - \alpha} \\
&= \frac{\frac{\mathbf{a}\cdot\mathbf{b} - \|\mathbf{b}\|^2}{\mathbf{a}\cdot\mathbf{b}}}{\frac{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2}{\|\mathbf{a}\|^2\mathbf{a}\cdot\mathbf{b}}} \\
&= \frac{\|\mathbf{a}\|^2\mathbf{a} \cdot \mathbf{b} - \|\mathbf{a}\|^2\|\mathbf{b}\|^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2}
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} + y\mathbf{A_b} &= \mathbf{b} + \frac{\|\mathbf{a}\|^2\mathbf{a}\cdot\mathbf{b} - \|\mathbf{a}\|^2\|\mathbf{b}\|^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2}\left(-\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} + \mathbf{b}\right) \\
&= \frac{\|\mathbf{b}\|^2\mathbf{a}\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2}\mathbf{a} + \frac{\|\mathbf{a}\|^2\mathbf{a}\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2}\mathbf{b} \\
&= \frac{\|\mathbf{a}\|^2\mathbf{a}\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2}\left(\frac{\|\mathbf{b}\|^2\mathbf{a}\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})^2}{\|\mathbf{a}\|^2\mathbf{a}\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})^2}\mathbf{a} + \mathbf{b}\right) \\
&= \frac{\|\mathbf{a}\|^2\mathbf{a}\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2}\left(\frac{\|\mathbf{b}\|^2 - \mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2 - \mathbf{a}\cdot\mathbf{b}}\mathbf{a} + \mathbf{b}\right) \\
&= \frac{\|\mathbf{a}\|^2\mathbf{a}\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2}\mathbf{A_o}
\end{aligned}$$

$\because$ The intersection of the two altitudes starting from $\mathbf{a}$ and $\mathbf{b}$ is a direction vector of the altitude starting from $\mathbf{o}$,
$\therefore$ The three altitudes are concurrent.

Q19

Find a vector equation and parametric equations for each of the following lines in $\mathbb{R}^3$.

Q19.a

The line passing through $(6,-5,2)$ and parallel to $\langle 3,9,-2\rangle$.

$$\begin{aligned}
&\text{Let } t \in \mathbb{R}. \\
\text{vector equation} &= \langle 6,-5,2\rangle + t\langle 3,9,-2\rangle \\
&= \langle 6+3t, -5+9t, 2-2t\rangle \\
\text{parametric equation} &= \begin{cases} x = 6+3t \\ y = -5+9t \\ z = 2-2t \end{cases}
\end{aligned}$$

Q19.b

The line segment with end-points with end-points $(4,-6,6)$ and $(2,3,1)$.

$$\begin{aligned}
\text{difference vector} &= \langle 4,-6,6\rangle - \langle 2,3,1\rangle \\
&= \langle 2,-9,5\rangle \\
&\text{Let } t \in [0,1]. \\
\text{vector equation} &= \langle 2,3,1\rangle + t\langle 2,-9,5\rangle \\
&= \langle 2+2t, 3-9t, 1+5t\rangle \\
\text{parametric equation} &= \begin{cases} x = 2+2t \\ y = 3-9t \\ z = 1+5t \end{cases}
\end{aligned}$$

Q19.c

The line passing through $(2,1,0)$ and perpendicular to both $\hat{\imath}+\hat{\jmath}$ and $\hat{\jmath}+\hat{k}$.

$$\begin{aligned}\text{direction vector} &= (\hat{i}+\hat{j})\times(\hat{j}+\hat{k})\\ &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 0\\ 0 & 1 & 1\end{vmatrix}\\ &= \hat{i}-\hat{j}+\hat{k}\\ &= \langle 1,-1,1\rangle\\ \text{Let } t\in\mathbb{R}.\\ \text{vector equation} &= \langle 2,1,0\rangle + t\langle 1,-1,1\rangle\\ &= \langle 2+t, 1-t, t\rangle\\ \text{parametric equation} &= \begin{cases} x = 2+t\\ y = 1-t\\ z = t\end{cases}\end{aligned}$$

Q19.d

The line passing through $(0,1,2)$ and orthogonally intersecting the line

$$x = 1+t \text{ and } y = 1-t \text{ and } z = 2t$$

.

$$\begin{aligned}\mathbf{p} &:= \langle 0,1,2\rangle\\ \mathbf{L}(t) &:= \langle 1+t, 1-t, 2t\rangle \qquad t\in\mathbb{R}\end{aligned}$$

$$\begin{aligned}\text{Set...}\\ (\mathbf{L}(t)-\mathbf{p})\cdot\text{direction vector of }\mathbf{L} &= 0\\ \langle 1+t, -t, 2t-2\rangle\cdot\langle 1,-1,2\rangle &= 0\\ (1+t)(1)+(-t)(-1)+(2t-2)(2) &= 0\\ 6t-3 &= 0\\ t &= 0.5\end{aligned}$$

$$\begin{aligned}\text{direction vector} &= \mathbf{L}(0.5)-\mathbf{p}\\ &= \langle 1+0.5, 1-0.5, 1\rangle - \langle 0,1,2\rangle\\ &= \langle 1.5, -0.5, -1\rangle\\ \text{Let } u\in\mathbb{R}.\\ \text{vector equation} &= \langle 0,1,2\rangle + u\langle 1.5,-0.5,-1\rangle\\ &= \langle 1.5u, 1-0.5u, 2-u\rangle\\ \text{parametric equation} &= \begin{cases} x = 1.5u\\ y = 1-0.5u\\ z = 2-u\end{cases}\end{aligned}$$

Q22

Q22.a

Let P be a point on a smooth curve $r=f(\theta)$ in $\mathbb{R}^2$ which is not the origin, and let α be the acute angle between the line OP and the tangent to the curve at P. Show that

$$\cos\alpha = \frac{|f'(\theta)|}{\sqrt{f(\theta)^2 + f'(\theta)^2}} \qquad 1$$

.

Rotate the curve such that $\theta = \frac{\pi}{2}$.
α does not change in this transformation by symmetry.
$(x, y) = (r\cos\theta, r\sin\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$

Then the line starting with $f'(\theta)$ as the slope is the tangent.
Consider the triangle enclosed by the horizontal, the vertical, and the line.

$$\begin{aligned}\cos\alpha &= \frac{f(\theta)}{\sqrt{f(\theta)^2 + \frac{f(\theta)^2}{f'(\theta)^2}}} \\ &= \frac{f(\theta)}{f(\theta)\frac{1}{|f'(\theta)|}\sqrt{f'(\theta)^2 + 1}}\end{aligned}$$

Q22.b

Using (a), show that at every point P on the curve $r = e^\theta$, the angle between the line OP and the tangent line to the curve at P is always $\pi/4$.

$$\begin{aligned}\cos a &= \frac{|f'(\theta)|}{\sqrt{f(\theta)^2 + f'(\theta)^2}} \\ &= \frac{|e^\theta|}{\sqrt{e^{2\theta} + e^{2\theta}}} \qquad 2 \\ &= \frac{|e^\theta|}{|e^\theta|\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \\ \alpha &= \frac{\pi}{4} \qquad (\alpha \text{ is acute})\end{aligned}$$

Q22.c

Let $r = f(\theta)$ be a smooth curve such that at every point P on it, the angle between the line OP and the tangent line to the curve at P is always a fixed constant. Show that there exist constants C and k such that $f(\theta) = Ce^{k\theta}$ for all θ.

$\boxed{\text{Let } f(\theta) = Ce^{k\theta}.}$

0

$$\begin{aligned}
\cos\alpha &= \frac{|Cke^{k\theta}|}{\sqrt{e^{2k\theta}+C^2k^2e^{2k\theta}}} \\
&= \frac{|Ck||e^{k\theta}|}{|e^{k\theta}|\sqrt{1+C^2k^2}} \\
&= \frac{|Ck|}{\sqrt{1+C^2k^2}}
\end{aligned}$$

The above shows that $\cos a$ does not depend on θ.
So the angle is constant when changing θ.

When $C = 1,$ set $k = 0$ to get $\cos\alpha = 0,$
and as $k \to +\infty,\ \cos\alpha \to 1.$
By the intermediate value theorem,
C, k can be set arbitrarily to get any real number in $[0, 1).$
$\cos\alpha = 1$ can be ignored because the tangent is parallel to OP,
which is impossible if $f(\theta)$ is a function.