Solution to Problem Set 3

1. (a) First observe that

$$\int_{-\pi}^{\pi}\cos mx\cos nx\,dx=\frac{1}{2}\int_{-\pi}^{\pi}\cos(m+n)x\,dx+\frac{1}{2}\int_{-\pi}^{\pi}\cos(m-n)x\,dx\,,$$

$$\int_{-\pi}^{\pi}\sin mx\sin nx\,dx=\frac{1}{2}\int_{-\pi}^{\pi}\cos(m-n)x\,dx-\frac{1}{2}\int_{-\pi}^{\pi}\cos(m+n)x\,dx\,,$$

$$\int_{-\pi}^{\pi}\cos mx\sin nx\,dx=\frac{1}{2}\int_{-\pi}^{\pi}\sin(m+n)x\,dx-\frac{1}{2}\int_{-\pi}^{\pi}\sin(m-n)x\,dx.$$

⊙ If $m=n=0$, then we have

$$\int_{-\pi}^{\pi}\cos(m\pm n)x\,dx=\int_{-\pi}^{\pi}1dx=2\pi \qquad \text{and} \qquad \int_{-\pi}^{\pi}\sin(m\pm n)x\,dx=\int_{-\pi}^{\pi}0dx=0.$$

⊙ If $m=n\neq 0$, then we have

$$\int_{-\pi}^{\pi}\cos(m-n)x\,dx=\int_{-\pi}^{\pi}1dx=2\pi, \qquad \int_{-\pi}^{\pi}\sin(m-n)x\,dx=\int_{-\pi}^{\pi}0dx=0,$$

$$\int_{-\pi}^{\pi}\cos(m+n)x\,dx=\left[\frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi}=0 \qquad \text{and} \qquad \int_{-\pi}^{\pi}\sin(m+n)x\,dx=\left[\frac{-\cos 2nx}{2n}\right]_{-\pi}^{\pi}=0.$$

⊙ If $m\neq n$, then we have

$$\int_{-\pi}^{\pi}\cos(m\pm n)x\,dx=\left[\frac{\sin(m\pm n)x}{m\pm n}\right]_{-\pi}^{\pi}=0 \qquad \text{and}$$

$$\int_{-\pi}^{\pi}\sin(m\pm n)x\,dx=\left[\frac{-\cos(m\pm n)x}{m\pm n}\right]_{-\pi}^{\pi}=0.$$

Therefore in summary, we have

$$\int_{-\pi}^{\pi}\cos mx\cos nx\,dx=\begin{cases}2\pi & \text{if } m=n=0\\ \pi & \text{if } m=n\neq 0,\\ 0 & \text{if } m\neq n\end{cases} \qquad \int_{-\pi}^{\pi}\sin mx\sin nx\,dx=\begin{cases}\pi & \text{if } m=n\neq 0\\ 0 & \text{otherwise}\end{cases},$$

and

$$\int_{-\pi}^{\pi}\cos mx\sin nx\,dx=0 \qquad \text{for every non-negative integers } m,n.$$

(b) For each $k\in\{1,2,\dots,n\}$, we have

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin kx\,dx&=\frac{1}{\pi}\int_{-\pi}^{\pi}(a_1\sin x+a_2\sin 2x+\cdots+a_n\sin nx)\sin kx\,dx\\&=\frac{1}{\pi}\underbrace{\left(a_1\int_{-\pi}^{\pi}\sin x\sin kx\,dx+a_2\int_{-\pi}^{\pi}\sin 2x\sin kx\,dx+\cdots+a_n\int_{-\pi}^{\pi}\sin nx\sin kx\,dx\right)}_{\text{only the } k\text{th term is nonzero by (a)}}\\&=\frac{1}{\pi}\left(a_k\int_{-\pi}^{\pi}\sin kx\sin kx\,dx\right)=\frac{1}{\pi}(a_k\pi)=a_k.\end{aligned}$$

2. (a) Taking antiderivatives by parts, we have

$$\int x^2 \arctan x \, dx = \int \arctan x \, d\left(\frac{x^3}{3}\right) = \frac{1}{3}x^3 \arctan x - \int \frac{x^3}{3}\frac{1}{1+x^2}\,dx$$

$$= \frac{1}{3}x^3 \arctan x - \frac{1}{6}\int \frac{x^2}{1+x^2}\,dx^2 = \frac{1}{3}x^3 \arctan x - \frac{1}{6}\int\left(1 - \frac{1}{1+x^2}\right)dx^2$$

$$= \frac{1}{3}x^3 \arctan x - \frac{1}{6}(x^2 - \ln(1+x^2)) + C = \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C,$$

where C is an arbitrary constant.

(b) Taking antiderivatives by parts, we have

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx\,,$$

so rearranging the terms we get

$$\int \sin(\ln x)\,dx = \frac{1}{2}x(\sin(\ln x) - \cos(\ln x)) + C,$$

where C is an arbitrary constant.

(c) First consider $\int e^{2x}\sin 2x\,dx$. Taking antiderivatives by parts, we have

$$\int e^{2x}\sin 2x\,dx = -\frac{e^{2x}\cos 2x}{2} + \int e^{2x}\cos 2x\,dx = -\frac{e^{2x}\cos 2x}{2} + \frac{e^{2x}\sin 2x}{2} - \int e^{2x}\sin 2x\,dx\,,$$

so rearranging the terms we get

$$\int e^{2x}\sin 2x\,dx = \frac{e^{2x}(\sin 2x - \cos 2x)}{4} + C.$$

Therefore the required antiderivative is

$$\int e^{2x}(\sin x + \cos x)^2 dx = \int e^{2x}(1 + \sin 2x)dx = \int e^{2x}dx + \int e^{2x}\sin 2x\,dx$$

$$= \frac{e^{2x}}{2} + \frac{e^{2x}(\sin 2x - \cos 2x)}{4} + C,$$

where C is an arbitrary constant.

(d) First consider $\int e^{x^2}dx$. Taking antiderivatives by parts, we have

$$\int e^{x^2}dx = xe^{x^2} - \int x\cdot 2xe^{x^2}dx = xe^{x^2} - \int 2x^2e^{x^2}dx\,,$$

so rearranging the terms we get

$$\int (2x^2+1)e^{x^2}dx = \int 2x^2e^{x^2}dx + \int e^{x^2}dx = xe^{x^2} + C,$$

where C is an arbitrary constant.

3. For each positive integer n, we have

$$\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}=\frac{1}{n}\sqrt[n]{(n+1)(n+2)\cdots(2n)}=\sqrt[n]{\frac{(n+1)(n+2)\cdots(2n)}{n^n}}=\sqrt[n]{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)},$$

so

$$\ln\left(\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}\right)=\frac{1}{n}\ln\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]=\frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\cdots+\ln\left(1+\frac{n}{n}\right)\right],$$

which is the right Riemann sum of the function $f(x)=\ln x$ with respect to the regular partition of $[1,2]$ into n subintervals. Since f is continuous on $[1,2]$, it is integrable on $[1,2]$; thus taking limits as $n\to+\infty$, we have

$$\lim_{n\to+\infty}\ln\left(\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}\right)=\int_1^2\ln x\,dx=[x\ln x-x]_1^2=2\ln 2-1.$$

Finally, since the exponential function is continuous at $2\ln 2-1$, the required limit is

$$\lim_{n\to+\infty}\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}=\lim_{n\to+\infty}e^{\ln\left(\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}\right)}=e^{\lim_{n\to+\infty}\ln\left(\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}\right)}=e^{2\ln 2-1}=\frac{4}{e}.$$

4. Since f is continuous, by the first version of Fundamental Theorem Calculus, its area function is differentiable and

$$\frac{d}{dx}\int_{-a}^{x}f(t)dt=f(x).$$

Therefore we apply integration by parts to get

$$\begin{aligned}\int_{-a}^{a}\left(\int_{-a}^{x}f(t)dt\right)dx&=\left[x\int_{-a}^{x}f(t)dt\right]_{x=-a}^{x=a}-\int_{-a}^{a}x\left(\frac{d}{dx}\int_{-a}^{x}f(t)dt\right)dx\\&=\left[a\underbrace{\int_{-a}^{a}f(t)dt}_{=0\text{ since }f\text{ is odd}}-(-a)\underbrace{\int_{-a}^{-a}f(t)dt}_{=0}\right]-\int_{-a}^{a}xf(x)dx\\&=\int_{-a}^{a}xf(x)dx.\end{aligned}$$

5. (a) The (explicit) substitution $u=\frac{1}{x}$ is non-differentiable (even worse, it is undefined) at the point $0\in(-1,1)$. So the substitution rule cannot be applied.

(b) When handling the integral $\int_a^b f(x)dx$ by the substitution rule, we apply the equality (cf. Theorem 5.68)

$$\int_?^? f(g(u))g'(u)du=\int_a^b f(x)dx$$

"from the right-hand side to the left-hand side". This requires that $f(x)$ can be expressed as a differentiable function of u, i.e. $f(x)=f(g(u))$. But the substitution $u=(x-a)(x-b)$ given in the "proof" does not have an inverse; so x, and consequently $f(x)$, **is not guaranteed to be expressible as a function of u**. So the substitution rule cannot be applied in general.

Remark: In fact, the substitution $u=(x-a)(x-b)$ is valid on **each** interval $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$ separately: We have $x=\frac{a+b}{2}-\sqrt{u+\frac{(b-a)^2}{4}}$ for $x\in\left[a,\frac{a+b}{2}\right]$; and $x=\frac{a+b}{2}+\sqrt{u+\frac{(b-a)^2}{4}}$ for $x\in\left[\frac{a+b}{2},b\right]$. If we apply the substitution rule on these two intervals separately, then we can observe explicitly that $\int_a^b f(x)dx$ does not necessarily equal to zero in general.

(c) The antiderivatives $\int\frac{f'(x)}{f(x)}dx$ on the two sides of the equation <u>may differ by a constant function</u>. So when we subtract such an antiderivative from both sides, the left-hand side should be left with a constant function instead of just 0. Note that the 1 on the right hand side is indeed a constant function.

6. (a) Applying integration by parts, we have

$$\begin{aligned}\int_a^b (x-a)f'(x)dx &= \int_a^b (x-a)df(x) = [(x-a)f(x)]_a^b - \int_a^b f(x)dx\\ &= (b-a)f(b) - \int_a^b f(x)dx = \int_a^b f(b)dx - \int_a^b f(x)dx\\ &= \int_a^b \big(f(b)-f(x)\big)dx.\end{aligned}$$

(b) According to the result from (a), we have

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left[f\left(\frac{k}{n}\right)-f(x)\right]dx = \int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(x-\frac{k-1}{n}\right)f'(x)dx\,.$$

Now since f' and $x-\frac{k-1}{n}$ are both continuous on $\left[\frac{k-1}{n},\frac{k}{n}\right]$ and since $x-\frac{k-1}{n}\geq 0$ for every $x\in\left[\frac{k-1}{n},\frac{k}{n}\right]$, by the generalized Mean Value Theorem for integrals, there exists $\omega_k\in\left[\frac{k-1}{n},\frac{k}{n}\right]$ such that

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(x-\frac{k-1}{n}\right)f'(x)dx = f'(\omega_k)\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(x-\frac{k-1}{n}\right)dx\,.$$

Note that $\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(x-\frac{k-1}{n}\right)dx=\frac{1}{2n^2}$ as it represents the area of a right triangle with base $\frac{1}{n}$ and height $\frac{1}{n}$; so

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left[f\left(\frac{k}{n}\right)-f(x)\right]dx = \frac{f'(\omega_k)}{2n^2}.$$

■

(c) For each $n\in\mathbb{N}$ and each integer $1\leq k\leq n$, we have $\frac{1}{n}f\left(\frac{k}{n}\right)=\int_{\frac{k-1}{n}}^{\frac{k}{n}}f\left(\frac{k}{n}\right)dx$. Therefore

$$\begin{aligned}E_n &= \frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-\int_0^1 f(x)dx = \sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}f\left(\frac{k}{n}\right)dx - \sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}f(x)dx\\ &= \sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left[f\left(\frac{k}{n}\right)-f(x)\right]dx\,.\end{aligned}$$

Now applying the result from (b), we see that for each $n \in \mathbb{N}$, there exist n numbers $\omega_1, \omega_2, \dots, \omega_n$ such that $\omega_k \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$ and

$$nE_n = n \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left[f\left(\frac{k}{n}\right) - f(x) \right] dx = n \sum_{k=1}^{n} \frac{f'(\omega_k)}{2n^2} = \frac{1}{2} \sum_{k=1}^{n} f'(\omega_k) \frac{1}{n}.$$

Now this is a Riemann sum of f' with respect to the regular partition of $[0,1]$ into n subintervals. Since f' is continuous on $[0,1]$, it is integrable on $[0,1]$. Thus taking limit as $n \to +\infty$, we have

$$\lim_{n\to+\infty} nE_n = \frac{1}{2}\int_0^1 f'(x)dx = \frac{f(1) - f(0)}{2}$$

by the second version of the Fundamental Theorem of Calculus. ■

7. (a) For every $x \in (0, +\infty)$, we have

$$f'(x) = e^x + xe^x > 0,$$

so f is strictly increasing on its domain $[0, +\infty)$.

(b) (i) The domain of g is same as the range of f, which is $[0, +\infty)$. For every $x \in (0, +\infty)$, we have

$$g'(x) = \frac{1}{f'\big(g(x)\big)} = \frac{1}{e^{g(x)} + g(x)e^{g(x)}} = \frac{1}{e^{g(x)} + x}$$

since $g(x)e^{g(x)} = f\big(g(x)\big) = x$.

(ii) Let $u = g(x)$. Then $du = g'(x)dx$, so taking antiderivatives by parts we have

$$\begin{aligned}\int g(x)dx &= xg(x) - \int xg'(x)dx = xg(x) - \int x \cdot \frac{1}{e^{g(x)} + x} dx \\ &= xg(x) - \int \left(1 - \frac{e^{g(x)}}{e^{g(x)} + x}\right) dx = xg(x) - \int 1dx + \int e^{g(x)} g'(x)dx \\ &= xg(x) - x + \int e^u du = xg(x) - x + e^u + C \\ &= xg(x) - x + e^{g(x)} + C,\end{aligned}$$

where C is an arbitrary constant.

(iii) It is clear that $f(0) = 0$ and $f(1) = e$, so $g(0) = 0$ and $g(e) = 1$. By (b)(ii) and the Fundamental Theorem of Calculus, we have

$$\begin{aligned}\int_0^e g(x)dx &= \left[xg(x) - x + e^{g(x)}\right]_0^e \\ &= (e \cdot 1 - e + e^1) - (0 \cdot 0 - 0 + e^0) \\ &= e - 1.\end{aligned}$$

Remark: The inverse of the function $f(x) = xe^x$ studied in this problem is called ***Lambert's W-function***, and is usually denoted by W_0 or simply W, i.e. $W(x) = f^{-1}(x)$.

8. (a) (i) The derivative of f is $f'(x) = n(x^2-1)^{n-1}(2x) = 2nx(x^2-1)^{n-1}$ for every $x \in \mathbb{R}$, so

$$(x^2-1)f'(x) - 2nxf(x) = (x^2-1)[2nx(x^2-1)^{n-1}] - 2nx(x^2-1)^n = 0$$

for every $x \in \mathbb{R}$. ■

(ii) We differentiate both sides of the result from (a) for $(n+1)$ times. According to Leibniz's rule, we have

$$\left[(x^2-1)f^{(n+2)}(x) + (n+1)(2x)f^{(n+1)}(x) + \frac{(n+1)(n)}{2}(2)f^{(n)}(x)\right] - \left[(2nx)f^{(n+1)}(x) + (n+1)(2n)f^{(n)}(x)\right] = 0$$

for every $x \in \mathbb{R}$, so

$$(x^2-1)f^{(n+2)}(x) + [(n+1)(2x) - (2nx)]f^{(n+1)}(x) + \left[\frac{(n+1)(n)}{2}(2) - (n+1)(2n)\right]f^{(n)}(x) = 0,$$

i.e. $(x^2-1)f^{(n+2)}(x) + 2xf^{(n+1)}(x) - n(n+1)f^{(n)}(x) = 0$ for every $x \in \mathbb{R}$. ■

(b) (i) For every non-negative integer n and every $x \in \mathbb{R}$, we have

$$\frac{d}{dx}[(x^2-1)p_n{}'(x)] = 2xp_n{}'(x) + (x^2-1)p_n{}''(x) = 2x \cdot \frac{1}{2^n n!}f^{(n+1)}(x) + (x^2-1) \cdot \frac{1}{2^n n!}f^{(n+2)}(x)$$
$$= \frac{1}{2^n n!}n(n+1)f^{(n)}(x) = n(n+1)p_n(x)$$

according to the result from (a) (ii). ■

(ii) Suppose that m and n are distinct non-negative integers. Then applying the result from (b) (i) and integration by parts, we have

$$\begin{aligned}
&\int_{-1}^{1}[m(m+1) - n(n+1)]p_m(x)p_n(x)dx \\
&= \int_{-1}^{1} m(m+1)p_m(x)p_n(x)dx - \int_{-1}^{1} n(n+1)p_n(x)p_m(x)dx \\
&= \int_{-1}^{1} p_n(x)d[(x^2-1)p_m{}'(x)] - \int_{-1}^{1} p_m(x)d[(x^2-1)p_n{}'(x)] \\
&= \underbrace{[(x^2-1)p_n(x)p_m{}'(x)]_{-1}^{1}}_{=0} - \int_{-1}^{1}(x^2-1)p_m{}'(x)p_n{}'(x)dx - \underbrace{[(x^2-1)p_m(x)p_n{}'(x)]_{-1}^{1}}_{=0} \\
&\qquad + \int_{-1}^{1}(x^2-1)p_n{}'(x)p_m{}'(x)dx \\
&= 0.
\end{aligned}$$

Since $m(m+1) - n(n+1) \neq 0$, we have $\int_{-1}^{1} p_m(x)p_n(x)dx = 0$. ■

Remark: The functions p_n studied in this problem are called the ***Legendre polynomials***. We can verify that each p_n is a polynomial of degree n. The sequence of polynomials (p_n) is said to be ***orthogonal*** because of the property we obtained in (b) (ii).

9. (a) For each positive integer n, we have

$$\frac{d}{dx}\cos^n x \sin nx = (-n\cos^{n-1} x \sin x)(\sin nx) + (\cos^n x)(n\cos nx)$$
$$= n\cos^{n-1} x\,(\cos nx \cos x - \sin nx \sin x) = n\cos^{n-1} x \cos(n+1)x\,;$$

so integration by parts gives

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx\, dx = \frac{1}{n}\int_0^{\frac{\pi}{2}} \cos^n x\, d\sin nx = \frac{1}{n}\underbrace{[\cos^n x \sin nx]_0^{\frac{\pi}{2}}}_{=0} - \frac{1}{n}\int_0^{\frac{\pi}{2}} \sin nx\,(-n\cos^{n-1} x \sin x)dx$$
$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} x\,(\sin nx \sin x)dx = \int_0^{\frac{\pi}{2}} \cos^{n-1} x\left(\frac{1}{2}[\cos(nx-x) - \cos(nx+x)]\right)dx$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(n-1)x\, dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(n+1)x\, dx$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(n-1)x\, dx - \frac{1}{2}\underbrace{[\cos^n x \sin nx]_0^{\frac{\pi}{2}}}_{=0} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(n-1)x\, dx = \frac{1}{2}I_{n-1}.$$

(b) According to the result from (a), for each positive integer n we have

$$I_n = \frac{1}{2}I_{n-1} = \frac{1}{2^2}I_{n-2} = \frac{1}{2^3}I_{n-3} = \cdots = \frac{1}{2^{n-1}}I_1 = \frac{1}{2^n}I_0 = \frac{1}{2^n}\int_0^{\frac{\pi}{2}} 1dx = \frac{\pi}{2^{n+1}}.$$

10. (a) For every non-negative integer n and every $t \in [0,1]$, we have $t^n e^0 \le t^n e^t \le t^n e^1$, so

$$\int_0^1 t^n dt \le \int_0^1 t^n e^t dt \le e\int_0^1 t^n dt\,,$$

i.e. $\frac{1}{n+1} \le I_n \le \frac{e}{1+n}$. ■

(b) For each positive integer n, integration by parts gives

$$I_n = \int_0^1 t^n e^t dt = \int_0^1 t^n de^t = [t^n e^t]_0^1 - \int_0^1 e^t(nt^{n-1})dt = -nI_{n-1} + e.$$

Now we prove the last equality by induction. For $n = 0$, the two sides both equal $e - 1$. Assume that

$$I_m = (-1)^{m+1}m! + e\sum_{k=0}^{m}(-1)^k\frac{m!}{(m-k)!}$$

for some non-negative integer m. Then for $n = m+1$, we have

$$I_{m+1} = -(m+1)I_m + e = -(m+1)\left((-1)^{m+1}m! + e\sum_{k=0}^{m}(-1)^k\frac{m!}{(m-k)!}\right) + e$$
$$= (-1)^{m+2}(m+1)! + e\sum_{k=0}^{m}(-1)^{k+1}\frac{(m+1)!}{(m-k)!} + e$$
$$= (-1)^{m+2}(m+1)! + e\left(\sum_{k=1}^{m+1}(-1)^k\frac{(m+1)!}{(m+1-k)!} + 1\right)$$
$$= (-1)^{m+2}(m+1)! + e\sum_{k=0}^{m+1}(-1)^k\frac{(m+1)!}{(m+1-k)!}.$$

So the equality is true. ■

(c) Suppose that e is a (positive) rational number. Then $e = \frac{p}{q}$ where p and q are positive integers. According to the result from (b), we have $I_n = (-1)^{n+1}n! + \frac{p}{q}\sum_{k=0}^{n}(-1)^k \frac{n!}{(n-k)!}$, so

$$qI_n = q(-1)^{n+1}n! + p\sum_{k=0}^{n}(-1)^k \frac{n!}{(n-k)!}$$

is an integer, for every non-negative integer n. Now the result from (a) implies that

$$\frac{q}{n+1} \le qI_n \le \frac{p}{n+1} \qquad \text{for every non-negative integer } n.$$

But when n is sufficiently large, we have $\frac{q}{n+1}, \frac{p}{n+1} \in (0,1)$, so in particular $qI_n \in (0,1)$ also. This is a contradiction because there is no integer in the open interval $(0,1)$. ■

11. (a) For every integers $m \ge 0$ and $n \ge 1$, integration by parts gives

$$\begin{aligned} B(m,n) &= \int_0^1 x^m(1-x)^n dx = \frac{1}{m+1}\int_0^1 (1-x)^n dx^{m+1} \\ &= \frac{1}{m+1}\underbrace{[x^{m+1}(1-x)^n]_0^1}_{=0} - \frac{1}{m+1}\int_0^1 x^{m+1}(-n(1-x)^{n-1})dx \\ &= 0 + \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}dx = \frac{n}{m+1}B(m+1,n-1). \end{aligned}$$

Therefore

$$\begin{aligned} B(m,n) &= \frac{n}{m+1}B(m+1,n-1) = \frac{n}{m+1}\frac{n-1}{m+2}B(m+2,n-2) = \frac{n}{m+1}\frac{n-1}{m+2}\frac{n-2}{m+3}B(m+3,n-3) \\ &= \cdots = \frac{n}{m+1}\frac{n-1}{m+2}\frac{n-2}{m+3}\cdots\frac{1}{m+n}B(m+n,0) = \frac{m!\,n!}{(m+n)!}\underbrace{\int_0^1 x^{m+n}dx}_{=\frac{1}{m+n+1}} = \frac{m!\,n!}{(m+n+1)!}. \end{aligned}$$

(b) First, we have

$$\begin{aligned} \int_0^1 \frac{x^4(x-1)^4}{x^2+1}dx &= \int_0^1 \frac{x^8-4x^7+6x^6-4x^5+x^4}{x^2+1}dx = \int_0^1 \left(x^6-4x^5+5x^4-4x^2+4-\frac{4}{x^2+1}\right)dx \\ &= \left[\frac{1}{7}x^7 - \frac{2}{3}x^6 + x^5 - \frac{4}{3}x^3 + 4x - 4\arctan x\right]_0^1 = \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\cdot\frac{\pi}{4} = \frac{22}{7} - \pi. \end{aligned}$$

On the other hand, we also have $\frac{1}{2}x^4(x-1)^4 < \frac{x^4(x-1)^4}{x^2+1} < x^4(x-1)^4$ for every $x \in (0,1)$. So

$$\frac{1}{2}\int_0^1 x^4(x-1)^4dx < \int_0^1 \frac{x^4(x-1)^4}{x^2+1}dx < \int_0^1 x^4(x-1)^4dx.$$

According to the result from (a), we have $\int_0^1 x^4(x-1)^4dx = \frac{4!4!}{9!} = \frac{1}{630}$. So the above inequality implies that

$$\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630},$$

i.e. $\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}$. ■

12. (a) Let $x = a\tan t$ where $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $\sec t > 0$, so

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \int \frac{1}{a\sec t} a\sec^2 t\,dt = \int \sec t\,dt = \ln|\sec t + \tan t| + C_0$$

$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_0 = \ln\left(\sqrt{x^2+a^2} + x\right) + C,$$

where $C = C_0 - \ln a$ is an arbitrary constant.

(b) Let $x = a\sec t$ where $t \in \left(0, \frac{\pi}{2}\right)$. Then $\tan t > 0$, so

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \int \frac{1}{a\tan t} a\sec t\tan t\,dt = \int \sec t\,dt = \ln|\sec t + \tan t| + C_0$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C_0 = \ln\left|x + \sqrt{x^2-a^2}\right| + C,$$

where $C = C_0 - \ln a$ is an arbitrary constant.

(c) Let $x = a\sin t$ where $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $\cos t > 0$, so

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{a\cos t} a\cos t\,dt = \int 1dt = t + C = \arcsin\frac{x}{a} + C,$$

where C is an arbitrary constant.

(d) Let $x = a\tan t$ where $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $\sec t > 0$, so

$$\int \sqrt{x^2+a^2}\,dx = \int (a\sec t)a\sec^2 t\,dt = a^2\int \sec^3 t\,dt.$$

Now since

$$\int \sec^3 t\,dt = \int \sec t\,d\tan t = \sec t\tan t - \int \tan t\,(\sec t\tan t)dt = \sec t\tan t - \int \sec^3 t\,dt + \int \sec t\,dt,$$

we have

$$\int \sec^3 t\,dt = \frac{1}{2}\sec t\tan t + \frac{1}{2}\int \sec t\,dt = \frac{1}{2}\sec t\tan t + \frac{1}{2}\ln|\sec t + \tan t| + C_0.$$

Therefore

$$\int \sqrt{x^2+a^2}\,dx = \frac{a^2}{2}\sec t\tan t + \frac{a^2}{2}\ln|\sec t + \tan t| + C_0$$

$$= \frac{a^2}{2}\frac{\sqrt{x^2+a^2}}{a}\frac{x}{a} + \frac{a^2}{2}\ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_0$$

$$= \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(\sqrt{x^2+a^2} + x\right) + C,$$

where $C = C_0 - \frac{a^2}{2}\ln a$ is an arbitrary constant.

(e) Let $x = a\sec t$ where $t \in \left(0, \frac{\pi}{2}\right)$. Then $\tan t > 0$, so

$$\int \sqrt{x^2 - a^2}\,dx = \int (a\tan t)a\sec t\tan t\,dt = a^2\int \sec t\tan^2 t\,dt\,.$$

Now from the steps in (d) we have

$$\int \sec t\tan^2 t\,dt = \int \sec^3 t\,dt - \int \sec t\,dt = \frac{1}{2}\sec t\tan t - \frac{1}{2}\ln|\sec t + \tan t| + C_0.$$

Therefore

$$\int \sqrt{x^2 - a^2}\,dx = \frac{a^2}{2}\sec t\tan t - \frac{a^2}{2}\ln|\sec t + \tan t| + C_0 = \frac{a^2}{2}\frac{x}{a}\frac{\sqrt{x^2 - a^2}}{a} - \frac{a^2}{2}\ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C_0$$

$$= \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left(x + \sqrt{x^2 - a^2}\right) + C,$$

where $C = C_0 + \frac{a^2}{2}\ln a$ is an arbitrary constant.

(f) Let $x = a\sin t$ where $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $\cos t > 0$, so

$$\int \sqrt{a^2 - x^2}\,dx = \int (a\cos t)a\cos t\,dt = a^2\int \cos^2 t\,dt = a^2\int \frac{1 + \cos 2t}{2}\,dt$$

$$= \frac{a^2}{2}t + \frac{a^2}{4}\sin 2t + C = \frac{a^2}{2}t + \frac{a^2}{2}\sin t\cos t + C = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{2}\frac{x}{a}\frac{\sqrt{a^2 - x^2}}{a} + C$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{1}{2}x\sqrt{a^2 - x^2} + C,$$

where C is an arbitrary constant.

Think: If a is allowed to be negative, how will these antiderivatives be affected?

13. (a) First observe that the function has domain $(-1, 1)$ and

$$\int \frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 - x^4}}\,dx = \int \frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2}\sqrt{1 - x^2}}\,dx = \int \frac{1}{\sqrt{1 - x^2}}\,dx + \int \frac{1}{\sqrt{1 + x^2}}\,dx\,.$$

Now we make the trigonometric substitution $x = \tan t$. Since $x \in (-1, 1)$, we may choose $t \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ so that $\sec t > 0$. Thus,

$$\int \frac{1}{\sqrt{1 + x^2}}\,dx = \int \frac{1}{\sqrt{1 + \tan^2 t}}\sec^2 t\,dt = \int \sec t\,dt = \ln|\sec t + \tan t| + C$$

$$= \ln\left(\sqrt{1 + x^2} + x\right) + C.$$

Therefore

$$\int \frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 - x^4}}\,dx = \int \frac{1}{\sqrt{1 - x^2}}\,dx + \int \frac{1}{\sqrt{1 + x^2}}\,dx = \arcsin x + \ln\left(\sqrt{1 + x^2} + x\right) + C,$$

where C is an arbitrary constant.

(b) Completing squares, we have $x^2+x+1=\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$, so we let $x=-\frac{1}{2}+\frac{\sqrt{3}}{2}\tan t$. Let's choose $t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ so that $\sec t>0$. Then $x^2+x+1=\frac{3}{4}\sec^2 t$ and $\sqrt{x^2+x+1}=\frac{\sqrt{3}}{2}\sec t$, and also

$$\sin t=\frac{x+\frac{1}{2}}{\sqrt{x^2+x+1}} \qquad \text{and} \qquad \cos t=\frac{\sqrt{3}}{2\sqrt{x^2+x+1}}.$$

Thus,

$$\begin{aligned}\int\frac{x+1}{(x^2+x+1)\sqrt{x^2+x+1}}dx &= \int\frac{\frac{1}{2}+\frac{\sqrt{3}}{2}\tan t}{\left(\frac{3}{4}\sec^2 t\right)\left(\frac{\sqrt{3}}{2}\sec t\right)}\left(\frac{\sqrt{3}}{2}\sec^2 t\right)dt\\
&=\int\left(\frac{2}{3}\cos t+\frac{2}{\sqrt{3}}\sin t\right)dt=\frac{2}{3}\sin t-\frac{2}{\sqrt{3}}\cos t+C\\
&=\frac{2x+1}{3\sqrt{x^2+x+1}}-\frac{1}{\sqrt{x^2+x+1}}+C,\end{aligned}$$

where C is an arbitrary constant.

(c) First let $x=\tan t$. Then $\sin t=\frac{x}{\sqrt{1+x^2}}$ and $\cos t=\frac{1}{\sqrt{1+x^2}}$

$$\begin{aligned}\int\frac{x^2-1}{(x^2+1)\sqrt{1+x^4}}dx &= \int\frac{\tan^2 t-1}{(\tan^2 t+1)\sqrt{1+\tan^4 t}}(\sec^2 t)dt=\int\frac{\tan^2 t-1}{\sqrt{1+\tan^4 t}}dt\\
&=\int\frac{\sin^2 t-\cos^2 t}{\sqrt{\cos^4 t+\sin^4 t}}dt=\int\frac{\sin^2 t-\cos^2 t}{\sqrt{\cos^4 t+\sin^4 t+2\sin^2 t\cos^2 t-2\sin^2 t\cos^2 t}}dt\\
&=\int\frac{\sin^2 t-\cos^2 t}{\sqrt{(\cos^2 t+\sin^2 t)^2-2\sin^2 t\cos^2 t}}dt=\int\frac{\sin^2 t-\cos^2 t}{\sqrt{1-2\sin^2 t\cos^2 t}}dt\\
&=\int\frac{-\cos 2t}{\sqrt{1-\frac{1}{2}\sin^2 2t}}dt.\end{aligned}$$

Next let $u=\sin 2t$. Then $du=\cos 2t\,dt$, so

$$\int\frac{-\cos 2t}{\sqrt{1-\frac{1}{2}\sin^2 2t}}dt=\int\frac{-1}{\sqrt{1-\frac{1}{2}u^2}}du=-\sqrt{2}\int\frac{1}{\sqrt{2-u^2}}du=-\sqrt{2}\arcsin\frac{u}{\sqrt{2}}+C.$$

Therefore

$$\begin{aligned}\int\frac{x^2-1}{(x^2+1)\sqrt{1+x^4}}dx &= -\sqrt{2}\arcsin\frac{u}{\sqrt{2}}+C=-\sqrt{2}\arcsin\left(\sqrt{2}\sin t\cos t\right)+C\\
&=-\sqrt{2}\arcsin\left(\sqrt{2}\cdot\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}}\right)+C=-\sqrt{2}\arcsin\frac{\sqrt{2}x}{1+x^2}+C,\end{aligned}$$

where C is an arbitrary constant.

14. (a) For each $n \geq 2$, taking antiderivatives by parts, we have

$$\begin{aligned}
I_n(x) &= \int \frac{x^n}{\sqrt{x^2+1}}\,dx = \int x^{n-1}\left(\frac{x}{\sqrt{x^2+1}}\,dx\right) = \int x^{n-1}\,d\sqrt{x^2+1} \\
&= x^{n-1}\sqrt{x^2+1} - \int \sqrt{x^2+1}(n-1)x^{n-2}\,dx \\
&= x^{n-1}\sqrt{x^2+1} - (n-1)\int \frac{x^{n-2}(x^2+1)}{\sqrt{x^2+1}}\,dx \\
&= x^{n-1}\sqrt{x^2+1} - (n-1)\int \frac{x^n}{\sqrt{x^2+1}}\,dx - (n-1)\int \frac{x^{n-2}}{\sqrt{x^2+1}}\,dx \\
&= x^{n-1}\sqrt{x^2+1} - (n-1)I_n(x) - (n-1)I_{n-2}(x).
\end{aligned}$$

Rearranging, we obtain

$$I_n(x) = \frac{1}{n}x^{n-1}\sqrt{x^2+1} - \frac{n-1}{n}I_{n-2}(x).$$

(b) Using the result from (a), we obtain

$$\begin{aligned}
\int_0^1 \frac{x^5}{\sqrt{x^2+1}}\,dx &= \frac{1}{5}\left[x^4\sqrt{x^2+1}\right]_0^1 - \frac{4}{5}\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx \\
&= \frac{1}{5}\left[x^4\sqrt{x^2+1}\right]_0^1 - \frac{4}{5}\left(\frac{1}{3}\left[x^2\sqrt{x^2+1}\right]_0^1 - \frac{2}{3}\int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx\right) \\
&= \frac{1}{5}\left[x^4\sqrt{x^2+1}\right]_0^1 - \frac{4}{5}\cdot\frac{1}{3}\left[x^2\sqrt{x^2+1}\right]_0^1 + \frac{4}{5}\cdot\frac{2}{3}\left[\sqrt{x^2+1}\right]_0^1 \\
&= \frac{7\sqrt{2}-8}{15}.
\end{aligned}$$

15. (a) Let $f\colon\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \to \mathbb{R}$ be the function $f(x) = \frac{4x}{\pi} - \tan x$. Then

$$f'(x) = \frac{4}{\pi} - \sec^2 x.$$

- Since f' is continuous on $\left[0,\frac{\pi}{4}\right]$, $f'(0) = \frac{4}{\pi} - 1 > 0$ and $f'\left(\frac{\pi}{4}\right) = \frac{4}{\pi} - \sqrt{2} < 0$, by Intermediate Value Theorem f' has at least one root in $\left(0,\frac{\pi}{4}\right)$.
- Since $\sec^2 x$ is strictly increasing on $\left[0,\frac{\pi}{4}\right]$, it follows that f' is strictly decreasing on $\left[0,\frac{\pi}{4}\right]$. So f' has at most one root in $\left(0,\frac{\pi}{4}\right)$.

Therefore f' has exactly one root in $\left(0,\frac{\pi}{4}\right)$, which means that f has exactly one critical number $c \in \left(0,\frac{\pi}{4}\right)$. Now we have $f'(x)\begin{cases} > 0 & \text{if } x \in [0,c) \\ < 0 & \text{if } x \in \left(c,\frac{\pi}{4}\right] \end{cases}$. Since $f(0) = f\left(\frac{\pi}{4}\right) = 0$, it follows that the global minimum value of f is 0. Thus $f(x) \geq f(0) = 0$ for every $x \in \left[0,\frac{\pi}{4}\right]$, i.e. $\tan x \leq \frac{4x}{\pi}$ for every $x \in \left[0,\frac{\pi}{4}\right]$.

(b) (i) For each non-negative integer n, since $\tan x \le \frac{4x}{\pi}$ for every $x \in \left[0, \frac{\pi}{4}\right]$ according to (a), we have

$$\tan^n x \le \left(\frac{4x}{\pi}\right)^n \qquad \text{for every } x \in \left[0, \frac{\pi}{4}\right],$$

so

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx \le \int_0^{\frac{\pi}{4}} \left(\frac{4x}{\pi}\right)^n dx = \left(\frac{4}{\pi}\right)^n \left[\frac{1}{n+1} x^{n+1}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4(n+1)}.$$

(ii) For each $n \ge 2$, we have

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{4}} \tan^n x\, dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \tan^2 x\, dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,(\sec^2 x - 1) dx \\ &= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, d\tan x - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, dx = \left[\frac{1}{n-1} \tan^{n-1} x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, dx \\ &= \frac{1}{n-1} - I_{n-2}, \end{aligned}$$

or equivalently $I_{n-2} = \frac{1}{n-1} - I_n$. Hence for each $k \in \mathbb{N}$, we have

$$\begin{aligned} I_0 &= 1 - I_2 = 1 - \frac{1}{3} + I_4 = 1 - \frac{1}{3} + \frac{1}{5} - I_6 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + I_8 = \cdots \\ &= 1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{k-1} \frac{1}{2k-1} + (-1)^k I_{2k}. \end{aligned}$$

(c) Using the result from (b)(ii), we have

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{k-1} \frac{1}{2k-1} = I_0 - (-1)^k I_{2k} \qquad \text{for every } k \in \mathbb{N}.$$

Now we have

- $I_0 = \int_0^{\pi/4} 1 dx = \frac{\pi}{4}$; and
- according to (b)(i) we have

$$-\frac{\pi}{4(2k+1)} \le (-1)^k I_{2k} \le \frac{\pi}{4(2k+1)} \qquad \text{for every } k \in \mathbb{N},$$

from which we obtain $\lim\limits_{k \to +\infty} (-1)^k I_{2k} = 0$ by Squeeze Theorem, as $\lim\limits_{k \to +\infty} -\frac{\pi}{4(2k+1)} = \lim\limits_{k \to +\infty} \frac{\pi}{4(2k+1)} = 0$.

Therefore

$$\begin{aligned} \lim_{k \to +\infty} \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{k-1} \frac{1}{2k-1}\right) &= \lim_{k \to +\infty} (I_0 - (-1)^k I_{2k}) \\ &= I_0 - \lim_{k \to +\infty} (-1)^k I_{2k} \\ &= \frac{\pi}{4} - 0 = \frac{\pi}{4}. \end{aligned}$$

16. (a) For every $m \in \mathbb{R} \setminus \{-1\}$ and every $n \in \mathbb{N}$, taking antiderivative by parts we have

$$I_{m,n}(x) = \int x^m (\ln x)^n dx = \frac{1}{m+1}\int (\ln x)^n dx^{m+1}$$

$$= \frac{1}{m+1}x^{m+1}(\ln x)^n - \frac{1}{m+1}\int x^{m+1}\left(n(\ln x)^{n-1}\frac{1}{x}\right)dx$$

$$= \frac{1}{m+1}x^{m+1}(\ln x)^n - \frac{n}{m+1}\int x^m(\ln x)^{n-1}dx$$

$$= \frac{1}{m+1}x^{m+1}(\ln x)^n - \frac{n}{m+1}I_{m,n-1}(x).$$

<u>*Alternative solution*</u>: For every $m \in \mathbb{R} \setminus \{-1\}$ and every $n \geq 0$, taking antiderivative by parts we have

$$I_{m,n}(x) = \int x^m(\ln x)^n dx = \int x^{m+1}(\ln x)^n \frac{1}{x}dx = \frac{1}{n+1}\int x^{m+1} d(\ln x)^{n+1}$$

$$= \frac{1}{n+1}x^{m+1}(\ln x)^{n+1} - \frac{1}{n+1}\int (\ln x)^{n+1}(m+1)x^m dx$$

$$= \frac{1}{n+1}x^{m+1}(\ln x)^{n+1} - \frac{m+1}{n+1}\int x^m(\ln x)^{n+1}dx$$

$$= \frac{1}{n+1}x^{m+1}(\ln x)^{n+1} - \frac{m+1}{n+1}I_{m,n+1}(x).$$

Rearranging the above equation we have $(n+1)I_{m,n}(x) = x^{m+1}(\ln x)^{n+1} - (m+1)I_{m,n+1}(x)$, and so

$$I_{m,n+1}(x) = \frac{1}{m+1}x^{m+1}(\ln x)^{n+1} - \frac{n+1}{m+1}I_{m,n}(x).$$

Renaming the positive integer $n+1$ as n, we have

$$I_{m,n}(x) = \frac{1}{m+1}x^{m+1}(\ln x)^n - \frac{n}{m+1}I_{m,n-1}(x)$$

for every $m \in \mathbb{R} \setminus \{-1\}$ and every positive integer n.

(b) Using the reduction formula obtained in (a), we have

$$\int \frac{(\ln x)^3}{x^4}dx = I_{-4,3}(x) = -\frac{(\ln x)^3}{3x^3} - \frac{3}{-3}I_{-4,2}(x) = -\frac{(\ln x)^3}{3x^3} + I_{-4,2}(x)$$

$$= -\frac{(\ln x)^3}{3x^3} + \left(-\frac{(\ln x)^2}{3x^3} - \frac{2}{-3}I_{-4,1}(x)\right) = -\frac{(\ln x)^3}{3x^3} - \frac{(\ln x)^2}{3x^3} + \frac{2}{3}I_{-4,1}(x)$$

$$= -\frac{(\ln x)^3}{3x^3} - \frac{(\ln x)^2}{3x^3} + \frac{2}{3}\left(-\frac{\ln x}{3x^3} - \frac{1}{-3}I_{-4,0}(x)\right) = -\frac{(\ln x)^3}{3x^3} - \frac{(\ln x)^2}{3x^3} - \frac{2\ln x}{9x^3} + \frac{2}{9}\int \frac{1}{x^4}dx$$

$$= -\frac{(\ln x)^3}{3x^3} - \frac{(\ln x)^2}{3x^3} - \frac{2\ln x}{9x^3} - \frac{2}{27x^3} + C,$$

where C is an arbitrary constant.