Solution to Problem Set 2

1. (a) Note that the function in the numerator has derivative

$$\frac{d}{dx}\int_0^{x^n}\cos(t^2)\,dt=\left[\frac{d}{dx^n}\int_0^{x^n}\cos(t^2)\,dt\right]\left(\frac{d}{dx}x^n\right)=[\cos(x^n)^2](nx^{n-1})$$

by the first version of the Fundamental Theorem of Calculus. Now in the given limit, the numerator and the denominator both tend to 0 as $x\to 0$. Applying l'Hôpital's rule, we get

$$\lim_{x\to 0}\frac{1}{x^n}\int_0^{x^n}\cos(t^2)\,dt=\lim_{x\to 0}\frac{[\cos(x^n)^2](nx^{n-1})}{nx^{n-1}}=\lim_{x\to 0}\cos(x^{2n})=\cos 0=1.$$

Alternative solution: Let $F:\mathbb{R}\to\mathbb{R}$ be the function $F(u)=\int_0^u\cos(t^2)\,dt$. Then F is differentiable by the first version of the Fundamental Theorem of Calculus. The given limit is in fact

$$\lim_{x\to 0}\frac{1}{x^n}\int_0^{x^n}\cos(t^2)\,dt=\lim_{x\to 0}\frac{F(x^n)-F(0)}{x^n-0}=\lim_{u\to 0}\frac{F(u)-F(0)}{u-0}=F'(0)=\cos(0^2)=1,$$

according to the definition of derivative.

(b) Using a substitution $u=x^2t$, the function in the numerator can be rewritten as

$$\int_0^{x^n}\cos(x^2t)\,dt=\frac{1}{x^2}\int_0^{x^{n+2}}\cos u\,du=\frac{1}{x^2}[\sin u]_0^{x^{n+2}}=\frac{\sin(x^{n+2})}{x^2}.$$

Thus

$$\lim_{x\to 0}\frac{1}{x^n}\int_0^{x^n}\cos(x^2t)\,dt=\lim_{x\to 0}\frac{\sin(x^{n+2})}{x^{n+2}}=\lim_{y\to 0}\frac{\sin y}{y}=1.$$

2. (a) Since f and g (and hence their product fg) are all continuous on $\mathbb{R}$, their area functions $\int_0^x f(t)dt$, $\int_0^x g(t)dt$ and $\int_0^x f(t)g(t)dt$ are all differentiable on $\mathbb{R}$ by the first version of the Fundamental Theorem of Calculus. Now F is a difference of products of differentiable functions, so it is differentiable on $\mathbb{R}$ too. By product rule, the derivative of F is given by

$$\begin{aligned}F'(x)&=\left(\frac{d}{dx}x\right)\left(\int_0^x f(t)g(t)dt\right)+\underbrace{(x)}_{=\int_0^x dt}\left(\frac{d}{dx}\int_0^x f(t)g(t)dt\right)-\left(\frac{d}{dx}\int_0^x f(t)dt\right)\left(\int_0^x g(t)dt\right)\\&\qquad-\left(\int_0^x f(t)dt\right)\left(\frac{d}{dx}\int_0^x g(t)dt\right)\\&=(1)\left(\int_0^x f(t)g(t)dt\right)+\left(\int_0^x dt\right)\big(f(x)g(x)\big)-f(x)\left(\int_0^x g(t)dt\right)-\left(\int_0^x f(t)dt\right)g(x)\\&=\int_0^x f(t)g(t)dt+\int_0^x f(x)g(x)dt-\int_0^x f(x)g(t)dt-\int_0^x f(t)g(x)dt\\&=\int_0^x\big(f(x)-f(t)\big)\big(g(x)-g(t)\big)dt\end{aligned}$$

for every $x\in\mathbb{R}$. ■

(b) According to the result from (a), we observe right away that $F'(0) = 0$. Also,

- If $x > 0$, then we have $f(x) \geq f(t)$ and $g(x) \geq g(t)$ for every $t \in [0, x]$, since f and g are increasing; thus $\big(f(x) - f(t)\big)\big(g(x) - g(t)\big) \geq 0$ for every $t \in [0, x]$, and so

$$F'(x) = \int_0^x \big(f(x) - f(t)\big)\big(g(x) - g(t)\big)dt \geq 0.$$

- If $x < 0$, then we have $f(x) \leq f(t)$ and $g(x) \leq g(t)$ for every $t \in [x, 0]$, since f and g are increasing; thus we still have $\big(f(x) - f(t)\big)\big(g(x) - g(t)\big) \geq 0$ for every $t \in [x, 0]$, and so

$$F'(x) = \int_0^x \big(f(x) - f(t)\big)\big(g(x) - g(t)\big)dt = -\int_x^0 \big(f(x) - f(t)\big)\big(g(x) - g(t)\big)dt \leq 0.$$

Therefore by the first derivative test, F is decreasing on $(-\infty, 0]$ and increasing on $[0, +\infty)$, and F attains its global minimum on $\mathbb{R}$ at 0; the global minimum value is $F(0) = 0$.

3. Since f is a polynomial, it is continuous on $[0, 1]$. Applying Mean Value Theorem for integrals to f, there exists $c \in (0, 1)$ such that

$$\begin{aligned} f(c) &= \frac{1}{1-0}\int_0^1 f(x)dx = \int_0^1 (a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n)dx \\ &= \left[a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_{n-1}}{n}x^n + \frac{a_n}{n+1}x^{n+1}\right]_0^1 \\ &= \left(a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_{n-1}}{n} + \frac{a_n}{n+1}\right) - 0 = 0. \end{aligned}$$

This number $c \in (0, 1)$ is therefore a root of f. ■

4. (a) The area of the shaded region is given by

$$\int_0^\pi \sin x \, dx = [-\cos x]_0^\pi = (-\cos \pi) - (-\cos 0) = 2.$$

(b) Since the area under the graph f between $x = a$ and $x = \pi - a$ is half of the whole shaded area, this part of the region has an area of 1 squared unit. Thus,

$$1 = \int_a^{\pi - a} \sin x \, dx = [-\cos x]_a^{\pi - a} = [-\cos(\pi - a)] - (-\cos a) = 2\cos a.$$

Since $a \in \left(0, \frac{\pi}{2}\right)$, this implies that $a = \arccos\frac{1}{2} = \frac{\pi}{3}$, and so

$$b = \sin a = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

5. (a) Since $t^3 - 3t^2 + 2t = t(t-1)(t-2)\begin{cases} \geq 0 & \text{if } t \in [0,1] \cup [2,+\infty) \\ < 0 & \text{if } t \in (-\infty, 0) \cup (1,2) \end{cases}$, we have

$$\begin{aligned}
&\int_{-1}^{3} |t^3 - 3t^2 + 2t| dt \\
&= -\int_{-1}^{0} (t^3 - 3t^2 + 2t)dt + \int_{0}^{1} (t^3 - 3t^2 + 2t)dt - \int_{1}^{2} (t^3 - 3t^2 + 2t)dt + \int_{2}^{3} (t^3 - 3t^2 + 2t)dt \\
&= -\left[\frac{1}{4}t^4 - t^3 + t^2\right]_{-1}^{0} + \left[\frac{1}{4}t^4 - t^3 + t^2\right]_{0}^{1} - \left[\frac{1}{4}t^4 - t^3 + t^2\right]_{1}^{2} + \left[\frac{1}{4}t^4 - t^3 + t^2\right]_{2}^{3} \\
&= -\left(0 - \frac{9}{4}\right) + \left(\frac{1}{4} - 0\right) - \left(0 - \frac{1}{4}\right) + \left(\frac{9}{4} - 0\right) = 5
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{1 - \sin^3\theta}{\cos^2\theta} d\theta &= \int_0^{\frac{\pi}{4}} \frac{1 - \sin\theta\,(1 - \cos^2\theta)}{\cos^2\theta} d\theta = \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2\theta} - \frac{\sin\theta}{\cos^2\theta} + \sin\theta\right) d\theta \\
&= \int_0^{\frac{\pi}{4}} (\sec^2\theta - \sec\theta\tan\theta + \sin\theta) d\theta = [\tan\theta - \sec\theta - \cos\theta]_0^{\frac{\pi}{4}} \\
&= \left(1 - \sqrt{2} - \frac{\sqrt{2}}{2}\right) - (0 - 1 - 1) = 3 - \frac{3\sqrt{2}}{2}
\end{aligned}$$

6. (a) For every $t > 0$, we have

$$f'(t) = 1 - \cos t \geq 0;$$

so f is increasing on $[0, +\infty)$. Thus for every $x \geq 0$, we have $f(x) \geq f(0)$, i.e.

$$x - \sin x \geq 0 - \sin 0\,,$$

which implies that $\sin x \leq x$. ■

(b) (i) Given any $x \geq 0$, from (a) we have $\sin t \leq t$ for every $t \in [0, x]$. Therefore, we have

$$\int_0^x \sin t\, dt \leq \int_0^x t dt$$

which is the same as

$$1 - \cos x \leq \frac{1}{2}x^2,$$

i.e. $\cos x \geq 1 - \frac{1}{2}x^2$. ■

(ii) Given any $x \geq 0$, from (b) (i), we have $\cos t \geq 1 - \frac{1}{2}t^2$ for every $t \in [0, x]$. Therefore, we have

$$\int_0^x \cos t\, dt \geq \int_0^x \left(1 - \frac{1}{2}t^2\right) dt$$

which is the same as $\sin x \geq x - \frac{x^3}{6}$. ■

(iii) Given any $x \geq 0$, from (b) (ii), we have $\sin t \geq t - \frac{t^3}{6}$ for every $t \in [0, x]$. Therefore, we have

$$\int_0^x \sin t \, dt \geq \int_0^x \left(t - \frac{t^3}{6}\right) dt$$

which is the same as

$$1 - \cos x \geq \frac{1}{2}x^2 - \frac{1}{24}x^4,$$

i.e. $\cos x \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$. ■

7. (a) Since f is a positive continuous function on $[a, b]$, both $\sqrt{f}$ and $1/\sqrt{f}$ are also continuous on $[a, b]$. Applying the Cauchy-Schwarz inequality to these two functions, we have

$$\left[\int_a^b \left(\sqrt{f(x)}\right)^2 dx\right]\left[\int_a^b \left(\frac{1}{\sqrt{f(x)}}\right)^2 dx\right] \geq \left(\int_a^b \sqrt{f(x)}\frac{1}{\sqrt{f(x)}} dx\right)^2,$$

i.e.

$$\left(\int_a^b f(x) dx\right)\left(\int_a^b \frac{1}{f(x)} dx\right) \geq \left(\int_a^b 1 \, dx\right)^2 = (b - a)^2.$$

■

(b) Let $f: [0, 2\pi] \to \mathbb{R}$ be the function

$$f(x) = \sqrt{1 - \frac{1}{2}\cos x}.$$

Then f is continuous on $[0, 2\pi]$ and $f(x) \geq \frac{1}{\sqrt{2}} > 0$ for every $x \in [0, 2\pi]$. Applying the result from (a) to this function f, we obtain

$$\left(\int_0^{2\pi} \sqrt{1 - \frac{1}{2}\cos x} \, dx\right)\left(\int_0^{2\pi} \frac{1}{\sqrt{1 - \frac{1}{2}\cos x}} dx\right) \geq (2\pi - 0)^2.$$

On the other hand, by Cauchy-Schwarz inequality we have

$$\left(\int_0^{2\pi} \underbrace{\sqrt{1 - \frac{1}{2}\cos x} \, dx}_{>0}\right)^2 \leq \underbrace{\left[\int_0^{2\pi} \left(1 - \frac{1}{2}\cos x\right) dx\right]}_{=2\pi}\underbrace{\left(\int_0^{2\pi} 1 \, dx\right)}_{2\pi} = (2\pi)^2,$$

so $0 < \int_0^{2\pi} \sqrt{1 - \frac{1}{2}\cos x} \, dx \leq 2\pi$. Therefore

$$\int_0^{2\pi} \frac{1}{\sqrt{1 - \frac{1}{2}\cos x}} dx \geq \frac{(2\pi - 0)^2}{2\pi} = 2\pi.$$

■

8. (a) By the Fundamental Theorem of Calculus, we have

$$f'(x) = \frac{1}{\sqrt{1 - m\sin^2 x}} \qquad \text{for every } x \in \mathbb{R}.$$

Since $m < 1$, this implies that $f'(x) \geq \frac{1}{\sqrt{1-m}} > 0$ for every $x \in \mathbb{R}$, so f is strictly increasing on $\mathbb{R}$. ■

(b) Since $0 < m < 1$ and $0 \leq \sin^2 t \leq 1$, it follows that $\frac{1}{\sqrt{1-m\sin^2 t}} \geq 1$ for every $t \in \mathbb{R}$. In particular, we have

$$f(x) = \int_0^x \frac{1}{\sqrt{1 - m\sin^2 t}}\,dt \geq \int_0^x 1\,dt = x \qquad \text{for every } x > 0.$$

Together with $\lim_{x\to+\infty} x = +\infty$, we have $\lim_{x\to+\infty} f(x) = +\infty$ by Squeeze Theorem. Similarly, we also have

$$f(x) = \int_0^x \frac{1}{\sqrt{1 - m\sin^2 t}}\,dt = -\int_x^0 \frac{1}{\sqrt{1 - m\sin^2 t}}\,dt \leq -\int_x^0 1\,dt = x \qquad \text{for every } x < 0.$$

Together with $\lim_{x\to-\infty} x = -\infty$, we have $\lim_{x\to-\infty} f(x) = -\infty$ by Squeeze Theorem. ■

(c) We aim to show that (i) **f is one-to-one** and (ii) **the range of f is $\mathbb{R}$**.

- ⊙ f is one-to-one because f is strictly increasing according to (a).
- ⊙ Let $a \in \mathbb{R}$ be arbitrary. According to (b) we have $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, so (by the definition of infinite limit) there exist real numbers $p < q$ such that $f(p) < a$ and $f(q) > a$. Now f is continuous on $\mathbb{R}$, so by **Intermediate Value Theorem** there exists $c \in (p, q)$ such that $f(c) = a$. This shows that the range of f is $\mathbb{R}$.

Since f is one-to-one and the range of f is $\mathbb{R}$, we conclude that f has an inverse $f^{-1}: \mathbb{R} \to \mathbb{R}$. ■

(d) For every $y \in \mathbb{R}$, we have

$$\begin{aligned}
p'(y) &= (\cos x)\frac{dx}{dy} = \frac{\cos x}{\frac{dy}{dx}} = \frac{\cos x}{f'(x)} = \frac{\cos x}{\frac{1}{\sqrt{1 - m\sin^2 x}}} \\
&= (\cos x)\sqrt{1 - m\sin^2 x} = q(y)r(y), \\
q'(y) &= (-\sin x)\frac{dx}{dy} = -\frac{\sin x}{\frac{dy}{dx}} = -\frac{\sin x}{f'(x)} = -\frac{\sin x}{\frac{1}{\sqrt{1 - m\sin^2 x}}} \\
&= -(\sin x)\sqrt{1 - m\sin^2 x} = -p(y)r(y), \\
r'(y) &= \frac{1}{2\sqrt{1 - m\sin^2 x}}(-2m\sin x\cos x)\frac{dx}{dy} = \frac{-m\sin x\cos x}{\sqrt{1 - m\sin^2 x}\left(\frac{dy}{dx}\right)} \\
&= \frac{-m\sin x\cos x}{\sqrt{1 - m\sin^2 x}\, f'(x)} = \frac{-m\sin x\cos x}{\sqrt{1 - m\sin^2 x}\cdot\frac{1}{\sqrt{1 - m\sin^2 x}}} \\
&= -m\sin x\cos x = -mp(y)q(y).
\end{aligned}$$

Remark: The functions p, q and r in part (d) are called ***Jacobi elliptic functions***. They are denoted by the symbols sn, cn and dn respectively.

9. (a) For every $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have

$$f'(x) = \frac{d}{dx}\int_1^x \sin(\cos t)\,dt = \sin(\cos x) > 0,$$

So f is strictly increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore f is one-to-one. ∎

(b) Since $f(1) = \int_1^1 \sin(\cos t)\,dt = 0$ and g is the inverse of f, we have $g(0) = 1$. Therefore

$$g'(0) = \frac{1}{f'\big(g(0)\big)} = \frac{1}{f'(1)} = \frac{1}{\sin(\cos 1)}.$$

10. (a) For every $x \neq 0$, we let $u = xt$. Then $du = xdt$, $u = 0$ when $t = 0$, and $u = x$ when $t = 1$. So,

$$F'(x) = \frac{d}{dx}\int_0^1 \cos xt\,dt = \frac{d}{dx}\left(\frac{1}{x}\int_0^x \cos u\,du\right) = \left(\frac{d}{dx}\frac{1}{x}\right)\left(\int_0^x \cos u\,du\right) + \left(\frac{1}{x}\right)\left(\frac{d}{dx}\int_0^x \cos u\,du\right)$$

$$= \left(-\frac{1}{x^2}\right)(\sin x - \sin 0) + \left(\frac{1}{x}\right)(\cos x) = \frac{x\cos x - \sin x}{x^2}.$$

For $x = 0$, we compute the derivative at 0 from definition. We have

$$F'(0) = \lim_{h\to 0}\frac{F(h) - F(0)}{h} = \lim_{h\to 0}\frac{\int_0^1 \cos ht\,dt - \int_0^1 dt}{h}$$

$$= \lim_{h\to 0}\frac{\int_0^1 \cos ht\,dt - 1}{h} = \lim_{h\to 0}\frac{\frac{d}{dh}\left(\int_0^1 \cos ht\,dt - 1\right)}{\frac{d}{dh}h} \qquad \text{(l'Hôpital's rule)}$$

$$= \lim_{h\to 0}\frac{\frac{h\cos h - \sin h}{h^2}}{1} = \lim_{h\to 0}\frac{h\cos h - \sin h}{h^2} = \lim_{h\to 0}\frac{-h\sin h}{2h} \qquad \text{(l'Hôpital's rule)}$$

$$= \lim_{h\to 0} -\frac{1}{2}\sin h = 0.$$

Alternative Solution: Since

$$F(x) = \int_0^1 \cos xt\,dt = \begin{cases}\left[\frac{1}{x}\sin xt\right]_0^1 & \text{if } x \neq 0\\ [t]_0^1 & \text{if } x = 0\end{cases} = \begin{cases}\frac{\sin x}{x} & \text{if } x \neq 0\\ 1 & \text{if } x = 0\end{cases},$$

we have

$$F'(x) = \frac{(\cos x)(x) - (\sin x)(1)}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

for every $x \neq 0$, and

$$F'(0) = \lim_{h\to 0}\frac{F(h) - F(0)}{h} = \lim_{h\to 0}\frac{\frac{\sin h}{h} - 1}{h} = \lim_{h\to 0}\frac{\sin h - h}{h^2} = \lim_{h\to 0}\frac{\cos h - 1}{2h} \qquad \text{(l'Hôpital's rule)}$$

$$= \lim_{h\to 0}\frac{-\sin h}{2} \qquad \text{(l'Hôpital's rule)}$$

$$= 0.$$

(b) For every $x > 0$, we let $u = xt$. Then $du = x\,dt$, $u = 1$ when $t = \frac{1}{x}$, and $u = x^2$ when $t = x$. So

$$\begin{aligned} G'(x) &= \frac{d}{dx}\int_{\frac{1}{x}}^{x} \cos\sqrt{xt}\,dt = \frac{d}{dx}\left(\frac{1}{x}\int_1^{x^2} \cos\sqrt{u}\,du\right) \\ &= \left(\frac{d}{dx}\frac{1}{x}\right)\left(\int_1^{x^2} \cos\sqrt{u}\,du\right) + \left(\frac{1}{x}\right)\left(\frac{d}{dx}\int_1^{x^2} \cos\sqrt{u}\,du\right) \\ &= \left(-\frac{1}{x^2}\right)\left(\int_1^{x^2} \cos\sqrt{u}\,du\right) + \left(\frac{1}{x}\right)\left(2x\cos\sqrt{x^2}\right) = -\frac{1}{x^2}\int_1^{x^2} \cos\sqrt{u}\,du + 2\cos x\,, \end{aligned}$$

and in particular

$$G'(1) = -\frac{1}{1^2}\int_1^{1^2} \cos\sqrt{u}\,du + 2\cos 1 = 2\cos 1.$$

11. (a) (i) Let $u = a - x$. Then $du = -dx$, $u = a$ when $x = 0$, and $u = 0$ when $x = a$. Thus we have

$$\int_0^a f(a-x)dx = -\int_a^0 f(u)du = \int_0^a f(u)du = \int_0^a f(x)dx\,.$$

■

Remark: The result in (a) (i) is also true for Riemann integrable functions on $[0, a]$. Under this setting, the proof will involve the Riemann sum definition of integrals instead.

(ii) Suppose that $f(x) + f(a-x) = c$ for every $x \in [0, a]$. Then with $x = \frac{a}{2}$ we have $f\left(\frac{a}{2}\right) + f\left(\frac{a}{2}\right) = c$, so $f\left(\frac{a}{2}\right) = \frac{c}{2}$. Moreover, we also have

$$\int_0^a f(x)dx + \int_0^a f(a-x)dx = \int_0^a \big(f(x) + f(a-x)\big)dx = \int_0^a c\,dx = ac.$$

But the two integrals on the left are the same according to (a), so

$$\int_0^a f(x)dx = \frac{ac}{2} = af\left(\frac{a}{2}\right).$$

(b) Let $f: [0, 2\pi] \to \mathbb{R}$ be the continuous function $f(x) = \frac{1}{e^{\sin^3 x}+1}$. Then we can verify that

$$\begin{aligned} f(x) + f(2\pi - x) &= \frac{1}{e^{\sin^3 x} + 1} + \frac{1}{e^{\sin^3(2\pi - x)} + 1} = \frac{1}{e^{\sin^3 x} + 1} + \frac{1}{e^{-\sin^3 x} + 1} \\ &= \frac{1}{e^{\sin^3 x} + 1} + \frac{e^{\sin^3 x}}{1 + e^{\sin^3 x}} = \frac{1 + e^{\sin^3 x}}{1 + e^{\sin^3 x}} = 1 \end{aligned}$$

for every $x \in [0, a]$. Thus applying the result from (a) (ii), we have

$$\int_0^{2\pi} \frac{1}{e^{\sin^3 x} + 1}dx = 2\pi \cdot \frac{1}{2} = \pi.$$

12. (a) We consider the following cases.

⦿ If $x = 0$, then

$$\int_0^{\pi} \frac{\sin t}{\sqrt{1 - 2x\cos t + x^2}}\,dt = \int_0^{\pi} \sin t\,dt = [-\cos t]_0^{\pi} = 2.$$

⦿ If $x \neq 0$, then we apply the substitution $u = 1 - 2x\cos t + x^2$. Then $du = 2x\sin t\,dt$, $u = (1-x)^2$ when $t = 0$, and $u = (1+x)^2$ when $t = \pi$. Thus

$$\begin{aligned}\int_0^{\pi} \frac{\sin t}{\sqrt{1 - 2x\cos t + x^2}}\,dt &= \frac{1}{x}\int_{(1-x)^2}^{(1+x)^2} \frac{1}{2\sqrt{u}}\,du = \frac{1}{x}\left[\sqrt{u}\right]_{(1-x)^2}^{(1+x)^2} \\ &= \frac{\sqrt{(1+x)^2} - \sqrt{(1-x)^2}}{x} = \frac{|1+x| - |1-x|}{x} \\ &= \begin{cases} 2/x & \text{if } x > 1 \\ 2 & \text{if } 0 < x < 1 \end{cases}.\end{aligned}$$

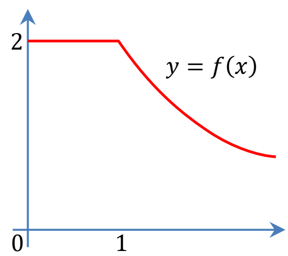
(b) According to the result from (a), we have

$$f(x) = \begin{cases} 2/x & \text{if } x > 1 \\ 2 & \text{if } 0 \leq x < 1 \end{cases}.$$

We have

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2 = 2 \qquad \text{and} \qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{2}{x} = \frac{2}{1} = 2.$$

In order that f is continuous at 1, we must require that $f(1) = 2$, and therefore $a = 2$. The following is a sketch of the graph of f:



13. (a) Let $u = \frac{1}{x}$. Then we have $x = \frac{1}{u}$, $dx = -\frac{1}{u^2}du$, $u = 2$ when $x = \frac{1}{2}$, and $u = \frac{1}{2}$ when $x = 2$. Thus,

$$\int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2}\,dx = \int_2^{\frac{1}{2}} \frac{\ln\frac{1}{u}}{1+\left(\frac{1}{u}\right)^2}\left(-\frac{1}{u^2}\right)du = -\int_2^{\frac{1}{2}} \frac{-\ln u}{u^2+1}\,du = -\int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2}\,dx.$$

Rearranging the above equation we get

$$2\int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2}\,dx = 0,$$

so $\int_{1/2}^{2} \frac{\ln x}{1+x^2}\,dx = 0$. ■

(b) Consider the regular partition P of $\left[\frac{1}{2}, 2\right]$ into $3n$ subintervals of width $\frac{1}{2n}$. Then $\|P\| = \frac{1}{2n} \to 0$ as $n \to +\infty$. Regarding the given sum as a right Riemann sum with respect to P, we have

$$\lim_{n\to+\infty} \sum_{k=1}^{3n} \frac{1}{2n} \frac{\ln\left[2\left(\frac{1}{2}+\frac{k}{2n}\right)\right]}{1+\left(\frac{1}{2}+\frac{k}{2n}\right)^2} = \int_{\frac{1}{2}}^{2} \frac{\ln 2x}{1+x^2}dx = \int_{\frac{1}{2}}^{2} \frac{\ln 2}{1+x^2}dx + \int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2}dx$$

$$= (\ln 2)[\arctan x]_{\frac{1}{2}}^{2} + 0 = (\ln 2)\left(\arctan 2 - \arctan\frac{1}{2}\right).$$

This final numerical answer can be further simplified to either $(\ln 2)\left(\arctan\frac{3}{4}\right)$ or $(\ln 2)\left(2\arctan 2 - \frac{\pi}{2}\right)$.

14. (a) Let $u = \csc x + \cot x$. Then $du = -(\csc x \cot x + \csc^2 x)dx$. Thus

$$\int \csc x\, dx = \int \frac{\csc x\,(\csc x + \cot x)}{\csc x + \cot x} dx = -\int \frac{1}{u} du$$

$$= -\ln|u| + C = -\ln|\csc x + \cot x| + C,$$

where C is an arbitrary constant.

(b)

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x}+1} = \int \frac{1}{e^{2x}+1} de^x$$

$$= \arctan e^x + C,$$

where C is an arbitrary constant.

(c)

$$\int \frac{\cos^5\theta}{\sin^7\theta} d\theta = \int \frac{\cos^5\theta}{\sin^5\theta}\csc^2\theta\, d\theta = -\int \cot^5\theta\, d\cot\theta$$

$$= -\frac{1}{6}\cot^6\theta + C,$$

where C is an arbitrary constant.

(d)

$$\int \frac{[\ln(u^2)]^2}{u} du = \int (2\ln|u|)^2 \cdot \frac{1}{u} du = 4\int (\ln|u|)^2 d\ln|u| = \frac{4}{3}(\ln|u|)^3 + C,$$

where C is an arbitrary constant.

<u>*Alternative Solution*</u>: Let $t = \ln(u^2)$. Then $dt = \frac{2}{u}du$, so

$$\int \frac{[\ln(u^2)]^2}{u} du = \frac{1}{2}\int [\ln(u^2)]^2 \cdot \frac{2}{u} du = \frac{1}{2}\int t^2 dt = \frac{1}{6}t^3 + C = \frac{1}{6}[\ln(u^2)]^3 + C,$$

where C is an arbitrary constant.

(e) Let $u = x^{\frac{3}{2}}$. Then $x^3 = u^2$ and $du = \frac{3}{2}x^{\frac{1}{2}}dx = \frac{3}{2}\sqrt{x}dx$, so

$$\int \frac{\sqrt{x}}{1+x^3}dx = \int \frac{1}{1+u^2}\cdot\frac{2}{3}du = \frac{2}{3}\int\frac{1}{1+u^2}du$$
$$= \frac{2}{3}\arctan u + C = \frac{2}{3}\arctan x^{\frac{3}{2}} + C,$$

where C is an arbitrary constant.

15. (a) Let $u = x - t$. Then $t = x - u$, $du = -dt$, $u = x$ when $t = 0$, and $u = 0$ when $t = x$. Therefore

$$\int_0^x tf(x-t)dt = \int_x^0 (x-u)f(u)(-du) = \int_0^x (x-u)f(u)du = \int_0^x (x-t)f(t)dt.$$

■

(b) Using the result from (a), the given equality can be written as $\int_0^x (x-t)f(t)dt = e^x - x - 1$, i.e.

$$x\int_0^x f(t)dt - \int_0^x tf(t)dt = e^x - x - 1$$

for every $x \in \mathbb{R}$. Differentiating both sides with respect to x, we have

$$\left[(1)\left(\int_0^x f(t)dt\right) + (x)\big(f(x)\big)\right] - xf(x) = e^x - 1,$$

so $\int_0^x f(t)dt = e^x - 1$ for every $x \in \mathbb{R}$. Differentiating both sides with respect to x again, we have

$$f(x) = e^x$$

for every $x \in \mathbb{R}$.

16. (a) Let $u = \pi - x$. Then $x = \pi - u$, $du = -dx$, $u = \pi$ when $x = 0$, and $u = 0$ when $x = \pi$. Therefore

$$\int_0^\pi f(x)\ln(1+e^{\cos x})\,dx$$
$$= \int_\pi^0 f(\pi-u)\ln\big(1+e^{\cos(\pi-u)}\big)\,(-du) = \int_0^\pi f(\pi-u)\ln\big(1+e^{\cos(\pi-u)}\big)\,du$$
$$= \int_0^\pi -f(u)\ln(1+e^{-\cos u})\,du = \int_0^\pi f(u)\left[\ln(e^{\cos u})\underbrace{-\ln(e^{\cos u}) - \ln(1+e^{-\cos u})}_{=-\ln(e^{\cos u}+1)}\right]du$$
$$= \int_0^\pi f(u)\ln(e^{\cos u})\,du - \int_0^\pi f(u)\ln(e^{\cos u}+1)\,du$$
$$= \int_0^\pi f(x)\ln(e^{\cos x})\,dx - \int_0^\pi f(x)\ln(1+e^{\cos x})\,dx.$$

This implies that

$$2\int_0^\pi f(x)\ln(1+e^{\cos x})\,dx = \int_0^\pi f(x)\ln(e^{\cos x})\,dx = \int_0^\pi f(x)\cos x\,dx,$$

and so $\int_0^\pi f(x)\ln(1+e^{\cos x})\,dx = \frac{1}{2}\int_0^\pi f(x)\cos x\,dx$. ■

(b) The derivative of g is given by

$$g'(x) = \frac{(-\sin x)(1+\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 1}{(1+\sin x)^2} = -\frac{1}{1+\sin x}$$

for every $x \in (0,\pi)$. Now we evaluate the given integral; let $f\colon [0,\pi] \to \mathbb{R}$ be the function

$$f(x) = \frac{\cos x}{(1+\sin x)^2}.$$

Then f is continuous on $[0,\pi]$, and we have

$$f(\pi - x) = \frac{\cos(\pi - x)}{(1+\sin(\pi - x))^2} = \frac{-\cos x}{(1+\sin x)^2} = -f(x)$$

for every $x \in [0,\pi]$. Thus according to the result from (a), we have

$$\int_0^\pi \frac{(\cos x)\ln(1+e^{\cos x})}{(1+\sin x)^2}\,dx = \int_0^\pi f(x)\ln(1+e^{\cos x})\,dx = \frac{1}{2}\int_0^\pi f(x)\cos x\,dx = \frac{1}{2}\int_0^\pi \frac{\cos^2 x}{(1+\sin x)^2}\,dx.$$

Now we observe that

$$\frac{\cos^2 x}{(1+\sin x)^2} = \frac{1-\sin^2 x}{(1+\sin x)^2} = \frac{(1-\sin x)(1+\sin x)}{(1+\sin x)^2} = \frac{1-\sin x}{1+\sin x} = \frac{2}{1+\sin x} - 1$$

for every $x \in [0,\pi]$, so

$$\int_0^\pi \frac{(\cos x)\ln(1+e^{\cos x})}{(1+\sin x)^2}\,dx = \frac{1}{2}\int_0^\pi \left(\frac{2}{1+\sin x} - 1\right)dx = \int_0^\pi \left(\frac{1}{1+\sin x} - \frac{1}{2}\right)dx$$

$$= \left[-g(x) - \frac{1}{2}x\right]_0^\pi = \left[-\frac{\cos x}{1+\sin x} - \frac{1}{2}x\right]_0^\pi = 2 - \frac{\pi}{2}.$$