Problem Set 1

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 1 covers materials from §5.1 – §5.3. <u>All problems in this set, including all the evaluations of integrals, are supposed to be solved **without using the Fundamental Theorem of Calculus**.</u>

1. Evaluate each of the following integrals by considering some simple geometric shape.

   (a) $\int_0^3 \lfloor x \rfloor dx$ (Recall the ***floor function*** in Definition 1.36)

   (b) $\int_0^3 (3 - |x-1| - |x-2|) dx$

   (c) $\int_0^a \sqrt{4-x^2} dx$, where $0 \le a \le 2$

   Hint: First sketch the graph of each integrand.

2. Find the pair of real numbers $a, b \in [0, 2\pi]$ with $a < b$ such that the integral
$$\int_a^b (\sin x - \cos x) dx$$
   (a) attains its maximum possible value,
   (b) attains its minimum possible value.
   Give justifications to your answers.

3. Let $f(x) = \cos(x^2)$ and $g(x) = \cos(x^3)$. For each of the following pairs of integrals, explain which one has a greater value.

   (a) $\int_0^1 f(x) dx$ and $\int_0^1 g(x) dx$

   (b) $\int_0^1 f(x) dx$ and $\int_{\cos 1}^1 f^{-1}(x) dx$

   (Note that f has an inverse defined on $[\cos 1, 1]$ because $f: [0,1] \to [\cos 1, 1]$ is strictly decreasing.)

4. Let $f: \mathbb{R} \to \mathbb{R}$ be the function
$$f(x) = x - \lfloor x \rfloor.$$
   (Recall the ***floor function*** in Definition 1.36; also see Example 1.48.)
   (a) Sketch the graph of f.
   (b) Using the graph obtained in (a), show that
$$\int_0^x f(t) dt = \frac{1}{2} \lfloor x \rfloor + \frac{1}{2} (x - \lfloor x \rfloor)^2$$
   for every $x \in \mathbb{R}$.
   Hint: Consider the case $x \in [n, n+1)$ for each integer n separately. What happens if $x < 0$?

5. (a) Let $a \geq 2$ be a real number. By considering appropriate trapeziums, show that

$$\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} \ln x\,dx \leq \ln a \qquad \text{and} \qquad \int_{a-1}^{a} \ln x\,dx \geq \frac{\ln(a-1)+\ln a}{2}.$$

(b) Using the result from (a), deduce that

$$\int_{\frac{3}{2}}^{n} \ln x\,dx \leq \ln(n!) - \frac{1}{2}\ln n \leq \int_{1}^{n} \ln x\,dx$$

for every integer $n \geq 2$.

6. Let $f: \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = e^{-x^2}.$$

(a) Let P be the regular partition of $[-1, 1]$ into 4 subintervals. Write explicitly

(i) the upper Darboux sum of f with respect to P;

(ii) the lower Darboux sum of f with respect to P;

(iii) the right Riemann sum of f with respect to P.

(b) Let S be the **mid-point** Riemann sum of f with respect to a certain partition of $[1, 3]$. Determine whether

$$S > \int_1^3 f(x)dx \qquad \text{or} \qquad S < \int_1^3 f(x)dx\,.$$

Explain your answer.

7. Express each of the following limits as an integral.

(a) $\displaystyle\lim_{n\to+\infty} \frac{e^{\frac{1}{n^2}} + e^{\frac{4}{n^2}} + e^{\frac{9}{n^2}} + \cdots + e^4}{n}$

(b) $\displaystyle\lim_{n\to+\infty} \sum_{k=n}^{2n+1014} \frac{n}{n^2+k^2}$

(c) $\displaystyle\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{(n+2k-1)^3}{n^4}$

(d) $\displaystyle\lim_{n\to+\infty} \frac{1^2+4^2+7^2+\cdots+(3n-2)^2}{n^3}$

8. Evaluate the following limits. Do not use the Fundamental Theorem of Calculus when computing any integral.

(a) $\displaystyle\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{\sqrt{n^2-k^2}}{n^2}$

(b) $\displaystyle\lim_{n\to+\infty} \frac{1}{n^2}\left(\sqrt{n-1} + \sqrt{2n-4} + \sqrt{3n-9} + \cdots + \sqrt{n^2-n^2}\right)$

9. Let $a < b$ be a pair of real numbers. Evaluate

$$\int_a^b x^3 dx$$

from the definition of a Riemann integral.

10. (a) Let t be a real number such that $\sin\frac{t}{2} \neq 0$ and let n be a positive integer. Show that

$$\sum_{k=1}^{n} \sin kt = \frac{\cos\frac{t}{2} - \cos\left(n + \frac{1}{2}\right)t}{2\sin\frac{t}{2}}.$$

(b) Hence for each $a > 0$, evaluate

$$\int_0^a \sin t\, dt$$

from the definition of a Riemann integral.

11. Let $a > 1$ be a real number. Evaluate

$$\int_1^a \ln x\, dx$$

from the definition of a Riemann integral.

Hint: Consider a partition of the interval as in Example 5.36.

12. Let $a < b$ be real numbers and let $f: [a, b] \to [0, +\infty)$ be a non-negative **continuous** function such that

$$f(c) > 0 \qquad \text{for some } c \in (a, b).$$

Show that

$$\int_a^b f(x)dx > 0.$$

13. Let m be a non-negative real number.

(a) Let $g: [0, 1] \to [0, +\infty)$ be a non-negative integrable function such that

$$g(x) \leq mx \qquad \text{and} \qquad g(x) \leq m(1 - x)$$

for every $x \in [0, 1]$. Show that

$$\int_0^1 g(x)dx \leq \frac{m}{4}.$$

(b) Let $f: [0, 1] \to \mathbb{R}$ be a function that is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Suppose that $f(0) = f(1) = 0$ and $|f'(x)| \leq m$ for every $x \in (0, 1)$. Using the result from (a), show that

$$\int_0^1 |f(x)|dx \leq \frac{m}{4}.$$

(c) Using the result from (b), show that

$$\int_0^1 \left|\sin\big(mx(x - 1)\big)\right|dx \leq \frac{m}{4}.$$

14. (a) Let $a < b$ be real numbers.

(i) By considering the area of a simple geometric shape, evaluate

$$\int_a^b (x-a)dx.$$

(ii) Let f be a function which is **continuously differentiable** on $[a,b]$. (See Remark 3.55 if you need to recall what this means.) Explain why the numbers

$$m = \min\{f'(x): x \in [a,b]\} \qquad \text{and} \qquad M = \max\{f'(x): x \in [a,b]\}$$

exist. Using Mean Value Theorem or otherwise, show that

$$\frac{m}{2}(b-a)^2 \leq \int_a^b \big(f(x) - f(a)\big)dx \leq \frac{M}{2}(b-a)^2.$$

(b) Using the result from (a) (ii), show that

$$\frac{1}{2\sqrt{e}} \leq \int_1^2 e^{-\frac{x^2}{2}} dx \leq \frac{1}{\sqrt{e}} - \frac{1}{e^2}.$$

15. Let $a < b$ be real numbers, let $f: [a,b] \to (0,+\infty)$ be a **positive** continuous function, and let $g: [a,b] \to \mathbb{R}$ be the function

$$g(x) = \int_a^x f(t)dt + \int_b^x \frac{1}{f(t)}dt.$$

(a) Show from definition (Definition 1.49) that g is **strictly** increasing on $[a,b]$.

(b) Suppose it is given that g is continuous on $[a,b]$ (This fact actually follows immediately from Lemma 5.50 in §5.4). Show that g has one and only one root in $[a,b]$.

16. (a) Let $p: [0,+\infty) \to \mathbb{R}$ be the function

$$p(t) = t^{\frac{1}{3}} - \frac{1}{3}t - \frac{2}{3}.$$

Show that $p(t) \leq 0$ for every $t \in [0,+\infty)$. Hence deduce that

$$x^{\frac{1}{3}} y^{\frac{2}{3}} \leq \frac{1}{3}x + \frac{2}{3}y \qquad \text{for every } x, y \in [0,+\infty).$$

(b) Let $f, g: [a,b] \to [0,+\infty)$ be non-negative continuous functions such that

$$\int_a^b f(x)dx = \int_a^b g(x)dx = 1.$$

Using the result from (a), show that

$$\int_a^b f(x)^{\frac{1}{3}} g(x)^{\frac{2}{3}} dx \leq 1.$$

(c) Let $F, G: [a,b] \to [0,+\infty)$ be non-negative continuous functions. Using the result from (b), show that

$$\int_a^b F(x)G(x)dx \leq \left(\int_a^b F(x)^3 dx\right)^{\frac{1}{3}} \left(\int_a^b G(x)^{\frac{3}{2}} dx\right)^{\frac{2}{3}}.$$