

# HKUST MATH 1014 L1 assignment 8 submission

MATH1014 Calculus II Problem Set 8

L01 (Spring 2024)

Problem Set 8

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 8 covers materials from chapter 9.

## Q2

Find the 6<sup>th</sup> order approximation of each of the following functions at 0.

### Q2.b

$$g(x) = e^{\cos x}$$

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$$i(x) := \cos x - 1$$

As  $x \rightarrow 0$ ,

$$\begin{aligned} i(x) &= \cos 0 - 1 + \frac{-\sin 0}{1!}x^1 + \frac{-\cos 0}{2!}x^2 + \frac{\sin 0}{3!}x^3 + \frac{\cos 0}{4!}x^4 + \frac{-\sin 0}{5!}x^5 + \frac{-\cos 0}{6!}x^6 + \frac{\sin 0}{7!}x^7 + O(x^8) \\ &= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^8) \end{aligned}$$

Note that  $i(x) = O(x^2)$ .

$$j(x) := e^x$$

As  $x \rightarrow 0$ ,

$$\begin{aligned} j(x) &= e^0 + \frac{e^0}{1!}x^1 + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + O(x^4) \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4) \end{aligned}$$

As  $x \rightarrow 0$ ,

$$\begin{aligned} g(x) &= e^{\cos x} \\ &= e \cdot e^{\cos x - 1} \\ &= e \cdot j(i(x)) \\ &= e \left( 1 + i(x) + \frac{i(x)^2}{2} + \frac{i(x)^3}{6} + O(i(x)^4) \right) \\ &= e \left( 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{1}{2} \left( \frac{x^4}{4} - \frac{x^6}{24} \right) + \frac{1}{6} \left( -\frac{x^6}{8} \right) + O(x^8) \right) \\ &= e \left( 1 - \frac{x^2}{2} + \frac{x^4}{6} - \frac{31x^6}{720} + O(x^8) \right) \\ &= e - \frac{ex^2}{2} + \frac{ex^4}{6} - \frac{31ex^6}{720} + O(x^8) \end{aligned}$$

### Q2.c

$$h(x) = \sec x$$

---

Using the same  $i(x)$  as in (b),

As  $x \rightarrow 0$ ,

$$i(x) = -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^8)$$

Note that  $i(x) = O(x^2)$ .

$$k(x) := \frac{1}{1-x}$$

As  $x \rightarrow 0$ ,

$$\begin{aligned} k(x) &= (1-x)^{-1} + \frac{1!(1-0)^{-2}}{1!}x^1 + \frac{2!(1-0)^{-3}}{2!}x^2 + \frac{3!(1-0)^{-4}}{3!}x^3 + O(x^4) \\ &= 1 + x + x^2 + x^3 + O(x^4) \end{aligned}$$

$$h(x) = \sec x$$

$$= \frac{1}{\cos x}$$

$$= \frac{1}{1+i(x)}$$

$$= k(-i(x))$$

$$= 1 - i(x) + i(x)^2 - i(x)^3 + O((-i(x))^4)$$

$$= 1 - \left(-\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}\right) + \left(\frac{x^4}{4} - \frac{x^6}{24}\right) - \left(-\frac{x^6}{8}\right) + O(x^8)$$

$$= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + O(x^8)$$

### Q3

Evaluate each of the following limits using polynomial approximations.

#### Q3.b

$$\lim_{x \rightarrow 0} \frac{\sin^2 x - \sin(x^2) + \frac{1}{3}x^4}{x^6}$$

$$f(x) := \sin x$$

As  $x \rightarrow 0$ ,

$$f(x) = \sin 0 + \frac{\cos 0}{1!}x + \frac{-\sin 0}{2!}x^2 + \frac{-\cos 0}{3!}x^3 + \frac{\sin 0}{4!}x^4 + \frac{\cos 0}{5!}x^5 + O(x^6)$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^6)$$

Note that  $f(x) = O(x)$ .

As  $x \rightarrow 0$ ,

$$\sin^2 x = f(x)^2$$

$$= x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \frac{x^4}{6} + \frac{x^6}{36} + \frac{x^6}{120} + O(x^7)$$

$$= x^2 - \frac{x^4}{3} + \frac{4x^6}{90} + O(x^7)$$

As  $x \rightarrow 0$ ,

$$\sin(x^2) = f(x^2)$$

$$= x^2 - \frac{x^6}{6} + O(x^{10})$$

$$\lim_{x \rightarrow 0} \frac{\sin^2 x - \sin(x^2) + \frac{1}{3}x^4}{x^6}$$

$$= \lim_{x \rightarrow 0} \frac{\left(x^2 - \frac{x^4}{3} + \frac{4x^6}{90} + O(x^7)\right) - \left(x^2 - \frac{x^6}{6} + O(x^{10})\right) + \frac{x^4}{3}}{x^6}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{19x^6}{90}}{x^6}$$

$$= \lim_{x \rightarrow 0} \frac{19}{90}$$

$$= \frac{19}{90}$$

$$\left( \lim_{x \rightarrow 0} \frac{O(x^y)}{x^6} = 0 \quad \forall y > 6 \right)$$

### Q3.c

$$\lim_{x \rightarrow +\infty} x^2 \left( e - \frac{e}{2x} - \left( 1 + \frac{1}{x} \right)^x \right)$$

$$f(x) := \ln(1+x)$$

As  $x \rightarrow 0$ ,

$$f(x) = \ln(1+0) + \frac{(1+0)^{-1}}{1!}x - \frac{1!(1+0)^{-2}}{2!}x^2 + \frac{2!(1+0)^{-3}}{3!}x^3 + O(x^4)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$$

$$\frac{1}{x}f(x) = 1 - \frac{x}{2} + \frac{x^2}{3} + O(x^3)$$

$$\frac{1}{x}f(x) - 1 = -\frac{x}{2} + \frac{x^2}{3} + O(x^3)$$

Note that  $\frac{1}{x}f(x) - 1 = O(x)$ .

$$g(x) := \exp x$$

As  $x \rightarrow 0$ ,

$$g(x) = \exp 0 + \frac{\exp 0}{1!}x^1 + \frac{\exp 0}{2!}x^2 + O(x^4)$$

$$= 1 + x + \frac{x^2}{2} + O(x^3)$$

As  $x \rightarrow 0$ ,

$$\exp \left( \frac{1}{x} \ln(1+x) \right)$$

$$= \exp \left( \frac{1}{x} f(x) - 1 + 1 \right)$$

$$= e \cdot g \left( \frac{1}{x} f(x) - 1 \right)$$

$$= e \left( 1 + \left( -\frac{x}{2} + \frac{x^2}{3} \right) + \frac{1}{2} \left( \frac{x^2}{4} \right) + O(x^3) \right)$$

$$= e \left( 1 - \frac{x}{2} + \frac{11x^2}{24} + O(x^3) \right)$$

$$= e - \frac{ex}{2} + \frac{11ex^2}{24} + O(x^3)$$

$$\lim_{x \rightarrow +\infty} x^2 \left( e - \frac{e}{2x} - \left( 1 + \frac{1}{x} \right)^x \right)$$

$$= \lim_{x \rightarrow 0^+} \frac{e - \frac{ex}{2} - (1+x)^{\frac{1}{x}}}{x^2}$$

(change of variables:  $\frac{1}{x} \mapsto x$ )

$$= \lim_{x \rightarrow 0^+} \frac{e - \frac{ex}{2} - \exp \left( \frac{1}{x} \ln(1+x) \right)}{x^2}$$

$$= \lim_{x \rightarrow 0^+} \frac{e - \frac{ex}{2} - e + \frac{ex}{2} - \frac{11ex^2}{24} + O(x^3)}{x^2}$$

$$= \lim_{x \rightarrow 0^+} \frac{e - \frac{ex}{2} - e + \frac{ex}{2} - \frac{11ex^2}{24}}{x^2}$$

$\left( \lim_{x \rightarrow 0} \frac{O(x^y)}{x^2} = 0 \quad \forall y > 2 \right)$

$$= \lim_{x \rightarrow 0^+} -\frac{11e}{24}$$

$$= -\frac{11e}{24}$$

### Q5

In Examples 9.31 and 9.32, we have seen that in some cases, Lagrange's remainder formula is not strong enough to show that  $\lim_{n \rightarrow +\infty} R_n(x) = 0$ . Let's develop another remainder formula.

#### Q5.a

Let  $a$  be real number and let  $x > a$ , let  $n$  be a non-negative integer and let  $f$  be a function such that  $f^{(n)}$  is continuous on  $[a, x]$  and differentiable on  $(a, x)$ .

### Q5.a.i

Let  $g: [a, x] \rightarrow \mathbb{R}$  be the function

$$g(t) = f(x) - \sum_{k=0}^n \frac{f^{(k)}(t)}{k!} (x-t)^k$$

.

Compute  $g'(t)$  for  $t \in (a, x)$

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$$\begin{aligned} & g(t) \\ &= f(x) - \sum_{k=0}^n \frac{f^{(k)}(t)}{k!} (x-t)^k \\ &= f(x) - f(t) - \sum_{k=1}^n \frac{f^{(k)}(t)}{k!} (x-t)^k \\ & g'(t) \\ &= -f'(t) - \sum_{k=1}^n \frac{1}{k!} \left( f^{(k+1)}(t)(x-t)^k - k f^{(k)}(t)(x-t)^{k-1} \right) \\ &= -f'(t) - \sum_{k=1}^n \frac{f^{(k+1)}(t)}{k!} (x-t)^k + \sum_{k=1}^n \frac{f^{(k)}(t)}{(k-1)!} (x-t)^{(k-1)} \\ &= -f'(t) - \sum_{k=1}^n \frac{f^{(k+1)}(t)}{k!} (x-t)^k + \sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!} (x-t)^k \\ &= -f'(t) - \frac{f^{(n+1)}(t)}{n!} (x-t)^n + f'(t) \\ &= -\frac{f^{(n+1)}(t)}{n!} (x-t)^n \end{aligned}$$

### Q5.a.ii

**(Cauchy's remainder formula)** By applying Mean Value Theorem to the function  $g$ , show that there exists a number  $c \in (a, x)$  such that

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{f^{(n+1)}(c)}{n!} (x-c)^n (x-a)$$

.

*Remark:* If we assume further that  $f^{(n+1)}$  is integrable on  $[a, x]$ , then another way of obtaining [\(a\)\(ii\)](#) is to use [Q4\(a\)](#) and then the MVT for integrals.

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$f^{(n)}(t)$  is continuous on  $[a, x]$ .  
 $\implies f^{(k)}(t)$  is continuous on  $[a, x]$  for all  $k \in \mathbb{Z}_{[0, n]}$ .  
 $g(t)$  is continuous on  $[a, x]$ .

Additionally,  $f^{(n+1)}(t)$  exists on  $(a, x)$ .  
 $g(t)$  is differentiable on  $(a, x)$ .

By the mean value theorem,  
 $\exists c \in (a, x)$

$$\begin{aligned}
 g'(c) &= \frac{g(x) - g(a)}{x - a} \\
 -\frac{f^{(n+1)}(c)}{n!}(x - c)^n &= \frac{f(x) - \sum_{k=0}^n \frac{f^{(k)}(x)}{k!}(x - x)^k - f(x) + \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x - a)^k}{x - a} \\
 &= \frac{-f(x) + \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x - a)^k}{x - a} \quad (0^0 = 1 \text{ in this context}) \\
 -\frac{f^{(n+1)}(c)}{n!}(x - c)^n(x - a) &= -f(x) + \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x - a)^k \\
 f(x) &= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x - a)^k + \frac{f^{(n+1)}(c)}{n!}(x - c)^n(x - a)
 \end{aligned}$$

## Q5.b

Using the result from (a)(ii) (which obviously still holds if  $x < a$ ), show that for each of the following functions, the remainder term at 0 satisfies

$$\lim_{n \rightarrow +\infty} R_n(x) = 0 \quad \text{for each fixed } x \in (-1, 1)$$

.

## Q5.b.i

(Example 9.31)

$$f(x) = \ln(1 + x)$$


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$$f(x) \in C^\infty((-1, 1], \mathbb{R})$$

By Q5.a.ii...

Set  $a = 0$ .

$$(\forall x \in (0, 1])(\exists c \in (0, x))$$

$$\begin{aligned} f(x) &= \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} (x-0)^k + \frac{f^{(n+1)}(c)}{n!} (x-c)^n (x-0) \\ &= \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(c)}{n!} (x-c)^n x \end{aligned}$$

$$(\forall x \in (-1, 0])(\exists c \in (x, 0))$$

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(c)}{n!} (x-c)^n x$$

$$\begin{aligned} &\lim_{n \rightarrow +\infty} R_n(x) \\ &= \lim_{n \rightarrow +\infty} \left( f(x) - \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k \right) \end{aligned}$$

When  $x = 0$ ,

$$\begin{aligned} &\lim_{n \rightarrow +\infty} R_n(0) \\ &= \lim_{n \rightarrow +\infty} \left( f(0) - \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} 0^k \right) \\ &= \lim_{n \rightarrow +\infty} (f(0) - f(0)) \quad (0^0 = 1 \text{ in this context}) \\ &= \lim_{n \rightarrow +\infty} 0 \\ &= 0 \end{aligned}$$

When  $x \in (-1, 1) \setminus \{0\}$ ,

$\exists c$  in between 0 and  $x$

$$\begin{aligned} &\lim_{n \rightarrow +\infty} R_n(x) \\ &= \lim_{n \rightarrow +\infty} \left( f(x) - \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k \right) \\ &= \lim_{n \rightarrow +\infty} \left( \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(c)}{n!} (x-c)^n x - \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k \right) \quad (n \text{ in the above formula is arbitrary}) \\ &= \lim_{n \rightarrow +\infty} \frac{f^{(n+1)}(c)}{n!} (x-c)^n x \\ &= \lim_{n \rightarrow +\infty} \frac{n!(-1)^{n+1}(1+c)^{-(n+1)}}{n!} (x-c)^n x \\ &= \lim_{n \rightarrow +\infty} (-1)^{n+1} \left( \frac{x-c}{1+c} \right)^n \frac{x}{1+x} \\ &= \frac{x}{1+x} \lim_{n \rightarrow +\infty} (-1)^{n+1} \left( \frac{x-c}{1+c} \right)^n \end{aligned}$$

When  $x \in (0, 1)$ ,  $c \in (0, x)$ , and

$$\begin{aligned} &\left| \frac{x-c}{1+c} \right| \\ &= \frac{x-c}{1+c} \quad (x > c, c > 0) \\ &< \frac{1-c}{1+c} \quad (x < 1, c > 0) \\ &< 1 \quad (c < x < 1) \end{aligned}$$

When  $x \in (-1, 0)$ ,  $c \in (x, 0)$  and

$$\begin{aligned} &\left| \frac{x-c}{1+c} \right| \\ &= \frac{c-x}{1+c} \quad (x < c, c > x > -1) \\ &< \frac{c+1}{1+c} \quad (-x < 1, c > x > -1) \\ &= 1 \end{aligned}$$

Therefore...

When  $x \in (0, 1)$ ,  $c \in (0, x)$ , and

$$\begin{aligned} &\lim_{n \rightarrow +\infty} R_n(x) \\ &= \frac{x}{1+x} \lim_{n \rightarrow +\infty} (-1)^{n+1} \left( \frac{x-c}{1+c} \right)^n \end{aligned}$$

$$\begin{aligned}
 &= \frac{x}{1+x} \cdot 0 \\
 &= 0
 \end{aligned}$$

$$\left(\left|\frac{x-c}{1+c}\right|<1\right)$$

$$\begin{aligned}
 &\forall x \in (-1,1) \\
 &\lim_{n \rightarrow +\infty} R_n(x) \\
 &= 0
 \end{aligned}$$

**Q7**

For each of the following, compute its Maclaurin series and find its radius of convergence.

**Q7.b**

$$f(x)=\int_0^x \frac{\sin t}{t} \, \mathrm{d}t$$


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$$\begin{aligned} & f'(x) \\ &= \frac{\sin x}{x} \end{aligned}$$

$$\begin{aligned} & \sin x \\ &= \sum_{k=0}^{+\infty} \left( \frac{\sin 0}{(4k)!} x^{4k} + \frac{\cos 0}{(4k+1)!} x^{4k+1} + \frac{-\sin 0}{(4k+2)!} x^{4k+2} + \frac{-\cos 0}{(4k+3)!} x^{4k+3} \right) \\ &= \sum_{k=0}^{+\infty} \left( \frac{x^{4k+1}}{(4k+1)!} - \frac{x^{4k+3}}{(4k+3)!} \right) \\ &= \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \end{aligned}$$

$$\begin{aligned} & f'(x) \\ &= \frac{\sin x}{x} \\ &= \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k}}{(2k+1)!} \end{aligned}$$

$$\begin{aligned} & f(x) \\ &= \int_0^x f'(x) \, dx \\ &= \int_0^x \left( \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k}}{(2k+1)!} \right) dx \\ &= \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} \int_0^x x^{2k} \, dx \\ &= \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!(2k+1)} x^{2k+1} \end{aligned}$$

(Maclaurin series can be integrated term-wise)

$$\begin{aligned} & f(x) \\ &= x \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!(2k+1)} (x^2)^k \end{aligned}$$

The center of the power series is  $x^2 = 0$ .

The coefficients are  $c_k = \frac{(-1)^k}{(2k+1)!(2k+1)}$ .

radius of convergence of the power series

$$\begin{aligned} &= \lim_{k \rightarrow +\infty} \left| \frac{\frac{(-1)^k}{(2k+1)!(2k+1)}}{\frac{(-1)^{k+1}}{(2k+3)!(2k+3)}} \right| \\ &= \lim_{k \rightarrow +\infty} \left| \frac{(2k+3)!(2k+3)}{(2k+1)!(2k+1)} \right| \\ &= \lim_{k \rightarrow +\infty} \left| \frac{(2k+2)(2k+3)^2}{2k+1} \right| \\ &= +\infty \end{aligned}$$

radius of convergence of the Maclurin series of  $f(x)$

$$\begin{aligned} &= \sqrt{+\infty} \\ &= +\infty \end{aligned}$$

## Q7.d

$$f(x) = \ln \left( x + \sqrt{1+x^2} \right)$$

Hint: In [Q7.d](#), first consider  $f'$ .



$$\begin{aligned}
 & f'(x) \\
 &= \frac{1 + \frac{x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} \\
 &= \frac{\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} \\
 &= \frac{1}{\sqrt{1+x^2}}
 \end{aligned}$$

$$\begin{aligned}
 & g(x) \\
 &:= \frac{1}{\sqrt{1+x}} \\
 & \text{Macluarin series of } g(x) \\
 &= \sum_{k=0}^{+\infty} \frac{g^{(k)}(0)}{k!} x^k \\
 &= \sum_{k=0}^{+\infty} \frac{(-1)^k \left( \prod_{i=0}^{k-1} \frac{2i+1}{2} \right) (1+0)^{-\frac{2k+1}{2}}}{k!} x^k \\
 &= \sum_{k=0}^{+\infty} (-1)^k \frac{\prod_{i=0}^{k-1} (2i+1)}{k! 2^k} x^k
 \end{aligned}$$

$$\begin{aligned}
 & f'(x) \\
 &= \frac{1}{\sqrt{1+x^2}} \\
 &= g(x^2) \\
 & \text{Macluarin series of } f'(x) \\
 &= \sum_{k=0}^{+\infty} (-1)^k \frac{\prod_{i=0}^{k-1} (2i+1)}{k! 2^k} x^{2k}
 \end{aligned}$$

$$\begin{aligned}
 & f(x) \\
 &= \int f'(x) \, dx \\
 & \text{Macluarin series of } f(x) \\
 &= \int \left( \sum_{k=0}^{+\infty} (-1)^k \frac{\prod_{i=0}^{k-1} (2i+1)}{k! 2^k} x^{2k} \right) dx \\
 &= \sum_{k=0}^{+\infty} (-1)^k \frac{\prod_{i=0}^{k-1} (2i+1)}{k! 2^k} \int x^{2k} \, dx \quad (\text{Macluarin series can be integrated term-wise}) \\
 &= \sum_{k=0}^{+\infty} (-1)^k \frac{\prod_{i=0}^{k-1} (2i+1)}{k! 2^k (2k+1)} x^{2k+1}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Macluarin series of } f(x) \\
 &= x \sum_{k=0}^{+\infty} (-1)^k \frac{\prod_{i=0}^{k-1} (2i+1)}{k! 2^k (2k+1)} (x^2)^k
 \end{aligned}$$

The center of the power series is  $x^2 = 0$ .

The coefficients are  $c_k = (-1)^k \frac{\prod_{i=0}^{k-1} (2i+1)}{k! 2^k (2k+1)}$ .

radius of convergence of the power series

$$\begin{aligned}
 &= \lim_{k \rightarrow +\infty} \left| \frac{(-1)^k \frac{\prod_{i=0}^{k-1} (2i+1)}{k! 2^k (2k+1)}}{(-1)^{k+1} \frac{\prod_{i=0}^k (2i+1)}{(k+1)! 2^{k+1} (2k+3)}} \right| \\
 &= \lim_{k \rightarrow +\infty} \left| \frac{2(k+1)(2k+3)}{(2k+1)^2} \right| \\
 &= \lim_{k \rightarrow +\infty} \left| \frac{\left(1 + \frac{1}{k}\right) \left(1 + \frac{3}{2k}\right)}{\left(1 + \frac{1}{2k}\right)^2} \right| \\
 &= 1
 \end{aligned}$$

radius of convergence of the Macluarin series of  $f(x)$

$$\begin{aligned}
 &= \sqrt{1} \\
 &= 1
 \end{aligned}$$

## Q9

Let  $f(x) = x^3 e^x$ . Using the Taylor series of  $f$ , compute...

for every positive integer  $n$ . (Do not try to really differentiate for  $n$  times!)

### Q9.a

...

$$f^{(n)}(0)$$

and

---

$$\begin{aligned} & \exp x \\ & \text{Maclaurin series of } \exp x \\ &= \sum_{n=0}^{+\infty} \frac{\exp 0}{n!} x^n \\ &= \sum_{n=0}^{+\infty} \frac{x^n}{n!} \end{aligned}$$

$$\begin{aligned} & f(x) \\ &= x^3 \exp x \\ & \text{Maclaurin series of } f(x) \\ &= x^3 \sum_{n=0}^{+\infty} \frac{x^n}{n!} \\ &= \sum_{n=0}^{+\infty} \frac{x^{n+3}}{n!} \\ &= \sum_{n=3}^{+\infty} \frac{x^n}{(n-3)!} \end{aligned}$$

$$\begin{aligned} & f(x) \\ & \text{Maclaurin series of } f(x) \\ &= \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{k!} x^k \end{aligned}$$

Since Taylor (Maclaurin) series is unique for a  $C^\infty$  function,  
by comparing coefficients:

When  $0 \leq n < 3$ ,

$$\frac{f^{(n)}(0)}{n!} = 0$$

$$f^{(n)}(0) = 0$$

When  $n \geq 3$ ,

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{(n-3)!}$$

$$\begin{aligned} f^{(n)}(0) &= \frac{n!}{(n-3)!} \\ &= n(n-1)(n-2) \end{aligned}$$

### Q9.b

$$f^{(n)}(1)$$

---

$\exp x$   
Taylor series of  $\exp x$  at 1

$$= \sum_{n=0}^{+\infty} \frac{\exp 1}{n!} (x-1)^n$$

$$= e \sum_{n=0}^{+\infty} \frac{(x-1)^n}{n!}$$

$f(x)$   
 $= x^3 \exp x$   
 $= ((x^3 - 3x^2 + 3x - 1) + 3x^2 - 3x + 1) \exp x$   
 $= ((x^3 - 3x^2 + 3x - 1) + 3(x^2 - 2x + 1) + 3x - 2) \exp x$   
 $= ((x^3 - 3x^2 + 3x - 1) + 3(x^2 - 2x + 1) + 3(x-1) + 1) \exp x$   
 $= ((x-1)^3 + 3(x-1)^2 + 3(x-1) + 1) \exp x$   
Taylor series of  $f(x)$  at 1

$$= e ((x-1)^3 + 3(x-1)^2 + 3(x-1) + 1) \sum_{n=0}^{+\infty} \frac{(x-1)^n}{n!}$$

$$= e \left( \sum_{n=0}^{+\infty} \frac{(x-1)^{n+3}}{n!} + 3 \sum_{n=0}^{+\infty} \frac{(x-1)^{n+2}}{n!} + 3 \sum_{n=0}^{+\infty} \frac{(x-1)^{n+1}}{n!} + \sum_{n=0}^{+\infty} \frac{(x-1)^n}{n!} \right)$$

$$= e \left( \sum_{n=3}^{+\infty} \frac{(x-1)^n}{(n-3)!} + 3 \sum_{n=2}^{+\infty} \frac{(x-1)^n}{(n-2)!} + 3 \sum_{n=1}^{+\infty} \frac{(x-1)^n}{(n-1)!} + \sum_{n=0}^{+\infty} \frac{(x-1)^n}{n!} \right)$$

$f(x)$   
Taylor series of  $f(x)$  at 1

$$= \sum_{k=0}^{+\infty} \frac{f^{(k)}(1)}{k!} (x-1)^k$$

Since Taylor series is unique for a  $C^\infty$  function,  
by comparing coefficients:

$$\frac{f^{(n)}(1)}{n!} = \begin{cases} \frac{e}{n!}, & n = 0 \\ \frac{e}{n!} + \frac{3e}{(n-1)!}, & n = 1 \\ \frac{e}{n!} + \frac{3e}{(n-1)!} + \frac{3e}{(n-2)!}, & n = 2 \\ \frac{e}{n!} + \frac{3e}{(n-1)!} + \frac{3e}{(n-2)!} + \frac{e}{(n-3)!}, & n \geq 3 \end{cases}$$

$$f^{(n)}(1) = \begin{cases} e, & n = 0 \\ e + 3en, & n = 1 \\ e + 3en + 3en(n-1), & n = 2 \\ e + 3en + 3en(n-1) + en(n-1)(n-2), & n \geq 3 \end{cases}$$

$$= \begin{cases} e, & n = 0 \\ 4e, & n = 1 \\ 13e, & n = 2 \\ e + 3en + 3en(n-1) + en(n-1)(n-2), & n \geq 3 \end{cases}$$

## Q11

Let  $a_0, a_1, \dots, a_n, b_1, \dots, b_n$  be real numbers and let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx)$$

be a trigonometric polynomial. (Note that  $f$  is a finite sum.) Show that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx = \frac{a_0^2}{2} + \sum_{k=1}^n (a_k^2 + b_k^2)$$

.

Let  $m, n$  be two integers.

$$\begin{aligned}
& \int_{-\pi}^{\pi} \cos mx \cos nx \, dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) + \cos((m-n)x)) \, dx \\
&= \begin{cases} \frac{1}{2} \int_{-\pi}^{\pi} (1+1) \, dx & m+n=0, m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (1+\cos((m-n)x)) \, dx & m+n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x)+1) \, dx & m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) + \cos((m-n)x)) \, dx & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{2} [2x]_{x=-\pi}^{x=\pi} & m+n=0, m-n=0 \\ \frac{1}{2} [x + \frac{1}{m-n} \sin((m-n)x)]_{x=-\pi}^{x=\pi} & m+n=0 \\ \frac{1}{2} [\frac{1}{m+n} \sin((m+n)x) + x]_{x=-\pi}^{x=\pi} & m-n=0 \\ \frac{1}{2} [\frac{1}{m+n} \sin((m+n)x) + \frac{1}{m-n} \sin((m-n)x)]_{x=-\pi}^{x=\pi} & \text{otherwise} \end{cases} \\
&= \begin{cases} 2\pi & m+n=0, m-n=0 \\ \pi & m+n=0 \\ \pi & m-n=0 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 2\pi & m=n=0 \\ \pi & m=-n \\ \pi & m=n \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

$$\begin{aligned}
& \int_{-\pi}^{\pi} \sin mx \sin nx \, dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m-n)x) - \cos((m+n)x)) \, dx \\
&= \begin{cases} \frac{1}{2} \int_{-\pi}^{\pi} (1-1) \, dx & m+n=0, m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m-n)x) - 1) \, dx & m+n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos((m+n)x)) \, dx & m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m-n)x) - \cos((m+n)x)) \, dx & \text{otherwise} \end{cases} \\
&= \begin{cases} 0 & m+n=0, m-n=0 \\ \frac{1}{2} [\frac{1}{m-n} \sin((m-n)x) - x]_{x=-\pi}^{x=\pi} & m+n=0 \\ \frac{1}{2} [x - \frac{1}{m+n} \sin((m+n)x)]_{x=-\pi}^{x=\pi} & m-n=0 \\ \frac{1}{2} [\frac{1}{m-n} \sin((m-n)x) - \frac{1}{m+n} \sin((m+n)x)]_{x=-\pi}^{x=\pi} & \text{otherwise} \end{cases} \\
&= \begin{cases} 0 & m+n=0, m-n=0 \\ -\pi & m+n=0 \\ \pi & m-n=0 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 0 & m=n=0 \\ -\pi & m=-n \\ \pi & m=n \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

$$\begin{aligned}
& \int_{-\pi}^{\pi} \cos mx \sin nx \, dx \\
&= \int_0^{\pi} \cos mx \sin nx \, dx + \int_{-\pi}^0 \cos mx \sin nx \, dx \\
&= \int_0^{\pi} \cos mx \sin nx \, dx - \int_{\pi}^0 \cos(-mu) \sin(-nu) \, du \quad (u := -x) \\
&= \int_0^{\pi} \cos mx \sin nx \, dx - \int_0^{\pi} \cos mu \sin nu \, du \\
&= 0
\end{aligned}$$

(The above is adapted from my own work in assignment 3 Q1.)

$$\begin{aligned}
& \int_{-\pi}^{\pi} f(x) \, dx \\
&= \int_{-\pi}^{\pi} \left( \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) \right) \, dx \\
&= \int_{-\pi}^{\pi} \frac{a_0}{2} \, dx + \sum_{k=1}^n a_k \int_{-\pi}^{\pi} \cos kx \, dx + \sum_{k=1}^n b_k \int_{-\pi}^{\pi} \sin kx \, dx \quad (\text{linearity}) \\
&= \pi a_0 + \sum_{k=1}^n \frac{a_k}{k} [\sin kx]_{x=-\pi}^{\pi} - \sum_{k=1}^n \frac{b_k}{k} [\cos kx]_{x=-\pi}^{\pi}
\end{aligned}$$

$$\begin{aligned}
& \overset{k=1}{=} \overset{\nu}{\pi a_0} + 0 - 0 \\
& = \pi a_0 \\
& \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx \\
& = \frac{1}{\pi} \int_{-\pi}^{\pi} \left( \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) \right)^2 \, dx \\
& = \frac{1}{\pi} \int_{-\pi}^{\pi} \left( a_0 f(x) - \frac{a_0^2}{4} + \sum_{1 \leq i, j \leq n} (a_i \cos ix + b_i \sin ix)(a_j \cos jx + b_j \sin jx) \right) \, dx \\
& = \frac{1}{\pi} \int_{-\pi}^{\pi} \left( a_0 f(x) - \frac{a_0^2}{4} + \sum_{1 \leq i, j \leq n} (a_i a_j \cos ix \cos jx + a_i b_j \cos ix \sin jx + \right. \\
& \quad \left. b_i a_j \sin ix \cos jx + b_i b_j \sin ix \sin jx) \right) \, dx \\
& = \frac{a_0}{\pi} \int_{-\pi}^{\pi} \left( f(x) - \frac{a_0}{4} \right) \, dx + \frac{1}{\pi} \sum_{1 \leq i, j \leq n} a_i a_j \int_{-\pi}^{\pi} \cos ix \cos jx \, dx \quad (\text{linearity}) \\
& \quad + \frac{1}{\pi} \sum_{1 \leq i, j \leq n} a_i b_j \int_{-\pi}^{\pi} \cos ix \sin jx \, dx \\
& \quad + \frac{1}{\pi} \sum_{1 \leq i, j \leq n} b_i a_j \int_{-\pi}^{\pi} \sin ix \cos jx \, dx \\
& \quad + \frac{1}{\pi} \sum_{1 \leq i, j \leq n} b_i b_j \int_{-\pi}^{\pi} \sin ix \sin jx \, dx \\
& = \frac{a_0}{\pi} \int_{-\pi}^{\pi} f(x) \, dx - \frac{a_0^2}{2} + \sum_{k=1}^n a_k^2 + 0 + 0 + \sum_{k=1}^n b_k^2 \quad (\text{apply equations above}) \\
& = a_0^2 - \frac{a_0^2}{2} + \sum_{k=1}^n (a_k^2 + b_k^2) \\
& = \frac{a_0^2}{2} + \sum_{k=1}^n (a_k^2 + b_k^2)
\end{aligned}$$

## Q13

Let  $a$  be a real number which is not an integer. Let  $f(x) = \cos ax$  be defined on  $[-\pi, \pi]$  and extended periodically to become a function with period  $2\pi$ .

### Q13.a

Compute the Fourier series of  $f$ .

---

$\therefore f$  is a periodic function of period  $2\pi$   
 $f$  is continuously differentiable on  $\mathbb{R}$   
 $\therefore f$  equals its Fourier series on  $\mathbb{R}$

$$\begin{aligned}
& f(x) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx + \sum_{n=1}^{+\infty} \left( \cos(nx) \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx + \sin(nx) \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx \right) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(ax) \, dx + \sum_{n=1}^{+\infty} \left( \cos(nx) \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(ax) \cos(nx) \, dx + \sin(nx) \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(ax) \sin(nx) \, dx \right) \\
&= \frac{1}{2a\pi} [\sin(ax)]_{x=-\pi}^{\pi} + \sum_{n=1}^{+\infty} \left( \cos(nx) \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cos((a-n)x) + \cos((a+n)x)) \, dx \right. \\
&\quad \left. + \sin(nx) \frac{1}{2\pi} \int_{-\pi}^{\pi} (\sin((a+n)x) - \sin((a-n)x)) \, dx \right) \\
&= \frac{\sin(a\pi)}{a\pi} + \sum_{n=1}^{+\infty} \left( \frac{\cos(nx)}{2\pi} \left[ \frac{\sin((a-n)x)}{a-n} + \frac{\sin((a+n)x)}{a+n} \right]_{x=-\pi}^{\pi} \right. \\
&\quad \left. - \frac{\sin(nx)}{2\pi} \left[ \frac{\cos((a+n)x)}{a+n} - \frac{\cos((a-n)x)}{a-n} \right]_{x=-\pi}^{\pi} \right) \quad (a \notin \mathbb{Z}) \\
&= \frac{\sin(a\pi)}{a\pi} + \frac{1}{\pi} \sum_{n=1}^{+\infty} \cos(nx) \left( \frac{\sin((a-n)\pi)}{a-n} + \frac{\sin((a+n)\pi)}{a+n} \right) \\
&= \frac{\sin(a\pi)}{a\pi} + \frac{1}{\pi} \sum_{n=1}^{+\infty} \cos(nx) \frac{(a+n) \sin((a-n)\pi) + (a-n) \sin((a+n)\pi)}{a^2 - n^2} \\
&= \frac{\sin(a\pi)}{a\pi} + \frac{1}{\pi} \sum_{n=1}^{+\infty} \cos(nx) \frac{2a \sin(a\pi) \cos(n\pi) + 2n \sin(n\pi) \cos(a\pi)}{a^2 - n^2} \\
&= \frac{\sin(a\pi)}{a\pi} + \frac{2a}{\pi} \sum_{n=1}^{+\infty} \cos(nx) \frac{(-1)^n \sin(a\pi)}{a^2 - n^2} \\
&= \frac{\sin(a\pi)}{a\pi} + \frac{2a \sin(a\pi)}{\pi} \sum_{n=1}^{+\infty} (-1)^n \frac{\cos(nx)}{a^2 - n^2}
\end{aligned}$$

### Q13.b

Using (a), prove that

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k^2 - a^2} = -\frac{1}{2a^2} + \frac{\pi}{2a \sin(a\pi)}$$

.

and in a similar way also compute

$$\sum_{k=1}^{+\infty} \frac{1}{k^2 - a^2}$$

.

$$f(x) = \frac{\sin(a\pi)}{a\pi} + \frac{2a \sin(a\pi)}{\pi} \sum_{n=1}^{+\infty} (-1)^n \frac{\cos(nx)}{a^2 - n^2} \quad (\text{Q13.a})$$

$$f(0) = \frac{\sin(a\pi)}{a\pi} + \frac{2a \sin(a\pi)}{\pi} \sum_{n=1}^{+\infty} (-1)^n \frac{\cos(0n)}{a^2 - n^2}$$

$$\cos(0a) = \frac{\sin(a\pi)}{a\pi} + \frac{2a \sin(a\pi)}{\pi} \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k^2 - a^2}$$

$$1 - \frac{\sin(a\pi)}{a\pi} = \frac{2a \sin(a\pi)}{\pi} \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k^2 - a^2}$$

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k^2 - a^2} = -\frac{1}{2a^2} + \frac{\pi}{2a \sin(a\pi)}$$

$$f(\pi) = \frac{\sin(a\pi)}{a\pi} + \frac{2a \sin(a\pi)}{\pi} \sum_{n=1}^{+\infty} (-1)^n \frac{\cos(n\pi)}{a^2 - n^2}$$

$$\cos(a\pi) = \frac{\sin(a\pi)}{a\pi} + \frac{2a \sin(a\pi)}{\pi} \sum_{k=1}^{+\infty} \frac{1}{a^2 - k^2}$$

$$\cos(a\pi) - \frac{\sin(a\pi)}{a\pi} = -\frac{2a \sin(a\pi)}{\pi} \sum_{k=1}^{+\infty} \frac{1}{k^2 - a^2}$$

$$\begin{aligned} \sum_{k=1}^{+\infty} \frac{1}{k^2 - a^2} &= \frac{1}{2a^2} - \frac{\pi \cos(a\pi)}{2a \sin(a\pi)} \\ &= \frac{1}{2a^2} - \frac{\pi}{2a} \cot(a\pi) \end{aligned}$$