Solution to Problem Set 4

1. (a)

$$\int \frac{1}{x^2+1}\,dx = \arctan x + C$$

(b)

$$\int \frac{1}{x^2+2x}\,dx = \int \frac{1}{x(x+2)}\,dx = \int \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+2}\right)dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C$$

(c)

$$\int \frac{1}{x^2+2x+1}\,dx = \int \frac{1}{(x+1)^2}\,dx = -\frac{1}{x+1} + C$$

(d)

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx = \arctan(x+1) + C$$

(e)

$$\int \frac{1}{x^2+2x+3}\,dx = \int \frac{1}{(x+1)^2+\left(\sqrt{2}\right)^2}\,dx = \frac{1}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}} + C$$

(f)

$$\int \frac{x}{x^2+2x+1}\,dx = \int \frac{(x+1)-1}{(x+1)^2}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{1}{x^2+2x+1}\,dx$$

$$= \ln|x+1| + \frac{1}{x+1} + C$$

(g)

$$\int \frac{x}{x^2+2x+2}\,dx = \int \frac{(x+1)-1}{(x+1)^2+1}\,dx = \frac{1}{2}\int \frac{1}{(x+1)^2+1}\,d(x+1)^2 - \int \frac{1}{(x+1)^2+1}\,dx$$

$$= \frac{1}{2}\ln((x+1)^2+1) - \arctan(x+1) + C$$

(h)

$$\int \frac{x+2}{x^2+2x+2}\,dx = \int \frac{(x+1)+1}{(x+1)^2+1}\,dx = \frac{1}{2}\int \frac{1}{(x+1)^2+1}\,d(x+1)^2 + \int \frac{1}{(x+1)^2+1}\,dx$$

$$= \frac{1}{2}\ln((x+1)^2+1) + \arctan(x+1) + C$$

(i) We first need a partial fraction decomposition of $\frac{1}{x^2(x+2)}$. Suppose that

$$\frac{1}{x^2(x+2)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+2} = \frac{ax(x+2) + b(x+2) + cx^2}{x^2(x+2)}.$$

Then we obtain the polynomial identity $ax(x+2) + b(x+2) + cx^2 = 1$.

- Putting $x = -2$ in the identity we obtain $c = \frac{1}{4}$;
- Comparing the coefficients of x^2 on both sides of the identity, we have $a + c = 0$, so $a = -\frac{1}{4}$;
- Comparing the constant term on both sides of the identity, we have $2b = 1$, so $b = \frac{1}{2}$.

Therefore,

$$\int \frac{1}{x^2(x+2)}\,dx = \int \left[-\frac{1}{4x} + \frac{1}{2x^2} + \frac{1}{4(x+2)}\right] dx = -\frac{1}{4}\ln|x| - \frac{1}{2x} + \frac{1}{4}\ln|x+2| + C.$$

(j) We first need a partial fraction decomposition of $\frac{1}{x(x+2)^2}$. Suppose that

$$\frac{1}{x(x+2)^2} = \frac{a}{x} + \frac{b}{x+2} + \frac{c}{(x+2)^2} = \frac{a(x+2)^2 + bx(x+2) + cx}{x(x+2)^2}.$$

Then we obtain the polynomial identity $a(x+2)^2 + bx(x+2) + cx = 1$.

- Putting $x = -2$ in the identity we obtain $c = -\frac{1}{2}$;
- Putting $x = 0$ in the identity we obtain $a = \frac{1}{4}$;
- Comparing the coefficient of x^2 on both sides of the identity, we have $a + b = 0$, so $b = -\frac{1}{4}$.

Therefore,

$$\int \frac{1}{x(x+2)^2}\,dx = \int \left[\frac{1}{4x} - \frac{1}{4(x+2)} - \frac{1}{2(x+2)^2}\right] dx = \frac{1}{4}\ln|x| - \frac{1}{4}\ln|x+2| + \frac{1}{2(x+2)} + C.$$

2. (a) Let $u = \sqrt{e^x + 1}$. Then $du = \frac{1}{2\sqrt{e^x+1}} e^x dx$, so $dx = \frac{2u}{u^2-1} du$. Thus,

$$\int \frac{1}{\sqrt{e^x+1}}\,dx = \int \frac{1}{u} \cdot \frac{2u}{u^2-1}\,du = \int \frac{2}{(u-1)(u+1)}\,du = \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du$$
$$= \ln|u-1| - \ln|u+1| + C = \ln\left(\sqrt{e^x+1} - 1\right) - \ln\left(\sqrt{e^x+1} + 1\right) + C$$
$$= \ln\frac{\sqrt{e^x+1} - 1}{\sqrt{e^x+1} + 1} + C = x - 2\ln\left(\sqrt{e^x+1} + 1\right) + C.$$

(b) Taking antiderivatives by parts, we have

$$\int \ln(x^3+1)\,dx = x\ln(x^3+1) - \int x \cdot \frac{3x^2}{x^3+1}\,dx = x\ln(x^3+1) - \int \frac{3x^3}{x^3+1}\,dx$$
$$= x\ln(x^3+1) - \int \left(3 - \frac{3}{x^3+1}\right) dx = x\ln(x^3+1) - 3x + \int \frac{3}{x^3+1}\,dx.$$

Now we suppose that

$$\frac{3}{x^3+1} = \frac{a}{x+1} + \frac{bx+c}{x^2-x+1} = \frac{a(x^2-x+1) + (bx+c)(x+1)}{x^3+1}.$$

Then we obtain the polynomial identity $a(x^2 - x + 1) + (bx + c)(x + 1) = 3$.

- Putting $x = -1$ in the identity we obtain $a = 1$;
- Comparing the coefficient of x^2 on both sides of the identity, we have $a + b = 0$, so $b = -1$.
- Comparing the constant term on both sides of the identity, we have $a + c = 3$, so $c = 2$.

Thus we have

$$\int \frac{3}{x^3+1}dx = \int \left(\frac{1}{x+1} + \frac{-x+2}{x^2-x+1}\right)dx = \int \left(\frac{1}{x+1} + \frac{-\frac{1}{2}(2x-1)+\frac{3}{2}}{x^2-x+1}\right)dx$$

$$= \int \frac{1}{x+1}dx - \frac{1}{2}\int \frac{1}{x^2-x+1}d(x^2-x+1) + \frac{3}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$= \ln|x+1| - \frac{1}{2}\ln(x^2-x+1) + \sqrt{3}\arctan\frac{2x-1}{\sqrt{3}} + C.$$

Therefore

$$\int \ln(x^3+1)\,dx = x\ln(x^3+1) - 3x + \ln|x+1| - \frac{1}{2}\ln(x^2-x+1) + \sqrt{3}\arctan\frac{2x-1}{\sqrt{3}} + C.$$

(c) Let $u = \sqrt{x+1}$. Then $x = u^2 - 1$, so $dx = 2u\,du$. Thus,

$$\int \frac{1}{x^2\sqrt{x+1}}dx = \int \frac{1}{(u^2-1)^2u}2udu = \int \frac{2}{(u-1)^2(u+1)^2}du.$$

Now we suppose that

$$\frac{2}{(u-1)^2(u+1)^2} = \frac{a}{u-1} + \frac{b}{(u-1)^2} + \frac{c}{u+1} + \frac{d}{(u+1)^2}$$

$$= \frac{a(u-1)(u+1)^2 + b(u+1)^2 + c(u+1)(u-1)^2 + d(u-1)^2}{(u-1)^2(u+1)^2}.$$

Then we obtain the identity $a(u-1)(u+1)^2 + b(u+1)^2 + c(u+1)(u-1)^2 + d(u-1)^2 = 2$.

- Putting $u = 1$ in the identity we obtain $b = \frac{1}{2}$;
- Putting $u = -1$ in the identity we obtain $d = \frac{1}{2}$;
- Comparing the constant term on both sides of the identity, we have $-a + b + c + d = 2$, so $c - a = 1$;
- Comparing the coefficient of u^3 on both sides, we have $a + c = 0$, so $c = \frac{1}{2}$ and $a = -\frac{1}{2}$.

Therefore

$$\int \frac{1}{x^2\sqrt{x+1}}dx = \int \left(-\frac{1}{2(u-1)} + \frac{1}{2(u-1)^2} + \frac{1}{2(u+1)} + \frac{1}{2(u+1)^2}\right)du$$

$$= -\frac{1}{2}\ln|u-1| - \frac{1}{2(u-1)} + \frac{1}{2}\ln|u+1| - \frac{1}{2(u+1)} + C$$

$$= -\frac{1}{2}\ln\left|\sqrt{x+1}-1\right| - \frac{1}{2\left(\sqrt{x+1}-1\right)} + \frac{1}{2}\ln\left(\sqrt{x+1}+1\right) - \frac{1}{2\left(\sqrt{x+1}+1\right)} + C.$$

(d) Let $u=\sqrt{\frac{1+x}{1-x}}$. Then $u^2=\frac{1+x}{1-x}=\frac{2}{1-x}-1$, so $x=1-\frac{2}{u^2+1}$, and $dx=\frac{4u}{(u^2+1)^2}du$. Thus

$$\int (x+2)\sqrt{\frac{1+x}{1-x}}\,dx=\int\left(3-\frac{2}{u^2+1}\right)u\frac{4u}{(u^2+1)^2}du=\int\left[\frac{12u^2}{(u^2+1)^2}-\frac{8u^2}{(u^2+1)^3}\right]du$$
$$=\int\left[\frac{(12u^2+12)-12}{(u^2+1)^2}-\frac{(8u^2+8)-8}{(u^2+1)^3}\right]du$$
$$=\int\left[\frac{12}{u^2+1}-\frac{20}{(u^2+1)^2}+\frac{8}{(u^2+1)^3}\right]du.$$

Now recalling the reduction formula obtained in Example 6.33, we have

$$\int\frac{1}{(u^2+1)^2}du=\frac{u}{2(u^2+1)}+\frac{1}{2}\int\frac{1}{u^2+1}du=\frac{u}{2(u^2+1)}+\frac{1}{2}\arctan u+C \qquad \text{and}$$
$$\int\frac{1}{(u^2+1)^3}du=\frac{u}{4(u^2+1)^2}+\frac{3}{4}\int\frac{1}{(u^2+1)^2}du=\frac{u}{4(u^2+1)^2}+\frac{3}{8}\frac{u}{u^2+1}+\frac{3}{8}\arctan u+C.$$

Therefore,

$$\int (x+2)\sqrt{\frac{1+x}{1-x}}\,dx=12\arctan u-\left(\frac{10u}{u^2+1}+10\arctan u\right)+\left(\frac{2u}{(u^2+1)^2}+\frac{3u}{u^2+1}+3\arctan u\right)+C$$
$$=\frac{2u}{(u^2+1)^2}-\frac{7u}{u^2+1}+5\arctan u+C$$
$$=\frac{1}{2}(1-x)^2\sqrt{\frac{1+x}{1-x}}-\frac{7}{2}(1-x)\sqrt{\frac{1+x}{1-x}}+5\arctan\sqrt{\frac{1+x}{1-x}}+C.$$

3. (a) We first need a partial fraction decomposition of $\frac{x^2}{x^4+1}$. Instead of the usual algorithm, one can observe that

$$\frac{1}{x^2-\sqrt{2}x+1}-\frac{1}{x^2+\sqrt{2}x+1}=\frac{\left(x^2+\sqrt{2}x+1\right)-\left(x^2-\sqrt{2}x+1\right)}{x^4+1}=\frac{2\sqrt{2}x}{x^4+1};$$

so

$$\frac{x^2}{x^4+1}=\frac{x}{2\sqrt{2}}\left(\frac{1}{x^2-\sqrt{2}x+1}-\frac{1}{x^2+\sqrt{2}x+1}\right)=\frac{1}{2\sqrt{2}}\cdot\frac{x}{x^2-\sqrt{2}x+1}-\frac{1}{2\sqrt{2}}\cdot\frac{x}{x^2+\sqrt{2}x+1}.$$

Thus,

$$\int\frac{x^2}{x^4+1}dx=\frac{1}{2\sqrt{2}}\int\frac{x}{x^2-\sqrt{2}x+1}dx-\frac{1}{2\sqrt{2}}\int\frac{x}{x^2+\sqrt{2}x+1}dx.$$

Now the first antiderivative is

$$\int\frac{x}{x^2-\sqrt{2}x+1}dx=\int\frac{\frac{1}{2}\left(2x-\sqrt{2}\right)+\frac{\sqrt{2}}{2}}{x^2-\sqrt{2}x+1}dx=\frac{1}{2}\int\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx+\frac{1}{\sqrt{2}}\int\frac{1}{x^2-\sqrt{2}x+1}dx$$
$$=\frac{1}{2}\int\frac{1}{x^2-\sqrt{2}x+1}d\left(x^2-\sqrt{2}x+1\right)+\frac{1}{\sqrt{2}}\int\frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}dx$$
$$=\frac{1}{2}\ln\left(x^2-\sqrt{2}x+1\right)+\arctan\left(\sqrt{2}x-1\right)+C.$$

In a similar way, the second antiderivative is

$$\int \frac{x}{x^2+\sqrt{2}x+1}dx = \int \frac{\frac{1}{2}(2x+\sqrt{2})-\frac{\sqrt{2}}{2}}{x^2+\sqrt{2}x+1}dx = \frac{1}{2}\int \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx - \frac{1}{\sqrt{2}}\int \frac{1}{x^2+\sqrt{2}x+1}dx$$
$$= \frac{1}{2}\int \frac{1}{x^2+\sqrt{2}x+1}d(x^2+\sqrt{2}x+1) - \frac{1}{\sqrt{2}}\int \frac{1}{(x+1/\sqrt{2})^2+(1/\sqrt{2})^2}dx$$
$$= \frac{1}{2}\ln(x^2+\sqrt{2}x+1) - \arctan(\sqrt{2}x+1) + C.$$

Therefore

$$\int \frac{x^2}{x^4+1}dx = \frac{1}{4\sqrt{2}}\ln\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} + \frac{1}{2\sqrt{2}}\left(\arctan(\sqrt{2}x-1)+\arctan(\sqrt{2}x+1)\right) + C.$$

(b) Let $u=\sqrt{\tan x}$. Then $u^2=\tan x$, so $2u\ du = \sec^2 x\,dx = (u^4+1)dx$, i.e. $dx = \frac{2u}{u^4+1}du$. Thus,

$$\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\,dx = \int_0^1 u\cdot\frac{2u}{u^4+1}du = \int_0^1 \frac{2u^2}{u^4+1}du$$
$$= \left[\frac{1}{2\sqrt{2}}\ln\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} + \frac{1}{\sqrt{2}}\left(\arctan(\sqrt{2}x-1)+\arctan(\sqrt{2}x+1)\right)\right]_0^1$$
$$= \frac{1}{2\sqrt{2}}\ln\frac{2-\sqrt{2}}{2+\sqrt{2}} + \frac{1}{\sqrt{2}}\underbrace{\left(\arctan(\sqrt{2}-1)+\arctan(\sqrt{2}+1)\right)}_{=\pi/2} = \frac{1}{2\sqrt{2}}\left(\pi+\ln\frac{2-\sqrt{2}}{2+\sqrt{2}}\right).$$

4. Note that the function $\frac{1}{(1+x^2)\sqrt{1-x^2}}$ has domain $(-1,1)$. Let $u=\sqrt{\frac{1-x}{1+x}}$. Then we have $u^2=\frac{1-x}{1+x}=\frac{2}{1+x}-1$, so $x=\frac{2}{u^2+1}-1$ and $dx=-\frac{4u}{(u^2+1)^2}du$. Since $1+x>0$, we have $\sqrt{1-x^2}=(1+x)\sqrt{\frac{1-x}{1+x}}=\frac{2u}{u^2+1}$. Thus,

$$\int \frac{1}{(1+x^2)\sqrt{1-x^2}}dx = \int \frac{1}{\left(1+\left(\frac{2}{u^2+1}-1\right)^2\right)\frac{2u}{u^2+1}}\frac{-4u}{(u^2+1)^2}du = -\int \frac{u^2+1}{u^4+1}du$$
$$= -\frac{1}{2}\left(\int \frac{1}{u^2-\sqrt{2}u+1}du + \int \frac{1}{u^2+\sqrt{2}u+1}du\right)$$
$$= -\frac{1}{\sqrt{2}}\left(\arctan(\sqrt{2}u-1)+\arctan(\sqrt{2}u+1)\right)+C$$
$$= -\frac{1}{\sqrt{2}}\arctan\frac{\sqrt{2}u}{1-u^2}+C = \frac{1}{\sqrt{2}}\arctan\frac{\sqrt{2}x}{\sqrt{1-x^2}}+C$$

$$\tan(a+b)=\frac{\tan a+\tan b}{1-\tan a\tan b}$$

5. (a) Let $f(x)=x^3+3x+1$. Then $f:\mathbb{R}\to\mathbb{R}$ is continuous since it is a polynomial. Since

$$f(-1)=-3<0 \quad\text{and}\quad f(0)=1>0,$$

f has a root in $(-1,0)$ by Intermediate Value Theorem. But since $f'(x)=3x^2+3>0$ for every $x\in\mathbb{R}$, it follows that f is strictly increasing on $\mathbb{R}$. In particular f is one-to-one, so it does not have any other real root.

(b) Let r be the only real root of x^3+3x+1. Then $(x-r)$ is a factor of the polynomial, so by long division we obtain the factorization $x^3+3x+1=(x-r)\left(x^2+rx+(r^2+3)\right)$. Now find a partial fraction decomposition of $\frac{1}{x^3+3x+1}$. Suppose that

$$\frac{1}{x^3+3x+1}=\frac{a}{x-r}+\frac{bx+c}{x^2+rx+(r^2+3)}=\frac{a\left(x^2+rx+(r^2+3)\right)+(bx+c)(x-r)}{x^3+3x+1}.$$

Then we obtain the polynomial identity $a\left(x^2+rx+(r^2+3)\right)+(bx+c)(x-r)=1$.

- Putting $x=r$ in the identity we obtain $a=\frac{1}{3(r^2+1)}$;
- Comparing the coefficient of x^2 on both sides of the identity, we have $a+b=0$, so $b=-\frac{1}{3(r^2+1)}$.
- Comparing the constant term on both sides, we have $(r^2+3)a-rc=1$, so $c=-\frac{2r}{3(r^2+1)}$.

Thus we have

$$\int \frac{1}{x^3+3x+1}\,dx$$

$$=\frac{1}{3(r^2+1)}\int\left(\frac{1}{x-r}-\frac{x+2r}{x^2+rx+(r^2+3)}\right)dx=\frac{1}{3(r^2+1)}\int\left(\frac{1}{x-r}-\frac{\frac{1}{2}(2x+r)+\frac{3r}{2}}{x^2+rx+(r^2+3)}\right)dx$$

$$=\frac{1}{3(r^2+1)}\int\frac{1}{x-r}\,dx-\frac{1}{6(r^2+1)}\int\frac{1}{x^2+rx+(r^2+3)}\,d\left(x^2+rx+(r^2+3)\right)$$

$$-\frac{r}{2(r^2+1)}\int\frac{1}{\left(x+\frac{r}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\sqrt{r^2+4}\right)^2}\,dx$$

$$=\frac{1}{3(r^2+1)}\ln|x-r|-\frac{1}{6(r^2+1)}\ln\left(x^2+rx+(r^2+3)\right)-\frac{r}{\sqrt{3}(r^2+1)\sqrt{r^2+4}}\arctan\frac{2x+r}{\sqrt{3}\sqrt{r^2+4}}+C.$$

6. (a) Let $t=\tan x$. Then

$$\int\frac{1}{1+2\sin x\cos x+\cos^2 x}\,dx=\int\frac{1}{1+2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)+\left(\frac{1}{\sqrt{1+t^2}}\right)^2}\frac{1}{1+t^2}\,dt$$

$$=\int\frac{1}{t^2+2t+2}\,dt=\int\frac{1}{(t+1)^2+1}\,dt=\arctan(t+1)+C$$

$$=\arctan(1+\tan x)+C=x+\arctan\left(\frac{1+\cos 2x}{2+\sin 2x}\right)+C.$$

(b) Let $t=\tan\frac{x}{2}$. Then

$$\int\frac{1}{3\sin x+4\cos x}\,dx=\int\frac{1}{3\left(\frac{2t}{1+t^2}\right)+4\left(\frac{1-t^2}{1+t^2}\right)}\frac{2}{1+t^2}\,dt=\int\frac{1}{2+3t-2t^2}\,dt$$

$$=\frac{1}{5}\int\left(\frac{2}{2t+1}-\frac{1}{t-2}\right)dt=\frac{1}{5}\ln|2t+1|-\frac{1}{5}\ln|t-2|+C$$

$$=\frac{1}{5}\ln\left|\frac{2\tan(x/2)+1}{\tan(x/2)-2}\right|+C.$$

<u>*Alternative solution*</u>:

$$\int \frac{1}{3\sin x + 4\cos x}\,dx = \int \frac{1}{5\sin\left(x + \arccos\frac{3}{5}\right)}\,dx = \frac{1}{5}\int \csc\left(x + \arccos\frac{3}{5}\right)dx$$

$$= -\frac{1}{5}\ln\left|\csc\left(x + \arccos\frac{3}{5}\right) + \cot\left(x + \arccos\frac{3}{5}\right)\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{3\sin x + 4\cos x}{3\cos x - 4\sin x + 5}\right| + C$$

(c) First, we have

$$\int \frac{1 + \sin x - 3\cos x}{2 + 2\sin x - \cos x}\,dx = \int \frac{(2 + 2\sin x - \cos x) - (2\cos x + \sin x) - 1}{2 + 2\sin x - \cos x}\,dx$$

$$= \int 1dx - \int \frac{1}{2 + 2\sin x - \cos x}\,d(2 + 2\sin x - \cos x) - \int \frac{1}{2 + 2\sin x - \cos x}\,dx$$

$$= x - \ln|2 + 2\sin x - \cos x| - \int \frac{1}{2 + 2\sin x - \cos x}\,dx\,.$$

To handle the last antiderivative, we let $t = \tan\frac{x}{2}$. Then

$$\int \frac{1}{2 + 2\sin x - \cos x}\,dx = \int \frac{1}{2 + 2\left(\frac{2t}{1+t^2}\right) - \frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt = \int \frac{2}{3t^2 + 4t + 1}\,dt$$

$$= \int \left(\frac{3}{3t+1} - \frac{1}{t+1}\right)dt = \ln|3t+1| - \ln|t+1| + C = \ln\left|\frac{3\tan(x/2) + 1}{\tan(x/2) + 1}\right| + C.$$

Therefore,

$$\int \frac{1 + \sin x - 3\cos x}{2 + 2\sin x - \cos x}\,dx = x - \ln|2 + 2\sin x - \cos x| - \ln\left|\frac{3\tan(x/2) + 1}{\tan(x/2) + 1}\right| + C.$$

7. Let $t = \tan\frac{x}{2}$. Then

$$\int \frac{1}{1 - a\sin x}\,dx = \int \frac{1}{1 - a\left(\frac{2t}{1+t^2}\right)}\frac{2}{1+t^2}\,dt = \int \frac{2}{t^2 - 2at + 1}\,dt.$$

(a) If $0 < a < 1$, then

$$\int \frac{2}{t^2 - 2at + 1}\,dt = \int \frac{2}{(t-a)^2 + (1-a^2)}\,dt = \frac{2}{\sqrt{1-a^2}}\arctan\frac{t-a}{\sqrt{1-a^2}} + C,$$

so

$$\int \frac{1}{1 - a\sin x}\,dx = \frac{2}{\sqrt{1-a^2}}\arctan\frac{\tan\frac{x}{2} - a}{\sqrt{1-a^2}} + C.$$

(b) If $a = 1$, then

$$\int \frac{2}{t^2 - 2at + 1}\,dt = \int \frac{2}{(t-1)^2}\,dt = -\frac{2}{t-1} + C,$$

so

$$\int \frac{1}{1 - a\sin x}\,dx = \frac{2}{1 - \tan(x/2)} + C.$$

(c) If $a > 1$, then

$$\int \frac{2}{t^2 - 2at + 1}dt = \int \frac{2}{\left(t - a - \sqrt{a^2-1}\right)\left(t - a + \sqrt{a^2-1}\right)}dt$$

$$= \frac{1}{\sqrt{a^2-1}}\int\left(\frac{1}{t-a-\sqrt{a^2-1}} - \frac{1}{t-a+\sqrt{a^2-1}}\right)dt$$

$$= \frac{1}{\sqrt{a^2-1}}\ln\left|\frac{t-a-\sqrt{a^2-1}}{t-a+\sqrt{a^2-1}}\right| + C,$$

so

$$\int \frac{1}{1 - a\sin x}dx = \frac{1}{\sqrt{a^2-1}}\ln\left|\frac{\tan(x/2)-a-\sqrt{a^2-1}}{\tan(x/2)-a+\sqrt{a^2-1}}\right| + C.$$

8. Let $t = \tan\frac{x}{2}$. Then

$$\int e^x\frac{1+\sin x}{1+\cos x}dx = \int e^{2\arctan t}\frac{1+\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt = \int e^{2\arctan t}\left(1+\frac{2t}{1+t^2}\right)dt$$

$$= \int e^{2\arctan t}dt + \int e^{2\arctan t}\frac{2t}{1+t^2}dt$$

$$= \left(te^{2\arctan t} - \int e^{2\arctan t}\frac{2t}{1+t^2}dt\right) + \int e^{2\arctan t}\frac{2t}{1+t^2}dt$$

$$= te^{2\arctan t} + C = e^x\tan(x/2) + C,$$

where C is an arbitrary constant.

9. (a) For each $t > 0$, we have

$$\int_0^t \frac{3-5x}{(1+x)(1-x+2x^2)}dx = \int_0^t\left(\frac{2}{x+1} - \frac{4x-1}{2x^2-x+1}\right)dx = [2\ln|x+1| - \ln(2x^2-x+1)]_0^t$$

$$= \ln\frac{(t+1)^2}{2t^2-t+1}.$$

Taking limit as $t \to +\infty$, we have

$\ln$ is continuous at $\frac{1}{2}$.

$$\int_0^{+\infty}\frac{3-5x}{(1+x)(1-x+2x^2)}dx = \lim_{t\to+\infty}\ln\frac{(t+1)^2}{2t^2-t+1} = \ln\lim_{t\to+\infty}\frac{\left(1+\frac{1}{t}\right)^2}{2-\frac{1}{t}+\frac{1}{t^2}} = \ln\frac{1}{2} = -\ln 2.$$

(b) For each $s \in (1,2)$, we have

$$\int_s^2 \frac{1}{x^2-1}dx = \frac{1}{2}\int_s^2\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx = \frac{1}{2}\left[\ln\left|\frac{x-1}{x+1}\right|\right]_s^2 = \frac{1}{2}\left(\ln\frac{1}{3} - \ln\frac{s-1}{s+1}\right).$$

Taking limit as $s \to 1^+$, we have

$$\lim_{s\to1^+}\int_s^2\frac{1}{x^2-1}dx = +\infty.$$

So the improper integral $\int_1^{+\infty}\frac{1}{x^2-1}dx$ diverges.

(c) First observe that the improper integral $\int_0^{\pi/2} \ln x\,dx$ converges because $\int_s^{\pi/2} \ln x\,dx = [x\ln x - x]_s^{\pi/2}$ has a finite limit as $s \to 0^+$. Now by l'Hôpital's rule, the limit

$$\lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln x} = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{1/x} = \left(\lim_{x\to 0^+} \frac{x}{\sin x}\right)\left(\lim_{x\to 0^+} \cos x\right) = 1 \cdot 1 = 1$$

exists and is finite, so the improper integral $I = \int_0^{\pi/2} \ln(\sin x)\,dx$ also converges by the limit comparison test. Now it remains to compute the value of the improper integral. With a substitution $u = \frac{\pi}{2} - x$, we have

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} \ln(\sin x)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\sin x)\,dx = \int_0^{\frac{\pi}{4}} \ln(\sin x)\,dx - \int_{\frac{\pi}{4}}^{0} \ln\left(\sin\left(\frac{\pi}{2} - u\right)\right)du \\ &= \int_0^{\frac{\pi}{4}} \ln(\sin x)\,dx + \int_0^{\frac{\pi}{4}} \ln(\cos u)\,du = \int_0^{\frac{\pi}{4}} (\ln(\sin x) + \ln(\cos x))dx \\ &= \int_0^{\frac{\pi}{4}} \ln(\sin x\cos x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{1}{2}\sin 2x\right)dx = \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,dx - \int_0^{\frac{\pi}{4}} \ln 2\,dx \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin v)\,dv - \frac{\pi}{4}\ln 2 = \frac{I}{2} - \frac{\pi}{4}\ln 2\,. \end{aligned}$$

Rearranging, we get $I = 2\left(-\frac{\pi}{4}\ln 2\right) = -\frac{\pi}{2}\ln 2$.

10. (a) For each $x \in (0, +\infty)$, we have

$$f'(x) = \frac{d}{dx}\int_1^{\sqrt{x}} e^{-t^2}dt = \frac{d\sqrt{x}}{dx} \cdot \frac{d}{d\sqrt{x}}\int_1^{\sqrt{x}} e^{-t^2}dt = \frac{1}{2\sqrt{x}} \cdot e^{-(\sqrt{x})^2} = \frac{1}{2\sqrt{x}}e^{-x}.$$

(b) The required integral is improper at 0. For each $s > 0$, we have

$$\int_s^1 \frac{f(x)}{\sqrt{x}}dx = 2\int_s^1 f(x)d\sqrt{x} = \left[2\sqrt{x}f(x)\right]_s^1 - 2\int_s^1 \sqrt{x}f'(x)dx = \left[2\sqrt{x}f(x)\right]_s^1 - \int_s^1 e^{-x}dx\,.$$

Now l'Hôpital's rule and (a) together give

$$\lim_{s\to 0^+} \sqrt{s}f(s) = \lim_{s\to 0^+} \frac{f(s)}{s^{-\frac{1}{2}}} = \lim_{s\to 0^+} \frac{f'(s)}{-\frac{1}{2}s^{-\frac{3}{2}}} = \lim_{s\to 0^+} \frac{\frac{1}{2\sqrt{s}}e^{-s}}{-\frac{1}{2}s^{-3/2}} = \lim_{s\to 0^+} se^{-s} = 0;$$

so taking limit as $s \to 0^+$, we have

$$\int_0^1 \frac{f(x)}{\sqrt{x}}dx = \lim_{s\to 0^+} \int_s^1 \frac{f(x)}{\sqrt{x}}dx = 2\underbrace{f(1)}_{=0} - 2\underbrace{\lim_{s\to 0^+} \sqrt{s}f(s)}_{=0} - \int_0^1 e^{-x}dx = [e^{-x}]_0^1 = \frac{1}{e} - 1.$$

11. For each $s > 0$, we have

$$\begin{aligned} \int_0^s \left(\frac{1}{\sqrt{x^2+1}} + \frac{a}{x+1}\right)dx &= \int_0^{\arctan s} \frac{1}{\sqrt{\sec^2 t}}\sec^2 t\,dt + a\int_0^s \frac{1}{x+1}dx \\ &= \int_0^{\arctan s} \sec t\,dt + a\int_0^s \frac{1}{x+1}dx \\ &= \ln\left(s + \sqrt{s^2+1}\right) + a\ln(s+1) \\ &= \ln\left[\left(s + \sqrt{s^2+1}\right)(s+1)^a\right]. \end{aligned}$$

In order that the improper integral converges, we require that $\lim_{s\to+\infty} \ln\left[\left(s+\sqrt{s^2+1}\right)(s+1)^a\right]$ is a finite real number, or equivalently $\lim_{s\to+\infty}\left(s+\sqrt{s^2+1}\right)(s+1)^a$ is a **positive** real number. This happens only if $a=-1$, in which case we have

$$\lim_{s\to+\infty}\left(s+\sqrt{s^2+1}\right)(s+1)^{-1} = \lim_{s\to+\infty}\frac{1+\sqrt{1+1/s^2}}{1+1/s} = 2$$

and since $\ln$ is continuous at 2, we obtain

$$\int_0^{+\infty}\left(\frac{1}{\sqrt{x^2+1}}-\frac{1}{x+1}\right)dx = \lim_{s\to+\infty}\ln\left[\left(s+\sqrt{s^2+1}\right)(s+1)^a\right] = \ln 2.$$

12. (a) For every $s\in(0,1)$, we have

$$\int_s^1 \ln x\,dx = [x\ln x - x]_s^1 = (s-1) - s\ln s\,.$$

Taking limit as $s\to 0^+$ we have

$$\lim_{s\to0^+} s\ln s = \lim_{s\to0^+}\frac{\ln s}{1/s} = \lim_{s\to0^+}\frac{1/s}{-1/s^2} = \lim_{s\to0^+} -s = 0$$

by l'Hôpital's rule; so the improper integral $\int_0^1 \ln x\,dx$ converges and

$$\int_0^1 \ln x\,dx = \lim_{s\to0^+}(s-1-s\ln s) = 0-1-0 = -1.$$

Now taking antiderivative by parts we have

$$\int (\ln x)^n dx = x(\ln x)^n - \int x\cdot n(\ln x)^{n-1}\frac{1}{x}dx = x(\ln x)^n - n\int(\ln x)^{n-1}dx.$$

Since for every non-negative integer n we have (cf. Example 4.38 (b))

$$\lim_{s\to0^+} s(\ln s)^n = \lim_{s\to0^+}\frac{(\ln s)^n}{\frac{1}{s}} = \lim_{s\to0^+}\frac{n(\ln s)^{n-1}\frac{1}{s}}{-\frac{1}{s^2}} = (-n)\lim_{s\to0^+}\frac{(\ln s)^{n-1}}{\frac{1}{s}} = \cdots = (-1)^{n-1}n!\lim_{s\to0^+} s\ln s = 0,$$

it follows that the improper integral $\int_0^1(\ln x)^n dx$ converges and

$$\int_0^1(\ln x)^n dx = \underbrace{[x(\ln x)^n]_0^1}_{=0} - n\int_0^1(\ln x)^{n-1}dx = (-n)\int_0^1(\ln x)^{n-1}dx$$

$$= \cdots = (-1)^{n-1}n!\underbrace{\int_0^1 \ln x\,dx}_{=-1} = (-1)^n n!.$$

(b) Let $u = x^\alpha$. Then $du = \alpha x^{\alpha-1}dx$, so

$$\int_t^1 x^{\alpha-1}(\ln x)^n dx = \frac{1}{\alpha}\int_{t^\alpha}^1\left(\ln u^{\frac{1}{\alpha}}\right)^n du = \frac{1}{\alpha}\int_{t^\alpha}^1\left(\frac{1}{\alpha}\ln u\right)^n du = \frac{1}{\alpha^{n+1}}\int_{t^\alpha}^1(\ln u)^n du = \frac{1}{\alpha^{n+1}}\int_{t^\alpha}^1(\ln x)^n dx.$$

Taking limit as $t\to0^+$, we have

$$\int_0^1 x^{\alpha-1}(\ln x)^n dx = \frac{1}{\alpha^{n+1}}\lim_{t\to0^+}\int_{t^\alpha}^1(\ln x)^n dx = \frac{1}{\alpha^{n+1}}\int_0^1(\ln x)^n dx = \frac{(-1)^n n!}{\alpha^{n+1}},$$

so the improper integral $\int_0^1 x^{\alpha-1}(\ln x)^n dx$ converges.

13. (a) If $|f(t)| \le Me^{at}$ for every $t \ge 0$ and if $s > a$, then we have

$$\int_0^{+\infty} e^{-st}|f(t)|dt \le \int_0^{+\infty} e^{-st}Me^{at}dt = M\int_0^{+\infty} e^{(a-s)t}dt = M\left[\frac{e^{(a-s)t}}{a-s}\right]_0^{+\infty} = \frac{M}{s-a},$$

so the improper integral $\int_0^{+\infty} e^{-st}|f(t)|dt$ converges. Hence $\int_0^{+\infty} e^{-st}f(t)dt$ also converges by absolute convergence test.

(b) (i) Note that $|1| \le e^{0t}$ for every $t \ge 0$, so by (a) $F(s)$ exists for $s > 0$. For every $s > 0$, we have

$$F(s) = \int_0^{+\infty} e^{-st} \cdot 1dt = \frac{1}{s}.$$

(ii) Note that $|e^t| \le e^{1t}$ for every $t \ge 0$, so by (a) $F(s)$ exists for $s > 1$. For every $s > 1$, we have

$$F(s) = \int_0^{+\infty} e^{-st}e^t dt = \frac{1}{s-1}.$$

(iii) Note that for each $a > 0$, we have $|t^2| \le e^{at}$ for every $t \ge 0$, so by (a) $F(s)$ exists for $s > a$; in other words, $F(s)$ exists for $s > 0$. For every $s > 0$, we have

$$F(s) = \int_0^{+\infty} e^{-st}t^2 dt = \lim_{r\to+\infty}\left(\left[-\frac{t^2e^{-st}}{s}\right]_0^r + \int_0^r \frac{e^{-st}}{s}2tdt\right) = \frac{2}{s}\int_0^{+\infty} te^{-st}dt$$
$$= \frac{2}{s}\lim_{r\to+\infty}\left(\left[-\frac{te^{-st}}{s}\right]_0^r + \int_0^r \frac{e^{-st}}{s}dt\right) = \frac{2}{s}\cdot\frac{1}{s}\int_0^{+\infty} e^{-st}dt = \frac{2}{s^3}.$$

(iv) Note that $|\cos t| \le e^{0t}$ for every $t \ge 0$, so by (a) $F(s)$ exists for $s > 0$. For every $s > 0$, we have

$$F(s) = \int_0^{+\infty} e^{-st}\cos t\, dt = \lim_{r\to+\infty}\left([e^{-st}\sin t]_0^r - \int_0^r \sin t\,(-se^{-st})dt\right) = s\int_0^{+\infty} e^{-st}\sin t\, dt$$
$$= s\lim_{r\to+\infty}\left([-e^{-st}\cos t]_0^r + \int_0^r \cos t\,(-se^{-st})dt\right) = s\left(1 - s\int_0^{+\infty} e^{-st}\cos t\, dt\right) = s - s^2F(s).$$

Rearranging, we get $F(s) = \frac{s}{s^2+1}$.

(c) For every $s > a$, both $F(s)$ and $G(s)$ exist according to (a); and integration by parts gives

$$G(s) = \int_0^{+\infty} e^{-st}f'(t)dt = \lim_{r\to+\infty}\left([e^{-st}f(t)]_0^r - \int_0^r -se^{-st}f(t)dt\right) = -f(0) + s\int_0^{+\infty} e^{-st}f(t)dt$$
$$= sF(s) - f(0).$$

14. (a) Let n be a positive integer. Applying l'Hôpital's rule successively for $(n-1)$ times we have

$$\lim_{t\to+\infty}\frac{t^{n-1}}{e^{\frac{1}{2}t}} = \lim_{t\to+\infty}\frac{(n-1)t^{n-2}}{(1/2)e^{\frac{1}{2}t}} = \cdots = \lim_{t\to+\infty}\frac{(n-1)!}{(1/2)^{n-1}e^{\frac{1}{2}t}} = 0;$$

so we have $0 \le t^{n-1} \le e^{\frac{1}{2}t}$ for every sufficiently large $t > 0$, which implies that

$$0 \le t^{n-1}e^{-t} \le e^{\frac{1}{2}t}e^{-t} = e^{-\frac{1}{2}t} \qquad \text{for every sufficiently large } t > 0.$$

Since $\int_0^{+\infty} e^{-\frac{1}{2}t}dt = \left[-2e^{-\frac{1}{2}t}\right]_0^{+\infty} = 2$ converges, $f(n) = \int_0^{+\infty} t^{n-1}e^{-t}dt$ also converges by comparison test.

Now for each $x \ge 1$, there exists a positive integer $n = \lfloor x \rfloor$ such that $n \le x < n+1$. Thus we have

$$0 \le t^{x-1}e^{-t} \le t^n e^{-t} \qquad \text{for every } t > 0.$$

Since $\int_0^{+\infty} t^n e^{-t}dt$ converges as proven, $f(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt$ also converges by comparison test. ∎

(b) Now for $x \in (0,1)$, we analyze the improper integrals $\int_1^{+\infty} t^{x-1}e^{-t}dt$ and $\int_0^1 t^{x-1}e^{-t}dt$ separately.

- ⊙ Since $\lim_{t\to+\infty} \frac{t^{x-1}}{e^{\frac{1}{2}t}} = 0$, we have $0 \le t^{x-1} \le e^{\frac{1}{2}t}$ for every sufficiently large $t \ge 1$, which implies that

$$0 \le t^{x-1}e^{-t} \le e^{-\frac{1}{2}t} \qquad \text{for every sufficiently large } t \ge 1.$$

Since $\int_1^{+\infty} e^{-\frac{1}{2}t}dt$ converges, $\int_1^{+\infty} t^{x-1}e^{-t}dt$ also converges by comparison test.

- ⊙ On the other hand, we have

$$0 \le t^{x-1}e^{-t} \le t^{x-1} \qquad \text{for every } t \in (0,1].$$

Since $\int_0^1 t^{x-1}dt = \left[\frac{1}{x}t^x\right]_0^1 = \frac{1}{x}$ converges, $\int_0^1 t^{x-1}e^{-t}dt$ also converges by comparison test.

Consequently, $f(x) = \int_0^1 t^{x-1}e^{-t}dt + \int_1^{+\infty} t^{x-1}e^{-t}dt$ converges also. ∎

(c) For each $x > 0$, integration by parts gives

$$f(x+1) = \int_0^{+\infty} t^x e^{-t}dt = \underbrace{\left[-t^x e^{-t}\right]_0^{+\infty}}_{=0} + \int_0^{+\infty} x t^{x-1}e^{-t}dt = x\int_0^{+\infty} t^{x-1}e^{-t}dt = xf(x).$$

- ⊙ For $n = 1$, we have $f(n) = f(1) = \int_0^{+\infty} e^{-t}dt = 1 = 0!$.
- ⊙ Suppose that $f(k) = (k-1)!$ for some positive integer k. Then

$$f(k+1) = kf(k) = k(k-1)! = k!$$

Hence by mathematical induction, we have $f(n) = (n-1)!$ for every positive integer n. ∎

Remark: The function f in this problem is called the ***Gamma function***. It generalizes the "factorial" to the context of real numbers greater than -1.

15. (a) Since $\cos x \ge -1$ for every $x \in \mathbb{R}$, we have

$$\frac{2+\cos x}{\sqrt{x+5}} \ge \frac{1}{\sqrt{x+5}} > 0 \qquad \text{for every } x \ge 1.$$

Since $\int_1^{+\infty} \frac{1}{\sqrt{x+5}}dx = \int_6^{+\infty} \frac{1}{\sqrt{u}}du$ diverges by p-test, the improper integral $\int_1^{+\infty} \frac{2+\cos x}{\sqrt{x+5}}dx$ also diverges by comparison test.

(b) We have

$$\lim_{x\to 0^+} \frac{\frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}}}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}} = \lim_{x\to 0^+} \frac{1}{\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{x^{3/2}}}}} = 0.$$

Since $\int_0^1 \frac{1}{\sqrt{x}}dx = \left[2\sqrt{x}\right]_0^1 = 2$ converges, the improper integral $\int_0^1 \frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}}dx$ also converges by limit comparison test.

(c) Since $0 \le \sin^2 x \le 1$ for every $x \in \mathbb{R}$, we have

$$\frac{x}{1+x^2\sin^2 x} \ge \frac{x}{1+x^2} \ge 0 \qquad \text{for every } x \ge 0.$$

Since $\lim\limits_{t\to+\infty}\int_0^t \frac{x}{1+x^2}dx = \lim\limits_{t\to+\infty}\frac{1}{2}\ln(1+t^2) = +\infty$, the improper integral $\int_0^{+\infty}\frac{x}{1+x^2}dx$ diverges; therefore the improper integral $\int_0^{+\infty}\frac{x}{1+x^2\sin^2 x}dx$ also diverges by comparison test.

(d) Since $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ is finite, the integrand can be extended to become a continuous function at 0. So $\int_0^1 \frac{\sin x}{x}dx$ exists. On the other hand, for every $t > 1$ we have

$$\int_1^t \frac{\sin x}{x}dx = \left[-\frac{\cos x}{x}\right]_1^t - \int_1^t \frac{\cos x}{x^2}dx = \frac{\cos 1}{1} - \frac{\cos t}{t} - \int_1^t \frac{\cos x}{x^2}dx\,.$$

Now

- Since $-\frac{1}{t} \le \frac{\cos t}{t} \le \frac{1}{t}$ for every $t > 0$ and $\lim\limits_{t\to+\infty} -\frac{1}{t} = \lim\limits_{t\to+\infty}\frac{1}{t} = 0$, $\lim\limits_{t\to+\infty}\frac{\cos t}{t} = 0$ by squeeze theorem.
- We have $\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$ for every $x > 0$. Since $\int_1^{+\infty}\frac{1}{x^2}dx$ converges by p-test, we have $\int_1^{+\infty}\left|\frac{\cos x}{x^2}\right|dx$ converges by comparison test. Therefore $\int_1^{+\infty}\frac{\cos x}{x^2}dx$ converges by absolute convergence test.

Therefore $\int_1^{+\infty}\frac{\sin x}{x}dx = \frac{\cos 1}{1} - \int_1^{+\infty}\frac{\cos x}{x^2}dx$ converges. Finally $\int_0^{+\infty}\frac{\sin x}{x}dx = \int_0^1\frac{\sin x}{x}dx + \int_1^{+\infty}\frac{\sin x}{x}dx$ also converges.

16. For every $t > 0$, we have

$$\left(\int_0^t |f(x)g(x)|dx\right)^2 \le \left(\int_0^t f(x)^2dx\right)\left(\int_0^t g(x)^2dx\right)$$

according to Cauchy-Schwarz inequality. Since both the improper integrals $\int_0^{+\infty} f(x)^2dx$ and $\int_0^{+\infty} g(x)^2dx$ converge, it follows that $\int_0^{+\infty}|f(x)g(x)|dx$ converges too. Hence $\int_0^{+\infty} f(x)g(x)dx$ converges by absolute convergence test. Finally,

$$\int_0^{+\infty}\left(f(x)+g(x)\right)^2dx = \int_0^{+\infty} f(x)^2dx + 2\int_0^{+\infty} f(x)g(x)dx + \int_0^{+\infty} g(x)^2dx$$

also converges. ■

17. (a) Let $x > 0$. Since f is decreasing on $[0,+\infty)$, we have $f(t) \ge f(x)$ for every $t \in \left[\frac{x}{2}, x\right]$. Therefore

$$\int_{\frac{x}{2}}^x f(t)dt \ge \int_{\frac{x}{2}}^x f(x)dt = \frac{x}{2}f(x)$$

This implies that

$$0 \le xf(x) \le 2\int_{\frac{x}{2}}^x f(t)dt \qquad \text{for every } x > 0.$$

Now if $\int_0^{+\infty} f(x)dx$ converges, say $\int_0^{+\infty} f(x)dx = L \in \mathbb{R}$, then

$$\lim_{x\to+\infty} 2\int_{\frac{x}{2}}^x f(t)dt = 2\left(\lim_{x\to+\infty}\int_0^x f(t)dt - \lim_{x\to+\infty}\int_0^{\frac{x}{2}} f(t)dt\right) = 2(L-L) = 0.$$

Therefore by Squeeze Theorem we must also have $\lim\limits_{x\to+\infty} xf(x) = 0$. ■

(b) Let $f: [0, +\infty) \to [0, +\infty)$ be the function

$$f(x) = \frac{1}{(x+2)\ln(x+2)}.$$

Then f is continuous and decreasing, with $\lim_{x\to+\infty} xf(x) = \lim_{x\to+\infty} \frac{x}{(x+2)\ln(x+2)} = 0$; but the improper integral

$$\int_0^{+\infty} f(x)dx = \int_0^{+\infty} \frac{1}{(x+2)\ln(x+2)} dx = [\ln\ln(x+2)]_0^{+\infty}$$

diverges. (Other possible examples include $f(x) = \begin{cases} 1/e & \text{if } x \in [0, e] \\ \frac{1}{x\ln x} & \text{if } x \in [e, +\infty) \end{cases}$, etc.)

(c) If $\lim_{x\to+\infty} \frac{x}{\ln(g(e^x))} = 0$, then for every sufficiently large x we have $\frac{x}{\ln(g(e^x))} < \frac{1}{2}$, so

$$2x < \ln\big(g(e^x)\big).$$

Since the exponential function is strictly increasing, we also have

$$e^{2x} < e^{\ln(g(e^x))} = g(e^x).$$

Dividing both sides by the positive number $e^x g(e^x)$, we obtain

$$\frac{e^x}{g(e^x)} < e^{-x}.$$

(d) We prove the contrapositive. Suppose that $\int_1^{+\infty} \frac{1}{x\ln g(x)} dx$ converges. Then with the (implicit) substitution $x = e^t$ we have

$$\int_1^{+\infty} \frac{1}{x\ln g(x)} dx = \int_0^{+\infty} \frac{1}{e^t \ln g(e^t)} e^t dt = \int_0^{+\infty} \frac{1}{\ln g(e^t)} dt,$$

so $\int_0^{+\infty} \frac{1}{\ln g(e^t)} dt$ converges too. Now since g is increasing on $[1, +\infty)$, it follows that $\frac{1}{\ln g(e^t)}$ is decreasing on $[0, +\infty)$; so according to the result from (a) we have

$$\lim_{x\to+\infty} \frac{x}{\ln g(e^x)} = 0.$$

Now according to the result from (c), there exists $r > 0$ such that

$$\frac{e^x}{g(e^x)} < e^{-x} \qquad \text{for every } x > r.$$

Thus the improper integral

$$\int_0^{+\infty} \frac{e^t}{g(e^t)} dt = \int_0^r \frac{e^t}{g(e^t)} dt + \int_r^{+\infty} \frac{e^t}{g(e^t)} dt \le \int_0^r \frac{e^t}{g(e^t)} dt + \int_r^{+\infty} e^{-t} dt$$

converges by comparison test. Finally with the substitution $x = e^t$ we have

$$\int_0^{+\infty} \frac{e^t}{g(e^t)} dt = \int_1^{+\infty} \frac{1}{g(x)} dx,$$

so the improper integral $\int_1^{+\infty} \frac{1}{g(x)} dx$ also converges. ■