Problem Set 3

Note: The problem sets serve as additional exercise problems for your own practice.   Problem Set 3 covers materials from §6.1 – §6.3.

1.   (a)   Let m and n be non-negative integers.   Evaluate the following integrals, distinguishing all possible cases for m and n.

   (i) $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \, dx$ (ii) $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx \, dx$ (iii) $\displaystyle\int_{-\pi}^{\pi} \cos mx \sin nx \, dx$

   (b)   Let n be a positive integer and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by
   $$f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx,$$
   where $a_1, a_2, \ldots, a_n$ are real numbers.   Show that we must have
   $$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx \qquad \text{for each } k \in \{1, 2, \ldots, n\}.$$

2.   Evaluate the following antiderivatives.

   (a) $\displaystyle\int x^2 \arctan x \, dx$

   (b) $\displaystyle\int \sin(\ln x) \, dx$

   (c) $\displaystyle\int e^{2x} (\sin x + \cos x)^2 \, dx$

   (d) $\displaystyle\int (2x^2 + 1) e^{x^2} \, dx$

   Hint: In (d), first consider $\int e^{x^2} \, dx$.

3.   Evaluate the limit
   $$\lim_{n \to +\infty} \frac{1}{n} \sqrt[n]{\frac{(2n)!}{n!}}.$$

   Hint: Take natural logarithm.

4.   Let $a > 0$ and let $f : [-a, a] \to \mathbb{R}$ be an **odd** continuous function.   Show that
   $$\int_{-a}^{a} \left(\int_{-a}^{x} f(t) dt \right) dx = - \int_{-a}^{a} x f(x) dx.$$

5.   The following are "proofs" of some obviously false statements.   Point out what is wrong in each of these "proofs".

   (a)   A "proof" of the statement that "$\pi = 0$".

   > *Proof.*   By Fundamental Theorem of Calculus, we have $\int_{-1}^{1} \frac{1}{1+x^2} dx = [\arctan x]_{-1}^{1} = \frac{\pi}{2}$.
   >
   > Let $u = \frac{1}{x}$.   Then $du = -\frac{1}{x^2} dx$, $u = 1$ when $x = 1$ and $u = -1$ when $x = -1$; so
   > $$\frac{\pi}{2} = \int_{-1}^{1} \frac{1}{1+x^2} dx = \int_{-1}^{1} \frac{-1}{(1/x)^2 + 1} \left(-\frac{1}{x^2} dx \right) = \int_{-1}^{1} \frac{-1}{u^2 + 1} du = -\frac{\pi}{2}.$$
   > Adding $\frac{\pi}{2}$ on both sides we obtain $\pi = 0$.   ∎

(b) A "proof" of the statement that "**every integral equals zero**":

> *Proof.*   Let $f:[a,b] \to \mathbb{R}$ be any continuous function, and consider the integral $\int_a^b f(x)dx$.   We let $u = (x-a)(x-b)$.   Then $u = 0$ when $x = a$ and $u = 0$ when $x = b$, so the integral becomes
> $$\int_a^b f(x)dx = \int_0^0 (\text{something})du = 0$$
> according to the substitution rule.   ∎

(c) A "proof" of the statement that "$0 = 1$".

> *Proof.*   Let f be a differentiable function whose value is never zero, and consider $\int \frac{f'(x)}{f(x)}dx$.   Let $u = \frac{1}{f(x)}$ and $v = f(x)$.   Taking antiderivatives by parts, we obtain
> $$\int \underbrace{\frac{f'(x)}{f(x)}}dx = \int \underbrace{\frac{1}{f(x)}}_{u}\underbrace{f'(x)dx}_{dv} = \underbrace{\frac{1}{f(x)}}_{u}\underbrace{f(x)}_{v} - \int \underbrace{f(x)}_{v}\underbrace{\left[-\frac{1}{(f(x))^2}f'(x)\right]dx}_{du}$$
> $$= 1 + \int \frac{f'(x)}{f(x)}dx.$$
> Therefore subtracting $\int \frac{f'(x)}{f(x)}dx$ from both sides we obtain $0 = 1$.   ∎

6. Let f be a function which is continuously differentiable on $[0,1]$.

(a) For every $a, b \in [0,1]$, show that
$$\int_a^b (x-a)f'(x)dx = \int_a^b \big(f(b) - f(x)\big)dx.$$

(b) Let $n \geq k \geq 1$ be integers.   Using (a) and the generalized Mean Value Theorem for integrals (Example 5.47 (a)), show that there exists $\omega_k \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$ such that
$$\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left[f\left(\frac{k}{n}\right) - f(x)\right]dx = \frac{f'(\omega_k)}{2n^2}.$$

(c) For each $n \in \mathbb{N}$, we let $E_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x)dx$.   Show that
$$E_n = \sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left[f\left(\frac{k}{n}\right) - f(x)\right]dx.$$

Hence deduce using the result from (b) that
$$\lim_{n\to+\infty} nE_n = \frac{f(1) - f(0)}{2}.$$

7.　Let $f\colon [0, +\infty) \to \mathbb{R}$ be the function defined by $f(x) = xe^x$.

　　(a)　Show that f is strictly increasing.

　　(b)　Now f is one-to-one according to (a), so we let g be the **inverse** of f, i.e. $g = f^{-1}$.

　　　　(i)　Write down the domain of g.　Show that
$$g'(x) = \frac{1}{x + e^{g(x)}}$$
　　　　　for every x in the interior of the domain of g.

　　　　(ii)　Using the result from (b) (i) or otherwise, evaluate the antiderivative $\int g(x)\,dx$, expressing your answer in terms of g and other elementary functions only.

　　　　(iii)　Hence, or otherwise, evaluate the integral $\int_0^e g(x)\,dx$.

8.　(a)　Let n be a non-negative integer, and let $f\colon \mathbb{R} \to \mathbb{R}$ be the polynomial
$$f(x) = (x^2 - 1)^n.$$
　　　　(i)　Show that $(x^2 - 1)f'(x) - 2nxf(x) = 0$ for every $x \in \mathbb{R}$.

　　　　(ii)　Hence, show that
$$(x^2 - 1)f^{(n+2)}(x) + 2xf^{(n+1)}(x) - n(n+1)f^{(n)}(x) = 0$$
　　　　　for every $x \in \mathbb{R}$.

　　Hint:　Recall "Leibniz rule" in chapter 3.　Part (a) is almost the same as Example 3.69.

　　(b)　For each non-negative integer n, let $p_n\colon \mathbb{R} \to \mathbb{R}$ be the function
$$p_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n.$$
　　　　(i)　Using the result from (a) (ii), show that
$$\frac{d}{dx}[(x^2 - 1)p_n'(x)] = n(n+1)p_n(x)$$
　　　　　for every non-negative integer n.

　　　　(ii)　Hence deduce that if m and n are distinct non-negative integers, then
$$\int_{-1}^1 p_m(x)p_n(x)\,dx = 0.$$

9.　For each non-negative integer n, let
$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx \, dx.$$

　　(a)　For each positive integer n, show that
$$\frac{d}{dx}\cos^n x \sin nx = n\cos^{n-1} x \cos(n+1)x.$$
　　　Hence show that
$$I_n = \frac{1}{2}I_{n-1}.$$

　　(b)　Using the result from (a), find the value of I_n in terms of n.

10. For each non-negative integer n, let

$$I_n = \int_0^1 t^n e^t dt.$$

(a) Show that $\dfrac{1}{n+1} \leq I_n \leq \dfrac{e}{n+1}$ for every non-negative integer n.

(b) Express I_n in terms of I_{n-1} for each $n \geq 1$.   Hence show that

$$I_n = (-1)^{n+1} n! + e \sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!}.$$

(c) Using (a) and (b), deduce that e is an irrational number.

   Hint: There is no integer in the open interval $(0, 1)$.

11. (a) For every pair of non-negative integers m and n, let

$$B(m, n) = \int_0^1 x^m (1-x)^n dx.$$

Show that $B(m, n) = \dfrac{n}{m+1} B(m+1, n-1)$ for every pair of integers $m \geq 0$ and $n \geq 1$.

Hence or otherwise, deduce that $B(m, n) = \dfrac{m! n!}{(m+n+1)!}$.

(b) Show that

$$\int_0^1 \frac{x^4 (x-1)^4}{x^2 + 1} dx = \frac{22}{7} - \pi.$$

Using this together with the result from (a), deduce that

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}.$$

12. Let a be a positive real number.   Evaluate the following antiderivatives.

(a) $\displaystyle \int \frac{1}{\sqrt{x^2 + a^2}} dx$ (d) $\displaystyle \int \sqrt{x^2 + a^2}\, dx$

(b) $\displaystyle \int \frac{1}{\sqrt{x^2 - a^2}} dx$ (e) $\displaystyle \int \sqrt{x^2 - a^2}\, dx$

(c) $\displaystyle \int \frac{1}{\sqrt{a^2 - x^2}} dx$ (f) $\displaystyle \int \sqrt{a^2 - x^2}\, dx$

13. Evaluate the following antiderivatives, using trigonometric substitutions when appropriate.

(a) $\displaystyle \int \frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 - x^4}} dx$

(b) $\displaystyle \int \frac{x + 1}{(x^2 + x + 1)\sqrt{x^2 + x + 1}} dx$

(c) $\displaystyle \int \frac{x^2 - 1}{(x^2 + 1)\sqrt{1 + x^4}} dx$ (*Hint*: $x = \tan t$)

14. (a) For each non-negative integer n, let

$$I_n(x) = \int \frac{x^n}{\sqrt{x^2 + 1}} dx$$

   which is defined up to addition by a constant function.   Find a reduction formula that connects I_n and I_{n-2} for $n \geq 2$.

   (b) Hence evaluate

$$\int_0^1 \frac{x^5}{\sqrt{x^2 + 1}} dx.$$

15. (a) By considering a suitable function, show that

$$\tan x \leq \frac{4x}{\pi} \qquad \text{for every } x \in \left[0, \frac{\pi}{4}\right].$$

   (b) For each non-negative integer n, let

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx.$$

   (i) Using the result from (a), show that

$$I_n \leq \frac{\pi}{4(n+1)} \qquad \text{for every non-negative integer } n.$$

   (ii) Show that $I_n = \frac{1}{n-1} - I_{n-2}$ for every $n \geq 2$.   Hence show that

$$I_0 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{k-1}\frac{1}{2k-1} + (-1)^k I_{2k} \qquad \text{for every } k \in \mathbb{N}.$$

   (c) Using the results from (b), evaluate the limit

$$\lim_{k \to +\infty} \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{k-1}\frac{1}{2k-1}\right).$$

16. For each pair of real numbers m and n, let

$$I_{m,n}(x) = \int x^m (\ln x)^n dx$$

   be defined up to addition by a constant function.

   (a) Show that

$$I_{m,n}(x) = \frac{1}{m+1}x^{m+1}(\ln x)^n - \frac{n}{m+1}I_{m,n-1}(x)$$

   for every $m \in \mathbb{R} \setminus \{-1\}$ and every $n \in \mathbb{N}$.

   (b) Hence evaluate the antiderivative

$$\int \frac{(\ln x)^3}{x^4} dx.$$