Problem Set 6

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 6 covers materials from §7.7 – §8.2.

1. Find the arc-length of each of the following curves in $\mathbb{R}^2$ defined by the given equation.
   (a) $y = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}$, where $x \in [1, 4]$.
   (b) $x = \frac{3}{2}y^{\frac{2}{3}}$, where $y \in [1, 8]$.
   (c) $y = \ln\left(x + \sqrt{x^2 - 1}\right)$, where $x \in \left[1, \sqrt{2}\right]$.
   (d) $y = \int_e^x \sqrt{(\ln t)^2 - 1}\, dt$, where $x \in [e, e^3]$.
2. Find the length of the portion of the polar curve
$$r = 1 + \sin\theta$$
that lies within Quadrants I, II and III of $\mathbb{R}^2$.
3. Find the areas of the following regions in $\mathbb{R}^2$.
   (a) The region bounded by the graphs of
$$f(x) = x^3 - x \qquad \text{and} \qquad g(x) = 3x.$$
   (b) The region bounded by the graphs of
$$f(x) = 8 - 2x, \qquad g(x) = x - 4 \qquad \text{and} \qquad h(x) = x^2 - 6x + 8.$$
4. Compute the area of the region in $\mathbb{R}^2$ that is inside the curve with polar equation
$$r = 1 + \cos\theta$$
but outside the curve with polar equation
$$r = 3\cos\theta.$$
Hint: Sketch the required region first.
5. Let $f: [0, \pi] \to [0, +\infty)$ be a continuously differentiable function, and consider the curve in $\mathbb{R}^2$ defined by the polar equation
$$r = f(\theta).$$
Such a curve can be viewed as a polar curve on the one hand, and as a parametrized curve on the other hand. Show that the area bounded between the curve and the x-axis evaluated using Theorem 7.98 (as a polar curve) is the same as that evaluated using Theorem 7.96 (as a parametrized curve).
6. Find the volume of each of the following solids in $\mathbb{R}^3$.
   (a) The solid lying between the planes $x = 0$ and $x = 1$, whose cross sections perpendicular to the x-axis are disks with diameters in the xy-plane running from the parabola $y = x^2$ to the parabola $y = \sqrt{x}$.

(b) The solid whose base is the region in the plane bounded by the line $y = x$ and the parabola $y = \frac{x^2}{4}$, and whose cross sections perpendicular to the y-axis are equilateral triangles all pointing toward the same side.

(c) The solid lying between the planes $x = 0$ and $x = 6$, whose cross sections perpendicular to the x-axis are squares with an edge in the xy-plane running from the x-axis up to the curve $\sqrt{x} + \sqrt{y} = \sqrt{6}$.

7. Consider the region in the plane bounded by the graph of

$$f(x) = \frac{4}{x^3}$$

and the two lines $x = 1$ and $y = 1/2$. Find the volumes of the solids which are obtained by revolving this region

(a) about the x-axis,

(b) about the y-axis,

(c) about the line $x = 2$, and

(d) about the line $y = 4$

respectively.

8. Consider the region in the coordinate plane bounded by the curve

$$y = e^{-x},$$

the x- and y-axes, and the line $x = \ln 2$. Find the volume of the solid obtained by revolving this region about the line $x = \ln 2$.

9. A circular hole of radius $\sqrt{3}$ is bored through the center of a solid ball of radius 2. Find the volume of material removed from the ball.

10. Let a be a fixed real number and let $f\colon [a, +\infty) \to [0, +\infty)$ be a continuous function. For each $t > a$, we consider the region in the coordinate plane bounded by the graph of f, the x-axis and the lines $x = a$ and $x = t$. If the solid obtained by revolving such a region about the x-axis always has volume $t^2 - at$, find $f(x)$.

11. Let $a, b > 0$ and consider the ellipse in $\mathbb{R}^2$ defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(a) Set up an integral that represents the arc-length of the ellipse. You do not need to evaluate the integral.

(b) Set up two integrals that represent respectively the area of the surface obtained by revolving the ellipse

(i) about the x-axis;

(ii) about the y-axis.

You do not need to evaluate the integrals.

12. Find the area of the surface obtained by revolving each of the following plane curves about the given axes of revolution.

(a) $y = \frac{x^3}{3}$ for $x \in [0, 1]$; about the x-axis.

(b) $x = \sqrt{4y - y^2}$ for $y \in [1, 2]$; about the y-axis.

(c) $y = \frac{1}{2}\ln\left(2x + \sqrt{4x^2 - 1}\right)$ for $x \in \left[\frac{1}{2}, \frac{17}{16}\right]$; about the y-axis.

13. (a) Let $0 \leq a < b \leq \pi$ and let $f: [a, b] \to [0, +\infty)$ be a continuously differentiable function. Show that the surface obtained by revolving the polar curve $r = f(\theta)$ about the x-axis has surface area

$$S = \int_a^b 2\pi f(\theta) \sin\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta.$$

(b) Using (a), compute the area of the surface obtained by revolving the polar curve

$$r^2 = \cos 2\theta$$

about the x-axis.

(c) Compute the area of the surface obtained by revolving the polar curve

$$r^2 = \cos 2\theta$$

about the y-axis.

14. Let $f: [1, +\infty) \to [0, +\infty)$ be the function

$$f(x) = \frac{1}{x}$$

and consider its graph in the plane.

(a) Consider the (unbounded) region under the graph of f and above the x-axis. Show that the solid obtained by revolving this region about the x-axis has a finite volume.

(b) Show that the surface obtained by revolving the graph of f about the x-axis has an infinite surface area.

15. Let $\mathbf{r}: [0, 2\pi] \to \mathbb{R}^2$ be the curve defined by

$$\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle.$$

(a) Find the arc-length of this curve.

(b) Find the area of the region in $\mathbb{R}^2$ bounded by this curve.

(c) Find the volume of the solid obtained by revolving this curve about the x-axis.

(d) Find the area of the surface obtained by revolving this curve about the x-axis.

16. Consider the cardioid in $\mathbb{R}^2$ defined by the polar equation

$$r = 1 + \sin\theta.$$

(a) Find the volume of the solid obtained by revolving this curve about the y-axis.

(b) Find the area of the surface obtained by revolving this curve about the y-axis.

Hint: The given cardioid is symmetric about the y-axis. To generate a solid or a surface by revolving about the y-axis, we only need the **right-half** of the cardioid.

17. Determine whether each of the following series of real numbers converges or diverges. Also compute its limit (i.e. the sum) if it converges.

(a) $\displaystyle\sum_{k=1}^{+\infty}\left(3^{\frac{1}{k}}-3^{\frac{1}{k+1}}\right)$

(b) $\displaystyle\sum_{k=1}^{+\infty}\frac{2}{k(k+1)(k+2)}$

(c) $\displaystyle\sum_{k=1}^{+\infty}\ln\left(1+\frac{1}{k}\right)$

(d) $1+\underbrace{\frac{1}{2}+\frac{1}{2}}_{\text{2 terms}}+\underbrace{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}_{\text{4 terms}}+\underbrace{\frac{1}{8}+\frac{1}{8}+\cdots+\frac{1}{8}}_{\text{8 terms}}+\cdots$

(e) $\displaystyle\sum_{k=2}^{+\infty}\frac{k}{2^{k-1}}$

Hint: $\frac{d}{dx}x^k=kx^{k-1}$.

18. (a) Prove that $\sin^3 x=\frac{3}{4}\sin x-\frac{1}{4}\sin 3x$ for every $x\in\mathbb{R}$.

(b) Using (a) and mathematical induction, show that

$$\sum_{k=1}^{n}3^{k-1}\sin^3\frac{1}{3^k}=\frac{3^n}{4}\sin\frac{1}{3^n}-\frac{1}{4}\sin 1$$

for every positive integer n.

(c) Using (b) or otherwise, evaluate the sum of the series

$$\sum_{k=1}^{+\infty}3^{k-1}\sin^3\frac{1}{3^k}.$$

19. A series $\sum_{k=1}^{+\infty}a_k$ of real numbers is said to ***satisfy Cauchy's criterion*** if for each $\varepsilon>0$, there exists $N>0$ such that

if m and n are integers with $n>m\geq N$, then $|a_{m+1}+a_{m+2}+\cdots+a_n|<\varepsilon$.

(a) Show that if $\sum_{k=1}^{+\infty}a_k$ converges, then $\sum_{k=1}^{+\infty}a_k$ satisfies Cauchy's criterion.

(b) Show that if $\sum_{k=1}^{+\infty}a_k$ satisfies Cauchy's criterion, then the sequence of partial sums $\left(\sum_{k=1}^{n}a_k\right)_{n\in\mathbb{N}}$ is bounded, i.e. there exists $M>0$ such that

$$|a_1+a_2+\cdots+a_n|<M \qquad \text{for every } n\in\mathbb{N}.$$