

HKUST MATH 1014 L1 assignment 4 submission

MATH1014 Calculus II Problem Set 4
L01 (Spring 2024)

Problem Set 4

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 4 covers materials from §6.4 – §6.6.

Q2

Evaluate the following antiderivatives.

Q2.b

$$\int \ln(x^3 + 1) \, dx$$

$$\begin{aligned} & \int \ln(x^3 + 1) \, dx \\ &= \int \ln((x+1)(x^2 - x + 1)) \, dx \\ &= \int (\ln(x+1) + \ln(x^2 - x + 1)) \, dx \\ &= \int \ln(x+1) \, dx + \int \ln(x^2 - x + 1) \, dx \\ &= \int \ln(x+1) \, d(x+1) + \int \ln(x^2 - x + 1) \, dx \\ &= (x+1)\ln(x+1) - \int (x+1) \, d\ln(x+1) + x\ln(x^2 - x + 1) - \int x \, d\ln(x^2 - x + 1) \\ &= (x+1)\ln(x+1) - \int dx + x\ln(x^2 - x + 1) - \int \frac{2x^2 - x}{x^2 - x + 1} \, dx \\ &= (x+1)\ln(x+1) - x + x\ln(x^2 - x + 1) - \int \left(2 + \frac{1}{2} \frac{2x-1}{x^2 - x + 1} - \frac{3}{2} \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} \right) \, dx \\ &= (x+1)\ln(x+1) - x + x\ln(x^2 - x + 1) - 2x - \frac{1}{2} \ln(x^2 - x + 1) + \frac{3}{2} \frac{\sqrt{3}}{2} \frac{4}{3} \arctan\left(\left(x - \frac{1}{2}\right) \frac{2}{\sqrt{3}}\right) + C \\ &= (x+1)\ln(x+1) - 3x + \left(x - \frac{1}{2}\right) \ln(x^2 - x + 1) + \sqrt{3} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C \quad (x > -1) \end{aligned}$$

Q3

Q3.a

Using the factorization $x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$, evaluate

$$\int \frac{x^2}{x^4 + 1} \, dx$$

.

$$\begin{aligned}
& \int \frac{x^2}{x^4+1} dx \\
&= \int \frac{x^2}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)} dx \\
&= \frac{1}{2\sqrt{2}} \int \left(\frac{x}{x^2-\sqrt{2}x+1} - \frac{x}{x^2+\sqrt{2}x+1} \right) dx \\
&= \frac{1}{2\sqrt{2}} \int \left(\frac{1}{2} \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1} + \frac{\sqrt{2}}{2} \frac{1}{x^2-\sqrt{2}x+1} - \frac{1}{2} \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{\sqrt{2}}{2} \frac{1}{x^2+\sqrt{2}x+1} \right) dx \\
&= \frac{1}{4\sqrt{2}} (\ln|x^2-\sqrt{2}x+1| - \ln|x^2+\sqrt{2}x+1|) + \frac{1}{4} \int \left(\frac{1}{x^2-\sqrt{2}x+1} + \frac{1}{x^2+\sqrt{2}x+1} \right) dx \\
&= \frac{1}{4\sqrt{2}} (\ln|x^2-\sqrt{2}x+1| - \ln|x^2+\sqrt{2}x+1|) + \frac{1}{4} \int \left(\frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} + \frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} \right) dx \\
&= \frac{1}{4\sqrt{2}} (\ln|x^2-\sqrt{2}x+1| - \ln|x^2+\sqrt{2}x+1|) + \frac{1}{4} \left(\frac{1}{\sqrt{2}} (2) \arctan \left(\left(x-\frac{1}{\sqrt{2}}\right)\sqrt{2} \right) + \frac{1}{\sqrt{2}} (2) \arctan \left(\left(x+\frac{1}{\sqrt{2}}\right)\sqrt{2} \right) \right) + C \\
&= \frac{1}{4\sqrt{2}} (\ln|x^2-\sqrt{2}x+1| - \ln|x^2+\sqrt{2}x+1|) + \frac{1}{4} \left(\sqrt{2} \arctan(\sqrt{2}x-1) + \sqrt{2} \arctan(\sqrt{2}x+1) \right) + C \\
&= \frac{1}{4\sqrt{2}} \ln|x^2-\sqrt{2}x+1| - \frac{1}{4\sqrt{2}} \ln|x^2+\sqrt{2}x+1| + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}x-1) + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}x+1) + C
\end{aligned}$$

Q3.b

Using [\(a\)](#) and the substitution $u = \sqrt{\tan x}$, evaluate

$$\int_0^{\frac{\pi}{4}} \sqrt{\tan x} \, dx$$

.

$$\begin{aligned}
& \int_0^{\frac{\pi}{4}} \sqrt{\tan x} \, dx \\
&= 2 \int_{\sqrt{\tan x}=0}^{\sqrt{\tan x}=1} \frac{\tan x}{\sec^2 x} \, d\sqrt{\tan x} \\
&= 2 \int_{\sqrt{\tan x}=0}^{\sqrt{\tan x}=1} \frac{\tan x}{\tan^2 x + 1} \, d\sqrt{\tan x} \\
&= 2 \int_0^1 \frac{u^2}{u^4 + 1} \, du \\
&= 2 \left[\frac{1}{4\sqrt{2}} \ln|u^2 - \sqrt{2}u + 1| - \frac{1}{4\sqrt{2}} \ln|u^2 + \sqrt{2}u + 1| + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}u - 1) + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}u + 1) \right]_0^1 \\
&= 2 \left(\frac{1}{4\sqrt{2}} \ln|1 - \sqrt{2} + 1| - \frac{1}{4\sqrt{2}} \ln|1 + \sqrt{2} + 1| + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2} - 1) + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2} + 1) \right) \\
&= \frac{1}{2\sqrt{2}} \ln|2 - \sqrt{2}| - \frac{1}{2\sqrt{2}} \ln|2 + \sqrt{2}| + \frac{1}{\sqrt{2}} \arctan(\sqrt{2} - 1) + \frac{1}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2} - 1}\right) \\
&= \frac{1}{2\sqrt{2}} \ln|2 - \sqrt{2}| - \frac{1}{2\sqrt{2}} \ln|2 + \sqrt{2}| + \frac{\pi}{2\sqrt{2}}
\end{aligned}$$

Q7

Let a be a positive real number. Evaluate

$$\int \frac{1}{1 - a \sin x} \, dx$$

for each of the following cases:

Q7.a

$0 < a < 1$ (for $x \in (-\pi, \pi)$),

$$\begin{aligned}
& \int \frac{1}{1-a \sin x} dx \\
&= \int \frac{1}{1-a \frac{2t}{1+t^2}} \frac{2 dt}{1+t^2} \\
&= \int \frac{1}{\frac{1+t^2-2at}{1+t^2}} \frac{2 dt}{1+t^2} \\
&= 2 \int \frac{1}{1+t^2-2at} dt \\
&= 2 \int \frac{1}{(t-a)^2+(1-a^2)} dt \\
&= 2 \int \frac{1}{(t-a)^2+(1-a^2)} dt \\
&= 2\sqrt{1-a^2} \frac{1}{1-a^2} \arctan \left((t-a) \frac{1}{\sqrt{1-a^2}} \right) + C \\
&= \frac{2}{\sqrt{1-a^2}} \arctan \left(\frac{t-a}{\sqrt{1-a^2}} \right) + C \\
&= \frac{2}{\sqrt{1-a^2}} \arctan \left(\frac{\tan \frac{x}{2} - a}{\sqrt{1-a^2}} \right) + C, x \in (-\pi, \pi)
\end{aligned}$$

$(t := \tan \frac{x}{2}, x \in (-\pi, \pi))$

 $(0 < a < 1 \implies 1-a^2 > 0)$

Q7.b

$a = 1,$

$$\begin{aligned}
& \int \frac{1}{1-a \sin x} dx \\
&= \int \frac{1}{1-\sin x} dx \\
&= \int \frac{1}{1-\frac{2t}{1+t^2}} \frac{2 dt}{1+t^2} \\
&= \int \frac{1}{\frac{1+t^2-2t}{1+t^2}} \frac{2 dt}{1+t^2} \\
&= 2 \int \frac{1}{1+t^2-2t} dt \\
&= 2 \int \frac{1}{(t-1)^2} dt \\
&= -\frac{2}{t-1} + C \\
&= -\frac{2}{\tan \frac{x}{2} - 1} + C
\end{aligned}$$

$(a = 1)$

 $(t := \tan \frac{x}{2}, x \in (-\pi, \pi))$

 $(x \in (-\pi, \pi))$

Notice that the integrand $\frac{1}{1-\sin x}$ is periodic with a period of 2π .

Further, the integrand has vertical asymptotes when $x = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$,

which separates each cycle, so we can extend the domain:

$$\begin{cases} -\frac{2}{\tan \frac{x}{2} - 1} + C & x \neq \pi + 2n\pi \\ \lim_{t \rightarrow \pm\infty} -\frac{2}{t-1} = 0 & x = \pi + 2n\pi, x \neq \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z} \end{cases}$$

Q7.c

$a > 1.$

$$\begin{aligned}
& \int \frac{1}{1-a \sin x} dx \\
&= \int \frac{1}{1-a \frac{2t}{1+t^2}} \frac{2 dt}{1+t^2} \\
&= \int \frac{1}{\frac{1+t^2-2at}{1+t^2}} \frac{2 dt}{1+t^2} \\
&= 2 \int \frac{1}{1+t^2-2at} dt \\
&= 2 \int \frac{1}{(t-a)^2+(1-a^2)} dt \\
&= 2 \int \frac{1}{(t-a)^2-\sqrt{a^2-1}^2} dt \\
&= 2 \int \frac{1}{(t-a-\sqrt{a^2-1})(t-a+\sqrt{a^2-1})} dt \\
&= \frac{1}{\sqrt{a^2-1}} \int \left(\frac{1}{t-a-\sqrt{a^2-1}} - \frac{1}{t-a+\sqrt{a^2-1}} \right) dt \\
&= \frac{1}{\sqrt{a^2-1}} \left(\ln |t-a-\sqrt{a^2-1}| + \ln |t-a+\sqrt{a^2-1}| \right) + C \\
&= \frac{1}{\sqrt{a^2-1}} \ln \left| \tan \frac{x}{2} - a - \sqrt{a^2-1} \right| + \frac{1}{\sqrt{a^2-1}} \ln \left| \tan \frac{x}{2} - a + \sqrt{a^2-1} \right| + C, \\
& x \neq \pi + 2n\pi, \tan \frac{x}{2} \neq a + \sqrt{a^2-1}, \tan \frac{x}{2} \neq a - \sqrt{a^2-1}
\end{aligned}$$

$(t := \tan \frac{x}{2}, x \in (-\pi, \pi))$

 $(a > 1 \implies a^2-1 > 0)$

Q10

Let $f: [0, +\infty) \rightarrow \mathbb{R}$ be the function

$$f(x) = \int_1^{\sqrt{x}} e^{-t^2} dt$$

.

Q10.a

Find $f'(x)$ for every $x \in (0, +\infty)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx} \int_1^{\sqrt{x}} e^{-t^2} dt \\ &= \frac{1}{2} \frac{d}{dx} \int_1^x \frac{e^{-u}}{\sqrt{u}} du \quad (u := t^2, u > 0) \\ &= \frac{e^{-u}}{2\sqrt{u}} \quad (\text{FTC}) \\ &= \frac{e^{-t^2}}{2t} \quad (u > 0) \end{aligned}$$

Q10.b

Evaluate the improper integral

$$\int_0^1 \frac{f(x)}{\sqrt{x}} dx$$

.

$$\begin{aligned} &\int_0^1 \frac{f(x)}{\sqrt{x}} dx \\ &= 2 \int_0^1 f(x) d\sqrt{x} \\ &= 2[f(x)\sqrt{x}]_0^1 - 2 \int_0^1 \sqrt{x} df(x) \\ &= 2(0 - 0) - 2 \int_0^1 \sqrt{x} f'(x) dx \\ &= - \int_0^1 \frac{e^{-t^2}}{\sqrt{x}} dx \\ &= -2 \int_0^1 e^{-u} du \quad (u := \sqrt{x}, u > 0) \\ &= -2[e^{-u}]_0^1 \\ &= -2(e^{-1} - 1) \\ &= 2 - \frac{2}{e} \end{aligned}$$

Q15

For each of the following improper integrals, determine whether it converges or not.

Q15.a

$$\int_1^{+\infty} \frac{2 + \cos x}{\sqrt{x+5}} dx$$

$$\begin{aligned} &\int_1^{+\infty} \frac{2 + \cos x}{\sqrt{x+5}} dx \\ &\geq \int_1^{+\infty} \frac{1}{\sqrt{x+5}} dx \quad (2 + \cos x \geq 1) \\ &= \int_6^{+\infty} \frac{1}{\sqrt{x}} dx \quad (\text{change of variables: } x+5 \mapsto x) \end{aligned}$$

By p -test, the last integral is divergent.

By comparison test, the original integral is divergent.

Q15.b

$$\int_0^1 \frac{1}{\sqrt{x + \sqrt{x + \sqrt{x}}}} dx$$

$$\begin{aligned}
& \lim_{x \rightarrow 0^+} \frac{\frac{1}{\sqrt{x + \sqrt{x + \sqrt{x}}}}}{\frac{1}{\sqrt{x}}} \\
&= \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x + \sqrt{x}}}} \\
&= \lim_{x \rightarrow 0^+} \frac{\sqrt{\frac{x}{x^{\frac{1}{4}}}}}{\sqrt{\sqrt{\frac{x}{x^{\frac{1}{4}}}} + \sqrt{\frac{x}{\sqrt{x}}} + \sqrt{\frac{x}{x}}}} \\
&= \lim_{x \rightarrow 0^+} \frac{x^{\frac{3}{8}}}{\sqrt{x^{\frac{3}{4}} + \sqrt{\sqrt{x} + 1}}} \\
&= \frac{0}{\sqrt{0 + \sqrt{0 + 1}}} \\
&= 0
\end{aligned}$$

By p -test, $\int_0^1 \frac{1}{\sqrt{x}} dx$ is convergent.

By limit comparison test, the original integral is convergent.

Q15.c

$$\int_0^{+\infty} \frac{x}{1 + x^2 \sin^2 x} dx$$

$$\begin{aligned}
& \int_0^{+\infty} \frac{x}{1 + x^2 \sin^2 x} dx \\
& \geq \int_0^{+\infty} \frac{x}{1 + x^2} dx & (\sin^2 x \leq 1) \\
& = \frac{1}{2} \int_1^{+\infty} \frac{1}{x} dx & (\text{change of variables: } 1 + x^2 \mapsto x)
\end{aligned}$$

By p -test, the last integral is divergent.

By comparison test, the original integral is divergent.

Q15.d

$$\int_0^{+\infty} \frac{\sin x}{x} dx$$

$$\begin{aligned}
& \int_0^{+\infty} \frac{\sin x}{x} dx \\
& \leq \int_0^{+\infty} \left| \frac{\sin x}{x} \right| dx & (\text{triangle inequality}) \\
& \leq \int_0^{+\infty} \left| \frac{1}{x} \right| dx \\
& = \int_0^{+\infty} \frac{1}{x} dx & (x \geq 0)
\end{aligned}$$

By p -test, the last integral is divergent.

By absolute convergence test, the second last integral is convergent.

By comparison test, the original integral is convergent.

Q17

Let $f: [0, +\infty) \rightarrow [0, +\infty)$ be a decreasing continuous function.

Q17.a

Show that if $\int_0^{+\infty} f(x) dx$ converges, then $\lim_{x \rightarrow +\infty} xf(x) = 0$.

Hint: Show that $0 \leq xf(x) \leq 2 \int_{\frac{x}{2}}^x f(t) dt$ and apply Squeeze Theorem.

Assume that if $\int_0^{+\infty} f(x) \, dx$ is convergent, then $\lim_{x \rightarrow +\infty} xf(x) \neq 0$.

$\int_1^{+\infty} \frac{1}{x} \, dx$ is divergent by p -test.

$$\begin{aligned} & \lim_{x \rightarrow +\infty} \frac{f(x)}{\frac{1}{x}} \\ &= \lim_{x \rightarrow +\infty} xf(x) \\ &> 0 \end{aligned}$$

(assumption, $f(x) \geq 0, x > 0$)

By limit comparison test, $\int_1^{+\infty} f(x) \, dx$ diverges.

$$\left(f(x) \geq 0, \frac{1}{x} > 0\right)$$

By comparison test, $\int_0^{+\infty} f(x) \, dx \geq \int_1^{+\infty} f(x) \, dx$ diverges.

$$(f(x) \geq 0)$$

This contradicts our initial assumption that $\int_0^{+\infty} f(x) \, dx$ is convergent.

By contradiction, if $\int_0^{+\infty} f(x) \, dx$ is convergent, then $\lim_{x \rightarrow +\infty} xf(x) = 0$.

Q17.b

Show that the converse of the result from (a) is not true, i.e. give an example of $f(x)$ so that $\lim_{x \rightarrow +\infty} xf(x) = 0$ but $\int_0^{+\infty} f(x) \, dx$ diverges.

$$f(x) := \begin{cases} \frac{1}{x \ln x} & x \geq e \\ \frac{1}{e} & x \in [0, e] \end{cases}$$

Because $x \ln x$ is increasing and nonnegative,
 $f(x)$ is decreasing and nonnegative.

$$\begin{aligned} \lim_{x \rightarrow e^+} \frac{1}{x \ln x} &= \frac{1}{e \ln e} \\ &= \frac{1}{e} \\ f(x) &\text{ is continuous.} \end{aligned}$$

$$\begin{aligned} \int_0^{+\infty} f(x) \, dx &= \int_0^e \frac{1}{e} \, dx + \int_e^{+\infty} \frac{1}{x \ln x} \, dx \\ &= 1 + \int_e^{+\infty} \frac{1}{x \ln x} \, dx \\ &= 1 + \int_e^{+\infty} \frac{1}{\ln x} \, d(\ln x) \\ &= 1 + [\ln(\ln x)]_e^{+\infty} \\ &= 1 + \lim_{x \rightarrow +\infty} \ln(\ln x) \\ &= +\infty \\ \text{So } \int_0^{+\infty} f(x) \, dx &\text{ diverges.} \end{aligned}$$

However,

$$\begin{aligned} \lim_{x \rightarrow +\infty} xf(x) &= \lim_{x \rightarrow +\infty} \frac{x}{x \ln x} \\ &= \lim_{x \rightarrow +\infty} \frac{1}{\ln x} \\ &= 0 \end{aligned}$$

Thus it is proved.

Now let $g: [1, +\infty) \rightarrow [e, +\infty)$ be an increasing continuous function.

Q17.c

If $\lim_{x \rightarrow +\infty} \frac{x}{\ln(g(e^x))} = 0$, show that for every sufficiently large $x > 0$ we have $\frac{e^x}{g(e^x)} < e^{-x}$.

Hint: In the definition of limit (Definition 2.91), take $\epsilon = \frac{1}{2}$.

$$\lim_{x \rightarrow +\infty} \frac{x}{\ln(g(e^x))} = 0$$

$$\lim_{x \rightarrow +\infty} (e^x - g(e^x)) = e^0$$

$$= 1$$

By limit definition, for large sufficiently $x > 0$,

$$\frac{x}{\ln(g(e^x))} < \frac{1}{2}$$

$$x < \frac{1}{2} \ln(g(e^x))$$

$$x - \ln(g(e^x)) < -\frac{1}{2} \ln(g(e^x))$$

$$x - \ln(g(e^x)) < -x \quad \left(\frac{x}{\ln(g(e^x))} < \frac{1}{2} \implies -x > -\frac{1}{2} \ln(g(e^x)) \right)$$

$$\frac{e^x}{g(e^x)} < e^{-x} \quad (e^* \text{ is strictly increasing})$$

Q17.d

Using the results from [\(a\)](#) and [\(c\)](#), show that if $\int_1^{+\infty} \frac{1}{g(x)} dx$ diverges, then $\int_1^{+\infty} \frac{1}{x \ln g(x)} dx$ also diverges.

For large enough $x \geq X > 0$,

$$\frac{e^x}{g(e^x)} < e^{-x}$$

$$e^{2x} < g(e^x)$$

$$g(e^x) > e^{2x}$$

$$\ln g(e^x) > 2x \quad (\ln * \text{ is strictly increasing})$$

$$\int_1^{+\infty} \frac{1}{g(x)} dx = \int_0^{+\infty} \frac{e^x}{g(e^x)} dx \quad (\text{change of variables: } \ln x \mapsto x)$$

$$\int_1^{+\infty} \frac{1}{x \ln g(x)} dx = \int_0^{+\infty} \frac{1}{\ln g(e^x)} dx \quad (\text{change of variables: } \ln x \mapsto x)$$