COMP 2711H - Honors Discrete Mathematics
Final Exam

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The HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed if students are found to have violated the regulations governing academic integrity and honesty.

Declaration of Academic Integrity

I confirm that I have answered the questions using only materials specifically approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination.

Name: Student ID: Signature:

Instructions

1. Write your name and student ID in the fields above and sign the declaration of academic integrity. You should also write your name and student ID on top of **every page** of this answer book.
2. Place your student card on the desk next to you.
3. This exam takes ***2.5 hours*** and consists of ***3 problems***. You should answer all problems.
4. This final exam accounts for 30% of your total grade, 10% per problem.
5. All answers should be written in this answer book. Do ***not*** use the same page for more than one problem. Do ***not*** use the problem pages for your solutions. You can use these pages for drafting. You will ***not*** be provided with any extra paper. Do ***not*** use pencils. The answer sheets will be scanned before being graded. You will lose marks if your answer is not readable in the scanned version, e.g. due to the use of a pencil, bad handwriting, untidiness, or writing too close to the edges of the paper.
6. Answers without proofs will get zero points. You have to always prove the correctness of your solutions.
7. If you cannot solve a problem or task completely, write your best attempt. You will very likely get partial credit for it.
8. This is a closed-book exam. You are not allowed to use any notes or electronic devices.
9. If your solution contains a *logical error or fallacy*, you will get zero points, no matter how close you are to a correct answer. This rule applies even if disregarding the logical error leads to a 100% correct solution.
10. **You are not allowed to use calculators.**
11. You are not allowed to leave the examination room during the first 30 minutes and last 30 minutes of the exam.

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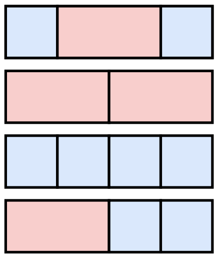
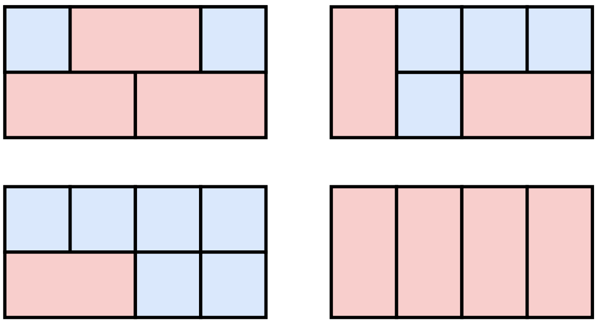
Problem 1. After grading 2711H midterm exam papers, Amir has decided that being a professor of computer science is too much work. Thus, in his infinite wisdom, he has changed careers and become a tiler, instead. In this new job, life is relatively easy. Amir just has to cover rectangular surfaces using smaller rectangles (tiles). The tiles can be rotated, but their sides must always be parallel to the axes.

(i) (2 points) Amir has two types of tiles: 1×1 and 1×2. He has infinitely many tiles of each type. All tiles of the same type are identical. He wants to tile a $1 \times n$ surface. In how many different ways can he do it? For example, if $n = 4$, some possible tilings are shown below.



(ii) (2 points) Amir's next job is more complicated. He now has to tile a $2 \times n$ surface. In how many different ways can he do it? For example, if $n = 4$, some possible tilings are shown below.



(iii) (2 points) Amir's life keeps getting harder and harder. He now has to tile a 3×4 surface. In how many different ways can he do it?

(iv) (2 points) Amir is having a nightmare in which he has to tile an infinite surface. This surface has one row, but infinitely many columns, one for each natural number. In how many ways can Amir do this? In other words, if A is the set of all possible tilings, what is $|A|$? Is A countable?

(v) (2 points) Amir's nightmares continue. He now has to tile an infinite by infinite surface. This surface has one row for every natural number and also one column for every natural number. Let B be the set of all possible tilings of this surface. How does $|B|$ compare with $|A|$ from the last subproblem?

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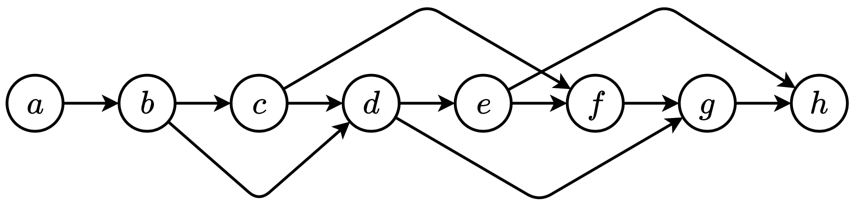
Problem 2. 2711H is over, so Xuran and Zhaorun finally have some free time and are now playing games.

(i) (2 points) In their first game, they have the DAG below. They choose a starting vertex $v_0 \in \{a, b, c, d, e, f, g, h\}$ and put a token on it. They take turns. Xuran plays first. In each turn, the player can choose one of the outgoing edges of the current vertex and move the token to its other endpoint, i.e. if the token is at u and there is an edge (u, v), the player can move the token to v. This is called a single jump. The player who cannot move in their turn loses the game and the other player wins. Both TAs are infinitely smart and they never make a mistake. From which starting vertices will Zhaorun win?



(ii) (3 points) Amir saw the game above and decided to make it more interesting. He changed the rules so that in each turn, the player can take either one or two edges of the graph. In other words, there are two types of moves:

- (Single jump) If the token is at u and there is an edge (u, v) in the graph, the player can move the token to v.
- (Double jump) Alternatively, if the token is at u and there are two edges (u, v) and (v, w) in the graph, the player can move the token to w.

When making a move, the player must announce the type of move and the exact edges they are using. A player who cannot move in their turn would lose the game. From which starting vertices will Zhaorun win?

(iii) (3 points) As you can imagine, even the previous game was not complicated enough for Amir. So, he introduced yet another type of move in addition to the two types above:

- (Reverse jump) If the token is at u and in the previous step the other player has done a double jump, then the current player can take an incoming edge to u and go to its source, i.e. find an edge (v, u) and move the token to v.

A player who cannot move in their turn loses the game. Prove that the game always ends and find the set of starting vertices from which Zhaorun will win.

(iv) (2 points) Zhaorun and Xuran decided to combine the problem in subtask (i) above with Nim. They have written three numbers on a board, the numbers are 2, 3, 4. They have also put a token in one of the vertices of the graph above. They take turns playing, starting with Xuran. In each turn, the player can do exactly one of the following moves:

- Decrease one of the three numbers, as long as the resulting number remains a non-negative integer (exactly as in Nim); or
- Do a single jump on the DAG.

The player who ***cannot*** make a move in their turn ***wins*** the game. For which starting vertices will Zhaorun win?

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Problem 3. Amir has decided that numbers do not matter. All that matters are their remainders modulo 7 and modulo 35. Find the following remainders:

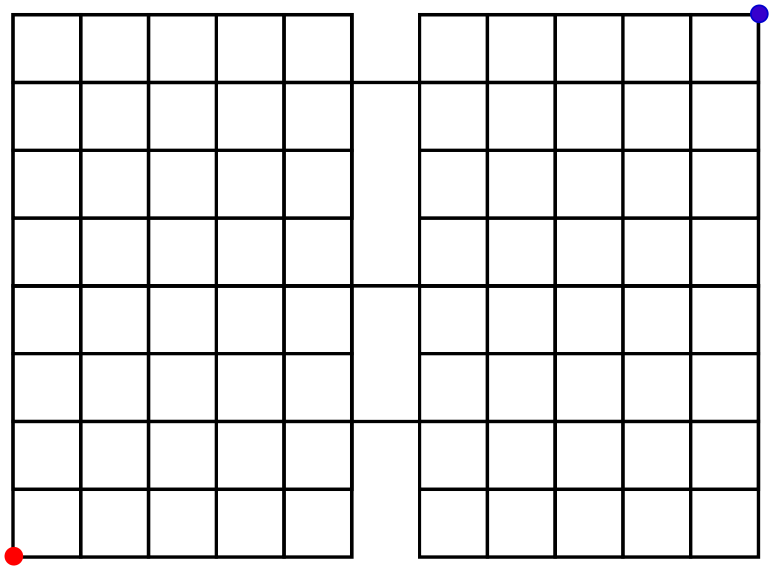
(i) $2^{2024} \times 3^{2711} \mod 7$

(ii) $2^{2024} \times 3^{2711} \mod 35$

(iii) $\binom{2711}{2701} \mod 7$

(iv) $\binom{2711}{2701} \mod 35$

(v) Number of paths from the bottom left corner to the top right corner of the figure below, assuming that we can only move right or up, $\mod 35$.



Each part carries 2 points. If you are asked for a remainder modulo 7, your answer should be in $\{0, 1, 2, 3, 4, 5, 6\}$. Similarly, if you are asked for a remainder modulo 35, your answer should be in $\{0, 1, 2, \ldots, 34\}$. Remember that you must prove the correctness of your result.

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