## Solution to Problem Set 7

1. (a) Suppose on the contrary that $(\cos n)$ converges to some real number $L$. Since

$$\cos(n+1) = \cos n \cos 1 - \sin n \sin 1 \qquad \text{for every } n \in \mathbb{N},$$

it follows that

$$\sin n = \frac{\cos n \cos 1 - \cos(n+1)}{\sin 1} \qquad \text{for every } n \in \mathbb{N}$$

and so $\lim\limits_{n\to+\infty} \sin n = \frac{L\cos 1 - L}{\sin 1}$ also exists as a real number; let's call this limit $M$. Now since

$$\cos^2 n + \sin^2 n = 1 \qquad \text{for every } n \in \mathbb{N},$$

we have $L^2 + M^2 = 1$. But on the other hand since

$$\cos 1 = \cos\big((n+1) - n\big) = \cos(n+1)\cos n - \sin(n+1)\sin n \qquad \text{for every } n \in \mathbb{N},$$

we have $L^2 + M^2 = \cos 1$. These together imply that $\cos 1 = 1$, which is a contradiction. ■

(b) Since $(\cos n)$ diverges according to (a), the series $\sum_{k=0}^{+\infty} \cos n$ diverges by term test.

2. (a) Since the exponential function is continuous, we have

$$\lim_{n\to+\infty} e^{\frac{1}{n^2}} = e^{\lim\limits_{n\to+\infty}\frac{1}{n^2}} = e^0 = 1 \neq 0.$$

Therefore the series $\sum_{k=1}^{+\infty} e^{\frac{1}{k^2}}$ diverges by term test.

(b) Since

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = \lim_{n\to+\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to+\infty} \frac{1}{\ln n} = 0,$$

the series $\sum_{k=2}^{+\infty} \frac{1}{(\ln k)^k}$ converges by root test. (Other possible solutions include ratio test, or comparison test with a geometric series.)

(c) Since the cosine and sine functions are continuous, we have

$$\lim_{n\to+\infty} \cos\left(\sin\frac{1}{n}\right) = \cos\left(\sin \lim_{n\to+\infty}\frac{1}{n}\right) = \cos(\sin 0) = 1 \neq 0.$$

Therefore the series $\sum_{k=1}^{+\infty} \cos\left(\sin\frac{1}{n}\right)$ diverges by term test.

(d) The terms of the series are all positive. Since

$$\lim_{n\to+\infty} \frac{1 - \cos\frac{1}{n}}{\frac{1}{n^2}} = \lim_{n\to+\infty} \frac{2\sin^2\frac{1}{n}}{\frac{1}{n^2}} = 2\left(\lim_{n\to+\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)^2 = 2,$$

which exists and is finite, and since $\sum_{k=1}^{+\infty} \frac{1}{k^2}$ converges (by $p$-test), the series $\sum_{k=1}^{+\infty} \left(1 - \cos\frac{1}{k}\right)$ converges by limit comparison test.

(e) Let $f:[1,+\infty)\to[0,+\infty)$ be the non-negative function $f(x)=xe^{-x^2}$. Since

$$f'(x)=(1-2x^2)e^{-x^2}<0 \qquad \text{for every } x>1,$$

$f$ is decreasing on $[1,+\infty)$. Since the improper integral

$$\int_1^{+\infty} f(x)dx=\int_1^{+\infty} xe^{-x^2}dx=\left[-\frac{1}{2}e^{-x^2}\right]_1^{+\infty}=\frac{1}{2e}$$

converges, the series $\sum_{k=1}^{+\infty} ke^{-k^2}$ also converges by integral test. (Other possible solutions include ratio test, root test and comparison test with a geometric series.)

(f) The terms of the series are all positive. Since

$$\lim_{n\to+\infty}\frac{\frac{\ln n}{n(n-1)}}{\frac{1}{n^{3/2}}}=\lim_{n\to+\infty}\frac{\ln n}{\sqrt{n}-\frac{1}{\sqrt{n}}}=0,$$

which exists and is finite, and since $\sum_{k=2}^{+\infty}\frac{1}{k^{3/2}}$ converges (by $p$-test), the series $\sum_{k=2}^{+\infty}\frac{\ln k}{k(k-1)}$ converges by limit comparison test.

(g) The terms of the series are all positive. Since

$$\lim_{n\to+\infty}\frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}}=\lim_{n\to+\infty}\frac{1}{n^{1/n}}=1,$$

(cf. Example (b) on page 2 of chapter 8) which exists and is positive, and since $\sum_{k=1}^{+\infty}\frac{1}{k}$ diverges (by $p$-test), the series $\sum_{k=1}^{+\infty}\frac{1}{k^{1+1/k}}$ diverges by limit comparison test. (Another possible solution is to apply comparison test with $\sum_{k=1}^{+\infty}\frac{1}{2k}$.)

(h) Since

$$\lim_{n\to+\infty}\sqrt[n]{|a_n|}=\lim_{n\to+\infty}\sqrt[n]{\left(\frac{1}{2}+\frac{1}{n}\right)^n}=\lim_{n\to+\infty}\left(\frac{1}{2}+\frac{1}{n}\right)=\frac{1}{2}<1,$$

the series $\sum_{k=1}^{+\infty}\left(\frac{1}{2}+\frac{1}{k}\right)^k$ converges by root test. (Other possible solutions include ratio test, or comparison test with a geometric series.)

(i) Since

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to+\infty}\frac{\frac{\left(2(n+1)\right)!}{(n+2)!\,n!}}{\frac{2n!}{(n+1)!\,(n-1)!}}=\lim_{n\to+\infty}\frac{(2n+1)(2n+2)}{(n+2)n}=\lim_{n\to+\infty}\frac{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}{\left(1+\frac{2}{n}\right)}=4>1,$$

the series $\sum_{k=1}^{+\infty}\frac{(2k)!}{(k+1)!(k-1)!}$ diverges by ratio test. (Other possible solutions include root test, or comparison test with a geometric series.)

(j) Since

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\frac{\frac{3^{n+1}+4^{n+1}}{2^{n+1}+5^{n+1}}}{\frac{3^n+4^n}{2^n+5^n}} = \lim_{n\to+\infty}\frac{3^{n+1}+4^{n+1}}{3^n+4^n}\frac{2^n+5^n}{2^{n+1}+5^{n+1}}$$

$$= \lim_{n\to+\infty}\frac{3\cdot\left(\frac{3}{4}\right)^n+4}{\left(\frac{3}{4}\right)^n+1}\frac{\left(\frac{2}{5}\right)^n+1}{2\cdot\left(\frac{2}{5}\right)^n+5} = \frac{4}{5} < 1,$$

the series $\sum_{k=1}^{+\infty}\frac{3^k+4^k}{2^k+5^k}$ converges by ratio test. (Other possible solutions include root test, or comparison test with a geometric series.)

3. (a) Suppose that $\sum_{k=1}^{+\infty} a_k$ converges. Then by term test, the sequence $(a_n)$ must converge to $0$. Since $(a_n)$ is a sequence of positive numbers, $\left(\frac{1}{a_n}\right)$ must diverge to $+\infty$. In particular, $\left(\frac{1}{a_n}\right)$ does not converge to $0$ and so $\sum_{k=1}^{+\infty}\frac{1}{a_k}$ diverges by term test. ∎

(b) If $\lim_{n\to+\infty} na_n = L > 0$, then (for $\varepsilon = \frac{L}{2} > 0$) there exists $N > 0$ such that for every integer $n \geq N$, we have

$$|na_n - L| < \frac{L}{2},$$

which implies that $na_n > \frac{L}{2}$. Therefore $a_n > \frac{L}{2n} > 0$ for every $n \geq N$. Now the series $\sum_{k=1}^{+\infty}\frac{L}{2k} = \frac{L}{2}\sum_{k=1}^{+\infty}\frac{1}{k}$ diverges by $p$-test, so $\sum_{k=1}^{+\infty} a_k$ diverges by comparison test. ∎

_Alternative solution_: Since $\lim_{n\to+\infty}\frac{a_n}{\frac{1}{n}} = \lim_{n\to+\infty} na_n = L > 0$ and the harmonic series $\sum_{k=1}^{+\infty}\frac{1}{k}$ diverges, the series $\sum_{k=1}^{+\infty} a_k$ also diverges by limit comparison test. ∎

(c) Suppose that $\sum_{k=1}^{+\infty} a_k$ converges. Then by term test, the sequence $(a_n)$ must converge to $0$. In particular, there exists $N > 0$ such that for every integer $n \geq N$, we have $0 < a_n < 1$. This implies that $0 < {a_n}^2 < a_n$ for every $n \geq N$. Recall again that $\sum_{k=1}^{+\infty} a_k$ converges, so $\sum_{k=1}^{+\infty} {a_k}^2$ converges by comparison test. ∎

The converse of this result from (c) is not true. To see this, let $a_n = \frac{1}{n}$. Then $\sum_{k=1}^{+\infty} {a_k}^2 = \sum_{k=1}^{+\infty}\frac{1}{k^2}$ converges but $\sum_{k=1}^{+\infty} a_k = \sum_{k=1}^{+\infty}\frac{1}{k}$ diverges.

(d) For each $n \in \mathbb{N}$, since $a_n > 0$, we have

$$\frac{1}{2}\left({a_n}^2 + \frac{1}{n^2}\right) \geq \sqrt{{a_n}^2\cdot\frac{1}{n^2}} = \left|\frac{a_n}{n}\right| = \frac{a_n}{n} > 0$$

by the AM-GM inequality. Now we know that $\sum_{k=1}^{+\infty}\frac{1}{k^2}$ converges. If $\sum_{k=1}^{+\infty} {a_k}^2$ also converges, then $\sum_{k=1}^{+\infty}\frac{1}{2}\left({a_k}^2 + \frac{1}{k^2}\right)$ converges, and so $\sum_{k=1}^{+\infty}\frac{a_k}{k}$ converges by comparison test. ∎

(e) For each $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (ka_k)\left(\frac{1}{k}\right) \le \sqrt{\sum_{k=1}^{n} k^2 {a_k}^2}\sqrt{\sum_{k=1}^{n} \frac{1}{k^2}}$$

by Cauchy-Schwarz inequality. (Let $\mathbf{u} = \langle 1a_1, 2a_2, \dots, na_n \rangle$ and $\mathbf{v} = \langle 1, \frac{1}{2}, \dots, \frac{1}{n} \rangle$ be vectors in $\mathbb{R}^n$ and apply Cauchy-Schwarz inequality $\mathbf{u} \cdot \mathbf{v} \le \|\mathbf{u}\|\|\mathbf{v}\|$ as in Theorem 7.33.) Now we know that $\sum_{k=1}^{+\infty} \frac{1}{k^2}$ converges. If $\sum_{k=1}^{+\infty} k^2 {a_k}^2$ also converges, then the above inequality shows that $\sum_{k=1}^{+\infty} a_k$ also converges. ■

4. (a) Note that for each $j \in \mathbb{N}$, the sum $\left(\sum_{k=N+1}^{+\infty} \frac{1}{p_k}\right)^j$, after expanding, consists of **all possible terms** of the form

$$\frac{1}{p_{N+m_1} p_{N+m_2} \cdots p_{N+m_j}},$$

in which there are exactly $j$ factors in the denominator which may or may not repeat, and all these factors come from the set $\{p_{N+1}, p_{N+2}, \dots\}$. Now each number $1+M$, $1+2M$, $1+3M$, … is not divisible by any of the primes $p_1, p_2, \dots, p_N$; so their prime factorizations take exactly the form of the above denominator. Therefore

$$\frac{1}{1+M} + \frac{1}{1+2M} + \cdots + \frac{1}{1+nM} \le \sum_{j=1}^{+\infty} \left( \sum_{k=N+1}^{+\infty} \frac{1}{p_k} \right)^j \qquad \text{for all } n \in \mathbb{N}.$$

(b) Suppose that $\sum_{k=1}^{+\infty} \frac{1}{p_k}$ converges. Then by definition, (for $\varepsilon = \frac{1}{2} > 0$) there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{+\infty} \frac{1}{p_k} < \frac{1}{2}.$$

According to (a), we have

$$\frac{1}{1+M} + \frac{1}{1+2M} + \cdots + \frac{1}{1+nM} \le \sum_{j=1}^{+\infty} \frac{1}{2^j} = 1 \qquad \text{for all } n \in \mathbb{N}.$$

This implies that $\sum_{k=1}^{+\infty} \frac{1}{1+kM}$ converges, which is impossible as it would imply that $\sum_{k=1}^{+\infty} \frac{1}{k}$ converges by limit comparison test (or by comparison test). ■

5. (a) For every $n \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \\
&= h_{2n} - h_n.
\end{aligned}$$

(b) According to (a), we have

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = h_{2n} - h_n = (h_{2n} - \ln 2n) - (h_n - \ln n) + \ln 2\,,$$

so with $\gamma$ denoting the Euler-Mascheroni constant, we have

$$\lim_{n\to+\infty} \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \lim_{n\to+\infty} (h_{2n} - \ln 2n) - \lim_{n\to+\infty} (h_n - \ln n) + \ln 2 = \gamma - \gamma + \ln 2 = \ln 2\,,$$

i.e. the $(2n)^{\text{th}}$ partial sum converges to $\ln 2$. On the other hand, we have

$$\lim_{n\to+\infty} \sum_{k=1}^{2n+1} \frac{(-1)^{k+1}}{k} = \lim_{n\to+\infty} \left( \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} - \frac{1}{2n+1} \right) = \left( \lim_{n\to+\infty} \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} \right) - \left( \lim_{n\to+\infty} \frac{1}{2n+1} \right)$$
$$= \ln 2 - 0 = \ln 2$$

the $(2n+1)^{\text{st}}$ partial sum also converges to $\ln 2$. Combining the odd and even terms in the sequence of partial sums, we conclude that $\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = \ln 2$; i.e. $\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k}$ converges to $\ln 2$. ■

<u>*Alternative solution*</u>: For each $n \in \mathbb{N}$, $h_{2n} - h_n$ can be treated as the right Riemann-sum of $f(x) = \frac{1}{x}$ with respect to the regular partition of $[1, 2]$ into $n$ subintervals. Therefore

$$\lim_{n\to+\infty} (h_{2n} - h_n) = \lim_{n\to+\infty} \sum_{k=n+1}^{2n} \frac{1}{k} = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=n+1}^{2n} \frac{1}{k/n} = \int_1^2 \frac{1}{x}\,dx = \ln 2\,.$$

Thus according to (a), we have $\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = \ln 2$; i.e. $\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k}$ converges to $\ln 2$. ■

6. For each $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} \left( \frac{1}{k} - \ln\left(1 + \frac{1}{k}\right) \right) = (1 - \ln 2) + \left( \frac{1}{2} - \ln\frac{3}{2} \right) + \left( \frac{1}{3} - \ln\frac{4}{3} \right) + \cdots + \left( \frac{1}{n} - \ln\frac{n+1}{n} \right)$$
$$= \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) - \left( \ln 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \cdots \cdot \frac{n+1}{n} \right) = \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) - \ln(n+1)$$
$$= \left( \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) - \ln n \right) + \ln\frac{n}{n+1}.$$

So

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \left( \frac{1}{k} - \ln\left(1 + \frac{1}{k}\right) \right) = \lim_{n\to+\infty} \left( \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) - \ln n \right) + \lim_{n\to+\infty} \ln\frac{n}{n+1} = \gamma + \ln 1 = \gamma.$$

In other words, the series $\sum_{k=1}^{+\infty} \left( \frac{1}{k} - \ln\left(1 + \frac{1}{k}\right) \right)$ converges to the Euler-Mascheroni constant $\gamma$.

7. (a) The terms of the series are all positive. Since

$$\lim_{n\to+\infty} \frac{n^2 \sin^p(1/n)}{1/n^{p-2}} = \left( \lim_{n\to+\infty} \frac{\sin(1/n)}{1/n} \right)^p = 1,$$

which exists and is in $(0, +\infty)$, according to limit comparison test, the series $\sum_{k=1}^{+\infty} k^2 \sin^p \frac{1}{k}$ converges if and only if $\sum_{k=1}^{+\infty} \frac{1}{k^{p-2}}$ converges. Now $\sum_{k=1}^{+\infty} \frac{1}{k^{p-2}}$ converges if and only if $p - 2 > 1$ according to $p$-test, so $\sum_{k=1}^{+\infty} k^2 \sin^p \frac{1}{k}$ converges if and only if $p > 3$.

(b) Let $f:[3,+\infty)\to[0,+\infty)$ be the non-negative function $f(x)=\frac{1}{x(\ln x)(\ln\ln x)^p}$. Then $f$ is decreasing on $[3,+\infty)$ regardless of the value of $p$. Since the improper integral

$$\int_3^{+\infty} f(x)dx=\int_3^{+\infty}\frac{1}{x(\ln x)(\ln\ln x)^p}dx=\int_3^{+\infty}\frac{1}{(\ln\ln x)^p}d\ln\ln x=\begin{cases}\left[\frac{1}{1-p}(\ln\ln x)^{1-p}\right]_3^{+\infty} & \text{if } p\neq 1\\ [\ln\ln\ln x]_3^{+\infty} & \text{if } p=1\end{cases}$$

converges if and only if $1-p<0$, according to integral test, the series $\sum_{k=3}^{+\infty}\frac{1}{k(\ln k)(\ln\ln k)^p}$ converges if and only if $1-p<0$, i.e. $p>1$.

(c) For each $n\in\mathbb{N}$, we have

$$\frac{1}{(\ln\ln n)^{p\ln n}}=\frac{1}{e^{p(\ln n)(\ln\ln\ln n)}}=\frac{1}{n^{p\ln\ln\ln n}}.$$

- If $p=0$, then $\lim\limits_{n\to+\infty}\frac{1}{n^{p\ln\ln\ln n}}=\lim\limits_{n\to+\infty}1=1$, so the series diverges by term test.
- If $p<0$, then $\lim\limits_{n\to+\infty}\frac{1}{n^{p\ln\ln\ln n}}=+\infty$, so the series also diverges by term test.
- If $p>0$, then since $\lim\limits_{n\to+\infty}p\ln\ln\ln n=+\infty$, there exists $N>0$ such that for every integer $n\geq N$ we have $p\ln\ln\ln n\geq 2$, i.e. $0<\frac{1}{n^{p\ln\ln\ln n}}\leq\frac{1}{n^2}$. Since $\sum_{k=3}^{+\infty}\frac{1}{k^2}$ converges by $p$-test, $\sum_{k=3}^{+\infty}\frac{1}{(\ln\ln k)^{p\ln k}}$ also converges by comparison test.

Therefore the series $\sum_{k=3}^{+\infty}\frac{1}{(\ln\ln k)^{p\ln k}}$ converges if and only if $p>0$.

(d) For every $p\in\mathbb{R}$, since

$$\lim_{n\to+\infty}\sqrt[n]{|a_n|}=\lim_{n\to+\infty}\sqrt[n]{\frac{n^p}{(\ln n)^n}}=\lim_{n\to+\infty}\frac{\left(\sqrt[n]{n}\right)^p}{\ln n}=0<1,$$

the series $\sum_{k=2}^{+\infty}\frac{k^p}{(\ln k)^k}$ converges by root test. Therefore the series converges for every $p\in\mathbb{R}$.

8. With $a_n^+:=\max\{a_n,0\}$ and $a_n^-:=\max\{-a_n,0\}$, we have

$$a_n=a_n^+-a_n^- \qquad\text{and}\qquad |a_n|=a_n^++a_n^- \qquad\text{for every } n.$$

(a) Suppose that $\sum_{k=1}^{+\infty}a_k$ converges absolutely. Then $\sum_{k=1}^{+\infty}|a_k|$ converges. Since

$$0\leq a_n^+\leq|a_n| \qquad\text{and}\qquad 0\leq a_n^-\leq|a_n| \qquad\text{for every } n,$$

we must have both $\sum_{k=1}^{+\infty}a_k^+$ and $\sum_{k=1}^{+\infty}a_k^-$ converge, by comparison test. ■

(b) Suppose that $\sum_{k=1}^{+\infty}a_k$ converges conditionally. Then $\sum_{k=1}^{+\infty}a_k$ converges but $\sum_{k=1}^{+\infty}|a_k|$ diverges.

- If $\sum_{k=1}^{+\infty}a_k^+$ converges, then $\sum_{k=1}^{+\infty}a_k^-=\sum_{k=1}^{+\infty}(a_k^+-a_k)$ also converges.
- If $\sum_{k=1}^{+\infty}a_k^-$ converges, then $\sum_{k=1}^{+\infty}a_k^+=\sum_{k=1}^{+\infty}(a_k^-+a_k)$ also converges.

In any case, the series

$$\sum_{k=1}^{+\infty}|a_k|=\sum_{k=1}^{+\infty}(a_k^++a_k^-)$$

converges, which is a contradiction. Therefore $\sum_{k=1}^{+\infty}a_k^+$ and $\sum_{k=1}^{+\infty}a_k^-$ must both diverge. ■

9. (a) Consider the series of absolute values $\sum_{k=1}^{+\infty}\left|\frac{\cos k}{k^3}\right|$. For each $n\in\mathbb{N}$, we have

$$\left|\frac{\cos n}{n^3}\right|\leq\frac{1}{n^3}.$$

Since $\sum_{k=0}^{+\infty}\frac{1}{k^3}$ converges by $p$-test, it follows that $\sum_{k=1}^{+\infty}\left|\frac{\cos k}{k^3}\right|$ converges by comparison test. Therefore we conclude that $\sum_{k=1}^{+\infty}\frac{\cos k}{k^3}$ **converges absolutely**.

(b) ⊙ Consider the series of absolute values $\sum_{k=0}^{+\infty}\left|(-1)^{k+1}\left(\sqrt{k+1}-\sqrt{k}\right)\right|=\sum_{k=0}^{+\infty}\left(\sqrt{k+1}-\sqrt{k}\right)$. Now for each $n\in\mathbb{N}$, we have the partial sum

$$\sum_{k=0}^{n}\left(\sqrt{k+1}-\sqrt{k}\right)=\left(\sqrt{1}-\sqrt{0}\right)+\left(\sqrt{2}-\sqrt{1}\right)+\cdots+\left(\sqrt{n+1}-\sqrt{n}\right)=\sqrt{n+1},$$

so $\sum_{k=0}^{+\infty}\left(\sqrt{k+1}-\sqrt{k}\right)$ diverges to $+\infty$, i.e. $\sum_{k=0}^{+\infty}(-1)^{k+1}\left(\sqrt{k+1}-\sqrt{k}\right)$ does not converge absolutely.

⊙ On the other hand, we have

$$\lim_{n\to+\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=\lim_{n\to+\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}=0.$$

Moreover, for each $n\in\mathbb{N}$ we have

$$\left(\sqrt{n+2}-\sqrt{n+1}\right)-\left(\sqrt{n+1}-\sqrt{n}\right)=\frac{1}{\sqrt{n+2}+\sqrt{n+1}}-\frac{1}{\sqrt{n+1}+\sqrt{n}}<0,$$

so the sequence $\left(\sqrt{n+1}-\sqrt{n}\right)$ is decreasing. Thus the series $\sum_{k=0}^{+\infty}(-1)^{k+1}\left(\sqrt{k+1}-\sqrt{k}\right)$ converges by alternating series test.

Therefore we conclude that $\sum_{k=0}^{+\infty}(-1)^{k+1}\left(\sqrt{k+1}-\sqrt{k}\right)$ **converges conditionally**.

(c) Note that $\cos k\pi=(-1)^k$ for every positive integer $k$.

⊙ Consider the series of absolute values $\sum_{k=1}^{+\infty}\left|\cos k\pi\sin\frac{1}{k\pi}\right|=\sum_{k=1}^{+\infty}\left|(-1)^k\sin\frac{1}{k\pi}\right|=\sum_{k=1}^{+\infty}\sin\frac{1}{k\pi}$. Since $\sum_{k=1}^{+\infty}\frac{1}{k}$ diverges (by $p$-test) and since

$$\lim_{n\to+\infty}\frac{\sin(1/n\pi)}{1/n}=\frac{1}{\pi}$$

which exists and is positive, it follows that $\sum_{k=1}^{+\infty}\sin\frac{1}{k\pi}$ also diverges by limit comparison test, i.e. $\sum_{k=1}^{+\infty}\cos k\pi\sin\frac{1}{k\pi}$ does not converge absolutely.

⊙ On the other hand, since $\sin$ is continuous at $0$ we have

$$\lim_{n\to+\infty}\sin\frac{1}{n\pi}=\sin\lim_{n\to+\infty}\frac{1}{n\pi}=0.$$

Moreover, for each $n\in\mathbb{N}$ we have $\frac{1}{(n+1)\pi}<\frac{1}{n\pi}$; since $\sin$ is (strictly) increasing on $\left[0,\frac{\pi}{2}\right]$, we have

$$\sin\frac{1}{(n+1)\pi}<\sin\frac{1}{n\pi},$$

and so the sequence $\left(\sin\frac{1}{n\pi}\right)$ is decreasing. Thus the series $\sum_{k=1}^{+\infty}\cos k\pi\sin\frac{1}{k\pi}=\sum_{k=1}^{+\infty}(-1)^k\sin\frac{1}{k\pi}$ converges by alternating series test.

Therefore we conclude that $\sum_{k=1}^{+\infty}\cos k\pi\sin\frac{1}{k\pi}$ **converges conditionally**.

(d) For each integer $n \geq 2$, we have the partial sum

$$\sum_{k=2}^{n} \frac{(-1)^k}{\sqrt{k}+(-1)^k} = \sum_{k=2}^{n} \frac{(-1)^k\left(\sqrt{k}-(-1)^k\right)}{k-1} = \sum_{k=2}^{n}(-1)^k \frac{1}{\sqrt{k}-1/\sqrt{k}} - \sum_{k=2}^{n} \frac{1}{k-1}.$$

Now consider the series $\sum_{k=2}^{+\infty}(-1)^k \frac{1}{\sqrt{k}-\frac{1}{\sqrt{k}}}$. We have

$$\lim_{n\to+\infty} \frac{1}{\sqrt{n}-\frac{1}{\sqrt{n}}} = 0.$$

Moreover, for each integer $n \geq 2$ we have $\sqrt{n+1} > \sqrt{n}$ and $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$, so $\left(\frac{1}{\sqrt{n}-\frac{1}{\sqrt{n}}}\right)$ is obviously a decreasing sequence. Thus the series $\sum_{k=2}^{+\infty}(-1)^k \frac{1}{\sqrt{k}-\frac{1}{\sqrt{k}}}$ converges by alternating series test. Finally we see that the given series $\sum_{k=2}^{+\infty} \frac{(-1)^k}{\sqrt{k}+(-1)^k}$ **diverges**, because otherwise if it converges, then the above would imply that the harmonic series

$$\sum_{k=2}^{+\infty} \frac{1}{k-1} = \sum_{k=2}^{+\infty}(-1)^k \frac{1}{\sqrt{k}-1/\sqrt{k}} - \sum_{k=2}^{+\infty} \frac{(-1)^k}{\sqrt{k}+(-1)^k}$$

converges, which is a contradiction.

*Remark*: Note that the **alternating series test does not apply** on the given series $\sum_{k=2}^{+\infty} \frac{(-1)^k}{\sqrt{k}+(-1)^k}$ in (d), because $\left(\frac{1}{\sqrt{n}+(-1)^n}\right)$ is not a decreasing sequence.

10. (a) (i) Note that $b_1 = B_1$, so for $n = 1$ we have

$$a_2 B_1 - \sum_{k=1}^{1} B_k(a_{k+1}-a_k) = a_2 B_1 - B_1(a_2 - a_1) = a_1 B_1 = a_1 b_1 = \sum_{k=1}^{1} a_k b_k\,.$$

Assume that for some positive integer $m$ we have

$$\sum_{k=1}^{m} a_k b_k = a_{m+1} B_m - \sum_{k=1}^{m} B_k(a_{k+1}-a_k)\,.$$

Then for $n = m+1$ we have

$$\begin{aligned}\sum_{k=1}^{m+1} a_k b_k &= a_{m+1} b_{m+1} + \sum_{k=1}^{m} a_k b_k = a_{m+1} b_{m+1} + a_{m+1} B_m - \sum_{k=1}^{m} B_k(a_{k+1}-a_k) \\ &= a_{m+1}(B_m + b_{m+1}) - \sum_{k=1}^{m} B_k(a_{k+1}-a_k) = a_{m+1} B_{m+1} - \sum_{k=1}^{m} B_k(a_{k+1}-a_k) \\ &= a_{m+2} B_{m+1} - B_{m+1}(a_{m+2}-a_{m+1}) - \sum_{k=1}^{m} B_k(a_{k+1}-a_k) = a_{m+2} B_{m+1} - \sum_{k=1}^{m+1} B_k(a_{k+1}-a_k)\,.\end{aligned}$$

So by mathematical induction, the given formula is true for every positive integer $n$. ∎

(ii) Since $(B_n)$ is bounded, there exists $M \geq 0$ such that $|B_n| \leq M$ for every $n \in \mathbb{N}$. Now for each $n \in \mathbb{N}$, summation by parts gives the $n^{\text{th}}$ partial sum

$$\sum_{k=1}^{n} a_k b_k = a_{n+1}B_n + \sum_{k=1}^{n} B_k(a_k - a_{k+1}).$$

⊙ Since $0 \leq |a_{n+1}B_n| \leq M|a_{n+1}|$ for every $n \in \mathbb{N}$ and since $\lim_{n\to+\infty}(M|a_{n+1}|) = M\left(\lim_{n\to+\infty}|a_{n+1}|\right) = 0$, we have $\lim_{n\to+\infty}|a_{n+1}B_n| = 0$ by squeeze theorem and therefore $\lim_{n\to+\infty} a_{n+1}B_n = 0$.

⊙ Since $(a_n)$ is decreasing, we have

$$|B_n(a_n - a_{n+1})| = |B_n|(a_n - a_{n+1}) \leq M(a_n - a_{n+1}) \qquad \text{for every } n \in \mathbb{N}.$$

But since the telescoping series $\sum_{k=1}^{+\infty} M(a_k - a_{k+1}) = Ma_1$ converges, so $\sum_{k=1}^{+\infty}|B_k(a_k - a_{k+1})|$ also converges by comparison test. Therefore $\sum_{k=1}^{+\infty} B_k(a_k - a_{k+1})$ converges (absolutely).

Combining the two paragraphs above, it follows that $\sum_{k=1}^{+\infty} a_k b_k$ converges. ∎

(b) We consider the following two cases.

⊙ If $t$ is an integer multiple of $\pi$, then every term in $\sum_{k=1}^{+\infty}\frac{\sin kt}{k}$ is just $0$, so the series converges to $0$.

⊙ If $t$ is not an integer multiple of $\pi$, then we let $a_n = \frac{1}{n}$ and $b_n = \sin nt$ for each $n \in \mathbb{N}$. Now $(a_n)$ is obviously decreasing and $\lim_{n\to+\infty} a_n = 0$. Since $\sin\frac{t}{2} \neq 0$, for every $n \in \mathbb{N}$ we have

$$|B_n| = \left|\sum_{k=1}^{n} \sin kt\right| = \left|\frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right| \leq \frac{\left|\cos\frac{t}{2}\right| + \left|\cos\left(n+\frac{1}{2}\right)t\right|}{\left|2\sin\frac{t}{2}\right|} \leq \frac{1+1}{2\left|\sin\frac{t}{2}\right|} \leq \frac{1}{\left|\sin\frac{t}{2}\right|},$$

so $(B_n)$ is a bounded sequence (Note that $t$ is fixed, the upper bound works for all $n$). Therefore applying the result from (a)(ii), it follows that the series $\sum_{k=1}^{+\infty}\frac{\sin kt}{k} = \sum_{k=1}^{+\infty} a_k b_k$ converges also.

*Remark*: The convergence test proved in (a)(ii) is called ***Dirichlet's test***. It is a generalization of the alternating series test, which is the case when $b_n = (-1)^n$.

11. (a) The coefficients of the given power series are given by $c_n = n^{\sqrt{n}}$. Now the limit

$$\lim_{n\to+\infty}\sqrt[n]{|c_n|} = \lim_{n\to+\infty} n^{\frac{\sqrt{n}}{n}} = \lim_{n\to+\infty} e^{\frac{\ln n}{\sqrt{n}}} = e^{\left(2\lim_{n\to+\infty}\frac{\ln\sqrt{n}}{\sqrt{n}}\right)} = e^0 = 1$$

exists, so the radius of convergence of the given power series is

$$R = \frac{1}{\lim_{n\to+\infty}\sqrt[n]{|c_n|}} = 1.$$

The center of the power series is $0$, so its interval of convergence has end-points $-1$ and $1$.

⊙ At $1$, the power series becomes $\sum_{k=1}^{+\infty} k^{\sqrt{k}}$ which diverges by term test.

⊙ At $-1$, the power series becomes $\sum_{k=1}^{+\infty} k^{\sqrt{k}}(-1)^k$ which diverges by term test also.

Therefore the interval of convergence of the given power series is $(-1, 1)$.

(b) The coefficients of the given power series are given by $c_n = \frac{1}{2^n n^2}$. Now the limit

$$\lim_{n\to+\infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n\to+\infty} \frac{2^{n+1}(n+1)^2}{2^n n^2} = \lim_{n\to+\infty} 2\left(1+\frac{1}{n}\right)^2 = 2(1+0)^2 = 2$$

exists, so the radius of convergence of the given power series is

$$R = \lim_{n\to+\infty} \frac{|c_n|}{|c_{n+1}|} = 2.$$

The center of the power series is $0$, so its interval of convergence has end-points $-2$ and $2$.

- At $2$, the power series becomes $\sum_{k=0}^{+\infty} \frac{2^k}{2^k k^2} = \sum_{k=0}^{+\infty} \frac{1}{k^2}$ which converges by $p$-test.
- At $-2$, the power series becomes $\sum_{k=0}^{+\infty} \frac{(-2)^k}{2^k k^2} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k^2}$ which converges by either alternating series test or by absolute convergence test.

Therefore the interval of convergence of the given power series is $[-2, 2]$.

(c) The coefficients of the given power series $\sum_{k=1}^{+\infty} \frac{(1-2x)^k}{k} = \sum_{k=1}^{+\infty} \frac{(-2)^k}{k}\left(x-\frac{1}{2}\right)^k$ are given by $c_n = \frac{(-2)^n}{n}$ (but not just $\frac{1}{n}$). Now the limit

$$\lim_{n\to+\infty} \sqrt[n]{|c_n|} = \lim_{n\to+\infty} \sqrt[n]{\frac{2^n}{n}} = \lim_{n\to+\infty} \frac{2}{\sqrt[n]{n}} = \frac{2}{1} = 2$$

exists, so the radius of convergence of the given power series is

$$R = \frac{1}{\lim_{n\to+\infty} \sqrt[n]{|c_n|}} = \frac{1}{2}.$$

The center of the power series is $\frac{1}{2}$, so its interval of convergence has end-points $0$ and $1$.

- At $0$, the power series becomes $\sum_{k=1}^{+\infty} \frac{1}{k}$ which diverges by $p$-test.
- At $1$, the power series becomes $\sum_{k=1}^{+\infty} \frac{(-1)^k}{k}$ which converges by alternating series test.

Therefore the interval of convergence of the given power series is $(0, 1]$.

(d) The coefficients of the given power series are given by $c_n = \frac{(-1)^{n+1}}{\sqrt{n!}}$. Now the limit

$$\lim_{n\to+\infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n\to+\infty} \frac{\sqrt{(n+1)!}}{\sqrt{n!}} = \lim_{n\to+\infty} \sqrt{n+1} = +\infty$$

exists, so the radius of convergence of the given power series is also

$$R = +\infty.$$

The interval of convergence of the given power series is $\mathbb{R}$.

12. The coefficients of the given power series are given by $c_n = a^n + b^n$. Now since $a > b > 0$, we have

$$a^n \le a^n + b^n \le a^n + a^n \qquad \text{for every } n \in \mathbb{N},$$

so taking $n^{\text{th}}$ root of each component we obtain

$$a \le \sqrt[n]{a^n + b^n} \le \sqrt[n]{2}a \qquad \text{for every } n \in \mathbb{N}.$$

Since $\lim_{n\to+\infty} \sqrt[n]{2} = 2^{\lim_{n\to+\infty}\frac{1}{n}} = 2^0 = 1$, the limit

$$\lim_{n\to+\infty} \sqrt[n]{|c_n|} = \lim_{n\to+\infty} \sqrt[n]{a^n + b^n} = a$$

exists by squeeze theorem. Therefore the radius of convergence of the given power series is

$$R = \frac{1}{\lim_{n\to+\infty} \sqrt[n]{|c_n|}} = \frac{1}{a}.$$

13. (a) Suppose that the power series $\sum_{k=0}^{+\infty} a_k x^k$ has radius of convergence $R$. Then the series $\sum_{k=0}^{+\infty} a_k t^k$ converges for every $t \in (-R, R)$.

- Now if $x \in \left(-R^{\frac{1}{m}}, R^{\frac{1}{m}}\right)$, then $x^m \in (-R, R)$, so the series $\sum_{k=0}^{+\infty} a_k x^{mk} = \sum_{k=0}^{+\infty} a_k (x^m)^k$ converges. This shows that the radius of convergence of $\sum_{k=0}^{+\infty} a_k x^{mk}$ is at least $R^{\frac{1}{m}}$.
- If the radius of convergence of $\sum_{k=0}^{+\infty} a_k x^{mk}$, call it $\rho$, is strictly greater than $R^{\frac{1}{m}}$, then we take a number $t \in \left(R^{\frac{1}{m}}, \rho\right)$. The series $\sum_{k=0}^{+\infty} a_k t^{mk}$ converges. But this implies that the series $\sum_{k=0}^{+\infty} a_k x^k$ converges at the number $x = t^m > R$, which contradicts with the fact that $\sum_{k=0}^{+\infty} a_k x^k$ has radius of convergence $R$.

Therefore the radius of convergence of $\sum_{k=0}^{+\infty} a_k x^{mk}$ is exactly $R^{\frac{1}{m}}$. ■

(b) (i) We consider the power series $\sum_{k=0}^{+\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^k$ whose coefficients are given by $c_n = \frac{(-1)^n}{2^{2n}(n!)^2}$. Since

$$\lim_{n\to+\infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n\to+\infty} \frac{2^{2(n+1)}\left((n+1)!\right)^2}{2^{2n}(n!)^2} = \lim_{n\to+\infty} 4(n+1)^2 = +\infty$$

exists, so the radius of convergence of the given power series is also $R = +\infty$. According to (a), the radius of convergence of $J_0(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k}$ is also $+\infty$.

(ii) We consider the power series $1 + \frac{1}{2\cdot 3}x + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^2 + \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}x^3 + \cdots$ whose coefficients are given by $c_n = \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdots(3n)}$. Since

$$\lim_{n\to+\infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n\to+\infty} \frac{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdots(3n)(3n+2)(3n+3)}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdots(3n)} = \lim_{n\to+\infty} (3n+2)(3n+3) = +\infty$$

exists, so the radius of convergence of the given power series is also $R = +\infty$. According to (a), the radius of convergence of $A(x) = 1 + \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 + \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}x^9 + \cdots$ is also $+\infty$.

(c) (i) For every $x \in \mathbb{R}$, we differentiate the power series $J_0(x)$ termwise to obtain

$$J_0{}'(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{2^{2k}(k!)^2} 2kx^{2k-1} = \sum_{k=1}^{+\infty} \frac{(-1)^k}{2^{2k-1}(k!)(k-1)!} x^{2k-1} = \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k+1}(k+1)!\,k!} x^{2k+1}.$$

Differentiating once more, we have

$$\begin{aligned} J_0{}''(x) &= \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k+1}(k+1)!\,k!}(2k+1)x^{2k} \\ &= \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k+1}(k+1)!\,k!}(2k+2)x^{2k} - \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k+1}(k+1)!\,k!} x^{2k} \\ &= \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k}(k!)^2} x^{2k} - \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k+1}(k+1)!\,k!} x^{2k}. \end{aligned}$$

Thus for every $x \in \mathbb{R}$ we have

$$\begin{aligned} xJ_0{}''(x) + J_0{}'(x) &= \left( \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k}(k!)^2} x^{2k+1} - \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k+1}(k+1)!\,k!} x^{2k+1} \right) + \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k+1}(k+1)!\,k!} x^{2k+1} \\ &= \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2^{2k}(k!)^2} x^{2k+1} = -x \sum_{k=0}^{+\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k} = -xJ_0(x), \end{aligned}$$

i.e. $xJ_0{}''(x) + J_0{}'(x) + xJ_0(x) = 0$. ■

(ii) For every $x \in \mathbb{R}$, we differentiate the power series $A(x)$ termwise to obtain

$$A'(x) = \frac{1}{2\cdot 3} 3x^2 + \frac{1}{2\cdot 3\cdot 5\cdot 6} 6x^5 + \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} 9x^8 + \cdots$$

$$\begin{aligned} A''(x) &= \frac{1}{2\cdot 3} 3\cdot 2x + \frac{1}{2\cdot 3\cdot 5\cdot 6} 6\cdot 5x^4 + \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} 9\cdot 8x^7 + \cdots \\ &= x + \frac{1}{2\cdot 3} x^4 + \frac{1}{2\cdot 3\cdot 5\cdot 6} x^7 + \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} x^{10} + \cdots \\ &= x\left(1 + \frac{1}{2\cdot 3} x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6} x^6 + \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} x^9 + \cdots\right) = xA(x), \end{aligned}$$

i.e. $A''(x) - xA(x) = 0$. ■

14. (a) Consider the geometric series

$$\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x} \qquad \text{for every } x \in (-1, 1).$$

Differentiating both sides, we have $\sum_{k=1}^{+\infty} kx^{k-1} = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$ for every $x \in (-1, 1)$. Multiplying both sides by $x$, we have

$$\sum_{k=1}^{+\infty} kx^k = \frac{x}{(1-x)^2} \qquad \text{for every } x \in (-1, 1).$$

Differentiating both sides again, we have $\sum_{k=1}^{+\infty} k^2x^{k-1} = \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^3}$ for every $x \in (-1, 1)$.

Multiplying both sides by $x$, we have

$$\sum_{k=1}^{+\infty} k^2 x^k = \frac{x+x^2}{(1-x)^3} \qquad \text{for every } x \in (-1,1).$$

At both the end-points $-1$ and $1$ of the interval of convergence, the series diverges by term test.

(b) Consider the geometric series

$$\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x} \qquad \text{for every } x \in (-1,1).$$

Shifting the indices of the series, we also have $\sum_{k=2}^{+\infty} x^{k-2} = \frac{1}{1-x}$ for every $x \in (-1,1)$. Now relabeling $x$ by $t$ and then integrating termwise from $0$ to $x$, we obtain

$$\sum_{k=2}^{+\infty} \frac{1}{k-1} x^{k-1} = \int_0^x \frac{1}{1-t}\,dt = -\ln(1-x) \qquad \text{for every } x \in (-1,1).$$

Integrating once more, we obtain

$$\sum_{k=2}^{+\infty} \frac{1}{k(k-1)} x^k = \int_0^x -\ln(1-t)\,dt = (1-x)\ln(1-x) + x \qquad \text{for every } x \in (-1,1).$$

Finally replacing $x$ by $x-1$, we have

$$\sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (x-1)^k = (2-x)\ln(2-x) + x - 1 \qquad \text{for every } x \in (0,2).$$

- At the end-point $2$ of the interval of convergence, the series becomes $\sum_{k=2}^{+\infty} \frac{1}{k(k-1)} = \sum_{k=2}^{+\infty} \left(\frac{1}{k-1} - \frac{1}{k}\right)$ which is a telescoping series which converges to $1$.
- At the end-point $0$ of the interval of convergence, the series becomes $\sum_{k=2}^{+\infty} \frac{(-1)^k}{k(k-1)}$ which converges by alternating series test. According to Abel's limit theorem, we have

$$\sum_{k=2}^{+\infty} \frac{(-1)^k}{k(k-1)} = \lim_{x\to 0^+} [(2-x)\ln(2-x) + x - 1] = 2\ln 2 - 1.$$

(c) Consider the geometric series

$$\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x} \qquad \text{for every } x \in (-1,1).$$

Shifting the indices of the series, we also have $\sum_{k=1}^{+\infty} x^{k-1} = \frac{1}{1-x}$ for every $x \in (-1,1)$. Now relabeling $x$ by $t$ and then integrating termwise from $0$ to $x$, we obtain

$$\sum_{k=1}^{+\infty} \frac{1}{k} x^k = \int_0^x \frac{1}{1-t}\,dt = -\ln(1-x) \qquad \text{for every } x \in (-1,1).$$

Integrating once more, we obtain

$$\sum_{k=1}^{+\infty} \frac{1}{k(k+1)} x^{k+1} = \int_0^x -\ln(1-t)\,dt = (1-x)\ln(1-x) + x \qquad \text{for every } x \in (-1,1).$$

Integrating once more, we have

$$\sum_{k=1}^{+\infty}\frac{1}{k(k+1)(k+2)}x^{k+2}=\int_0^x[(1-t)\ln(1-t)+t]dt=\frac{3}{4}x^2-\frac{1}{2}x-\frac{1}{2}(1-x)^2\ln(1-x)$$

for every $x\in(-1,1)$. Dividing by $x^2$, we get

$$\sum_{k=1}^{+\infty}\frac{1}{k(k+1)(k+2)}x^k=\frac{3}{4}-\frac{1}{2x}-\frac{1}{2}\left(\frac{1}{x}-1\right)^2\ln(1-x)$$

for every $x\in(-1,1)\setminus\{0\}$. For $x=0$, the series becomes $0$ as all terms are $0$.

- ⊙ At the end-point $1$ of the interval of convergence, the series becomes $\sum_{k=1}^{+\infty}\frac{1}{k(k+1)(k+2)}$ which converges by comparing to the $p$-series $\sum_{k=1}^{+\infty}\frac{1}{k^3}$. According to Abel's limit theorem, we have

$$\sum_{k=1}^{+\infty}\frac{1}{k(k+1)(k+2)}=\lim_{x\to1^-}\left[\frac{3}{4}-\frac{1}{2x}-\frac{1}{2}\left(\frac{1}{x}-1\right)^2\ln(1-x)\right]=\frac{1}{4}.$$

- ⊙ At the end-point $-1$ of the interval of convergence, the series becomes $\sum_{k=1}^{+\infty}\frac{(-1)^k}{k(k+1)(k+2)}$ which converges by alternating series test. According to Abel's limit theorem, we have

$$\sum_{k=1}^{+\infty}\frac{(-1)^k}{k(k+1)(k+2)}=\lim_{x\to-1^+}\left[\frac{3}{4}-\frac{1}{2x}-\frac{1}{2}\left(\frac{1}{x}-1\right)^2\ln(1-x)\right]=\frac{5}{4}-2\ln 2.$$

(d) Consider the geometric series

$$\sum_{k=0}^{+\infty}x^{2k}=\frac{1}{1-x^2}\qquad\text{for every } x\in(-1,1).$$

Relabeling $x$ by $t$ and then integrating termwise from $0$ to $x$, we obtain

$$\sum_{k=0}^{+\infty}\frac{1}{2k+1}x^{2k+1}=\int_0^x\frac{1}{1-t^2}dt=\frac{1}{2}\ln(1+x)-\frac{1}{2}\ln(1-x)\qquad\text{for every } x\in(-1,1).$$

- ⊙ At the end-point $1$ of the interval of convergence, the series becomes $\sum_{k=0}^{+\infty}\frac{1}{2k+1}$ which diverges by comparing to the harmonic series $\sum_{k=0}^{+\infty}\frac{1}{2k+2}=\frac{1}{2}\sum_{k=1}^{+\infty}\frac{1}{k}$.
- ⊙ At the end-point $-1$ of the interval of convergence, the series becomes $\sum_{k=0}^{+\infty}\frac{-1}{2k+1}=-\sum_{k=0}^{+\infty}\frac{1}{2k+1}$ which also diverges.

(e) Consider the geometric series

$$\sum_{k=0}^{+\infty}x^k=\frac{1}{1-x}\qquad\text{for every } x\in(-1,1).$$

Differentiating both sides, we have $\sum_{k=1}^{+\infty}kx^{k-1}=\frac{d}{dx}\frac{1}{1-x}=\frac{1}{(1-x)^2}$ for every $x\in(-1,1)$. Multiplying both sides by $x$, we have

$$\sum_{k=1}^{+\infty}kx^k=\frac{x}{(1-x)^2}\qquad\text{for every } x\in(-1,1).$$

Now relabeling $x$ by $t$ and then integrating termwise from $0$ to $x$, we obtain

$$\sum_{k=1}^{+\infty} \frac{k}{k+1} x^{k+1} = \int_0^x \frac{t}{(1-t)^2}\,dt = \ln(1-x) + \frac{x}{1-x} \qquad \text{for every } x \in (-1,1).$$

Dividing by $x$, we get

$$\sum_{k=1}^{+\infty} \frac{k}{k+1} x^{k} = \frac{\ln(1-x)}{x} + \frac{1}{1-x} \qquad \text{for every } x \in (-1,1) \setminus \{0\};$$

while for $x = 0$ the series converges to $0$ since all terms become $0$.

At both the end-points $-1$ and $1$ of the interval of convergence, the series diverges by term test.

15. (a) Consider the power series

$$\sum_{k=1}^{+\infty} (x^{3k-1} - x^{3k+1}) = x^2 - x^4 + x^5 - x^7 + x^8 - x^{10} + \cdots,$$

which converges absolutely for $x \in (-1,1)$. By the sum of geometric series we have

$$\sum_{k=1}^{+\infty} (x^{3k-1} - x^{3k+1}) = \frac{x^2}{1-x^3} - \frac{x^4}{1-x^3} = \frac{x^2(1-x^2)}{1-x^3} = \frac{x^2+x^3}{1+x+x^2} \qquad \text{for every } x \in (-1,1).$$

Relabeling $x$ by $t$ and then integrating termwise from $0$ to $x$, we obtain

$$\begin{aligned} f(x) = \sum_{k=1}^{+\infty} \left( \frac{1}{3k} x^{3k} - \frac{1}{3k+2} x^{3k+2} \right) &= \int_0^x \frac{t^2+t^3}{1+t+t^2}\,dt \\ &= \int_0^x \left( t - \frac{1}{2}\frac{2t+1}{1+t+t^2} + \frac{1}{2}\frac{1}{\left(t+\frac{1}{2}\right)^2 + \frac{3}{4}} \right) dt \\ &= \frac{1}{2}x^2 - \frac{1}{2}\ln(1+x+x^2) + \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} - \frac{\pi}{6\sqrt{3}} \end{aligned}$$

for every $x \in (-1,1)$.

(b) Since

$$0 \le \frac{1}{3n^2+2n} \le \frac{1}{3n^2}$$

for every $n \in \mathbb{N}$ and since $\sum_{k=1}^{+\infty} \frac{1}{3k^2} = \frac{1}{3}\sum_{k=1}^{+\infty} \frac{1}{k^2}$ converges by $p$-test, the given series $\sum_{k=1}^{+\infty} \frac{1}{3k^2+2k}$ converges by comparison test. Now according to (a) and Abel's limit theorem, we have

$$\begin{aligned} \sum_{k=1}^{+\infty} \frac{1}{3k^2+2k} = \frac{3}{2}\sum_{k=1}^{+\infty}\left(\frac{1}{3k} - \frac{1}{3k+2}\right) = \frac{3}{2} f(1) &= \frac{3}{2}\lim_{x\to 1^-} f(x) \\ &= \frac{3}{2}\lim_{x\to 1^-}\left[\frac{1}{2}x^2 - \frac{1}{2}\ln(1+x+x^2) + \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} - \frac{\pi}{6\sqrt{3}}\right] \\ &= \frac{3}{4} - \frac{3}{4}\ln 3 + \frac{\pi}{4\sqrt{3}}. \end{aligned}$$