Problem Set 5

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 5 covers materials from §7.1 – §7.6.

1. Apart from the definition from our lecture, an ellipse and a hyperbola can also be constructed using a **fixed line** and a **fixed point** not on the line, in a similar fashion as a parabola. Let $F(4,0)$ be a point in $\mathbb{R}^2$.
   (a) Show that all the points $P(x,y)$ such that
   $$\frac{\text{distance from } P \text{ to } F}{\text{distance from } P \text{ to the } y\text{-axis}} = \frac{1}{3}$$
   form an ellipse in $\mathbb{R}^2$.
   (b) Show that all the points $P(x,y)$ such that
   $$\frac{\text{distance from } P \text{ to } F}{\text{distance from } P \text{ to the } y\text{-axis}} = 3$$
   form a hyperbola in $\mathbb{R}^2$.
2. Let a and b be non-zero real numbers, and consider the ellipse in $\mathbb{R}^2$ defined by
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$
   Let m be a real number. Find the equations of all the lines with slope m that are tangent to the above ellipse.
3. Suppose that a point in $\mathbb{R}^2$ has rectangular coordinate (x,y) and polar coordinate (r,θ), where $r > 0$ and $\theta \in (-\pi, \pi]$. Express r and θ in terms of x and y.
4. Let a and b be non-zero real numbers. Show that the polar equation
   $$r = a\sin\theta + b\cos\theta$$
   represents a circle in $\mathbb{R}^2$. What are the center and the radius of this circle?
5. Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^3$ such that $\|\mathbf{u}\| = 4$, $\|\mathbf{v}\| = 3$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{3}$.
   (a) Find $\mathbf{u} \cdot \mathbf{v}$.
   (b) Find the real number k such that the vectors $\mathbf{u} + k\mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$ are orthogonal.
   (c) Let $\mathbf{a} = 3\mathbf{u} + 4\mathbf{v}$ and $\mathbf{b} = -2\mathbf{u} - \mathbf{v}$. Find the area of the parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as two adjacent edges.
6. (**Parallelogram Law**) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in the same dimension. Show that
   $$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2).$$

7. Determine whether the following statements are true or false. In statements (b) – (f), $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are arbitrary vectors of the same dimension.
   (a) $\mathbb{R}^2$ is a subset of $\mathbb{R}^3$.
   (b) If $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
   (c) If $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
   (d) If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.
   (e) If $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\mathbf{v}$ is orthogonal to $\mathbf{w}$, then $\mathbf{u}$ is orthogonal to $\mathbf{w}$.
   (f) If $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{u}$ is parallel to $\mathbf{v}$ and $\mathbf{v}$ is parallel to $\mathbf{w}$, then $\mathbf{u}$ is parallel to $\mathbf{w}$.

8. Let $\mathbf{a} = \langle a_1, a_2 \rangle$, $\mathbf{b} = \langle b_1, b_2 \rangle$ be fixed vectors in $\mathbb{R}^2$, and let $\mathbf{r} = \langle x, y \rangle$ be a variable vector.
   (a) Show that the vector equation
   $$(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$$
   represents a circle in $\mathbb{R}^2$, and find the center and the radius of the circle.
   (b) What geometric object does the vector equation
   $$(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = (\mathbf{r} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$$
   represent in $\mathbb{R}^2$?

9. Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors of the same dimension. Show that the vector
   $$\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$$
   bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

10. Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors of the same dimension. If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, show that
   $$\mathrm{proj}_{\mathbf{u}}\mathbf{v} \cdot \mathrm{proj}_{\mathbf{v}}\mathbf{u} = (\mathbf{u} \cdot \mathbf{v}) \cos^2 \theta.$$

11. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^n$ such that
   $$\mathbf{a} \cdot \mathbf{a} = 1, \qquad \mathbf{b} \cdot \mathbf{b} = 1 \qquad \text{and} \qquad \mathbf{a} \cdot \mathbf{b} = 0.$$
   Let $S = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u} = x\mathbf{a} + y\mathbf{b} \text{ for some } x, y \in \mathbb{R}\}$.
   (a) Show that for every $\mathbf{u} \in S$, we have
   $$\mathbf{u} = (\mathbf{u} \cdot \mathbf{a})\mathbf{a} + (\mathbf{u} \cdot \mathbf{b})\mathbf{b}.$$
   (b) For each $\mathbf{v} \in \mathbb{R}^n$, let $\mathbf{w} = (\mathbf{v} \cdot \mathbf{a})\mathbf{a} + (\mathbf{v} \cdot \mathbf{b})\mathbf{b}$. Show that $\mathbf{v} - \mathbf{w}$ is orthogonal to every $\mathbf{u} \in S$.

12. Let A, B and C be points in $\mathbb{R}^3$ whose position vectors are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. If
   $$\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} = \mathbf{0},$$
   show that
   (a) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar, and
   (b) A, B and C are collinear.

13. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be three-dimensional vectors.
   (a) Show that
   $$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$
   (b) Hence show that
   $$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}.$$

Q14 – Q18 are some problems in geometry. Try to solve these problems **using vectors**.

14. Find the area of the triangle in $\mathbb{R}^2$ with vertices $A(-3,0)$, $B(-1,3)$ and $C(5,2)$.
15. Let $A(3,-5,1)$, $B(0,2,-2)$, $C(3,1,1)$ and $O(0,0,0)$ be points in $\mathbb{R}^3$. Are they coplanar?
16. Three lines are said to be ***concurrent*** if they pass through the same point.
    (a) A ***median*** of a triangle is a line that passes through both a vertex of the triangle and the mid-point of the edge opposite the vertex. Prove that the three medians of a triangle are concurrent.
    (b) Prove that the three altitudes of a triangle are concurrent.
    (c) Prove that the three perpendicular bisectors of a triangle are concurrent.
17. Prove that the diagonals of a rhombus are perpendicular to each other.
18. Let $ABCD$ be a parallelogram. Let X and Y be the mid-points of BC and CD respectively. Prove that the line segments AX and AY divide the diagonal BD into three portions of equal length.
19. Find a vector equation and parametric equations for each of the following lines in $\mathbb{R}^3$.
    (a) The line passing through $(6,-5,2)$ and parallel to $\langle 3,9,-2\rangle$.
    (b) The line segment with end-points $(4,-6,6)$ and $(2,3,1)$.
    (c) The line passing through $(2,1,0)$ and perpendicular to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{j}+\mathbf{k}$
    (d) The line passing through $(0,1,2)$ and orthogonally intersecting the line
    $$x = 1+t \quad \text{and} \quad y = 1-t \quad \text{and} \quad z = 2t.$$
20. Let $\mathbf{r}_0$ and $\mathbf{r}_1$ be vectors in $\mathbb{R}^3$, and let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^3$. Let P_1 be the point in $\mathbb{R}^3$ with position vector $\mathbf{r}_1$ and let L be the line in $\mathbb{R}^3$ with equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. Show that the distance between the point P_1 and the line L is given by
    $$d(P_1, L) = \frac{\|(\mathbf{r}_1 - \mathbf{r}_0) \times \mathbf{v}\|}{\|\mathbf{v}\|}.$$
21. Let $a > 0$. Show that the two curves with polar equations $r = a\sin\theta$ and $r = a\cos\theta$ intersect at right angles.
    Hint: What are their tangent vectors at a point of intersection?
22. (a) Let P be a point on a smooth curve $r = f(\theta)$ in $\mathbb{R}^2$ which is not the origin, and let α be the acute angle between the line OP and the tangent to the curve at P. Show that
    $$\cos\alpha = \frac{|f'(\theta)|}{\sqrt{f(\theta)^2 + f'(\theta)^2}}.$$
    (b) Using (a), show that at every point P on the curve $r = e^{\theta}$, the angle between the line OP and the tangent line to the curve at P is always $\pi/4$.
    (c) Let $r = f(\theta)$ be a smooth curve such that at every point P on it, the angle between the line OP and the tangent line to the curve at P is always a fixed constant. Show that there exist constants C and k such that $f(\theta) = Ce^{k\theta}$ for all θ.