Solution to Problem Set 3

1. (a) First observe that

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)x \, dx,$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)x \, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x \, dx,$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(m+n)x \, dx - \frac{1}{2}\int_{-\pi}^{\pi} \sin(m-n)x \, dx.$$

⊙ If $m = n = 0$, then we have

$$\int_{-\pi}^{\pi} \cos(m \pm n)x \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi \qquad \text{and} \qquad \int_{-\pi}^{\pi} \sin(m \pm n)x \, dx = \int_{-\pi}^{\pi} 0 \, dx = 0.$$

⊙ If $m = n \neq 0$, then we have

$$\int_{-\pi}^{\pi} \cos(m-n)x \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi, \qquad \int_{-\pi}^{\pi} \sin(m-n)x \, dx = \int_{-\pi}^{\pi} 0 \, dx = 0,$$

$$\int_{-\pi}^{\pi} \cos(m+n)x \, dx = \left[\frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = 0 \qquad \text{and} \qquad \int_{-\pi}^{\pi} \sin(m+n)x \, dx = \left[\frac{-\cos 2nx}{2n}\right]_{-\pi}^{\pi} = 0.$$

⊙ If $m \neq n$, then we have

$$\int_{-\pi}^{\pi} \cos(m \pm n)x \, dx = \left[\frac{\sin(m \pm n)x}{m \pm n}\right]_{-\pi}^{\pi} = 0 \qquad \text{and}$$

$$\int_{-\pi}^{\pi} \sin(m \pm n)x \, dx = \left[\frac{-\cos(m \pm n)x}{m \pm n}\right]_{-\pi}^{\pi} = 0.$$

Therefore in summary, we have

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 2\pi & \text{if } m = n = 0 \\ \pi & \text{if } m = n \neq 0, \\ 0 & \text{if } m \neq n \end{cases} \qquad \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} \pi & \text{if } m = n \neq 0 \\ 0 & \text{otherwise} \end{cases},$$

and

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0 \qquad \text{for every non-negative integers } m, n.$$

(b) For each $k \in \{1, 2, \dots, n\}$, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx) \sin kx \, dx$$

$$= \frac{1}{\pi}\underbrace{\left(a_1 \int_{-\pi}^{\pi} \sin x \sin kx \, dx + a_2 \int_{-\pi}^{\pi} \sin 2x \sin kx \, dx + \cdots + a_n \int_{-\pi}^{\pi} \sin nx \sin kx \, dx\right)}_{\text{only the } k\text{th term is nonzero by (a)}}$$

$$= \frac{1}{\pi}\left(a_k \int_{-\pi}^{\pi} \sin kx \sin kx \, dx\right) = \frac{1}{\pi}(a_k \pi) = a_k.$$

2. (a) Taking antiderivatives by parts, we have

$$\int x^2 \arctan x \, dx = \int \arctan x \, d\left(\frac{x^3}{3}\right) = \frac{1}{3}x^3 \arctan x - \int \frac{x^3}{3}\frac{1}{1+x^2}dx$$

$$= \frac{1}{3}x^3 \arctan x - \frac{1}{6}\int \frac{x^2}{1+x^2}dx^2 = \frac{1}{3}x^3 \arctan x - \frac{1}{6}\int \left(1 - \frac{1}{1+x^2}\right)dx^2$$

$$= \frac{1}{3}x^3 \arctan x - \frac{1}{6}(x^2 - \ln(1+x^2)) + C = \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C,$$

where C is an arbitrary constant.

(b) Taking antiderivatives by parts, we have

$$\int \sin(\ln x) \, dx = x \sin(\ln x) - \int \cos(\ln x) \, dx = x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) \, dx \,,$$

so rearranging the terms we get

$$\int \sin(\ln x) \, dx = \frac{1}{2}x(\sin(\ln x) - \cos(\ln x)) + C,$$

where C is an arbitrary constant.

(c) First consider $\int e^{2x} \sin 2x \, dx$. Taking antiderivatives by parts, we have

$$\int e^{2x} \sin 2x \, dx = -\frac{e^{2x}\cos 2x}{2} + \int e^{2x}\cos 2x \, dx = -\frac{e^{2x}\cos 2x}{2} + \frac{e^{2x}\sin 2x}{2} - \int e^{2x}\sin 2x \, dx \,,$$

so rearranging the terms we get

$$\int e^{2x} \sin 2x \, dx = \frac{e^{2x}(\sin 2x - \cos 2x)}{4} + C.$$

Therefore the required antiderivative is

$$\int e^{2x}(\sin x + \cos x)^2 dx = \int e^{2x}(1 + \sin 2x)dx = \int e^{2x}dx + \int e^{2x}\sin 2x \, dx$$

$$= \frac{e^{2x}}{2} + \frac{e^{2x}(\sin 2x - \cos 2x)}{4} + C,$$

where C is an arbitrary constant.

(d) First consider $\int e^{x^2} dx$. Taking antiderivatives by parts, we have

$$\int e^{x^2}dx = xe^{x^2} - \int x \cdot 2xe^{x^2}dx = xe^{x^2} - \int 2x^2 e^{x^2}dx \,,$$

so rearranging the terms we get

$$\int (2x^2 + 1)e^{x^2}dx = \int 2x^2 e^{x^2}dx + \int e^{x^2}dx = xe^{x^2} + C,$$

where C is an arbitrary constant.

3. For each positive integer n, we have

$$\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}} = \frac{1}{n}\sqrt[n]{(n+1)(n+2)\cdots(2n)} = \sqrt[n]{\frac{(n+1)(n+2)\cdots(2n)}{n^n}} = \sqrt[n]{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)},$$

so

$$\ln\left(\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}\right) = \frac{1}{n}\ln\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right] = \frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right) + \ln\left(1+\frac{2}{n}\right) + \cdots + \ln\left(1+\frac{n}{n}\right)\right],$$

which is the right Riemann sum of the function $f(x) = \ln x$ with respect to the regular partition of $[1, 2]$ into n subintervals. Since f is continuous on $[1, 2]$, it is integrable on $[1, 2]$; thus taking limits as $n \to +\infty$, we have

$$\lim_{n\to+\infty}\ln\left(\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}\right) = \int_1^2 \ln x \, dx = [x\ln x - x]_1^2 = 2\ln 2 - 1.$$

Finally, since the exponential function is continuous at $2\ln 2 - 1$, the required limit is

$$\lim_{n\to+\infty}\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}} = \lim_{n\to+\infty} e^{\ln\left(\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}\right)} = e^{\lim_{n\to+\infty}\ln\left(\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}\right)} = e^{2\ln 2-1} = \frac{4}{e}.$$

4. Since f is continuous, by the first version of Fundamental Theorem Calculus, its area function is differentiable and

$$\frac{d}{dx}\int_{-a}^x f(t)dt = f(x).$$

Therefore we apply integration by parts to get

$$\int_{-a}^a \left(\int_{-a}^x f(t)dt\right)dx = \left[x\int_{-a}^x f(t)dt\right]_{x=-a}^{x=a} - \int_{-a}^a x\left(\frac{d}{dx}\int_{-a}^x f(t)dt\right)dx$$

$$= \left[a\underbrace{\int_{-a}^a f(t)dt}_{=0 \text{ since } f \text{ is odd}} - (-a)\underbrace{\int_{-a}^{-a} f(t)dt}_{=0}\right] - \int_{-a}^a xf(x)dx$$

$$= \int_{-a}^a xf(x)dx.$$

5. (a) The (explicit) substitution $u = \frac{1}{x}$ is non-differentiable (even worse, it is undefined) at the point $0 \in (-1, 1)$. So the substitution rule cannot be applied.

    (b) When handling the integral $\int_a^b f(x)dx$ by the substitution rule, we apply the equality (cf. Theorem 5.68)

    $$\int_?^? f\big(g(u)\big)g'(u)du = \int_a^b f(x)dx$$

    "from the right-hand side to the left-hand side". This requires that $f(x)$ can be expressed as a differentiable function of u, i.e. $f(x) = f\big(g(u)\big)$. But the substitution $u = (x-a)(x-b)$ given in the "proof" does not have an inverse; so x, and consequently $f(x)$, **is not guaranteed to be expressible as a function of u**. So the substitution rule cannot be applied in general.

Remark: In fact, the substitution $u = (x-a)(x-b)$ is valid on **each** interval $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$

separately: We have $x = \frac{a+b}{2} - \sqrt{u + \frac{(b-a)^2}{4}}$ for $x \in \left[a, \frac{a+b}{2}\right]$; and $x = \frac{a+b}{2} + \sqrt{u + \frac{(b-a)^2}{4}}$

for $x \in \left[\frac{a+b}{2}, b\right]$.   If we apply the substitution rule on these two intervals separately, then we

can observe explicitly that $\int_a^b f(x)dx$ does not necessarily equal to zero in general.

(c) The antiderivatives $\int \frac{f'(x)}{f(x)} dx$ on the two sides of the equation <u>may differ by a constant function</u>.   So when

we subtract such an antiderivative from both sides, the left-hand side should be left with a constant function

instead of just 0.   Note that the 1 on the right hand side is indeed a constant function.

6. (a) Applying integration by parts, we have

$$\int_a^b (x-a)f'(x)dx = \int_a^b (x-a)df(x) = [(x-a)f(x)]_a^b - \int_a^b f(x)dx$$

$$= (b-a)f(b) - \int_a^b f(x)dx = \int_a^b f(b)dx - \int_a^b f(x)dx$$

$$= \int_a^b \big(f(b) - f(x)\big)dx.$$

(b) According to the result from (a), we have

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}} \left[f\left(\frac{k}{n}\right) - f(x)\right] dx = \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(x - \frac{k-1}{n}\right) f'(x)dx.$$

Now since f' and $x - \frac{k-1}{n}$ are both continuous on $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ and since $x - \frac{k-1}{n} \geq 0$ for every $x \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$,
by the generalized Mean Value Theorem for integrals, there exists $\omega_k \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$ such that

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(x - \frac{k-1}{n}\right) f'(x)dx = f'(\omega_k) \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(x - \frac{k-1}{n}\right) dx.$$

Note that $\int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(x - \frac{k-1}{n}\right) dx = \frac{1}{2n^2}$ as it represents the area of a right triangle with base $\frac{1}{n}$ and height $\frac{1}{n}$; so

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}} \left[f\left(\frac{k}{n}\right) - f(x)\right] dx = \frac{f'(\omega_k)}{2n^2}.$$

∎

(c) For each $n \in \mathbb{N}$ and each integer $1 \leq k \leq n$, we have $\frac{1}{n}f\left(\frac{k}{n}\right) = \int_{\frac{k-1}{n}}^{\frac{k}{n}} f\left(\frac{k}{n}\right) dx.$ Therefore

$$E_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x)dx = \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f\left(\frac{k}{n}\right) dx - \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)dx$$

$$= \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left[f\left(\frac{k}{n}\right) - f(x)\right] dx.$$

Now applying the result from (b), we see that for each $n \in \mathbb{N}$, there exist n numbers $\omega_1, \omega_2, \ldots, \omega_n$ such that $\omega_k \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$ and

$$nE_n = n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left[f\left(\frac{k}{n}\right) - f(x)\right]dx = n\sum_{k=1}^{n}\frac{f'(\omega_k)}{2n^2} = \frac{1}{2}\sum_{k=1}^{n}f'(\omega_k)\frac{1}{n}.$$

Now this is a Riemann sum of f' with respect to the regular partition of $[0,1]$ into n subintervals. Since f' is continuous on $[0,1]$, it is integrable on $[0,1]$. Thus taking limit as $n \to +\infty$, we have

$$\lim_{n \to +\infty} nE_n = \frac{1}{2}\int_0^1 f'(x)dx = \frac{f(1) - f(0)}{2}$$

by the second version of the Fundamental Theorem of Calculus.   ∎

7. (a) For every $x \in (0, +\infty)$, we have

$$f'(x) = e^x + xe^x > 0,$$

so f is strictly increasing on its domain $[0, +\infty)$.

(b) (i) The domain of g is same as the range of f, which is $[0, +\infty)$. For every $x \in (0, +\infty)$, we have

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{e^{g(x)} + g(x)e^{g(x)}} = \frac{1}{e^{g(x)} + x}$$

since $g(x)e^{g(x)} = f(g(x)) = x$.

(ii) Let $u = g(x)$. Then $du = g'(x)dx$, so taking antiderivatives by parts we have

$$\int g(x)dx = xg(x) - \int xg'(x)dx = xg(x) - \int x \cdot \frac{1}{e^{g(x)} + x}dx$$

$$= xg(x) - \int\left(1 - \frac{e^{g(x)}}{e^{g(x)} + x}\right)dx = xg(x) - \int 1dx + \int e^{g(x)}g'(x)dx$$

$$= xg(x) - x + \int e^u du = xg(x) - x + e^u + C$$

$$= xg(x) - x + e^{g(x)} + C,$$

where C is an arbitrary constant.

(iii) It is clear that $f(0) = 0$ and $f(1) = e$, so $g(0) = 0$ and $g(e) = 1$. By (b)(ii) and the Fundamental Theorem of Calculus, we have

$$\int_0^e g(x)dx = \left[xg(x) - x + e^{g(x)}\right]_0^e$$

$$= (e \cdot 1 - e + e^1) - (0 \cdot 0 - 0 + e^0)$$

$$= e - 1.$$

Remark: The inverse of the function $f(x) = xe^x$ studied in this problem is called ***Lambert's W-function***, and is usually denoted by W_0 or simply W, i.e. $W(x) = f^{-1}(x)$.

8.   (a)   (i)   The derivative of f is $f'(x) = n(x^2 - 1)^{n-1}(2x) = 2nx(x^2 - 1)^{n-1}$ for every $x \in \mathbb{R}$, so

$$(x^2 - 1)f'(x) - 2nxf(x) = (x^2 - 1)[2nx(x^2 - 1)^{n-1}] - 2nx(x^2 - 1)^n = 0$$

for every $x \in \mathbb{R}$.   ∎

(ii)   We differentiate both sides of the result from (a) for $(n + 1)$ times.   According to Leibniz's rule, we have

$$\left[(x^2 - 1)f^{(n+2)}(x) + (n + 1)(2x)f^{(n+1)}(x) + \frac{(n + 1)(n)}{2}(2)f^{(n)}(x)\right]$$

$$- \left[(2nx)f^{(n+1)}(x) + (n + 1)(2n)f^{(n)}(x)\right] = 0$$

for every $x \in \mathbb{R}$, so

$$(x^2 - 1)f^{(n+2)}(x) + [(n + 1)(2x) - (2nx)]f^{(n+1)}(x) + \left[\frac{(n + 1)(n)}{2}(2) - (n + 1)(2n)\right]f^{(n)}(x) = 0,$$

i.e. $(x^2 - 1)f^{(n+2)}(x) + 2xf^{(n+1)}(x) - n(n + 1)f^{(n)}(x) = 0$ for every $x \in \mathbb{R}$.   ∎

(b)   (i)   For every non-negative integer n and every $x \in \mathbb{R}$, we have

$$\frac{d}{dx}[(x^2 - 1)p_n'(x)] = 2xp_n'(x) + (x^2 - 1)p_n''(x) = 2x \cdot \frac{1}{2^n n!}f^{(n+1)}(x) + (x^2 - 1) \cdot \frac{1}{2^n n!}f^{(n+2)}(x)$$

$$= \frac{1}{2^n n!}n(n + 1)f^{(n)}(x) = n(n + 1)p_n(x)$$

according to the result from (a) (ii).   ∎

(ii)   Suppose that m and n are distinct non-negative integers.   Then applying the result from (b) (i) and integration by parts, we have

$$\int_{-1}^{1}[m(m + 1) - n(n + 1)]p_m(x)p_n(x)dx$$

$$= \int_{-1}^{1} m(m + 1)p_m(x)p_n(x)dx - \int_{-1}^{1} n(n + 1)p_n(x)p_m(x)dx$$

$$= \int_{-1}^{1} p_n(x)d[(x^2 - 1)p_m'(x)] - \int_{-1}^{1} p_m(x)d[(x^2 - 1)p_n'(x)]$$

$$= \underbrace{[(x^2 - 1)p_n(x)p_m'(x)]_{-1}^{1}}_{=0} - \int_{-1}^{1}(x^2 - 1)p_m'(x)p_n'(x)dx - \underbrace{[(x^2 - 1)p_m(x)p_n'(x)]_{-1}^{1}}_{=0}$$

$$+ \int_{-1}^{1}(x^2 - 1)p_n'(x)p_m'(x)dx$$

$$= 0.$$

Since $m(m + 1) - n(n + 1) \neq 0$, we have $\int_{-1}^{1} p_m(x)p_n(x)dx = 0$.   ∎

Remark: The functions p_n studied in this problem are called the ***Legendre polynomials***.   We can verify that each p_n is a polynomial of degree n.   The sequence of polynomials (p_n) is said to be ***orthogonal*** because of the property we obtained in (b) (ii).

9. (a) For each positive integer n, we have

$$\frac{d}{dx}\cos^n x \sin nx = (-n\cos^{n-1} x \sin x)(\sin nx) + (\cos^n x)(n \cos nx)$$

$$= n\cos^{n-1} x\,(\cos nx \cos x - \sin nx \sin x) = n\cos^{n-1} x \cos(n+1)x\,;$$

so integration by parts gives

$$I_n = \int_0^{\frac{\pi}{2}}\cos^n x \cos nx\, dx = \frac{1}{n}\int_0^{\frac{\pi}{2}}\cos^n x\, d\sin nx = \frac{1}{n}\underbrace{[\cos^n x \sin nx]_0^{\frac{\pi}{2}}}_{=0} - \frac{1}{n}\int_0^{\frac{\pi}{2}}\sin nx\,(-n\cos^{n-1} x \sin x)dx$$

$$= \int_0^{\frac{\pi}{2}}\cos^{n-1} x\,(\sin nx \sin x)dx = \int_0^{\frac{\pi}{2}}\cos^{n-1} x\left(\frac{1}{2}[\cos(nx-x) - \cos(nx+x)]\right)dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^{n-1} x \cos(n-1)x\, dx - \frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^{n-1} x \cos(n+1)x\, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^{n-1} x \cos(n-1)x\, dx - \frac{1}{2}\underbrace{[\cos^n x \sin nx]_0^{\frac{\pi}{2}}}_{=0} = \frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^{n-1} x \cos(n-1)x\, dx = \frac{1}{2}I_{n-1}.$$

(b) According to the result from (a), for each positive integer n we have

$$I_n = \frac{1}{2}I_{n-1} = \frac{1}{2^2}I_{n-2} = \frac{1}{2^3}I_{n-3} = \cdots = \frac{1}{2^{n-1}}I_1 = \frac{1}{2^n}I_0 = \frac{1}{2^n}\int_0^{\frac{\pi}{2}}1dx = \frac{\pi}{2^{n+1}}.$$

10. (a) For every non-negative integer n and every $t \in [0,1]$, we have $t^n e^0 \le t^n e^t \le t^n e^1$, so

$$\int_0^1 t^n dt \le \int_0^1 t^n e^t dt \le e\int_0^1 t^n dt\,,$$

i.e. $\frac{1}{n+1} \le I_n \le \frac{e}{1+n}$.   ∎

(b) For each positive integer n, integration by parts gives

$$I_n = \int_0^1 t^n e^t dt = \int_0^1 t^n de^t = [t^n e^t]_0^1 - \int_0^1 e^t(nt^{n-1})dt = -nI_{n-1} + e.$$

Now we prove the last equality by induction.   For $n = 0$, the two sides both equal $e - 1$.   Assume that

$$I_m = (-1)^{m+1}m! + e\sum_{k=0}^m (-1)^k \frac{m!}{(m-k)!}$$

for some non-negative integer m.   Then for $n = m+1$, we have

$$I_{m+1} = -(m+1)I_m + e = -(m+1)\left((-1)^{m+1}m! + e\sum_{k=0}^m (-1)^k \frac{m!}{(m-k)!}\right) + e$$

$$= (-1)^{m+2}(m+1)! + e\sum_{k=0}^m (-1)^{k+1}\frac{(m+1)!}{(m-k)!} + e$$

$$= (-1)^{m+2}(m+1)! + e\left(\sum_{k=1}^{m+1} (-1)^k \frac{(m+1)!}{(m+1-k)!} + 1\right)$$

$$= (-1)^{m+2}(m+1)! + e\sum_{k=0}^{m+1} (-1)^k \frac{(m+1)!}{(m+1-k)!}.$$

So the equality is true.   ∎

(c) Suppose that e is a (positive) rational number. Then $e = \frac{p}{q}$ where p and q are positive integers.

According to the result from (b), we have $I_n = (-1)^{n+1}n! + \frac{p}{q}\sum_{k=0}^{n}(-1)^k\frac{n!}{(n-k)!}$, so

$$qI_n = q(-1)^{n+1}n! + p\sum_{k=0}^{n}(-1)^k\frac{n!}{(n-k)!}$$

is an integer, for every non-negative integer n.   Now the result from (a) implies that

$$\frac{q}{n+1} \leq qI_n \leq \frac{p}{n+1} \qquad\qquad \text{for every non-negative integer } n.$$

But when n is sufficiently large, we have $\frac{q}{n+1}, \frac{p}{n+1} \in (0,1)$, so in particular $qI_n \in (0,1)$ also.   This is a contradiction because there is no integer in the open interval $(0,1)$.   ∎

11. (a) For every integers $m \geq 0$ and $n \geq 1$, integration by parts gives

$$B(m,n) = \int_0^1 x^m(1-x)^n dx = \frac{1}{m+1}\int_0^1 (1-x)^n dx^{m+1}$$

$$= \frac{1}{m+1}\underbrace{[x^{m+1}(1-x)^n]_0^1}_{=0} - \frac{1}{m+1}\int_0^1 x^{m+1}(-n(1-x)^{n-1})dx$$

$$= 0 + \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}dx = \frac{n}{m+1}B(m+1, n-1).$$

Therefore

$$B(m,n) = \frac{n}{m+1}B(m+1, n-1) = \frac{n}{m+1}\frac{n-1}{m+2}B(m+2, n-2) = \frac{n}{m+1}\frac{n-1}{m+2}\frac{n-2}{m+3}B(m+3, n-3)$$

$$= \cdots = \frac{n}{m+1}\frac{n-1}{m+2}\frac{n-2}{m+3}\cdots\frac{1}{m+n}B(m+n, 0) = \frac{m!\,n!}{(m+n)!}\underbrace{\int_0^1 x^{m+n}dx}_{=\frac{1}{m+n+1}} = \frac{m!\,n!}{(m+n+1)!}.$$

(b) First, we have

$$\int_0^1 \frac{x^4(x-1)^4}{x^2+1}dx = \int_0^1 \frac{x^8 - 4x^7 + 6x^6 - 4x^5 + x^4}{x^2+1}dx = \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}\right)dx$$

$$= \left[\frac{1}{7}x^7 - \frac{2}{3}x^6 + x^5 - \frac{4}{3}x^3 + 4x - 4\arctan x\right]_0^1 = \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\cdot\frac{\pi}{4} = \frac{22}{7} - \pi.$$

On the other hand, we also have $\frac{1}{2}x^4(x-1)^4 < \frac{x^4(x-1)^4}{x^2+1} < x^4(x-1)^4$ for every $x \in (0,1)$.   So

$$\frac{1}{2}\int_0^1 x^4(x-1)^4 dx < \int_0^1 \frac{x^4(x-1)^4}{x^2+1}dx < \int_0^1 x^4(x-1)^4 dx.$$

According to the result from (a), we have $\int_0^1 x^4(x-1)^4 dx = \frac{4!4!}{9!} = \frac{1}{630}$.   So the above inequality implies that

$$\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630},$$

i.e. $\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}$.   ∎

12. (a) Let $x = a \tan t$ where $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.   Then $\sec t > 0$, so

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \int \frac{1}{a \sec t} a \sec^2 t\, dt = \int \sec t\, dt = \ln|\sec t + \tan t| + C_0$$

$$= \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| + C_0 = \ln\left(\sqrt{x^2 + a^2} + x\right) + C,$$

where $C = C_0 - \ln a$ is an arbitrary constant.

(b) Let $x = a \sec t$ where $t \in \left(0, \frac{\pi}{2}\right)$.   Then $\tan t > 0$, so

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{1}{a \tan t} a \sec t \tan t\, dt = \int \sec t\, dt = \ln|\sec t + \tan t| + C_0$$

$$= \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C_0 = \ln\left|x + \sqrt{x^2 - a^2}\right| + C,$$

where $C = C_0 - \ln a$ is an arbitrary constant.

(c) Let $x = a \sin t$ where $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.   Then $\cos t > 0$, so

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{1}{a \cos t} a \cos t\, dt = \int 1 dt = t + C = \arcsin\frac{x}{a} + C,$$

where C is an arbitrary constant.

(d) Let $x = a \tan t$ where $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.   Then $\sec t > 0$, so

$$\int \sqrt{x^2 + a^2}\, dx = \int (a \sec t) a \sec^2 t\, dt = a^2 \int \sec^3 t\, dt.$$

Now since

$$\int \sec^3 t\, dt = \int \sec t\, d \tan t = \sec t \tan t - \int \tan t\,(\sec t \tan t) dt = \sec t \tan t - \int \sec^3 t\, dt + \int \sec t\, dt,$$

we have

$$\int \sec^3 t\, dt = \frac{1}{2} \sec t \tan t + \frac{1}{2} \int \sec t\, dt = \frac{1}{2} \sec t \tan t + \frac{1}{2} \ln|\sec t + \tan t| + C_0.$$

Therefore

$$\int \sqrt{x^2 + a^2}\, dx = \frac{a^2}{2} \sec t \tan t + \frac{a^2}{2} \ln|\sec t + \tan t| + C_0$$

$$= \frac{a^2}{2} \frac{\sqrt{x^2 + a^2}}{a} \frac{x}{a} + \frac{a^2}{2} \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| + C_0$$

$$= \frac{1}{2} x\sqrt{x^2 + a^2} + \frac{a^2}{2} \ln\left(\sqrt{x^2 + a^2} + x\right) + C,$$

where $C = C_0 - \frac{a^2}{2} \ln a$ is an arbitrary constant.

(e) Let $x = a \sec t$ where $t \in \left(0, \frac{\pi}{2}\right)$.   Then $\tan t > 0$, so

$$\int \sqrt{x^2 - a^2}\, dx = \int (a \tan t) a \sec t \tan t\, dt = a^2 \int \sec t \tan^2 t\, dt\,.$$

Now from the steps in (d) we have

$$\int \sec t \tan^2 t\, dt = \int \sec^3 t\, dt - \int \sec t\, dt = \frac{1}{2} \sec t \tan t - \frac{1}{2} \ln|\sec t + \tan t| + C_0.$$

Therefore

$$\int \sqrt{x^2 - a^2}\, dx = \frac{a^2}{2} \sec t \tan t - \frac{a^2}{2} \ln|\sec t + \tan t| + C_0 = \frac{a^2}{2} \frac{x}{a} \frac{\sqrt{x^2 - a^2}}{a} - \frac{a^2}{2} \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C_0$$

$$= \frac{1}{2} x \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln\left(x + \sqrt{x^2 - a^2}\right) + C,$$

where $C = C_0 + \frac{a^2}{2} \ln a$ is an arbitrary constant.

(f) Let $x = a \sin t$ where $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.   Then $\cos t > 0$, so

$$\int \sqrt{a^2 - x^2}\, dx = \int (a \cos t) a \cos t\, dt = a^2 \int \cos^2 t\, dt = a^2 \int \frac{1 + \cos 2t}{2}\, dt$$

$$= \frac{a^2}{2} t + \frac{a^2}{4} \sin 2t + C = \frac{a^2}{2} t + \frac{a^2}{2} \sin t \cos t + C = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{a^2}{2} \frac{x}{a} \frac{\sqrt{a^2 - x^2}}{a} + C$$

$$= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{1}{2} x \sqrt{a^2 - x^2} + C,$$

where C is an arbitrary constant.

Think: If a is allowed to be negative, how will these antiderivatives be affected?

13. (a) First observe that the function has domain $(-1, 1)$ and

$$\int \frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 - x^4}}\, dx = \int \frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2}\sqrt{1 - x^2}}\, dx = \int \frac{1}{\sqrt{1 - x^2}}\, dx + \int \frac{1}{\sqrt{1 + x^2}}\, dx\,.$$

Now we make the trigonometric substitution $x = \tan t$.   Since $x \in (-1, 1)$, we may choose $t \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ so
that $\sec t > 0$.   Thus,

$$\int \frac{1}{\sqrt{1 + x^2}}\, dx = \int \frac{1}{\sqrt{1 + \tan^2 t}} \sec^2 t\, dt = \int \sec t\, dt = \ln|\sec t + \tan t| + C$$

$$= \ln\left(\sqrt{1 + x^2} + x\right) + C.$$

Therefore

$$\int \frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 - x^4}}\, dx = \int \frac{1}{\sqrt{1 - x^2}}\, dx + \int \frac{1}{\sqrt{1 + x^2}}\, dx = \arcsin x + \ln\left(\sqrt{1 + x^2} + x\right) + C,$$

where C is an arbitrary constant.

(b) Completing squares, we have $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$, so we let $x = -\frac{1}{2} + \frac{\sqrt{3}}{2}\tan t$. Let's choose

$t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ so that $\sec t > 0$. Then $x^2 + x + 1 = \frac{3}{4}\sec^2 t$ and $\sqrt{x^2 + x + 1} = \frac{\sqrt{3}}{2}\sec t$, and also

$$\sin t = \frac{x + \dfrac{1}{2}}{\sqrt{x^2 + x + 1}} \qquad \text{and} \qquad \cos t = \frac{\sqrt{3}}{2\sqrt{x^2 + x + 1}}.$$

Thus,

$$\int \frac{x + 1}{(x^2 + x + 1)\sqrt{x^2 + x + 1}}dx = \int \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}\tan t}{\left(\frac{3}{4}\sec^2 t\right)\left(\frac{\sqrt{3}}{2}\sec t\right)}\left(\frac{\sqrt{3}}{2}\sec^2 t\right) dt$$

$$= \int \left(\frac{2}{3}\cos t + \frac{2}{\sqrt{3}}\sin t\right) dt = \frac{2}{3}\sin t - \frac{2}{\sqrt{3}}\cos t + C$$

$$= \frac{2x + 1}{3\sqrt{x^2 + x + 1}} - \frac{1}{\sqrt{x^2 + x + 1}} + C,$$

where C is an arbitrary constant.

(c) First let $x = \tan t$. Then $\sin t = \frac{x}{\sqrt{1+x^2}}$ and $\cos t = \frac{1}{\sqrt{1+x^2}}$

$$\int \frac{x^2 - 1}{(x^2 + 1)\sqrt{1 + x^4}}dx = \int \frac{\tan^2 t - 1}{(\tan^2 t + 1)\sqrt{1 + \tan^4 t}}(\sec^2 t)dt = \int \frac{\tan^2 t - 1}{\sqrt{1 + \tan^4 t}}dt$$

$$= \int \frac{\sin^2 t - \cos^2 t}{\sqrt{\cos^4 t + \sin^4 t}}dt = \int \frac{\sin^2 t - \cos^2 t}{\sqrt{\cos^4 t + \sin^4 t + 2\sin^2 t \cos^2 t - 2\sin^2 t \cos^2 t}}dt$$

$$= \int \frac{\sin^2 t - \cos^2 t}{\sqrt{(\cos^2 t + \sin^2 t)^2 - 2\sin^2 t \cos^2 t}}dt = \int \frac{\sin^2 t - \cos^2 t}{\sqrt{1 - 2\sin^2 t \cos^2 t}}dt$$

$$= \int \frac{-\cos 2t}{\sqrt{1 - \frac{1}{2}\sin^2 2t}}dt.$$

Next let $u = \sin 2t$. Then $du = \cos 2t\, dt$, so

$$\int \frac{-\cos 2t}{\sqrt{1 - \frac{1}{2}\sin^2 2t}}dt = \int \frac{-1}{\sqrt{1 - \frac{1}{2}u^2}}du = -\sqrt{2}\int \frac{1}{\sqrt{2 - u^2}}du = -\sqrt{2}\arcsin\frac{u}{\sqrt{2}} + C.$$

Therefore

$$\int \frac{x^2 - 1}{(x^2 + 1)\sqrt{1 + x^4}}dx = -\sqrt{2}\arcsin\frac{u}{\sqrt{2}} + C = -\sqrt{2}\arcsin\left(\sqrt{2}\sin t \cos t\right) + C$$

$$= -\sqrt{2}\arcsin\left(\sqrt{2}\cdot\frac{x}{\sqrt{1 + x^2}}\cdot\frac{1}{\sqrt{1 + x^2}}\right) + C = -\sqrt{2}\arcsin\frac{\sqrt{2}x}{1 + x^2} + C,$$

where C is an arbitrary constant.

14. (a) For each $n \geq 2$, taking antiderivatives by parts, we have

$$I_n(x) = \int \frac{x^n}{\sqrt{x^2+1}}\,dx = \int x^{n-1}\left(\frac{x}{\sqrt{x^2+1}}\,dx\right) = \int x^{n-1}d\sqrt{x^2+1}$$

$$= x^{n-1}\sqrt{x^2+1} - \int \sqrt{x^2+1}(n-1)x^{n-2}dx$$

$$= x^{n-1}\sqrt{x^2+1} - (n-1)\int \frac{x^{n-2}(x^2+1)}{\sqrt{x^2+1}}\,dx$$

$$= x^{n-1}\sqrt{x^2+1} - (n-1)\int \frac{x^n}{\sqrt{x^2+1}}\,dx - (n-1)\int \frac{x^{n-2}}{\sqrt{x^2+1}}\,dx$$

$$= x^{n-1}\sqrt{x^2+1} - (n-1)I_n(x) - (n-1)I_{n-2}(x).$$

Rearranging, we obtain

$$I_n(x) = \frac{1}{n}x^{n-1}\sqrt{x^2+1} - \frac{n-1}{n}I_{n-2}(x).$$

(b) Using the result from (a), we obtain

$$\int_0^1 \frac{x^5}{\sqrt{x^2+1}}\,dx = \frac{1}{5}\left[x^4\sqrt{x^2+1}\right]_0^1 - \frac{4}{5}\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$$

$$= \frac{1}{5}\left[x^4\sqrt{x^2+1}\right]_0^1 - \frac{4}{5}\left(\frac{1}{3}\left[x^2\sqrt{x^2+1}\right]_0^1 - \frac{2}{3}\int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx\right)$$

$$= \frac{1}{5}\left[x^4\sqrt{x^2+1}\right]_0^1 - \frac{4}{5}\cdot\frac{1}{3}\left[x^2\sqrt{x^2+1}\right]_0^1 + \frac{4}{5}\cdot\frac{2}{3}\left[\sqrt{x^2+1}\right]_0^1$$

$$= \frac{7\sqrt{2}-8}{15}.$$

15. (a) Let $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \to \mathbb{R}$ be the function $f(x) = \frac{4x}{\pi} - \tan x$. Then

$$f'(x) = \frac{4}{\pi} - \sec^2 x.$$

⊙ Since f' is continuous on $\left[0,\frac{\pi}{4}\right]$, $f'(0) = \frac{4}{\pi} - 1 > 0$ and $f'\left(\frac{\pi}{4}\right) = \frac{4}{\pi} - \sqrt{2} < 0$, by Intermediate Value Theorem f' has at least one root in $\left(0,\frac{\pi}{4}\right)$.

⊙ Since $\sec^2 x$ is strictly increasing on $\left[0,\frac{\pi}{4}\right]$, it follows that f' is strictly decreasing on $\left[0,\frac{\pi}{4}\right]$. So f' has at most one root in $\left(0,\frac{\pi}{4}\right)$.

Therefore f' has exactly one root in $\left(0,\frac{\pi}{4}\right)$, which means that f has exactly one critical number $c \in \left(0,\frac{\pi}{4}\right)$.

Now we have $f'(x)\begin{cases} > 0 & \text{if } x \in [0,c) \\ < 0 & \text{if } x \in \left(c,\frac{\pi}{4}\right] \end{cases}$. Since $f(0) = f\left(\frac{\pi}{4}\right) = 0$, it follows that the global minimum value

of f is 0. Thus $f(x) \geq f(0) = 0$ for every $x \in \left[0,\frac{\pi}{4}\right]$, i.e. $\tan x \leq \frac{4x}{\pi}$ for every $x \in \left[0,\frac{\pi}{4}\right]$.

(b) (i) For each non-negative integer n, since $\tan x \le \frac{4x}{\pi}$ for every $x \in \left[0, \frac{\pi}{4}\right]$ according to (a), we have

$$\tan^n x \le \left(\frac{4x}{\pi}\right)^n \qquad \text{for every } x \in \left[0, \frac{\pi}{4}\right],$$

so

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx \le \int_0^{\frac{\pi}{4}} \left(\frac{4x}{\pi}\right)^n dx = \left(\frac{4}{\pi}\right)^n \left[\frac{1}{n+1} x^{n+1}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4(n+1)}.$$

(ii) For each $n \ge 2$, we have

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \tan^2 x \, dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\sec^2 x - 1) dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, d \tan x - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx = \left[\frac{1}{n-1} \tan^{n-1} x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$$

$$= \frac{1}{n-1} - I_{n-2},$$

or equivalently $I_{n-2} = \frac{1}{n-1} - I_n$.   Hence for each $k \in \mathbb{N}$, we have

$$I_0 = 1 - I_2 = 1 - \frac{1}{3} + I_4 = 1 - \frac{1}{3} + \frac{1}{5} - I_6 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + I_8 = \cdots$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{k-1} \frac{1}{2k-1} + (-1)^k I_{2k}.$$

(c) Using the result from (b)(ii), we have

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{k-1} \frac{1}{2k-1} = I_0 - (-1)^k I_{2k} \qquad \text{for every } k \in \mathbb{N}.$$

Now we have

⊙ $I_0 = \int_0^{\pi/4} 1 \, dx = \frac{\pi}{4}$; and

⊙   according to (b)(i) we have

$$-\frac{\pi}{4(2k+1)} \le (-1)^k I_{2k} \le \frac{\pi}{4(2k+1)} \qquad \text{for every } k \in \mathbb{N},$$

from which we obtain $\lim\limits_{k \to +\infty} (-1)^k I_{2k} = 0$ by Squeeze Theorem, as $\lim\limits_{k \to +\infty} -\frac{\pi}{4(2k+1)} = \lim\limits_{k \to +\infty} \frac{\pi}{4(2k+1)} = 0$.

Therefore

$$\lim_{k \to +\infty} \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{k-1} \frac{1}{2k-1}\right) = \lim_{k \to +\infty} (I_0 - (-1)^k I_{2k})$$

$$= I_0 - \lim_{k \to +\infty} (-1)^k I_{2k}$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

16. (a) For every $m \in \mathbb{R} \setminus \{-1\}$ and every $n \in \mathbb{N}$, taking antiderivative by parts we have

$$I_{m,n}(x) = \int x^m (\ln x)^n dx = \frac{1}{m+1} \int (\ln x)^n dx^{m+1}$$

$$= \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{1}{m+1} \int x^{m+1} \left(n(\ln x)^{n-1} \frac{1}{x} \right) dx$$

$$= \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx$$

$$= \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1} I_{m,n-1}(x).$$

Alternative solution: For every $m \in \mathbb{R} \setminus \{-1\}$ and every $n \geq 0$, taking antiderivative by parts we have

$$I_{m,n}(x) = \int x^m (\ln x)^n dx = \int x^{m+1} (\ln x)^n \frac{1}{x} dx = \frac{1}{n+1} \int x^{m+1} d(\ln x)^{n+1}$$

$$= \frac{1}{n+1} x^{m+1} (\ln x)^{n+1} - \frac{1}{n+1} \int (\ln x)^{n+1} (m+1) x^m dx$$

$$= \frac{1}{n+1} x^{m+1} (\ln x)^{n+1} - \frac{m+1}{n+1} \int x^m (\ln x)^{n+1} dx$$

$$= \frac{1}{n+1} x^{m+1} (\ln x)^{n+1} - \frac{m+1}{n+1} I_{m,n+1}(x).$$

Rearranging the above equation we have $(n+1) I_{m,n}(x) = x^{m+1} (\ln x)^{n+1} - (m+1) I_{m,n+1}(x)$, and so

$$I_{m,n+1}(x) = \frac{1}{m+1} x^{m+1} (\ln x)^{n+1} - \frac{n+1}{m+1} I_{m,n}(x).$$

Renaming the positive integer $n+1$ as n, we have

$$I_{m,n}(x) = \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1} I_{m,n-1}(x)$$

for every $m \in \mathbb{R} \setminus \{-1\}$ and every positive integer n.

(b) Using the reduction formula obtained in (a), we have

$$\int \frac{(\ln x)^3}{x^4} dx = I_{-4,3}(x) = -\frac{(\ln x)^3}{3x^3} - \frac{3}{-3} I_{-4,2}(x) = -\frac{(\ln x)^3}{3x^3} + I_{-4,2}(x)$$

$$= -\frac{(\ln x)^3}{3x^3} + \left(-\frac{(\ln x)^2}{3x^3} - \frac{2}{-3} I_{-4,1}(x) \right) = -\frac{(\ln x)^3}{3x^3} - \frac{(\ln x)^2}{3x^3} + \frac{2}{3} I_{-4,1}(x)$$

$$= -\frac{(\ln x)^3}{3x^3} - \frac{(\ln x)^2}{3x^3} + \frac{2}{3} \left(-\frac{\ln x}{3x^3} - \frac{1}{-3} I_{-4,0}(x) \right) = -\frac{(\ln x)^3}{3x^3} - \frac{(\ln x)^2}{3x^3} - \frac{2 \ln x}{9x^3} + \frac{2}{9} \int \frac{1}{x^4} dx$$

$$= -\frac{(\ln x)^3}{3x^3} - \frac{(\ln x)^2}{3x^3} - \frac{2 \ln x}{9x^3} - \frac{2}{27x^3} + C,$$

where C is an arbitrary constant.