

Solution to Problem Set 6

1. (a) The given curve is the graph of $f: [1, 4] \rightarrow \mathbb{R}$ defined by $f(x) = x^{\frac{1}{2}} - (1/3)x^{\frac{3}{2}}$. Since

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2}\sqrt{x}$$

for every $x \in (1, 4)$, the given curve has arc-length

$$\begin{aligned} l &= \int_1^4 \sqrt{1 + [f'(x)]^2} dx = \int_1^4 \sqrt{1 + \left(\frac{1}{2\sqrt{x}} - \frac{1}{2}\sqrt{x}\right)^2} dx = \int_1^4 \sqrt{\left(\frac{1}{2\sqrt{x}} + \frac{1}{2}\sqrt{x}\right)^2} dx \\ &= \int_1^4 \left(\frac{1}{2\sqrt{x}} + \frac{1}{2}\sqrt{x}\right) dx = \left[x^{\frac{1}{2}} + \frac{1}{3}x^{\frac{3}{2}}\right]_1^4 = \frac{10}{3}. \end{aligned}$$

- (b) Since $\frac{dx}{dy} = y^{-\frac{1}{3}}$ for every $y \in (1, 8)$, the given curve has arc-length

$$l = \int_1^8 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_1^8 \sqrt{1 + \left(y^{-\frac{1}{3}}\right)^2} dy = \int_1^8 \sqrt{1 + y^{-\frac{2}{3}}} dy.$$

With the substitution $u = y^{\frac{2}{3}}$, we have $y = u^{\frac{3}{2}}$ and $dy = (3/2)u^{\frac{1}{2}} du$. Therefore

$$l = \int_1^4 \sqrt{1 + u^{-1}} \cdot \frac{3}{2}u^{\frac{1}{2}} du = \frac{3}{2} \int_1^4 \sqrt{u+1} du = \left[(u+1)^{\frac{3}{2}}\right]_1^4 = 5\sqrt{5} - 2\sqrt{2}.$$

- (c) The given curve is the graph of $f: [1, \sqrt{2}] \rightarrow \mathbb{R}$ defined by $f(x) = \ln(x + \sqrt{x^2 - 1})$. Since

$$f'(x) = \frac{1}{x + \sqrt{x^2 - 1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x\right) = \frac{1}{\sqrt{x^2 - 1}}$$

for every $x \in (1, \sqrt{2})$, the given curve has arc-length

$$l = \int_1^{\sqrt{2}} \sqrt{1 + [f'(x)]^2} dx = \int_1^{\sqrt{2}} \sqrt{1 + \left(\frac{1}{\sqrt{x^2 - 1}}\right)^2} dx = \int_1^{\sqrt{2}} \frac{x}{\sqrt{x^2 - 1}} dx = \left[\sqrt{x^2 - 1}\right]_1^{\sqrt{2}} = 1.$$

Alternative solution: The equation $y = \ln(x + \sqrt{x^2 - 1})$ of the curve for $x \in [1, \sqrt{2}]$ can be rewritten as

$$x = \frac{e^y + e^{-y}}{2}$$

for $y \in [0, \ln(1 + \sqrt{2})]$. Since

$$\frac{dx}{dy} = \frac{e^y - e^{-y}}{2}$$

for every $y \in (0, \ln(1 + \sqrt{2}))$, the given curve has arc-length

$$\begin{aligned} l &= \int_0^{\ln(1+\sqrt{2})} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^{\ln(1+\sqrt{2})} \sqrt{1 + \left(\frac{e^y - e^{-y}}{2}\right)^2} dy = \int_0^{\ln(1+\sqrt{2})} \sqrt{\left(\frac{e^y + e^{-y}}{2}\right)^2} dy \\ &= \int_0^{\ln(1+\sqrt{2})} \left|\frac{e^y + e^{-y}}{2}\right| dy = \int_0^{\ln(1+\sqrt{2})} \frac{e^y + e^{-y}}{2} dy = \left[\frac{e^y - e^{-y}}{2}\right]_0^{\ln(1+\sqrt{2})} = 1. \end{aligned}$$

- (d) The given curve is the graph of $f: [e, e^3] \rightarrow \mathbb{R}$ defined by $f(x) = \int_e^x \sqrt{(\ln t)^2 - 1} dt$. By the first version of the Fundamental Theorem of Calculus we have

$$f'(x) = \sqrt{(\ln x)^2 - 1}$$

for every $x \in (e, e^3)$, so the given curve has arc-length

$$\begin{aligned} l &= \int_e^{e^3} \sqrt{1 + [f'(x)]^2} dx = \int_e^{e^3} \sqrt{1 + (\sqrt{(\ln x)^2 - 1})^2} dx = \int_e^{e^3} \sqrt{(\ln x)^2} dx \\ &= \int_e^{e^3} |\ln x| dx = \int_e^{e^3} \ln x dx = [x \ln x - x]_e^{e^3} = 2e^3. \end{aligned}$$

2. Let f be the function $f(\theta) = 1 + \sin \theta$. The required arc-length is given by

$$\begin{aligned} l &= \int_0^{\frac{3\pi}{2}} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_0^{\frac{3\pi}{2}} \sqrt{(1 + \sin \theta)^2 + (\cos \theta)^2} d\theta \\ &= \int_0^{\frac{3\pi}{2}} \sqrt{2 + 2 \sin \theta} d\theta = \int_0^{\frac{3\pi}{2}} \sqrt{2 + 2 \cos\left(\frac{\pi}{2} - \theta\right)} d\theta. \end{aligned}$$

With a substitution $u = \frac{\pi}{2} - \theta$, we obtain

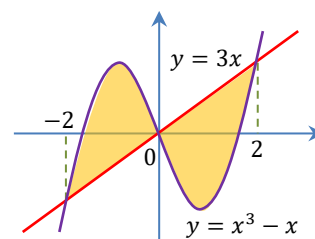
$$\begin{aligned} l &= \int_{-\pi}^{\frac{\pi}{2}} \sqrt{2 + 2 \cos u} du = \int_{-\pi}^{\frac{\pi}{2}} \sqrt{4 \cos^2 \frac{u}{2}} du = \int_{-\pi}^{\frac{\pi}{2}} \left| 2 \cos \frac{u}{2} \right| du \\ &= \int_{-\pi}^{\frac{\pi}{2}} 2 \cos \frac{u}{2} du = \left[4 \sin \frac{u}{2} \right]_{-\pi}^{\frac{\pi}{2}} = 4 + 2\sqrt{2}. \end{aligned}$$

3. (a) We first find the x -coordinates of the points of intersection of the two graphs by solving

$$\begin{cases} y = x^3 - x \\ y = 3x \end{cases}.$$

The solutions are $x = -2$, $x = 0$ and $x = 2$. So the required area is

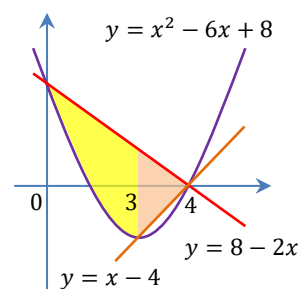
$$\begin{aligned} A &= \int_{-2}^0 [(x^3 - x) - (3x)] dx + \int_0^2 [(3x) - (x^3 - x)] dx \\ &= \left[\frac{1}{4} x^4 - 2x^2 \right]_{-2}^0 + \left[2x^2 - \frac{1}{4} x^4 \right]_0^2 \\ &= 8. \end{aligned}$$



- (b) The x -coordinates of the points of intersection of the graphs are 0, 3 and 4.

The area of the required region is given by

$$\begin{aligned} A &= \int_0^3 [(8 - 2x) - (x^2 - 6x + 8)] dx + \int_3^4 [(8 - 2x) - (x - 4)] dx \\ &= \left[-\frac{1}{3} x^3 + 2x^2 \right]_0^3 + \left[-\frac{3}{2} x^2 + 12x \right]_3^4 \\ &= \frac{21}{2}. \end{aligned}$$

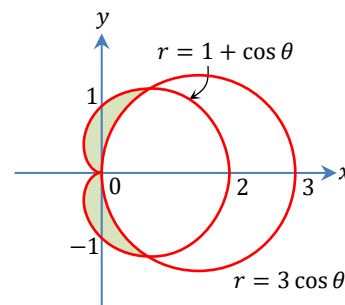


4. On solving the inequality $1 + \cos \theta \geq 3 \cos \theta$, we find that $\theta \in \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$. But we note that $3 \cos \theta \geq 0$ only when θ is in Quadrant I or IV, so the required region can be written as

$$\left\{(r, \theta): \theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, \frac{5\pi}{3}\right] \text{ and } 3 \cos \theta \leq r \leq 1 + \cos \theta\right\} \cup \left\{(r, \theta): \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \text{ and } 0 \leq r \leq 1 + \cos \theta\right\}.$$

Its area is given by

$$\begin{aligned} A &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2} [(1 + \cos \theta)^2 - (3 \cos \theta)^2] d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2} (1 + \cos \theta)^2 d\theta + \int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}} \frac{1}{2} [(1 + \cos \theta)^2 - (3 \cos \theta)^2] d\theta \\ &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2} [(1 + \cos \theta)^2 - (3 \cos \theta)^2] d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2} (1 + \cos \theta)^2 d\theta \\ &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(-\frac{3}{2} + \cos \theta - 2 \cos 2\theta\right) d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{9}{4} + \frac{9}{4} \cos 2\theta\right) d\theta \\ &= \left[-\frac{3}{2}\theta + \sin \theta - \sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} + \left[\frac{9}{4}\theta + \frac{9}{8} \sin 2\theta\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \frac{\pi}{4}. \end{aligned}$$



5. The polar curve $r = f(\theta)$ can be parametrized by the vector-valued function $\mathbf{r}: [0, \pi] \rightarrow \mathbb{R}^2$ defined by

$$\mathbf{r}(t) = \langle f(t) \cos t, f(t) \sin t \rangle.$$

According to Theorem 7.96, the area between the x -axis and this parametrized curve is

$$\begin{aligned} A &= - \int_{t=0}^{t=\pi} y dx = - \int_0^\pi (f(t) \sin t)(f'(t) \cos t - f(t) \sin t) dt \\ &= \int_0^\pi f(t)^2 \sin^2 t dt - \int_0^\pi f(t) f'(t) \sin t \cos t dt. \end{aligned}$$

Our goal is to show that this area equals to $\int_0^\pi \frac{1}{2} [f(t)]^2 dt$ as in Theorem 7.98. Now in the second integral on the right-hand side, we integrate by parts to obtain

$$\begin{aligned} A &= \int_0^\pi f(t)^2 \sin^2 t dt - \frac{1}{2} \int_0^\pi \sin t \cos t d[f(t)]^2 \\ &= \int_0^\pi f(t)^2 \sin^2 t dt - \frac{1}{2} \underbrace{[f(t)^2 \sin t \cos t]_0^\pi}_{=0} + \frac{1}{2} \int_0^\pi f(t)^2 (\cos^2 t - \sin^2 t) dt \\ &= \int_0^\pi f(t)^2 \left(\sin^2 t + \frac{1}{2} (\cos^2 t - \sin^2 t) \right) dt \\ &= \int_0^\pi f(t)^2 \left(\frac{1}{2} (\cos^2 t + \sin^2 t) \right) dt = \int_0^\pi \frac{1}{2} [f(t)]^2 dt \end{aligned}$$

as desired. Therefore the area evaluated using the two approaches are the same. ■

6. (a) At each $x \in [0, 1]$, the cross-section of the solid perpendicular to the x -axis is a disk of diameter $\sqrt{x} - x^2$,

whose area is $A(x) = \pi \left(\frac{\sqrt{x} - x^2}{2} \right)^2$. So the given solid has volume

$$V = \int_0^1 \pi \left(\frac{\sqrt{x} - x^2}{2} \right)^2 dx = \pi \int_0^1 \left(\frac{1}{4}x - \frac{1}{2}x^{\frac{5}{2}} + \frac{1}{4}x^4 \right) dx = \pi \left[\frac{1}{8}x^2 - \frac{1}{7}x^{\frac{7}{2}} + \frac{1}{20}x^5 \right]_0^1 = \frac{9\pi}{280}.$$

- (b) The y -coordinates of the points of intersection of the line $y = x$ and the parabola $y = x^2/4$ are 0 and 4, so the given solid is bounded between the planes $y = 0$ and $y = 4$. At each $y \in [0, 4]$, the cross-section of the solid perpendicular to the y -axis is an equilateral triangle having an edge with length $2\sqrt{y} - y$, so its area is $A(y) = (\sqrt{3}/4)(2\sqrt{y} - y)^2$. Therefore the given solid has volume

$$V = \int_0^4 \frac{\sqrt{3}}{4} (2\sqrt{y} - y)^2 dy = \sqrt{3} \int_0^4 \left(y - y^{\frac{3}{2}} + \frac{1}{4}y^2 \right) dy = \sqrt{3} \left[\frac{1}{2}y^2 - \frac{2}{5}y^{\frac{5}{2}} + \frac{1}{12}y^3 \right]_0^4 = \frac{8\sqrt{3}}{15}.$$

- (c) At each $x \in [0, 6]$, the cross-section of the solid perpendicular to the x -axis is a square having an edge with length $(\sqrt{6} - \sqrt{x})^2$, so its area is $A(x) = (\sqrt{6} - \sqrt{x})^4$. Therefore the given solid has volume

$$\begin{aligned} V &= \int_0^6 (\sqrt{6} - \sqrt{x})^4 dx = \int_0^6 \left(36 - 24\sqrt{6}x^{\frac{1}{2}} + 36x - 4\sqrt{6}x^{\frac{3}{2}} + x^2 \right) dx \\ &= \left[36x - 16\sqrt{6}x^{\frac{3}{2}} + 18x^2 - \frac{8}{5}\sqrt{6}x^{\frac{5}{2}} + \frac{1}{3}x^3 \right]_0^6 = \frac{72}{5}. \end{aligned}$$

7. The curve $y = \frac{4}{x^3}$ and the lines $x = 1$ and $y = \frac{1}{2}$ intersect at the three points
(1, 4), (1, 1/2) and (2, 1/2).

- (a) The solid obtained by revolving the region about the x -axis has volume

$$V = \int_1^2 \pi \left[\left(\frac{4}{x^3} \right)^2 - \left(\frac{1}{2} \right)^2 \right] dx = \pi \int_1^2 \left(\frac{16}{x^6} - \frac{1}{4} \right) dx = \pi \left[-\frac{16}{5x^5} - \frac{1}{4}x \right]_1^2 = \frac{57\pi}{20}.$$

- (b) (Shell method) The solid obtained by revolving the region about the y -axis has volume

$$V = \int_1^2 2\pi x \left(\frac{4}{x^3} - \frac{1}{2} \right) dx = \pi \int_1^2 \left(\frac{8}{x^2} - x \right) dx = \pi \left[-\frac{8}{x} - \frac{1}{2}x^2 \right]_1^2 = \frac{5\pi}{2}.$$

(Slice method) The equation of the curve can be written as $x = 4^{\frac{1}{3}}y^{-\frac{1}{3}}$, so the solid obtained by revolving the region about the y -axis has volume

$$V = \int_{\frac{1}{2}}^4 \pi \left[\left(4^{\frac{1}{3}}y^{-\frac{1}{3}} \right)^2 - 1^2 \right] dy = \pi \int_{\frac{1}{2}}^4 \left(4^{\frac{2}{3}}y^{-\frac{2}{3}} - 1 \right) dy = \pi \left[4^{\frac{2}{3}} \cdot 3y^{\frac{1}{3}} - y \right]_{\frac{1}{2}}^4 = \frac{5\pi}{2}.$$

- (c) (Shell method) The solid obtained by revolving the region about the line $x = 2$ has volume

$$V = \int_1^2 2\pi(2 - x) \left(\frac{4}{x^3} - \frac{1}{2} \right) dx = \pi \int_1^2 \left(\frac{16}{x^3} - \frac{8}{x^2} - 2 + x \right) dx = \pi \left[-\frac{8}{x^2} + \frac{8}{x} - 2x + \frac{1}{2}x^2 \right]_1^2 = \frac{3\pi}{2}.$$

(Slice method) The equation of the curve can be written as $x = 4^{\frac{1}{3}}y^{-\frac{1}{3}}$, so the solid obtained by revolving the region about the line $x = 2$ has volume

$$\begin{aligned} V &= \int_{\frac{1}{2}}^4 \pi \left[(2 - 1)^2 - \left(2 - 4^{\frac{1}{3}}y^{-\frac{1}{3}} \right)^2 \right] dy = \pi \int_{\frac{1}{2}}^4 \left(-3 + 4^{\frac{1}{3}}y^{-\frac{1}{3}} - 4^{\frac{2}{3}}y^{-\frac{2}{3}} \right) dy \\ &= \pi \left[-3y + 4^{\frac{4}{3}} \cdot \frac{3}{2}y^{\frac{2}{3}} - 4^{\frac{2}{3}} \cdot 3y^{\frac{1}{3}} \right]_{\frac{1}{2}}^4 = \frac{3\pi}{2}. \end{aligned}$$

(d) The solid obtained by revolving the region about the line $y = 4$ has volume

$$\begin{aligned} V &= \int_1^2 \pi \left[\left(4 - \frac{1}{2}\right)^2 - \left(4 - \frac{4}{x^3}\right)^2 \right] dx = \pi \int_1^2 \left(-\frac{15}{4} + \frac{32}{x^3} - \frac{16}{x^6} \right) dx \\ &= \pi \left[-\frac{15}{4}x - \frac{16}{x^2} + \frac{16}{5x^5} \right]_1^2 = \frac{103\pi}{20}. \end{aligned}$$

8. We divide the solid obtained by revolving the region about the line $x = \ln 2$ into cylindrical shells. Now for each $x \in [0, \ln 2]$, the shell of radius $\ln 2 - x$ has height e^{-x} , so the whole solid has volume

$$V = \int_0^{\ln 2} 2\pi(\ln 2 - x)e^{-x} dx = 2\pi \int_0^{\ln 2} (x - \ln 2)(-e^{-x}) dx.$$

To handle this integral, we let $f(x) = x - \ln 2$ and $g'(x) = -e^{-x}$ (so that we may take $g(x) = e^{-x}$). Then integrating by parts we get

$$\begin{aligned} V &= 2\pi[(x - \ln 2)e^{-x}]_0^{\ln 2} - 2\pi \int_0^{\ln 2} e^{-x} dx \\ &= 2\pi[(x - \ln 2)e^{-x} + e^{-x}]_0^{\ln 2} = 2\pi \ln 2 - \pi. \end{aligned}$$

Alternative solution: We may also use the usual slice method. The point of intersection of the curve $y = e^{-x}$ and the line $x = \ln 2$ has y -coordinate $\frac{1}{2}$. For each $y \in [0, \frac{1}{2}]$, the cross-section of the required solid of revolution is a disk of radius $\ln 2$, while for each $y \in [\frac{1}{2}, 1]$, the cross section is the difference of two concentric disks with outer radius $\ln 2$ and inner radius $\ln 2 + \ln y$ (note that $\ln y$ is negative). So the whole solid has volume

$$\begin{aligned} V &= \int_0^{\frac{1}{2}} \pi(\ln 2)^2 dy + \int_{\frac{1}{2}}^1 \pi((\ln 2)^2 - (\ln 2 + \ln y)^2) dy \\ &= \int_0^{\frac{1}{2}} \pi(\ln 2)^2 dy - \int_{\frac{1}{2}}^1 \pi(\ln(2y))^2 dy = \pi(\ln 2)^2 - \frac{\pi}{2} \int_1^2 (\ln u)^2 du \\ &= \pi(\ln 2)^2 - \frac{\pi}{2} [u(\ln u)^2]_1^2 + \frac{\pi}{2} \int_1^2 u \cdot \frac{2 \ln u}{u} du \\ &= \pi(\ln 2)^2 - \frac{\pi}{2} [u(\ln u)^2 - 2u \ln u + 2u]_1^2 = 2\pi \ln 2 - \pi. \end{aligned}$$

9. We first compute the remaining volume. Since the curves $y = \sqrt{2^2 - x^2}$ and $y = \sqrt{3}$ intersect at the points $(-1, \sqrt{3})$ and $(1, \sqrt{3})$, the remaining volume is given by

$$V_0 = \int_{-1}^1 \pi \left[(2^2 - x^2) - (\sqrt{3})^2 \right] dx = \pi \int_{-1}^1 (1 - x^2) dx = \pi \left[x - \frac{1}{3}x^3 \right]_{-1}^1 = \frac{4\pi}{3}.$$

Therefore the volume of material removed from the ball is

$$\begin{aligned} V &= (\text{Volume of a ball of radius 2}) - V_0 \\ &= \frac{4}{3}\pi(2)^3 - \frac{4\pi}{3} = \frac{28\pi}{3}. \end{aligned}$$

10. The volume of the given solid is $\int_a^t \pi[f(x)]^2 dx$. So now we are given that

$$\int_a^t \pi[f(x)]^2 dx = t^2 - at \quad \text{for every } t > a.$$

Differentiating both sides with respect to t , we obtain

$$\pi[f(t)]^2 = 2t - a$$

by the first version of the Fundamental Theorem of Calculus. Noting that f is non-negative, we have

$$f(x) = \sqrt{\frac{2x - a}{\pi}}$$

for every $x \in [a, +\infty)$.

11. (a) The ellipse can be parametrized by the vector-valued function $\mathbf{r}: [0, 2\pi] \rightarrow \mathbb{R}$ defined by

$$\mathbf{r}(t) = \langle a \cos t, b \sin t \rangle,$$

whose derivative is $\mathbf{r}'(t) = \langle -a \sin t, b \cos t \rangle$. So the arc-length of the ellipse is

$$l = \int_0^{2\pi} \|\mathbf{r}'(t)\| dt = \int_0^{2\pi} \sqrt{(-a \sin t)^2 + (b \cos t)^2} dt = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt.$$

(b) (i) The upper-half of the ellipse can be regarded as the graph of the function $f: [-a, a] \rightarrow [0, +\infty)$ given by

$$f(x) = b \sqrt{1 - \frac{x^2}{a^2}} = \frac{b}{a} \sqrt{a^2 - x^2}.$$

The derivative of this function is

$$f'(x) = \frac{b}{a} \cdot \frac{1}{2\sqrt{a^2 - x^2}} \cdot (-2x) = -\frac{b}{a} \frac{x}{\sqrt{a^2 - x^2}}$$

for every $x \in (-a, a)$. The required surface is obtained by revolving the graph of f about the x -axis, so its surface area is

$$\begin{aligned} S &= \int_{-a}^a 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx = \int_{-a}^a 2\pi \frac{b}{a} \sqrt{a^2 - x^2} \sqrt{1 + \left(-\frac{b}{a} \frac{x}{\sqrt{a^2 - x^2}}\right)^2} dx \\ &= \frac{2\pi b}{a^2} \int_{-a}^a \sqrt{a^2(a^2 - x^2) + b^2 x^2} dx. \end{aligned}$$

(ii) The right-half of the ellipse has equation

$$x = \frac{a}{b} \sqrt{b^2 - y^2} \quad \text{for } y \in [-b, b].$$

Now

$$\frac{dx}{dy} = \frac{a}{b} \cdot \frac{1}{2\sqrt{b^2 - y^2}} \cdot (-2y) = -\frac{a}{b} \frac{y}{\sqrt{b^2 - y^2}}$$

for every $y \in (-b, b)$. With a similar computation as in (b)(i), the required surface area is

$$S = \int_{-b}^b 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \frac{2\pi a}{b^2} \int_{-b}^b \sqrt{b^2(b^2 - y^2) + a^2 y^2} dy.$$

12. (a) The given curve is the graph of $f: [0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = \frac{x^3}{3}$. Since

$$f'(x) = x^2$$

for every $x \in (0, 1)$, the area of the given surface is

$$\begin{aligned} S &= \int_0^1 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx = \int_0^1 2\pi \frac{x^3}{3} \sqrt{1 + (x^2)^2} dx = \frac{\pi}{6} \int_0^1 \sqrt{1 + x^4} dx^4 \\ &= \frac{\pi}{6} \left[\frac{2}{3} (1 + x^4)^{\frac{3}{2}} \right]_0^1 = \frac{\pi}{9} (2\sqrt{2} - 1). \end{aligned}$$

- (b) Since

$$\frac{dx}{dy} = \frac{1}{2\sqrt{4y - y^2}} \cdot (4 - 2y) = \frac{2 - y}{\sqrt{4y - y^2}}$$

for every $y \in (1, 2)$, the area of the given surface is

$$S = \int_1^2 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_1^2 2\pi \sqrt{4y - y^2} \sqrt{1 + \left(\frac{2 - y}{\sqrt{4y - y^2}}\right)^2} dy = \int_1^2 4\pi dy = 4\pi.$$

- (c) The equation $y = \frac{1}{2} \ln(2x + \sqrt{4x^2 - 1})$ of the curve for $x \in \left[\frac{1}{2}, \frac{17}{16}\right]$ can be rewritten as

$$x = \frac{e^{2y} + e^{-2y}}{4}$$

for $y \in [0, \ln 2]$. Since

$$\frac{dx}{dy} = \frac{e^{2y} - e^{-2y}}{2}$$

for every $y \in (0, \ln 2)$, the area of the given surface is

$$\begin{aligned} S &= \int_0^{\ln 2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^{\ln 2} 2\pi \frac{e^{2y} + e^{-2y}}{4} \sqrt{1 + \left(\frac{e^{2y} - e^{-2y}}{2}\right)^2} dy \\ &= \int_0^{\ln 2} 2\pi \frac{e^{2y} + e^{-2y}}{4} \sqrt{\left(\frac{e^{2y} + e^{-2y}}{2}\right)^2} dy = \frac{\pi}{4} \int_0^{\ln 2} (e^{2y} + e^{-2y})^2 dy \\ &= \frac{\pi}{4} \int_0^{\ln 2} (e^{4y} + 2 + e^{-4y}) dy = \frac{\pi}{4} \left[\frac{1}{4} e^{4y} + 2y - \frac{1}{4} e^{-4y} \right]_0^{\ln 2} = \left(\frac{255}{256} + \frac{1}{2} \ln 2 \right) \pi. \end{aligned}$$

13. (a) The polar curve $r = f(\theta)$ can be parametrized by the vector-valued function $\mathbf{r}: [a, b] \rightarrow \mathbb{R}^2$ defined by

$$\mathbf{r}(t) = \langle f(t) \cos t, f(t) \sin t \rangle.$$

The parametric equations of this curve are

$$x = f(t) \cos t \quad \text{and} \quad y = f(t) \sin t,$$

so

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t) \sin t + f(t) \cos t}{f'(t) \cos t - f(t) \sin t}$$

Revolving this curve about the x -axis, we obtain a surface whose area is

$$\begin{aligned} S &= \int_{t=a}^{t=b} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b 2\pi f(t) \sin t \sqrt{1 + \left(\frac{f'(t) \sin t + f(t) \cos t}{f'(t) \cos t - f(t) \sin t}\right)^2} (f'(t) \cos t - f(t) \sin t) dt \\ &= \int_a^b 2\pi f(t) \sin t \sqrt{(f'(t) \cos t - f(t) \sin t)^2 + (f'(t) \sin t + f(t) \cos t)^2} dt \\ &= \int_a^b 2\pi f(t) \sin t \sqrt{(f(t))^2 + (f'(t))^2} dt. \end{aligned}$$

- (b) Let $f(\theta) = \sqrt{\cos 2\theta}$. Note that $\cos 2\theta \geq 0$ for $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, and by symmetry only $\theta \in \left[0, \frac{\pi}{4}\right]$ is needed to generate the required surface. Thus the surface area is given by

$$\begin{aligned} S &= \int_0^{\frac{\pi}{4}} 2\pi f(\theta) \sin \theta \sqrt{(f(\theta))^2 + (f'(\theta))^2} d\theta = \int_0^{\frac{\pi}{4}} 2\pi \sqrt{\cos 2\theta} \sin \theta \sqrt{\cos 2\theta + \left(\frac{-2 \sin 2\theta}{2\sqrt{\cos 2\theta}}\right)^2} d\theta \\ &= \int_0^{\frac{\pi}{4}} 2\pi \sin \theta d\theta = 2\pi [-\cos \theta]_0^{\frac{\pi}{4}} = (2 - \sqrt{2})\pi. \end{aligned}$$

- (c) By a similar computation as in (a), the area of the surface obtained by revolving the polar curve $r = f(\theta)$ about the y -axis is given by

$$S = \int_{t=a}^{t=b} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_a^b 2\pi f(t) \cos t \sqrt{(f(t))^2 + (f'(t))^2} dt.$$

Now with $f(\theta) = \sqrt{\cos 2\theta}$, this time we need all of $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ to generate the required surface. The surface area is given by

$$\begin{aligned} S &= \int_{-\pi/4}^{\pi/4} 2\pi f(\theta) \cos \theta \sqrt{(f(\theta))^2 + (f'(\theta))^2} d\theta = \int_{-\pi/4}^{\pi/4} 2\pi \sqrt{\cos 2\theta} \cos \theta \sqrt{\cos 2\theta + \left(\frac{-2 \sin 2\theta}{2\sqrt{\cos 2\theta}}\right)^2} d\theta \\ &= \int_{-\pi/4}^{\pi/4} 2\pi \cos \theta d\theta = 2\pi [\sin \theta]_{-\pi/4}^{\pi/4} = 2\sqrt{2}\pi. \end{aligned}$$

14. (a) The required volume is given by the improper integral

$$V = \int_1^{+\infty} \pi [f(x)]^2 dx = \pi \int_1^{+\infty} \frac{1}{x^2} dx = \pi \left[-\frac{1}{x}\right]_1^{+\infty} = \pi$$

which is finite.

- (b) The required surface area is given by the improper integral

$$S = \int_1^{+\infty} 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx = 2\pi \int_1^{+\infty} \frac{1}{x} \sqrt{1 + (-1/x^2)^2} dx = 2\pi \int_1^{+\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx.$$

Since $\frac{1}{x} \sqrt{1 + \frac{1}{x^4}} > \frac{1}{x} > 0$ for every $x \in [1, +\infty)$ and since the improper integral $\int_1^{+\infty} \frac{1}{x} dx$ diverges by p -test,

the improper integral $\int_1^{+\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx$ diverges by comparison test. Hence the surface area is infinite.

15. Given $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$, we have $\mathbf{r}'(t) = \langle -3 \cos^2 t \sin t, 3 \sin^2 t \cos t \rangle$ for every $t \in (0, 2\pi)$.

(a) The arc-length of the curve is

$$\begin{aligned} l &= \int_0^{2\pi} \|\mathbf{r}'(t)\| dt = \int_0^{2\pi} \sqrt{(-3 \cos^2 t \sin t)^2 + (3 \sin^2 t \cos t)^2} dt \\ &= \int_0^{2\pi} \sqrt{9 \sin^2 t \cos^2 t (\cos^2 t + \sin^2 t)} dt = \int_0^{2\pi} |3 \sin t \cos t| dt \\ &= 4 \int_0^{\frac{\pi}{2}} 3 \sin t \cos t dt = [6 \sin^2 t]_0^{\frac{\pi}{2}} = 6. \end{aligned}$$

(b) The required area is

$$A = \int_{t=0}^{t=2\pi} x dy = \int_0^{2\pi} \cos^3 t (3 \sin^2 t \cos t) dt = 3 \int_0^{2\pi} \sin^2 t \cos^4 t dt.$$

Recall from Example 6.18 (c) we have

$$\int \sin^2 t \cos^4 t dt = \frac{1}{16} t + \frac{1}{64} \sin 2t - \frac{1}{64} \sin 4t - \frac{1}{192} \sin 6t + C;$$

so

$$A = 3 \left[\frac{1}{16} t + \frac{1}{64} \sin 2t - \frac{1}{64} \sin 4t - \frac{1}{192} \sin 6t \right]_0^{2\pi} = \frac{3\pi}{8}.$$

(c) The required volume is

$$V = \int_{x=-1}^{x=1} \pi y^2 dx = \int_{t=\pi}^{t=0} \pi y^2 dx = \int_{\pi}^0 \pi (\sin^3 t)^2 (-3 \cos^2 t \sin t) dt = 3\pi \int_{\pi}^0 \cos^2 t \sin^6 t d \cos t.$$

With a substitution $u = \cos t$, we obtain

$$V = 3\pi \int_{-1}^1 u^2 (1 - u^2)^3 du = 3\pi \int_{-1}^1 (u^2 - 3u^4 + 3u^6 - u^8) du = 3\pi \left[\frac{1}{3} u^3 - \frac{3}{5} u^5 + \frac{3}{7} u^7 - \frac{1}{9} u^9 \right]_{-1}^1 = \frac{32\pi}{105}.$$

(d) The required surface area is

$$\begin{aligned} S &= \int_{x=-1}^{x=1} 2\pi y \underbrace{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}_{= \|\mathbf{r}'(t)\| dt} dx = \int_{t=0}^{t=\pi} 2\pi (\sin^3 t) \|\mathbf{r}'(t)\| dt \\ &= \int_0^{\pi} 2\pi (\sin^3 t) \sqrt{(-3 \cos^2 t \sin t)^2 + (3 \sin^2 t \cos t)^2} dt \\ &= \int_0^{\pi} 2\pi (\sin^3 t) |3 \sin t \cos t| dt = 2 \int_0^{\frac{\pi}{2}} 2\pi (\sin^3 t) (3 \sin t \cos t) dt \\ &= 12\pi \int_0^{\frac{\pi}{2}} \sin^4 t d \sin t = 12\pi \left[\frac{1}{5} \sin^5 t \right]_0^{\frac{\pi}{2}} = \frac{12\pi}{5}. \end{aligned}$$

16. (a) Note that the right-half of the cardioid can be parametrized by the vector-valued function

$$\mathbf{r}(t) = \langle (1 + \sin t) \cos t, (1 + \sin t) \sin t \rangle$$

for $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The required volume is given by

$$V = \int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} \pi x^2 dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi ((1 + \sin t) \cos t)^2 ((1 + 2 \sin t) \cos t) dt.$$

With a substitution $u = \sin t$, we obtain

$$\begin{aligned} V &= \pi \int_{-1}^1 (1 + u)^2 (1 - u^2) (1 + 2u) du = \pi \int_{-1}^1 \left(1 + \underbrace{4u}_{\text{odd}} + \underbrace{4u^2}_{\text{odd}} - \underbrace{2u^3}_{\text{odd}} - 5u^4 - \underbrace{2u^5}_{\text{odd}} \right) du \\ &= \pi \int_{-1}^1 (1 + 4u^2 - 5u^4) du = \pi \left[u + \frac{4}{3}u^3 - u^5 \right]_{-1}^1 = \frac{8\pi}{3}. \end{aligned}$$

- (b) First note that

$$\mathbf{r}'(t) = \langle \cos^2 t - (1 + \sin t) \sin t, (1 + 2 \sin t) \cos t \rangle = \langle -\sin t + \cos 2t, \cos t + \sin 2t \rangle$$

for every $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and so

$$\|\mathbf{r}'(t)\| = \sqrt{(-\sin t + \cos 2t)^2 + (\cos t + \sin 2t)^2} = \sqrt{2(1 + \sin t)}$$

for every $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The required surface area is

$$\begin{aligned} S &= \int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} 2\pi x \underbrace{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}}_{=\|\mathbf{r}'(t)\|} dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi (1 + \sin t) \cos t \|\mathbf{r}'(t)\| dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi (1 + \sin t) \cos t \sqrt{2(1 + \sin t)} dt. \end{aligned}$$

With a substitution $u = \sin t$, we obtain

$$\begin{aligned} S &= \int_{-1}^1 2\pi (1 + u) \sqrt{2(1 + u)} du = 2\sqrt{2}\pi \int_{-1}^1 (1 + u)^{\frac{3}{2}} du \\ &= 2\sqrt{2}\pi \left[\frac{2}{5} (1 + u)^{\frac{5}{2}} \right]_{-1}^1 = \frac{32\pi}{5}. \end{aligned}$$

17. (a) For each $n \in \mathbb{N}$, we have

$$\sum_{k=1}^n \left(3^{\frac{1}{k}} - 3^{\frac{1}{k+1}} \right) = \left(3 - 3^{\frac{1}{2}} \right) + \left(3^{\frac{1}{2}} - 3^{\frac{1}{3}} \right) + \left(3^{\frac{1}{3}} - 3^{\frac{1}{4}} \right) + \cdots + \left(3^{\frac{1}{n}} - 3^{\frac{1}{n+1}} \right) = 3 - 3^{\frac{1}{n+1}}.$$

This implies that

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^n \left(3^{\frac{1}{k}} - 3^{\frac{1}{k+1}} \right) = \lim_{n \rightarrow +\infty} \left(3 - 3^{\frac{1}{n+1}} \right) = 3 - 3^{\lim_{n \rightarrow +\infty} \frac{1}{n+1}} = 3 - 3^0 = 2$$

(because the function $f(x) = 3^x$ is continuous). Therefore $\sum_{k=1}^{+\infty} \left(3^{\frac{1}{k}} - 3^{\frac{1}{k+1}} \right)$ converges to 2.

(b) For each $n \in \mathbb{N}$, we have

$$\begin{aligned} & \sum_{k=1}^n \frac{2}{k(k+1)(k+2)} \\ &= \sum_{k=1}^n \left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2} \right) \\ &= \left(\frac{1}{1} - \frac{2}{2} + \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right) + \cdots + \left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2} \right) \\ &= \frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n+1} - \frac{2}{n+1} + \frac{1}{n+2} \\ &= \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}. \end{aligned}$$

Thus

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^n \frac{2}{k(k+1)(k+2)} = \lim_{n \rightarrow +\infty} \left(\frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2} \right) = \frac{1}{2}.$$

Therefore $\sum_{k=1}^{+\infty} \frac{2}{k(k+1)(k+2)}$ converges to $\frac{1}{2}$.

(c) For each $n \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{k=1}^n \ln \left(1 + \frac{1}{k} \right) &= \ln \left(1 + \frac{1}{1} \right) + \ln \left(1 + \frac{1}{2} \right) + \ln \left(1 + \frac{1}{3} \right) + \cdots + \ln \left(1 + \frac{1}{n} \right) \\ &= \ln \frac{2}{1} + \ln \frac{3}{2} + \ln \frac{4}{3} + \cdots + \ln \frac{n+1}{n} = \ln \left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \cdots \cdot \frac{n+1}{n} \right) \\ &= \ln(n+1). \end{aligned}$$

Thus

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^n \ln \left(1 + \frac{1}{k} \right) = \lim_{n \rightarrow +\infty} \ln(n+1) = +\infty.$$

Therefore $\sum_{k=1}^{+\infty} \ln \left(1 + \frac{1}{k} \right)$ diverges to $+\infty$.

(d) For each $n \in \mathbb{N}$, let s_n be the n^{th} partial sum. Then

- ⊙ The sequence (s_n) is increasing because each term is positive.
- ⊙ Moreover, for each $m \in \mathbb{N}$ we have

$$\begin{aligned} s_{2^m-1} &= 1 + \left(\frac{1}{2} + \frac{1}{2} \right) + \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right) + \cdots + \underbrace{\left(\frac{1}{2^{m-1}} + \cdots + \frac{1}{2^{m-1}} \right)}_{2^{m-1} \text{ terms}} \\ &= \underbrace{1 + 1 + 1 + \cdots + 1}_{m \text{ terms}} = m, \end{aligned}$$

which shows that (s_n) is unbounded.

Therefore (s_n) diverges (to $+\infty$) by the Monotone Sequence Theorem.

(e) For each integer $n \geq 2$, we have

$$\begin{aligned}\sum_{k=2}^n \frac{k}{2^{k-1}} &= \sum_{k=2}^n \left(\frac{d}{dx} x^k \Big|_{x=\frac{1}{2}} \right) = \left(\frac{d}{dx} \sum_{k=2}^n x^k \right) \Big|_{x=\frac{1}{2}} \\ &= \left(\frac{d}{dx} \frac{x^2 - x^{n+1}}{1-x} \right) \Big|_{x=\frac{1}{2}} = \frac{(2x - (n+1)x^n)(1-x) - (x^2 - x^{n+1})(-1)}{(1-x)^2} \Big|_{x=\frac{1}{2}} \\ &= 3 - \frac{n+2}{2^{n-1}}.\end{aligned}$$

Thus

$$\lim_{n \rightarrow +\infty} \sum_{k=2}^n \frac{k}{2^{k-1}} = \lim_{n \rightarrow +\infty} \left(3 - \frac{n+2}{2^{n-1}} \right) = 3.$$

Therefore $\sum_{k=2}^{+\infty} \frac{k}{2^{k-1}}$ converges to 3.

Remark: (For those who are taking / will take MATH2411) The probability that it takes exactly k births to get babies of both sex is $\frac{1}{2^{k-1}}$. The sum of this series in (e) therefore represents the **expected number of births** to get babies of both sex.

18. (a) For every $x \in \mathbb{R}$, we have

$$\begin{aligned}\sin 3x &= \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x \\ &= (2 \sin x \cos x)(\cos x) + (1 - 2 \sin^2 x)(\sin x) \\ &= 2 \sin x (1 - \sin^2 x) + (1 - 2 \sin^2 x)(\sin x) \\ &= 3 \sin x - 4 \sin^3 x.\end{aligned}$$

Therefore $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$. ■

(b) When $n = 1$, we have

$$\sum_{k=1}^1 3^{k-1} \sin^3 \frac{1}{3^k} = \sin^3 \frac{1}{3} = \frac{3}{4} \sin \frac{1}{3} - \frac{1}{4} \sin 1,$$

according to (a) with $x = \frac{1}{3}$. Assume that for some positive integer m we have

$$\sum_{k=1}^m 3^{k-1} \sin^3 \frac{1}{3^k} = \frac{3^m}{4} \sin \frac{1}{3^m} - \frac{1}{4} \sin 1.$$

Then when $n = m + 1$, we have

$$\begin{aligned}\sum_{k=1}^{m+1} 3^{k-1} \sin^3 \frac{1}{3^k} &= 3^m \sin^3 \frac{1}{3^{m+1}} + \sum_{k=1}^m 3^{k-1} \sin^3 \frac{1}{3^k} = 3^m \sin^3 \frac{1}{3^{m+1}} + \frac{3^m}{4} \sin \frac{1}{3^m} - \frac{1}{4} \sin 1 \\ &= 3^m \left(\sin^3 \frac{1}{3^{m+1}} + \frac{1}{4} \sin \frac{1}{3^m} \right) - \frac{1}{4} \sin 1 = 3^m \left(\frac{3}{4} \sin \frac{1}{3^{m+1}} \right) - \frac{1}{4} \sin 1 \\ &= \frac{3^{m+1}}{4} \sin \frac{1}{3^{m+1}} - \frac{1}{4} \sin 1,\end{aligned}$$

according to (a) with $x = \frac{1}{3^{m+1}}$. So by induction, the equality is true for every positive integer n . ■

(c) According to the result from (b), we have

$$\begin{aligned}\lim_{n \rightarrow +\infty} \sum_{k=1}^n 3^{k-1} \sin^3 \frac{1}{3^k} &= \lim_{n \rightarrow +\infty} \left(\frac{3^n}{4} \sin \frac{1}{3^n} - \frac{1}{4} \sin 1 \right) \\ &= \frac{1}{4} \left(\lim_{n \rightarrow +\infty} \frac{\sin \frac{1}{3^n}}{\frac{1}{3^n}} \right) - \frac{1}{4} \sin 1 \\ &= \frac{1}{4} (1 - \sin 1).\end{aligned}$$

Therefore the series $\sum_{k=1}^{+\infty} 3^{k-1} \sin^3 \frac{1}{3^k}$ converges to $\frac{1}{4}(1 - \sin 1)$.

19. (a) Suppose that $\sum_{k=1}^{+\infty} a_k$ converges to a number L . Then for each $\varepsilon > 0$, there exists $N > 0$ such that if p is an integer with $p \geq N$, then $|(a_1 + a_2 + \cdots + a_p) - L| < \frac{\varepsilon}{2}$.

Now if m and n are integers with $n > m \geq N$, then

$$|(a_1 + a_2 + \cdots + a_m) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad |(a_1 + a_2 + \cdots + a_n) - L| < \frac{\varepsilon}{2}.$$

Thus by triangle inequality,

$$\begin{aligned}|a_{m+1} + a_{m+2} + \cdots + a_n| &= |(a_1 + a_2 + \cdots + a_n) - L + L - (a_1 + a_2 + \cdots + a_m)| \\ &\leq |(a_1 + a_2 + \cdots + a_n) - L| + |L - (a_1 + a_2 + \cdots + a_m)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This shows that $\sum_{k=1}^{+\infty} a_k$ satisfies Cauchy's criterion. ■

- (b) Suppose that $\sum_{k=1}^{+\infty} a_k$ satisfies Cauchy's criterion. Then (for $\varepsilon = 1$) there exists $N > 0$ such that if m and p are integers with $p > m \geq N$, then $|a_{m+1} + a_{m+2} + \cdots + a_p| < 1$.

Now let

$$M = 1 + \max \underbrace{\{|a_1|, |a_1 + a_2|, |a_1 + a_2 + a_3|, \dots, |a_1 + a_2 + \cdots + a_N|\}}_{\text{a set of } N \text{ elements}}.$$

Then for each positive integer n ,

- ⊙ If $n \leq N$, then $|a_1 + a_2 + \cdots + a_n| < M$ because $|a_1 + a_2 + \cdots + a_n|$ is a member of the set above.
- ⊙ If $n > N$, then by triangle inequality we also have

$$\begin{aligned}|a_1 + a_2 + \cdots + a_n| &= |a_1 + a_2 + \cdots + a_N + a_{N+1} + a_{N+2} + \cdots + a_n| \\ &\leq \underbrace{|a_1 + a_2 + \cdots + a_N|}_{\leq \max\{|a_1|, \dots, |a_1 + \cdots + a_N|\}} + \underbrace{|a_{N+1} + a_{N+2} + \cdots + a_n|}_{< 1} \\ &< \max\{|a_1|, |a_1 + a_2|, |a_1 + a_2 + a_3|, \dots, |a_1 + a_2 + \cdots + a_N|\} + 1 \\ &= M.\end{aligned}$$

Therefore the sequence of partial sums $(\sum_{k=1}^n a_k)_{n \in \mathbb{N}}$ is bounded. ■