Problem Set 3

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 3 covers materials from §6.1 – §6.3.

1. (a) Let m and n be non-negative integers. Evaluate the following integrals, distinguishing all possible cases for m and n.

   (i) $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \, dx$ (ii) $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx \, dx$ (iii) $\displaystyle\int_{-\pi}^{\pi} \cos mx \sin nx \, dx$

   (b) Let n be a positive integer and let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by
$$f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx \,,$$
   where $a_1, a_2, \ldots, a_n$ are real numbers. Show that we must have
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx \qquad \text{for each } k \in \{1, 2, \ldots, n\}.$$

2. Evaluate the following antiderivatives.

   (a) $\displaystyle\int x^2 \arctan x \, dx$

   (b) $\displaystyle\int \sin(\ln x) \, dx$

   (c) $\displaystyle\int e^{2x} (\sin x + \cos x)^2 dx$

   (d) $\displaystyle\int (2x^2 + 1) e^{x^2} dx$

   Hint: In (d), first consider $\int e^{x^2} dx$.

3. Evaluate the limit
$$\lim_{n \to +\infty} \frac{1}{n} \sqrt[n]{\frac{(2n)!}{n!}}.$$
   Hint: Take natural logarithm.

4. Let $a > 0$ and let $f: [-a, a] \to \mathbb{R}$ be an **odd** continuous function. Show that
$$\int_{-a}^{a} \left(\int_{-a}^{x} f(t) dt \right) dx = -\int_{-a}^{a} x f(x) dx.$$

5. The following are "proofs" of some obviously false statements. Point out what is wrong in each of these "proofs".

   (a) A "proof" of the statement that "$\pi = 0$".

> *Proof.* By Fundamental Theorem of Calculus, we have $\int_{-1}^{1} \frac{1}{1+x^2} dx = [\arctan x]_{-1}^{1} = \frac{\pi}{2}$.
>
> Let $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2} dx$, $u = 1$ when $x = 1$ and $u = -1$ when $x = -1$; so
> $$\frac{\pi}{2} = \int_{-1}^{1} \frac{1}{1 + x^2} dx = \int_{-1}^{1} \frac{-1}{(1/x)^2 + 1} \left(-\frac{1}{x^2} dx \right) = \int_{-1}^{1} \frac{-1}{u^2 + 1} du = -\frac{\pi}{2}.$$
> Adding $\frac{\pi}{2}$ on both sides we obtain $\pi = 0$. ■

(b) A "proof" of the statement that "**every integral equals zero**":

Proof. Let $f:[a,b]\to\mathbb{R}$ be any continuous function, and consider the integral $\int_a^b f(x)dx$. We let $u=(x-a)(x-b)$. Then $u=0$ when $x=a$ and $u=0$ when $x=b$, so the integral becomes

$$\int_a^b f(x)dx=\int_0^0 (\text{something})du=0$$

according to the substitution rule. ■

(c) A "proof" of the statement that "$0=1$".

Proof. Let f be a differentiable function whose value is never zero, and consider $\int \frac{f'(x)}{f(x)}dx$. Let $u=\frac{1}{f(x)}$ and $v=f(x)$. Taking antiderivatives by parts, we obtain

$$\int \frac{f'(x)}{f(x)}dx=\int \underbrace{\frac{1}{f(x)}}_{u}\underbrace{f'(x)dx}_{dv}=\underbrace{\frac{1}{f(x)}}_{u}\underbrace{f(x)}_{v}-\int \underbrace{f(x)}_{v}\underbrace{\left[-\frac{1}{(f(x))^2}f'(x)\right]dx}_{du}$$

$$=1+\int \frac{f'(x)}{f(x)}dx.$$

Therefore subtracting $\int \frac{f'(x)}{f(x)}dx$ from both sides we obtain $0=1$. ■

6. Let f be a function which is continuously differentiable on $[0,1]$.

(a) For every $a,b\in[0,1]$, show that

$$\int_a^b (x-a)f'(x)dx=\int_a^b \big(f(b)-f(x)\big)dx.$$

(b) Let $n\ge k\ge 1$ be integers. Using (a) and the generalized Mean Value Theorem for integrals (Example 5.47 (a)), show that there exists $\omega_k\in\left[\frac{k-1}{n},\frac{k}{n}\right]$ such that

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left[f\left(\frac{k}{n}\right)-f(x)\right]dx=\frac{f'(\omega_k)}{2n^2}.$$

(c) For each $n\in\mathbb{N}$, we let $E_n=\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)-\int_0^1 f(x)dx$. Show that

$$E_n=\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left[f\left(\frac{k}{n}\right)-f(x)\right]dx.$$

Hence deduce using the result from (b) that

$$\lim_{n\to+\infty} nE_n=\frac{f(1)-f(0)}{2}.$$

7. Let $f: [0, +\infty) \to \mathbb{R}$ be the function defined by $f(x) = xe^x$.
   (a) Show that f is strictly increasing.
   (b) Now f is one-to-one according to (a), so we let g be the **inverse** of f, i.e. $g = f^{-1}$.
      (i) Write down the domain of g. Show that
      $$g'(x) = \frac{1}{x + e^{g(x)}}$$
      for every x in the interior of the domain of g.
      (ii) Using the result from (b) (i) or otherwise, evaluate the antiderivative $\int g(x)dx$, expressing your answer in terms of g and other elementary functions only.
      (iii) Hence, or otherwise, evaluate the integral $\int_0^e g(x)dx$.

8. (a) Let n be a non-negative integer, and let $f: \mathbb{R} \to \mathbb{R}$ be the polynomial
   $$f(x) = (x^2 - 1)^n.$$
      (i) Show that $(x^2 - 1)f'(x) - 2nxf(x) = 0$ for every $x \in \mathbb{R}$.
      (ii) Hence, show that
      $$(x^2 - 1)f^{(n+2)}(x) + 2xf^{(n+1)}(x) - n(n+1)f^{(n)}(x) = 0$$
      for every $x \in \mathbb{R}$.

   Hint: Recall "Leibniz rule" in chapter 3. Part (a) is almost the same as Example 3.69.

   (b) For each non-negative integer n, let $p_n: \mathbb{R} \to \mathbb{R}$ be the function
   $$p_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n.$$
      (i) Using the result from (a) (ii), show that
      $$\frac{d}{dx}[(x^2 - 1)p_n{}'(x)] = n(n+1)p_n(x)$$
      for every non-negative integer n.
      (ii) Hence deduce that if m and n are distinct non-negative integers, then
      $$\int_{-1}^{1} p_m(x)p_n(x)dx = 0.$$

9. For each non-negative integer n, let
   $$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx \, dx.$$
   (a) For each positive integer n, show that
   $$\frac{d}{dx}\cos^n x \sin nx = n\cos^{n-1} x \cos(n+1)x\,.$$
   Hence show that
   $$I_n = \frac{1}{2}I_{n-1}.$$
   (b) Using the result from (a), find the value of I_n in terms of n.

10. For each non-negative integer n, let

$$I_n = \int_0^1 t^n e^t dt.$$

(a) Show that $\frac{1}{n+1} \le I_n \le \frac{e}{n+1}$ for every non-negative integer n.

(b) Express I_n in terms of I_{n-1} for each $n \ge 1$. Hence show that

$$I_n = (-1)^{n+1} n! + e \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!}.$$

(c) Using (a) and (b), deduce that e is an irrational number.

Hint: There is no integer in the open interval $(0, 1)$.

11. (a) For every pair of non-negative integers m and n, let

$$B(m,n) = \int_0^1 x^m (1-x)^n dx\,.$$

Show that $B(m,n) = \frac{n}{m+1} B(m+1, n-1)$ for every pair of integers $m \ge 0$ and $n \ge 1$.

Hence or otherwise, deduce that $B(m,n) = \frac{m!n!}{(m+n+1)!}$.

(b) Show that

$$\int_0^1 \frac{x^4(x-1)^4}{x^2+1} dx = \frac{22}{7} - \pi.$$

Using this together with the result from (a), deduce that

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}.$$

12. Let a be a positive real number. Evaluate the following antiderivatives.

(a) $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}} dx$

(b) $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}} dx$

(c) $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}} dx$

(d) $\displaystyle\int \sqrt{x^2+a^2}\, dx$

(e) $\displaystyle\int \sqrt{x^2-a^2}\, dx$

(f) $\displaystyle\int \sqrt{a^2-x^2}\, dx$

13. Evaluate the following antiderivatives, using trigonometric substitutions when appropriate.

(a) $\displaystyle\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}} dx$

(b) $\displaystyle\int \frac{x+1}{(x^2+x+1)\sqrt{x^2+x+1}} dx$

(c) $\displaystyle\int \frac{x^2-1}{(x^2+1)\sqrt{1+x^4}} dx$ (*Hint*: $x = \tan t$)

14. (a) For each non-negative integer n, let

$$I_n(x) = \int \frac{x^n}{\sqrt{x^2+1}}\,dx$$

which is defined up to addition by a constant function. Find a reduction formula that connects I_n and I_{n-2} for $n \geq 2$.

(b) Hence evaluate

$$\int_0^1 \frac{x^5}{\sqrt{x^2+1}}\,dx.$$

15. (a) By considering a suitable function, show that

$$\tan x \leq \frac{4x}{\pi} \qquad \text{for every } x \in \left[0, \frac{\pi}{4}\right].$$

(b) For each non-negative integer n, let

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx.$$

(i) Using the result from (a), show that

$$I_n \leq \frac{\pi}{4(n+1)} \qquad \text{for every non-negative integer } n.$$

(ii) Show that $I_n = \frac{1}{n-1} - I_{n-2}$ for every $n \geq 2$. Hence show that

$$I_0 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{k-1}\frac{1}{2k-1} + (-1)^k I_{2k} \qquad \text{for every } k \in \mathbb{N}.$$

(c) Using the results from (b), evaluate the limit

$$\lim_{k\to+\infty}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{k-1}\frac{1}{2k-1}\right).$$

16. For each pair of real numbers m and n, let

$$I_{m,n}(x) = \int x^m (\ln x)^n dx$$

be defined up to addition by a constant function.

(a) Show that

$$I_{m,n}(x) = \frac{1}{m+1}x^{m+1}(\ln x)^n - \frac{n}{m+1}I_{m,n-1}(x)$$

for every $m \in \mathbb{R} \setminus \{-1\}$ and every $n \in \mathbb{N}$.

(b) Hence evaluate the antiderivative

$$\int \frac{(\ln x)^3}{x^4}\,dx.$$