HKUST MATH 1014 L1 assignment 8 submission

MATH1014 Calculus II Problem Set 8
L01 (Spring 2024)

Problem Set 8

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 8 covers materials from chapter 9.

Q2

Find the 6^{th} order approximation of each of the following functions at 0.

Q2.b

$$g(x) = e^{\cos x}$$

$$\begin{aligned}
&i(x) := \cos x - 1\\
&\quad \text{As } x \to 0,\\
&i(x) = \cos 0 - 1 + \frac{-\sin 0}{1!}x^1 + \frac{-\cos 0}{2!}x^2 + \frac{\sin 0}{3!}x^3 + \frac{\cos 0}{4!}x^4 + \frac{-\sin 0}{5!}x^5 + \frac{-\cos 0}{6!}x^6 + \frac{\sin 0}{7!}x^7 + O\left(x^8\right)\\
&\quad = -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O\left(x^8\right)\\
&\quad \text{Note that } i(x) = O\left(x^2\right).
\end{aligned}$$

$$\begin{aligned}
&j(x) := e^x\\
&\quad \text{As } x \to 0,\\
&j(x) = e^0 + \frac{e^0}{1!}x^1 + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + O\left(x^4\right)\\
&\quad = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O\left(x^4\right)
\end{aligned}$$

$$\begin{aligned}
&\quad \text{As } x \to 0,\\
&g(x) = e^{\cos x}\\
&\quad = e \cdot e^{\cos x - 1}\\
&\quad = e \cdot j(i(x))\\
&\quad = e\left(1 + i(x) + \frac{i(x)^2}{2} + \frac{i(x)^3}{6} + O\left(i(x)^4\right)\right)\\
&\quad = e\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{1}{2}\left(\frac{x^4}{4} - \frac{x^6}{24}\right) + \frac{1}{6}\left(-\frac{x^6}{8}\right) + O\left(x^8\right)\right)\\
&\quad = e\left(1 - \frac{x^2}{2} + \frac{x^4}{6} - \frac{31x^6}{720} + O\left(x^8\right)\right)\\
&\quad = e - \frac{ex^2}{2} + \frac{ex^4}{6} - \frac{31ex^6}{720} + O\left(x^8\right)
\end{aligned}$$

Q2.c

$$h(x) = \sec x$$

Using the same $i(x)$ as in (b),
As $x \to 0$,

$$i(x) = -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O\left(x^8\right)$$

Note that $i(x) = O\left(x^2\right)$.

$$k(x) := \frac{1}{1-x}$$

As $x \to 0$,

$$\begin{aligned} k(x) &= (1-x)^{-1} + \frac{1!(1-0)^{-2}}{1!}x^1 + \frac{2!(1-0)^{-3}}{2!}x^2 + \frac{3!(1-0)^{-4}}{3!}x^3 + O\left(x^4\right) \\ &= 1 + x + x^2 + x^3 + O\left(x^4\right) \end{aligned}$$

$$\begin{aligned} h(x) &= \sec x \\ &= \frac{1}{\cos x} \\ &= \frac{1}{1+i(x)} \\ &= k(-i(x)) \\ &= 1 - i(x) + i(x)^2 - i(x)^3 + O\left((-i(x))^4\right) \\ &= 1 - \left(-\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}\right) + \left(\frac{x^4}{4} - \frac{x^6}{24}\right) - \left(-\frac{x^6}{8}\right) + O\left(x^8\right) \\ &= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + O\left(x^8\right) \end{aligned}$$

Q3

Evaluate teach of the following limits using polynomial approximations.

Q3.b

$$\lim_{x\to 0} \frac{\sin^2 x - \sin\left(x^2\right) + \frac{1}{3}x^4}{x^6}$$

$$f(x) := \sin x$$

As $x \to 0$,

$$\begin{aligned} f(x) &= \sin 0 + \frac{\cos 0}{1!}x + \frac{-\sin 0}{2!}x^2 + \frac{-\cos 0}{3!}x^3 + \frac{\sin 0}{4!}x^4 + \frac{\cos 0}{5!}x^5 + O\left(x^6\right) \\ &= x - \frac{x^3}{6} + \frac{x^5}{120} + O\left(x^6\right) \end{aligned}$$

Note that $f(x) = O(x)$.

As $x \to 0$,

$$\begin{aligned} \sin^2 x &= f(x)^2 \\ &= x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \frac{x^4}{6} + \frac{x^6}{36} + \frac{x^6}{120} + O\left(x^7\right) \\ &= x^2 - \frac{x^4}{3} + \frac{4x^6}{90} + O\left(x^7\right) \end{aligned}$$

As $x \to 0$,

$$\begin{aligned} \sin\left(x^2\right) &= f\left(x^2\right) \\ &= x^2 - \frac{x^6}{6} + O\left(x^{10}\right) \end{aligned}$$

$$\begin{aligned} &\lim_{x\to 0} \frac{\sin^2 x - \sin\left(x^2\right) + \frac{1}{3}x^4}{x^6} \\ &= \lim_{x\to 0} \frac{\left(x^2 - \frac{x^4}{3} + \frac{4x^6}{90} + O\left(x^7\right)\right) - \left(x^2 - \frac{x^6}{6} + O\left(x^{10}\right)\right) + \frac{x^4}{3}}{x^6} \\ &= \lim_{x\to 0} \frac{\frac{19x^6}{90}}{x^6} \qquad \left(\lim_{x\to 0} \frac{O(x^y)}{x^6} = 0 \quad \forall y > 6\right) \\ &= \lim_{x\to 0} \frac{19}{90} \\ &= \frac{19}{90} \end{aligned}$$

Q3.c

$$\lim_{x\to+\infty} x^2\left(e-\frac{e}{2x}-\left(1+\frac{1}{x}\right)^x\right)$$

$$\begin{aligned}
f(x) &:= \ln(1+x) \\
&\text{As } x\to 0, \\
f(x) &= \ln(1+0)+\frac{(1+0)^{-1}}{1!}x-\frac{1!(1+0)^{-2}}{2!}x^2+\frac{2!(1+0)^{-3}}{3!}x^3+O\left(x^4\right) \\
&= x-\frac{x^2}{2}+\frac{x^3}{3}+O\left(x^4\right) \\
\frac{1}{x}f(x) &= 1-\frac{x}{2}+\frac{x^2}{3}+O\left(x^3\right) \\
\frac{1}{x}f(x)-1 &= -\frac{x}{2}+\frac{x^2}{3}+O\left(x^3\right) \\
&\text{Note that } \frac{1}{x}f(x)-1=O(x).
\end{aligned}$$

$$\begin{aligned}
g(x) &:= \exp x \\
&\text{As } x\to 0, \\
g(x) &= \exp 0+\frac{\exp 0}{1!}x^1+\frac{\exp 0}{2!}x^2+O\left(x^4\right) \\
&= 1+x+\frac{x^2}{2}+O\left(x^3\right)
\end{aligned}$$

$$\begin{aligned}
&\text{As } x\to 0, \\
&\quad \exp\left(\frac{1}{x}\ln(1+x)\right) \\
&= \exp\left(\frac{1}{x}f(x)-1+1\right) \\
&= e\cdot g\left(\frac{1}{x}f(x)-1\right) \\
&= e\left(1+\left(-\frac{x}{2}+\frac{x^2}{3}\right)+\frac{1}{2}\left(\frac{x^2}{4}\right)+O\left(x^3\right)\right) \\
&= e\left(1-\frac{x}{2}+\frac{11x^2}{24}+O\left(x^3\right)\right) \\
&= e-\frac{ex}{2}+\frac{11ex^2}{24}+O\left(x^3\right)
\end{aligned}$$

$$\begin{aligned}
&\quad \lim_{x\to+\infty} x^2\left(e-\frac{e}{2x}-\left(1+\frac{1}{x}\right)^x\right) \\
&= \lim_{x\to0^+}\frac{e-\frac{ex}{2}-(1+x)^{\frac{1}{x}}}{x^2} && \left(\text{change of variables: } \frac{1}{x}\mapsto x\right) \\
&= \lim_{x\to0^+}\frac{e-\frac{ex}{2}-\exp\left(\frac{1}{x}\ln(1+x)\right)}{x^2} \\
&= \lim_{x\to0^+}\frac{e-\frac{ex}{2}-e+\frac{ex}{2}-\frac{11ex^2}{24}+O\left(x^3\right)}{x^2} \\
&= \lim_{x\to0^+}\frac{e-\frac{ex}{2}-e+\frac{ex}{2}-\frac{11ex^2}{24}}{x^2} && \left(\lim_{x\to0}\frac{O(x^y)}{x^2}=0 \quad \forall y>2\right) \\
&= \lim_{x\to0^+} -\frac{11e}{24} \\
&= -\frac{11e}{24}
\end{aligned}$$

Q5

In Examples 9.31 and 9.32, we have seen that in some cases, Lagrange's remainder formula is not strong enough to show that $\lim_{n\to+\infty}R_n(x)=0$. Let's develop another remainder formula.

Q5.a

Let a be real number and let $x>a$, let n be a non-negative integer and let f be a function such that $f^{(n)}$ is continuous on $[a,x]$ and differentiable on (a,x).

Q5.a.i

Let $g : [a, x] \to \mathbb{R}$ be the function

$$g(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$$

.

Compute $g'(t)$ for $t \in (a, x)$

$$\begin{aligned}
&g(t) \\
&= f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k \\
&= f(x) - f(t) - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k \\
&g'(t) \\
&= -f'(t) - \sum_{k=1}^{n} \frac{1}{k!}\left(f^{(k+1)}(t)(x-t)^k - kf^{(k)}(t)(x-t)^{k-1}\right) \\
&= -f'(t) - \sum_{k=1}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{(k-1)} \\
&= -f'(t) - \sum_{k=1}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!}(x-t)^k \\
&= -f'(t) - \frac{f^{(n+1)}(t)}{n!}(x-t)^n + f'(t) \\
&= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n
\end{aligned}$$

Q5.a.ii

(Cauchy's remainder formula) By applying Mean Value Theorem to the function g, show that there exists a number $c \in (a, x)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a)$$

.

Remark: If we assume further that $f^{(n+1)}$ is integrable on $[a, x]$, then another way of obtaining (a) (ii) is to use Q4(a) and then the MVT for integrals.

$f^{(n)}(t)$ is continuous on $[a, x]$.
$\implies f^{(k)}(t)$ is continuous on $[a, x]$ for all $k \in \mathbb{Z}_{[0,n]}$.
$g(t)$ is continuous on $[a, x]$.

Additionally, $f^{(n+1)}(t)$ exists on (a, x).
$g(t)$ is differentiable on (a, x).

By the mean value theorem,
$\exists c \in (a, x)$

$$\begin{aligned}
g'(c) &= \frac{g(x) - g(a)}{x - a} \\
-\frac{f^{(n+1)}(c)}{n!}(x-c)^n &= \frac{f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(x-x)^k - f(x) + \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}{x-a} \\
&= \frac{-f(x) + \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}{x-a} \qquad (0^0 = 1 \text{ in this context}) \\
-\frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a) &= -f(x) + \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\
f(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a)
\end{aligned}$$

Q5.b

Using the result from (a) (ii) (which obviously still holds if $x < a$), show that for each of the following functions, the remainder term at 0 satisfies

$$\lim_{n \to +\infty} R_n(x) = 0 \qquad \text{for each fixed } x \in (-1, 1)$$

.

Q5.b.i

(Example 9.31)

$$f(x) = \ln(1 + x)$$

$f(x) \in C^{\infty}((-1,1],\mathbb{R})$

By Q5.a.ii...

Set $a = 0$.

$(\forall x \in (0,1])(\exists c \in (0,x))$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k + \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-0)$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n+1)}(c)}{n!}(x-c)^n x$$

$(\forall x \in (-1,0))(\exists c \in (x,0))$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n+1)}(c)}{n!}(x-c)^n x$$

$$\lim_{n\to+\infty} R_n(x)$$
$$= \lim_{n\to+\infty}\left(f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k\right)$$

When $x = 0$,

$$\lim_{n\to+\infty} R_n(0)$$
$$= \lim_{n\to+\infty}\left(f(0) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}0^k\right)$$
$$= \lim_{n\to+\infty}(f(0) - f(0)) \qquad (0^0 = 1 \text{ in this context})$$
$$= \lim_{n\to+\infty} 0$$
$$= 0$$

When $x \in (-1,1) \setminus \{0\}$,

$\exists c$ in between 0 and x

$$\lim_{n\to+\infty} R_n(x)$$
$$= \lim_{n\to+\infty}\left(f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k\right)$$
$$= \lim_{n\to+\infty}\left(\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n+1)}(c)}{n!}(x-c)^n x - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k\right) \qquad (n \text{ in the above formula is arbitrary})$$
$$= \lim_{n\to+\infty} \frac{f^{(n+1)}(c)}{n!}(x-c)^n x$$
$$= \lim_{n\to+\infty} \frac{n!(-1)^{n+1}(1+c)^{-(n+1)}}{n!}(x-c)^n x$$
$$= \lim_{n\to+\infty} (-1)^{n+1}\left(\frac{x-c}{1+c}\right)^n \frac{x}{1+x}$$
$$= \frac{x}{1+x}\lim_{n\to+\infty} (-1)^{n+1}\left(\frac{x-c}{1+c}\right)^n$$

When $x \in (0,1)$, $c \in (0,x)$, and

$$\left|\frac{x-c}{1+c}\right|$$
$$= \frac{x-c}{1+c} \qquad (x > c, c > 0)$$
$$< \frac{1-c}{1+c} \qquad (x < 1, c > 0)$$
$$< 1 \qquad (c < x < 1)$$

When $x \in (-1,0)$, $c \in (x,0)$ and

$$\left|\frac{x-c}{1+c}\right|$$
$$= \frac{c-x}{1+c} \qquad (x < c, c > x > -1)$$
$$< \frac{c+1}{1+c} \qquad (-x < 1, c > x > -1)$$
$$= 1$$

Therefore...

When $x \in (0,1)$, $c \in (0,x)$, and

$$\lim_{n\to+\infty} R_n(x)$$
$$= \frac{x}{1+x}\lim_{n\to+\infty} (-1)^{n+1}\left(\frac{x-c}{1+c}\right)^n$$

$$= \frac{x}{1+x} \cdot 0 \qquad \left(\left|\frac{x-c}{1+c}\right| < 1\right)$$
$$= 0$$

$$\forall x \in (-1, 1)$$
$$\lim_{n\to+\infty} R_n(x)$$
$$= 0$$

Q7

For each of the following, compute its Maclaurin series and find its radius of convergence.

Q7.b

$$f(x) = \int_0^x \frac{\sin t}{t}\, \mathrm{d}t$$

$$f'(x) = \frac{\sin x}{x}$$

$$\begin{aligned}\sin x &= \sum_{k=0}^{+\infty}\left(\frac{\sin 0}{(4k)!}x^{4k} + \frac{\cos 0}{(4k+1)!}x^{4k+1} + \frac{-\sin 0}{(4k+2)!}x^{4k+2} + \frac{-\cos 0}{(4k+3)!}x^{4k+3}\right)\\ &= \sum_{k=0}^{+\infty}\left(\frac{x^{4k+1}}{(4k+1)!} - \frac{x^{4k+3}}{(4k+3)!}\right)\\ &= \sum_{k=0}^{+\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\end{aligned}$$

$$\begin{aligned}f'(x) &= \frac{\sin x}{x}\\ &= \sum_{k=0}^{+\infty}(-1)^k\frac{x^{2k}}{(2k+1)!}\end{aligned}$$

$$\begin{aligned}f(x) &= \int_0^x f'(x)\,\mathrm{d}x\\ &= \int_0^x\left(\sum_{k=0}^{+\infty}(-1)^k\frac{x^{2k}}{(2k+1)!}\right)\mathrm{d}x\\ &= \sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}\int_0^x x^{2k}\,\mathrm{d}x && \text{(Maclaurin series can be integrated term-wise)}\\ &= \sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!(2k+1)}x^{2k+1}\end{aligned}$$

$$f(x) = x\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!(2k+1)}\left(x^2\right)^k$$

The center of the power series is $x^2 = 0$.

The coefficients are $c_k = \dfrac{(-1)^k}{(2k+1)!(2k+1)}$.

$$\begin{aligned}&\text{radius of convergence of the power series}\\ &= \lim_{k\to+\infty}\left|\frac{\frac{(-1)^k}{(2k+1)!(2k+1)}}{\frac{(-1)^{k+1}}{(2k+3)!(2k+3)}}\right|\\ &= \lim_{k\to+\infty}\left|\frac{(2k+3)!(2k+3)}{(2k+1)!(2k+1)}\right|\\ &= \lim_{k\to+\infty}\left|\frac{(2k+2)(2k+3)^2}{2k+1}\right|\\ &= +\infty\end{aligned}$$

$$\begin{aligned}&\text{radius of convergence of the Maclurin series of } f(x)\\ &= \sqrt{+\infty}\\ &= +\infty\end{aligned}$$

Q7.d

$$f(x) = \ln\left(x + \sqrt{1+x^2}\right)$$

Hint: In (d), first consider f'.

$$
\begin{aligned}
& f'(x) \\
&= \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} \\
&= \frac{\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} \\
&= \frac{1}{\sqrt{1+x^2}}
\end{aligned}
$$

$$
\begin{aligned}
& g(x) \\
&:= \frac{1}{\sqrt{1+x}} \\
& \text{Macluarin series of } g(x) \\
&= \sum_{k=0}^{+\infty} \frac{g^{(k)}(0)}{k!} x^k \\
&= \sum_{k=0}^{+\infty} \frac{(-1)^k \left(\prod_{i=0}^{k-1} \frac{2i+1}{2}\right)(1+0)^{-\frac{2k+1}{2}}}{k!} x^k \\
&= \sum_{k=0}^{+\infty} (-1)^k \frac{\prod_{i=0}^{k-1}(2i+1)}{k! 2^k} x^k
\end{aligned}
$$

$$
\begin{aligned}
& f'(x) \\
&= \frac{1}{\sqrt{1+x^2}} \\
&= g(x^2) \\
& \text{Macluarin series of } f'(x) \\
&= \sum_{k=0}^{+\infty} (-1)^k \frac{\prod_{i=0}^{k-1}(2i+1)}{k! 2^k} x^{2k}
\end{aligned}
$$

$$
\begin{aligned}
& f(x) \\
&= \int f'(x) \,\mathrm{d}x \\
& \text{Macluarin series of } f(x) \\
&= \int \left(\sum_{k=0}^{+\infty} (-1)^k \frac{\prod_{i=0}^{k-1}(2i+1)}{k! 2^k} x^{2k} \right) \mathrm{d}x \\
&= \sum_{k=0}^{+\infty} (-1)^k \frac{\prod_{i=0}^{k-1}(2i+1)}{k! 2^k} \int x^{2k} \,\mathrm{d}x && \text{(Macluarin series can be integrated term-wise)} \\
&= \sum_{k=0}^{+\infty} (-1)^k \frac{\prod_{i=0}^{k-1}(2i+1)}{k! 2^k (2k+1)} x^{2k+1}
\end{aligned}
$$

$$
\begin{aligned}
& \text{Macluarin series of } f(x) \\
&= x \sum_{k=0}^{+\infty} (-1)^k \frac{\prod_{i=0}^{k-1}(2i+1)}{k! 2^k (2k+1)} \left(x^2\right)^k
\end{aligned}
$$

The center of the power series is $x^2 = 0$.

The coefficients are $c_k = (-1)^k \dfrac{\prod_{i=0}^{k-1}(2i+1)}{k! 2^k (2k+1)}$.

$$
\begin{aligned}
& \text{radius of convergence of the power series} \\
&= \lim_{k\to+\infty} \left| \frac{(-1)^k \frac{\prod_{i=0}^{k-1}(2i+1)}{k! 2^k (2k+1)}}{(-1)^{k+1} \frac{\prod_{i=0}^{k}(2i+1)}{(k+1)! 2^{k+1} (2k+3)}} \right| \\
&= \lim_{k\to+\infty} \left| \frac{2(k+1)(2k+3)}{(2k+1)^2} \right| \\
&= \lim_{k\to+\infty} \left| \frac{\left(1+\frac{1}{k}\right)\left(1+\frac{3}{2k}\right)}{(1+\frac{1}{2k})^2} \right| \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
& \text{radius of convergence of the Macluarin series of } f(x) \\
&= \sqrt{1} \\
&= 1
\end{aligned}
$$

Q9

Let $f(x) = x^3 e^x$. Using the Taylor series of f, compute...

for every positive integer n. (Do not try to really differentiate for n times!)

Q9.a

...

$$f^{(n)}(0)$$

and

$$\begin{aligned}
&\exp x\\
&\text{Macluarin series of } \exp x\\
&= \sum_{n=0}^{+\infty} \frac{\exp 0}{n!} x^n\\
&= \sum_{n=0}^{+\infty} \frac{x^n}{n!}
\end{aligned}$$

$$\begin{aligned}
&f(x)\\
&= x^3 \exp x\\
&\text{Macluarin series of } f(x)\\
&= x^3 \sum_{n=0}^{+\infty} \frac{x^n}{n!}\\
&= \sum_{n=0}^{+\infty} \frac{x^{n+3}}{n!}\\
&= \sum_{n=3}^{+\infty} \frac{x^n}{(n-3)!}
\end{aligned}$$

$$\begin{aligned}
&f(x)\\
&\text{Macluarin series of } f(x)\\
&= \sum_{k=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n
\end{aligned}$$

Since Taylor (Macluarin) series is unique for a C^∞ function,
by comparing coefficients:
When $0 \le n < 3$,

$$\frac{f^{(n)}(0)}{n!} = 0$$

$$f^{(n)}(0) = 0$$

When $n \ge 3$,

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{(n-3)!}$$

$$\begin{aligned}
f^{(n)}(0) &= \frac{n!}{(n-3)!}\\
&= n(n-1)(n-2)
\end{aligned}$$

Q9.b

$$f^{(n)}(1)$$

$$
\begin{aligned}
&\exp x\\
&\text{Taylor series of } \exp x \text{ at } 1\\
&= \sum_{n=0}^{+\infty} \frac{\exp 1}{n!}(x-1)^n\\
&= e\sum_{n=0}^{+\infty} \frac{(x-1)^n}{n!}
\end{aligned}
$$

$$
\begin{aligned}
&f(x)\\
&= x^3 \exp x\\
&= \left(\left(x^3-3x^2+3x-1\right)+3x^2-3x+1\right)\exp x\\
&= \left(\left(x^3-3x^2+3x-1\right)+3\left(x^2-2x+1\right)+3x-2\right)\exp x\\
&= \left(\left(x^3-3x^2+3x-1\right)+3\left(x^2-2x+1\right)+3(x-1)+1\right)\exp x\\
&= \left((x-1)^3+3(x-1)^2+3(x-1)+1\right)\exp x\\
&\text{Taylor series of } f(x) \text{ at } 1\\
&= e\left((x-1)^3+3(x-1)^2+3(x-1)+1\right)\sum_{n=0}^{+\infty}\frac{(x-1)^n}{n!}\\
&= e\left(\sum_{n=0}^{+\infty}\frac{(x-1)^{n+3}}{n!}+3\sum_{n=0}^{+\infty}\frac{(x-1)^{n+2}}{n!}+3\sum_{n=0}^{+\infty}\frac{(x-1)^{n+1}}{n!}+\sum_{n=0}^{+\infty}\frac{(x-1)^{n}}{n!}\right)\\
&= e\left(\sum_{n=3}^{+\infty}\frac{(x-1)^{n}}{(n-3)!}+3\sum_{n=2}^{+\infty}\frac{(x-1)^{n}}{(n-2)!}+3\sum_{n=1}^{+\infty}\frac{(x-1)^{n}}{(n-1)!}+\sum_{n=0}^{+\infty}\frac{(x-1)^{n}}{n!}\right)
\end{aligned}
$$

$$
\begin{aligned}
&f(x)\\
&\text{Taylor series of } f(x) \text{ at } 1\\
&= \sum_{k=0}^{+\infty}\frac{f^{(n)}(1)}{n!}(x-1)^n
\end{aligned}
$$

Since Taylor series is unique for a C^∞ function,
by comparing coefficients:

$$
\frac{f^{(n)}(1)}{n!} = \begin{cases}
\frac{e}{n!}, & n=0\\
\frac{e}{n!}+\frac{3e}{(n-1)!}, & n=1\\
\frac{e}{n!}+\frac{3e}{(n-1)!}+\frac{3e}{(n-2)!}, & n=2\\
\frac{e}{n!}+\frac{3e}{(n-1)!}+\frac{3e}{(n-2)!}+\frac{e}{(n-3)!}, & n\geq 3
\end{cases}
$$

$$
\begin{aligned}
f^{(n)}(1) &= \begin{cases}
e, & n=0\\
e+3en, & n=1\\
e+3en+3en(n-1), & n=2\\
e+3en+3en(n-1)+en(n-1)(n-2), & n\geq 3
\end{cases}\\
&= \begin{cases}
e, & n=0\\
4e, & n=1\\
13e, & n=2\\
e+3en+3en(n-1)+en(n-1)(n-2), & n\geq 3
\end{cases}
\end{aligned}
$$

Q11

Let $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$ be real numbers and let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k \cos kx + b_k \sin kx\right)$$

be a trigonometric polynomial. (Note that f is a finite sum.) Show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\, \mathrm{d}x = \frac{a_0^2}{2} + \sum_{k=1}^{n}\left(a_k^2 + b_k^2\right)$$

.

Let m, n be two integers.

$$
\begin{aligned}
&\int_{-\pi}^{\pi} \cos mx \cos nx \,\mathrm{d}x \\
&= \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) + \cos((m-n)x)) \,\mathrm{d}x \\
&= \begin{cases} \frac{1}{2}\int_{-\pi}^{\pi}(1+1)\,\mathrm{d}x & m+n=0, m-n=0 \\ \frac{1}{2}\int_{-\pi}^{\pi}(1+\cos((m-n)x))\,\mathrm{d}x & m+n=0 \\ \frac{1}{2}\int_{-\pi}^{\pi}(\cos((m+n)x)+1)\,\mathrm{d}x & m-n=0 \\ \frac{1}{2}\int_{-\pi}^{\pi}(\cos((m+n)x)+\cos((m-n)x))\,\mathrm{d}x & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{2}[2x]_{x=-\pi}^{x=\pi} & m+n=0, m-n=0 \\ \frac{1}{2}\left[x + \frac{1}{m-n}\sin((m-n)x)\right]_{x=-\pi}^{x=\pi} & m+n=0 \\ \frac{1}{2}\left[\frac{1}{m+n}\sin((m+n)x) + x\right]_{x=-\pi}^{x=\pi} & m-n=0 \\ \frac{1}{2}\left[\frac{1}{m+n}\sin((m+n)x) + \frac{1}{m-n}\sin((m-n)x)\right]_{x=-\pi}^{x=\pi} & \text{otherwise} \end{cases} \\
&= \begin{cases} 2\pi & m+n=0, m-n=0 \\ \pi & m+n=0 \\ \pi & m-n=0 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 2\pi & m=n=0 \\ \pi & m=-n \\ \pi & m=n \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\int_{-\pi}^{\pi} \sin mx \sin nx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m-n)x) - \cos((m+n)x)) \,\mathrm{d}x \\
&= \begin{cases} \frac{1}{2}\int_{-\pi}^{\pi}(1-1)\,\mathrm{d}x & m+n=0, m-n=0 \\ \frac{1}{2}\int_{-\pi}^{\pi}(\cos((m-n)x)-1)\,\mathrm{d}x & m+n=0 \\ \frac{1}{2}\int_{-\pi}^{\pi}(1-\cos((m+n)x))\,\mathrm{d}x & m-n=0 \\ \frac{1}{2}\int_{-\pi}^{\pi}(\cos((m-n)x)-\cos((m+n)x))\,\mathrm{d}x & \text{otherwise} \end{cases} \\
&= \begin{cases} 0 & m+n=0, m-n=0 \\ \frac{1}{2}\left[\frac{1}{m-n}\sin((m-n)x) - x\right]_{x=-\pi}^{x=\pi} & m+n=0 \\ \frac{1}{2}\left[x - \frac{1}{m+n}\sin((m+n)x)\right]_{x=-\pi}^{x=\pi} & m-n=0 \\ \frac{1}{2}\left[\frac{1}{m-n}\sin((m-n)x) - \frac{1}{m+n}\sin((m+n)x)\right]_{x=-\pi}^{x=\pi} & \text{otherwise} \end{cases} \\
&= \begin{cases} 0 & m+n=0, m-n=0 \\ -\pi & m+n=0 \\ \pi & m-n=0 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 0 & m=n=0 \\ -\pi & m=-n \\ \pi & m=n \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\int_{-\pi}^{\pi} \cos mx \sin nx \,\mathrm{d}x \\
&= \int_{0}^{\pi} \cos mx \sin nx \,\mathrm{d}x + \int_{-\pi}^{0} \cos mx \sin nx \,\mathrm{d}x \\
&= \int_{0}^{\pi} \cos mx \sin nx \,\mathrm{d}x - \int_{\pi}^{0} \cos(-mu) \sin(-nu) \,\mathrm{d}u && (u := -x) \\
&= \int_{0}^{\pi} \cos mx \sin nx \,\mathrm{d}x - \int_{0}^{\pi} \cos mu \sin nu \,\mathrm{d}u \\
&= 0
\end{aligned}
$$

(The above is adapted from my own work in assignment 3 Q1.)

$$
\begin{aligned}
&\int_{-\pi}^{\pi} f(x) \,\mathrm{d}x \\
&= \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \right) \mathrm{d}x \\
&= \int_{-\pi}^{\pi} \frac{a_0}{2} \,\mathrm{d}x + \sum_{k=1}^{n} a_k \int_{-\pi}^{\pi} \cos kx \,\mathrm{d}x + \sum_{k=1}^{n} b_k \int_{-\pi}^{\pi} \sin kx \,\mathrm{d}x && \text{(linearity)} \\
&= \pi a_0 + \sum^{n} \frac{a_k}{k} [\sin kx]_{x=-\pi}^{\pi} - \sum^{n} \frac{b_k}{k} [\cos kx]_{x=-\pi}^{\pi}
\end{aligned}
$$

$$
\scriptstyle k=1 \qquad\qquad\qquad\qquad k=1
$$

$$
\begin{aligned}
&= \pi a_0 + 0 - 0 \\
&= \pi a_0
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 \,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k \cos kx + b_k \sin kx\right)\right)^2 \mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(a_0 f(x) - \frac{a_0^2}{4} + \sum_{1\le i,j\le n}\left(a_i \cos ix + b_i \sin ix\right)\left(a_j \cos jx + b_j \sin jx\right)\right) \mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(a_0 f(x) - \frac{a_0^2}{4} + \sum_{1\le i,j\le n}\left(a_i a_j \cos ix \cos jx + a_i b_j \cos ix \sin jx + \right.\right. \\
&\qquad \left.\left. b_i a_j \sin ix \cos jx + b_i b_j \sin ix \sin jx\right)\right) \mathrm{d}x \\
&= \frac{a_0}{\pi}\int_{-\pi}^{\pi}\left(f(x) - \frac{a_0}{4}\right)\mathrm{d}x + \frac{1}{\pi}\sum_{1\le i,j\le n} a_i a_j \int_{-\pi}^{\pi} \cos ix \cos jx \,\mathrm{d}x && \text{(linearity)} \\
&\quad + \frac{1}{\pi}\sum_{1\le i,j\le n} a_i b_j \int_{-\pi}^{\pi} \cos ix \sin jx \,\mathrm{d}x \\
&\quad + \frac{1}{\pi}\sum_{1\le i,j\le n} b_i a_j \int_{-\pi}^{\pi} \sin ix \cos jx \,\mathrm{d}x \\
&\quad + \frac{1}{\pi}\sum_{1\le i,j\le n} b_i b_j \int_{-\pi}^{\pi} \sin ix \sin jx \,\mathrm{d}x \\
&= \frac{a_0}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x - \frac{a_0^2}{2} + \sum_{k=1}^{n} a_k^2 + 0 + 0 + \sum_{k=1}^{n} b_k^2 && \text{(apply equations above)} \\
&= a_0^2 - \frac{a_0^2}{2} + \sum_{k=1}^{n}\left(a_k^2 + b_k^2\right) \\
&= \frac{a_0^2}{2} + \sum_{k=1}^{n}\left(a_k^2 + b_k^2\right)
\end{aligned}
$$

Q13

Let a be a real number which is not an integer. Let $f(x) = \cos ax$ be defined on $[-\pi, \pi]$ and extended periodically to become a function with period 2π.

Q13.a

Compute the Fourier series of f.

$\because f$ is a periodic function of period 2π
f is continuously differentiable on $\mathbb{R}$
$\therefore f$ equals its Fourier series on $\mathbb{R}$

$$
\begin{aligned}
& f(x) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x + \sum_{n=1}^{+\infty}\left(\cos(nx)\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x + \sin(nx)\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(ax)\,\mathrm{d}x + \sum_{n=1}^{+\infty}\left(\cos(nx)\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(ax)\cos(nx)\,\mathrm{d}x + \sin(nx)\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(ax)\sin(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{2a\pi}[\sin(ax)]_{x=-\pi}^{\pi} + \sum_{n=1}^{+\infty}\left(\cos(nx)\frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos((a-n)x)+\cos((a+n)x))\,\mathrm{d}x\right. \\
&\quad\left. + \sin(nx)\frac{1}{2\pi}\int_{-\pi}^{\pi}(\sin((a+n)x)-\sin((a-n)x))\,\mathrm{d}x\right) \\
&= \frac{\sin(a\pi)}{a\pi} + \sum_{n=1}^{+\infty}\left(\frac{\cos(nx)}{2\pi}\left[\frac{\sin((a-n)x)}{a-n}+\frac{\sin((a+n)x)}{a+n}\right]_{x=-\pi}^{\pi}\right. && (a \notin \mathbb{Z}) \\
&\quad\left. - \frac{\sin(nx)}{2\pi}\left[\frac{\cos((a+n)x)}{a+n}-\frac{\cos((a-n)x)}{a-n}\right]_{x=-\pi}^{\pi}\right) \\
&= \frac{\sin(a\pi)}{a\pi} + \frac{1}{\pi}\sum_{n=1}^{+\infty}\cos(nx)\left(\frac{\sin((a-n)\pi)}{a-n}+\frac{\sin((a+n)\pi)}{a+n}\right) \\
&= \frac{\sin(a\pi)}{a\pi} + \frac{1}{\pi}\sum_{n=1}^{+\infty}\cos(nx)\frac{(a+n)\sin((a-n)\pi)+(a-n)\sin((a+n)\pi)}{a^2-n^2} \\
&= \frac{\sin(a\pi)}{a\pi} + \frac{1}{\pi}\sum_{n=1}^{+\infty}\cos(nx)\frac{2a\sin(a\pi)\cos(n\pi)+2n\sin(n\pi)\cos(a\pi)}{a^2-n^2} \\
&= \frac{\sin(a\pi)}{a\pi} + \frac{2a}{\pi}\sum_{n=1}^{+\infty}\cos(nx)\frac{(-1)^n\sin(a\pi)}{a^2-n^2} \\
&= \frac{\sin(a\pi)}{a\pi} + \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{+\infty}(-1)^n\frac{\cos(nx)}{a^2-n^2}
\end{aligned}
$$

Q13.b

Using (a), prove that

$$\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k^2-a^2} = -\frac{1}{2a^2} + \frac{\pi}{2a\sin(a\pi)}$$

.

and in a similar way also compute

$$\sum_{k=1}^{+\infty}\frac{1}{k^2-a^2}$$

.

$$f(x) = \frac{\sin(a\pi)}{a\pi} + \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{+\infty}(-1)^n\frac{\cos(nx)}{a^2-n^2} \tag{Q13.a}$$

$$\begin{aligned}
f(0) &= \frac{\sin(a\pi)}{a\pi} + \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{+\infty}(-1)^n\frac{\cos(0n)}{a^2-n^2}\\
\cos(0a) &= \frac{\sin(a\pi)}{a\pi} + \frac{2a\sin(a\pi)}{\pi}\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k^2-a^2}\\
1-\frac{\sin(a\pi)}{a\pi} &= \frac{2a\sin(a\pi)}{\pi}\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k^2-a^2}\\
\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k^2-a^2} &= -\frac{1}{2a^2} + \frac{\pi}{2a\sin(a\pi)}
\end{aligned}$$

$$\begin{aligned}
f(\pi) &= \frac{\sin(a\pi)}{a\pi} + \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{+\infty}(-1)^n\frac{\cos(n\pi)}{a^2-n^2}\\
\cos(a\pi) &= \frac{\sin(a\pi)}{a\pi} + \frac{2a\sin(a\pi)}{\pi}\sum_{k=1}^{+\infty}\frac{1}{a^2-k^2}\\
\cos(a\pi)-\frac{\sin(a\pi)}{a\pi} &= -\frac{2a\sin(a\pi)}{\pi}\sum_{k=1}^{+\infty}\frac{1}{k^2-a^2}\\
\sum_{k=1}^{+\infty}\frac{1}{k^2-a^2} &= \frac{1}{2a^2} - \frac{\pi\cos(a\pi)}{2a\sin(a\pi)}\\
&= \frac{1}{2a^2} - \frac{\pi}{2a}\cot(a\pi)
\end{aligned}$$