Solution to Problem Set 8

1. (a) Since

$$\begin{aligned}
\lim_{x\to 0}\frac{e^x - P_n(x)}{x^n} &= \lim_{x\to 0}\frac{e^x - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n\right)}{x^n} \\
&= \lim_{x\to 0}\frac{e^x - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{(n-1)!}x^{n-1}\right)}{nx^{n-1}} && \text{(l'Hôpital's rule)} \\
&= \lim_{x\to 0}\frac{e^x - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{(n-2)!}x^{n-2}\right)}{n(n-1)x^{n-2}} && \text{(l'Hôpital's rule)} \\
&= \cdots \\
&= \lim_{x\to 0}\frac{e^x - (1+x)}{n!\,x} && \text{(l'Hôpital's rule)} \\
&= \lim_{x\to 0}\frac{e^x - 1}{n!} && \text{(l'Hôpital's rule)} \\
&= 0,
\end{aligned}$$

it follows that P_n is the n^{th} order approximation of f at 0. ■

(b) We have $f(x) = e^x = e\cdot e^{x-1}$. Replacing x by $x-1$ in (a), we have

$$\lim_{x\to 1}\frac{e^{x-1} - P_n(x-1)}{(x-1)^n} = 0, \qquad \text{i.e. } \lim_{x\to 1}\frac{f(x) - e\cdot P_n(x-1)}{(x-1)^n} = 0;$$

so the n^{th} order approximation of f at 1 is given by

$$e\cdot P_n(x-1) = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \cdots + \frac{e}{n!}(x-1)^n.$$

2. (a) Since

$$\begin{aligned}
e^x\cos x &= \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + o(x^6)\right)\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + o(x^6)\right) \\
&= 1 + x + \left(\frac{1}{2!} - \frac{1}{2!}\right)x^2 + \left(\frac{1}{3!} - \frac{1}{2!}\right)x^3 + \left(\frac{1}{4!} - \frac{1}{2!}\cdot\frac{1}{2!} + \frac{1}{4!}\right)x^4 + \left(\frac{1}{5!} - \frac{1}{3!}\frac{1}{2!} + \frac{1}{4!}\right)x^5 \\
&\quad + \left(\frac{1}{6!} - \frac{1}{4!}\frac{1}{2!} + \frac{1}{2!}\frac{1}{4!} - \frac{1}{6!}\right)x^6 + o(x^6) \\
&= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{30}x^5 + o(x^6)
\end{aligned}$$

as $x\to 0$, the 6^{th} order approximation of f at 0 is $P_6(x) = 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{30}x^5$.

(b) Since

$$\begin{aligned}
e^{\cos x} &= e^{1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+o(x^6)} = e\cdot e^{-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+o(x^6)} \\
&= e\cdot\left[1+\left(-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+o(x^6)\right)+\frac{1}{2!}\left(-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+o(x^6)\right)^2\right. \\
&\qquad\left.+\frac{1}{3!}\left(-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+o(x^6)\right)^3+o(x^6)\right] \\
&= e\cdot\left[1+\left(-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6\right)+\frac{1}{2!}\left(\frac{1}{(2!)^2}x^4-\frac{1}{2!}\frac{1}{4!}x^6-\frac{1}{2!}\frac{1}{4!}x^6\right)+\frac{1}{3!}\left(-\frac{1}{(2!)^3}x^6\right)+o(x^6)\right] \\
&= e-\frac{e}{2}x^2+\frac{e}{6}x^4-\frac{31e}{720}x^6+o(x^6)
\end{aligned}$$

as $x\to 0$, the 6th order approximation of g at 0 is $P_6(x)=e-\frac{e}{2}x^2+\frac{e}{6}x^4-\frac{31e}{720}x^6$.

(c) Since

$$\begin{aligned}
\sec x &= \frac{1}{\cos x} = \frac{1}{1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+o(x^6)} \\
&= 1+\left(\frac{1}{2!}x^2-\frac{1}{4!}x^4+\frac{1}{6!}x^6+o(x^6)\right)+\left(\frac{1}{2!}x^2-\frac{1}{4!}x^4+\frac{1}{6!}x^6+o(x^6)\right)^2 \\
&\qquad+\left(\frac{1}{2!}x^2-\frac{1}{4!}x^4+\frac{1}{6!}x^6+o(x^6)\right)^3+o(x^6) \\
&= 1+\left(\frac{1}{2!}x^2-\frac{1}{4!}x^4+\frac{1}{6!}x^6\right)+\left(\frac{1}{(2!)^2}x^4-\frac{1}{2!}\frac{1}{4!}x^6-\frac{1}{2!}\frac{1}{4!}x^6\right)+\left(\frac{1}{(2!)^3}x^6\right)+o(x^6) \\
&= 1+\frac{1}{2}x^2+\frac{5}{24}x^4+\frac{61}{720}x^6+o(x^6)
\end{aligned}$$

as $x\to 0$, the 6th order approximation of h at 0 is $P_6(x)=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\frac{61}{720}x^6$.

3. (a) Since

$$\begin{aligned}
e^{\sin x} &= e^{x-\frac{1}{6}x^3+o(x^3)} \\
&= 1+\left(x-\frac{1}{6}x^3+o(x^3)\right)+\frac{1}{2!}\left(x-\frac{1}{6}x^3+o(x^3)\right)^2+\frac{1}{3!}\left(x-\frac{1}{6}x^3+o(x^3)\right)^3+o(x^3) \\
&= 1+\left(x-\frac{1}{6}x^3\right)+\frac{1}{2!}(x^2)+\frac{1}{3!}(x^3)+o(x^3) \\
&= 1+x+\frac{1}{2}x^2+o(x^3)
\end{aligned}$$

as $x\to 0$, we have

$$\begin{aligned}
\lim_{x\to 0}\frac{e^x-e^{\sin x}}{x^3} &= \lim_{x\to 0}\frac{\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3\right)-\left(1+x+\frac{1}{2}x^2\right)+o(x^3)}{x^3} \\
&= \lim_{x\to 0}\frac{\frac{1}{6}x^3+o(x^3)}{x^3}=\frac{1}{6}.
\end{aligned}$$

(b) Since

$$\begin{aligned}\sin^2 x - \sin(x^2) &= \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + o(x^6)\right)^2 - \left(x^2 - \frac{1}{3!}x^6 + o(x^6)\right)\\ &= \left(x^2 - \frac{2}{3!}x^4 + \left(\frac{2}{5!} + \frac{1}{(3!)^2}\right)x^6\right) - \left(x^2 - \frac{1}{3!}x^6\right) + o(x^6)\\ &= -\frac{1}{3}x^4 + \frac{19}{90}x^6 + o(x^6)\end{aligned}$$

as $x \to 0$, we have

$$\lim_{x\to 0} \frac{\sin^2 x - \sin(x^2) + \frac{1}{3}x^4}{x^6} = \lim_{x\to 0} \frac{\frac{19}{90}x^6 + o(x^6)}{x^6} = \frac{19}{90}.$$

(c) First note that

$$\lim_{x\to+\infty} x^2\left(e - \frac{e}{2x} - \left(1 + \frac{1}{x}\right)^x\right) = \lim_{x\to 0^+} \frac{e - \frac{e}{2}x - (1+x)^{\frac{1}{x}}}{x^2}.$$

Now since

$$\begin{aligned}(1+x)^{\frac{1}{x}} &= e^{\frac{1}{x}\ln(1+x)} = e^{\frac{1}{x}\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)\right)} = e^{1 - \frac{1}{2}x + \frac{1}{3}x^2 + o(x^2)} = e \cdot e^{-\frac{1}{2}x + \frac{1}{3}x^2 + o(x^2)}\\ &= e\left(1 + \left(-\frac{1}{2}x + \frac{1}{3}x^2 + o(x^2)\right) + \frac{1}{2!}\left(-\frac{1}{2}x + \frac{1}{3}x^2 + o(x^2)\right)^2 + o(x^2)\right)\\ &= e\left(1 + \left(-\frac{1}{2}x + \frac{1}{3}x^2\right) + \frac{1}{2!}\left(\frac{1}{2^2}x^2\right)\right) + o(x^2)\\ &= e - \frac{e}{2}x + \frac{11e}{24}x^2 + o(x^2)\end{aligned}$$

as $x \to 0$, we have

$$\lim_{x\to+\infty} x^2\left(e - \frac{e}{2x} - \left(1 + \frac{1}{x}\right)^x\right) = \lim_{x\to 0^+} \frac{-\frac{11e}{24}x^2 + o(x^2)}{x^2} = -\frac{11e}{24}.$$

4. (a) By Fundamental Theorem of Calculus we have

$$f(x) - f(a) = \int_a^x f'(t)dt$$

since f' is continuous (and thus integrable) on $[a, x]$. Integrating by parts for n times, we obtain

$$\begin{aligned}f(x) &= f(a) + \int_a^x f'(t)dt\\ &= f(a) - [f'(t)(x-t)]_a^x + \int_a^x f''(t)(x-t)dt\\ &= f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)dt\end{aligned}$$

$$= f(a) + f'(a)(x-a) + \left[-\frac{1}{2}f''(t)(x-t)^2\right]_a^x - \int_a^x f'''(t)\left[-\frac{1}{2}(x-t)^2\right]dt$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{1}{2}\int_a^x f'''(t)(x-t)^2 dt$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \frac{1}{6}\int_a^x f^{(4)}(t)(x-t)^3 dt$$

$$= \cdots = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt,$$

since $f^{(n+1)}$ is continuous (and thus integrable) on $[a,x]$. ■

(b) Both $f^{(n+1)}(t)$ and $(x-t)^n$ are continuous functions of t on the interval $[a,x]$. Since $(x-t)^n \geq 0$ for every $t \in [a,x]$, by the generalized Mean Value Theorem for integrals (Example 5.47 (a)), there exists $c \in (a,x)$ such that

$$\int_a^x f^{(n+1)}(t)(x-t)^n dt = f^{(n+1)}(c)\int_a^x (x-t)^n dt.$$

Now

$$\int_a^x (x-t)^n dt = \left[-\frac{1}{n+1}(x-t)^{n+1}\right]_a^x = \frac{1}{n+1}(x-a)^{n+1},$$

so according to (a), this number c satisfies that

$$f(x) - \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt = \frac{f^{(n+1)}(c)}{n!}\cdot\frac{1}{n+1}(x-a)^{n+1}$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

which proves Lagrange's remainder formula. ■

5. (a) (i) For every $t \in (a,x)$, we have

$$g'(t) = -\sum_{k=0}^{n}\left[\frac{f^{(k+1)}(t)}{k!}(x-t)^k - \frac{f^{(k)}(t)}{k!}k(x-t)^{k-1}\right]$$

$$= \underbrace{-f'(t)}_{k=0} + \sum_{k=1}^{n}\left[\frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} - \frac{f^{(k+1)}(t)}{k!}(x-t)^k\right]$$

$$= -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right]$$

$$+ \left[\frac{f'''(t)}{2!}(x-t)^2 - \frac{f'''(t)}{3!}(x-t)^3\right] + \cdots$$

$$+ \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right]$$

$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

(ii) Note that $g(x) = f(x) - f(x) = 0$. Since g is continuous on $[a, x]$ and differentiable on (a, x), according to Mean Value Theorem there exists $c \in (a, x)$ such that $g(x) - g(a) = g'(c)(x - a)$, i.e.

$$0 - \left[f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right] = \left[-\frac{f^{(n+1)}(c)}{n!}(x-c)^n\right](x-a).$$

Rearranging the terms, we get

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a)$$

as desired. ■

(b) (i) Let $f(x) = \ln(1+x)$. Then $f^{(n+1)}(x) = \frac{(-1)^n n!}{(1+x)^{n+1}}$ for every non-negative integer n. Now according to Cauchy's remainder formula in (a) (ii), for each $x \in (-1, 1)$ there exists c between 0 and x such that

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{n!}(x-c)^n x\right| = \left|\frac{(-1)^n}{(1+c)^{n+1}}(x-c)^n x\right| = \frac{|x|}{1+c}\left(\frac{|x-c|}{1+c}\right)^n.$$

Since $x \in (-1, 1)$, we must have $\frac{|x-c|}{1+c} \leq |x|$ (if $x \in (0, 1)$, then we have $0 < c < x < 1$, so $\frac{|x-c|}{1+c} = \frac{x-c}{1+c} < x - c < x = |x|$; if $x \in (-1, 0)$, then $-1 < x < c < 0$, so $\frac{|x-c|}{1+c} = \frac{|x|-|c|}{1-|c|} < \frac{|x|-|x||c|}{1-|c|} = |x|$); thus

$$|R_n(x)| \leq \frac{1}{1+c}|x|^{n+1} \leq \frac{1}{\min\{1+x, 1\}}|x|^{n+1}.$$

c depends on n as well.

Since $\lim_{n\to+\infty} |x|^{n+1} = 0$, by squeeze theorem we have $\lim_{n\to+\infty} |R_n(x)| = 0$.

(ii) Let $f(x) = (1+x)^p$. Then $f^{(n+1)}(x) = p(p-1)(p-2)\cdots(p-n)(1+x)^{p-n-1}$. Now according to Cauchy's remainder formula in (a) (ii), for each $x \in (-1, 1)$ there exists c between 0 and x such that

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{n!}(x-c)^n x\right| = \left|\frac{p(p-1)(p-2)\cdots(p-n)(1+c)^{p-n-1}}{n!}(x-c)^n x\right|$$

$$= |px|(1+c)^{p-1}\left|\frac{(p-1)(p-2)\cdots(p-n)}{n!}\right|\left(\frac{|x-c|}{1+c}\right)^n.$$

c depends on n as well.

Since $x \in (-1, 1)$, we must have $\frac{|x-c|}{1+c} \leq |x|$ (for the same reason as in (b)(i)); thus

$$|R_n(x)| \leq \underbrace{|px|(1+|x|)^{p-1}}_{\text{independent of } n}\left|\frac{(p-1)(p-2)\cdots(p-n)}{n!}\right||x|^n$$

Now let a_n denote the right-hand side of the above inequality. For sufficiently large n, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1-p}{n+1}|x|.$$

Thus

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\frac{n+1-p}{n+1}|x| = |x| < 1,$$

which implies that $\lim_{n\to+\infty} a_n = 0$, and so $\lim_{n\to+\infty} |R_n(x)| = 0$ by squeeze theorem. ■

6. (a) At 1, the given series becomes $\sum_{k=0}^{+\infty}\frac{p(p-1)\cdots(p-k+1)}{k!}$. Let $a_n=\frac{p(p-1)\cdots(p-n+1)}{n!}$. Then we have

$$\frac{a_{n+1}}{a_n}=\frac{p-n}{n+1} \qquad \text{for every } n.$$

Let $N>p$ be a non-negative integer. Then for $n\geq N$ the terms have alternating signs.

- If $p>-1$, then for every integer $n\geq N>p$ we have

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{|p-n|}{n+1}=\frac{n-p}{n+1}=1-\frac{p+1}{n+1}<1,$$

so $(|a_n|)$ is decreasing. Moreover, since $\frac{p+1}{n+1}\in(0,1)$, we also obtain

$$\ln\left|\frac{a_{n+1}}{a_n}\right|=\ln\left(1-\frac{p+1}{n+1}\right)<-\frac{p+1}{n+1};$$

and so

$$|a_n|\leq|a_N|\left|\frac{a_{N+1}}{a_N}\right|\left|\frac{a_{N+2}}{a_{N+1}}\right|\cdots\left|\frac{a_n}{a_{n-1}}\right|\leq|a_N|e^{-\frac{p+1}{N+1}}e^{-\frac{p+1}{N+2}}\cdots e^{-\frac{p+1}{n}}=|a_N|e^{-(p+1)\sum_{k=N+1}^{n}\frac{1}{k}}.$$

Since the harmonic series diverges to $+\infty$, it follows that $\lim_{n\to+\infty}e^{-(p+1)\sum_{k=N+1}^{n}\frac{1}{k}}=0$ and so $(|a_n|)$ converges to 0 by squeeze theorem. Therefore $\sum_{k=0}^{+\infty}a_k$ converges by alternating series test.

- If $p\leq-1$, then we have

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{|p-n|}{n+1}=\frac{n-p}{n+1}\geq1$$

for every n. This implies that (a_n) does not converge to 0, so $\sum_{k=0}^{+\infty}a_k$ diverges by term test.

(b) At -1, the given series becomes $\sum_{k=0}^{+\infty}\frac{p(p-1)\cdots(p-k+1)}{k!}(-1)^k$. Let $a_n=\frac{p(p-1)\cdots(p-n+1)}{n!}(-1)^n$. Then

$$\frac{a_{n+1}}{a_n}=\frac{n-p}{n+1}=1-\frac{p+1}{n+1} \qquad \text{for every } n.$$

Let $N>p$ be a non-negative integer. Then for $n\geq N$ the terms all have the same sign. Now consider

$$\left|\frac{a_n}{a_N}\right|=\frac{a_{N+1}}{a_N}\frac{a_{N+2}}{a_{N+1}}\cdots\frac{a_n}{a_{n-1}}=\left(1-\frac{p+1}{N+1}\right)\left(1-\frac{p+1}{N+2}\right)\cdots\left(1-\frac{p+1}{n}\right).$$

- If $p>0$, then for every $n\geq N$ we have

$$\ln\left|\frac{a_n}{a_N}\right|=\ln\left(1-\frac{p+1}{N+1}\right)+\ln\left(1-\frac{p+1}{N+2}\right)\cdots+\ln\left(1-\frac{p+1}{n}\right)\leq-\left(\frac{p+1}{N+1}+\frac{p+1}{N+2}\cdots+\frac{p+1}{n}\right)$$

$$\leq-(p+1)\int_{N+1}^{n}\frac{1}{x}dx=-(p+1)\ln\frac{n}{N+1}.$$

This implies that

$$|a_n|\leq\frac{|a_N|(N+1)^{p+1}}{n^{p+1}}.$$

Since $\sum_{k=N}^{+\infty}\frac{1}{k^{p+1}}$ converges by p-test, $\sum_{k=0}^{+\infty}a_k$ converges (absolutely) by comparison test.

- If $p < 0$, then for every $n \geq N$ we have

$$\left|\frac{a_n}{a_N}\right| = \left(1 - \frac{p+1}{N+1}\right) \cdots \left(1 - \frac{p+1}{n}\right) \geq \left(1 - \frac{1}{N+1}\right) \cdots \left(1 - \frac{1}{n}\right) = \frac{N}{n}.$$

This implies that

$$|a_n| \geq \frac{N|a_N|}{n}.$$

Since the harmonic series $\sum_{k=N}^{+\infty} \frac{1}{k}$ diverges, $\sum_{k=N}^{+\infty} a_k$ diverges by comparison test (recall that for $n \geq N$ the terms all have the same sign).

7. (a) Note that $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ for every $x \in \mathbb{R}$. Now we know that

$$\cos 2x = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!}(2x)^{2k} \qquad \text{for every } x \in \mathbb{R},$$

so

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\left(\sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!}(2x)^{2k}\right) = \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}2^{2k-1}}{(2k)!}x^{2k} \qquad \text{for every } x \in \mathbb{R}.$$

The radius of convergence of this Maclaurin series is $+\infty$.

(b) We have

$$\sin x = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1} \qquad \text{for every } x \in \mathbb{R},$$

so

$$\frac{\sin x}{x} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!}x^{2k} \qquad \text{for every } x \in \mathbb{R} \setminus \{0\},$$

and the series converges to 1 at 0. Now termwise integration from 0 to x gives

$$\int_0^x \frac{\sin t}{t}\,dt = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!\,(2k+1)}x^{2k+1} \qquad \text{for every } x \in \mathbb{R}.$$

The radius of convergence of this Maclaurin series is $+\infty$.

(c) The binomial series gives

$$(1-x^2)^{-\frac{1}{2}} = \sum_{k=0}^{+\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-k+\frac{1}{2}\right)}{k!}(-x^2)^k = \sum_{k=0}^{+\infty} \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\left(k-\frac{1}{2}\right)}{k!}x^{2k}$$

for every $x \in (-1, 1)$. Termwise integration from 0 to x gives

$$\arcsin x = \sum_{k=0}^{+\infty} \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\left(k-\frac{1}{2}\right)}{k!\,(2k+1)}x^{2k+1} = \sum_{k=0}^{+\infty} \frac{1^2\cdot 3^2\cdot 5^2\cdots(2k-1)^2}{(2k+1)!}x^{2k+1}$$

for every $x \in (-1, 1)$. The radius of convergence of this Maclaurin series is 1.

(d) Given $f(x) = \ln\left(x + \sqrt{1+x^2}\right)$, we have

$$f'(x) = \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \frac{x}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}}.$$

Now the binomial series gives

$$(1+x^2)^{-\frac{1}{2}} = \sum_{k=0}^{+\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-k+\frac{1}{2}\right)}{k!}(x^2)^k \qquad \text{for every } x \in [-1,1].$$

Now termwise integration from 0 to x gives

$$\ln\left(x+\sqrt{1+x^2}\right) = \sum_{k=0}^{+\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-k+\frac{1}{2}\right)}{k!\,(2k+1)} x^{2k+1} = \sum_{k=0}^{+\infty} \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2k-1)^2(-1)^k}{(2k+1)!} x^{2k+1}$$

for every $x \in (-1,1)$. The radius of convergence of this Maclaurin series is 1.

8. (a) First consider the power series $\sum_{k=0}^{+\infty} \frac{(-1)^k}{4k^2-1} x^k$. The radius of convergence of this series is

$$\lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\left|\frac{(-1)^n}{4n^2-1}\frac{4(n+1)^2-1}{(-1)^{n+1}}\right| = \lim_{n\to+\infty}\frac{4\left(1+\frac{1}{n}\right)^2 - \frac{1}{n^2}}{4-\frac{1}{n^2}} = 1.$$

Now the radius of convergence of the given power series $f(x) = \sum_{k=0}^{+\infty}\frac{(-1)^k}{4k^2-1}x^{2k+1}$ is the square root of that of the above series, which is also 1. Termwise differentiation then gives

$$f'(x) = \sum_{k=0}^{+\infty}\frac{(-1)^k}{4k^2-1}(2k+1)x^{2k} = \sum_{k=0}^{+\infty}\frac{(-1)^k}{2k-1}x^{2k} \qquad \text{for every } x \in (-1,1).$$

On the other hand, we also have

$$\sum_{k=0}^{+\infty}\frac{(-1)^k}{2k-1}x^{2k} = -1 + \sum_{k=1}^{+\infty}\frac{(-1)^k}{2k-1}x^{2k} = -1 + \sum_{k=0}^{+\infty}\frac{(-1)^{k+1}}{2k+1}x^{2k+2} = -1 - x\arctan x$$

for every $x \in (-1,1)$, so $f'(x) = -1 - x\arctan x$ for $x \in (-1,1)$. Integrating from 0 to x, we have

$$\begin{aligned} f(x) - \underbrace{f(0)}_{=0} &= \int_0^x (-1 - t\arctan t)\,dt \\ &= -x - \frac{1}{2}x^2\arctan x + \frac{1}{2}\int_0^x \frac{t^2}{1+t^2}\,dt \\ &= -x - \frac{1}{2}x^2\arctan x + \frac{1}{2}(x - \arctan x), \end{aligned}$$

so the given power series $f(x)$ has sum

$$\sum_{k=0}^{+\infty}\frac{(-1)^k}{4k^2-1}x^{2k+1} = -\frac{1}{2}x - \frac{x^2+1}{2}\arctan x \qquad \text{for every } x \in (-1,1).$$

- ⊙ At the end-point -1, the power series $f(-1)$ becomes $\sum_{k=0}^{+\infty}\frac{(-1)^k}{4k^2-1}(-1)^{2k+1}=\sum_{k=0}^{+\infty}\frac{-1}{4k^2-1}$ which converges by limit comparison test with $\sum_{k=1}^{+\infty}\frac{1}{k^2}$.
- ⊙ At the end-point 1, the power series $f(1)$ becomes $\sum_{k=0}^{+\infty}\frac{(-1)^k}{4k^2-1}$ which converges by alternating series test.

So by Abel's limit theorem, the power series $f(x)$ has sum

$$\sum_{k=0}^{+\infty}\frac{(-1)^k}{4k^2-1}x^{2k+1}=-\frac{1}{2}x-\frac{x^2+1}{2}\arctan x \qquad \text{for every } x\in[-1,1].$$

(b) The given power series $f(x)=\sum_{k=1}^{+\infty}a_kx^k$ can be rewritten as

$$\begin{aligned} f(x)&=\sum_{k=0}^{+\infty}\frac{1}{3k+2}x^{3k+2}+\sum_{k=0}^{+\infty}\frac{1}{3k+1}x^{3k+1}-\sum_{k=1}^{+\infty}\frac{2}{3k}x^{3k}\\ &=\sum_{k=0}^{+\infty}\frac{1}{3k+2}x^{3k+2}+\sum_{k=0}^{+\infty}\frac{1}{3k+1}x^{3k+1}+\left(\sum_{k=1}^{+\infty}\frac{1}{3k}x^{3k}-\sum_{k=1}^{+\infty}\frac{3}{3k}x^{3k}\right)\\ &=\sum_{k=1}^{+\infty}\frac{1}{k}x^k-\sum_{k=1}^{+\infty}\frac{1}{k}x^{3k}, \end{aligned}$$

which has radius of convergence 1. Now for every $x\in(-1,1)$, we also have $x^3\in(-1,1)$; so the power series $f(x)$ has sum

$$\sum_{k=1}^{+\infty}a_kx^k=-\ln(1-x)+\ln(1-x^3)=\ln\frac{1-x^3}{1-x}=\ln(1+x+x^2).$$

- ⊙ At the end-point 1, the power series $f(1)$ becomes $\sum_{k=1}^{+\infty}a_k$. Let $s_n=\sum_{k=1}^{n}a_k$. We see that

$$s_{3n}=\sum_{k=1}^{3n}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{k}=\sum_{k=n+1}^{3n}\frac{1}{k}$$

for every $n\in\mathbb{N}$, so $\lim\limits_{n\to+\infty}s_{3n}=\int_1^3\frac{1}{x}dx=\ln 3$. Moreover, $\lim\limits_{n\to+\infty}s_{3n+1}=\lim\limits_{n\to+\infty}s_{3n}+\lim\limits_{n\to+\infty}\frac{1}{3n+1}=\ln 3$ and $\lim\limits_{n\to+\infty}s_{3n+2}=\lim\limits_{n\to+\infty}s_{3n}+\lim\limits_{n\to+\infty}\frac{1}{3n+1}+\lim\limits_{n\to+\infty}\frac{1}{3n+2}=\ln 3$.
- ⊙ At the end-point -1, the power series $f(-1)$ becomes $\sum_{k=1}^{+\infty}a_k(-1)^k$. Let $s_n=\sum_{k=1}^{n}a_k(-1)^k$. We see that

$$s_{3n}=\sum_{k=1}^{3n}\frac{(-1)^k}{k}-\sum_{k=1}^{n}\frac{(-1)^k}{k}$$

for every $n\in\mathbb{N}$. Since both sums converges to $-\ln 2$, $\lim\limits_{n\to+\infty}s_{3n}=(-\ln 2)-(-\ln 2)=0$. Moreover,

$\lim\limits_{n\to+\infty}s_{3n+1}=\lim\limits_{n\to+\infty}s_{3n}+\lim\limits_{n\to+\infty}\frac{(-1)^n}{3n+1}=0$ and $\lim\limits_{n\to+\infty}s_{3n+2}=\lim\limits_{n\to+\infty}s_{3n}+\lim\limits_{n\to+\infty}\frac{(-1)^n}{3n+1}+\lim\limits_{n\to+\infty}\frac{(-1)^n}{3n+2}=0$.

Therefore the power series $f(x)$ has sum

$$\sum_{k=1}^{+\infty}a_kx^k=\ln(1+x+x^2) \qquad \text{for every } x\in[-1,1].$$

9. (a) Using the Maclaurin series of e^x, we have

$$x^3e^x = x^3\sum_{k=0}^{+\infty}\frac{1}{k!}x^k = \sum_{k=0}^{+\infty}\frac{1}{k!}x^{k+3} = \sum_{k=3}^{+\infty}\frac{1}{(k-3)!}x^k$$

for every $x \in \mathbb{R}$. On the other hand, the Maclaurin series of f is also given by

$$x^3e^x = \sum_{k=0}^{+\infty}\frac{f^{(k)}(0)}{k!}x^k.$$

These two series are the same series for every $x \in \mathbb{R}$, i.e.

$$\sum_{k=0}^{+\infty}\frac{f^{(k)}(0)}{k!}x^k = \sum_{k=3}^{+\infty}\frac{1}{(k-3)!}x^k.$$

Comparing the coefficients of the two series, we have

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n \in \{0,1,2\} \\ \dfrac{n!}{(n-3)!} & \text{if } n \geq 3 \end{cases} = n(n-1)(n-2).$$

(b) Note that

$$x^3 = \big((x-1)+1\big)^3 = (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1.$$

Using the Taylor series of e^x at 1, we have

$$\begin{aligned} x^3e^x &= [(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1]e \cdot e^{x-1} \\ &= e[(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1]\sum_{k=0}^{+\infty}\frac{1}{k!}(x-1)^k \\ &= \sum_{k=0}^{+\infty}\frac{e}{k!}(x-1)^{k+3} + \sum_{k=0}^{+\infty}\frac{3e}{k!}(x-1)^{k+2} + \sum_{k=0}^{+\infty}\frac{3e}{k!}(x-1)^{k+1} + \sum_{k=0}^{+\infty}\frac{e}{k!}(x-1)^k \\ &= \sum_{k=3}^{+\infty}\frac{e}{(k-3)!}(x-1)^k + \sum_{k=2}^{+\infty}\frac{3e}{(k-2)!}(x-1)^k + \sum_{k=1}^{+\infty}\frac{3e}{(k-1)!}(x-1)^k + \sum_{k=0}^{+\infty}\frac{e}{k!}(x-1)^k \end{aligned}$$

for every $x \in \mathbb{R}$. On the other hand, the Taylor series of f at 1 is also given by

$$x^3e^x = \sum_{k=0}^{+\infty}\frac{f^{(k)}(1)}{k!}(x-1)^k.$$

These two series are the same series for every $x \in \mathbb{R}$, i.e.

$$\begin{aligned} &\sum_{k=0}^{+\infty}\frac{f^{(k)}(1)}{k!}(x-1)^k \\ &= \sum_{k=3}^{+\infty}\frac{e}{(k-3)!}(x-1)^k + \sum_{k=2}^{+\infty}\frac{3e}{(k-2)!}(x-1)^k + \sum_{k=1}^{+\infty}\frac{3e}{(k-1)!}(x-1)^k + \sum_{k=0}^{+\infty}\frac{1}{k!}(x-1)^k. \end{aligned}$$

Comparing the coefficients of the two series, we have

$$\begin{aligned} f^{(n)}(1) &= \begin{cases} e & \text{if } n = 0 \\ e\left(3\cdot\dfrac{1!}{0!} + \dfrac{1!}{1!}\right) & \text{if } n = 1 \\ e\left(3\cdot\dfrac{2!}{0!} + 3\cdot\dfrac{2!}{1!} + \dfrac{2!}{2!}\right) & \text{if } n = 2 \\ e\left(\dfrac{n!}{(n-3)!} + 3\cdot\dfrac{n!}{(n-2)!} + 3\cdot\dfrac{n!}{(n-1)!} + 1\right) & \text{if } n \geq 3 \end{cases} \\ &= e(n^3 + 2n + 1). \end{aligned}$$

10. (a) We have

$$\begin{aligned}(1-x-x^2)f(x) &= (1-x-x^2)(a_0+a_1x+a_2x^2+a_3x^3+\cdots)\\ &= a_0+(a_1-a_0)x+(a_2-a_1-a_0)x^2+(a_3-a_2-a_1)x^3+\cdots\\ &= a_0+(a_1-a_0)x+\sum_{k=1}^{+\infty}(a_{k+1}-a_k-a_{k-1})x^{k+1}\end{aligned}$$

for every $x \in (-R,R)$, where $R>0$ is the radius of convergence of the Maclaurin series. On the other hand,

$$(1-x-x^2)f(x) = (1-x-x^2)\frac{x}{1-x-x^2} = x$$

for every x near 0. These two power series are the same series, i.e.

$$a_0+(a_1-a_0)x+\sum_{k=1}^{+\infty}(a_{k+1}-a_k-a_{k-1})x^{k+1} = x.$$

Comparing the coefficients of the two series, we have

$$a_0=0, \qquad a_1-a_0=1 \qquad \text{and} \qquad a_{n+1}-a_n-a_{n-1}=0$$

for every $n\in\mathbb{N}$. That is $a_0=0$, $a_1=1$ and $a_{n+1}=a_n+a_{n-1}$ for every $n\in\mathbb{N}$. Therefore (a_n) is the Fibonacci sequence.

(b) p and q are the roots of the polynomial $1-x-x^2$. If we take $p>q$, then $p=\frac{\sqrt{5}-1}{2}$ and $q=\frac{-\sqrt{5}-1}{2}$.

Now suppose that the partial fraction decomposition of f is given by

$$f(x)=\frac{x}{1-x-x^2}=\frac{A}{x-p}+\frac{B}{x-q}=\frac{A(x-q)+B(x-p)}{(x-p)(x-q)}.$$

Then we obtain the polynomial identity $A(x-q)+B(x-p)=-x$.

- Putting $x=p$ in the identity we obtain $A=-\frac{p}{p-q}=-\frac{p}{\sqrt{5}}$;
- Putting $x=q$ in the identity we obtain $B=\frac{q}{p-q}=\frac{q}{\sqrt{5}}$.

Now if $|x|<p$, i.e. $x\in\left(\frac{1-\sqrt{5}}{2},\frac{\sqrt{5}-1}{2}\right)$, then we have both $\left|\frac{x}{q}\right|<1$ and $\left|\frac{x}{p}\right|<1$, so

$$\begin{aligned}f(x) &= -\frac{p}{\sqrt{5}}\frac{1}{x-p}+\frac{q}{\sqrt{5}}\frac{1}{x-q}=\frac{1}{\sqrt{5}}\frac{1}{1-\frac{x}{p}}-\frac{1}{\sqrt{5}}\frac{1}{1-\frac{x}{q}}\\ &= \frac{1}{\sqrt{5}}\sum_{k=0}^{+\infty}\left(\frac{x}{p}\right)^k-\frac{1}{\sqrt{5}}\sum_{k=0}^{+\infty}\left(\frac{x}{q}\right)^k=\sum_{k=0}^{+\infty}\frac{1}{\sqrt{5}}\left(\frac{1}{p^k}-\frac{1}{q^k}\right)x^k.\end{aligned}$$

This must be the same as the Maclaurin series of f, i.e.

$$\sum_{k=0}^{+\infty}a_kx^k=\sum_{k=0}^{+\infty}\frac{1}{\sqrt{5}}\left(\frac{1}{p^k}-\frac{1}{q^k}\right)x^k.$$

Comparing the coefficients of the two series, we have

$$a_n=\frac{1}{\sqrt{5}}\left(\frac{1}{p^n}-\frac{1}{q^n}\right) \qquad \text{for every non-negative integer } n.$$

11. Recall from Lemma 9.40 that when f is the given trigonometric polynomial, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = a_0, \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a_n \qquad \text{and} \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = b_n$$

for every $n \in \mathbb{N}$. Thus,

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2dx &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\left[\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)\right]dx \\ &= \frac{a_0}{2}\underbrace{\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx\right)}_{=a_0} + \sum_{k=1}^{n}\left[a_k\underbrace{\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx\right)}_{=a_k} + b_k\underbrace{\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx\right)}_{=b_k}\right] \\ &= \frac{{a_0}^2}{2}+\sum_{k=1}^{n}\left({a_k}^2+{b_k}^2\right).\end{aligned}$$

Note that we can "swap the summation sign and integration sign" because it is just a finite sum.

12. (a) The Fourier coefficients of f are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|dx = \frac{2}{\pi}\int_{0}^{\pi} x\,dx = \pi, \qquad \text{and}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx = \frac{2[(-1)^n-1]}{n^2\pi} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{4}{n^2\pi} & \text{if } n \text{ is odd}\end{cases},$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin nx\,dx = 0$$

for every positive integer n; so the Fourier series of f is

$$f(x) \sim \frac{\pi}{2} - \sum_{k=1}^{+\infty}\frac{4}{(2k-1)^2\pi}\cos(2k-1)x = \frac{\pi}{2}-\frac{4}{\pi}\cos x - \frac{4}{3^2\pi}\cos 3x - \frac{4}{5^2\pi}\cos 5x - \cdots.$$

(b) (i) For every $x \in [-\pi, \pi]$, f is continuous at x and has bounded one-sided derivative at x (which equals either 1 or -1); so according to Theorem 9.47 we have the equality

$$|x| = \frac{\pi}{2} - \sum_{k=1}^{+\infty}\frac{4}{(2k-1)^2\pi}\cos(2k-1)x = \frac{\pi}{2}-\frac{4}{\pi}\cos x - \frac{4}{3^2\pi}\cos 3x - \frac{4}{5^2\pi}\cos 5x - \cdots.$$

⊙ When $x = 0$, we obtain

$$0 = \frac{\pi}{2} - \sum_{k=1}^{+\infty}\frac{4}{(2k-1)^2\pi} = \frac{\pi}{2}-\frac{4}{\pi}-\frac{4}{3^2\pi}-\frac{4}{5^2\pi}-\cdots.$$

⊙ Alternatively, when $x = \pi$ or $x = -\pi$, we obtain

$$\pi = \frac{\pi}{2} - \sum_{k=1}^{+\infty}\frac{4}{(2k-1)^2\pi}(-1) = \frac{\pi}{2}+\frac{4}{\pi}+\frac{4}{3^2\pi}+\frac{4}{5^2\pi}+\cdots.$$

With either one of the above, rearranging the terms we get

$$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots.$$

(ii) The series $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots$ converges absolutely, so rearranging its terms does not affect its sum. Now according to the result from (b) (i), we have

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)+\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\cdots\right)$$

$$=\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)+\frac{1}{4}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right).$$

Rearranging, we get

$$\frac{3}{4}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8},$$

so

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{4}{3}\cdot\frac{\pi^2}{8}=\frac{\pi^2}{6}.$$

13. (a) The Fourier coefficients of f are given by

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos ax\,dx=\frac{2\sin(a\pi)}{a\pi}, \qquad \text{and}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos ax\cos nx\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}[\cos(a+n)x+\cos(a-n)x]dx$$

$$=\frac{1}{\pi}\left(\frac{\sin(a+n)\pi}{a+n}+\frac{\sin(a-n)\pi}{a-n}\right)=\frac{(-1)^n 2a\sin(a\pi)}{\pi(a^2-n^2)},$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos ax\sin nx\,dx=0$$

for every positive integer n; so the Fourier series of f is

$$f(x)\sim\frac{\sin(a\pi)}{a\pi}-\sum_{k=1}^{+\infty}\frac{(-1)^k 2a\sin(a\pi)}{\pi(k^2-a^2)}\cos kx.$$

(b) For every $x\in[-\pi,\pi]$, f is continuous at x and has bounded one-sided derivatives at x; so according to Theorem 9.47 we have the equality

$$\cos ax=\frac{\sin(a\pi)}{a\pi}-\sum_{k=1}^{+\infty}\frac{(-1)^k 2a\sin(a\pi)}{\pi(k^2-a^2)}\cos kx\,.$$

When $x=0$, we obtain

$$1=\frac{\sin(a\pi)}{a\pi}-\sum_{k=1}^{+\infty}\frac{(-1)^k 2a\sin(a\pi)}{\pi(k^2-a^2)}.$$

Thus

$$\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k^2-a^2}=\frac{\pi}{2a\sin(a\pi)}\left(-\frac{\sin(a\pi)}{a\pi}+1\right)=-\frac{1}{2a^2}+\frac{\pi}{2a\sin(a\pi)}.$$

When $x = \pi$ (or when $x = -\pi$), we obtain

$$\cos(a\pi) = \frac{\sin(a\pi)}{a\pi} - \sum_{k=1}^{+\infty} \frac{(-1)^k 2a\sin(a\pi)}{\pi(k^2 - a^2)}(-1)^k = \frac{\sin(a\pi)}{a\pi} - \sum_{k=1}^{+\infty} \frac{2a\sin(a\pi)}{\pi(k^2 - a^2)}.$$

Thus

$$\sum_{k=1}^{+\infty} \frac{1}{k^2 - a^2} = \frac{\pi}{2a\sin(a\pi)}\left(\frac{\sin(a\pi)}{a\pi} - \cos(a\pi)\right) = \frac{1}{2a^2} - \frac{\pi}{2a\tan(a\pi)}.$$

14. (a) The Fourier coefficients of f are given by

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^x dx = \frac{1}{\pi}(e^{2\pi} - 1), \qquad \text{and}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx\,dx,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx\,dx$$

for every positive integer n. Since

$$\int_0^{2\pi} e^x \cos nx\,dx = \left[\frac{e^x \sin nx}{n} + \frac{e^x \cos nx}{n^2}\right]_0^{2\pi} - \frac{1}{n^2}\int_0^{2\pi} e^x \cos nx\,dx$$

$$= \frac{e^{2\pi} - 1}{n^2} - \frac{1}{n^2}\int_0^{2\pi} e^x \cos nx\,dx,$$

we have

$$\int_0^{2\pi} e^x \cos nx\,dx = \frac{1}{1 + \frac{1}{n^2}}\frac{e^{2\pi} - 1}{n^2} = \frac{e^{2\pi} - 1}{n^2 + 1}$$

and

$$\int_0^{2\pi} e^x \sin nx\,dx = \left[-\frac{e^x \cos nx}{n}\right]_0^{2\pi} + \frac{1}{n}\int_0^{2\pi} e^x \cos nx\,dx$$

$$= \frac{1 - e^{2\pi}}{n} + \frac{1}{n}\frac{e^{2\pi} - 1}{n^2 + 1} = \frac{n(1 - e^{2\pi})}{n^2 + 1};$$

so

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx\,dx = \frac{1}{\pi}\frac{e^{2\pi} - 1}{n^2 + 1} \qquad \text{and} \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx\,dx = \frac{1}{\pi}\frac{n(1 - e^{2\pi})}{n^2 + 1}$$

for every positive integer n. Therefore the Fourier series of f is

$$f(x) \sim \frac{1}{2\pi}(e^{2\pi} - 1) + \sum_{k=1}^{+\infty}\left(\frac{1}{\pi}\frac{e^{2\pi} - 1}{k^2 + 1}\cos kx + \frac{1}{\pi}\frac{k(1 - e^{2\pi})}{k^2 + 1}\sin kx\right).$$

(b) (i) At $x = 0$, we have $f(0^-) = e^{2\pi}$, $f(0^+) = e^0 = 1$, $\lim_{x\to 0^-} f'(x) = e^{2\pi}$ and $\lim_{x\to 0^+} f'(x) = e^0 = 1$ all exist as real numbers; so according to Theorem 9.47 we have

$$\frac{1}{2\pi}(e^{2\pi}-1) + \sum_{k=1}^{+\infty} \frac{1}{\pi}\frac{e^{2\pi}-1}{k^2+1} = \frac{f(0^-)+f(0^+)}{2} = \frac{e^{2\pi}+1}{2}.$$

Rearranging the terms, we get

$$\sum_{k=1}^{+\infty} \frac{1}{k^2+1} = \frac{\pi}{2}\frac{e^{2\pi}+1}{e^{2\pi}-1} - \frac{1}{2}.$$

Adding 1 on both sides, we get

$$\sum_{k=0}^{+\infty} \frac{1}{k^2+1} = \frac{\pi}{2}\frac{e^{2\pi}+1}{e^{2\pi}-1} + \frac{1}{2}.$$

(ii) For every $x \in (0, 2\pi)$, f is differentiable at x; so according to Theorem 9.47 we have the equality

$$e^x = \frac{1}{2\pi}(e^{2\pi}-1) + \sum_{k=1}^{+\infty}\left(\frac{1}{\pi}\frac{e^{2\pi}-1}{k^2+1}\cos kx + \frac{1}{\pi}\frac{k(1-e^{2\pi})}{k^2+1}\sin kx\right).$$

At $x = \pi$, we obtain

$$e^\pi = \frac{1}{2\pi}(e^{2\pi}-1) + \sum_{k=1}^{+\infty} \frac{1}{\pi}\frac{e^{2\pi}-1}{k^2+1}(-1)^k.$$

Rearranging the terms, we get

$$\sum_{k=1}^{+\infty} \frac{(-1)^k}{k^2+1} = \frac{\pi e^\pi}{e^{2\pi}-1} - \frac{1}{2}.$$

Adding 1 on both sides, we get

$$\sum_{k=0}^{+\infty} \frac{(-1)^k}{k^2+1} = \frac{\pi e^\pi}{e^{2\pi}-1} + \frac{1}{2} = \frac{\pi}{e^\pi - e^{-\pi}} + \frac{1}{2}.$$