# Problem Set 7

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 7 covers materials from §8.2 – §8.4.

1. (a) Show that the sequence $(\cos n)$ diverges.

   *Hint*: Assume that $(\cos n)$ converges. Deduce that $(\sin n)$ also converges using $\cos(n+1) = \cos n \cos 1 - \sin n \sin 1$. Find a contradiction by considering the sum of squares of the limits of $(\cos n)$ and of $(\sin n)$.

   (b) Hence deduce that the series $\sum_{k=0}^{+\infty} \cos n$ diverges.

2. For each of the following series, determine whether it converges or diverges.

   (a) $\displaystyle\sum_{k=1}^{+\infty} e^{\frac{1}{k^2}}$

   (b) $\displaystyle\sum_{k=2}^{+\infty} \frac{1}{(\ln k)^k}$

   (c) $\displaystyle\sum_{k=1}^{+\infty} \cos\left(\sin\frac{1}{k}\right)$

   (d) $\displaystyle\sum_{k=1}^{+\infty} \left(1-\cos\frac{1}{k}\right)$

   (e) $\displaystyle\sum_{k=1}^{+\infty} ke^{-k^2}$

   (f) $\displaystyle\sum_{k=2}^{+\infty} \frac{\ln k}{k(k-1)}$

   (g) $\displaystyle\sum_{k=1}^{+\infty} \frac{1}{k^{1+\frac{1}{k}}}$

   (h) $\displaystyle\sum_{k=1}^{+\infty} \left(\frac{1}{2}+\frac{1}{k}\right)^k$

   (i) $\displaystyle\sum_{k=1}^{+\infty} \frac{(2k)!}{(k+1)!\,(k-1)!}$

   (j) $\displaystyle\sum_{k=0}^{+\infty} \frac{3^k+4^k}{2^k+5^k}$

3. Let $(a_n)$ be a sequence of positive real numbers.

   (a) Show that if $\sum_{k=1}^{+\infty} a_k$ converges, then $\sum_{k=1}^{+\infty} \frac{1}{a_k}$ diverges.

   (b) Show that if $\lim\limits_{n\to+\infty} na_n = L > 0$, then $\sum_{k=1}^{+\infty} a_k$ diverges.

   (c) Show that if $\sum_{k=1}^{+\infty} a_k$ converges, then $\sum_{k=1}^{+\infty} {a_k}^2$ converges. Is the converse true?

   (d) Show that if $\sum_{k=1}^{+\infty} {a_k}^2$ converges, then $\sum_{k=1}^{+\infty} \frac{a_k}{k}$ converges.

   *Hint*: AM-GM inequality.

   (e) Show that if $\sum_{k=1}^{+\infty} k^2{a_k}^2$ converges, then $\sum_{k=1}^{+\infty} a_k$ converges.

   *Hint*: Cauchy-Schwarz inequality (Theorem 7.33).

4. For each $n \in \mathbb{N}$, let $p_n$ be the $n^{\text{th}}$ prime number, i.e. $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7, \dots$.

   (a) Let $N \in \mathbb{N}$ be fixed and let $M \coloneqq p_1 p_2 \cdots p_N$. Deduce that

$$\sum_{j=1}^{+\infty} \left( \sum_{k=N+1}^{+\infty} \frac{1}{p_k} \right)^j \geq \sum_{m=1}^{n} \frac{1}{1+mM} \qquad \text{for all } n \in \mathbb{N}$$

   by considering the prime factorization of each number $1 + mM$.

   (b) Using the result from (a), show that the series

$$\sum_{k=1}^{+\infty} \frac{1}{p_k}$$

   diverges.

   *Hint*: Suppose that the series converges. Then by definition, there exists $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{+\infty} \frac{1}{p_k} < \frac{1}{2}$.

5. Consider the series $\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k}$. For each $n \in \mathbb{N}$, we let $h_n \coloneqq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ denote the $n^{\text{th}}$ partial sum of the harmonic series. Recall from Example 8.30 that

$$\lim_{n \to +\infty} (h_n - \ln n) = \gamma$$

   where $\gamma$ is the Euler-Mascheroni constant.

   (a) Show that $\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = h_{2n} - h_n$ for every $n \in \mathbb{N}$.

   (b) Using the result from (a), show that

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

6. Show that the series

$$\sum_{k=1}^{+\infty} \left( \frac{1}{k} - \ln\left(1 + \frac{1}{k}\right) \right)$$

   converges and compute its limit.

7. For each of the following series, find all the values of $p \in \mathbb{R}$ such that the series converges.

   (a) $\displaystyle\sum_{k=1}^{+\infty} k^2 \sin^p \frac{1}{k}$

   (b) $\displaystyle\sum_{k=3}^{+\infty} \frac{1}{k(\ln k)(\ln \ln k)^p}$

   (c) $\displaystyle\sum_{k=2}^{+\infty} \frac{1}{(\ln \ln k)^{p \ln k}}$

   (d) $\displaystyle\sum_{k=2}^{+\infty} \frac{k^p}{(\ln k)^k}$

8. Let $(a_n)$ be a sequence of real numbers, and define

$$a_n^+ \coloneqq \max\{a_n, 0\} \qquad \text{and} \qquad a_n^- \coloneqq \max\{-a_n, 0\}$$

for every $n$. Show that

(a) If $\sum_{k=1}^{+\infty} a_k$ **converges absolutely**, then both $\sum_{k=1}^{+\infty} a_k^+$ and $\sum_{k=1}^{+\infty} a_k^-$ converge.

(b) If $\sum_{k=1}^{+\infty} a_k$ **converges conditionally**, then both $\sum_{k=1}^{+\infty} a_k^+$ and $\sum_{k=1}^{+\infty} a_k^-$ diverge.

9. For each of the following series, determine whether it diverges, converges absolutely or converges conditionally.

(a) $\displaystyle\sum_{k=1}^{+\infty} \frac{\cos k}{k^3}$

(b) $\displaystyle\sum_{k=0}^{+\infty} (-1)^{k+1}\left(\sqrt{k+1} - \sqrt{k}\right)$

(c) $\displaystyle\sum_{k=1}^{+\infty} \cos k\pi \sin \frac{1}{k\pi}$

(d) $\displaystyle\sum_{k=2}^{+\infty} \frac{(-1)^k}{\sqrt{k} + (-1)^k}$

10. (a) Let $(a_n)$ and $(b_n)$ be sequences of real numbers, and let

$$B_n \coloneqq \sum_{k=1}^{n} b_k = b_1 + b_2 + \cdots + b_n.$$

(i) Using mathematical induction, prove the ***summation by parts*** formula

$$\sum_{k=1}^{n} a_k b_k = a_{n+1} B_n - \sum_{k=1}^{n} B_k (a_{k+1} - a_k)$$

for every positive integer $n$.

(ii) Suppose that the sequence $(a_n)$ is decreasing with $\lim\limits_{n\to+\infty} a_n = 0$, and that $(B_n)$ is a bounded sequence. Using (a)(i), show that the series $\sum_{k=1}^{+\infty} a_k b_k$ converges.

(b) Let $t$ be a fixed real number. Using (a)(ii) and the result from Q10(a) of Problem Set 1, deduce that the series $\sum_{k=1}^{+\infty} \frac{\sin kt}{k}$ converges.

11. Find the radius and interval of convergence for each of the following power series.

(a) $\displaystyle\sum_{k=1}^{+\infty} k^{\sqrt{k}} x^k$

(b) $\displaystyle\sum_{k=1}^{+\infty} \frac{x^k}{2^k k^2}$

(c) $\displaystyle\sum_{k=0}^{+\infty} \frac{(1-2x)^k}{k}$

(d) $\displaystyle\sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{\sqrt{k!}} x^k$

12. Let $a > b > 0$. What is the radius of convergence of the power series

$$\sum_{k=0}^{+\infty} (a^k + b^k) x^k\,?$$

13. (a) Let $(a_n)$ be a sequence of real numbers and let $m$ be a positive integer. If the power series $\sum_{k=0}^{+\infty} a_k x^k$ has radius of convergence $R$, show that the power series $\sum_{k=0}^{+\infty} a_k x^{mk}$ has radius of convergence $R^{1/m}$.

    (b) Using the result from (a), find the radius of convergence and the interval of convergence of each of the following power series.

    (i) The ***Bessel function***

$$J_0(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k}.$$

    (ii) The ***Airy function***

$$A(x) = 1 + \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \cdots.$$

    (c) (i) Show that the Bessel function in (b)(i) satisfies $xJ_0''(x) + J_0'(x) + xJ_0(x) = 0$ for every $x$ in its interval of convergence.

    (ii) Show that the Airy function in (b)(ii) satisfies $A''(x) - xA(x) = 0$ for every $x$ in its interval of convergence.

14. For each of the following power series, evaluate its sum whenever it converges. What happens at the end-points of its interval of convergence?

    (a) $\displaystyle\sum_{k=1}^{+\infty} k^2 x^k$

    (b) $\displaystyle\sum_{k=2}^{+\infty} \frac{1}{k(k-1)} (x-1)^k$

    (c) $\displaystyle\sum_{k=1}^{+\infty} \frac{1}{k(k+1)(k+2)} x^k$

    (d) $\displaystyle\sum_{k=0}^{+\infty} \frac{1}{2k+1} x^{2k+1}$

    (e) $\displaystyle\sum_{k=1}^{+\infty} \frac{k}{k+1} x^k$

    *Hint*: In each part, apply termwise differentiation or integration on some power series whose sum is well-known.

15. Let $f$ be the power series

$$f(x) = \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{6}x^6 - \frac{1}{8}x^8 + \frac{1}{9}x^9 - \frac{1}{11}x^{11} + \cdots.$$

    (a) Evaluate the sum of the power series for every $x \in (-1, 1)$.

    *Hint*: Apply termwise integration on some power series to get $f(x)$.

    (b) Using (a) and Abel's limit theorem, evaluate the sum of the series

$$\sum_{k=1}^{+\infty} \frac{1}{3k^2 + 2k}.$$