

HKUST MATH 1014 L1 assignment 3 submission

1 Q1: 6

1.a

Let m and n be non-negative integers. Evaluate the following integrals, distinguishing all possible cases for m and n .

1.a.i

$$\begin{aligned} & \int_{-\pi}^{\pi} \cos mx \cos nx \, dx \\ &= \int_{-\pi}^{\pi} \cos mx \cos nx \, dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) + \cos((m-n)x)) \, dx \\ &= \begin{cases} \frac{1}{2} \int_{-\pi}^{\pi} (1+1) \, dx & m+n=0, m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (1+\cos((m-n)x)) \, dx & m+n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x)+1) \, dx & m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) + \cos((m-n)x)) \, dx & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{2} [2x]_{x=-\pi}^{x=\pi} & m+n=0, m-n=0 \\ \frac{1}{2} \left[x + \frac{1}{m-n} \sin((m-n)x) \right]_{x=-\pi}^{x=\pi} & m+n=0 \\ \frac{1}{2} \left[\frac{1}{m+n} \sin((m+n)x) + x \right]_{x=-\pi}^{x=\pi} & m-n=0 \\ \frac{1}{2} \left[\frac{1}{m+n} \sin((m+n)x) + \frac{1}{m-n} \sin((m-n)x) \right]_{x=-\pi}^{x=\pi} & \text{otherwise} \end{cases} \\ &= \begin{cases} 2\pi & m+n=0, m-n=0 \\ \pi & m+n=0 \\ \pi & m-n=0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

1.a.ii

$$\begin{aligned} & \int_{-\pi}^{\pi} \sin mx \sin nx \, dx \\ &= \int_{-\pi}^{\pi} \sin mx \sin nx \, dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) - \cos((m-n)x)) \, dx \\ &= \begin{cases} \frac{1}{2} \int_{-\pi}^{\pi} (1-1) \, dx & m+n=0, m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} > & \\ \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos((m-n)x)) \, dx & m+n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) - 1) \, dx & m-n=0 \\ \frac{1}{2} \int_{-\pi}^{\pi} (\cos((m+n)x) - \cos((m-n)x)) \, dx & \text{otherwise} \end{cases} \\ &= \begin{cases} 0 & m+n=0, m-n=0 \\ \frac{1}{2} \left[x - \frac{1}{m-n} \sin((m-n)x) \right]_{x=-\pi}^{x=\pi} & m+n=0 \\ \frac{1}{2} \left[\frac{1}{m+n} \sin((m+n)x) - x \right]_{x=-\pi}^{x=\pi} & m-n=0 \\ \frac{1}{2} \left[\frac{1}{m+n} \sin((m+n)x) - \frac{1}{m-n} \sin((m-n)x) \right]_{x=-\pi}^{x=\pi} & \text{otherwise} \end{cases} \\ &= \begin{cases} 0 & m+n=0, m-n=0 \\ \pi & m+n=0 \\ -\pi & m-n=0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

1.a.iii

$$\begin{aligned} & \int_{-\pi}^{\pi} \cos mx \sin nx \, dx \\ &= \int_{-\pi}^{\pi} \cos mx \sin nx \, dx \\ &= \int_0^{\pi} \cos mx \sin nx \, dx + \int_{-\pi}^0 \cos mx \sin nx \, dx \\ &= \int_0^{\pi} \cos mx \sin nx \, dx - \int_{-\pi}^0 \cos(-mu) \sin(-nu) \, du \quad (u := -x) \\ &= \int_0^{\pi} \cos mx \sin nx \, dx - \int_0^{\pi} \cos mu \sin nu \, du \\ &= 0 \end{aligned}$$

Evaluate the following antiderivatives.

2.a Q2: 7

$$\int x^2 \arctan x \, dx$$

$$\begin{aligned} & \int x^2 \arctan x \, dx \\ &= \frac{1}{3} x^3 \arctan x - \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx \\ &= \frac{1}{3} x^3 \arctan x - \frac{1}{6} \int \frac{x^2}{1+x^2} \, dx^2 \\ &= \frac{1}{3} x^3 \arctan x - \frac{1}{6} \left(\int dx^2 - \int \frac{1}{1+x^2} \, dx^2 \right) \\ &= \frac{1}{3} x^3 \arctan x - \frac{1}{6} (x^2 - \ln(1+x^2)) + C \quad (1+x^2 > 0) \\ &= \frac{1}{3} x^3 \arctan x - \frac{1}{6} x^2 + \frac{1}{6} \ln(1+x^2) + C \end{aligned}$$

2.d

$$\int (2x^2 + 1)e^{x^2} \, dx$$

$$\begin{aligned} \frac{d}{dx} e^{x^2} &= 2xe^{x^2} \\ \frac{d}{dx} (2xe^{x^2}) &= 2e^{x^2} + 4x^2 e^{x^2} \\ &= 2(2x^2 + 1)e^{x^2} \\ \int (2x^2 + 1)e^{x^2} \, dx &= xe^{x^2} + C \end{aligned}$$

7

Let $f: [0, +\infty) \rightarrow \mathbb{R}$ be the function defined by $f(x) = xe^x$.

7.a

Show that f is strictly increasing.

$$\begin{aligned} f(x) &= xe^x \\ f'(x) &= e^x + xe^x \\ &= (1+x)e^x \end{aligned}$$

$$\begin{aligned} \forall x \in [0, +\infty) \\ 1+x &\geq 1 \\ e^x &\geq 1 \\ (1+x)e^x &\geq 1 \\ f'(x) &\geq 1 \\ &> 0 \end{aligned} \quad \implies f(x) \text{ is strictly increasing}$$

7.b

Now f is one-to-one according to (a), so we let g be the **inverse** of f , i.e. $g = f^{-1}$.

7.b.i

Write down the domain of g . Show that

$$g'(x) = \frac{1}{x + e^{g(x)}}$$

for every x in the interior of the domain of g .

The domain of g is $[0, +\infty)$.

$$\begin{aligned} f'(x) &= (1+x)e^x = e^x + f(x) \in C^0([0, +\infty), [0, +\infty)) \\ \implies f(x) &\in C^1((0, +\infty), (0, +\infty)) \\ \implies g'(x) &= \frac{1}{f'(g(x))} \quad \forall x \in (0, +\infty) \quad (\text{inverse function theorem}) \end{aligned}$$

$\forall x \in (0, +\infty)$

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ &= \frac{1}{f(g(x)) + e^{g(x)}} \\ &= \frac{1}{x + e^{g(x)}} \end{aligned}$$

7.b.ii

Using the result from 7.b.i or otherwise, evaluate the antiderivative $\int g(x) dx$, expressing your answer in terms of g and other elementary functions only.

$$\begin{aligned} \forall x \in (0, +\infty) \\ \int g(x) dx &= xg(x) - \int x dg(x) \\ &= xg(x) - \int \frac{x}{x + e^{g(x)}} dx \\ &= xg(x) - \left(\int dx - \int \frac{e^{g(x)}}{x + e^{g(x)}} dx \right) \\ &= xg(x) - \left(x - \int e^{g(x)} g'(x) dx \right) \\ &= xg(x) - x + \int e^{g(x)} dg(x) \\ &= xg(x) - x + e^{g(x)} + C \end{aligned}$$

7.b.iii

Hence, or otherwise, evaluate the integral $\int_0^e g(x) dx$.

$$\begin{aligned} f(1) &= 1e^1 = e \\ g(e) &= 1 \\ f(0) &= 0e^0 = 0 \\ g(0) &= 0 \\ \int_0^e g(x) dx &= \left[xg(x) - x + e^{g(x)} \right]_0^e \\ &= eg(e) - e + e^{g(e)} - e^{g(0)} \\ &= e - e + e^1 - e^0 \\ &= e - 1 \end{aligned}$$

10

For each non-negative integer n , let

$$I_n = \int_0^1 t^n e^t dt$$

Q10 a) 2/2

t

10.a

Show that $\frac{1}{n+1} \leq I_n \leq \frac{e}{n+1}$ for every non-negative integer n .

$$\begin{aligned} \int_0^1 t^n e^0 dt &\leq \int_0^1 t^n e^t dt \leq \int_0^1 t^n e^1 dt \quad ((\forall t \in [0, 1]) (e^0 \leq e^t \leq e^1)) \\ \int_0^1 t^n dt &\leq I_n \leq e \int_0^1 t^n dt \\ \left[\frac{t^{n+1}}{n+1} \right]_{t=0}^{t=1} &\leq I_n \leq e \left[\frac{t^{n+1}}{n+1} \right]_{t=0}^{t=1} \\ \frac{1}{n+1} &\leq I_n \leq \frac{e}{n+1} \end{aligned}$$

10.b

Express I_n in terms of I_{n-1} for each $n \geq 1$. Hence show that

$$I_n = (-1)^{n+1}n! + e \sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!}$$

$$\begin{aligned} I_n &= \int_0^1 t^n e^t dt \\ &= [t^n e^t]_{t=0}^{t=1} - n \int_0^1 t^{n-1} e^t dt \\ &= e - nI_{n-1} \end{aligned}$$

$$I_0 = \int_0^1 t^0 e^t dt = [e^t]_0^1 = e - 1 = (-1)^{0+1}0! + e \sum_{k=0}^0 (-1)^k \frac{0!}{(0-k)!}$$

$$\text{Assume } I_m = (-1)^{m+1}m! + e \sum_{k=0}^m (-1)^k \frac{m!}{(m-k)!}.$$

$$\begin{aligned} I_{m+1} &= e - (m+1)I_m & (m+1 \geq 1) \\ &= e - (m+1) \left((-1)^{m+1}m! + e \sum_{k=0}^m (-1)^k \frac{m!}{(m-k)!} \right) \\ &= e \left(1 - (m+1) \sum_{k=0}^m (-1)^k \frac{m!}{(m-k)!} \right) + (-1)^{m+2}(m+1)! \\ &= e \left(1 + \sum_{k=0}^m (-1)^{k+1} \frac{(m+1)!}{(m-k)!} \right) + (-1)^{m+2}(m+1)! \\ &= e \left((-1)^0 \frac{(m+1)!}{(m+1-0)!} + \sum_{k=1}^{m+1} (-1)^k \frac{(m+1)!}{(m+1-k)!} \right) + (-1)^{m+2}(m+1)! \\ &= e \sum_{k=0}^{m+1} (-1)^k \frac{(m+1)!}{(m+1-k)!} + (-1)^{m+2}(m+1)! \\ &= (-1)^{(m+1)+1}(m+1)! + e \sum_{k=0}^{m+1} (-1)^k \frac{(m+1)!}{((m+1)-k)!} \end{aligned}$$

By induction, $\forall n \in \mathbb{Z}_{\geq 0}$

$$I_n = (-1)^{n+1}n! + e \sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!}$$

10.c

Using [10.a](#) and [10.b](#), deduce that e is an irrational number.

Hint: There is no integer in the open interval $(0,1)$.

Assume e is rational, then $e = \frac{a}{b}$ where $a, b \in \mathbb{Z}, b \neq 0$.

$$\forall n \in \mathbb{Z}_{\geq 0}$$

$$I_n = (-1)^{n+1} n! + e \sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!}$$

$$e = \frac{I_n - (-1)^{n+1} n!}{\sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!}}$$

$$\frac{a}{b} = \frac{I_n - (-1)^{n+1} n!}{\sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!}}$$

$$a \sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!} = b (I_n - (-1)^{n+1} n!)$$

$$= b I_n - b (-1)^{n+1} n!$$

$$c = b I_n - d$$

$(c, d \in \mathbb{Z})$

$$b I_n = c + d$$

$$b I_n \in \mathbb{Z}$$

$(\mathbb{Z} \text{ is closed under addition})$

$$\frac{1}{n+1} \leq I_n \leq \frac{e}{n+1}$$

$$\frac{1}{n+1} \leq I_n < \frac{3}{n+1}$$

$(e < 3)$

$$\frac{|b|}{n+1} \leq |b| I_n < \frac{3|b|}{n+1}$$

$$\frac{|b|}{3|b|+1} \leq |b| I_{3|b|} < \frac{3|b|}{3|b|+1}$$

(set $n = 3|b|$)

$$|b| I_{3|b|} \notin \mathbb{Z}$$

$(|b| < |3b+1|, |3b| < |3b+1|)$

$$b I_{3|b|} \notin \mathbb{Z}$$

$$(\forall n \in \mathbb{Z}_{\geq 0})(b I_n \notin \mathbb{Z}) \Rightarrow \Leftarrow b I_{3|b|} \notin \mathbb{Z}$$

$\Rightarrow e$ is irrational.



11

11.a

For every pair of non-negative integers m and n , let

$$B(m, n) = \int_0^1 x^m (1-x)^n dx$$

.

Show that $B(m, n) = \frac{n}{m+1} B(m+1, n-1)$ for every pair of integers $m \geq 0$ and $n \geq 1$.

Hence or otherwise, deduce that $B(m, n) = \frac{m!n!}{(m+n+1)!}$.

$$\begin{aligned}
& \forall (m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1} \\
& B(m, n) \\
&= \int_0^1 x^m (1-x)^n \, dx \\
&= \left[\frac{1}{m+1} x^{m+1} (1-x)^n \right]_{x=0}^{x=1} + \frac{n}{m+1} \int_0^1 x^{m+1} (1-x)^{n-1} \, dx \\
&= 0 + \frac{n}{m+1} B(m+1, n-1) \\
&= \frac{n}{m+1} B(m+1, n-1) \\
& \\
& B(m, n) \\
&= \frac{n}{m+1} B(m+1, n-1) \\
&= \frac{n}{m+1} \frac{n-1}{m+2} B(m+2, n-2) \\
&\vdots \\
&= \frac{n!}{\frac{(m+n)!}{m!}} B(m+n, 0) \\
&= \frac{m!n!}{(m+n)!} B(m+n, 0) \\
&= \frac{m!n!}{(m+n)!} \int_0^1 x^{m+n} (1-x)^0 \, dx \\
&= \frac{m!n!}{(m+n)!} \left[\frac{1}{m+n+1} x^{m+n+1} \right]_{x=0}^{x=1} \\
&= \frac{m!n!}{(m+n)!} \frac{1}{m+n+1} \\
&= \frac{m!n!}{(m+n+1)!}
\end{aligned}$$

11.b

Show that

$$\int_0^1 \frac{x^4(x-1)^4}{x^2+1} \, dx = \frac{22}{7} - \pi$$

.

Using this together with the result from [11.a](#), deduce that

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}$$

.

$$\begin{aligned}
& \int_0^1 \frac{x^4(x-1)^4}{x^2+1} dx \\
&= \int_0^1 \frac{x^4(x^4-4x^3+6x^2-4x+1)}{x^2+1} dx \\
&= \int_0^1 \frac{x^8-4x^7+6x^6-4x^5+x^4}{x^2+1} dx \\
&= \int_0^1 \left(x^6-4x^5+5x^4-4x^2+4-\frac{4}{x^2+1} \right) dx \\
&= \left[\frac{1}{7}x^7 - \frac{2}{3}x^6 + x^5 - \frac{4}{3}x^3 + 4x - 4\arctan x \right]_0^1 \\
&= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - \pi \\
&= \frac{22}{7} - \pi
\end{aligned}$$

$$\begin{aligned}
& \frac{x^4(x-1)^4}{1^2+1} < \frac{x^4(x-1)^4}{x^2+1} < \frac{x^4(x-1)^4}{0^2+1} \quad (x \in (0,1)) \\
& \int_0^1 \frac{x^4(x-1)^4}{1^2+1} dx < \int_0^1 \frac{x^4(x-1)^4}{x^2+1} dx < \int_0^1 \frac{x^4(x-1)^4}{0^2+1} dx \\
& \frac{1}{2} \int_0^1 x^4(x-1)^4 dx < \frac{22}{7} - \pi < \int_0^1 x^4(x-1)^4 dx \\
& \frac{1}{2} B(4,4) < \frac{22}{7} - \pi < B(4,4) \\
& \frac{1}{2} \frac{4!4!}{9!} < \frac{22}{7} - \pi < \frac{4!4!}{9!} \\
& \frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630} \\
& -\frac{1}{1260} > \pi - \frac{22}{7} > -\frac{1}{630} \\
& \frac{22}{7} - \frac{1}{1260} > \pi > \frac{22}{7} - \frac{1}{630} \\
& \frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}
\end{aligned}$$

13

Evaluate the following antiderivatives, using trigonometric substitutions when appropriate.

13.a

$$\int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^4}} dx$$

Q13 a) 4/4

t

$$\begin{aligned}
& \forall x \in (-1,1) \\
& \int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^4}} dx \\
&= \int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1+x^2}} dx \\
&= \int \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1+x^2}} \right) dx \\
&= \arcsin x + \int \frac{1}{\sqrt{1+x^2}} dx \\
&= \arcsin x + \int \frac{\sec^2 \theta}{\sqrt{1+\tan^2 \theta}} d\theta \quad \left(x := \tan \theta, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right) \\
&= \arcsin x + \int \sec \theta d\theta \\
&= \arcsin x + \int \frac{(\sec \theta)(\sec \theta + \tan \theta)}{\sec \theta + \tan \theta} d\theta \\
&= \arcsin x + \int \frac{\sec^2 \theta + \tan \theta \sec \theta}{\sec \theta + \tan \theta} d\theta \\
&= \arcsin x + \ln |\tan \theta + \sec \theta| + C \\
&= \arcsin x + \ln |x + \sqrt{1+x^2}| + C \quad \left(\sec \theta > 0 \quad \forall \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right) \\
&= \arcsin x + \ln (x + \sqrt{1+x^2}) + C \quad \left(x + \sqrt{1+x^2} > 0 \quad \forall x \in (-1,1) \right)
\end{aligned}$$

13.b

$$\int \frac{x+1}{(x^2+x+1)\sqrt{x^2+x+1}} dx$$

$$\begin{aligned}
& \int \frac{x+1}{(x^2+x+1)\sqrt{x^2+x+1}} dx \\
&= \frac{1}{2} \int \frac{2x+1}{(x^2+x+1)\sqrt{x^2+x+1}} dx + \frac{1}{2} \int \frac{1}{(x^2+x+1)\sqrt{x^2+x+1}} dx \\
&= \frac{1}{2} \int \frac{1}{(x^2+x+1)^{1.5}} d(x^2+x+1) dx + \frac{1}{2} \int \frac{1}{((x+0.5)^2+0.75)\sqrt{(x+0.5)^2+0.75}} dx \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{1}{2\sqrt{0.75}} \int \frac{\sec^2 \theta}{(0.75 \tan^2 \theta + 0.75) \sqrt{0.75 \tan^2 \theta + 0.75}} d\theta \quad \left(x := \sqrt{0.75} \tan \theta - 0.5, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right) \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{1}{2\sqrt{0.75}} \int \frac{1}{0.75 \sqrt{0.75} \sec \theta} d\theta \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3} \int \cos \theta d\theta \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3} \sin \theta + C \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3} \operatorname{sgn}(x+0.5) \frac{1}{\sqrt{\frac{1}{\frac{4}{3}(x+0.5)^2} + 1}} + C \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3} \operatorname{sgn}(x+0.5) \frac{1}{\sqrt{\frac{3+4(x+0.5)^2}{4(x+0.5)^2}}} + C \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2}{3} \operatorname{sgn}(x+0.5) \sqrt{\frac{4(x+0.5)^2}{4x^2+4x+4}} + C \\
&= -\frac{1}{\sqrt{x^2+x+1}} + \frac{2(x+0.5)}{3\sqrt{x^2+x+1}} + C \\
&= \frac{2x+1-3}{3\sqrt{x^2+x+1}} + C \\
&= \frac{2x-2}{3\sqrt{x^2+x+1}} + C
\end{aligned}$$

14

14.a Q14: 8

For each non-negative integer n , let

$$I_n(x) = \int \frac{x^n}{\sqrt{x^2+1}} dx$$

which is defined up to addition by a constant function. Find a reduction formula that connects I_n and I_{n-2} for $n \geq 2$.

$$\begin{aligned}
& \forall n \geq 2 \\
& I_n(x) = \int \frac{x^n}{\sqrt{x^2+1}} dx \\
&= x^{n-1} \sqrt{x^2+1} - (n-1) \int x^{n-2} \sqrt{x^2+1} dx \\
&= x^{n-1} \sqrt{x^2+1} - (n-1) \left(\int x^{n-2} \left(\sqrt{x^2+1} - \frac{1}{\sqrt{x^2+1}} \right) dx + I_{n-2}(x) \right) \\
&= x^{n-1} \sqrt{x^2+1} - (n-1) \left(\int x^{n-2} \frac{x^2}{\sqrt{x^2+1}} dx + I_{n-2}(x) \right) \\
&= x^{n-1} \sqrt{x^2+1} - (n-1) \left(\int \frac{x^n}{\sqrt{x^2+1}} dx + I_{n-2}(x) \right) \\
&= x^{n-1} \sqrt{x^2+1} - (n-1) (I_n(x) + I_{n-2}(x)) \\
&= x^{n-1} \sqrt{x^2+1} - (n-1) I_n(x) - (n-1) I_{n-2}(x) \\
& n I_n(x) = x^{n-1} \sqrt{x^2+1} - (n-1) I_{n-2}(x) \\
& I_n(x) = \frac{1}{n} x^{n-1} \sqrt{x^2+1} - \frac{n-1}{n} I_{n-2}(x)
\end{aligned}$$

14.b

Hence evaluate

$$\int_0^1 \frac{x^5}{\sqrt{x^2+1}} dx$$

$$\begin{aligned}
& I_1(x) \\
&= \int \frac{x}{\sqrt{x^2+1}} \, dx \\
&= \frac{1}{2} \int \frac{1}{\sqrt{x^2+1}} \, dx^2 \\
&= \frac{1}{2} \left(2\sqrt{x^2+1} \right) + C \\
&= \sqrt{x^2+1} + C
\end{aligned}$$

$$\begin{aligned}
& I_5(x) \\
&= \frac{1}{5} x^4 \sqrt{x^2+1} - \frac{4}{5} I_3(x) \\
&= \frac{1}{5} x^4 \sqrt{x^2+1} - \frac{4}{5} \frac{1}{3} x^2 \sqrt{x^2+1} + \frac{4}{5} \frac{2}{3} I_1(x) \\
&= \frac{1}{5} x^4 \sqrt{x^2+1} - \frac{4}{15} x^2 \sqrt{x^2+1} + \frac{8}{15} \sqrt{x^2+1} + C
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 \frac{x^5}{\sqrt{x^2+1}} \, dx \\
&= I_5(1) - I_5(0) \\
&= \left[\frac{1}{5} x^4 \sqrt{x^2+1} - \frac{4}{15} x^2 \sqrt{x^2+1} + \frac{8}{15} \sqrt{x^2+1} \right]_0^1 \\
&= \frac{1}{5} \sqrt{2} - \frac{4}{15} \sqrt{2} + \frac{8}{15} \sqrt{2} - \frac{8}{15} \\
&= \frac{7}{15} \sqrt{2} - \frac{8}{15}
\end{aligned}$$