1. (a) ~~symbol E_1~~ 2018-12-31

Let E_i denote symbol energy of S_i (Signal is real)

$$E_1 = \int_{-\infty}^{\infty}(-3Ap(t))^2\,dt = 9A^2 \underbrace{\int_0^T (p(t))^2\,dt}_{\text{unit-energy}} = 9A^2$$

$$E_2 = \int_{-\infty}^{\infty}(-Ap(t))^2\,dt = A^2 \underbrace{\int_0^T (p(t))^2\,dt}_{\text{unit energy}} = A^2$$

$$E_3 = \int_{-\infty}^{\infty}(Ap(t))^2\,dt = A^2 \underbrace{\int_0^T (p(t))^2\,dt}_{\text{unit-energy}} = A^2$$

$$E_4 = \int_{-\infty}^{\infty}(3Ap(t))^2\,dt = 9A^2 \underbrace{\int_0^T (p(t))^2\,dt}_{\text{unit-energy}} = 9A^2$$

Equiprobable => ~~average~~ sy

$$\Rightarrow E_s = \frac{E_1+E_2+E_3+E_4}{4} = 5A^2$$

(b) ~~Correlator: $v(t) = p(t) * y(t) = \int_0^T$~~

~~Define sy~~

Define signal ~~$s($ $s(t$~~ $s(t;x) = xAp(t)$

~~$s(t;A) =$~~

Correlator using $p(t)$ ~~$= v(t) = p(t) * s(t;A)$~~ ~~(noise is)~~ ~~(noise has zero mean)~~

~~$h(t) = p($~~

$V(t) = p(t+T) * s(t;A)$ (ignore noise, has zero mean)

$$= \int_{-\infty}^{\infty} p(\tau+T-t)\, s(\tau;A)\,d\tau$$

~~$= \int_0^t xA(p(t))^2\,dt$~~

~~$= xA\int_0^T (p(t))^2\,dt$~~ unit-energy

~~$= xA$~~

$= f_2$

$$V(T) = \int_{-\infty}^{\infty} p(\tau+T-T)\, s(\tau;A)\,d\tau$$

~~$V(t) = p(t) * s(t;A)$~~

~~$V(t) = \int_{-\infty}^{\infty} p(t-\tau)\, p(T-t)\, s(\tau;A)\,dt\,d\tau$~~

$$= \int_{-\infty}^{\infty} p(\tau)\, s(\tau;A)\,d\tau$$

~~$V(T) = \int_{-\infty}^{\infty} p(T-(\tau-T))\, s(\tau;A)\,d\tau$~~

$$= xA \underbrace{\int_0^T (p(\tau))^2\,d\tau}_{\text{unit energy}}$$

~~$= \int_{-\infty}^{\infty}\int_0^T p(t-T)\,s$~~

$= xA$

~~$= \int_{-\infty}^{\infty} p(\tau-T)\, xAp(\tau)\,d\tau$~~

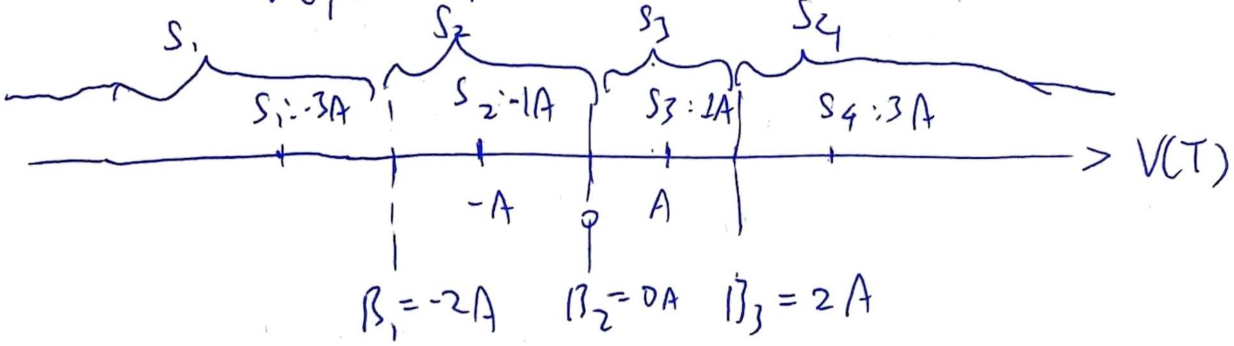
~~($p(t)$ is symmetric even across $t = \frac{T}{2}$)~~

~~$= xA\int_0^T (p(\tau))^2\,d\tau = \frac{p(t)}{xA}$ (unit energy)~~

(b) (continue)

$S_1 \Rightarrow x = -3$ $S_2 \Rightarrow x = -1$ $S_3 \Rightarrow x = 1$ $S_4 \Rightarrow x = 3$

$V(T) = -3A$ $V(T) = -A$ $V(T) = A$ $V(T) = 3A$



boundary should be in the middle of two signals adjacent signals as they are equiprobable.

Decision rule:

$$\hat{s}(t) = \begin{cases} s_1(t) & v(t) \le -2A \\ s_2(t) & -2A < v(t) \le 0A \\ s_3(t) & 0A < v(t) \le 2A \\ s_4(t) & 2A < v(t) \end{cases}$$

(c) noise over time T!

$\cancel{P_{e1} = Q}$ $\sigma = \sqrt{\frac{N_0}{2}T}$

$P_{e|s_1}$ $P_{e_1} = \cancel{Q(}\ Q\left(\frac{A}{\sigma}\right)$

$P_{e|s_2} = 2Q\left(\frac{A}{\sigma}\right)$

$P_{e|s_3} = 2Q\left(\frac{A}{\sigma}\right)$

$P_{e|s_4} = Q\left(\frac{A}{\sigma}\right)$

$$P_e = \frac{P_{e|s_1} + P_{e|s_2} + P_{e|s_3} + P_{e|s_4}}{4} \quad \text{(equiprobable)}$$

$$= \frac{3}{2} Q\left(\frac{A}{\sigma}\right) = \frac{3}{2} Q\left(\sqrt{\frac{2A^2}{N_0 T}}\right) \cancel{= \frac{3}{2} Q\left(\sqrt{\quad}\right)}$$

$$= \frac{3}{2} Q\left(\sqrt{\frac{2E_s}{5N_0 T}}\right)$$

2、(a)

$$d_{12} = \sqrt{(A - A\cos\alpha)^2 + (0 - \cancel{\sin} A\sin\alpha)^2}$$

$$\cancel{= A\sqrt{(1-\cos\alpha)^2 + \sin^2\alpha}}$$

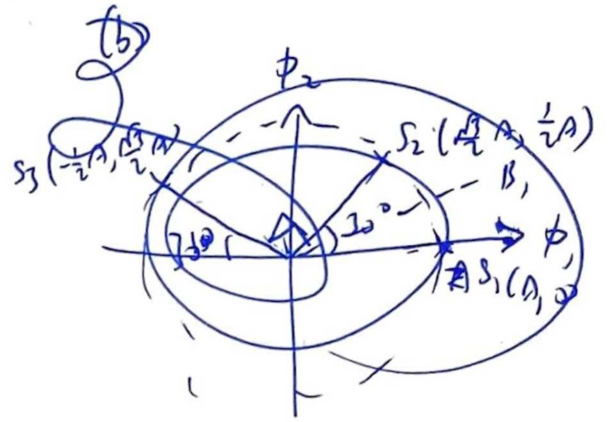
$$= A\sqrt{\cancel{1}\ 2\sin^2\alpha}$$

$$\cancel{= \sqrt{2}A\sin\alpha}$$

$$\cancel{= A\sqrt{2}(\cos)^2}$$

$$\cancel{= \frac{A}{\sqrt{2}}}$$

(b)



Signal coordinates

$S_1 : (A, 0)$

$S_2 : (A\cos\alpha = \frac{\sqrt{3}}{2}A,\ A\sin\alpha = \frac{1}{2}A)$

$S_3 : (-A\sin\alpha = -\frac{1}{2}A,\ A\cos\alpha = \frac{\sqrt{3}}{2}A)$

$$d_{12} = \cancel{\sqrt{A^2(1-\tfrac{\sqrt{3}}{2})^2 + \tfrac{1}{4}A}} \sqrt{(A - \tfrac{\sqrt{3}}{2}A)^2 + (0 - \tfrac{1}{2}A)^2}$$

$$= A\sqrt{1 - \sqrt{3} + \frac{3}{4} + \frac{1}{4}}$$

$$= A\sqrt{2-\sqrt{3}}$$

$$d_{13} = \cancel{\sqrt{A^2(1+\tfrac{1}{2})^2 + (0 - \tfrac{\sqrt{3}}{2}A)^2}}$$

$$\sqrt{(A + \tfrac{1}{2}A)^2 + (0 - \tfrac{\sqrt{3}}{2}A)^2}$$

$$\cancel{= \sqrt{A^2}}$$

$$= A\sqrt{1 + 1 + \frac{1}{4} + \frac{3}{4}}$$

$$= A\sqrt{3}$$

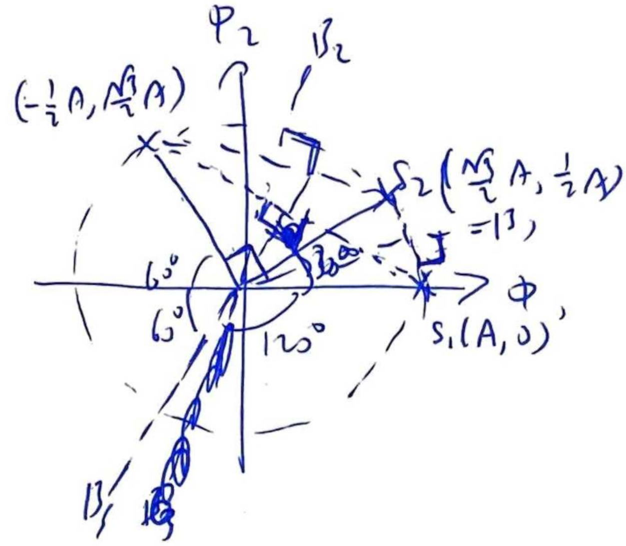
$$d_{23} = \cancel{\sqrt{(A\cos\alpha + A\sin}}$$

$$= \sqrt{(\tfrac{\sqrt{3}}{2}A + \tfrac{1}{2}A)^2 + (\tfrac{1}{2}A - \tfrac{\sqrt{3}}{2}A)^2}$$

$$= A\sqrt{\frac{3}{4} + \frac{\sqrt{3}}{2} + \frac{1}{4} + \frac{1}{4} - \frac{\sqrt{3}}{2} + \frac{3}{4}}$$

$$= \cancel{A\sqrt{2}}\ A\sqrt{2}$$

(b)



perpendicular ~~angle~~

The optimal decision boundaries should be bisector of the ~~signal~~ two adjacent signals, when all signals are equiprobable.

~~$B_1: y = \tan(\frac{\pi}{12} x) \ (x \geq 0)$~~

~~$B_2: y = \tan(\frac{5\pi}{12} x) \ (x \geq 0)$~~

~~$B_3: y = \tan$~~

$$B_1: y = x\tan\left(\frac{\pi}{12}\right) = x\frac{1-\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} = x\frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = (2-\sqrt{3})x \quad (x \geq 0)$$

$$B_2: y = x\tan\left(\frac{5\pi}{12}\right) = x\frac{1}{\tan(\frac{\pi}{12})} = \frac{1}{2-\sqrt{3}}x = (2+\sqrt{3})x \quad (x \geq 0)$$

$$B_3: y = x\tan\left(\frac{4\pi}{3}\right) = x\tan\left(\frac{\pi}{3}\right) = \sqrt{3}x \quad (x \leq 0)$$

(c) Equiprobable ~~$\sigma = \sqrt{\frac{N_0 T}{2}}$~~ noise per unit time: $\sigma = \sqrt{\frac{N_0}{2}}$

$$P_e \leq \frac{P_{e|s_1} + P_{e|s_2} + P_{e|s_3}}{3}$$

$$P_{e|s_1} \leq Q\left(\frac{d_{12}}{2\sigma}\right) + Q\left(\frac{d_{13}}{2\sigma}\right)$$

$$P_{e|s_2} \leq Q\left(\frac{d_{12}}{2\sigma}\right) + Q\left(\frac{d_{23}}{2\sigma}\right)$$

$$P_{e|s_3} \leq Q\left(\frac{d_{13}}{2\sigma}\right) + Q\left(\frac{d_{23}}{2\sigma}\right)$$

$$= \frac{2}{3}\left(Q\left(\frac{d_{12}}{2\sigma}\right) + Q\left(\frac{d_{13}}{2\sigma}\right) + Q\left(\frac{d_{23}}{2\sigma}\right)\right)$$

$$= \frac{2}{3}\left(Q\left(\frac{A\sqrt{2-\sqrt{3}}}{\sqrt{2N_0}}\right) + Q\left(\frac{A\sqrt{3}}{\sqrt{2N_0}}\right) + Q\left(\frac{A\sqrt{2}}{\sqrt{2N_0}}\right)\right)$$

Assume symbol time is T.

3. (a) The optimum receiver consists of M ~~fi~~ correlators

~~consists for~~ using $s_m(t)$ as the template

more specifically, each $s_m(t)$ correlator does

the following: cross-correlation:

$$\forall \quad \underbrace{s_m(t+T)}_{\text{advanced}} \star y(t).$$

When $t=T$ (assume $t=0$ is when symbol is received),

The result ~~of the above cross-correl~~

$$\left(s_m(t+T) \star y(t)\right)(T) \text{ is saved.}$$

This result is also the projection of the signal

during $[0, T]$ to each of $s_m(t)$ (up to scaling,

which is the same for all signals as $\|s_m\|^2 = E_s$).

To fix the scaling, just divide by $\sqrt{E_s}$. Let this

~~Then Finally,~~ i.e. ~~finally, collecting this number~~

into ~~multiple a tuple to~~ result be x_m

Finally, collect the M results into a coordinate

in the signal space

$: (x_1, ~~x_2~~ \ldots, x_M)$.

Each ~~sig~~ orthogonal signal $s_m(t)$ has the

coordinate $\overbrace{(0, \ldots, 0, \sqrt{E_s}, 0, \ldots, 0)}^{M \text{ numbers}}$

m-th entry

Compute distance of $(x_1, \ldots, x_M)$ to all orthogonal

signal, and choose $s_m(t)$ ~~such~~ with the smallest distance

(b) ~~Assume sy~~

Each orthogonal signal $s_m(t)$ has the coordinate in signal space

$$\underbrace{(0, \ldots, 0, \sqrt{E_s}, 0, \ldots, 0)}^{M \text{ numbers}}$$

m-th entry

$\therefore$ distance between any two $s_i(t)$ and $s_j(t)$ $(i \neq j)$

$$= \sqrt{E_s + E_s} = \sqrt{2E_s}$$

noise per unit time : $\sigma = \sqrt{\frac{N_0}{2}}$

$$P_{eM} = P_{e|s_m} \leq (M-1)\, Q\left(\frac{\sqrt{2E_s}}{2\sigma}\right)$$

$$= (M-1)\, Q\left(\sqrt{\frac{E}{N_0}}\right)$$

~~Equiprobable~~

M orthogonal modulation,

$$P_e \leq \frac{M}{2} \frac{P_{eM}}{M-1}$$

$$= \frac{M}{2} Q\left(\sqrt{\frac{E}{N_0}}\right)$$

4. (a) $\cancel{\langle s_a, s_b\rangle = \int_0^T \sqrt{\ }}$

Assume $n \neq m$, $n \neq -m$, $n \neq 0$, $m \neq 0$

$p(n)$ ~~d~~ has unit energy

$$\langle s_a, s_b\rangle = \int_0^T \cancel{\sqrt{\ }}\, 2E_s \cos(2\pi nt/T)\, \cancel{p(t)} \cos(2\pi mt/T) (p(t))^2\, dt$$

$$= \int_0^T \underbrace{E_s}_{\text{constant}} \Big(\underbrace{\cos(2\pi(n+m)t/T)}_{\text{integrates to zero over } [0,T] \text{ as } n \neq -m} + \underbrace{\cos(2\pi(n-m)t/T)}_{\text{integrates to zero over } [0,T] \text{ as } n \neq m} \Big) \underbrace{(p(t))^2}_{\text{constant within } [0,T]} dt$$

$$= 0$$

~~For $\langle s_b, s_c\rangle$~~

$$\langle s_b, s_c\rangle = \int_0^T 2E_s \cos(2\pi mt/T) \cos(2\pi nt/T + \theta) (p(t))^2\, dt$$

$$= \int_0^T \underbrace{E_s}_{\text{const}} \Big(\underbrace{\cos(2\pi(m+n)t/T + \theta)}_{\text{int to zero over } [0,T] \text{ as } n \neq -m} + \underbrace{\cos(2\pi(m-n)t/T + \theta)}_{\text{int to zero over } [0,T] \text{ as } n \neq m} \Big) \underbrace{(p(t))^2}_{\text{const over } [0,T]} dt$$

$$\langle s_a, s_c\rangle = \int_0^T 2E_s \cos(2\pi nt/T) \cos(2\pi nt/T + \theta) (p(t))^2\, dt$$

$$= \int_0^T \underbrace{E_s}_{\text{const}} \Big(\underbrace{\cos(4\pi nt/T + \theta)}_{\text{int to zero over } [0,T] \text{ as } n \neq 0} + \cos(-\theta) \Big) \underbrace{(p(t))^2}_{\text{const over } [0,T]} dt$$

$$= \int_0^T E_s \cos\theta\, (p(t))^2\, dt$$

$$= E_s \cos\theta \underbrace{\int_0^T (p(t))^2\, dt}_{\text{unit energy}}$$

$$= E_s \cos\theta$$

(b)

~~(b)~~ $\|s_a(t)\|^2 = \int_0^T 2E_s \cos^2(2\pi n t/T)\,(p(t))^2\,dt$

$$= \int_0^T E_s \big(\underbrace{\cos(4\pi n t/T)}_{\text{int to 0 over } [0,T] \text{ as } n\neq 0} + \underbrace{1}_{\text{const}}\big)\underbrace{(p(t))^2}_{\text{const over } [0,T]}\,dt$$

$$= E_s \underbrace{\int_0^T (p(t))^2\,dt}_{\text{unit energy}}$$

$$= E_s$$

Similarly for s_b, s_c (and same result)

so $\|s_a(t)\| = \|s_b(t)\| = \|s_c(t)\| = \sqrt{E_s}$

$$d_{ab} = \sqrt{\|s_a - s_b\|^2}$$
$$= \sqrt{\|s_a\|^2 - 2\langle s_a, s_b\rangle + \|s_b\|^2}$$
$$= \sqrt{E_s \cancel{+0} - 0 + E_s}$$
$$= \sqrt{2E_s}$$

$$d_{bc} = \sqrt{\|s_b - s_c\|^2}$$
$$= \sqrt{\|s_b\|^2 - 2\langle s_b, s_c\rangle + \|s_c\|^2}$$
$$= \sqrt{E_s - 2\cancel{E_s \cos 0}\,0 + E_s}$$
$$= \sqrt{2E_s}$$

$$d_{ac} = \sqrt{\|s_a - s_c\|^2}$$
$$= \sqrt{\|s_a\|^2 - 2\langle s_a, s_c\rangle + \|s_c\|^2}$$
$$= \sqrt{E_s - 2E_s\cos\theta + E_s}$$
$$= \sqrt{2E_s - 2E_s\cos\theta}$$

(b) $\overset{\text{we}}{}$ noise per unit time, $\sigma = \sqrt{\frac{N_0}{2}}$ $\qquad 2\sigma = \sqrt{2N_0}$

$$Pe|s_a \le Q\left(\frac{d_{ab}}{2\sigma}\right) + Q\left(\frac{d_{ac}}{2\sigma}\right)$$

$$Pe|s_b \le Q\left(\frac{d_{ab}}{2\sigma}\right) + Q\left(\frac{d_{bc}}{2\sigma}\right)$$

$$Pe|s_c \le Q\left(\frac{d_{ac}}{2\sigma}\right) + Q\left(\frac{d_{bc}}{2\sigma}\right)$$

~~&~~ A Equiprobable

$$\Rightarrow Pe \le \frac{2}{3}\cancel{\frac{1}{Q}}\left(Pe|s_a + Pe|s_b + Pe|s_c \right.$$

$$= \frac{2}{3}\left(Q\left(\frac{d_{ab}}{2\sigma}\right) + Q\left(\frac{d_{ac}}{2\sigma}\right) + Q\left(\frac{d_{bc}}{2\sigma}\right)\right)$$

$$= \frac{2}{3}\left(2Q\left(\sqrt{\frac{E_s}{N_0}}\right) + Q\left(\sqrt{\frac{E_s - E_s\cos\theta}{N_0}}\right)\right)\cancel{+Q\left(\frac{E_s}{E_s}\right))}$$

(c) $$Pe \le \frac{2}{3}\left(2Q(\sqrt{10}) + Q\left(\sqrt{10(1-\cos(\tfrac{\pi}{3}))}\right)\right)$$

$$= \frac{2}{3}\left(2Q(\sqrt{10}) + \cancel{Q\left(\sqrt{10}\sqrt{5}\right)}\ Q(\sqrt{5})\right)$$

$$\approx \frac{2}{3}\left(2(0.000\overset{27}{78\cancel{88}}) + 0.01267\overset{37}{\cancel{53}}\right)$$

$$\approx 0.0044927\cancel{333}$$