Problem Set 2

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 2 covers materials from §5.3 – §6.1.

1. Let n be a positive integer. Evaluate each of the following limits.

   (a) $\displaystyle\lim_{x\to 0}\frac{1}{x^n}\int_0^{x^n}\cos(t^2)\,dt$ (b) $\displaystyle\lim_{x\to 0}\frac{1}{x^n}\int_0^{x^n}\cos(x^2 t)\,dt$

2. Let $f, g: \mathbb{R} \to \mathbb{R}$ be **increasing** continuous functions, and let $F: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$F(x) = x\int_0^x f(t)g(t)dt - \left(\int_0^x f(t)dt\right)\left(\int_0^x g(t)dt\right).$$

   (a) Show that F is differentiable on $\mathbb{R}$ and

$$F'(x) = \int_0^x \big(f(x) - f(t)\big)\big(g(x) - g(t)\big)dt.$$

   (b) Using the result from (a), find the global minimum value of F on $\mathbb{R}$.

3. Let f be a polynomial given by

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n.$$

   If the coefficients of f satisfy that

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_{n-1}}{n} + \frac{a_n}{n+1} = 0,$$

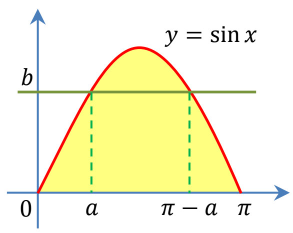
   show that f has a root in the interval $(0, 1)$.

   Hint: This problem is identical to Q2, Problem Set 7 of MATH1013; we used Rolle's Theorem to prove this result back then. This time let's try to use **Mean Value Theorem for integrals** to prove it again.

4. The diagram below shows the graph of

$$f(x) = \sin x$$

   on the interval $[0, \pi]$.



   (a) Find the area of the shaded region.

   (b) A horizontal line $y = b$ intersects the graph of f at two points (a, b) and $(\pi - a, b)$, so that the area under the graph of f between the vertical lines $x = a$ and $x = \pi - a$ is exactly half of the whole shaded area. Find the value of b.

5. Evaluate the following integrals.

(a) $\displaystyle\int_{-1}^{3} |t^3 - 3t^2 + 2t| dt$

(b) $\displaystyle\int_{0}^{\frac{\pi}{4}} \frac{1 - \sin^3\theta}{\cos^2\theta} d\theta$

6. (a) By considering the function $f(x) = x - \sin x$, show that

$$\sin x \leq x \qquad \text{for every } x \geq 0.$$

(b) Using the result from (a) and integration, show that each of the following inequalities holds for every $x \geq 0$.

(i) $\cos x \geq 1 - \dfrac{x^2}{2}$

(ii) $\sin x \geq x - \dfrac{x^3}{6}$

(iii) $\cos x \leq 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$

7. (a) Let $a < b$ be real numbers and let $f: [a, b] \to (0, +\infty)$ be a **positive** continuous function. Using Cauchy-Schwarz inequality, show that

$$\left(\int_a^b f(x)dx\right)\left(\int_a^b \frac{1}{f(x)}dx\right) \geq (b-a)^2.$$

(b) Using the result from (a) and Cauchy-Schwarz inequality again, show that

$$\int_0^{2\pi} \frac{1}{\sqrt{1 - \frac{1}{2}\cos x}} dx \geq 2\pi.$$

8. Let $m \in (0, 1)$ be a fixed number, and let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \int_0^x \frac{1}{\sqrt{1 - m\sin^2 t}} dt.$$

(a) Show that f is strictly increasing on $\mathbb{R}$.

(b) Show that $f(x) \geq x$ for every $x > 0$. Hence deduce that

$$\lim_{x\to+\infty} f(x) = +\infty \qquad \text{and} \qquad \lim_{x\to-\infty} f(x) = -\infty.$$

(c) Using the results from (a) and (b), deduce that f has an inverse which is defined on $\mathbb{R}$.

(d) For each $y \in \mathbb{R}$, let's write $x \coloneqq f^{-1}(y)$ and define three functions $p, q, r: \mathbb{R} \to \mathbb{R}$ by

$$\begin{cases} p(y) = \sin x \\ q(y) = \cos x \\ r(y) = \sqrt{1 - m\sin^2 x} \end{cases}.$$

Show that

$$p'(y) = q(y)r(y) \qquad \text{for every } y \in \mathbb{R}.$$

In a similar way, also compute $q'(y)$ and $r'(y)$ in terms of $p(y)$, $q(y)$ and $r(y)$.

9. Let $f\colon \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \to \mathbb{R}$ be the function defined by

$$f(x) = \int_1^x \sin(\cos t)\,dt.$$

(a) Show that f is one-to-one.

(b) If g is the inverse of f, find $g'(0)$.

10. (a) Let $F\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$F(x) = \int_0^1 \cos xt\,dt.$$

Find the derivative of F.

(b) Let $G\colon (0,+\infty) \to \mathbb{R}$ be the function defined by

$$G(x) = \int_{\frac{1}{x}}^{x} \cos\sqrt{xt}\,dt\,.$$

Find $G'(1)$.

11. (a) Let $a > 0$ and let $f\colon [0,a] \to \mathbb{R}$ be a continuous function.

(i) Prove that

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx.$$

(ii) If there exists a real constant c such that

$$f(x) + f(a-x) = c \qquad \text{for every } x \in [0,a],$$

show that

$$\int_0^a f(x)dx = af\left(\frac{a}{2}\right).$$

(b) Using the result from (a), evaluate the integral

$$\int_0^{2\pi} \frac{1}{e^{\sin^3 x} + 1}\,dx.$$

12. (a) Let x be a fixed non-negative number with $x \neq 1$. Evaluate the integral

$$\int_0^{\pi} \frac{\sin t}{\sqrt{1 - 2x\cos t + x^2}}\,dt$$

in terms of x.

(b) Let $f\colon [0,+\infty) \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \displaystyle\int_0^{\pi} \frac{\sin t}{\sqrt{1 - 2x\cos t + x^2}}\,dt & \text{if } x \neq 1 \\ a & \text{if } x = 1 \end{cases}.$$

Using the result from (a), find the value of a so that f is a continuous function. Hence sketch the graph of f.

13. (a) Using the substitution $u = \frac{1}{x}$, show that

$$\int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2}\,dx = 0.$$

(b) Using (a) or otherwise, evaluate the limit

$$\lim_{n\to+\infty} \sum_{k=1}^{3n} \frac{1}{2n} \frac{\ln\left[2\left(\frac{1}{2}+\frac{k}{2n}\right)\right]}{1+\left(\frac{1}{2}+\frac{k}{2n}\right)^2}.$$

14. Evaluate the following antiderivatives.

(a) $\displaystyle\int \csc x\,dx$

(b) $\displaystyle\int \frac{1}{e^x+e^{-x}}\,dx$

(c) $\displaystyle\int \frac{\cos^5\theta}{\sin^7\theta}\,d\theta$

(d) $\displaystyle\int \frac{[\ln(u^2)]^2}{u}\,du$

(e) $\displaystyle\int \frac{\sqrt{x}}{1+x^3}\,dx$

15. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function.

(a) Show that

$$\int_0^x tf(x-t)dt = \int_0^x (x-t)f(t)dt \qquad \text{for every } x \in \mathbb{R}.$$

(b) Suppose that

$$\int_0^x tf(x-t)dt = e^x - x - 1 \qquad \text{for every } x \in \mathbb{R}.$$

Using differentiation or otherwise, find a formula for $f(x)$.

16. (a) Let $f\colon [0,\pi] \to \mathbb{R}$ be a continuous function such that

$$f(\pi - x) = -f(x) \qquad \text{for every } x \in [0,\pi].$$

Using the substitution $u = \pi - x$, show that

$$\int_0^\pi f(x)\ln(1+e^{\cos x})\,dx = \frac{1}{2}\int_0^\pi f(x)\cos x\,dx.$$

(b) Compute the derivative of the function $g\colon [0,\pi] \to \mathbb{R}$ defined by

$$g(x) = \frac{\cos x}{1+\sin x}.$$

Using this together with the result from (a), evaluate the integral

$$\int_0^\pi \frac{(\cos x)\ln(1+e^{\cos x})}{(1+\sin x)^2}\,dx.$$