

# MATH 1014 L1 assignment 2 submission

## 1

Let  $n$  be a positive integer. Evaluate each of the following limits.

### 1.a

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1}{x^n} \int_0^{x^n} \cos(t^2) dt &= \lim_{x \rightarrow 0} \int_0^1 \cos(x^{2n} t^2) dt && \left( \text{change of variable } \frac{t}{x^n} \mapsto t \right) \\ &= \int_0^1 \lim_{x \rightarrow 0} \cos(x^{2n} t^2) dt && (\text{dominated convergence theorem; dominated by } f(x) = 1 \text{ on } [0, 1]) \\ &= \int_0^1 \cos\left(\left(\lim_{x \rightarrow 0} x\right)^{2n} t^2\right) dt && (f(x) = \cos(x^{2n} t) \text{ continuous at } 0) \\ &= \int_0^1 \cos 0 dt && (n > 0) \\ &= \int_0^1 dt \\ &= 1\end{aligned}$$

### 1.b

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1}{x^n} \int_0^{x^n} \cos(x^2 t) dt &= \lim_{x \rightarrow 0} \int_0^1 \cos(x^{n+2} t) dt && \left( \text{change of variable } \frac{t}{x^n} \mapsto t \right) \\ &= \int_0^1 \lim_{x \rightarrow 0} \cos(x^{n+2} t) dt && (\text{dominated convergence theorem; dominated by } f(x) = 1 \text{ on } [0, 1]) \\ &= \int_0^1 \cos\left(\left(\lim_{x \rightarrow 0} x\right)^{n+2} t\right) dt && (f(x) = \cos(x^{n+2} t) \text{ is continuous at } 0) \\ &= \int_0^1 \cos 0 dt && (n + 2 > 0) \\ &= \int_0^1 dt \\ &= 1\end{aligned}$$

## 2

Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  be **increasing** continuous functions, and let  $F: \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by

$$F(x) = x \int_0^x f(t)g(t) dt - \left( \int_0^x f(t) dt \right) \left( \int_0^x g(t) dt \right)$$

.

### 2.a

Show that  $F$  is differentiable on  $\mathbb{R}$  and

$$F'(x) = \int_0^x (f(x) - f(t))(g(x) - g(t)) dt$$

.

$$\begin{aligned}
f(t) \in C(\mathbb{R}, \mathbb{R}) &\implies X(x) := \int_0^x f(t) \, dt \in C^1(\mathbb{R}, \mathbb{R}) && \text{(first fundamental theorem of calculus)} \\
g(t) \in C(\mathbb{R}, \mathbb{R}) &\implies Y(x) := \int_0^x g(t) \, dt \in C^1(\mathbb{R}, \mathbb{R}) && \text{(first fundamental theorem of calculus)} \\
f(t), g(t) \in C(\mathbb{R}, \mathbb{R}) &\implies f(t)g(t) \in C(\mathbb{R}, \mathbb{R}) \\
&\implies Z(x) := \int_0^x f(t)g(t) \, dt \in C^1(\mathbb{R}, \mathbb{R}) && \text{(first fundamental theorem of calculus)} \\
x, Z(x) \in C^1(\mathbb{R}, \mathbb{R}) &\implies xZ(x) \in C^1(\mathbb{R}, \mathbb{R}) && \text{(product rule)} \\
X(x), Y(x) \in C^1(\mathbb{R}, \mathbb{R}) &\implies X(x)Y(x) \in C^1(\mathbb{R}, \mathbb{R}) && \text{(product rule)} \\
xZ(x), X(x)Y(x) \in C^1(\mathbb{R}, \mathbb{R}) &\implies F(x) = xZ(x) - X(x)Y(x) \in C^1(\mathbb{R}, \mathbb{R}) && \text{(linearity of differentiation)} \\
\therefore F(x) &\text{ is differentiable on } \mathbb{R}. \\
F(x) &= x \int_0^x f(t)g(t) \, dt - \left( \int_0^x f(t) \, dt \right) \left( \int_0^x g(t) \, dt \right) \\
&= xZ(x) - X(x)Y(x) \\
F'(x) &= Z(x) + xZ'(x) - X'(x)Y(x) - X(x)Y'(x) \\
&= \int_0^x f(t)g(t) \, dt + xf(x)g(x) - f(x) \int_0^x g(t) \, dt - g(x) \int_0^x f(t) \, dt && \text{(first fundamental theorem of calculus)} \\
&= \int_0^x f(t)g(t) \, dt + \int_0^x f(x)g(x) \, dt - \int_0^x f(x)g(t) \, dt - \int_0^x f(t)g(x) \, dt \\
&= \int_0^x (f(t)g(t) + f(x)g(x) - f(x)g(t) - f(t)g(x)) \, dt && \text{(linearity of integral)} \\
&= \int_0^x (f(x)(g(x) - g(t)) + f(t)(g(t) - g(x))) \, dt \\
&= \int_0^x (f(x) - f(t))(g(x) - g(t)) \, dt
\end{aligned}$$

## 2.b

Using the result from 2.a, find the global minimum of  $F$  on  $\mathbb{R}$ .

Solve for  $F'(x) = 0$ .

When  $x = 0$ , obviously  $F'(x) = 0$ .

When  $x > 0$ ,  $t \in [0, x]$ , then

$$\begin{aligned}
x \geq t &\implies f(x) - f(t) \geq 0 && (f \text{ is increasing}) \\
x \geq t &\implies g(x) - g(t) \geq 0 && (g \text{ is increasing}) \\
f(x) - f(t) \geq 0 \wedge g(x) - g(t) \geq 0 &\implies (f(x) - f(t))(g(x) - g(t)) \geq 0 \\
&\implies \int_0^x (f(x) - f(t))(g(x) - g(t)) \, dt \geq 0 && \text{(integrand is nonnegative; } x > 0) \\
&\implies F'(x) \geq 0
\end{aligned}$$

When  $x < 0$ ,  $t \in [x, 0]$ , then

$$\begin{aligned}
x \leq t &\implies f(x) - f(t) \leq 0 && (f \text{ is increasing}) \\
x \leq t &\implies g(x) - g(t) \leq 0 && (g \text{ is increasing}) \\
f(x) - f(t) \leq 0 \wedge g(x) - g(t) \leq 0 &\implies (f(x) - f(t))(g(x) - g(t)) \geq 0 \\
&\implies \int_x^0 (f(x) - f(t))(g(x) - g(t)) \, dt \geq 0 && \text{(integrand is nonnegative; } 0 > x) \\
&\implies \int_0^x (f(x) - f(t))(g(x) - g(t)) \, dt \leq 0 \\
&\implies F'(x) \leq 0
\end{aligned}$$

The above shows that  $F(x)$  is decreasing on  $(-\infty, 0)$ , stationary on  $\{0\}$ , and increasing on  $(0, +\infty)$ .

Then  $x = 0$  is one of the global minima of  $F(x)$ .

So the global minimum value is  $F(0) = 0$ .

## 6

### 6.a

By considering the function  $f(x) = x - \sin x$ , show that

$$\sin x \leq x$$

for every  $x \geq 0$ .

$$\begin{aligned}
 f(x) &= x - \sin x \\
 f(x) &\in C^\infty(\mathbb{R}, \mathbb{R}) \\
 f(0) &= 0 - \sin 0 = 0
 \end{aligned}$$

$$\begin{aligned}
 f'(x) &= 1 - \cos x \\
 &\geq 0 && (\cos x \leq 1) \\
 f'(x) \geq 0 &\implies (x \geq y \implies f(x) \geq f(y)) && (\text{increasing})
 \end{aligned}$$

$$\begin{aligned}
 \forall x &\geq 0 \\
 x \geq 0 &\implies f(x) \geq f(0) \\
 &\implies f(x) \geq 0 \\
 f(x) &\geq 0 \\
 x - \sin x &\geq 0 \\
 x &\geq \sin x \\
 \sin x &\leq x
 \end{aligned}$$

## 6.b

Using the result of [6.a](#) and integration, show that each of the following inequalities holds for every  $x \geq 0$ .

### 6.b.i

$$\begin{aligned}
 \cos x &\geq 1 - \frac{x^2}{2} \\
 \forall x &\geq 0 \\
 \sin x &\leq x \\
 \int_0^x \sin \xi \, d\xi &\leq \int_0^x \xi \, d\xi \\
 [-\cos \xi]_0^x &\leq \left[ \frac{\xi^2}{2} \right]_0^x \\
 \cos 0 - \cos x &\leq \frac{x^2}{2} - \frac{0^2}{2} \\
 1 - \cos x &\leq \frac{x^2}{2} \\
 -\cos x &\leq -1 + \frac{x^2}{2} \\
 \cos x &\geq 1 - \frac{x^2}{2}
 \end{aligned}$$

### 6.b.ii

$$\begin{aligned}
 \sin x &\geq x - \frac{x^3}{6} \\
 \forall x &\geq 0 \\
 \cos x &\geq 1 - \frac{x^2}{2} \\
 \int_0^x \cos \xi \, d\xi &\geq \int_0^x \left( 1 - \frac{\xi^2}{2} \right) d\xi \\
 [\sin \xi]_0^x &\geq \left[ \xi - \frac{\xi^3}{6} \right]_0^x \\
 \sin x - \sin 0 &\geq x - \frac{x^3}{6} - 0 + \frac{0^3}{6} \\
 \sin x &\geq x - \frac{x^3}{6}
 \end{aligned}$$

### 6.b.iii

$$\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$\begin{aligned}
& \forall x \geq 0 \\
& \sin x \geq x - \frac{x^3}{6} \\
& \int_0^x \sin \xi \, d\xi \geq \int_0^x \left( \xi - \frac{\xi^3}{6} \right) d\xi \\
& [-\cos \xi]_0^x \geq \left[ \frac{\xi^2}{2} - \frac{\xi^4}{24} \right]_0^x \\
& \cos 0 - \cos x \geq \frac{x^2}{2} - \frac{x^4}{24} - \frac{0^2}{2} + \frac{0^4}{24} \\
& 1 - \cos x \geq \frac{x^2}{2} - \frac{x^4}{24} \\
& -\cos x \geq -1 + \frac{x^2}{2} - \frac{x^4}{24} \\
& \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}
\end{aligned}$$

7

7.a

Let  $a < b$  be real numbers and let  $f: [a, b] \rightarrow (0, +\infty)$  be a **positive** continuous function. Using Cauchy-Schwarz inequality, show that

$$\left( \int_a^b f(x) \, dx \right) \left( \int_a^b \frac{1}{f(x)} \, dx \right) \geq (b-a)^2$$

.

$$g(x) := \frac{1}{f(x)}$$

$$F(x) := \sqrt{f(x)}$$

$$G(x) := \sqrt{g(x)} = \frac{1}{\sqrt{f(x)}}$$

$f(x)$  is integrable on  $[a, b]$  because it is bounded and continuous.

$g(x)$  is integrable on  $[a, b]$  because it is bounded and continuous.  $(f(x) > 0 \text{ and is bounded}) \implies g(x) = \frac{1}{f(x)} > 0 \text{ and is bounded}$

$F(x)$  is integrable on  $[a, b]$  because it is bounded and continuous.  $(f(x) > 0 \text{ and is bounded}) \implies F(x) = \sqrt{f(x)} > 0 \text{ and is bounded}$

$G(x)$  is integrable on  $[a, b]$  because it is bounded and continuous.  $(g(x) > 0 \text{ and is bounded}) \implies G(x) = \sqrt{g(x)} > 0 \text{ and is bounded}$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \left( \int_a^b (F(x))^2 \, dx \right) \left( \int_a^b (G(x))^2 \, dx \right) \geq \left( \int_a^b F(x)G(x) \, dx \right)^2 \\
& \left( \int_a^b \left( \sqrt{f(x)} \right)^2 \, dx \right) \left( \int_a^b \left( \frac{1}{\sqrt{f(x)}} \right)^2 \, dx \right) \geq \left( \int_a^b \sqrt{f(x)} \frac{1}{\sqrt{f(x)}} \, dx \right)^2 \\
& \left( \int_a^b |f(x)| \, dx \right) \left( \int_a^b \frac{1}{|f(x)|} \, dx \right) \geq \left( \int_a^b 1 \, dx \right)^2 \\
& \geq (b-a)^2 \\
& \left( \int_a^b f(x) \, dx \right) \left( \int_a^b \frac{1}{f(x)} \, dx \right) \geq (b-a)^2 \quad (f(x) > 0)
\end{aligned}$$

7.b

Using the result from 7.a and Cauchy-Schwarz inequality again, show that

$$\int_0^{2\pi} \frac{1}{\sqrt{1 - \frac{1}{2}\cos x}} \, dx \geq 2\pi$$

.

$$\begin{aligned}
f(x) &:= \sqrt{1 - \frac{1}{2} \cos x} & x \in \mathbb{R} \\
f(x) &\text{ is continuous.} \\
\cos x &\in [-1, 1] \\
1 - \frac{1}{2} \cos x &\in [0.5, 1.5] \\
\sqrt{1 - \frac{1}{2} \cos x} &> 0 \\
f(x) &> 0
\end{aligned}$$

$$\begin{aligned}
\left( \int_0^{2\pi} dx \right) \left( \int_0^{2\pi} (f(x))^2 dx \right) &\geq \left( \int_0^{2\pi} f(x) dx \right)^2 & (\text{Cauchy-Schwarz inequality}) \\
2\pi \left( \int_0^{2\pi} \left( 1 - \frac{1}{2} \cos x \right) dx \right) &\geq \left( \int_0^{2\pi} f(x) dx \right)^2 \\
2\pi \left[ x - \frac{1}{2} \sin x \right]_0^{2\pi} &\geq \left( \int_0^{2\pi} f(x) dx \right)^2 \\
2\pi \left( 2\pi - \frac{1}{2} \sin(2\pi) - 0 + \frac{1}{2} \sin 0 \right) &\geq \left( \int_0^{2\pi} f(x) dx \right)^2 \\
4\pi^2 &\geq \left( \int_0^{2\pi} f(x) dx \right)^2 \\
2\pi &\geq \left| \int_0^{2\pi} f(x) dx \right| \\
2\pi &\geq \int_0^{2\pi} f(x) dx & (f(x) > 0, 2\pi > 0)
\end{aligned}$$

$$\begin{aligned}
\left( \int_0^{2\pi} f(x) dx \right) \left( \int_0^{2\pi} \frac{1}{f(x)} dx \right) &\geq (2\pi - 0)^2 & (7.a) \\
&\geq 4\pi^2 \\
\int_0^{2\pi} \frac{1}{f(x)} dx &\geq \frac{4\pi^2}{\int_0^{2\pi} f(x) dx} \\
&\geq \frac{4\pi^2}{2\pi} & \left( 2\pi \geq \int_0^{2\pi} f(x) dx \right) \\
\int_0^{2\pi} \frac{1}{\sqrt{1 - \frac{1}{2} \cos x}} dx &\geq 2\pi
\end{aligned}$$

## 8

Let  $m \in (0, 1)$  be a fixed number, and let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by

$$f(x) = \int_0^x \frac{1}{\sqrt{1 - m \sin^2 t}} dt$$

.

### 8.a

Show that  $f$  is strictly increasing on  $\mathbb{R}$ .

$$\begin{aligned}
\forall t \in \mathbb{R} \\
\sin t &\in [-1, 1] \\
\sin^2 t &\in [0, 1] \\
m \sin^2 t &\in [0, 1] & (m \in (0, 1)) \\
1 - m \sin^2 t &\in (0, 1] \\
\sqrt{1 - m \sin^2 t} &\in (0, 1] \\
\frac{1}{\sqrt{1 - m \sin^2 t}} &\in [1, +\infty) \\
&> 0 \\
f(x) &= \int_0^x \frac{1}{\sqrt{1 - m \sin^2 t}} dt \\
f'(x) &= \frac{1}{\sqrt{1 - m \sin^2 x}} & (\text{first fundamental theorem of calculus}) \\
&> 0 \\
f'(x) > 0 &\implies f \text{ is strictly increasing on } \mathbb{R}.
\end{aligned}$$

### 8.b

Show that  $f(x) \geq x$  for every  $x > 0$ . Hence deduce that

$$\lim_{x \rightarrow +\infty} f(x) = +\infty$$

and

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

.

$$\begin{aligned} & \forall x > 0 \\ & \frac{1}{\sqrt{1 - m \sin^2 x}} \in [1, +\infty) \\ & \geq 1 \\ & \int_0^x \frac{1}{\sqrt{1 - m \sin^2 \xi}} d\xi \geq \int_0^x d\xi \quad (x > 0) \\ & f(x) \geq [\xi]_0^x \\ & \geq x \end{aligned}$$

$$\begin{aligned} & \forall x < 0 \\ & \frac{1}{\sqrt{1 - m \sin^2 x}} \in [1, +\infty) \\ & \geq 1 \\ & \int_x^0 \frac{1}{\sqrt{1 - m \sin^2 \xi}} d\xi \geq \int_x^0 d\xi \quad (0 > x) \\ & -f(x) \geq [\xi]_x^0 \\ & \geq -x \\ & f(x) \leq x \end{aligned}$$

When  $x > 0$ ,

$$\begin{aligned} & f(x) \geq x \\ & \lim_{x \rightarrow +\infty} f(x) \geq \lim_{x \rightarrow +\infty} x \quad (x > 0) \\ & \geq +\infty \\ & \lim_{x \rightarrow +\infty} f(x) = +\infty \end{aligned}$$

When  $x < 0$ ,

$$\begin{aligned} & f(x) \leq x \\ & \lim_{x \rightarrow -\infty} f(x) \leq \lim_{x \rightarrow -\infty} x \quad (x < 0) \\ & \leq -\infty \\ & \lim_{x \rightarrow -\infty} f(x) = -\infty \end{aligned}$$

### 8.c

Using the results from (a) and (b), deduce that  $f$  has an inverse which is defined on  $\mathbb{R}$ .

The range of  $f$  is  $(-\infty, +\infty)$ , i.e.  $\mathbb{R}$ . (8.b)

$f$  is invertible as it is strictly increasing on all of its domain. (8.a)

The domain of the inverse of  $f$  is the range of  $f$ , which is  $\mathbb{R}$ .

### 8.d

For each  $y \in \mathbb{R}$ , let's write  $x := f^{-1}(y)$  and define three functions  $p, q, r: \mathbb{R} \rightarrow \mathbb{R}$  by

$$\begin{cases} p(y) = \sin x \\ q(y) = \cos x \\ r(y) = \sqrt{1 - m \sin^2 x} \end{cases}$$

.

Show that

$$p'(y) = q(y)r(y)$$

for every  $y \in \mathbb{R}$ .

In a similar way, also compute  $q'(y)$  and  $r'(y)$  in terms of  $p(y)$ ,  $q(y)$ , and  $r(y)$ .

$$f(x) = \int_0^x \frac{1}{\sqrt{1-m\sin^2 t}} dt$$

$$f'(x) = \frac{1}{\sqrt{1-m\sin^2 x}} > 0$$

(first fundamental theorem of calculus)

$f(x)$  is continuously differentiable with nonzero derivative everywhere on  $\mathbb{R}$ , thus  $f^{-1}(y)$  is continuously differentiable on  $\mathbb{R}$  by the inverse function theorem.

$$p(y) = \sin(f^{-1}(y))$$

$$p'(y) = (\cos x)(f^{-1}(y))'$$

(chain rule)

$$= \frac{\cos x}{f'(x)}$$

(inverse function theorem)

$$= (\cos x) \left( \sqrt{1-m\sin^2 x} \right)$$

$$= q(y)r(y)$$

$$q(y) = \cos(f^{-1}(y))$$

$$q'(y) = (-\sin x)(f^{-1}(y))'$$

(chain rule)

$$= -\frac{\sin x}{f'(x)}$$

(inverse function theorem)

$$= -(\sin x) \left( \sqrt{1-m\sin^2 x} \right)$$

$$= -p(y)r(y)$$

$$r(y) = \sqrt{1-m\sin^2(f^{-1}(y))}$$

$$r'(y) = \frac{-2m\sin x \cos x (f^{-1}(y))'}{2\sqrt{1-m\sin^2 x}}$$

(chain rule)

$$r'(y) = \frac{-2m\sin x \cos x}{2\sqrt{1-m\sin^2 x} f'(x)}$$

(inverse function theorem)

$$r'(y) = -m\sin x \cos x$$

$$= -mp(y)q(y)$$

## 12

### 12.a

Let  $x$  be a fixed non-negative number with  $x \neq 1$ . Evaluate the integral

$$\int_0^\pi \frac{\sin t}{\sqrt{1-2x\cos t+x^2}} dt$$

in terms of  $x$ .

$$\begin{aligned} x &\neq 0 \\ \int_0^\pi \frac{\sin t}{\sqrt{1-2x\cos t+x^2}} dt &= -\int_1^{-1} \frac{1}{\sqrt{1-2xt+x^2}} dt && \text{(change of variable } \cos t \mapsto t) \\ &= \int_{-1}^1 \frac{1}{\sqrt{1+x^2-2xt}} dt \\ &= -\frac{1}{2x} \int_{1+2x+x^2}^{1-2x+x^2} \frac{1}{\sqrt{t}} dt && \text{(change of variable } 1+x^2-2xt \mapsto t, x \neq 0) \\ &= \frac{1}{x} \left[ \sqrt{t} \right]_{1-2x+x^2}^{1+2x+x^2} \\ &= \frac{1}{x} \left[ \sqrt{t} \right]_{(x-1)^2}^{(x+1)^2} \\ &= \frac{|x+1| - |x-1|}{x} \\ &= \begin{cases} \frac{(x+1)-(x-1)}{x} & x > 1 \\ \frac{(x+1)-(1-x)}{x} & 0 < x < 1 \end{cases} \\ &= \begin{cases} \frac{2}{x} & x > 1 \\ 2 & 0 < x < 1 \end{cases} \end{aligned}$$

$$\begin{aligned} x &= 0 \\ \int_0^\pi \frac{\sin t}{\sqrt{1-2x\cos t+x^2}} dt &= \int_0^\pi \sin t dt \\ &= [-\cos t]_0^\pi \\ &= 2 \end{aligned}$$

$$\int_0^\pi \frac{\sin t}{\sqrt{1-2x\cos t+x^2}} dt = \begin{cases} \frac{2}{x} & x > 1 \\ 2 & 0 \leq x < 1 \end{cases}$$

### 12.b

Let  $f: [0, +\infty) \rightarrow \mathbb{R}$  be the function defined by

$$f(x) = \begin{cases} \int_0^\pi \frac{\sin t}{\sqrt{1-2x \cos t + x^2}} dt & \text{if } x \neq 1 \\ a & \text{if } x = 1 \end{cases}$$

.

Using the result from [12.a](#), find the value of  $a$  so that  $f$  is a continuous function. Hence sketch the graph of  $f$ .

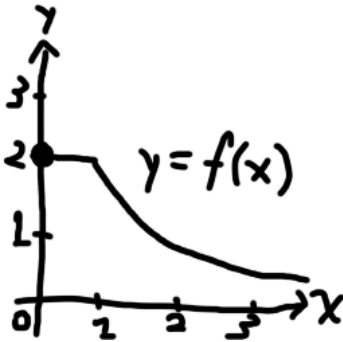
Using the definition in 12.a,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 2 = 2$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{2}{x} = 2$$

$$\therefore \lim_{x \rightarrow 1} f(x) = 2$$

$$\therefore \text{To make } f \text{ continuous, } a = \lim_{x \rightarrow 1} f(x) = 2.$$



13

13.a

Using the substitution  $u = \frac{1}{x}$ , show that

$$\int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2} dx = 0$$

.

$$\begin{aligned} \int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2} dx &= - \int_2^{\frac{1}{2}} \frac{\ln x}{1+x^{-2}} du && \text{(change of variable)} \\ &= \int_2^{\frac{1}{2}} \frac{\ln u}{1+u^2} du \\ &= \int_2^{\frac{1}{2}} \frac{\ln x}{1+x^2} dx && \text{(rename dummy variable)} \\ &= - \int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2} dx \\ 2 \int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2} dx &= 0 \\ \int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2} dx &= 0 \end{aligned}$$

13.b

Using [13.a](#) or otherwise, evaluate the limit

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^{3n} \frac{1}{2n} \frac{\ln \left( 2 \left( \frac{1}{2} + \frac{k}{2n} \right) \right)}{1 + \left( \frac{1}{2} + \frac{k}{2n} \right)^2}$$

.



$$\begin{aligned}
\lim_{n \rightarrow +\infty} \sum_{k=1}^{3n} \frac{1}{2n} \frac{\ln \left( 2 \left( \frac{1}{2} + \frac{k}{2n} \right) \right)}{1 + \left( \frac{1}{2} + \frac{k}{2n} \right)^2} &= \lim_{n \rightarrow +\infty} \sum_{k=1}^{3n} \frac{\frac{3}{2}}{3n} \frac{\ln \left( 2 \left( \frac{1}{2} + \frac{k}{2n} \right) \right)}{1 + \left( \frac{1}{2} + \frac{k}{2n} \right)^2} \\
&= \int_{\frac{1}{2}}^2 \frac{\ln 2x}{1+x^2} dx && \left( \text{the integrand is bounded and continuous on } \left[ \frac{1}{2}, 2 \right], \text{ so it is integrable} \right) \\
&= \int_{\frac{1}{2}}^2 \frac{\ln 2 + \ln x}{1+x^2} dx \\
&= (\ln 2) \int_{\frac{1}{2}}^2 \frac{1}{1+x^2} dx + \int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2} dx \\
&= (\ln 2) [\arctan x]_{\frac{1}{2}}^2 + 0 \\
&= (\ln 2) \left( \arctan 2 - \arctan \frac{1}{2} \right) \\
&= (\ln 2) \left( \arctan 2 - \left( \frac{\pi}{2} - \arctan 2 \right) \right) \\
&= (\ln 2) \left( 2 \arctan 2 - \frac{\pi}{2} \right)
\end{aligned}$$

## 16

### 16.a

Let  $f: [0, \pi] \rightarrow \mathbb{R}$  be a continuous function such that

$$f(\pi - x) = -f(x)$$

for every  $x \in [0, \pi]$ .

Using the substitution  $u = \pi - x$ , show that

$$\int_0^\pi f(x) \ln(1 + e^{\cos x}) dx = \frac{1}{2} \int_0^\pi f(x) \cos x dx$$

.

$$\begin{aligned}
\int_0^\pi f(x) \ln(1 + e^{\cos x}) dx &= - \int_\pi^0 f(\pi - u) \ln(1 + e^{\cos(\pi - u)}) du && \text{(change of variable)} \\
&= - \int_0^\pi f(u) \ln(1 + e^{-\cos u}) du \\
&= - \int_0^\pi f(x) \ln(1 + e^{-\cos x}) dx && \text{(rename dummy variable)} \\
2 \int_0^\pi f(x) \ln(1 + e^{\cos x}) dx &= \int_0^\pi f(x) \ln(1 + e^{\cos x}) dx - \int_0^\pi f(x) \ln(1 + e^{-\cos x}) dx \\
&= \int_0^\pi f(x) (\ln(1 + e^{\cos x}) - \ln(1 + e^{-\cos x})) dx \\
&= \int_0^\pi f(x) \ln \left( \frac{1 + e^{\cos x}}{e^{-\cos x} + 1} \right) dx \\
&= \int_0^\pi f(x) \ln(e^{\cos x}) dx \\
&= \int_0^\pi f(x) \cos x dx \\
\int_0^\pi f(x) \ln(1 + e^{\cos x}) dx &= \frac{1}{2} \int_0^\pi f(x) \cos x dx
\end{aligned}$$

### 16.b

Compute the derivative of the function  $g: [0, \pi] \rightarrow \mathbb{R}$  defined by

$$g(x) = \frac{\cos x}{1 + \sin x}$$

.

Using this together with the result from [16.a](#), evaluate the integral

$$\int_0^\pi \frac{(\cos x) \ln(1 + e^{\cos x})}{(1 + \sin x)^2} dx$$

.

$$\begin{aligned}
g(x) &= \frac{\cos x}{1 + \sin x} \\
g'(x) &= \frac{-(\sin x)(1 + \sin x) - \cos^2 x}{(1 + \sin x)^2} && \text{(quotient rule)} \\
&= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} \\
&= -\frac{1 + \sin x}{(1 + \sin x)^2} \\
&= -\frac{1}{1 + \sin x} \\
g''(x) &= -\frac{-\cos x}{(1 + \sin x)^2} && \text{(quotient rule)} \\
&= \frac{\cos x}{(1 + \sin x)^2}
\end{aligned}$$

$$\begin{aligned}
f &: [0, \pi] \rightarrow \mathbb{R} \\
f(x) &:= \frac{\cos x}{(1 + \sin x)^2} \\
f(x) &\text{ is continuous on } [0, \pi]. \\
f(\pi - x) &= \frac{\cos(\pi - x)}{(1 + \sin(\pi - x))^2} \\
&= -\frac{\cos x}{(1 + \sin x)^2} \\
&= -f(x)
\end{aligned}$$

$$\begin{aligned}
\int_0^\pi \frac{(\cos x) \ln(1 + e^{\cos x})}{(1 + \sin x)^2} dx &= \int_0^\pi f(x) \ln(1 + e^{\cos x}) dx \\
&= \frac{1}{2} \int_0^\pi f(x) \cos x dx && (16.a) \\
&= \frac{1}{2} \int_0^\pi \frac{\cos^2 x}{(1 + \sin x)^2} dx \\
&= \frac{1}{2} \int_0^\pi g''(x) \cos x dx \\
&= \frac{1}{2} \int_{x=0}^{x=\pi} \cos x dg'(x) \\
&= \frac{1}{2} \left( [g'(x) \cos x]_0^\pi - \int_{x=0}^{x=\pi} g'(x) d(\cos x) \right) && \text{(integration by parts)} \\
&= \frac{1}{2} \left( -g'(\pi) - g'(0) + \int_0^\pi \frac{\sin x}{1 + \sin x} dx \right) \\
&= \frac{1}{2} \left( 2 + \int_0^\pi g'(x) \sin x dx \right) \\
&= \frac{1}{2} \left( 2 + \int_{x=0}^{x=\pi} \sin x dg(x) \right) \\
&= \frac{1}{2} \left( 2 + [g'(x) \sin x]_0^\pi - \int_{x=0}^{x=\pi} g(x) d(\sin x) \right) && \text{(integration by parts)} \\
&= \frac{1}{2} \left( 2 - \int_0^\pi g(x) \cos x dx \right) \\
&= \frac{1}{2} \left( 2 - \int_0^\pi \frac{\cos^2 x}{1 + \sin x} dx \right) \\
&= \frac{1}{2} \left( 2 - \int_0^\pi \frac{(1 - \sin x)(1 + \sin x)}{1 + \sin x} dx \right) \\
&= \frac{1}{2} \left( 2 - \int_0^\pi (1 - \sin x) dx \right) \\
&= \frac{1}{2} (2 - [x + \cos x]_0^\pi) \\
&= \frac{1}{2} (2 - \pi + 1 + 0 + 1) \\
&= 2 - \frac{\pi}{2}
\end{aligned}$$