ELEC 4110 Homework Fall 2025: Homework 1 with Solutions

1 LTI Systems and Fourier Transform

1. (10 pts) Compute the Fourier transform of the following signals.

   (a) (5 pts) $x_1(t) = e^{-2|t-1|}$.
   We want to find the Fourier transform of $x_1(t) = e^{-2|t-1|}$.
   Recall the Fourier transform of $e^{-a|t|}$,

   $$\mathcal{F}\{e^{-a|t|}\} = \frac{2a}{a^2 + \omega^2}$$

   For $a = 2$, we have:

   $$\mathcal{F}\{e^{-2|t|}\} = \frac{4}{4 + \omega^2}$$

   Apply the time-shifting property of the Fourier transform,

   $$\mathcal{F}\{f(t - t_0)\} = e^{-j\omega t_0} F(j\omega)$$

   Therefore,

   $$\mathcal{F}\{e^{-2|t-1|}\} = e^{-j\omega \cdot 1} \cdot \frac{4}{4 + \omega^2} = \frac{4e^{-j\omega}}{4 + \omega^2}$$

   (b) (5 pts) $x_2(t) = \frac{\sin(\pi t)}{\pi t} \cdot \cos(4\pi t)$.
   Solution: The Fourier transform of $\frac{\sin(\pi t)}{\pi t}$ is:

   $$\mathcal{F}\left[\frac{\sin(\pi t)}{\pi t}\right] = G(j\omega) = \begin{cases} 1 & |\omega| < \pi \\ 0 & |\omega| > \pi \end{cases}$$

   Now, $x_2(t) = g(t)\cos(4\pi t)$, and $\cos(4\pi t) = \frac{e^{j4\pi t} + e^{-j4\pi t}}{2}$. Using the frequency shift property,

   $$X_2(j\omega) = \frac{1}{2}\left[G(j(\omega - 4\pi)) + G(j(\omega + 4\pi))\right]$$

   Since $G(j\omega)$ is non-zero only for $|\omega| < \pi$, $G(j(\omega - 4\pi))$ is non-zero for $|\omega - 4\pi| < \pi$, i.e., $3\pi < \omega < 5\pi$, and $G(j(\omega + 4\pi))$ is non-zero for $|\omega + 4\pi| < \pi$, i.e., $-5\pi < \omega < -3\pi$. Therefore,

   $$X_2(j\omega) = \begin{cases} \frac{1}{2}, & \text{for } |\omega| \in [3\pi, 5\pi] \\ 0, & \text{otherwise} \end{cases}$$

2. (20 pts) Consider a LTI system described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + x(t).$$

(a) (5 pts) Find the frequency response of this system.

Solution: Take the Fourier transform of both sides with zero initial conditions:

$$(j\omega)^2Y(j\omega) + 4j\omega Y(j\omega) + 4Y(j\omega) = j\omega X(j\omega) + X(j\omega)$$

$$[(j\omega)^2 + 4j\omega + 4]Y(j\omega) = [j\omega + 1]X(j\omega)$$

Thus, the frequency response is:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 1}{(j\omega)^2 + 4j\omega + 4} = \frac{j\omega + 1}{(j\omega + 2)^2}$$

(b) (5 pts) Find the impulse response of this system.

Solution: Since

$$H(s) = \frac{1}{j\omega + 2} - \frac{1}{(j\omega + 2)^2},$$

The inverse Fourier transform is

$$h(t) = \mathcal{F}^{-1}\left[\frac{1}{j\omega + 2}\right] - \mathcal{F}^{-1}\left[\frac{1}{(j\omega + 2)^2}\right] = e^{-2t}u(t) - te^{-2t}u(t) = (1-t)e^{-2t}u(t)$$

(c) (10 pts) For input $x(t) = e^{-2t}u(t)$, find the output $y(t)$.

Solution: First, find the Fourier transform of $x(t)$:

$$X(j\omega) = \mathcal{F}[e^{-2t}u(t)] = \frac{1}{2 + j\omega}$$

Since

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{j\omega + 1}{(j\omega + 2)^2} \cdot \frac{1}{2 + j\omega} = \frac{j\omega + 1}{(j\omega + 2)^3},$$

Thus,

$$Y(s) = \frac{1}{(j\omega + 2)^2} - \frac{1}{(j\omega + 2)^3}$$

Take inverse Fourier transform

$$y(t) = \mathcal{F}^{-1}\left[\frac{1}{(s + 2)^2}\right] - \mathcal{F}^{-1}\left[\frac{1}{(s + 2)^3}\right] = te^{-2t}u(t) - \frac{t^2}{2}e^{-2t}u(t) = e^{-2t}\left(t - \frac{t^2}{2}\right)u(t)$$

2 Random Variables and Random Processes

2. (15 pts) Two random variables are distributed by

$$f_{X,Y}(x, y) = \begin{cases} Kxe^{-y}, & \text{for } 0 \le x \le 1, x \le y < \infty \\ 0, & \text{otherwise} \end{cases}$$

(a) (5 pts) Find the constant K.

Solution: Integrate over the domain:

$$\int_0^1 \int_x^\infty Kxe^{-y}\,dy\,dx = 1$$

First, integrate with respect to y,

$$\int_x^\infty e^{-y}\,dy = \left[-e^{-y}\right]_x^\infty = e^{-x}$$

So

$$\int_0^1 Kxe^{-x}\,dx = K\left[-xe^{-x} - e^{-x}\right]_0^1 = K[(-1\cdot e^{-1} - e^{-1}) - (0-1)] = K[1-2e^{-1}]$$

Thus,

$$K(1-2e^{-1}) = 1 \Rightarrow K = \frac{1}{1-2/e} = \frac{e}{e-2}$$

(b) (5 pts) Find the marginal density functions $f_X(x)$ and $f_Y(y)$.

Solution: For $f_X(x)$:

$$f_X(x) = \int_x^\infty f_{X,Y}(x,y)\,dy = \int_x^\infty Kxe^{-y}\,dy = Kxe^{-x}, \quad \text{for } 0 \le x \le 1$$

So

$$f_X(x) = \begin{cases} \frac{e}{e-2}xe^{-x}, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

For $f_Y(y)$:
If $y < 0$, $f_Y(y) = 0$.
If $0 \le y \le 1$,

$$f_Y(y) = \int_0^y Kxe^{-y}\,dx = Ke^{-y}\int_0^y x\,dx = Ke^{-y}\frac{y^2}{2}$$

If $y > 1$,

$$f_Y(y) = \int_0^1 Kxe^{-y}\,dx = Ke^{-y}\int_0^1 x\,dx = \frac{1}{2}Ke^{-y}$$

Thus,

$$f_Y(y) = \begin{cases} \frac{e}{e-2}e^{-y}\frac{y^2}{2}, & 0 \le y \le 1 \\ \frac{e}{2(e-2)}e^{-y}, & y > 1 \\ 0, & \text{otherwise} \end{cases}$$

(c) (5 pts) Compute $P(X > 0.5 \mid Y = 2)$.

Solution: For $y = 2 > 1$,

$$f_Y(2) = \frac{1}{2}Ke^{-2} = \frac{e}{2(e-2)}e^{-2}$$

Therefore,

$$f_{X|Y}(x|2) = \frac{f_{X,Y}(x,2)}{f_Y(2)} = \frac{Kxe^{-2}}{\frac{1}{2}Ke^{-2}} = 2x, \quad \text{for } 0 \le x \le 1$$

Then,

$$P(X > 0.5 \mid Y = 2) = \int_{0.5}^{1} f_{X|Y}(x|2)\,dx = \int_{0.5}^{1} 2x\,dx = \left[x^2\right]_{0.5}^{1} = 1 - 0.25 = 0.75$$

3. (15 pts) Define the random process

$$Z(t) = X\cos(\omega_0 t) + Y\sin(\omega_0 t),$$

where X and Y are independent Gaussian random variables with mean 0 and variance σ^2.

(a) (5 pts) Find $m_Z(t)$.
Solution:

$$E[Z(t)] = E[X\cos(\omega_0 t) + Y\sin(\omega_0 t)] = \cos(\omega_0 t)E[X] + \sin(\omega_0 t)E[Y] = 0$$

So $m_Z(t) = 0$.

(b) (5 pts) Compute the autocorrelation $R_Z(t_1, t_2)$.
Solution:

$$R_Z(t_1, t_2) = E[Z(t_1)Z(t_2)]$$

$$\begin{aligned} Z(t_1)Z(t_2) &= [X\cos(\omega_0 t_1) + Y\sin(\omega_0 t_1)][X\cos(\omega_0 t_2) + Y\sin(\omega_0 t_2)] \\ &= X^2\cos(\omega_0 t_1)\cos(\omega_0 t_2) + XY\cos(\omega_0 t_1)\sin(\omega_0 t_2) \\ &\quad + YX\sin(\omega_0 t_1)\cos(\omega_0 t_2) + Y^2\sin(\omega_0 t_1)\sin(\omega_0 t_2) \end{aligned}$$

Thus,

$$R_Z(t_1, t_2) = \sigma^2\cos(\omega_0 t_1)\cos(\omega_0 t_2) + \sigma^2\sin(\omega_0 t_1)\sin(\omega_0 t_2) = \sigma^2\cos(\omega_0(t_1 - t_2))$$

(c) (5 pts) Determine whether $Z(t)$ is wide-sense stationary.
Solution: Yes. Mean is constant (0) and autocorrelation depends only on the time difference $t_1 - t_2$.

3 Digital Communication

4. (20 pts) **Digital Modulation and Optimum Receiver.** A binary communication system transmits signals

$$s_0(t) = A, \quad s_1(t) = -A, \quad 0 \le t \le T.$$

Bits "0" and "1" are equally likely. The channel is AWGN with two-sided power spectral density $N_0/2$.

(a) (10 pts) Derive the minimum bit error probability P_e.

Solution: The distance between signals $d = \|s_0 - s_1\| = 2A\sqrt{T}$. The error probability is

$$P_e = Q\left(\sqrt{\frac{d^2}{2N_0}}\right) = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right).$$

(b) (5 pts) Determine the impulse response $h(t)$ of the optimal matched filter.

Solution: The optimal matched filter for binary detection is given by $h(t) = s_1(T-t) - s_0(T-t)$. Given $s_0(t) = A$ and $s_1(t) = -A$ for $0 \le t \le T$, we have:

$$s_0(T-t) = A \quad \text{for } 0 \le t \le T,$$

$$s_1(T-t) = -A \quad \text{for } 0 \le t \le T.$$

Thus,

$$h(t) = s_1(T-t) - s_0(T-t) = -A - A = -2A \quad \text{for } 0 \le t \le T,$$

and $h(t) = 0$ otherwise.

(c) (5 pts) Specify the optimal sampling time and decision threshold.

Solution: The optimal sampling time is at $t = T$. The decision threshold is 0.

5. (20 pts) **Signal Space Representation.** Consider four signals $s_1(t), s_2(t), s_3(t), x(t)$ defined on $0 \le t \le 3$, each piecewise constant on intervals $[0,1)$, $[1,2)$, $[2,3]$:

$$\begin{aligned} s_1(t) &: \ [1,\ 1,\ 1], \\ s_2(t) &: \ [1,\ -1,\ -1], \\ s_3(t) &: \ [1,\ 1,\ -1], \\ x(t) &: \ [1,\ -1,\ 1]. \end{aligned}$$

(a) (10 pts) Apply Gram–Schmidt to obtain an orthonormal basis $\{\varphi_i(t)\}$ with $s_1(t), s_2(t), s_3(t)$.

Solution: Step 1: $u_1 = s_1 = [1,1,1]^T$, $\|u_1\| = \sqrt{3}$, so

$$\phi_1 = \frac{u_1}{\|u_1\|} = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^T$$

Step 2: Project s_2 onto u_1,

$$\text{proj}_{u_1} s_2 = \frac{s_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \frac{[1,-1,-1] \cdot [1,1,1]}{3}[1,1,1] = \frac{-1}{3}[1,1,1] = \left[-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right]$$

$$u_2 = s_2 - \text{proj}_{u_1} s_2 = [1,-1,-1] - \left[-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right] = \left[\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right]$$

$$\|u_2\| = \sqrt{\left(\frac{4}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{16}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{\frac{24}{9}} = \frac{2\sqrt{6}}{3}$$

$$\phi_2 = \frac{u_2}{\|u_2\|} = \frac{[4/3, -2/3, -2/3]}{2\sqrt{6}/3} = \left[\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right]^T$$

Step 3: Project s_3 onto u_1 and u_2,

$$\text{proj}_{u_1} s_3 = \frac{s_3 \cdot u_1}{u_1 \cdot u_1} u_1 = \frac{[1,1,-1]\cdot[1,1,1]}{3}[1,1,1] = \frac{1}{3}[1,1,1] = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$$

$$\text{proj}_{u_2} s_3 = \frac{s_3 \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{[1,1,-1]\cdot[4/3,-2/3,-2/3]}{8/3} u_2 = \frac{4/3}{8/3} u_2 = \frac{1}{2} u_2 = \left[\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right]$$

$$u_3 = s_3 - \text{proj}_{u_1} s_3 - \text{proj}_{u_2} s_3 = [1,1,-1] - \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right] - \left[\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right] = [0, 1, -1]$$

$$\|u_3\| = \sqrt{0^2 + 1^2 + (-1)^2} = \sqrt{2}$$

$$\phi_3 = \frac{u_3}{\|u_3\|} = \left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]^T$$

Thus, the orthonormal basis is

$$\phi_1 = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^T, \quad \phi_2 = \left[\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right]^T, \quad \phi_3 = \left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]^T$$

(b) (10 pts) Express $x(t)$ as a coordinate vector in the obtained basis $\{\varphi_i(t)\}$.
Solution: The coordinates are the projections onto the basis vectors.

$$\alpha_1 = x \cdot \phi_1 = [1,-1,1] \cdot \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right] = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\alpha_2 = x \cdot \phi_2 = [1,-1,1] \cdot \left[\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right] = \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} = \frac{2}{\sqrt{6}}$$

$$\alpha_3 = x \cdot \phi_3 = [1,-1,1] \cdot \left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right] = 0 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

So the coordinate vector is

$$\left[\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{6}}, -\sqrt{2}\right] = \left[\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}, -\sqrt{2}\right]$$