

HKUST MATH 1014 L1 assignment 5 submission

MATH1014 Calculus II Problem Set 5

L01 (Spring 2024)

Problem Set 5

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 5 covers materials from §7.1 – §7.6.

Q5

Let \mathbf{u} and \mathbf{v} be two vectors in \mathbb{R}^3 such that $\|\mathbf{u}\| = 4$, $\|\mathbf{v}\| = 3$, and the angle between \mathbf{u} and \mathbf{v} is $\frac{\pi}{3}$.

Q5.a

Find $\mathbf{u} \cdot \mathbf{v}$.

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \|\mathbf{v}\| \cos \frac{\pi}{3} \\ &= 4 \cdot 3 \cdot \frac{1}{2} \\ &= 6\end{aligned}$$

Q5.b

Find the real number k such that the vectors $\mathbf{u} + k\mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$ are orthogonal.

Set the dot product of the vectors to 0.

$$\begin{aligned}(\mathbf{u} + k\mathbf{v}) \cdot (\mathbf{u} - 2\mathbf{v}) &= 0 \\ \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + k\mathbf{v} \cdot \mathbf{u} - 2k\mathbf{v} \cdot \mathbf{v} &= 0 \\ \|\mathbf{u}\|^2 + (k - 2)\mathbf{u} \cdot \mathbf{v} - 2k\|\mathbf{v}\|^2 &= 0 \\ 4^2 + (k - 2)6 - 2k \cdot 3^2 &= 0 \\ 16 + 6k - 12 - 18k &= 0 \\ -12k &= -4 \\ k &= \frac{1}{3}\end{aligned}$$

Q5.c

Let $\mathbf{a} = 3\mathbf{u} + 4\mathbf{v}$ and $\mathbf{b} = -2\mathbf{u} - \mathbf{v}$. Find the area of the parallelogram with \mathbf{a} and \mathbf{b} as two adjacent edges.

$$\begin{aligned}
& \|\mathbf{a}\|^2 \\
&= \mathbf{a} \cdot \mathbf{a} \\
&= (3\mathbf{u} + 4\mathbf{v}) \cdot (3\mathbf{u} + 4\mathbf{v}) \\
&= 9\|\mathbf{u}\|^2 + 24\mathbf{u} \cdot \mathbf{v} + 16\|\mathbf{v}\|^2 \\
&= 9 \cdot 4^2 + 24 \cdot 6 + 16 \cdot 3^2 \\
&= 144 + 144 + 144 \\
&= 432 \\
& \|\mathbf{b}\|^2 \\
&= \mathbf{b} \cdot \mathbf{b} \\
&= (-2\mathbf{u} - \mathbf{v}) \cdot (-2\mathbf{u} - \mathbf{v}) \\
&= 4\|\mathbf{u}\|^2 + 4\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \\
&= 4 \cdot 4^2 + 4 \cdot 6 + 3^2 \\
&= 64 + 24 + 9 \\
&= 97 \\
& \mathbf{a} \cdot \mathbf{b} \\
&= (3\mathbf{u} + 4\mathbf{v}) \cdot (-2\mathbf{u} - \mathbf{v}) \\
&= -6\mathbf{u} \cdot \mathbf{u} - 3\mathbf{u} \cdot \mathbf{v} - 8\mathbf{v} \cdot \mathbf{u} - 4\mathbf{v} \cdot \mathbf{v} \\
&= -6\|\mathbf{u}\|^2 - 11\mathbf{u} \cdot \mathbf{v} - 4\|\mathbf{v}\|^2 \\
&= -6 \cdot 4^2 - 11 \cdot 6 - 4 \cdot 3^2 \\
&= -96 - 66 - 36 \\
&= -198
\end{aligned}$$

Let θ be the angle between \mathbf{a} and \mathbf{b} .

$$\begin{aligned}
& \text{area} \\
&= \|\mathbf{a} \times \mathbf{b}\| \\
&= \sqrt{\|\mathbf{a} \times \mathbf{b}\|^2} \\
&= \sqrt{\|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \sin^2 \theta} \\
&= \sqrt{\|\mathbf{a}\|^2 \|\mathbf{b}\|^2 (1 - \cos^2 \theta)} \\
&= \sqrt{\|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2} \\
&= \sqrt{432 \cdot 97 - 198^2} \\
&= \sqrt{41904 - 39204} \\
&= \sqrt{2700} \\
&= 30\sqrt{3}
\end{aligned}$$

Q9

Let \mathbf{u} and \mathbf{v} be non-zero vectors of the same dimension. Show that the vector

$$\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$$

bisects the angle between \mathbf{u} and \mathbf{v} .

Let $\theta \in [0, \pi]$ be the angle between \mathbf{u} and \mathbf{w} .

$$\mathbf{u} \cdot \mathbf{w} = \|\mathbf{u}\| \|\mathbf{w}\| \cos \theta$$

$$\|\mathbf{u}\| \mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\| \|\mathbf{u}\|^2 = \|\mathbf{u}\| \|\mathbf{w}\| \cos \theta$$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\| \|\mathbf{v}\|}{\|\mathbf{w}\|}$$

Let $\phi \in [0, \pi]$ be the angle between \mathbf{v} and \mathbf{w} .

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \phi$$

$$\|\mathbf{u}\| \|\mathbf{v}\|^2 + \|\mathbf{v}\| \mathbf{u} \cdot \mathbf{v} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \phi$$

$$\cos \phi = \frac{\|\mathbf{u}\| \|\mathbf{v}\| + \mathbf{u} \cdot \mathbf{v}}{\|\mathbf{w}\|}$$

$$= \cos \theta$$

$$\phi = \theta$$

($\cos(*)$ is injective on $[0, \pi]$)

$\therefore \mathbf{w}$ bisects the angle between \mathbf{u} and \mathbf{v} .

Q11

Let \mathbf{a} and \mathbf{b} be vectors in \mathbb{R}^n such that

$$\mathbf{a} \cdot \mathbf{a} = 1, \mathbf{b} \cdot \mathbf{b} = 1 \text{ and } \mathbf{a} \cdot \mathbf{b} = 0$$

.

Let $S = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u} = x\mathbf{a} + y\mathbf{b} \text{ for some } x, y \in \mathbb{R}\}$.

Q11.a

Show that for every $\mathbf{u} \in S$, we have

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{a})\mathbf{a} + (\mathbf{u} \cdot \mathbf{b})\mathbf{b}$$

.

$$\begin{aligned} & (\mathbf{u} \cdot \mathbf{a})\mathbf{a} + (\mathbf{u} \cdot \mathbf{b})\mathbf{b} \\ &= ((x\mathbf{a} + y\mathbf{b}) \cdot \mathbf{a})\mathbf{a} + ((x\mathbf{a} + y\mathbf{b}) \cdot \mathbf{b})\mathbf{b} \\ &= (x\mathbf{a} \cdot \mathbf{a} + y\mathbf{b} \cdot \mathbf{a})\mathbf{a} + (x\mathbf{a} \cdot \mathbf{b} + y\mathbf{b} \cdot \mathbf{b})\mathbf{b} \\ &= (x + 0)\mathbf{a} + (0 + y)\mathbf{b} \\ &= x\mathbf{a} + y\mathbf{b} \\ &= \mathbf{u} \end{aligned}$$

Q11.b

For each $\mathbf{v} \in \mathbb{R}^n$, let $\mathbf{w} = (\mathbf{v} \cdot \mathbf{a})\mathbf{a} + (\mathbf{v} \cdot \mathbf{b})\mathbf{b}$. Show that $\mathbf{v} - \mathbf{w}$ is orthogonal to every $\mathbf{u} \in S$.

$$\begin{aligned}
& (\mathbf{v} - \mathbf{w}) \cdot \mathbf{u} \\
&= \mathbf{v} \cdot \mathbf{u} - \mathbf{w} \cdot \mathbf{u} \\
&= \mathbf{v} \cdot \mathbf{u} - ((\mathbf{v} \cdot \mathbf{a})\mathbf{a} + (\mathbf{v} \cdot \mathbf{b})\mathbf{b}) \cdot ((\mathbf{u} \cdot \mathbf{a})\mathbf{a} + (\mathbf{u} \cdot \mathbf{b})\mathbf{b}) \\
&= \mathbf{v} \cdot \mathbf{u} - (\mathbf{v} \cdot \mathbf{a})(\mathbf{u} \cdot \mathbf{a})\mathbf{a} \cdot \mathbf{a} - (\mathbf{v} \cdot \mathbf{a})(\mathbf{u} \cdot \mathbf{b})\mathbf{a} \cdot \mathbf{b} \\
&\quad - (\mathbf{v} \cdot \mathbf{b})(\mathbf{u} \cdot \mathbf{a})\mathbf{b} \cdot \mathbf{a} - (\mathbf{v} \cdot \mathbf{b})(\mathbf{u} \cdot \mathbf{b})\mathbf{b} \cdot \mathbf{b} \\
&= \mathbf{v} \cdot \mathbf{u} - (\mathbf{v} \cdot \mathbf{a})(\mathbf{u} \cdot \mathbf{a}) - (\mathbf{v} \cdot \mathbf{b})(\mathbf{u} \cdot \mathbf{b}) \\
&= \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot ((\mathbf{u} \cdot \mathbf{a})\mathbf{a} + (\mathbf{u} \cdot \mathbf{b})\mathbf{b}) \\
&= \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} \\
&= 0 \\
&\quad (\mathbf{v} - \mathbf{w}) \cdot \mathbf{u} = 0 \\
&\implies \mathbf{v} - \mathbf{w} \perp \mathbf{u}
\end{aligned}$$

Q16

Three lines are said to be **concurrent** if they pass through the same point.

Q16.a

A **median** of a triangle is a line that passes through both a vertex of the triangle and the mid-point of the edge opposite the vertex. Prove that the three medians of a triangle are concurrent.

$$n \in \mathbb{Z}_{\geq 2}$$

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ be the position vectors of vertices of a triangle.

$$\mathbf{x} := \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}.$$

$$\mathbf{x}_a := \mathbf{x} - \mathbf{a} = \frac{-2\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}$$

$$\mathbf{x}_b := \mathbf{x} - \mathbf{b} = \frac{\mathbf{a} - 2\mathbf{b} + \mathbf{c}}{3}$$

$$\mathbf{x}_c := \mathbf{x} - \mathbf{c} = \frac{\mathbf{a} + \mathbf{b} - 2\mathbf{c}}{3}$$

$$\begin{aligned} & \frac{3}{2}\mathbf{x}_a \\ &= \frac{-6\mathbf{a} + 3\mathbf{b} + 3\mathbf{c}}{6} \\ &= -\mathbf{a} + \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} \\ &= \frac{\mathbf{b} + \mathbf{c}}{2} - \mathbf{a} \end{aligned}$$

$\Rightarrow \mathbf{x}_a$ is a direction vector of the median starting from \mathbf{a} .

$$\begin{aligned} & \frac{3}{2}\mathbf{x}_b \\ &= \frac{3\mathbf{a} - 6\mathbf{b} + 3\mathbf{c}}{6} \\ &= \frac{1}{2}\mathbf{a} - \mathbf{b} + \frac{1}{2}\mathbf{c} \\ &= \frac{\mathbf{a} + \mathbf{c}}{2} - \mathbf{b} \end{aligned}$$

$\Rightarrow \mathbf{x}_b$ is a direction vector of the median starting from \mathbf{b} .

$$\begin{aligned} & \frac{3}{2}\mathbf{x}_c \\ &= \frac{3\mathbf{a} + 3\mathbf{b} - 6\mathbf{c}}{6} \\ &= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} - \mathbf{c} \\ &= \frac{\mathbf{a} + \mathbf{b}}{2} - \mathbf{c} \end{aligned}$$

$\Rightarrow \mathbf{x}_c$ is a direction vector of the median starting from \mathbf{c} .

$$\therefore \mathbf{a} + \mathbf{x}_a = \mathbf{x}$$

$$\mathbf{b} + \mathbf{x}_b = \mathbf{x}$$

$$\mathbf{c} + \mathbf{x}_c = \mathbf{x}$$

\therefore The three medians are concurrent.

Q16.b

Prove that the three altitudes of a triangle are concurrent.

$$n \in \mathbb{Z}_{\geq 2}$$

Let the origin \mathbf{o} be one of the vertex of the triangle.

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ be the position vectors of the other two vertices of the triangle.

$\mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$ because otherwise two vertices are the same position,

making it no longer a triangle.

\mathbf{a} and \mathbf{b} are linearly independent because otherwise the three vertices are collinear, making it no longer a triangle.

Consider the 2-dimensional linear span S generated by \mathbf{a} and \mathbf{b} .

The triangle is on S as its three vertices are on S .

So its altitudes are on S .

Let \mathbf{A}_* be the direction vector (not necessarily normalized) of the altitude of the triangle starting from $*$.

$$\mathbf{A}_o := \omega \mathbf{a} + \mathbf{b} \quad \omega \in \mathbb{R}$$

$$\mathbf{A}_a := \alpha \mathbf{a} + \mathbf{b} \quad \alpha \in \mathbb{R}$$

$$\mathbf{A}_b := \beta \mathbf{a} + \mathbf{b} \quad \beta \in \mathbb{R}$$

$$\mathbf{A}_o \cdot (\mathbf{b} - \mathbf{a}) = 0$$

$$(\omega \mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) = 0$$

$$(\omega - 1)\mathbf{a} \cdot \mathbf{b} - \omega \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = 0$$

$$\omega = \frac{\mathbf{a} \cdot \mathbf{b} - \|\mathbf{b}\|^2}{\mathbf{a} \cdot \mathbf{b} - \|\mathbf{a}\|^2}$$

$$\mathbf{A}_a \cdot \mathbf{b} = 0$$

$$(\alpha \mathbf{a} + \mathbf{b}) \cdot \mathbf{b} = 0$$

$$\alpha \mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2 = 0$$

$$\alpha = -\frac{\|\mathbf{b}\|^2}{\mathbf{a} \cdot \mathbf{b}}$$

$$\mathbf{A}_b \cdot \mathbf{a} = 0$$

$$(\beta \mathbf{a} + \mathbf{b}) \cdot \mathbf{a} = 0$$

$$\beta \|\mathbf{a}\|^2 + \mathbf{a} \cdot \mathbf{b} = 0$$

$$\beta = -\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}$$

Find the intersection between the two medians starting from \mathbf{a} and \mathbf{b} , which is always possible on a plane, considering the medians are not parallel.

$$\mathbf{a} + x\mathbf{A}_a = \mathbf{b} + y\mathbf{A}_b \quad (x, y \in \mathbb{R})$$

$$\mathbf{a} + x\alpha \mathbf{a} + x\mathbf{b} = \mathbf{b} + y\beta \mathbf{a} + y\mathbf{b}$$

$$\begin{cases} 1 + x\alpha = y\beta \\ x = 1 + y \end{cases}$$

$$1 + \alpha + y\alpha = y\beta$$

$$y = \frac{1 + \alpha}{\beta - \alpha}$$

$$= \frac{\frac{\mathbf{a} \cdot \mathbf{b} - \|\mathbf{b}\|^2}{\mathbf{a} \cdot \mathbf{b}}}{\frac{\|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2}{\|\mathbf{a}\|^2 \mathbf{a} \cdot \mathbf{b}}}$$

$$= \frac{\|\mathbf{a}\|^2 \mathbf{a} \cdot \mathbf{b} - \|\mathbf{a}\|^2 \|\mathbf{b}\|^2}{\|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2}$$

$$\begin{aligned}
\mathbf{b} + y\mathbf{A}_b &= \mathbf{b} + \frac{\|\mathbf{a}\|^2\mathbf{a} \cdot \mathbf{b} - \|\mathbf{a}\|^2\|\mathbf{b}\|^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2} \left(-\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} + \mathbf{b} \right) \\
&= \frac{\|\mathbf{b}\|^2\mathbf{a} \cdot \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2}\mathbf{a} + \frac{\|\mathbf{a}\|^2\mathbf{a} \cdot \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2}\mathbf{b} \\
&= \frac{\|\mathbf{a}\|^2\mathbf{a} \cdot \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2} \left(\frac{\|\mathbf{b}\|^2\mathbf{a} \cdot \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})^2}{\|\mathbf{a}\|^2\mathbf{a} \cdot \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})^2}\mathbf{a} + \mathbf{b} \right) \\
&= \frac{\|\mathbf{a}\|^2\mathbf{a} \cdot \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2} \left(\frac{\|\mathbf{b}\|^2 - \mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2 - \mathbf{a} \cdot \mathbf{b}}\mathbf{a} + \mathbf{b} \right) \\
&= \frac{\|\mathbf{a}\|^2\mathbf{a} \cdot \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2}\mathbf{A}_o
\end{aligned}$$

- \therefore The intersection of the two altitudes starting from \mathbf{a} and \mathbf{b}
 is a direction vector of the altitude starting from \mathbf{o} ,
 \therefore The three altitudes are concurrent.

Q19

Find a vector equation and parametric equations for each of the following lines in \mathbb{R}^3 .

Q19.a

The line passing through $(6, -5, 2)$ and parallel to $\langle 3, 9, -2 \rangle$.

Let $t \in \mathbb{R}$.

$$\begin{aligned}
\text{vector equation} &= \langle 6, -5, 2 \rangle + t\langle 3, 9, -2 \rangle \\
&= \langle 6 + 3t, -5 + 9t, 2 - 2t \rangle
\end{aligned}$$

$$\text{parametric equation} = \begin{cases} x = 6 + 3t \\ y = -5 + 9t \\ z = 2 - 2t \end{cases}$$

Q19.b

The line segment with end-points with end-points $(4, -6, 6)$ and $(2, 3, 1)$.

$$\begin{aligned}
\text{difference vector} &= \langle 4, -6, 6 \rangle - \langle 2, 3, 1 \rangle \\
&= \langle 2, -9, 5 \rangle
\end{aligned}$$

Let $t \in [0, 1]$.

$$\begin{aligned}
\text{vector equation} &= \langle 2, 3, 1 \rangle + t\langle 2, -9, 5 \rangle \\
&= \langle 2 + 2t, 3 - 9t, 1 + 5t \rangle
\end{aligned}$$

$$\text{parametric equation} = \begin{cases} x = 2 + 2t \\ y = 3 - 9t \\ z = 1 + 5t \end{cases}$$

Q19.c

The line passing through $(2, 1, 0)$ and perpendicular to both $\hat{i} + \hat{j}$ and $\hat{j} + \hat{k}$.

$$\begin{aligned}
 \text{direction vector} &= (\hat{i} + \hat{j}) \times (\hat{j} + \hat{k}) \\
 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \\
 &= \hat{i} - \hat{j} + \hat{k} \\
 &= \langle 1, -1, 1 \rangle
 \end{aligned}$$

Let $t \in \mathbb{R}$.

$$\begin{aligned}
 \text{vector equation} &= \langle 2, 1, 0 \rangle + t\langle 1, -1, 1 \rangle \\
 &= \langle 2 + t, 1 - t, t \rangle
 \end{aligned}$$

$$\text{parametric equation} = \begin{cases} x = 2 + t \\ y = 1 - t \\ z = t \end{cases}$$

Q19.d

The line passing through $(0, 1, 2)$ and orthogonally intersecting the line

$$x = 1 + t \text{ and } y = 1 - t \text{ and } z = 2t$$

.

$$\begin{aligned}
 \mathbf{p} &:= \langle 0, 1, 2 \rangle \\
 \mathbf{L}(t) &:= \langle 1 + t, 1 - t, 2t \rangle \quad t \in \mathbb{R}
 \end{aligned}$$

Set...

$$\begin{aligned}
 (\mathbf{L}(t) - \mathbf{p}) \cdot \text{direction vector of } \mathbf{L} &= 0 \\
 \langle 1 + t, -t, 2t - 2 \rangle \cdot \langle 1, -1, 2 \rangle &= 0 \\
 (1 + t)(1) + (-t)(-1) + (2t - 2)(2) &= 0 \\
 6t - 3 &= 0 \\
 t &= 0.5
 \end{aligned}$$

$$\begin{aligned}
 \text{direction vector} &= \mathbf{L}(0.5) - \mathbf{p} \\
 &= \langle 1 + 0.5, 1 - 0.5, 1 \rangle - \langle 0, 1, 2 \rangle \\
 &= \langle 1.5, -0.5, -1 \rangle
 \end{aligned}$$

Let $u \in \mathbb{R}$.

$$\begin{aligned}
 \text{vector equation} &= \langle 0, 1, 2 \rangle + u\langle 1.5, -0.5, -1 \rangle \\
 &= \langle 1.5u, 1 - 0.5u, 2 - u \rangle
 \end{aligned}$$

$$\text{parametric equation} = \begin{cases} x = 1.5u \\ y = 1 - 0.5u \\ z = 2 - u \end{cases}$$

Q22

Q22.a

Let P be a point on a smooth curve $r = f(\theta)$ in \mathbb{R}^2 which is not the origin, and let α be the acute angle between the line OP and the

tangent to the curve at P . Show that

$$\cos \alpha = \frac{|f'(\theta)|}{\sqrt{f(\theta)^2 + f'(\theta)^2}}$$

.

$$\begin{aligned}\mathbf{p} &:= \langle \cos \theta, \sin \theta \rangle \\ \mathbf{p} \cdot \mathbf{p} &= 1\end{aligned}$$

$$\begin{aligned}\mathbf{p}' &= \langle -\sin \theta, \cos \theta \rangle \\ \mathbf{p}' \cdot \mathbf{p}' &= 1\end{aligned}$$

$$\begin{aligned}\mathbf{p} \cdot \mathbf{p}' &= -\sin \theta \cos \theta + \sin \theta \cos \theta \\ &= 0\end{aligned}$$

$$\begin{aligned}\mathbf{P} &= \langle r \cos \theta, r \sin \theta \rangle \\ &= \langle f(\theta) \cos \theta, f(\theta) \sin \theta \rangle \\ &= f(\theta) \langle \cos \theta, \sin \theta \rangle \\ &= f(\theta) \mathbf{p} \\ \mathbf{P} \cdot \mathbf{P} &= f(\theta)^2 \mathbf{p} \cdot \mathbf{p} \\ &= f(\theta)^2\end{aligned}$$

$$\begin{aligned}\mathbf{P}' &= f'(\theta) \mathbf{p} + f(\theta) \mathbf{p}' \\ \mathbf{P}' \cdot \mathbf{P}' &= f'(\theta)^2 + f(\theta)^2\end{aligned}$$

$$\mathbf{P} \cdot \mathbf{P}' = f(\theta) f'(\theta)$$

$$\begin{aligned}\cos \alpha &= \frac{|\mathbf{P} \cdot \mathbf{P}'|}{\|\mathbf{P}\| \|\mathbf{P}'\|} && (\alpha \text{ is acute}) \\ &= \frac{|f(\theta) f'(\theta)|}{|f(\theta)| \sqrt{f(\theta)^2 + f'(\theta)^2}} \\ &= \frac{|f'(\theta)|}{\sqrt{f(\theta)^2 + f'(\theta)^2}}\end{aligned}$$

Q22.b

Using [\(a\)](#), show that at every point P on the curve $r = e^\theta$, the angle between the line OP and the tangent line to the curve at P is always $\pi/4$.

$$\begin{aligned}
 \cos \alpha &= \frac{|f'(\theta)|}{\sqrt{f(\theta)^2 + f'(\theta)^2}} \\
 &= \frac{|e^\theta|}{\sqrt{e^{2\theta} + e^{2\theta}}} \\
 &= \frac{|e^\theta|}{|e^\theta|\sqrt{2}} \\
 &= \frac{1}{\sqrt{2}} \\
 \alpha &= \frac{\pi}{4} \quad (\alpha \text{ is acute})
 \end{aligned}$$

Q22 . c

Let $r = f(\theta)$ be a **smooth** curve such that at every point P on it, the angle between the line OP and the tangent line to the curve at P is always a fixed constant. Show that there exist constants C and k such that $f(\theta) = Ce^{k\theta}$ for all θ .

$$\begin{aligned}
 \text{Let } f(\theta) &= Ce^{k\theta}. \\
 \cos \alpha &= \frac{|Cke^{k\theta}|}{\sqrt{e^{2k\theta} + C^2k^2e^{2k\theta}}} \\
 &= \frac{|Ck||e^{k\theta}|}{|e^{k\theta}|\sqrt{1 + C^2k^2}} \\
 &= \frac{|Ck|}{\sqrt{1 + C^2k^2}}
 \end{aligned}$$

The above shows that $\cos \alpha$ does not depend on θ .

So the angle is constant when changing θ .

When $C = 1$, set $k = 0$ to get $\cos \alpha = 0$,
and as $k \rightarrow +\infty$, $\cos \alpha \rightarrow 1$.

By the intermediate value theorem,

C, k can be set arbitrarily to get any real number in $[0, 1)$.

$\cos \alpha = 1$ can be ignored because the tangent is parallel to OP ,
which is impossible if $f(\theta)$ is a function.