COMP 2711H - Honors Discrete Mathematics
Midterm Exam

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The HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed if students are found to have violated the regulations governing academic integrity and honesty.

Declaration of Academic Integrity

I confirm that I have answered the questions using only materials specifically approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination.

Name: Student ID: Signature:

Instructions

1. Write your name and student ID in the fields above and sign the declaration of academic integrity. You should also write your name and student ID on top of **every page** of this answer book.
2. Place your student card on the desk next to you.
3. This exam takes ***3 hours*** and consists of four problems. You should *choose and answer* ***three*** *out of the four problems.* Do ***not*** answer all four problems. At the end of the exam, mark your chosen problems below. Only the chosen problems will be graded. You should answer all parts of each chosen problem.

Problem 1	Problem 2	Problem 3	Problem 4

4. This exam accounts for 30% of your total grade, 10% per problem.
5. All answers should be written in this answer book. Do ***not*** use the same page for more than one problem. Do ***not*** use the problem pages for your solutions. You can use these pages for drafting. You will ***not*** be provided with any extra paper. Do ***not*** use pencils. The answer sheets will be scanned before being graded. You will lose marks if your answer is not readable in the scanned version, e.g. due to the use of a pencil, bad handwriting, untidiness, or writing too close to the edges of the paper.
6. Answers without proofs will get zero points. You have to always prove the correctness of your solutions.
7. If you cannot solve a problem or task completely, write your best attempt. You will very likely get partial credit for it.
8. This is a closed-book exam. You are not allowed to use any notes or electronic devices.
9. If your solution contains a *logical error or fallacy*, you will get zero points, no matter how close you are to a correct answer. This rule applies even if disregarding the logical error leads to a 100% correct solution.
10. You are not allowed to leave the examination room during the first 30 minutes and last 30 minutes of the exam.

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Problem 1. Amir is walking on a directed graph $G = (V, E)$. He starts at a vertex $v_0 \in V$ and keeps walking until he is satisfied that he has seen enough of G. At every timeslot t, if he is in vertex $v \in V$ and v has outgoing edges to $u_1, u_2, \dots, u_k$, Amir chooses one of these outgoing edges and ends up at the corresponding u_i at time $t + 1$. Amir is very curious about his graph and wants to ensure that he sees every part of the graph. Thus, he decides on the following strategy for his walk:

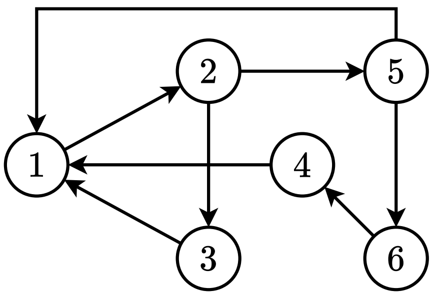
- When choosing between the edges $(v, u_1), (v, u_2), \dots, (v, u_k)$, Amir always takes the least-recently-used (LRU) edge, i.e. an edge that was previously taken as far before in the past as possible. If there are several such edges, e.g. if there are several available edges that were never taken before, Amir chooses one of them randomly.

For example, in the graph G below, with $v_0 = 1$, Amir may take a walk starting with

$$\langle 1, 2, 3, 1, 2, 5, 6, 4, 1 \rangle,$$

but will not take a walk starting with

$$\langle 1, 2, 3, 1, 2, 3 \rangle.$$



i. (2 points) Suppose Amir's goal is to see every vertex of G. Find an example of a graph G and a vertex v_0 for which Amir's strategy fails, i.e. when he follows the strategy above, he will never see all vertices of G, no matter how long he walks.

ii. (6 points) Suppose Amir's goal is to see every vertex and take every edge of G at least once. Prove that if G is strongly connected, then the strategy above guarantees that Amir reaches his goal.

iii. (2 points) Prove that if G is strongly connected and Amir keeps walking forever, he will see every vertex infinitely many times and take every edge infinitely many times.

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Problem 2. Consider a permutation π of the set $[n] = \{1, 2, 3, \dots, n\}$. An *inversion* in π is a pair $(i, j) \in [n] \times [n]$ such that $i < j$ but $\pi[i] > \pi[j]$. For example, if $n = 5$ and $\pi = \langle 4, 1, 3, 2, 5 \rangle$, then $\pi[1] = 4 > 1 = \pi[2]$ and thus $(1, 2)$ is an inversion.

i. (2 points) Xuran writes down all the possible permutations of $[n]$, counts the number of inversions in each permutation, and sums up these numbers. What is the result of Xuran's sum?

ii. (2 points) Zhaorun has a permutation π. He first finds an inversion (i, j) and then swaps $\pi[i]$ and $\pi[j]$. For example, in the π above, he finds the inversion $(1, 3)$ and swaps $\pi[1]$ with $\pi[3]$, obtaining $\langle 3, 1, 4, 2, 5 \rangle$. He keeps repeating this for as long as he can. Prove that Zhaorun will stop at some point and find the permutation he will end up with.

iii. (2 points) Zhaorun has a permutation π. He first finds an inversion (i, j) such that $j = i + 1$ and then swaps $\pi[i]$ and $\pi[j]$. For example, in the π above, he finds the inversion $(1, 2)$ and swaps $\pi[1]$ with $\pi[2]$, obtaining $\langle 1, 4, 3, 2, 5 \rangle$. He keeps repeating this for as long as he can. Prove that Zhaorun will stop at some point and find the permutation he will end up with.

iv. (2 points) Amir extends the notion of inversions to k-inversions. A k-inversion is a tuple $(i_1, i_2, \dots, i_k) \in [n]^k$ such that $i_1 < i_2 < \dots < i_k$ but $\pi[i_1] > \pi[i_2] > \dots > \pi[i_k]$. Let $\eta(\pi, k)$ be the number of k-inversions of the permutation π. Amir measures the beauty of π as $\sum_{k=2}^{n} \eta(\pi, k)$. Prove that there are 2024 permutations of $[2024]$ that are all equally beautiful.

v. (2 points) Xuran hears of Amir's extension. He writes down all the possible permutations of $[n]$, calculates the beauty of each permutation, and sums up these numbers. What is the result of Xuran's sum?

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Problem 3. Xuran and Zhaorun have all of their time consumed by grading COMP 2711H homeworks. Thus, they cannot attend their own classes. To avoid failing, they have decided to recruit k friends who would attend the classes on their behalf. Xuran and Zhaorun must attend n classes during the week. Class i starts at time t_i (this is the number of seconds from 00:00:00 on Monday to the starting time of the class) and ends at time t'_i. If two classes intersect, they cannot be attended by the same friend.

i. (2 points) Each friend charges Xuran and Zhaorun exactly 1000 HKD to attend classes for them. When they recruit a friend, they can ask that friend to attend as many classes as they like. What is the minimum number of friends the TAs have to recruit?

ii. (2 points) Design an algorithm that takes n and the starting and end times of all classes as its input and outputs an assignment of classes to friends. Your algorithm should use the minimum possible number of friends, i.e. the same number you have found in the previous part.

iii. (2 points) It turns out not all friends are equally friendly. Indeed, if Xuran and Zhaorun decide to recruit friend i, they have to pay them a cost of c_i. Design an algorithm that takes the class times and c_i's as input and assigns a friend to each class. Your algorithm should minimize the total cost for the TAs.

iv. (4 points) Xuran and Zhaorun are not happy with the costs of this operation. Thus, they change their classes to make sure no two of them collide. They also find better friends who do not charge them for taking classes. Unfortunately, when you do not pay people, you cannot expect them to take any class! Indeed, friend number i is only willing to take at most n_i classes for Xuran and Zhaorun. Moreover, each friend has a list of classes that they are absolutely not willing to take. Xuran and Zhaorun wonder if it is possible to cover every class with these restrictions. Design an algorithm that takes the number of friends, number of classes, n_i's and the lists of classes each friend is not willing to take, as input and outputs an assignment of friends to classes such that every class is taken by at least one friend. If no such assignment is possible, your algorithm should report this.

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Problem 4. (10 points) Find the number of trees on the vertex set $\{1, 2, \ldots, 2024\}$ that do not have a perfect matching.

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