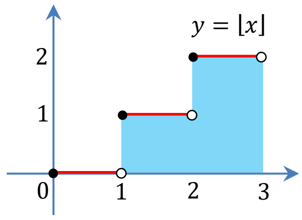
Solution to Problem Set 1

1. (a) Recall that

$$\lfloor x \rfloor = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2 \\ 2 & \text{if } 2 \le x < 3 \end{cases}.$$



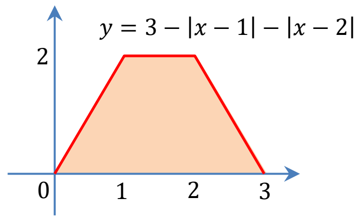
So the given integral represents the total area of a unit square and a rectangle with height 2 units and base width 1 unit, i.e.

$$\int_0^3 \lfloor x \rfloor\, dx = 0 + 1 \cdot 1 + 2 \cdot 1 = 3.$$

(b) We have

$$3 - |x-1| - |x-2| = \begin{cases} 3 + (x-1) + (x-2) & \text{if } 0 \le x < 1 \\ 3 - (x-1) + (x-2) & \text{if } 1 \le x < 2 \\ 3 - (x-1) - (x-2) & \text{if } 2 \le x < 3 \end{cases}$$

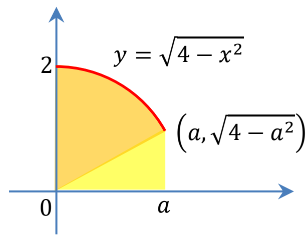
$$= \begin{cases} 2x & \text{if } 0 \le x < 1 \\ 2 & \text{if } 1 \le x < 2 \\ 6 - 2x & \text{if } 2 \le x < 3 \end{cases}.$$



Referring to the graph of the integrand as in the diagram, we see that the given integral represents the area of a trapezoid (trapezium) with base widths 1 and 3 units and height 2 units, so

$$\int_0^3 (3 - |x-1| - |x-2|)\, dx = \frac{1+3}{2} \cdot 2 = 4.$$

(c) The given integral represents the total area of a circular sector of angle $\arcsin\frac{a}{2}$ and of radius 2 centered at the origin, together with a right-angled triangle beneath it, as shown in the diagram. So

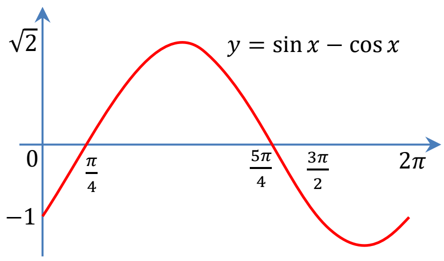


$$\int_0^a \sqrt{4 - x^2}\, dx = \underbrace{\frac{\arcsin\frac{a}{2}}{2\pi} \cdot \pi(2)^2}_{\text{area of sector}} + \underbrace{\frac{1}{2}(a)\left(\sqrt{2^2 - a^2}\right)}_{\text{area of triangle}}$$

$$= 2\arcsin\frac{a}{2} + \frac{1}{2}a\sqrt{4 - a^2}.$$

2. Denote the integrand of the given integral by $f(x)=\sin x-\cos x$. We observe that on $[0,2\pi]$ we have

$$f(x)=\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)\begin{cases}\geq 0 & \text{if } x\in\left[\frac{\pi}{4},\frac{5\pi}{4}\right]\\ <0 & \text{if } x\in\left[0,\frac{\pi}{4}\right)\cup\left(\frac{5\pi}{4},2\pi\right]\end{cases}.$$



(a) According to Corollary 5.19, the intervals $\left[0,\frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4},2\pi\right]$ have **negative contribution** to the integral of f; therefore $\int_a^b f(x)\,dx$ attains its maximum possible value if $a=\frac{\pi}{4}$ and $b=\frac{5\pi}{4}$.

(b) According to Corollary 5.19, the interval $\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$ have positive contribution to the integral of f; therefore $\int_a^b f(x)\,dx$ can possibly attain its minimum value if either $a=0$ and $b=\frac{\pi}{4}$ or $a=\frac{5\pi}{4}$ and $b=2\pi$. We observe by symmetry that the negative area represented by $\int_0^{\pi/4} f(x)\,dx$ is the same as that represented by $\int_{5\pi/4}^{3\pi/2} f(x)\,dx$. Therefore

$$\int_{5\pi/4}^{2\pi} f(x)\,dx=\int_{5\pi/4}^{3\pi/2} f(x)\,dx+\int_{3\pi/2}^{2\pi} f(x)\,dx=\int_0^{\pi/4} f(x)\,dx+\underbrace{\int_{3\pi/2}^{2\pi} f(x)\,dx}_{<0}<\int_0^{\pi/4} f(x)\,dx\,,$$

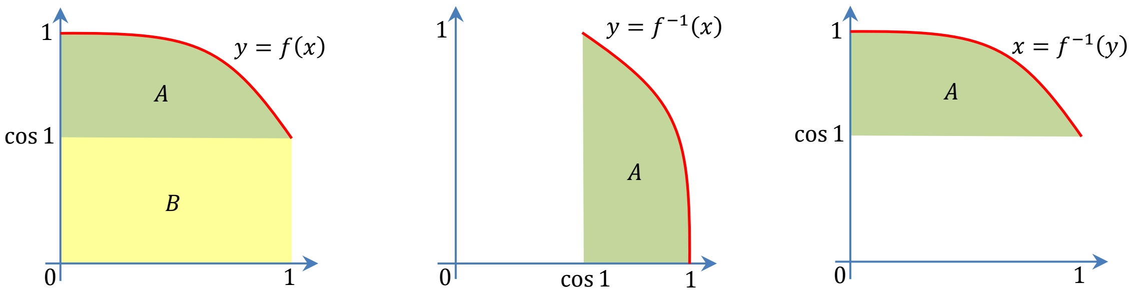
from which we conclude that $\int_a^b f(x)\,dx$ attains its minimum possible value if $a=\frac{5\pi}{4}$ and $b=2\pi$.

3. (a) For every $x\in[0,1]$, we have $x^3\leq x^2$. Since the function $\cos$ is strictly decreasing on $[0,1]$, we have

$$\cos(x^3)\geq\cos(x^2)\qquad\qquad \text{for every } x\in[0,1],$$

i.e. $g(x)\geq f(x)$ for every $x\in[0,1]$. Therefore $\int_0^1 g(x)dx\geq\int_0^1 f(x)dx$ by Theorem 5.8. Note that the inequality $\geq$ can actually be replaced by a strict inequality $>$ because in fact $x^3<x^2$ for every $x\in(0,1)$.

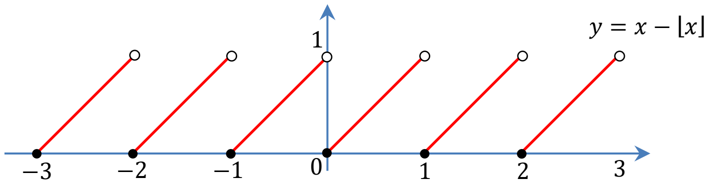
(b) Note that the function $f(x)=\cos(x^2)$ is strictly decreasing on $[0,1]$, and recall from Remark 1.71 that the graph of f^{-1} is the "mirror reflection" of the graph of f across the line $y=x$.



Now refer to the above diagrams with two regions whose areas are A and B. The integral $\int_0^1 f(x)dx$ represents the area $A+B$, while $\int_{\cos 1}^1 f^{-1}(x)dx$ represents the area of a region that is congruent to A only. Therefore $\int_0^1 f(x)dx>\int_{\cos 1}^1 f^{-1}(x)dx$.

Remark: Note that the reasoning in (b) applies not only for the function $f(x)=\cos(x^2)$, but also for any strictly decreasing function with appropriate domain and range.

4. (a) The following is a sketch of the graph of f.



(b) First we observe from the graph of f obtained in (a) that for every integer k, we have

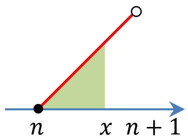
$$\int_k^{k+1} f(t)dt = \frac{1}{2}$$

as it represents the area of a right-angled triangle with base width 1 unit and height 1 unit. Now for each $x \in \mathbb{R}$, we let n be the integer such that $x \in [n, n+1)$, so that $\lfloor x \rfloor = n$. Consider the following cases:

⦿ If $x \geq 0$, then n is a non-negative integer and so

$$\int_0^x f(t)dt = \underbrace{\int_0^1 f(t)dt + \int_1^2 f(t)dt + \cdots + \int_{n-1}^n f(t)dt}_{n \text{ triangles with area } \frac{1}{2} \text{ each}} + \underbrace{\int_n^x f(t)dt}_{=\frac{1}{2}(x-n)^2}$$

$$= \frac{1}{2}n + \frac{1}{2}(x-n)^2 = \frac{1}{2}\lfloor x \rfloor + \frac{1}{2}(x - \lfloor x \rfloor)^2.$$



⦿ If $x < 0$, then n is a negative integer and so

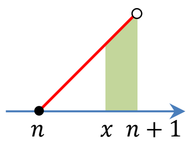
$$\int_0^x f(t)dt = -\int_x^0 f(t)dt$$

$$= -\left(\int_x^{n+1} f(t)dt + \int_{n+1}^{n+2} f(t)dt + \int_{n+2}^{n+3} f(t)dt + \cdots + \int_{-2}^{-1} f(t)dt + \int_{-1}^{0} f(t)dt\right)$$

$$= -\underbrace{\int_x^{n+1} f(t)dt}_{=\frac{1}{2}-\frac{1}{2}(x-n)^2} - \underbrace{\int_{n+1}^{n+2} f(t)dt - \int_{n+2}^{n+3} f(t)dt - \cdots - \int_{-2}^{-1} f(t)dt - \int_{-1}^{0} f(t)dt}_{(-n-1) \text{ triangles with area } \frac{1}{2} \text{ each}}$$

$$= -\left[\frac{1}{2} - \frac{1}{2}(x-n)^2\right] - \frac{1}{2}(-n-1)$$

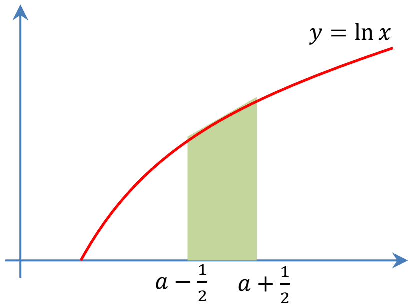
$$= \frac{1}{2}n + \frac{1}{2}(x-n)^2 = \frac{1}{2}\lfloor x \rfloor + \frac{1}{2}(x - \lfloor x \rfloor)^2$$



also. ■

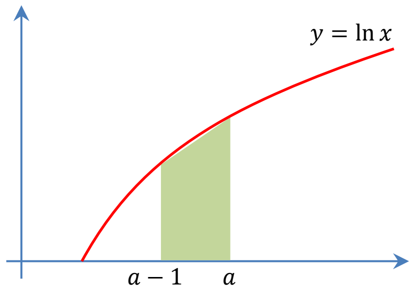
5. Let $f:(0,+\infty)\to\mathbb{R}$ be the function $f(x)=\ln x$. Since $f''(x)=-\frac{1}{x^2}<0$ for every $x>0$, we observe that the graph of f is concave downward on $(0,+\infty)$.

(a) Now let $a\geq 2$ be given. We consider the trapezium bounded by the tangent line to the graph of f at the point $(a,\ln a)$, the x-axis, and the vertical lines $x=a-\frac{1}{2}$ and $x=a+\frac{1}{2}$. The area of this trapezium is $\ln a$. Since the graph of f is concave downward on $\left[a-\frac{1}{2},a+\frac{1}{2}\right]$, the region represented by $\int_{a-1/2}^{a+1/2}\ln x\,dx$ lies completely inside the trapezium (This follows from Q9 of Problem Set 8 in MATH1013). Therefore



$$\int_{a-\frac{1}{2}}^{a+\frac{1}{2}}\ln x\,dx\leq \ln a\,.$$

On the other hand, we consider the trapezium with vertices $(a,\ln a)$, $(a-1,\ln(a-1))$, $(a-1,0)$ and $(a,0)$. This trapezium has area $\frac{\ln(a-1)+\ln a}{2}$. Again since the graph of f is concave downward on $[a-1,a]$, the trapezium lies completely inside the region represented by $\int_{a-1}^{a}\ln x\,dx$ (This follows from Example 4.30 in MATH1013). So



$$\int_{a-1}^{a}\ln x\,dx\geq\frac{\ln(a-1)+\ln a}{2}.$$

(b) Let $n\geq 2$ be an integer. According to the first inequality obtained in (a), we have

$$\begin{aligned}\int_{\frac{3}{2}}^{n}\ln x\,dx&=\int_{\frac{3}{2}}^{\frac{5}{2}}\ln x\,dx+\int_{\frac{5}{2}}^{\frac{7}{2}}\ln x\,dx+\cdots+\int_{n-\frac{3}{2}}^{n-\frac{1}{2}}\ln x\,dx+\int_{n-\frac{1}{2}}^{n}\ln x\,dx\\ &\leq \ln 2+\ln 3+\cdots+\ln(n-1)+\int_{n-\frac{1}{2}}^{n}\ln n\,dx=\ln 2+\ln 3+\cdots+\ln(n-1)+\frac{1}{2}\ln n\\ &=\ln 2+\ln 3+\cdots+\ln(n-1)+\ln n-\frac{1}{2}\ln n=\ln(2\cdot 3\cdot\cdots\cdot(n-1)\cdot n)-\frac{1}{2}\ln n\\ &=\ln(n!)-\frac{1}{2}\ln n\,.\end{aligned}$$

On the other hand, according to the second inequality obtained in (a), we have

$$\begin{aligned}\int_{1}^{n}\ln x\,dx&=\int_{1}^{2}\ln x\,dx+\int_{2}^{3}\ln x\,dx+\cdots+\int_{n-1}^{n}\ln x\,dx\\ &\geq\frac{\ln 1+\ln 2}{2}+\frac{\ln 2+\ln 3}{2}+\cdots+\frac{\ln(n-1)+\ln n}{2}\\ &=\ln 2+\ln 3+\cdots+\ln(n-1)+\frac{1}{2}\ln n=\ln(n!)-\frac{1}{2}\ln n\,.\end{aligned}$$

Combining the two inequalities obtained, the desired result follows. ■

6. (a) (i) The upper Darboux sum of f with respect to P is

$$f\left(-\frac{1}{2}\right)\left[\left(-\frac{1}{2}\right)-(-1)\right]+f(0)\left[0-\left(-\frac{1}{2}\right)\right]+f(0)\left(\frac{1}{2}-0\right)+f\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)$$

$$=e^{-\left(-\frac{1}{2}\right)^2}\left(\frac{1}{2}\right)+e^{-(0)^2}\left(\frac{1}{2}\right)+e^{-(0)^2}\left(\frac{1}{2}\right)+e^{-\left(\frac{1}{2}\right)^2}\left(\frac{1}{2}\right).$$

(ii) The lower Darboux sum of f with respect to P is

$$f(-1)\left[\left(-\frac{1}{2}\right)-(-1)\right]+f\left(-\frac{1}{2}\right)\left[0-\left(-\frac{1}{2}\right)\right]+f\left(\frac{1}{2}\right)\left(\frac{1}{2}-0\right)+f(1)\left(1-\frac{1}{2}\right)$$

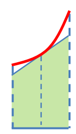
$$=e^{-(-1)^2}\left(\frac{1}{2}\right)+e^{-\left(-\frac{1}{2}\right)^2}\left(\frac{1}{2}\right)+e^{-\left(\frac{1}{2}\right)^2}\left(\frac{1}{2}\right)+e^{-(1)^2}\left(\frac{1}{2}\right).$$

(iii) The right Riemann sum of f with respect to P is

$$f\left(-\frac{1}{2}\right)\left[\left(-\frac{1}{2}\right)-(-1)\right]+f(0)\left[0-\left(-\frac{1}{2}\right)\right]+f\left(\frac{1}{2}\right)\left(\frac{1}{2}-0\right)+f(1)\left(1-\frac{1}{2}\right)$$

$$=e^{-\left(-\frac{1}{2}\right)^2}\left(\frac{1}{2}\right)+e^{-(0)^2}\left(\frac{1}{2}\right)+e^{-\left(\frac{1}{2}\right)^2}\left(\frac{1}{2}\right)+e^{-(1)^2}\left(\frac{1}{2}\right).$$

(b) Since $f''(x)=(4x^2-2)e^{-x^2}>0$ for every $x\in(1,3)$, we see that the graph of f is **concave upward** on $(1,3)$. The mid-point Riemann sum S consists of areas of rectangles that share the same areas with **trapeziums** bounded from above by the **tangent line to the graph of f at the mid-point** instead. These trapeziums are contained completely inside the region under the graph of f as the graph is concave upward; therefore the trapeziums have smaller area than the area under the graph of f, i.e.



Area of rectangle at mid-pt.
= Area of trapezium

$$S<\int_1^3 f(x)dx.$$

7. (a) The given sum

$$\frac{e^{\frac{1}{n^2}}+e^{\frac{4}{n^2}}+e^{\frac{9}{n^2}}+\cdots+e^4}{n}=\left[e^{\left(\frac{1}{n}\right)^2}+e^{\left(\frac{2}{n}\right)^2}+e^{\left(\frac{3}{n}\right)^2}+\cdots+e^{\left(\frac{2n}{n}\right)^2}\right]\left(\frac{1}{n}\right)$$

is the right Riemann sum of the function $f(x)=e^{x^2}$ with respect to the regular partition

$$P=\left\{0,\frac{1}{n},\frac{2}{n},\frac{3}{n},\ldots,\frac{2n}{n}\right\}$$

of $[0,2]$ into $2n$ subintervals of equal width $\Delta x=\frac{1}{n}$. Since f is continuous on $[0,2]$, it is integrable on $[0,2]$ and so

$$\lim_{n\to+\infty}\frac{e^{\frac{1}{n^2}}+e^{\frac{4}{n^2}}+e^{\frac{9}{n^2}}+\cdots+e^4}{n}=\int_0^2 e^{x^2}dx.$$

(b) Note that each term $\frac{n}{n^2+k^2}$ has limit 0 as $n \to +\infty$, so we can discard **finitely many** terms in the given sum without changing the limit. In particular,

$$\lim_{n\to+\infty} \sum_{k=n}^{2n+1014} \frac{n}{n^2+k^2} = \lim_{n\to+\infty} \sum_{k=n}^{2n-1} \frac{n}{n^2+k^2}.$$

Now observe that the remaining sum

$$\sum_{k=n}^{2n-1} \frac{n}{n^2+k^2} = \sum_{k=n}^{2n-1} \frac{1}{1+(k/n)^2}\left(\frac{1}{n}\right)$$

can be treated as the left Riemann sum of the function $f(x) = \frac{1}{1+x^2}$ with respect to the regular partition

$$P = \left\{\frac{n}{n}, \frac{n+1}{n}, \frac{n+2}{n}, \dots, \frac{2n}{n}\right\}$$

of $[1,2]$ into n subintervals of equal width $\Delta x = \frac{1}{n}$. Since f is continuous on $[1,2]$, it is integrable on $[1,2]$ and so

$$\lim_{n\to+\infty} \sum_{k=n}^{2n+1014} \frac{n}{n^2+k^2} = \int_1^2 \frac{1}{1+x^2}\,dx.$$

(c) Observe that the given sum

$$\sum_{k=1}^{n} \frac{(n+2k-1)^3}{n^4} = \sum_{k=1}^{n} \left(1+\frac{2k-1}{n}\right)^3\left(\frac{1}{n}\right) = \sum_{k=1}^{n} \frac{1}{2}\left(1+\frac{2k-1}{n}\right)^3\left(\frac{2}{n}\right)$$

can be treated as the **mid-point** Riemann sum of the function $f(x) = \frac{1}{2}(1+x)^3$ with respect to the regular partition

$$P = \left\{0, \frac{2}{n}, \frac{4}{n}, \frac{6}{n}, \dots, \frac{2n}{n}\right\}$$

of $[0,2]$ into n subintervals of equal width $\Delta x = \frac{2}{n}$. Since f is continuous on $[0,2]$, it is integrable on $[0,2]$ and so

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{(n+2k-1)^3}{n^4} = \int_0^2 \frac{1}{2}(1+x)^3\,dx.$$

Alternative solution: The given sum

$$\sum_{k=1}^{n} \frac{(n+2k-1)^3}{n^4} = \sum_{k=1}^{n} 8\left(\frac{1}{2}+\frac{2k-1}{2n}\right)^3\left(\frac{1}{n}\right)$$

can be treated as the **mid-point** Riemann sum of the function $f(x) = 8\left(\frac{1}{2}+x\right)^3$ with respect to the regular partition $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n}{n}\right\}$ of $[0,1]$ into n subintervals of equal width $\Delta x = \frac{1}{n}$. Since f is continuous on $[0,1]$, it is integrable on $[0,1]$ and so

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{(n+2k-1)^3}{n^4} = \int_0^1 8\left(\frac{1}{2}+x\right)^3 dx.$$

(d) Observe that the given sum

$$\frac{1^2+4^2+7^2+\cdots+(3n-2)^2}{n^3} = \left(\left(\frac{1}{n}\right)^2+\left(\frac{4}{n}\right)^2+\left(\frac{7}{n}\right)^2+\cdots+\left(\frac{3n-2}{n}\right)^2\right)\frac{1}{n}$$
$$= \left(\frac{1}{3}\left(\frac{1}{n}\right)^2+\frac{1}{3}\left(\frac{4}{n}\right)^2+\frac{1}{3}\left(\frac{7}{n}\right)^2+\cdots+\frac{1}{3}\left(\frac{3n-2}{n}\right)^2\right)\frac{3}{n}$$

can be treated as the Riemann sum of the function $f(x)=\frac{1}{3}x^2$ with respect to the regular tagged partition $P=\left\{0,\frac{3}{n},\frac{6}{n},\frac{9}{n},\dots,\frac{3n}{n}\right\}$ of $[0,3]$ into n subintervals, with the choice of sample points

$$\omega_1=\frac{1}{n}\in\left[0,\frac{3}{n}\right],\quad \omega_2=\frac{4}{n}\in\left[\frac{3}{n},\frac{6}{n}\right],\quad \omega_3=\frac{7}{n}\in\left[\frac{6}{n},\frac{9}{n}\right],\quad \dots,\quad \omega_n=\frac{3n-2}{n}\in\left[\frac{3n-3}{n},\frac{3n}{n}\right].$$

The width of each subinterval is $\Delta x=\frac{3}{n}$. Since f is continuous on $[0,3]$, it is integrable on $[0,3]$ and so

$$\lim_{n\to+\infty}\frac{1^2+4^2+7^2+\cdots+(3n-2)^2}{n^3}=\int_0^3\frac{1}{3}x^2dx\,.$$

Alternative solution: The given sum

$$\frac{1^2+4^2+7^2+\cdots+(3n-2)^2}{n^3} = \left(9\left(\frac{1}{3n}\right)^2+9\left(\frac{4}{3n}\right)^2+9\left(\frac{7}{3n}\right)^2+\cdots+9\left(\frac{3n-2}{3n}\right)^2\right)\frac{1}{n}$$

can also be treated as the Riemann sum of the function $f(x)=9x^2$ with respect to the regular tagged partition $P=\left\{0,\frac{1}{n},\frac{2}{n},\frac{3}{n},\dots,\frac{n}{n}\right\}$ of $[0,1]$ into n subintervals, with the choice of sample points

$$\omega_1=\frac{1}{3n}\in\left[0,\frac{1}{n}\right],\quad \omega_2=\frac{4}{3n}\in\left[\frac{1}{n},\frac{2}{n}\right],\quad \omega_3=\frac{7}{3n}\in\left[\frac{2}{n},\frac{3}{n}\right],\quad \dots,\quad \omega_n=\frac{3n-2}{3n}\in\left[\frac{n-1}{n},\frac{n}{n}\right].$$

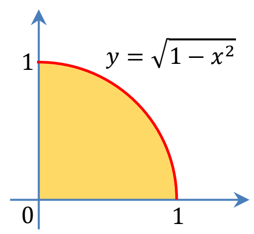
The width of each subinterval is $\Delta x=\frac{1}{n}$. Since f is continuous on $[0,1]$, it is integrable on $[0,1]$ and so

$$\lim_{n\to+\infty}\frac{1^2+4^2+7^2+\cdots+(3n-2)^2}{n^3}=\int_0^1 9x^2dx.$$

8. (a) Since the function $f(x)=\sqrt{1-x^2}$ is continuous on $[0,1]$, it is integrable on $[0,1]$. Thus,

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n^2}=\lim_{n\to+\infty}\sum_{k=1}^{n}\sqrt{1-\left(\frac{k}{n}\right)^2}\cdot\frac{1}{n}=\int_0^1\sqrt{1-x^2}dx.$$

Now the integral represents the area of a quarter circular disk with radius 1 and centered at the origin. Therefore



$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{\sqrt{n^2-k^2}}{n^2}=\frac{1}{4}\pi(1)^2=\frac{\pi}{4}.$$

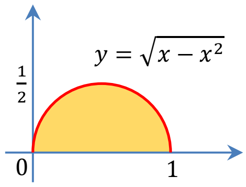
(b) Since the function $f(x)=\sqrt{x-x^2}$ is continuous on $[0,1]$, it is integrable on $[0,1]$. Thus,

$$\lim_{n\to+\infty}\frac{1}{n^2}\left(\sqrt{n-1}+\sqrt{2n-4}+\sqrt{3n-9}+\cdots+\sqrt{n^2-n^2}\right)$$

$$=\lim_{n\to+\infty}\frac{1}{n}\left(\sqrt{\frac{1}{n}-\left(\frac{1}{n}\right)^2}+\sqrt{\frac{2}{n}-\left(\frac{2}{n}\right)^2}+\sqrt{\frac{3}{n}-\left(\frac{3}{n}\right)^2}+\cdots+\sqrt{\frac{n}{n}-\left(\frac{n}{n}\right)^2}\right)=\int_0^1\sqrt{x-x^2}\,dx.$$

Now the integral represents the area of the upper half circular disk with radius $\frac{1}{2}$ centered at the point $\left(\frac{1}{2},0\right)$. Therefore

$$\lim_{n\to+\infty}\frac{1}{n^2}\left(\sqrt{n-1}+\sqrt{2n-4}+\sqrt{3n-9}+\cdots+\sqrt{n^2-n^2}\right)=\frac{1}{2}\pi\left(\frac{1}{2}\right)^2=\frac{\pi}{8}.$$



9. Let $f(x)=x^3$. Since f is a polynomial, it is continuous on $[a,b]$ and thus integrable on $[a,b]$. For each positive integer n, we consider the right Riemann sum of f with respect to the regular partition P of $[a,b]$ into n subintervals. Then $\|P\|=\frac{b-a}{n}\to 0$ as $n\to+\infty$, so

$$\int_a^b x^3dx=\lim_{n\to+\infty}\sum_{k=1}^{n}\left[a+\frac{k}{n}(b-a)\right]^3\left(\frac{b-a}{n}\right)$$

$$=\lim_{n\to+\infty}\sum_{k=1}^{n}\left[a^3+\frac{3a^2(b-a)}{n}k+\frac{3a(b-a)^2}{n^2}k^2+\frac{(b-a)^3}{n^3}k^3\right]\left(\frac{b-a}{n}\right)$$

$$=\lim_{n\to+\infty}\left[\frac{a^3(b-a)}{n}\sum_{k=1}^{n}1+\frac{3a^2(b-a)^2}{n^2}\sum_{k=1}^{n}k+\frac{3a(b-a)^3}{n^3}\sum_{k=1}^{n}k^2+\frac{(b-a)^4}{n^4}\sum_{k=1}^{n}k^3\right]$$

$$=\lim_{n\to+\infty}\left[\frac{a^3(b-a)}{n}\cdot n+\frac{3a^2(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2}+\frac{3a(b-a)^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{(b-a)^4}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$=\lim_{n\to+\infty}\left[a^3(b-a)+\frac{3}{2}a^2(b-a)^2\left(1+\frac{1}{n}\right)+\frac{1}{2}a(b-a)^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+\frac{(b-a)^4}{4}\left(1+\frac{1}{n}\right)^2\right]$$

$$=a^3(b-a)+\frac{3}{2}a^2(b-a)^2+a(b-a)^3+\frac{(b-a)^4}{4}$$

$$=\frac{1}{4}(b^4-a^4).$$

10. (a) Given $\sin\frac{t}{2}\neq 0$ and $n\in\mathbb{N}$, we have

$$\sum_{k=1}^{n}\sin kt=\frac{1}{\sin\frac{t}{2}}\sum_{k=1}^{n}\sin kt\sin\frac{t}{2}=\frac{1}{\sin\frac{t}{2}}\sum_{k=1}^{n}\frac{1}{2}\left[\cos\left(kt-\frac{t}{2}\right)-\cos\left(kt+\frac{t}{2}\right)\right]$$

$$=\frac{1}{2\sin\frac{t}{2}}\left[\sum_{k=1}^{n}\cos\left(k-\frac{1}{2}\right)t-\sum_{k=1}^{n}\cos\left(k+\frac{1}{2}\right)t\right]=\frac{1}{2\sin\frac{t}{2}}\left[\sum_{k=0}^{n-1}\cos\left(k+\frac{1}{2}\right)t-\sum_{k=1}^{n}\cos\left(k+\frac{1}{2}\right)t\right]$$

$$=\frac{1}{2\sin\frac{t}{2}}\left[\cos\frac{t}{2}+\sum_{k=1}^{n-1}\cos\left(k+\frac{1}{2}\right)t-\sum_{k=1}^{n-1}t-\cos\left(n+\frac{1}{2}\right)t\right]=\frac{\cos\frac{t}{2}-\cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}.$$

(b) For each $a > 0$, the function $f(x) = \sin x$ is continuous on $[0, a]$, so it is integrable on $[0, a]$. For each positive integer n, we consider the right Riemann sum of f with respect to the regular partition $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \frac{3a}{n}, \dots, \frac{na}{n}\right\}$ of $[0, a]$ into n subintervals. Then $\|P\| = \frac{a}{n} \to 0$ as $n \to +\infty$, so

$$\int_0^a \sin x\, dx = \lim_{n\to+\infty} \sum_{k=1}^{n} \left(\sin\frac{ka}{n}\right)\left(\frac{a}{n}\right) = \lim_{n\to+\infty} \frac{a}{n} \sum_{k=1}^{n} \sin\frac{ka}{n}$$

$$= \lim_{n\to+\infty} \frac{a}{n} \frac{\cos\frac{a}{2n} - \cos\left(n + \frac{1}{2}\right)\frac{a}{n}}{2\sin\frac{a}{2n}} = \lim_{n\to+\infty} \frac{\frac{a}{2n}}{\sin\frac{a}{2n}} \left[\cos\frac{a}{2n} - \cos\left(1 + \frac{1}{2n}\right)a\right]$$

$$= 1 \cdot (\cos 0 - \cos a) = 1 - \cos a.$$

11. For each $a > 1$, the function $f(x) = \ln x$ is continuous on $[1, a]$, so it is integrable on $[1, a]$. For each positive integer n, we consider the right Riemann sum of f with respect to the partition $P = \left\{1, a^{\frac{1}{n}}, a^{\frac{2}{n}}, a^{\frac{3}{n}}, \dots, a^{\frac{n}{n}}\right\}$ of $[1, a]$ into n subintervals. Then $\|P\| = a^{\frac{n}{n}} - a^{\frac{n-1}{n}} \to 0$ as $n \to +\infty$, so

$$\int_1^a \ln x\, dx = \lim_{n\to+\infty} \sum_{k=1}^{n} \left(\ln a^{\frac{k}{n}}\right)\left(a^{\frac{k}{n}} - a^{\frac{k-1}{n}}\right) = (\ln a) \lim_{n\to+\infty} \frac{1}{n}\left(a^{\frac{1}{n}} - 1\right) \sum_{k=1}^{n} k a^{\frac{k-1}{n}}$$

$$= (\ln a) \lim_{n\to+\infty} \frac{1}{n}\left(a^{\frac{1}{n}} - 1\right) \sum_{k=1}^{n} \frac{d}{dx} x^k \bigg|_{x=a^{\frac{1}{n}}} = (\ln a) \lim_{n\to+\infty} \frac{1}{n}\left(a^{\frac{1}{n}} - 1\right) \left(\frac{d}{dx} \sum_{k=1}^{n} x^k \bigg|_{x=a^{\frac{1}{n}}}\right)$$

$$= (\ln a) \lim_{n\to+\infty} \frac{1}{n}\left(a^{\frac{1}{n}} - 1\right) \frac{d}{dx} \frac{x(x^n - 1)}{x - 1}\bigg|_{x=a^{\frac{1}{n}}} = (\ln a) \lim_{n\to+\infty} \frac{1}{n}\left(a^{\frac{1}{n}} - 1\right) \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}\bigg|_{x=a^{\frac{1}{n}}}$$

$$= (\ln a) \lim_{n\to+\infty} \frac{na^{\frac{n+1}{n}} - (n+1)a + 1}{n\left(a^{\frac{1}{n}} - 1\right)} = (\ln a) \frac{a\left[\lim_{n\to+\infty} n\left(a^{\frac{1}{n}} - 1\right)\right] - a + 1}{\lim_{n\to+\infty} n\left(a^{\frac{1}{n}} - 1\right)}$$

$$= (\ln a) \frac{a\left(\lim_{t\to0^+} \frac{a^t - 1}{t}\right) - a + 1}{\lim_{t\to0^+} \frac{a^t - 1}{t}} = (\ln a) \frac{a\ln a - a + 1}{\ln a} = a\ln a - a + 1.$$

12. It is given that f is continuous at $c \in (a, b)$ and $f(c) > 0$. According to the **sign-preserving property** (Lemma 2.108) (or by the ε-δ definition of limits: choose the positive number $\varepsilon = \frac{f(c)}{2}$), there exists $\delta > 0$ such that

$$f(x) > \frac{f(c)}{2} \qquad \text{for every } x \in [c - \delta, c + \delta].$$

By choosing a smaller $\delta > 0$ if necessary, we may assume that $[c - \delta, c + \delta] \subseteq [a, b]$. Thus by Theorem 5.8,

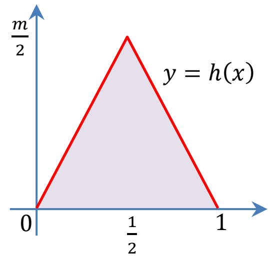
$$\int_a^b f(x) dx = \int_a^{c-\delta} \underbrace{f(x)}_{\geq 0}\, dx + \int_{c-\delta}^{c+\delta} \underbrace{f(x)}_{\geq f(c)/2}\, dx + \int_{c+\delta}^{b} \underbrace{f(x)}_{\geq 0}\, dx$$

$$\geq \frac{f(c)}{2}[(c + \delta) - (c - \delta)] = f(c) \cdot \delta > 0.$$

■

13. (a) Let $h: [0,1] \to [0,+\infty)$ be the piecewise defined function

$$h(x) = \begin{cases} mx & \text{if } x \in [0,1/2] \\ m(1-x) & \text{if } x \in (1/2,1] \end{cases}.$$



Then h is non-negative and continuous on $[0,1]$, and $\int_0^1 h(x)dx$ represents the area of the isosceles triangle with vertices $(0,0)$, $(1,0)$ and $\left(\frac{1}{2}, \frac{m}{2}\right)$, so

$$\int_0^1 h(x)dx = \frac{1}{2} \cdot 1 \cdot \frac{m}{2} = \frac{m}{4}.$$

Now by the given condition we have $g(x) \le h(x)$ for every $x \in [0,1]$. So by Theorem 5.41 (ii) we have

$$\int_0^1 g(x)dx \le \int_0^1 h(x)dx = \frac{m}{4}.$$

■

(b) Let $x \in (0,1)$. Since f is continuous on $[0,x]$ and differentiable on $(0,x)$, according to Mean Value Theorem there exists $c \in (0,x)$ such that

$$f(x) - f(0) = f'(c)(x-0).$$

Since $f(0) = 0$ and $|f'(c)| \le m$, this implies that

$$|f(x)| = |f'(c)(x-0)| = |f'(c)|x \le mx.$$

In a similar way, since f is continuous on $[x,1]$ and differentiable on $(x,1)$, according to Mean Value Theorem again, there exists $d \in (x,1)$ such that

$$f(x) - f(1) = f'(d)(x-1).$$

Since $f(1) = 0$ and $|f'(d)| \le m$, this implies that

$$|f(x)| = |f'(d)(x-1)| = |f'(d)|(1-x) \le m(1-x).$$

Now we have shown that $|f|$ is a non-negative integrable function on $[0,1]$ which satisfy both $|f(x)| \le mx$ and $|f(x)| \le m(1-x)$ for every $x \in [0,1]$, so according to the result from (a) we have

$$\int_0^1 |f(x)|dx \le \frac{m}{4}.$$

■

(c) Let $f: [0,1] \to \mathbb{R}$ be the function

$$f(x) = \sin\big(mx(x-1)\big).$$

Then obviously f is continuous on $[0,1]$ and differentiable on $(0,1)$, and $f(0) = f(1) = 0$. For every $x \in (0,1)$, we have $f'(x) = \cos\big(mx(x-1)\big) \cdot (2mx - m)$ and so

$$|f'(x)| = \underbrace{\big|\cos\big(mx(x-1)\big)\big|}_{\le 1} \cdot m \underbrace{|2x-1|}_{\substack{<1 \\ \text{since } x \in (0,1)}} \le m.$$

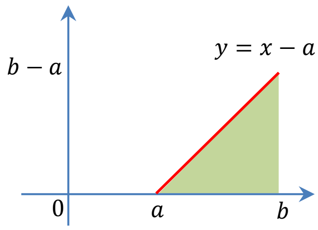
Therefore according to the result from (b), we have

$$\int_0^1 \big|\sin\big(mx(x-1)\big)\big|dx = \int_0^1 |f(x)|dx \le \frac{m}{4}.$$

■

14. (a) (i) The integral $\int_a^b (x-a)dx$ represents the area of a triangle with vertices $(a,0)$, $(b,0)$ and $(b,b-a)$. Such a triangle has base width $(b-a)$ units and height $(b-a)$ units, so

$$\int_a^b (x-a)dx = \frac{1}{2}(b-a)^2.$$



(ii) Since f' is continuous on the bounded closed interval $[a,b]$, it has a global maximum and a global minimum on $[a,b]$ according to Extreme Value Theorem, i.e. the numbers

$$m = \min\{f'(x): x \in [a,b]\} \qquad \text{and} \qquad M = \max\{f'(x): x \in [a,b]\}$$

exist. Now for each $x \in [a,b]$, the function f is differentiable at every number in $[a,x]$, so according to Mean Value Theorem there exists $c \in [a,x]$ such that

$$f(x) - f(a) = f'(c)(x-a).$$

Since $m \le f'(c) \le M$, the above implies that

$$m(x-a) \le f(x) - f(a) \le M(x-a)$$

for every $x \in [a,b]$. So by Theorem 5.41 (ii) we have

$$\int_a^b m(x-a)dx \le \int_a^b \big(f(x) - f(a)\big)dx \le \int_a^b M(x-a)dx.$$

By the result obtained from (a), this becomes

$$\frac{m}{2}(b-a)^2 \le \int_a^b \big(f(x)-f(a)\big)dx \le \frac{M}{2}(b-a)^2$$

as desired. ■

(b) Let $f(x) = e^{-x^2/2}$. Then f is continuously differentiable on $[1,2]$. We have

$$f'(x) = -xe^{-\frac{x^2}{2}} \qquad \text{and} \qquad f''(x) = (x^2-1)e^{-\frac{x^2}{2}}.$$

Since $f''(x) > 0$ for every $x \in (1,2)$, it follows that f' is strictly increasing on $[1,2]$ and so

$$m = \min\{f'(x): x \in [1,2]\} = f'(1) = -e^{-1/2},$$
$$M = \max\{f'(x): x \in [1,2]\} = f'(2) = -2e^{-2}.$$

Thus according to the result from (a) (ii) we have

$$\frac{-e^{-1/2}}{2}(2-1)^2 \le \int_1^2 \left(e^{-\frac{x^2}{2}} - e^{-\frac{1^2}{2}}\right)dx \le \frac{-2e^{-2}}{2}(2-1)^2.$$

Adding $\int_1^2 e^{-\frac{1^2}{2}}dx = e^{-\frac{1}{2}}$ to each component, we obtain $\frac{-e^{-1/2}}{2} + e^{-1/2} \le \int_1^2 e^{-\frac{x^2}{2}}dx \le -e^{-2} + e^{-\frac{1}{2}}$, i.e.

$$\frac{1}{2\sqrt{e}} \le \int_1^2 e^{-\frac{x^2}{2}}dx \le \frac{1}{\sqrt{e}} - \frac{1}{e^2}$$

as desired. ■

15. (a) Let $x, y \in [a, b]$ be real numbers with $x < y$. Then

$$\begin{aligned} g(y) - g(x) &= \left(\int_a^y f(t)dt + \int_b^y \frac{1}{f(t)}dt\right) - \left(\int_a^x f(t)dt + \int_b^x \frac{1}{f(t)}dt\right) \\ &= \left(\int_a^x f(t)dt + \int_x^y f(t)dt - \int_y^b \frac{1}{f(t)}dt\right) - \left(\int_a^x f(t)dt - \int_x^y \frac{1}{f(t)}dt - \int_y^b \frac{1}{f(t)}dt\right) \\ &= \int_x^y f(t)dt + \int_x^y \frac{1}{f(t)}dt. \end{aligned}$$

Now since f and $1/f$ are both continuous on the bounded closed interval $[a, b]$, they both attain their global minima on $[a, b]$ by Extreme Value Theorem. Since f and $1/f$ are both **positive**, their global minima must be positive also. In other words, there exist $m, M > 0$ such that

$$f(t) \geq m \qquad \text{and} \qquad \frac{1}{f(t)} \geq M$$

for every $t \in [a, b]$. Therefore by Theorem 5.41 (iii) we have

$$\int_x^y f(t)dt \geq m(y - x) > 0 \qquad \text{and} \qquad \int_x^y \frac{1}{f(t)}dt \geq M(y - x) > 0,$$

which shows that $g(y) - g(x) > 0$. Therefore g is strictly increasing on $[a, b]$. ■

(b) ⊙ If g has two distinct roots in $[a, b]$, then $g(x) = g(y) = 0$ with $x > y$, which contradicts with the result from (a) that g is strictly increasing on $[a, b]$; so g must have at most one root in $[a, b]$.

⊙ On the other hand, if it is also given that g is continuous on $[a, b]$, then since

$$g(a) = \int_a^a f(t)dt + \int_b^a \frac{1}{f(t)}dt = -\int_a^b \frac{1}{f(t)}dt < 0$$

and

$$g(b) = \int_a^b f(t)dt + \int_b^b \frac{1}{f(t)}dt = \int_a^b f(t)dt > 0,$$

it follows that g must have at least one root in (a, b), according to the Intermediate Value Theorem. ■

16. (a) The derivative of the function p is given by

$$p'(t) = \frac{1}{3}t^{-\frac{2}{3}} - \frac{1}{3} = \frac{1}{3}\left(t^{-\frac{2}{3}} - 1\right) \qquad \text{for every } t \in (0, +\infty).$$

Since $p'(t) \begin{cases} < 0 & \text{if } t > 1 \\ > 0 & \text{if } 0 < t < 1 \end{cases}$, p is strictly increasing on $[0, 1]$ and strictly decreasing on $[1, +\infty)$, so p attains global maximum at 1. In other words, we have

$$p(t) \leq p(1) = 1^{\frac{1}{3}} - \frac{1}{3} \cdot 1 - \frac{2}{3} = 0$$

for every $t \in [0, +\infty)$.

Now let x and y be non-negative numbers.

- If $y = 0$, then it is obvious that $x^{\frac{1}{3}}y^{\frac{2}{3}} = 0 \le \frac{1}{3}x + \frac{2}{3}y$.
- If $y > 0$, then x/y is a non-negative number, so $p(x/y) \le 0$. This gives

$$\left(\frac{x}{y}\right)^{\frac{1}{3}} - \frac{1}{3}\left(\frac{x}{y}\right) - \frac{2}{3} \le 0.$$

Multiplying both sides by the positive number y, we obtain $x^{\frac{1}{3}}y^{\frac{2}{3}} - \frac{1}{3}x - \frac{2}{3}y \le 0$, i.e. $x^{\frac{1}{3}}y^{\frac{2}{3}} \le \frac{1}{3}x + \frac{2}{3}y$.■

(b) For every $x \in [a, b]$, $f(x)$ and $g(x)$ are both non-negative numbers, so using the result from (a), we have

$$f(x)^{\frac{1}{3}}g(x)^{\frac{2}{3}} \le \frac{1}{3}f(x) + \frac{2}{3}g(x).$$

So by Theorem 5.41 (ii), we have

$$\begin{aligned}\int_a^b f(x)^{\frac{1}{3}}g(x)^{\frac{2}{3}}dx &\le \int_a^b \left[\frac{1}{3}f(x) + \frac{2}{3}g(x)\right]dx \\ &= \frac{1}{3}\int_a^b f(x)dx + \frac{2}{3}\int_a^b g(x)dx = \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot 1 = 1.\end{aligned}$$

■

(c) Given the function F and G, we consider the following two cases.

- If either F or G is the constant function zero, then both sides of the desired inequality are zero, so the inequality is trivially true.
- If F and G are both not the constant function zero, then we let $f, g: [a, b] \to [0, +\infty)$ be the functions defined by

$$f(x) = \frac{F(x)^3}{\int_a^b F(t)^3 dt} \qquad \text{and} \qquad g(x) = \frac{G(x)^{3/2}}{\int_a^b G(t)^{3/2} dt}.$$

Note that their denominators are just positive numbers, which are independent of the variable x. Now f and g are non-negative, continuous, and

$$\int_a^b f(x)dx = \frac{1}{\int_a^b F(t)^3 dt}\int_a^b F(x)^3 dx = 1 \qquad \text{and} \qquad \int_a^b g(x)dx = \frac{1}{\int_a^b G(t)^{3/2} dt}\int_a^b G(x)^{3/2} dx = 1.$$

So according to the result from (b) we have $\int_a^b f(x)^{\frac{1}{3}}g(x)^{\frac{2}{3}}dx \le 1$, i.e.

$$\int_a^b \left[\frac{F(x)^3}{\int_a^b F(t)^3 dt}\right]^{1/3} \left[\frac{G(x)^{3/2}}{\int_a^b G(t)^{3/2} dt}\right]^{2/3} dx \le 1.$$

Rearranging, we get

$$\int_a^b F(x)G(x)dx \le \left(\int_a^b F(t)^3 dt\right)^{\frac{1}{3}}\left(\int_a^b G(t)^{\frac{3}{2}}dt\right)^{\frac{2}{3}} = \left(\int_a^b F(x)^3 dx\right)^{\frac{1}{3}}\left(\int_a^b G(x)^{\frac{3}{2}}dx\right)^{\frac{2}{3}}.$$

■