Decomposition of Unitary Matrices Using Fourier Transforms and Phase Masks

Vincent Girouard[1] and Nicolás Quesada[2]

[1]vincent-2.girouard@polymtl.ca
[2]nicolas.quesada@polymtl.ca

August 28, 2025

Abstract

The ability to perform general linear transformation on a set of N optical modes has a lot of applications in communication and information processing. In 2021, López Pastor, Lundeen and Marquardt proposed a new constructive algorithm to decompose arbitrary $N \times N$ unitary matrices into a sequence of $6N + 1$ phase masks interleaved with discrete Fourier transforms [2]. Challenges faced during the implementation of this algorithm led us to rework the derivation to make it work with a different sequence of masks. We provide the result of this work, along with a code implementation of this new version.

1 Derivation

A lossless linear transformation on a set of N optical modes can be described by a $N \times N$ unitary matrix. Clements *et al.* showed how to decompose such matrices into a rectangular mesh of beam splitters and phase shifters [1]. The unit cell of this design consists of a phase shifter on the upper mode followed by a Mach-Zenhder interferometer (MZI), as shown in figure 1. The transfer matrix T of this unit cell acting on two consecutive modes m and $n = m + 1$ is given by

$$\begin{aligned}\begin{pmatrix}[T(\theta,\phi)]_{m,m} & [T(\theta,\phi)]_{m,n}\\ [T(\theta,\phi)]_{n,m} & [T(\theta,\phi)]_{n,n}\end{pmatrix} :=& X\begin{pmatrix}e^{i2\theta} & 0\\ 0 & 1\end{pmatrix} X \begin{pmatrix}e^{i\phi} & 0\\ 0 & 1\end{pmatrix}\\ =& e^{i\theta}\begin{pmatrix}e^{i\phi}\cos\theta & i\sin\theta\\ e^{i\phi} i\sin\theta & \cos\theta\end{pmatrix}.\end{aligned} \tag{1}$$

T acts as the identity on all the other modes. In (1), X represents a 50:50 beam splitter of the form

$$X := \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}. \tag{2}$$

The matrix defined in (1) is different from the one used in Clements *et al.* [1], as the matrix used in this paper does not correspond to the decomposition into phase shifters and beam splitters given by the unit cell in Figure 1. This error also propagated in Lundeen *et al.* [2], where they use the matrix from [1] with the decomposition in (1) to decompose unitaries into phase masks and Fourier transforms. However, the Clements algorithm can be easily adapted to use the matrix in (1).

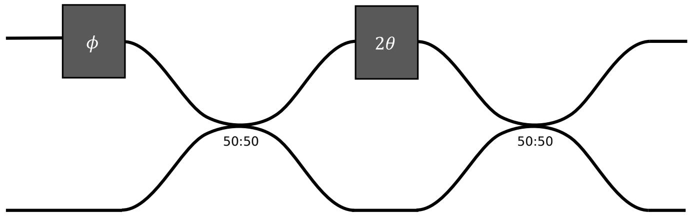


Figure 1: Unit cell of the design by Clements *et al.* to construct universal multimode interferometers. The unit cell consists of a phase shifter of ϕ on the first mode, followed by a MZI with a phase shifter of 2θ on the first mode.

Using the Clements scheme, any matrix U can be decomposed into a lattice of $N(N-1)/2$ unit cells distributed in N layers, plus a diagonal phase mask at the output of the interferometer. This makes a total of N^2 free parameters, which is the theoretical lower bound required to characterize a $N \times N$ unitary. The Clements decomposition is represented in Figure 2 for a 6 modes transformation.

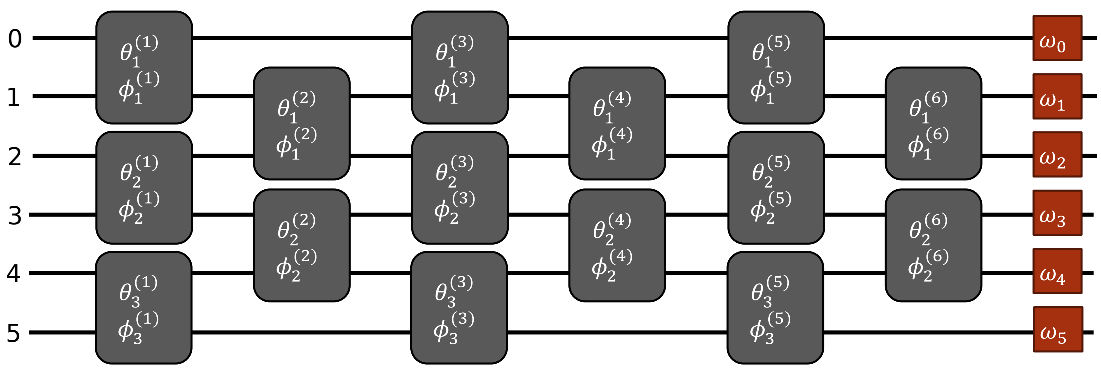


Figure 2: Rectangular mesh of unit cells generated using the Clements decomposition algorithm. This layout allows the implementation of any linear transformation on N optical modes.

We want to start from this decomposition to obtain a new decomposition of a unitary matrix U in terms of diagonal phase masks $D^{(i)}$ and discrete Fourier transforms (DFTs) such that

$$U = D^{(0)} \prod_{i=1}^{L} F D^{(i)}, \tag{3}$$

where F is the DFT matrix on N modes, whose elements are given by

$$F_{j,k} = \frac{1}{\sqrt{N}} e^{-i2\pi jk/N}. \tag{4}$$

In 2021, Pastor, Lundeen and Marquardt proposed a constructive procedure to find the phase masks needed to carry on this decomposition [2]. The method works by first relabelling the channels of the interferometer as

$$\{0,1,2,3,...\} \longrightarrow \left\{0, \frac{N}{2}, 1, \frac{N}{2}+1, \ \dots \right\}, \tag{5}$$

which can be achieved using the permutation matrix K given by

$$K_{j,k} = \begin{cases} 1 & k = 2j, \quad j \leq \frac{N}{2} - 1 \\ 1 & k = 2j+1-N, \quad j > \frac{N}{2}-1 \\ 0 & \text{otherwise} \end{cases} . \tag{6}$$

The effect of this permutation on the channels of the interferometer can be observed in figure 3a. With this relabelling, in the odd layers of the interferometer, each channel j of the second half ($j \geq \frac{N}{2}$) interacts with the channel $j - \frac{N}{2}$. In the even layers, each channels j in the second half ($j \geq \frac{N}{2}$) interacts with the channel $j - \frac{N}{2} + 1 \pmod{N/2}$. The transformation performed by an odd layer i can be expressed as

$$T^{(i)}_{\text{odd}} = X\Xi(\theta^{(i)})X\Xi(\phi^{(i)}). \tag{7}$$

X represents the layer made of the $N/2$ beam splitters acting between channels j ($j < N/2$) and $j + N/2$, which is given by the block matrix

$$X \coloneqq \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}, \tag{8}$$

where I is the $\frac{N}{2} \times \frac{N}{2}$ identity matrix. In (7), Ξ corresponds to a layer of phase shifters applied on the first half of the channels. It is a diagonal matrix of the form

$$\Xi(\phi^{(i)}) \coloneqq \text{diag}\{e^{i\phi^{(i)}_0}, \ \dots, \ e^{i\phi^{(i)}_{N/2-1}}, \ 1, \ \dots, \ 1\}. \tag{9}$$

For the even layers, where each channel j ($j \geq \frac{N}{2}$) interacts with channel $j - \frac{N}{2} + 1$, the transformation performed on the optical modes can be described by the same sequence as (7), but with some additional permutations. First, we need to apply a cyclic shift on the first half of the channels so that channel $j - \frac{N}{2} + 1 \pmod{N/2}$ is mapped to $j - \frac{N}{2}$. This permutation is displayed in Figure 3b. It can be performed with the permutation matrix P given by

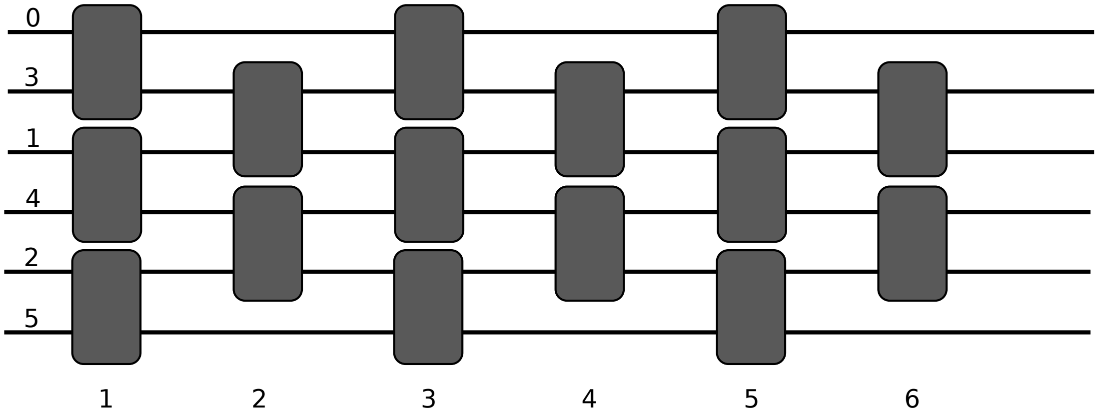
$$P_{j,k} \coloneqq \begin{cases} 1 & k = j+1 \ (\text{mod} \frac{N}{2}), \quad j \leq \frac{N}{2} - 1 \\ 1 & k = j, \quad j > \frac{N}{2} - 1 \\ 0 & \text{otherwise} \end{cases} . \tag{10}$$

Also, since the unit cell used in the mesh is not symmetrical between the upper and the lower channel (see Figure 1), we need to swap the first and second half of the channels in the interferometer. This transformation is shown in Figure 3c and can be done using the permutation matrix S given by

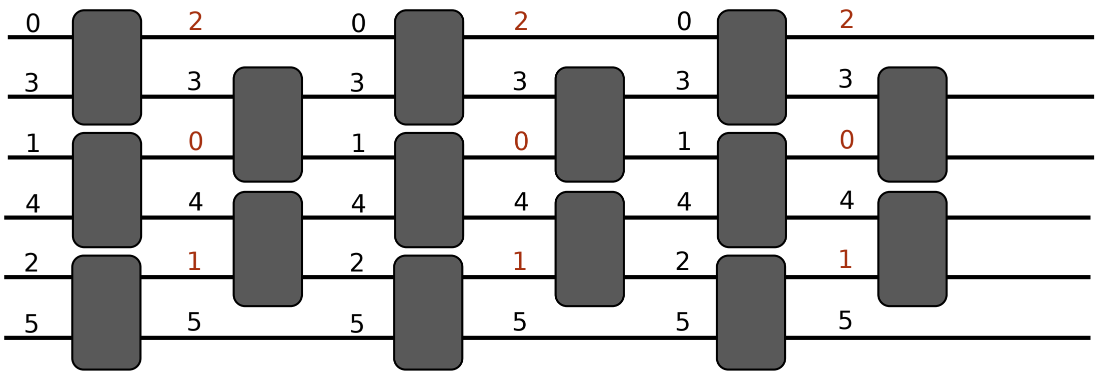
$$S \coloneqq \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{11}$$

Therefore, each even channel can be expressed as

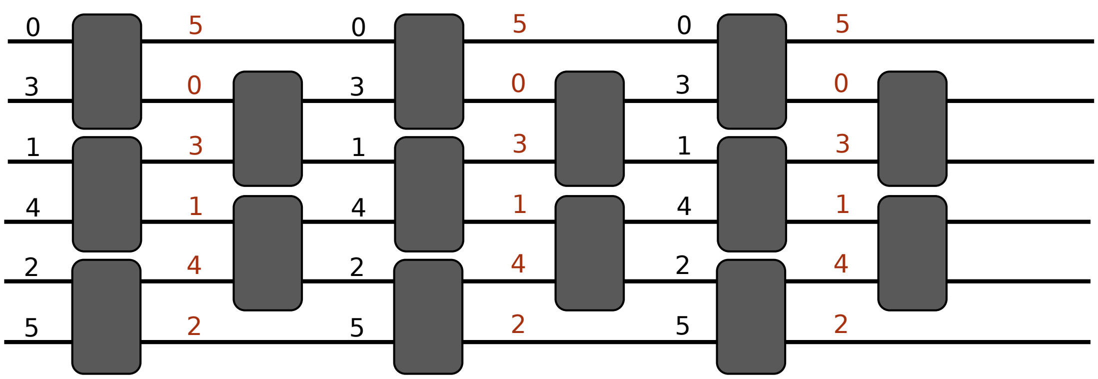
$$T^{(i)}_{\text{even}} = P^T S X \Xi(\chi^{(i)}) X \Xi(\eta^{(i)}) S P. \tag{12}$$



(a) Relabelling of the channels in the Clements universal multimode interferometer to apply the algorithm described in [2].



(b) Cyclic permutation of the first half of the channels in the even layers using the permutation matrix P in equation (10). For each layer, two-mode interactions occur between the same channels.



(c) Permutation of the first and second half of the channels using the permutation matrix S in equation (11). Transformation are represented by the same matrices in the odd and even layers.

Figure 3: Channel permutations required to decompose a unitary matrix U into a sequence of phase masks and discrete Fourier transforms.

If we insert identity $I = SS$ between the matrices in (12), we get

$$T^{(i)}_{\text{even}} = P^T X' \Xi'(\chi^{(i)}) X' \Xi'(\eta^{(i)}) P, \tag{13}$$

where the prime notation represents a swap transformation, i.e.

$$X' \coloneqq SXS = \frac{1}{\sqrt{2}} \begin{pmatrix} -I & I \\ I & I \end{pmatrix}, \tag{14}$$

and

$$\Xi'(\eta^{(i)}) \coloneqq S\Xi(\eta^{(i)})S = \text{diag}\{1,\ \ldots,\ 1,\ e^{i\eta_0^{(i)}},\ \ldots,\ e^{i\eta_{N/2-1}^{(i)}}\}. \tag{15}$$

It is convenient to re-express (14) as $X' = -ZXZ$, where Z is a diagonal matrix given by

$$Z \coloneqq \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{16}$$

Using the expressions for $T^{(i)}_{\text{even}}$ and $T^{(i)}_{\text{odd}}$ in equations (13) and (7), any unitary matrix U can thus be expressed as

$$U = D \prod_{i=1}^{N/2} T^{(i)}_{\text{even}} T^{(i)}_{\text{odd}}. \tag{17}$$

The next step is to express (17) as a sequence of phase masks interleaved with DFTs. To do so, we use the fact that circulant matrices are diagonalized by discrete Fourier transforms. A matrix is circulant if each of its columns is given by a cyclic permutation of the first column, with an offset given by the column index. If C is circulant, there exist a diagonal matrix D such that $C = F^\dagger D F$. An interesting property of the DFT matrix is that its inverse $F^\dagger$ is related to F by a permutation, i.e. $F^\dagger = \Pi F = F \Pi$, where Π is the permutation matrix given by

$$\Pi_{j,k} = \begin{cases} 1 & j = k = 0 \\ 1 & j = N - k \\ 0 & \text{otherwise} \end{cases}. \tag{18}$$

Furthermore, for any diagonal matrix U, $\Pi U \Pi$ is also diagonal. This allows to get a sequence that is made only from F and diagonal matrices. In the decomposition in (17), X can be decomposed as

$$X = GYG, \tag{19}$$

where

$$G \coloneqq \begin{pmatrix} I & 0 \\ 0 & iI \end{pmatrix} \tag{20}$$

and

$$Y \coloneqq \frac{1}{\sqrt{2}} \begin{pmatrix} I & -iI \\ -iI & I \end{pmatrix}. \tag{21}$$

G is already diagonal, and Y is circulant and can therefore be diagonalized using DFTs into the E matrix, whose elements are given by

$$E_{jj} = \frac{1}{\sqrt{2}} \left[1 - i(-1)^j\right]. \tag{22}$$

X can thus be expressed as $GF^\dagger EFG$. Since $X' = -ZXZ$ and $ZG = GZ = G^*$, we also have that $X' = -G^* F^\dagger E F G^*$.

The S matrix from (11) is also circulant. It can be diagonalized as $S = F^\dagger V F$, where V is the diagonal matrix defined as

$$V_{jj} = (-1)^j. \tag{23}$$

Finally, P can be decomposed as a product of diagonal and circulant matrices such that

$$P = X \Lambda X, \tag{24}$$

where Λ is a circulant matrix defined as

$$\Lambda \coloneqq \frac{1}{2} \begin{pmatrix} C + I & C - I \\ C - I & C + I \end{pmatrix}, \tag{25}$$

and $C \coloneqq \delta_{j,j+1 \pmod{N/2}}$. Λ can be diagonalized using DFTs into the H matrix, whose elements are given by

$$H_{jj} = \frac{1}{2} \left[1 - (-1)^j\right] + \frac{1}{2} \left[1 + (-1)^j\right] e^{i2\pi j/N}. \tag{26}$$

Therefore, P can be written using only phase masks and DFTs. Substituting the expression for P and X in (17), we get an expression of U that contains only Fourier transforms and phase masks:

$$U = DG \left[\prod_{i=1}^{N/2} B^{(i)} A^{(i)} \right] G^\dagger, \tag{27}$$

where $A^{(i)}$ and $B^{(i)}$ are sequences of phase masks given by

$$B^{(i)} = \left\{ E, p(G), H^* V, G\Xi'(\chi^{(i)}), E, p\left(\Xi'(\eta^{(i)})\right) \right\}_F, \tag{28}$$

$$A^{(i)} = \left\{ E, G, H^*, p\left(G\Xi(\theta^{(i)})\right), E, Z\Xi(\phi^{(i)}) \right\}_F. \tag{29}$$

In (28) and (29), we used the notations $\{D_1, \ldots, D_N\}_F \coloneqq \prod_{i=1}^N F D_i$ and $p(U) \coloneqq \Pi U \Pi$. The derivation for A and B is given in Appendix A.

To apply the algorithm on a matrix U, we must first relabel the channels of U by computing $U_p = K^T U K$, where K is the permutation matrix of equation (6). We can then find the parameters $\left\{\theta^{(i)}, \phi^{(i)}, \chi^{(i)}, \eta^{(i)}\right\}$ for the $N/2$ bi-layers of unit cells and the diagonal matrix D using the Clements scheme from [1] adapted for the transfer matrix of equation (1). We can then use the decomposition in (27) with the mask sequences in (28) and (29) to reconstruct the initial matrix U. This algorithm provides a sequence of $6N + 1$ phase masks to decompose a $N \times N$ unitary. The sequence is slightly different from the one prescribed by Lundeen *et al.* in their paper [2].

2 Challenges with the approach of López Pastor *et al.*

Some challenges were encountered when trying to implement the algorithm from López Pastor, Lundeen and Marquardt. First, the unit cell matrix used in their paper (see eq. (1) in [2]) cannot be constructed from the phase shifters and 50:50 beam splitters in the unit cell they provide. This error probably originates from Clements *et al.* where they use the same matrix form. The Clements algorithm can however be easily adapted to use the matrix of equation (1) by some small phase corrections.

A more significant difference between the López Pastor *et al.* approach and this current approach is the sequence of phase masks used for even layers in the interferometer. In their derivation, they use this expression as a starting point to derive the sequence of phase masks for even layers:

$$T^{(i)}_{\text{even}} = P^T X \Xi(\chi^{(i)}) X \Xi(\eta^{(i)}) P. \tag{30}$$

Using (30), and following the procedure described in sec. 1, we find the $6N+1$ layers prescribed in equations (7) and (8) of [2]. However, this sequence for even layers did not appear to work in our implementation.

Equation (30) suggests that the even layer has the same structure as the odd layer after a cyclic permutation of the first half of the channels. We argue that this is not the case, as we also need to swap the first and second half of the channels to have the same structure for odd and even layers. This stems from the fact that the unit cell of Figure 1 is not invariant under channel inversion. After the application of the permutation matrix P, the two-mode interactions in the even layers occur between the same pairs of channels as in the odd layers. However, these pairs are reversed, since now channels $\{0, \ldots, \frac{N}{2}-1\}$ are on the lower port in each unit cell as seen in Figure 3b. To fix this, we introduced a swap operation between the first and second half of the channels in even layers. After this swap, the two-mode interactions occur between the same pairs of channels and in the same order as in the odd layers, as seen in Figure 3c. The introduction of those swap operations, however, generates a different sequence of phase masks from the one in [2]. These changes were necessary for the functioning of our implementation.

3 Code implementation

A Python implementation of the algorithm described in this paper is available here, under the `lplm_interferometer` module.

A Derivation of the sequence of phase masks in (28) and (29)

Consider one bi-layer from (17):

$$T^{(i)}_{\text{even}}T^{(i)}_{\text{odd}} = P^T X'\Xi'(\chi^{(i)})X'\Xi'(\eta^{(i)})PX\Xi(\theta^{(i)})X\Xi(\phi^{(i)}), \tag{31}$$

where the prime notation stands for a swap operation, i.e. $X' := SXS$ Using the decomposition for P given in (24) into (31), we get

$$T^{(i)}_{\text{even}}T^{(i)}_{\text{odd}} = X\Lambda^T XX'\Xi'(\chi^{(i)})X'\Xi'(\eta^{(i)})X\Lambda\Xi(\theta^{(i)})X\Xi(\phi^{(i)}), \tag{32}$$

where we used the fact that $XX = I$ and $X^T = X$. Using the expression for X' in terms of X and Z ($X' = -ZXZ$) into (32), we find

$$T^{(i)}_{\text{even}}T^{(i)}_{\text{odd}} = X\Lambda^T XZX\Xi'(\chi^{(i)})XZ\Xi'(\eta^{(i)})X\Lambda\Xi(\theta^{(i)})X\Xi(\phi^{(i)}), \tag{33}$$

where we used the fact that $ZZ = I$. Using equation (19) to express X as a product of diagonal and circulant matrices, we find

$$T^{(i)}_{\text{even}}T^{(i)}_{\text{odd}} = GYG\Lambda^T S\Xi'(\chi^{(i)})GYGZ\Xi'(\eta^{(i)})GYG\Lambda\Xi(\theta^{(i)})GYG\Xi(\phi^{(i)}). \tag{34}$$

In (34), we used the fact that $XZX = S$. We can now diagonalize Y, S and Λ using equations (22), (23) and (26):

$$T^{(i)}_{\text{even}}T^{(i)}_{\text{odd}} = GF^\dagger EFGF^\dagger H^*VF\Xi'(\chi^{(i)})GF^\dagger EFGZ\Xi'(\eta^{(i)})GF^\dagger EFGF^\dagger HF\Xi(\theta^{(i)})GF^\dagger EFG\Xi(\phi^{(i)}) \tag{35}$$

$$= GFEF\Pi G\Pi FH^*VF\Xi'(\chi^{(i)})GFEF\Pi\Xi'(\eta^{(i)})\Pi FEFGFH^*F\Pi\Xi(\theta^{(i)})G\Pi FEFG\Xi(\phi^{(i)}) \tag{36}$$

$$= GFEFp(G)FH^*VF\Xi'(\chi^{(i)})GFEFp\left(\Xi'(\eta^{(i)})\right)FEFGFH^*Fp\left(\Xi(\theta^{(i)})G\right)FEFG\Xi(\phi^{(i)}) \tag{37}$$

$$= GFEFp(G)FH^*VF\Xi'(\chi^{(i)})GFEFp\left(\Xi'(\eta^{(i)})\right)FEFGFH^*Fp\left(\Xi(\theta^{(i)})G\right)FEFZ\Xi(\phi^{(i)})G^\dagger \tag{38}$$

where we used the fact that $\Pi F = F\Pi = F^\dagger$, $\Pi E = E\Pi$, $H\Pi = \Pi H^*$, and $GG = Z$.

Equation (38) generates the lists of phase masks $B^{(i)}$ and $A^{(i)}$ in (28) and (29), for a total of $6N + 1$ layers for a $N \times N$ unitary.

References

[1] William R. Clements, Peter C. Humphreys, Benjamin J. Metcalf, W. Steven Kolthammer, and Ian A. Walmsley. Optimal design for universal multiport interferometers. *Optica*, 3(12):1460–1465, Dec 2016.

[2] Víctor López Pastor, Jeff Lundeen, and Florian Marquardt. Arbitrary optical wave evolution with fourier transforms and phase masks. *Opt. Express*, 29(23):38441–38450, Nov 2021.