PERFECT MATCHING FREE SUBGRAPH PROBLEM

Abstract. Lacroix, Mahjoub, Martin and Picoleau have a polynomial time reduction from one-in-three SAT to PMFSP. We survey their result. Given a bipartite graph $G = (U \cup V, E)$, such that $|U| = |V|$ and every edge is labelled true or false or both, the perfect matching free subgraph problem is to determine whether or not there exists a subgraph of G containing, for each node u of U, either all the edges labelled true or all the edges labelled false incident to u, and which does not contain a perfect matching. In their joint paper, the authors first show that the stable set problem in a restricted case of tripartite graphs can be reduced to the perfect matching free subgraph problem. Then, a reduction from one-in-three SAT to the stable set problem in a restricted case of tripartite graphs is present to finish the proof of NP-completeness of the perfect matching free subgraph problem.

1. Introduction

Theorem 1. *Given a bipartite graph, if M is a maximum cardinality matching and S is a maximum stable set, then $|M| + |S| = |U \cup V|$.* □

Let $G = (U \cup V, E)$ be a bipartite graph such that $|U| = V| = n$. Let $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$. Suppose that every edge of E is labelled *true* or *false*, where an edge may have both true and false labels. For a node $u_i \in U$, let E_i^t and E_i^f demote the sets of edges incident to u_i labelled true and false, respectively. The *perfect matching free subgraph problem* (PMFSP) in G is to determine whether or not there exists a subgraph containing for each node $u_i \in U$ either E_i^t or E_i^f (but not both), and which is perfect matching free.

2. PMFSP and stable sets

Let $H = (V^1 \cup V^2 \cup V^3, F)$ be a tripartite graph where $|V^1| = |V^2| = |V^3| = n$, $V^j = \{v_1^j, \ldots, v_n^j\}$ for $j = 1, 2, 3$ and V^1 and V^2 are connected by the perfect matching $M = \{v_1^1v_1^2, v_2^1v_2^2, \ldots, v_n^1v_n^2\}$. We will consider the following problem: does there exist a stable set in H of size $n + 1$? We will call this problem the *tripartite stable set with perfect matching problem* (TSSPMP).

Theorem 2. *TSSPMP reduces to PMFSP in polynomial time.*

Proof. We show that an instance of TSSPMP can be transformed into an instance of PMFSP. For an edge $v_i^1v_i^2$ of the perfect matching where $v_i^1 \in V^1$ and $v_i^2 \in V^2$, we consider a node u_i in U. And for a node v_i^3 of V^3 we consider a node u_i in U. And for node v_i^3 of V^3 we consider a node v_i in V. Moreover, if $v_i^1v_k^3$ (resp. $v_i^2v_k^3$) is in F for some $i, k \in \{1, \ldots, n\}$, then we add an edge u_iv_k in E with label true (resp. false).

In what follows, we will show that there exists a stable set in H of size $n + 1$ if and only if there exists a subgraph $G' = (U \cup V, E')$ of G such that for each node $u_i \in U$, either $E_i^t \subset E'$ or $E_i^f \subset E'$, and G' is perfect matching free. In fact, suppose first that there exists a subgraph G' of G that satisfies the required properties. Since G' is perfect matching free, this implies that a maximum cardinality matching in G' contains fewer than n edges. As $|U \cup V| = 2n$ by Theorem 1 there exists a stable set in G', say S', of size $|S'| \geq n + 1$. Now from S', we are going to construct a stable set in H with the same cardinality. Let S be the node subset of H obtained as follows. For every node $v_j \in V \cap S'$, add node v_j^3 in S. And for every node $u_i \in U \cap S'$, add node v_i^1 in S if $E_i^t \subseteq E'$ and node v_2^i if $E_i^f \subseteq E'$. As $|S'| \geq n + 1$, we have $|S| \geq n + 1$. Now it is not hard to see that S is a stable set. Moreover from a stable set in H of size greater or equal to $n + 1$ one can obtain along the same line a stable set in G with the same cardinality. □

3. The NP-completeness of PMFSP

Theorem 3. *TSSPMP is NP-complete.*

Proof. Suppose we are given an instance of 1-in-3 3SAT with a set of n literals $L = \{l_1, \ldots, l_n\}$ and a set of m clauses $C = \{C_1, \ldots, C_m\}$. We shall construct an instance of TSSPMP on a graph $H = (V^1 \cup V^2 \cup V^3, F)$ where $|V^1| = |V^2| = |V^3| = p = 3n + m - 1$ and the set of edges between V^1 and V^2 correspond exactly to a perfect matching. We will show that H has a stable set of size $p + 1$ if and only if 1-in-3 3SAT admits a truth assignment. The construction will be done in 4 steps. First, with each literal $l_i \in L$, we associate the nodes $v_i^1, \bar{v}_i^1 \in V^1$, $v_i^2, \bar{v}_i^2 \in V^2$ and $v_i^3, \bar{v}_i^3 \in V^3$, along with the edges $v_i^1\bar{v}_i^2, \bar{v}_i^2v_i^3, v_i^3\bar{v}_i^1, \bar{v}_i^1v_i^2, v_i^2\bar{v}_i^3, \bar{v}_i^3v_i^1$ in F. These nodes will be called *literal nodes* and the edges *literal edges*. Note that these edges form a cycle Γ_i. Moreover, if the stable set contains three nodes, these must be either $\{v_i^1, v_i^2, v_i^3\}$ or $\{\bar{v}_i^1, \bar{v}_i^2, \bar{v}_i^3\}$. For every $j \in \{1, \ldots, m\}$ and $k \in \{1, 2, 3\}$, we associate with x_j^k the node set $\{v_i^1, v_i^2, v_i^3\}$ (resp. $\{\bar{v}_i^1, \bar{v}_i^2, \bar{v}_i^3\}$) if $x_j^k = l_i$ (resp. $x_j^k = \bar{l}_i$) for some $i \in \{1, \ldots, m\}$. As a consequence we have that $\{\bar{v}_i^1, \bar{v}_i^2, \bar{v}_i^3\}$ (resp. v_i^1, v_i^2, v_i^3) is associated with $\bar{x}_j^k$ if $x_j^k = l_i$ (resp. $x_j^k = \bar{l}_i$) for some $i \in \{1, \ldots, m\}$.

Next, we add for each clause C_j, $j = 1, \ldots, m$ the nodes $w_j^1 \in V^1$, $w_j^2 \in V^2$, $w_j^3 \in V^3$ along with the edges $w_j^1 w_j^2$, $w_j^2 w_j^3$, $w_j^3 w_j^1$. These nodes will be called *clause nodes*, the edges *clause edges*. Note that these edges form a triangle, which will be denoted by T_j, for $j = 1, \ldots, m$.

In the next step, for each $C_j = (x_j^1, x_j^2, x_j^3)$, we add edges between w_j^1 (resp. w_j^2) and all the nodes associated with $\bar{x}_j^1$, x_j^2 and x_j^3 (resp. x_j^1, $\bar{x}_j^2$ and x_j^3) which belong to V^3. Moreover, we add edges between w_j^3 and all the nodes associated with x_j^1, x_j^2 and $\bar{x}_j^3$ which belong to V^1 and V^2. All these edges will be called *satisfiability edges*.

Finally, we add the nodes $z_q^1 \in V^1$, $z_q^2 \in V^2$, $z_q^3 \in V^3$ for $q = 1, \ldots, n-1$. These nodes will be called *fictitious nodes*. For each fictitious node in $V^1 \cup V^2$, we add edges to connect this node to the all nodes in V^3. And for each fictitious node V^3, we add edges to connect this node to the all non fictitious nodes in $V^1 \cup V^2$. We also add the edges $z_q^1 z_q^2$ for $q = 1, \ldots, n-1$. Observe that $|V^1| = |V^2| = |V^3| = p$. Moreover, the edges between V^1 and V^2 form a perfect matching given by the edges $v_i^1 \bar{v}_i^2$, $\bar{v}_i^1 v_i^2$, $i = 1, \ldots, n$, $w_j^1 w_j^2$, $j = 1, \ldots, m$, and $z_q^1 z_q^2$, $q = 1, \ldots, n-1$.

Thus, from an instance of the 1-in-3 3SAT with n literals and m clauses, we obtain a tripartite graph with $9n + 3m - 3$ nodes and $10n^2 + 4nm - 5n + 14m + 1$ edges.

Claim 4. *Any stable set in H cannot contain more than $3n + m$ nodes. Moreover, if a stable set contains $3n + m$ nodes, then it does not contain any fictitious node.* □

In what follows we show that there exists in H a stable set of size $3n + m$ if and only if 1-in-3 3SAT admits a solution.
($\Rightarrow$) Let S be a stable set in H of size $3n + m$. By Claim 4, S does not contain any fictitious node. Thus, as $|S| = 3n + m$, S intersects each cycle Γ_i in exactly three nodes and each triangle T_j in exactly one node. Moreover, we have either $S \cap \Gamma_i = \{v_i^1, v_i^2, v_i^3\}$ or $S \cap \Gamma_i = \{\bar{v}_i^1, \bar{v}_i^2, \bar{v}_i^3\}$, for $i = 1, \ldots, n$. Consider the solution I for 1-in-3 3SAT defined as follows. If $v_i^k \in S$ (resp. $\bar{v}_i^k \in S$), $k = 1, 2, 3$, then associate the true (resp. false) value to the literal l_i, for $i = 1, \ldots, n$. In what follows we will show that for each clause $C_j = (x_j^1, x_j^2, x_j^3)$, we have exactly one variable with true value. For this it suffices to show that a clause node of T_j is in S if and only if the corresponding variable is of true value. Indeed, suppose that $w_j^1 \in S$. We may suppose that $x_j^1 = l_i$, the case where $x_j^1 = \bar{l}_i$ is similar. By construction of H, as the satisfiability edge $w_j^1 \bar{v}_i^3$ belongs to F, it follows that $\bar{v}_i^3 \notin S$. By the remark above, this implies that v_i^1, v_i^2, v_i^3 belong to S. Therefore literal l_i has a true value in solution I. Thus x_j^1 has a true value. It is similar for w_j^2 and w_j^3. Conversely, if $x_j^1 = l_i$, then by definition of I, v_i^1, v_i^2, $v_i^3 \in S$. Moreover, the satisfiability edges $w_j^2 v_i^3$, $w_j^3 v_i^2$ belong to F. As $|S \cap T_j| = 1$, it follows that $w_j^1 \in S$.

As a consequence, as S contains exactly one clause node from each T_i, it follows that each clause has exactly one true variable.
($\Leftarrow$) Suppose that there exists a solution I of 1-in-3 3SAT. Let S be the node set obtained as follows. If l_i has true value in I, then add v_i^1, v_i^2, v_i^3 to S, otherwise add $\bar{v}_i^1$, $\bar{v}_i^2$, $\bar{v}_i^3$ to S. For each clause $C_j = (x_j^1, x_j^2, x_j^3)$, $j = 1, \ldots, m$, add node w_j^k to S if x_j^k has true value, for $k \in \{1, 2, 3\}$. We have that $|S| = 3n + m$. We now show that S is a stable set. For this, it suffices to show that no clause node in S is adjacent to a literal node. Consider $j \in \{1, \ldots, m\}$ and suppose that x_j^1 has a true value in C_j. (The case where x_j^k, $k \in \{2, 3\}$, has a true value is similar.) Node w_j^1 associated with x_j^1, which is in S, is adjacent to exactly three literal nodes, namely the ones associated with $\bar{x}_j^1$, x_j^2 and x_j^3, in V_3. As I is a solution of the 1-in-3 3SAT, and in consequence, these variables have all false values, by construction, the nodes corresponding to these variables do not belong to S. Therefore w_j^1 cannot be adjacent to one of these nodes, and hence the proof is complete. □