2016 IMO 第二階段選訓營

【獨立研究一】2016 年 4 月 9 日 10:40~12:30

1. 設a,b為兩正整數，並且$a!b!$為$a!+b!$的倍數。證明: $3a\geq 2b+2$。

   Let a,b be positive integers such that $a!b!$ is a multiple of $a!+b!$. Prove that $3a\geq 2b+2$.

2. 令$\mathbb{Z}$為所有整數所成的集合。求所有函數$f:\mathbb{Z}\to\mathbb{Z}$，滿足:

   $$f\left(f(x)+f(y)\right)+f(x)f(y)=f(x+y)f(x-y)$$

   對所有整數x,y都成立。

   Let $\mathbb{Z}$ be the set of all integers. Determine all function $f:\mathbb{Z}\to\mathbb{Z}$ such that

   $$f\left(f(x)+f(y)\right)+f(x)f(y)=f(x+y)f(x-y)$$

   holds for all $x,y\in\mathbb{Z}$.

【獨立研究二】2016 年 4 月 9 日 16:10~18:00

1. 設三角形 ABC 的外心為 O 點，並令三角形 BOC 的外接圓為 ω。直線 AO 與圓 ω 的第二個交點為 G 點。設 BC 邊的中點為 M，且 BC 中垂線交 ω 於 O,N 兩點。

   證明:線段 AN 的中點，位於三角形 OMG 的外接圓與以 AO 為直徑的圓的根軸上。

   Let O be the circumcenter of triangle ABC, and ω be the circumcircle of triangle BOC. Line AO intersects with circle ω again at the point G. Let M be the midpoint of side BC, and the perpendicular bisector of BC meets circle ω at the points O and N.

   Prove that the midpoint of the segment AN lies on the radical axis of the circumcircle of triangle OMG, and the circle whose diameter is AO.

2. 設 $\langle f_n \rangle$ 為費氏數列，亦即: $f_0=0$， $f_1=1$，且對所有非負整數 n， $f_{n+2}=f_{n+1}+f_n$ 均成立。

   試找出所有的正整數對 (a,b) 滿足 $a<b$，並且對任意的正整數 n， $f_n-2n\cdot a^n$ 總能被 b 整除。

   Let $\langle f_n \rangle$ be the Fibonacci sequence, that is, $f_0=0$, $f_1=1$, and that $f_{n+2}=f_{n+1}+f_n$ holds for all nonnegative integers n.

   Find all pairs (a,b) of positive integers with $a<b$ such that $f_n-2n\cdot a^n$ is divisible by b for all positive integers n.

【獨立研究三】2016 年 4 月 10 日 8:30~10:20

1. 設 n 為正整數，一條東西向的路上從東到西有 n 個城鎮。每個城鎮都派出兩頭犀牛，一隻從城鎮往東出發，另一隻從城鎮往西出發(犀牛不會轉向)，而這 $2n$ 頭犀牛的大小都不一樣。當兩頭犀牛面對面相撞時，大隻的會把小隻的撞出道路；但如果一隻犀牛從背後撞向另一頭犀牛，不論犀牛的大小，從背後被撞的犀牛都會被撞出道路。

   假設有兩個城鎮 A 和 B，其中 B 在 A 的東邊。如果 A 的東進犀牛可以一路抵達 B 而把其間的犀牛全部撞飛，則稱 A 城輾過 B 城。反之，如果 B 城的西進犀牛可以一路抵達 A 並把其間的犀牛全部撞飛，則稱 B 輾過 A。

   證明:恰有一個城鎮不會被任何其他城鎮輾過。

   Let n be a positive integer. There are n towns arranged on an East-West road. Each town has two rhinoceroses, one heading East from the town, and the other heading West from the town (rhinoceroses cannot turn into another direction.) These $2n$ rhinoceroses all have different sizes. When two rhinoceroses confront face to face, the larger one would knock the smaller one out of the road. However, if a rhinoceros is bumped by another form its rear end, the bumped rhinoceros is knocked out of the road, regardless of their sizes.

   Let A and B be two towns, with B being East to A. We say that A tramples B, if the East-heading rhinoceros of A can reach B by knocking off all rhinoceroses in between. Similarly, we say that B tramples A if the West-heading rhinoceros of B can reach A by knocking off all rhinoceroses in between.

   Prove that there is exactly one town that would not be trampled by any other town.

2. 已知 x, y 為滿足 $x+y=1$ 的正實數。試證:

$$\frac{x}{x^2+y^3}+\frac{y}{x^3+y^2}\le 2\left(\frac{x}{x+y^2}+\frac{y}{x^2+y}\right).$$

   Let x, y be positive real numbers such that $x+y=1$. Prove that

$$\frac{x}{x^2+y^3}+\frac{y}{x^3+y^2}\le 2\left(\frac{x}{x+y^2}+\frac{y}{x^2+y}\right).$$

【模擬競賽 Day 1】2016 年 4 月 11 日 8:30~13:00

1. 設 ABC 為銳角三角形，H 為其垂心。取點 D 使得四邊形 $HABD$ 為平行四邊形(其中 $AB // HD$ 及 $AH // BD$)。取 E 為直線 DH 上一點，使得直線 AC 通過線段 HE 的中點。令 F 為直線 AC 與三角形 DCE 的外接圓的另一個交點。

   證明: $EF = AH$ 。

   Let ABC be an acute triangle with orthocentre H. Let D be the point such that the quadrilateral $HABD$ is a parallelogram (with $AB // HD$ and $AH // BD$). Let E be the point on the line DH such that the line AC passes through the midpoint of the segment HE. Let F be the second point of intersection of the line AC and the circumcircle of triangle DCE.

   Prove that $EF = AH$.

2. 設 m, n 為正整數，且 $m > n$ 。對 $k = 1, 2, ..., n+1$ ，定義 $x_k = \dfrac{m+k}{n+k}$ 。證明:若 $x_1, x_2, ..., x_{n+1}$ 都是整數，則 $x_1 x_2 ... x_{n+1} - 1$ 能被某個奇質數整除。

   Let m and n be positive integers such that $m > n$. Define $x_k = \dfrac{m+k}{n+k}$ for $k = 1, 2, ..., n+1$. Prove that if $x_1, x_2, ..., x_{n+1}$ are integers, then $x_1 x_2 ... x_{n+1} - 1$ is divisible by an odd prime.

3. 平面上有一個正三角形網格，相鄰兩格點的距離為 1。有一個邊長為 n 的正三角形，其三個頂點都在格點上，三邊都落在格線上。現在，將此正三角形分割成 n^2 個面積相等的小三角形(不需為正三角形)，使得每個小三角形的三個頂點都在格點上。

   證明:其中至少有 n 個小三角形是正三角形。

   There is a grid of equilateral triangles with a distance 1 between any two neighboring grid points. An equilateral triangle with side length n lies on the grid so that all of its vertices are grid points, and all of its sides match the grid. Now, let us decompose this equilateral triangle into n^2 smaller triangles (not necessarily equilateral triangles) so that the vertices of all these smaller triangles are all grid points, and all these small triangles have equal areas.

   Prove that there are at least n equilateral triangles among these smaller triangles.

【模擬競賽 Day 2】2016 年 4 月 12 日 8:30~13:00

4. 假設正實數數列$a_1,a_2,\dots$滿足:對每一個正整數 k，都有

$$a_{k+1}\ge\frac{ka_k}{a_k^2+(k-1)}$$

試證:對每一個正整數$n\ge 2$，恆有

$$a_1+a_2+\dots+a_n\ge n.$$

Suppose that a sequence $a_1,a_2,\dots$ of positive real numbers satisfies

$$a_{k+1}\ge\frac{ka_k}{a_k^2+(k-1)}$$

for every positive integer *k*. Prove that $a_1+a_2+\dots+a_n\ge n$ for every $n\ge 2$.

5. 令 n 為一正整數，並在黑板上寫下$1,2,\dots,n$ 等數字。阿發和小李輪流從黑板上選擇一個數字，規則如下:

(i)你不能選擇之前被任何人選過的數字。

(ii)如果你之前選過 k，你不能選$k-1$或$k+1$。

(iii)如果所有數字被選完則雙方平手；否則，先沒有數字可選的人輸。

假設阿發先選。試求所有小李有必勝法的正整數 n。

Let *n* be a positive integer, and write down $1,2,\dots,n$ on the blackboard. Alpha and Lee take turn choosing a number from the board according to the following rules:

(i)You cannot choose any number that was previous selected by either player.

(ii)If you have chosen *k*, you cannot choose $k-1$ or $k+1$.

(iii)The game is a draw if all numbers are chosen. Otherwise, the player who cannot choose any number first lose the game.

Suppose Alpha chooses first. Determine all *n* such that Lee has a winning strategy.

6. 設凸五邊形 *AXYZB* 內接於一個以 *AB* 為直徑的半圓。令 *K* 為 *Y* 對 *AB* 的垂足，且令 *O* 為 *AB* 的中點，令 *L* 為 *XZ* 與 *YO* 的交點。在直線 *KL* 上取一點 *M* 使得 $MA=MB$，及設 *I* 為 *O* 對直線 *XZ* 的對稱點。

證明:若四邊形 *XKOZ* 內接於一圓，則四邊形 *YOMI* 也內接於一圓。

Let *AXYZB* be a convex pentagon inscribed in a semicircle with diameter *AB*, and let *K* be the foot of the altitude from *Y* to *AB*. Let *O* denote the midpoint of *AB* and *L* be the intersection of *XZ* with *YO*. Select a point *M* on line *KL* with $MA=MB$, and finally, let *I* be the reflection of *O* across *XZ*.

Prove that if quadrilateral *XKOZ* is cyclic then so is quadrilateral *YOMI*.