2016 IMO 第三階段選訓營

【獨立研究一】2016 年 4 月 23 日 8:30~10:20

1. 已知 n 是正整數，求多項式 $(x^2-x+1)^n$ 的奇係數的個數。

   Let n be a positive integer. Find the number of odd coefficients of the polynomial $(x^2-x+1)^n$.

2. 給定一正整數 k。設正整數數列 $a_0,a_1,...,a_n(n>0)$ 滿足下列所有條件:

   (i) $a_0=a_n=1$；

   (ii)對任何的 $i=1,2,...,n-1$，都有 $2\le a_i\le k$；

   (iii)對任何的 $j=2,3,...,k$，j 在 $a_0,a_1,...,a_n$ 中皆出現 $\varphi(j)$ 次($\varphi(j)$ 代表不超過 j 且與 j 互質之正整數的個數)；

   (iv)對任何的 $i=1,2,...,n-1$， $\gcd(a_{i-1},a_i)=1=\gcd(a_i,a_{i+1})$，並且 a_i 整除 $a_{i-1}+a_{i+1}$。

   現另有一整數數列 $b_0,b_1,...,b_n$ 滿足:對所有的 $i=0,1,...,n-1$，都有 $\dfrac{b_{i+1}}{a_{i+1}}>\dfrac{b_i}{a_i}$。試求 b_n-b_0 的最小值。

   Let k be a positive integer. A sequence $a_0,a_1,...,a_n(n>0)$ of positive integers satisfies the following conditions:

   (i) $a_0=a_n=1$;

   (ii) $2\le a_i\le k$ for each $i=1,2,...,n-1$;

   (iii) For each $j=2,3,...,k$, the number j appears $\varphi(j)$ times in the sequence $a_0,a_1,...,a_n$ ($\varphi(j)$ is the number of positive integers that do not exceed j and are coprime to j);

   (iv) For any $i=1,2,...,n-1$, $\gcd(a_{i-1},a_i)=1=\gcd(a_i,a_{i+1})$, and a_i divides $a_{i-1}+a_{i+1}$.

   There is another sequence $b_0,b_1,...,b_n$ of integers such that $\dfrac{b_{i+1}}{a_{i+1}}>\dfrac{b_i}{a_i}$ for all $i=0,1,...,n-1$. Find the minimum value for b_n-b_0.

【獨立研究二】2016 年 4 月 23 日 16:10~18:00

1. 已知正實數 x, y, z 滿足 $x+y+z=1$。試求使不等式

$$\frac{x^2y^2}{1-z}+\frac{y^2z^2}{1-x}+\frac{z^2x^2}{1-y}\le k-3xyz$$

恆成立的實數 k 的最小值。

Let x, y, z be positive real numbers satisfying $x+y+z=1$. Find the smallest k such that

$$\frac{x^2y^2}{1-z}+\frac{y^2z^2}{1-x}+\frac{z^2x^2}{1-y}\le k-3xyz.$$

2. 平面上有一個凸 $3n$ 邊形，其每個頂點上都有一台機器人，每台機器人都射出一道雷射光指向另一台機器人。你每次操作可以選取一台機器人，叫它順時鐘旋轉，直到它的雷射光指向一台新的機器人為止。當三台機器人 A、B、C，其中 A 的雷射光射向 B，B 的雷射光射向 C，而 C 的雷射光射向 A 時，我們稱這三台機器人構成一個三角形。試問:至少要多少次操作，才能保證平面上出現 n 個三角形?

There’s a convex $3n$-polygon on the plane with a robot on each of its vertices. Each robot fires a laser beam toward another robot. On each of your move, you select a robot to rotate clockwise until its laser points at a new robot. Three robots A,B and C form a triangle, if A’s laser points at B, B’s laser points at C, and C’s laser points at A. Find the minimum number of moves that can guarantee n triangles on the plane.

【獨立研究三】2016 年 4 月 24 日 8:30~10:20

1. 設ΔABC為銳角三角形，其中$\angle B \neq \angle C$。設 M 為 BC 邊中點，E、F 分別為過 B、C 點的高的垂足，令 K、L 分別為線段 ME、MF 中點。在直線 KL 上取一點 T 使得 $AT//BC$。

   證明: $TA = TM$ 。

   Let ΔABC be an acute-angled triangle, with $\angle B \neq \angle C$. Let M be the midpoint of side BC, and E,F be the feet of the altitude from B,C respectively. Denote by K,L the midpoints of segments ME,MF, respectively. Suppose T is a point on the line KL such that $AT//BC$.

   Prove that $TA = TM$.

2. 令所有正實數所成的集合為$\mathbb{R}^+$。求所有函數 $f:\mathbb{R}^+ \to \mathbb{R}^+$，滿足:

$$f\left(x+y+f(y)\right) = 4030x - f(x) + f(2016y)$$

   對所有正實數x, y都成立。

   Let $\mathbb{R}^+$ be the set of all positive real numbers. Determine all functions $f:\mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$f\left(x+y+f(y)\right) = 4030x - f(x) + f(2016y), \forall x, y \in \mathbb{R}^+ .$$

【模擬競賽 Day 1】2016 年 4 月 25 日 8:30~13:00

1. 設$\lambda > 0$為滿足方程式$\lambda = \lambda^{2/3} + 1$的正實數。證明:存在正整數$M$使得

$$\left|M - \lambda^{300}\right| < 4^{-100}.$$

Let $\lambda > 0$ be a positive real number satisfying $\lambda = \lambda^{2/3} + 1$. Show that there exists a positive integer M such that

$$\left|M - \lambda^{300}\right| < 4^{-100}.$$

2. 設三角形ABC中，$CA \neq CB$。令點D、F、G分別為三邊AB、AC、BC的中點。一圓Γ通過C點，並與AB切於D點。設圓Γ分別與線段AF及BG交於H、I點。令點H'為H對F的對稱點，又令點I'為I對G的對稱點。直線$H'I'$分別與CD、FG交於Q、M點。設直線CM與圓Γ再交於P點。

證明: $CQ = QP$。

Let ABC be a triangle with $CA \neq CB$. Let D,F and G be the midpoints of the sides AB,AC, and BC, respectively. A circle Γ passing through C and tangents to AB at D meets the segment AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

3. 你受託幫神盾局辦餐會，但局內有若干對員工是仇人。對一群至少包含 3 個人且人數為奇數的員工們而言，只要可以讓他們圍著一個圓桌入坐，使得任何相鄰的兩個員工都是仇人的話，就稱這群員工為<u>復仇者聯盟</u>。

你發現:如果想要將所有員工分坐若干桌，使得同桌的任兩人都不是仇人的話，至少需要 11 張桌子。試證:神盾局內至少可找到$2^{10} - 11$個復仇者聯盟。

You are responsible for arranging a banquet for an agency. In the agency, some pairs of agents are enemies. A group of agents are called *avengers*, if and only if the number of agents in the group is odd and at least 3, and it is possible to arrange all of them around a round table so that every two neighbors are enemies.

You figure out a way to assign all agents to 11 tables so that any two agents on the same tables are not enemies, and that's the minimum number of tables you can get. Prove that there are at least $2^{10} - 11$ avengers in the agency.

【模擬競賽 Day 2】2016 年 4 月 26 日 8:30~13:00

4. 對於一個由有限多個正整數所成的集合 A，我們將它分割成兩個非空的子集 A_1 和 A_2。我們稱 (A_1, A_2) 是個好分割，若且唯若 A_1 所有元素的最小公倍數等於 A_2 所有元素的最大公因數。試求最小的 n，使得存在一個由 n 個正整數所成的集合，其恰好有 2015 個好分割。

   For a finite set A of positive integers, we call a partition of disjoint nonempty subsets A_1 and A_2 are *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set n positive integers with exactly 2015 good partitions.

5. 已知非常數的整係數多項式 $f(x)$ 滿足

$$(x^3+4x^2+4x+3)f(x)=(x^3-2x^2+2x-1)f(x+1)$$

   證明:對所有正整數 $n(n\geq 8)$， $f(n)$ 至少有五個不同的質因數。

   Let $f(x)$ be the polynomial with integer coefficients ($f(x)$ is not constant) such that

$$(x^3+4x^2+4x+3)f(x)=(x^3-2x^2+2x-1)f(x+1)$$

   Prove that for each positive integer $n(n\geq 8)$, $f(n)$ has at least five distinct prime divisors.

6. 設 $ABCD$ 為凸四邊形，點 P,Q,R,S 分別在邊 AB,BC,CD,DA 上。直線 PR 與 QS 交於 O 點。設四個四邊形 $APOS,BQOP,CROQ,DSOR$ 都有內切圓。證明:直線 AC,PQ,RS 共點或是兩兩互相平行。

   Let $ABCD$ be a convex quadrilateral, and let P,Q,R,S be points on the sides of AB,BC,CD, and DA, respectively. Let the line segments PR and QS meet at O. Suppose that each of the quadrilateral $APOS,BQOP,CROQ$, and $DSOR$ has an incircle. Prove that the lines AC,PQ, and RS are either concurrent or parallel to each other.