IDS 702

Linear Regression - 1

September 1, 2022
Dr. Andrea Lane

Agenda

1. Reading poll
2. Big picture review
3. SLR review
4. EDA/SLR activity
5. MLR

Learning Objectives

By the end of today's class, you should be able to:

- Identify when SLR and MLR are useful (e.g., what kind of data?)
- Describe ordinary least squares (OLS) estimation
- Generate EDA plots in R
- Generate an SLR model in R

1. Reading poll

Sakai $\longrightarrow$ Polls

2. Big picture review

Data analysis depends on the data

Types of variables	Examples

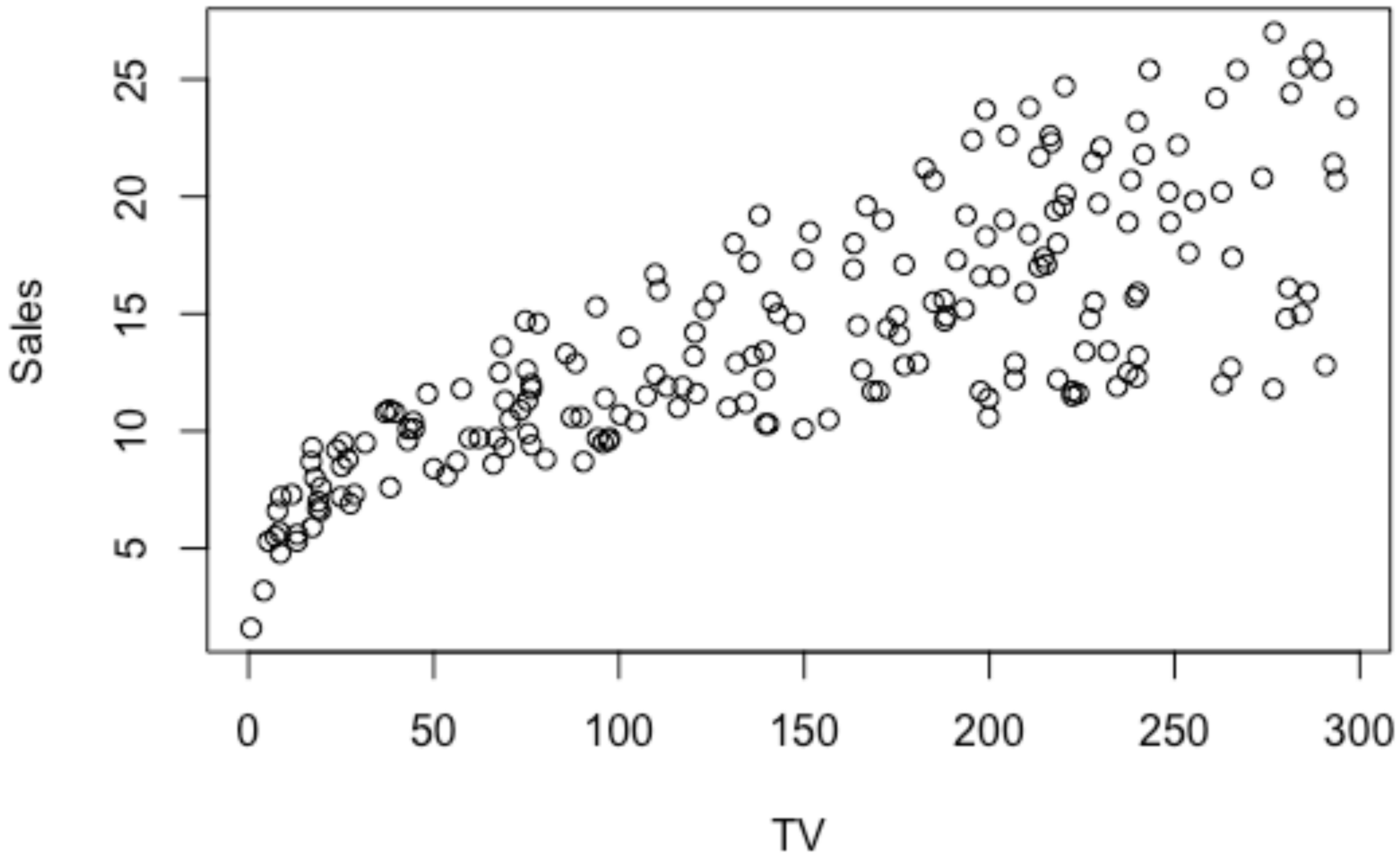
Type of model depends on the response variable

Types of variables	Model

3. SLR Review

Simple Linear Regression

Goal: Examine the relationship between two continuous variables



SLR Model

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i; \epsilon_i \overset{\text{iid}}{\sim} N(0, \sigma^2), i = 1, \ldots, n$$

SLR Model

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i; \epsilon_i \overset{\text{iid}}{\sim} N(0, \sigma^2), i = 1, \ldots, n$$

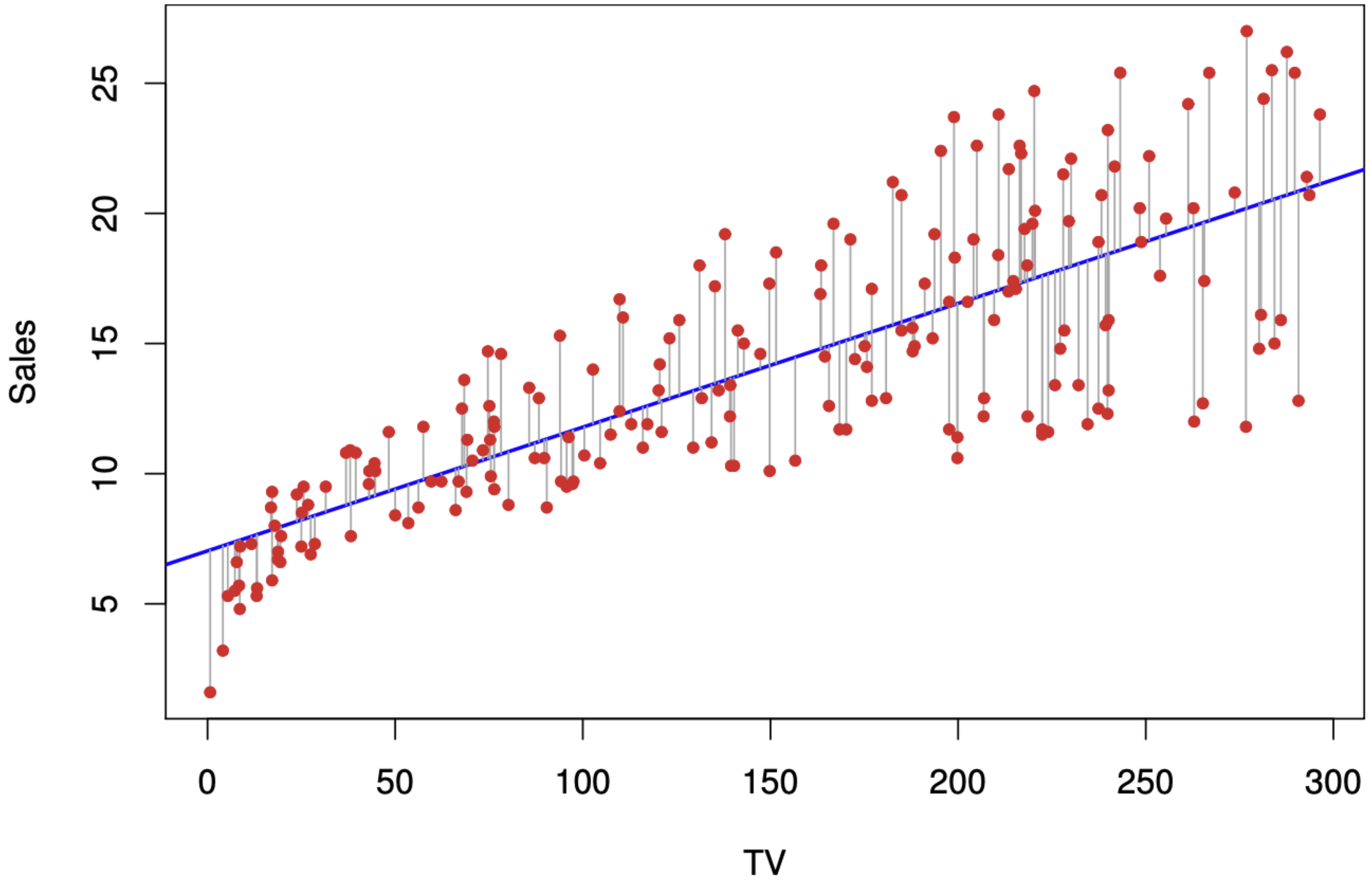
Goals: Estimation and Inference

Assumptions for Linear Regression

- Linear relationship between X and Y
- Independence of errors
- Equal variance of errors
- Normality of errors

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i; \epsilon_i \overset{\text{iid}}{\sim} N(0, \sigma^2), i = 1, \ldots, n$$

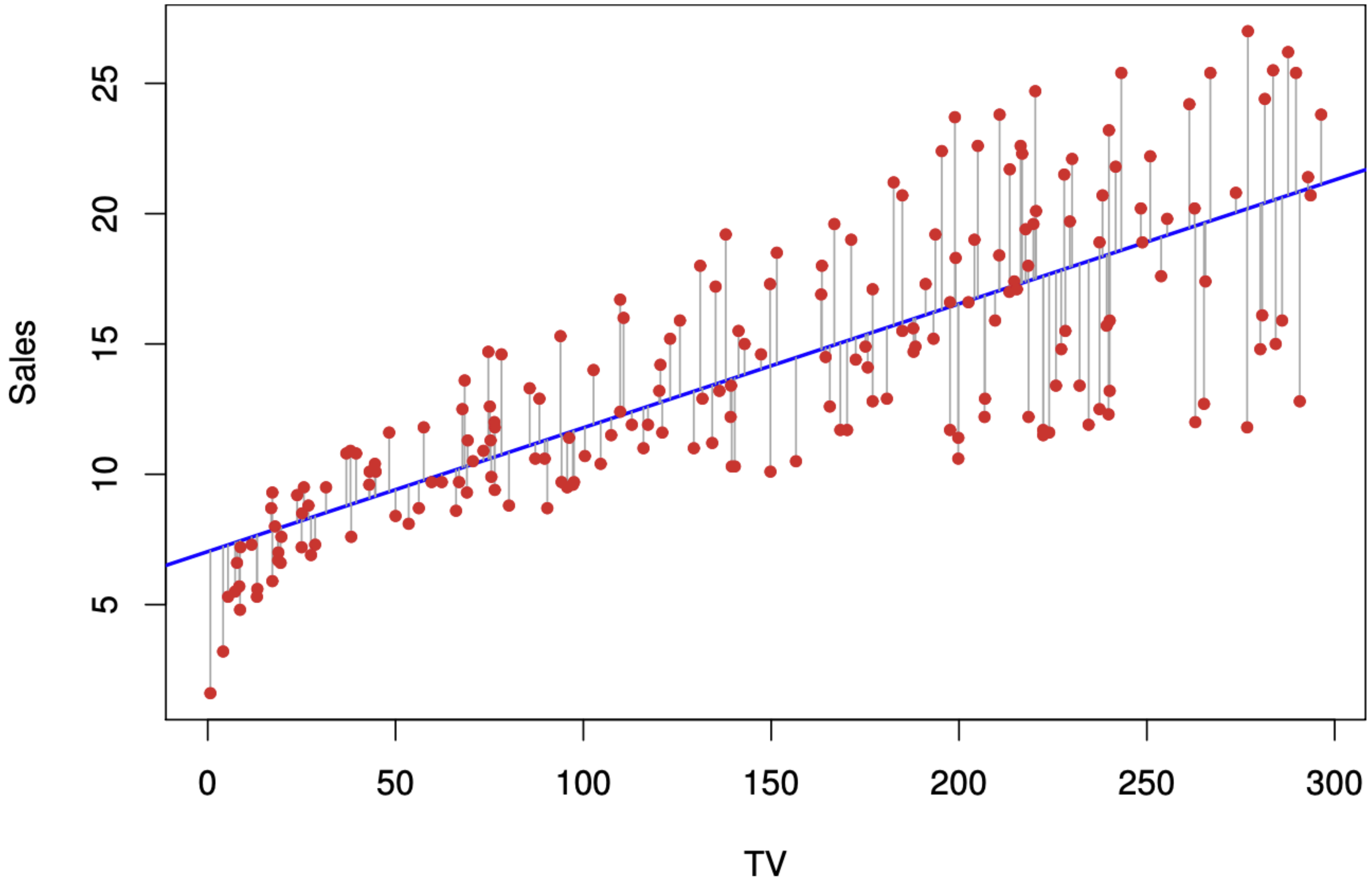
Estimation: Ordinary Least Squares



OLS chooses estimates that minimize the **residual sum of squares**

$$\sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

Estimated Regression Line



$$\hat{Y} = 7.032 + 0.048X$$

Inference

$$H_0 : \beta_1 = 0$$

$$t = \frac{\hat{\beta}_1}{SE(\hat{\beta}_1)}$$

Assessing the accuracy of the model

- Residual standard error (RSE)
  - Estimate of the standard error of ϵ
  - $\sqrt{\frac{1}{n-2}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}$

- R^2 statistic
  - Proportion of variance explained
  - $1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$

Let’s see it in R!

- Advertising.csv located in Sakai (Resources — datasets or Lessons)

4. EDA/SLR Activity

Your turn!

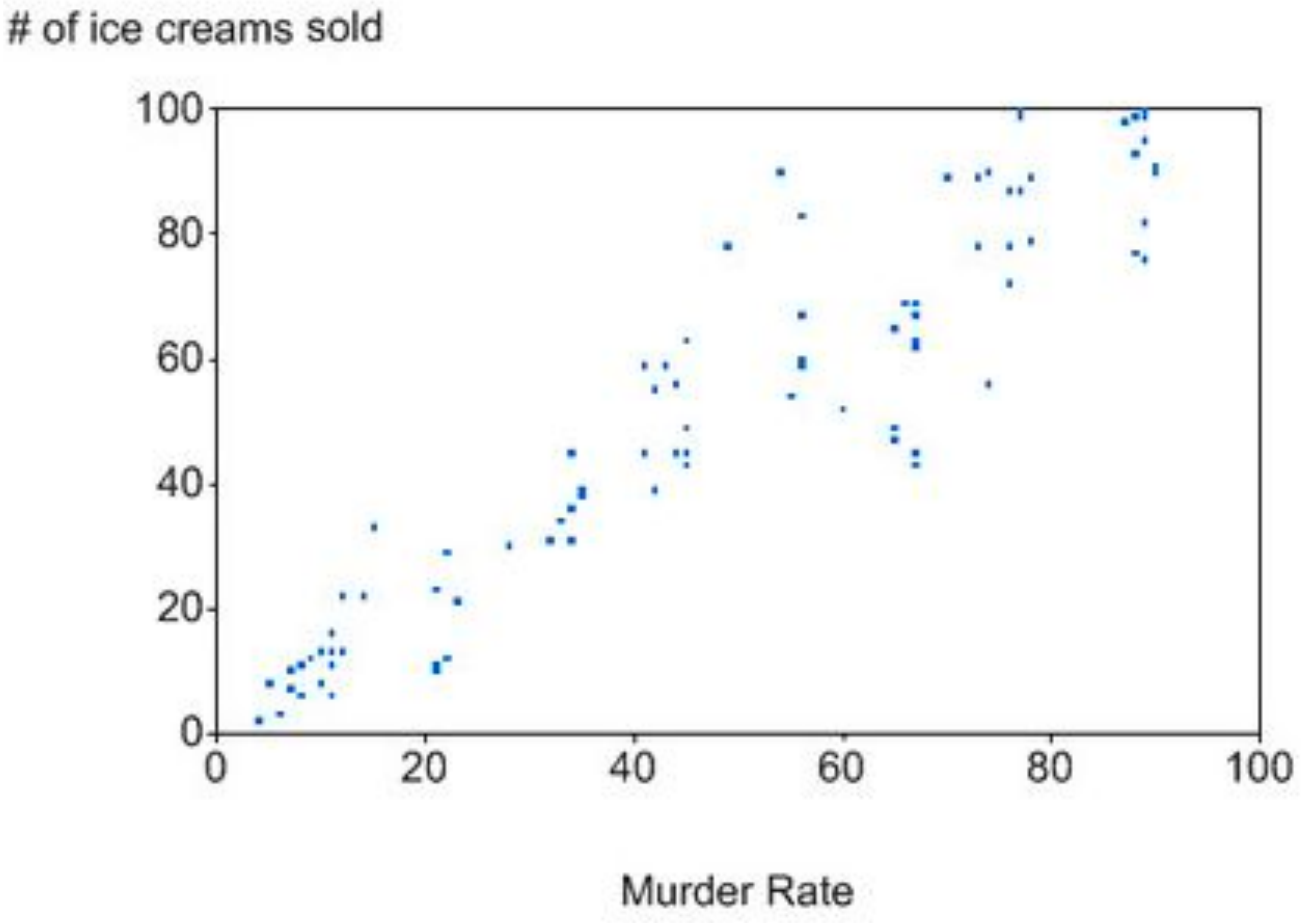
In a group of 2-3, generate a SLR model with the “Boston” dataset

- Install the “ISLR2” package
- data(“Boston”)
- Select two (continuous) variables you are interested in
    - Generate a histogram for each variable
    - Generate a scatter plot to visually assess the relationship
    - Generate the SLR model and note the estimates, relevant p-value, RSE, and R^2

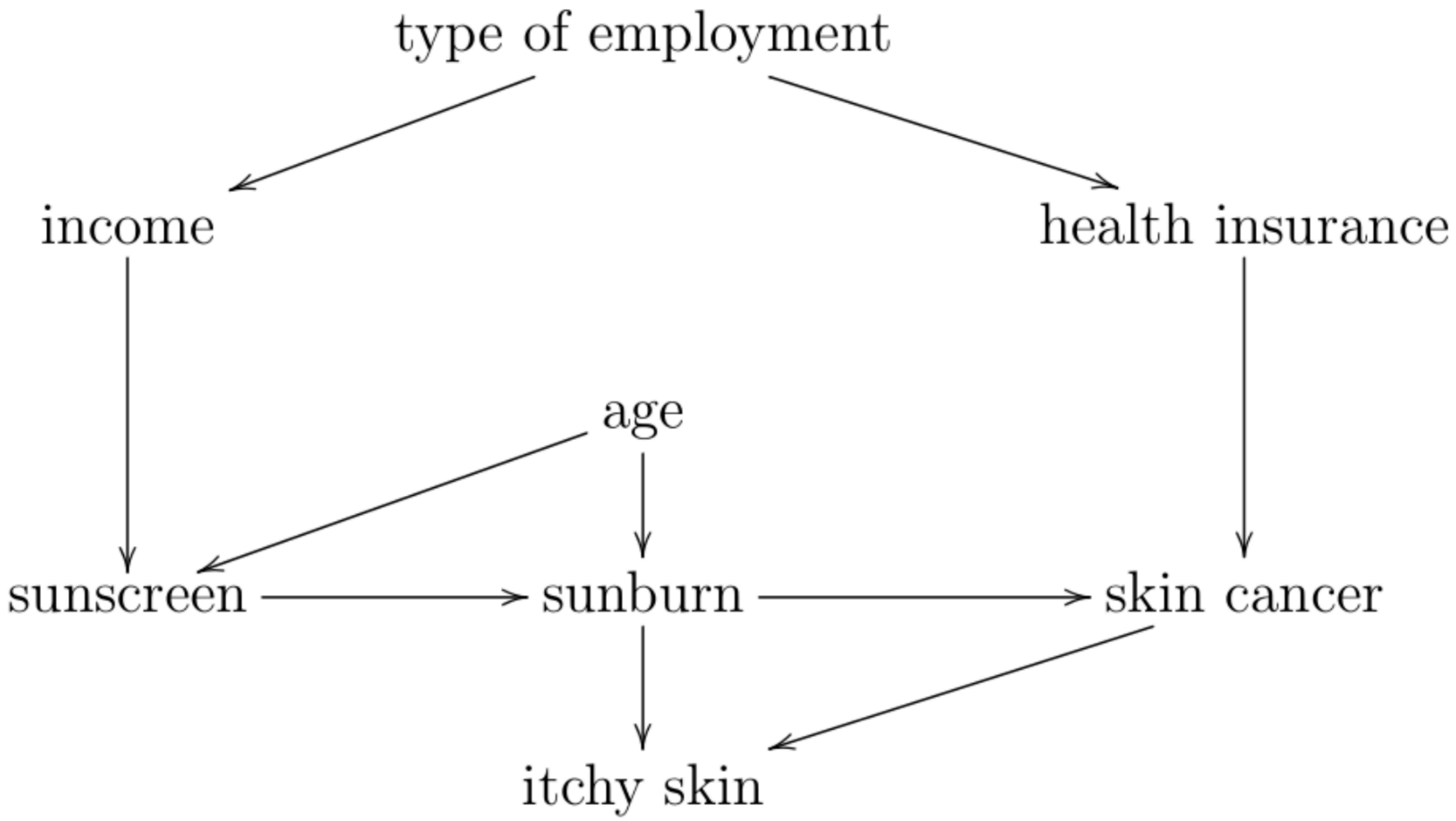
5. MLR

Most relationships cannot be fully explained by two variables

- **Confounding variables** are related to both variables of interest and explain (at least) some of the relationship between them



Directed Acyclic Graph (DAG)



https://health.ucdavis.edu/ctsc/area/Resource_Library/documents/directed-acyclic-graphs20220209.pdf

Multiple Linear Regression Model

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip} + \epsilon_i; \epsilon_i \overset{\text{iid}}{\sim} N(0, \sigma^2), i = 1, \ldots n$$

We can also write the model as:

$$y_i \overset{\text{iid}}{\sim} N(\beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip}, \sigma^2)$$

$$p(y_i \mid x_i) = N(\beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip}, \sigma^2)$$

MLR Assumptions

- Linear relationship between EACH X and Y
- Independence of errors
- Equal variance of errors
- Normality of errors
- No multicollinearity

Estimation: Ordinary Least Squares

Coefficient estimates are obtained by taking partial derivatives of the sum of squares of the errors with respect to each parameter

$$\sum_{i=1}^{n} (y_i - [\beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip}])^2$$

Matrix Representation

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon; \epsilon \sim \mathbf{N}(\mathbf{0}, \sigma^2\mathbf{I})$$

Matrix Representation

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon; \epsilon \sim \mathbf{N}(\mathbf{0}, \sigma^2\mathbf{I})$$

Then the OLS estimates are:

$$\hat{\beta} = (\mathbf{X}^\mathrm{T}\mathbf{X})^{-1}\mathbf{X}^\mathrm{T}\mathbf{Y}$$

For more on LR matrix representation: “Plane Answers to Complex Questions” by Ronald Christensen

Matrix Representation

The predictions can be written as:

$$\hat{\mathbf{Y}} = \mathbf{X}\hat{\beta} = \mathbf{X}[(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathbf{T}}\mathbf{Y}]$$

And the residuals can be written as:

$$\mathbf{e} = \mathbf{Y} - \hat{\mathbf{Y}} = \mathbf{Y} - [\mathbf{X}(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathbf{T}}]\mathbf{Y} = [\mathbf{1}_{\mathbf{n}} - \mathbf{X}(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathbf{T}}]\mathbf{Y}$$

Hat matrix/Projection matrix:

$$\mathbf{H} = \mathbf{X}(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathbf{T}}$$

Matrix Representation: SE

$$s_e^2 = \sum_{i=1}^{n} \frac{(y_i - \hat{y}_i)^2}{n - (p+1)} = \frac{(\mathbf{Y} - \mathbf{X}\hat{\beta})^{\mathbf{T}}(\mathbf{Y} - \mathbf{X}\hat{\beta})}{n - (p+1)} = \frac{\mathbf{e}^{\mathbf{T}}\mathbf{e}}{n - (p+1)}$$

The variance of the OLS estimates of all (p+1) coefficients is

$$\mathbf{V}[\hat{\beta}] = \sigma^2(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-1}$$

Note that this is a **covariance matrix**; the square root of the diagonal elements give us the standard errors for each coefficient, which we can use for hypothesis testing

Wrap-up

- Statistical Reflection I due Friday (9/2) 11:55 PM
- Reading for next week will be posted by Friday (9/2) 11:55 PM
- First data analysis assignment will be posted by Tues (9/6) at the latest
    - Due Sept 16