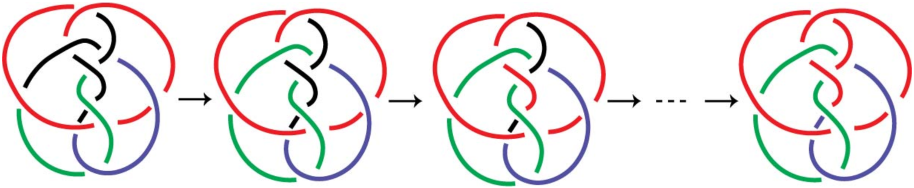

An Algebraic Definition of the Bridge Number of a Knot

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USTARS 2018 at Reed College

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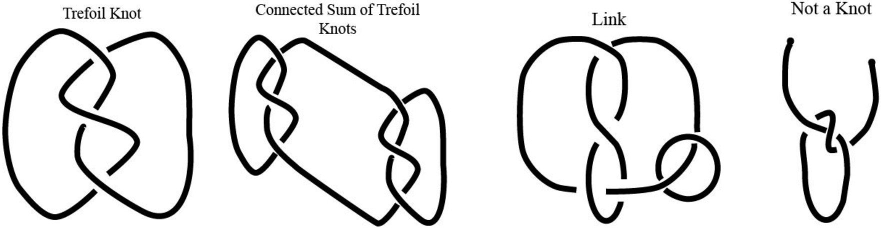
Outline

- Knot theory basics
- Define Wirtinger number, $\omega(K)$
- Main theorem: $\beta(K) = \omega(K)$
- An implementation
- Some results
- Future Work

What is a Knot?

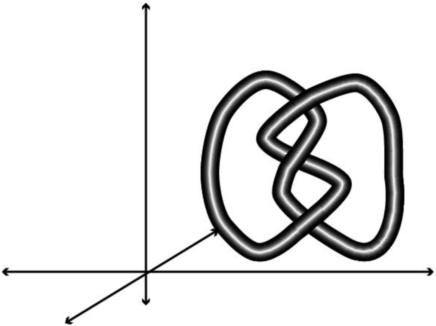
To create a knot, take a piece of string, tangle it up, and then glue the ends together.

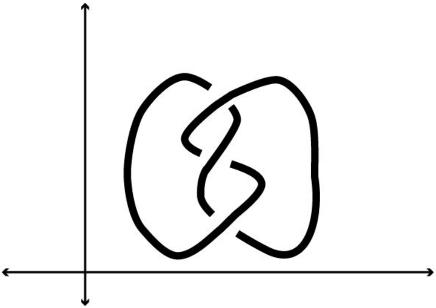
More formally, a **knot** is a smooth embedding of the circle into $\mathbb{R}^3$.



Knot Diagrams

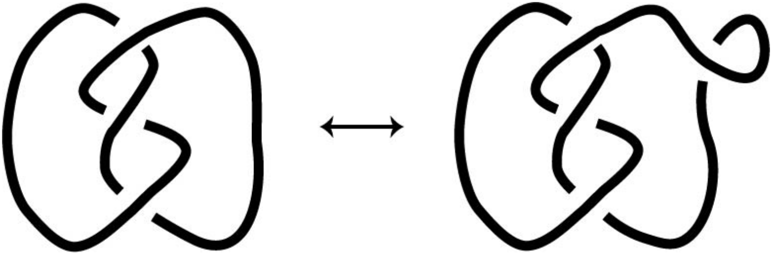
We can project a knot K into the plane, together with crossing information, to produce a knot diagram D.





Knot Equivalence

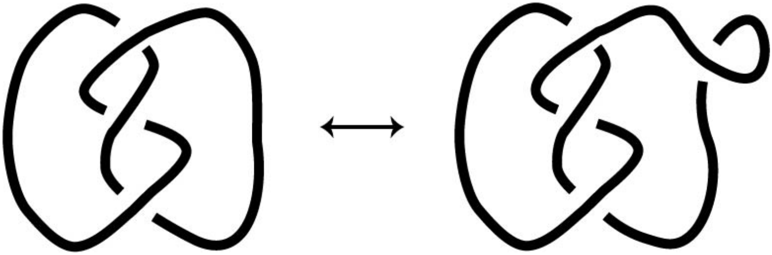
Intuitively, two knots are equivalent if one can be bent, tangled, or stretched to coincide exactly with the other.



More rigorously, two knots are **equivalent** if there is an ambient isotopy taking one to the other.

Knot Invariants

A **knot invariant** is some algebraic object assigned to a knot that is the same regardless of the presentation of the knot.



Example: the **crossing number** of a knot is the minimal number of crossings in a knot diagram across all equivalent representations of a knot.

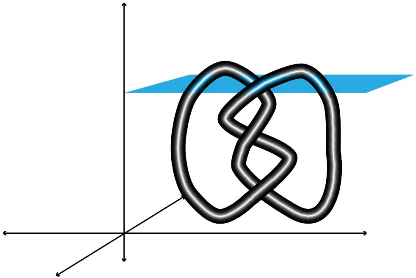
For instance, the crossing number of the trefoil is 3.

Bridge Number

The bridge number of a knot is a classical knot invariant first studied by German mathematician Horst Schubert in the 1950's.

There are many equivalent definitions of the bridge number of a knot. We will use the following definition:

The **bridge number** of a knot K, denoted $\beta(K)$, is the minimal number of local maxima across all smooth embeddings of K into $\mathbb{R}^3$.



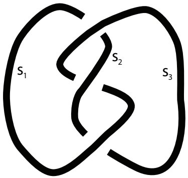
The Wirtinger Presentation

Another way to represent a knot is with its Wirtinger presentation.

The Wirtinger presentation is a group of the form

$$\langle g_1, g_2, \ldots, g_n \mid r_1, r_2, \ldots r_n \rangle,$$

where each of the g_i correspond to a strand in a knot diagram and each of the r_j is a relation that arises from each of the diagram's crossing.



For example, the Wirtinger presentation of this trefoil is

$$\langle s_1, s_2, s_3 \mid s_1 s_2 s_1^{-1} s_3^{-1}, s_3 s_1 s_3^{-1} s_2^{-1}, s_2 s_3 s_2^{-1} s_1^{-1} \rangle.$$

Generating Sets

We can manipulate the relations and show that a generating set for this presentation is

$$\langle s_1, s_2 \mid (s_2 s_1 s_2)^2 = (s_2 s_1)^3 \rangle.$$

As the number of crossings of the knot increases, finding the appropriate relations becomes more tedious.

However, determining whether a set of strands *is* a generating set is far simpler.

Generating Sets

Pick some set of strands S and visit each crossing of the diagram.

If the overstrand and one of the understrands of that crossing is contained in S, then we can generate it via a Wirtinger relation - add that new strand to S.

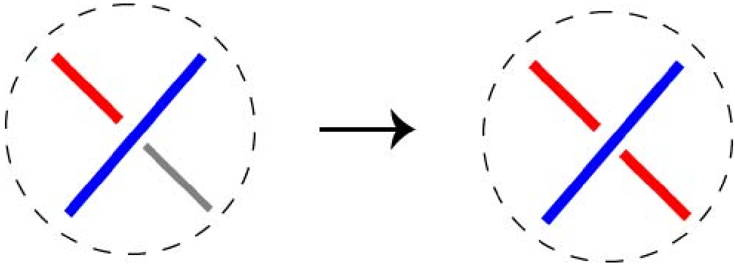
If we can repeat this process until all the strands in the diagram are contained in S, then the original set of strands is a generating set.

We formalize this process with the notion of coloring moves.

Coloring Move

We begin by coloring some of the strands of a knot diagram.

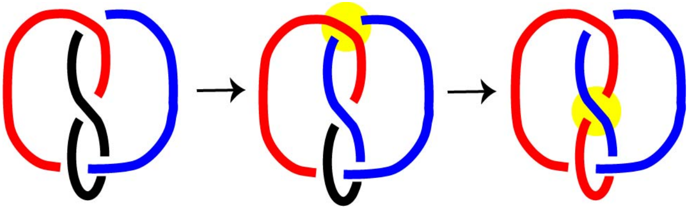
At each crossing, if the overstrand is colored and *one* of the understrands is colored, then we can extend the coloring to the other understrand.



This is analogous to generating a strand via a Wirtinger relation.

Coloring a Knot

For some initial choices of strands, we can repeat this process to color the entire diagram.

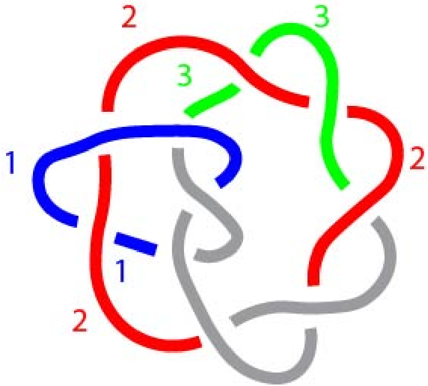


This analogous to finding a generating set for the diagram.

We are interested in the minimal number of colors needed to color a diagram.

We call a knot diagram D **k-partially colored** if there exists a subset A of the strands of D and a coloring function $f : A \to \{1, 2, \ldots, k\}$.

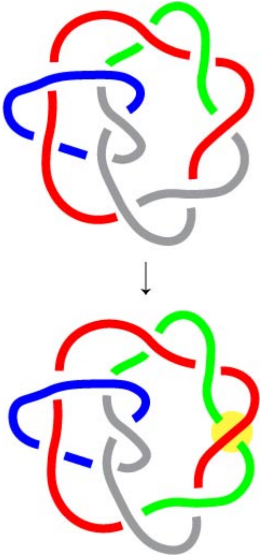
We refer to this partial coloring by (A, f).



Coloring Move

Given a knot diagram D and two partial colorings (A_1, f_1) and (A_2, f_2), we say (A_2, f_2) is the result of a **coloring move** on (A_1, f_1), denoted $(A_1, f_1) \to (A_2, f_2)$, if each of the following hold:

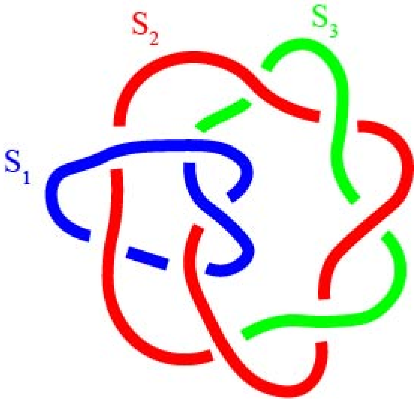
- $A_1 \subset A_2$ and $A_2 \setminus A_1 = \{s_j\}$ for some $s_j \in D$;
- $f_2|_{A_1} = f_1$;
- s_j is adjacent to some s_i at some crossing c of D, and $s_i \in A_1$;
- the overstrand s_k of c is an element of A_1;
- and $f_2(s_j) = f_1(s_i)$.



k-colorable

If there exists a k-partial coloring $f : \{s_1, \ldots, s_k\} \to \{1, \ldots, k\}$ such that $f(s_i) = i$, and a sequence of coloring moves that colors all of the strands of D, we say that D is k-**colorable.**

We refer to the initial set of strands $\{s_1, \ldots, s_k\}$ as **seed strands.**



Wirtinger Number

Definition: Wirtinger Number

The **Wirtinger Number** of a knot diagram D, denoted $\omega(D)$, is the smallest integer k such that D is k-colorable.

The **Wirtinger Number** of a knot K, denoted $\omega(K)$, is the smallest integer k such that K has a diagram that is k-colorable.

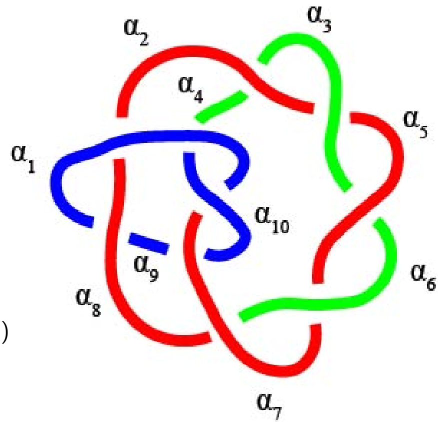
Coloring Sequence

Suppose D is k-colorable.

Then there is a sequence of coloring moves $(A_0, f_0) \to \cdots \to (A_{c(D)-k}, f_{c(D)-k})$.

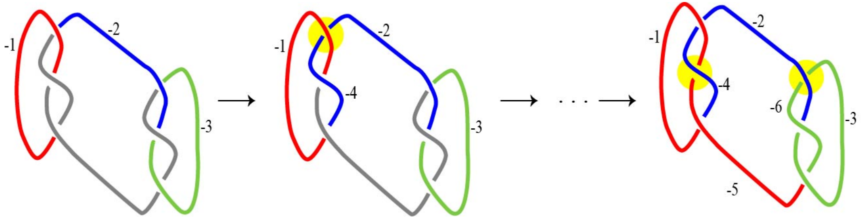
We define the **coloring sequence** $\{\alpha_j\}_{j=1}^{c(D)}$ by

$$\alpha_j = \begin{cases} s_i, & 1 \leq j \leq k \\ \alpha_j \in A_{j-k} \setminus A_{j-(k+1)}, & k+1 \leq j \leq c(D) \end{cases}$$



Standard Height Function

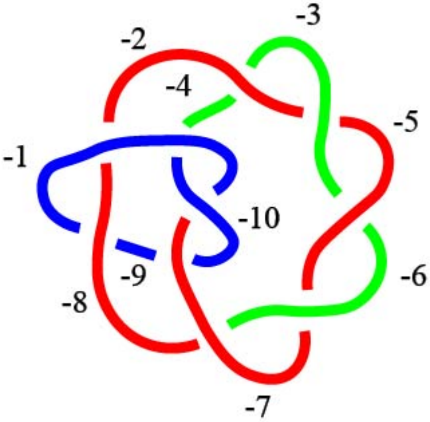
We define the **standard height function** $h : s(D) \to \mathbb{R}$ by $h(\alpha_j) = -j$.



A Couple Lemmas

Lemma 1: The set of strands corresponding to each of the *k* colors has a unique local maximum.

Lemma 2: If the two understrands at a crossing are the same color, the height of the overstrand is greater than the height of at least one of the understrands.



Main Theorem

Theorem

For any knot K, $\omega(K) = \beta(K)$.

Blair and Kjuchukova showed that $\beta(K) \geq \omega(K)$, so it suffices to show $\omega(K) \geq \beta(K)$.

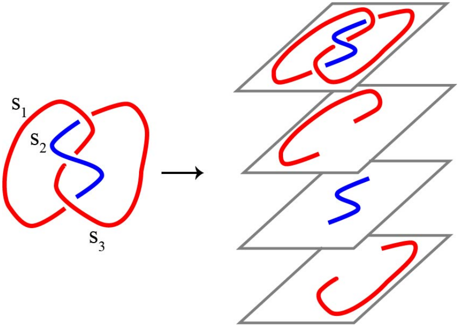
Our strategy is to demonstrate that we can construct a knot embedding with $\omega(K)$ local maxima for any given knot K.

Lifts

Let K be a a knot with $\omega(K) = n$. Then there exists a diagram D that is n-colorable.

Assign heights to each of the strands of D via the standard height function h.

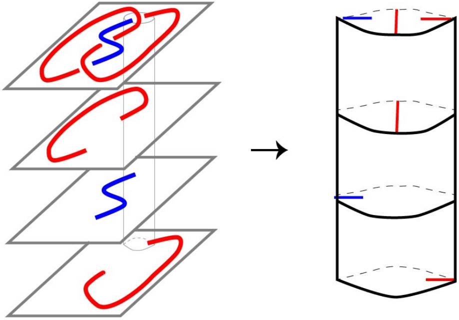
Position D into the xy-plane, and embed copies of each of the strands s_i into $z = h(s_i)$.



Constructing an embedding of K

At each crossing c of D, construct an infinite cylinder with cross-section a suitable ε-neighborhood of c in the xy-plane.

There are two cases to consider.

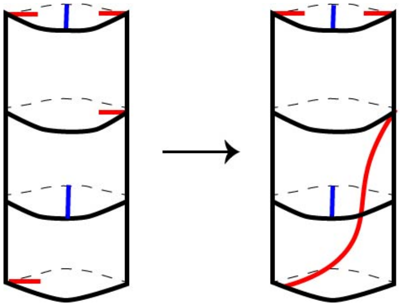


Case 1

The two understrands are the same color.

By **Lemma 2**, the height of the overstrand is greater than the height of at least one of the understrands.

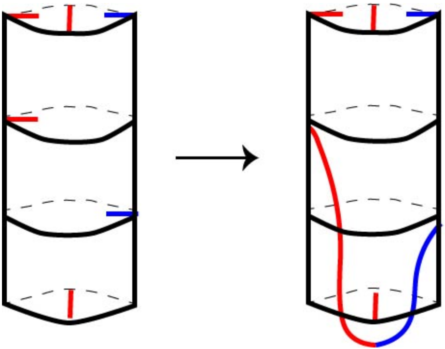
We can connect the two understrands as shown.



Case 2

The two understrands are different colors.

We can connect them to some point beneath the projection of the overstrand as shown.



Recap

We've constructed an embedding of K that has as many local maxima as D does with respect to h.

By **Lemma 1**, every collection of colored strands has a unique local maximum.

Thus, the embedding of K we constructed has $\omega(K)$ local maxima.

So $\omega(K) \geq \beta(K)$, and $\omega(K) = \beta(K)$. ■

Some Facts

Blair, Kjuchukova, and Makoto showed that the Wirtinger number isn't realized in every reduced, minimal crossing diagram [2].

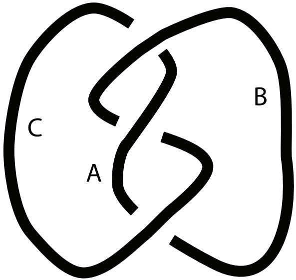
Pongtanapaisan showed that this result holds for virtual knots [3].

The Knot Dictionary

The **knot dictionary** of a diagram is a dictionary whose keys are the strands of that diagram and whose values are subsets of that diagrams crossings.

This diagram has knot dictionary

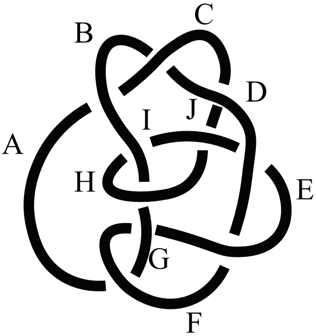
$$\begin{aligned} D_{3_1} = \{A &: \{(B, C)\}, \\ B &: \{(A, C)\}, \\ C &: \{(A, B)\}\} \end{aligned}$$



Another Example

For the 10_{146} knot pictured to the right, the corresponding knot dictionary is

$$D_{10_{146}} = \{A : \{\}, B : \{(A, C), (H, I)\}, C : \{(B, D)\}, D : \{(C, J), (E, I)\}, E : \{(A, G)\}, F : \{(A, G)\}, G : \{(E, F)\}, H : \{(G, I)\}, I : \{(H, J)\}, J : \{\}\}$$



The IS-VALID-COLORING function

Once we have our knot dictionary, we can determine if a set of strands S leads to a valid coloring in the following way:

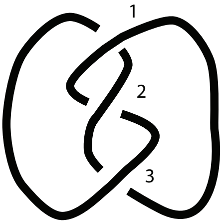
- Make a copy of S called *colored-strands*
- Pick some key $s_i \in \textit{colored-strands}$. For each $c = (s_j, s_k) \in D(s_i)$, if s_j is in *colored-strands* but s_k isn't, then add s_k to *colored-strands* (and vice versa)
- Repeat until entire iteration is completed without adding a new strand.
- If *colored-strands* is equal to the set of keys of the knot dictionary, then S was a generating set.

Gauss Code

We can derive a knot dictionary from the Gauss code of a knot diagram.

To generate a Gauss code, walk along the knot diagram and label the crossings with integers.

Walk along the diagram once more, recording the number of crossings you go over and the negative of the crossings you go under.

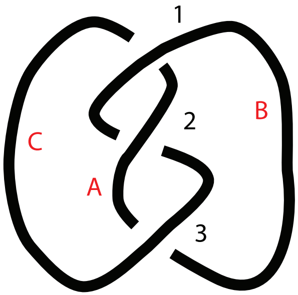


$G_{3_1} = [-1, 2, -3, 1, -2, 3]$

Converting from a Gauss code to a Knot Dictionary

The negative numbers in the Gauss code correspond to a strand going under a crossing. Since each strand is under two crossings, we see that each strand is described by a subsequence of G_K beginning and ending with a negative integer, "wrapping around" the sequence if necessary. We assign letters to each of these subsequences.

For example, G_{3_1}, we have the three subsequences $A = [-1, 2, -3], B = [-3, 1, -2]$, and $C = [-2, 3, -1]$.



$G_{3_1} = [-1, 2, -3, 1, -2, 3]$

The CALC-WIRT function

The CALC-WIRT function takes a Gauss code and calculates the Wirtinger number of the diagram.

CALC-WIRT(G_K)

Input: Gauss code G_K representing some knot diagram D.
Output: The Wirtinger number of D

$D_K \leftarrow$ CREATE-KNOT-DICTIONARY(G_K)
for $n \leftarrow 1$ **to** D_K.*length*
  for *seed-strand-set* **in** COMBINATIONS(D_K.*keys*, n)
    if IS-VALID-COLORING(D_K, *seed-strand-set*)
      return *seed-strand-set*.*size*

More information and a command line script are available at
https://tinyurl.com/calc-wirt.

The Number of Prime Knots of n Crossings

We used CALC-WIRT and referenced other work (de Wit, Jang) to find bridge numbers for all the 2,176 12 crossing knots.

We were able to find the bridge number of approximately 48,000 knots of 13 and 14 crossings.

We also have Wirtinger numbers for every prime knot up to 16 crossings. We suspect that these are in fact the bridge numbers.

Crossing Number	1	2	3	4	5	6	7	8	9	10	11	12
Number of Knots	0	0	1	1	2	3	7	21	49	165	552	2176

Crossing Number	13	14	15	16
Number of Knots	9988	46972	253293	1388705

You can find spreadsheets with all these Wirtinger numbers at https://tinyurl.com/wirt-db.

Future Work

This work arose from an attempt to find a counterexample to Shaneson and Cappel's Meridional Rank Conjecture, which asks if the meridional rank of a knot is equal to its bridge number.

Each of the generators of the Wirtinger presentation correspond to a meridion of the knot. Thus, the Wirtinger number is an upper bound on the meridional rank.

The Wirtinger number of a knot only considers generating sets where the only relations allowed are Wirtinger relations.

Can we define other coloring moves (relations) to bring the Wirtinger number and meridional rank closer together?

Acknowledgements

- USTARS for giving me the opportunity to present
- Dr. Ryan Blair for his guidance
- Roman Velazquez for being the homie
- NSF for funding this research

Thanks!

This presentation and supporting material available at
https://tinyurl.com/ustars-wirt-talk

References:

1. Ryan Blair, Alexandra Kjuchukova, Roman Velazquez, and Paul Villanueva, *Wirtinger systems of generators of knot groups*, arXiv:1705.03108

2. Ryan Blair, Alexandra Kjuchukova, and Makoto Ozawa, *The incompatibility of crossing number and bridge number for knot diagrams*, arXiv:1710.11327

3. Puttipong Pongtanapaisan, *Wirtinger Numbers for Virtual Links*, arXiv:1801.02923