GROUP REPRESENTATIONS AND CHARACTER THEORY

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Abstract. In this paper, we provide an introduction to the representation theory of finite groups. We begin by defining representations, G-linear maps, and other essential concepts before moving quickly towards initial results on irreducibility and Schur's Lemma. We then consider characters, class functions, and show that the character of a representation uniquely determines it up to isomorphism. Orthogonality relations are introduced shortly afterwards. Finally, we construct the character tables for a few familiar groups.

Contents

1. Introduction 1
2. Preliminaries 1
3. Group Representations 2
4. Maschke's Theorem and Complete Reducibility 4
5. Schur's Lemma and Decomposition 5
6. Character Theory 7
7. Character Tables for S_4 and $\mathbb{Z}_3$ 12
Acknowledgments 13
References 14

1. Introduction

The primary motivation for the study of group representations is to simplify the study of groups. Representation theory offers a powerful approach to the study of groups because it reduces many group theoretic problems to basic linear algebra calculations. To this end, we assume that the reader is already quite familiar with linear algebra and has had some exposure to group theory. With this said, we begin with a preliminary section on group theory.

2. Preliminaries

Definition 2.1. A group is a set G with a binary operation satisfying

(1) $\forall\ g, h, i \in G, (gh)i = g(hi)$ $\quad$ (*associativity*)
(2) $\exists\ 1 \in G$ such that $1g = g1 = g,\ \forall g\ \in G$ $\quad$ (*identity*)
(3) $\forall\ g \in G,\ \exists\ g^{-1}$ such that $gg^{-1} = g^{-1}g = 1$ $\quad$ (*inverses*)

Definition 2.2. Let G be a group. We call H a subgroup of G if H is a subset of G and is itself a group under the binary operation inherited from G.

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Definition 2.3. Let G and H be groups. A homomorphism from G to H is a map $\phi : G \to H$ satisfying

$$\phi(gh) = \phi(g)\phi(h) \quad \forall\ g, h \in G$$

If ϕ is a bijection, then ϕ is an isomorphism. We then also call G and H isomorphic, and we write $G \cong H$.

Definition 2.4. A subgroup N of a group G is called a normal subgroup if

$$Ng = gN \quad (\forall\ g \in G) \tag{2.5}$$

Definition 2.6. A group G is a *simple* group if it is a nontrivial group ($G \neq \{1\}$) and if its only normal subgroups are the trivial group and G.

Definition 2.7. The *center* $Z(G)$ of a group G is the set of elements in G that commute with G. In other words,

$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\} \tag{2.8}$$

This next concept in group theory will be especially important when we consider character theory later in this paper.

Definition 2.9. If $x, y \in G$, we say x is *conjugate* to y if there exists a $g \in G$ such that

$$gy = xg$$

We call the set of all elements conjugate to x the *conjugacy class* of x, denoted x^G. That is, $x^G = \{gxg^{-1} \mid g \in G\}$. The conjugacy classes of a group are disjoint and the union of all the conjugacy classes forms the group.

That is enough group theory. Let's dive into the topic of this paper.

3. Group Representations

Definition 3.1. Let V be a vector space over a field F. The general linear group *GL(V)* is the set of all automorphisms of V viewed as a group under composition.

If V has finite dimension n, then $GL(V) = GL(n, F)$, which is simply the group of invertible n x n matrices with entries in F.

Definition 3.2. A *representation* of a group G is a homomorphism ρ from G to *GL(V)*.

$$\rho : G \to GL(V) \tag{3.3}$$

Definition 3.4. We say a representation ρ is $faithful$ if it is injective. We call a representation ρ the trivial representation if $\rho = 1$.

Definition 3.5. An *FG-module* is a vector space V over a field F together with group action. That is, $\forall\ g \in G$, $\alpha \in F$, $u, v \in V$, the operation $g \cdot v$ is defined and satisfies

(1) $g \cdot (\alpha v) = \alpha(g \cdot v)$
(2) $g \cdot (u + v) = g \cdot u + g \cdot v$

We now let gv := ρ(g)v, and we can say that ρ gives V the structure of an FG-module.

Remark 3.6. For shorthand, we often call the FG-module the representation of G. That is, when we take the map $\rho : G \to GL(V)$, we sometimes call V the representation of G instead of ρ.

Definition 3.7. Let V be a representation. We say W is a subrepresentation of V if W is a subspace of V that is invariant under G. That is, for all $g \in G$ and $w \in W$, $gw \in W$.

Definition 3.8. A representation V is said to be *irreducible* if the only subrepresentations of V are $\{0\}$ and V itself.

Definition 3.9. Let V and W be representations. A function ϕ: $V \to W$ is called a G-linear map if ϕ is a linear transformation and it satisfies

$$\phi(gv) = g\phi(v) \quad \forall g \in G, v \in V.$$

Proposition 3.10. *Let V and W be representations and let ϕ: $V \to W$ be a G-linear map. Then Ker ϕ is a subrepresentation of V and Im ϕ is a subrepresentation of W.*

Proof. Since ϕ is a linear transformation, it follows that Ker ϕ is a subspace of V and Im ϕ is a subspace of W. Now take any $u \in$ Ker ϕ and $g \in G$. Then,

$$\phi(gu) = g\phi(u) = g \cdot 0 = 0.$$
$$\iff gu \in \text{ Ker } \phi$$

Thus, Ker ϕ is a subrepresentation of V. Similarly, take any $w \in$ Im ϕ and $g \in G$. Then ϕ(v) $= w$ for some $v \in V$. Thus,

$$\phi(gv) = g\phi(v) = gw$$

Since $gw \in$ W, Im ϕ is a subrepresentation of W. □

Definition 3.11. We call two representations V and W isomorphic if there exists a G-linear map ϕ: $V \to W$ that is invertible, i.e. ϕ is a representation isomorphism. We write $V \cong W$.

Definition 3.12. Let FG be a vector space over the field F with a basis g_1, ... , g_n, where g_1, ... , g_n are all the elements of a finite group G. Then every element v $\in FG$ can be written in the form

$$v = \sum_{i=1}^{n} \lambda_i g_i \quad \text{where every } \lambda_i \in F$$

or equivalently,

$$v = \sum_{g \in G} \lambda_g g \quad \text{where every } \lambda_g \in F$$

Notice now we can define multiplication in FG in a natural way. If

$$u = \Big(\sum_{g \in G} \lambda_g g \Big) \text{ and } v = \Big(\sum_{h \in G} \mu_h h \Big) \quad (\lambda_g, \mu_h \in F)$$

then let

$$uv = \Big(\sum_{g \in G} \lambda_g g \Big) \Big(\sum_{h \in G} \mu_h h \Big) = \Big(\sum_{g,h \in G} \lambda_g \mu_h gh \Big)$$

With multiplication defined in this way, we call FG a *group algebra* of G.

Definition 3.13. Let FG be the group algebra of a finite group G. The vector space FG, together with group action, is called the *regular* FG-module or the *regular* representation.

Remark 3.14. From here on, we will only work with finite groups. Furthermore, all representations that we use will be $\mathbb{C}G$-modules. The reason is that most of the group representation results we are concerned with only hold for complex representations.

4. Maschke's Theorem and Complete Reducibility

We are ready to establish a few preliminary results. In this section, we will show that every complex representation can be decomposed into a direct sum of irreducible subrepresentations. In this way, we can think of every representation as being "built" from these irreducible representations.

Theorem 4.1. *(Maschke's Theorem) Let V be the representation of a finite group G. Then if there exists a subrepresentation U of V, there must also be a subrepresentation W of V such that*

$$V = U \oplus W.$$

Proof. Let U be a subrepresentation of V. We choose any complementary subspace W_0 of V such that $V = U \oplus W_0$. For all $v \in V$, $v = u + w$ for unique vectors $u \in U$ and $w \in W_0$. Let π_0: $V \to U$ be the projection of V given by $v \mapsto u$. We then average π_0 over G to create a G-linear map. Let

$$\pi(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi_0(gv) \tag{4.2}$$

We need to show that $\pi(v)$ is a G-linear map. Let $g, h, x \in G$ such that $h = gx$. Then for all $v \in V$,

$$\begin{aligned}\pi(xv) &= \frac{1}{|G|}\sum_{g\in G} g^{-1}\pi_0(gxv)\\ &= \frac{1}{|G|}\sum_{h\in G} xh^{-1}\pi_0(hv)\\ &= x\Big(\frac{1}{|G|}\sum_{h\in G} h^{-1}\pi_0(hv)\Big)\\ &= x\pi(v)\end{aligned}$$

Hence π is a G-linear map. Note that for all $u \in U$, $\pi(u) = u$ since $\pi_0(gu) = gu$. This means Im $\pi = U$. By proposition 2.11, Ker π is also a subrepresentation of V. Let Ker $\pi = W$. Then,

$$V = U \oplus W$$

□

Definition 4.3. We say a representation V is *completely reducible* if V can be written as the direct sum of irreducible representations.

Corollary 4.4. *Every representation of a finite group is completely reducible.*

Proof. We prove this by induction. Let V be a representation. If dim $V = 0$ or 1, V is irreducible, and the result holds. Suppose V is not irreducible. Then V has a subrepresentation U aside from 0 and V. By Maschke's Theorem, there exists another subrepresentation W so that $V = U \oplus W$. It is clear that dim U and dim W are less than dim V. Then by induction, we can find subrepresentations of U and W such that

$$U = U_1 \oplus ... \oplus U_r \text{ and } W = W_1 \oplus ... \oplus W_s$$

where $U_1 \oplus ... \oplus U_r$, $W_1 \oplus ... \oplus W_s$ are irreducible representations. Then

$$V = U_1 \oplus ... \oplus U_r \oplus W_1 \oplus ... \oplus W_s$$

□

Remark 4.5. Remember that we are only working with representations over $\mathbb{C}$. As it turns out, the result remains true for any field of characteristic prime to the order of G.

5. Schur's Lemma and Decomposition

Theorem 5.1. *(Schur's Lemma) Let V and W be irreducible representations of a group G and let φ: $V \to W$ be a G-linear map. Then,*

(1) Either φ is an isomorphism or $\varphi = 0$.
(2) If $V = W$, then $\varphi = \lambda I_V$ for some $\lambda \in \mathbb{C}$

Proof. (1) Suppose $\varphi \neq 0$. Then Ker $\varphi \neq V$. By proposition 3.10, Ker φ is a subrepresentation of V. Since V is irreducible, and Ker $\varphi \neq V$, we deduce that Ker $\varphi = \{0\}$. Likewise, Im $\varphi \neq \{0\}$ and W is irreducible, so Im $\varphi = W$. Hence φ is a bijective G-linear map, or an isomorphism.

(2) Let $V = W$. Since $\mathbb{C}$ is algebraically closed, φ has an eigenvalue $\lambda \in \mathbb{C}$ for which Ker $(\varphi - \lambda I_V) \neq \{0\}$. V is irreducible, so this implies Ker $(\varphi - \lambda I_V) = V$. Then λ is an eigenvalue for all $v \in V$. Thus, $\varphi = \lambda I_V$.

□

Corollary 5.2. *Let V be a representation of a group G. Then V is irreducible if and only if every automorphism A of V that is also a G-linear map takes the form $A = \lambda I_V$ for some $\lambda \in \mathbb{C}$.*

Proof. Take any G-linear map A that is an automorphism of V. Suppose that every such A can be written in the form $A = \lambda I_V$ for some $\lambda \in \mathbb{C}$. Now assume for contradiction that V is reducible. Then V has a nonzero subrepresentation U that is not V. By Maschke's Theorem, there exists another subrepresentation W such that

$$V = U \oplus W$$

Let $\pi : V \to V$ be the projection defined by $v \mapsto u$, where $v = u + w$ for unique vectors $u \in U$, $w \in W$. Clearly π is a G-linear map that cannot be written as $\pi = \lambda I_V$ for some $\lambda \in \mathbb{C}$. Thus V is irreducible.

Conversely, suppose V is irreducible. Then by Schur's Lemma, every G-linear map that is an automorphism of V can be expressed as $\pi = \lambda I_V$.

□

Schur's Lemma produces a fairly interesting result for representations of finite abelian groups.

Theorem 5.3. *Every irreducible representation of a finite abelian group has dimension 1.*

Proof. Let G be a finite abelian group and let V be an irreducible representation of G. For all $g, h \in G$ and $v \in V$,

$$h(gv) = (hg)v = g(hv).$$

This shows that every $g \in G$ is a G-linear map $g : V \to V$. By Schur's Lemma, g takes the form $g = \lambda I_V$ for some $\lambda \in \mathbb{C}$. If W is a subspace of V, then for all $g \in G, w \in W, gw = \lambda w \in W$. But this means that every subspace of V is invariant under G. Since V is irreducible, this implies dim $V = 1$.

□

Corollary 5.4. *Let $\mathbb{C}G$ be the regular representation. By Corollary 4.4, $\mathbb{C}G$ can be written as:*

$$\mathbb{C}G = U_1 \oplus ... \oplus U_r$$

where $U_1, ..., U_r$ are irreducible representations. Let W be any irreducible representation. Then $W \cong U_i$ for some i.

Proof. Take any $w \in W$. Since $\mathbb{C}G$ is a group algebra, for all $g \in \mathbb{C}G$, gw is defined. Moreover, $\{gw | g \in \mathbb{C}G\}$ is a subrepresentation of W. Since W is irreducible, $W = \{gw | g \in \mathbb{C}G\}$. Let the function $\phi : \mathbb{C}G \to W$ be given by $g \mapsto gw$. We note that ϕ is G-linear map and that Im $\phi = W$. By Maschke's Theorem, we can find a subrepresentation U such that

$$\mathbb{C}G = U \oplus \text{Ker } \phi$$

Since we are taking a direct sum, $U \cong \text{Im } \phi = W$. Hence W is a subrepresentation of $\mathbb{C}G$. For all $w \in W$, $w = u_1 + ... + u_r$ for unique vectors in $U_1, ..., U_r$. Define a function $\pi : W \to U_i$ by $w \to u_i$. π is a G-linear map. Choose i such that $U_i \neq 0$. By Schur's Lemma, $W \cong U_i$.

□

Remark 5.5. This corollary shows that every irreducible representation is isomorphic to an element of a finite set of irreducible representations. This is significant enough for us to record it as a separate corollary.

Corollary 5.6. *Every finite group has only finitely many non-isomorphic irreducible representations.*

We now establish another important feature of representations that will be exploited later on.

Theorem 5.7. *Every representation V of a finite group G has a decomposition*

$$V \cong {U_1}^{\oplus a_1} \oplus ... \oplus {U_r}^{\oplus a_r} \tag{5.8}$$

where $U_1, ..., U_r$ are nonisomorphic irreducible representations, and each U_i has a_i multiplicities.

Proof. Let V be the direct sum of irreducible representations $W_1 \oplus ... \oplus W_s$. Take W_1 and all $W_{j_1}, ...W_{j_k}$ such that $W_1 \cong W_{j_1} \cong ... \cong W_{j_k}$. Let $W_1 = U_1$. Then

$$W_1 \oplus W_{j_1} \oplus ... \oplus W_{j_k} \cong {U_1}^{\oplus a_1}, \text{ where } a_1 = 1 + k.$$

Now consider S = $\{W_2, ..., W_s\} \setminus \{W_1, W_{j_1}, ..., W_{j_k}\}$. Take any $W_l \in S$ and find all W_i in S that are isomorphic to W_l. Allow $W_l = U_2$. Then we can find a_2 such that

$$\sum(\text{all } W_i \in S \text{ isomorphic to } W_l) \cong {U_2}^{\oplus a_2}$$

We can repeat this process until we arrive at

$$V \cong {U_1}^{\oplus a_1} \oplus ... \oplus {U_r}^{\oplus a_r}$$

where $U_1, ..., U_r$ are nonisomorphic and irreducible. □

$U_1, ..., U_r$ are called the complete set of nonisomorphic irreducible representations of G.

6. Character Theory

For the remainder of this paper we will work with characters, which are essential tools for the study of finite group representations. As we shall see, the character of a representation is intimately tied with the conjugacy classes of the group. This fact will allow us to prove that for a finite group, the number of non-isomorphic irreducible representations is equal to the number of conjugacy classes. In addition, we will show that a representation is uniquely determined up for isomorphism by its character, a rather remarkable result.

Definition 6.1. Let V be a representation of a group G. The character of V is a map $\chi : G \to \mathbb{C}$ given by

$$\chi(g) = \mathrm{tr}(g|_v), \tag{6.2}$$

where $\mathrm{tr}(g|_v)$ is the trace of the action of g on V.

The character χ does not depend on our choice of basis for V. Furthermore, χ is a *class function*, meaning that χ is constant along conjugacy classes.

Definition 6.3. We say a character is *irreducible* if it is the character of an irreducible representation.

Proposition 6.4. *Let V be a representation of G written as*

$$V = U_1 \oplus ... \oplus U_r$$

where $U_1, ..., U_r$ are irreducible representations. Then

$$\chi_v = \chi_{u_1} + ... + \chi_{u_r} \tag{6.5}$$

where χ_v is the character of V and χ_{u_i} is the character of U_i.

Proof. For each U_i, select a basis β_i. Call these vectors $u_{i_1}, ..., u_{i_k}$. Then by property of direct sums, $u_{1_1}, ..., u_{1_k}, u_{2_1}, ..., u_{2_k}, \ldots, u_{r_1}, ..., u_{r_k}$ is a basis for V. Call this basis β_v. Then for all $g \in G$, we can write the action of g on V as a diagonal matrix

$$[g]_{\beta_v} = \begin{bmatrix} [g]_{\beta_1} & & 0 \\ & \ddots & \\ 0 & & [g]_{\beta_r} \end{bmatrix}$$

where $[g]_{\beta_i}$ is the action of g on U_i. From here, we can observe that the trace of $[g]_{\beta_v}$ is simply the sum of the traces of $[g]_{\beta_i}$ for all i. Hence,

$$\chi_v = \chi_{u_1} + ... + \chi_{u_r}$$

□

Corollary 6.6. *Suppose now that V is isomorphic to a direct sum of irreducible representations $U_i, \ldots, U_r$ so that*

$$V \cong U_1 + ... + U_r$$

Then it still follows that

$$\chi_v = \chi_{u_1} + ... + \chi_{u_r}$$

Proof. Since V is isomorphic to $U_1 \oplus ... \oplus U_r$, it is possible to find a basis in $U_1 \oplus ... \oplus U_r$ that is also a basis for V. This basis will be a collection of bases for $U_1, \ldots, U_r$. For the rest of the proof, we can then follow the same approach we took in proposition 6.4.

□

Definition 6.7. A *class function* on G is a map $\phi : G \to \mathbb{C}$ such that for any two conjugate elements $g, h \in G$, $\phi(g) = \phi(h)$. That is, ϕ is constant along conjugacy classes.

Definition 6.8. Let $\mathbb{C}_{\text{class}}(G)$ be the set of all class functions from G to $\mathbb{C}$. Then we can define a Hermitian inner product $\langle\ ,\ \rangle$ on $\mathbb{C}_{\text{class}}(G)$ in the following way. For all $\chi, \psi \in \mathbb{C}_{\text{class}}(G)$, let

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}. \tag{6.9}$$

Remark 6.10. One can easily show that $\langle\ ,\ \rangle$ is an inner product and that additionally, $\langle \chi, \psi \rangle = \langle \psi, \chi \rangle$.

Theorem 6.11. *The irreducible characters of a group G are orthonormal. That is, if V and W are irreducible representations with characters χ and ψ,*

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{if } V \ncong W. \end{cases} \tag{6.12}$$

Proof. Let V be any representation. We start by letting $V^G = \{v \in V \mid gv = v, \forall g \in G\}$. We then define a projection $\varphi : V \to V^G$ by

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} gv \qquad (v \in V)$$

We need to show that φ is the desired projection. For all $h \in G$, $v \in V$,

$$h\varphi(v) = \frac{1}{|G|} \sum_{g \in G} hgv = \frac{1}{|G|} \sum_{g \in G} gv = \varphi(v) \qquad (v \in V)$$

Hence Im $\varphi \subseteq V^G$. Conversely, for all $v \in V^G$,

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} v = v.$$

This shows $V^G \subseteq \text{Im } \varphi$. Thus Im $\varphi = V^G$ as required. Next, we note that $\varphi(v) = v$ for all $v \in V^G$ implies that dim $V^G = \text{Trace}(\varphi)$. Thus,

$$\text{dim} V^G = \text{Trace}(\varphi) = \frac{1}{|G|} \sum_{g \in G} Tr(g|_v) \tag{6.13}$$

Let $\text{Hom}(V, W)$ be the set of all G-linear maps from V to W. We can talk about addition and scalar multiplication on $\text{Hom}(V, W)$: for $\phi, \psi \in \text{Hom}(V, W)$, define $\phi + \psi$ and $\lambda\phi$ as:

(1) $(\phi + \psi)v = \phi v + \psi v$
(2) $\phi(\lambda v) = \lambda\phi(v)$

for all $v \in V$ and $\lambda \in \mathbb{C}$. Then $\text{Hom}(V, W)$ is a vector space over $\mathbb{C}$. We let our group G act on $\text{Hom}(V, W)$ by conjugation. That is, for all $g, h \in G$ and $v \in \text{Hom}(V, W)$, let $gh(v) = gh(g^{-1}v)$.

Now suppose V and W are irreducible representations of G. Then

$$\dim\,(\mathrm{Hom}(V,W)) = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{if } V \ncong W. \end{cases}$$

This follows immediately from Schur's Lemma.

Define the dual representation $V^* = \mathrm{Hom}\,(V, \mathbb{C})$ of V as $\rho^* : G \to GL(V^*)$ where $\rho^*(g) = \rho(g^{-1})^T$. Then the character χ_{Hom} of the representation Hom $(V, W) = V^* \otimes W$ is

$$\chi_{Hom}(g) = \overline{\chi(g)} \cdot \psi(g) \qquad (\forall g \in G) \tag{6.14}$$

Applying this to 5.10, we find that

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \psi(g) = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{if } V \ncong W. \end{cases} \tag{6.15}$$

This is simply the inner product of $\langle \chi, \psi \rangle$, and we are done. □

We can now deduce some basic facts about characters from what we already know about inner products. Let V be a representation written as

$$V = {U_1}^{\oplus a_1} \oplus ... \oplus {U_r}^{\oplus a_r} \tag{6.16}$$

where $U_1, ..., U_r$ are nonisomorphic irreducible representations, and each U_i has a_i multiplicities. Then if χ_i is the character of U_i, the character ψ of V is

$$\psi = a_1 \chi_i + \ldots + a_r \chi_r \tag{6.17}$$

by proposition 6.4. By property of inner products, this then means that

$$(1)\ \langle \chi_i, \psi \rangle = a_i$$

$$(2)\ \langle \psi, \psi \rangle = \sum_{i=1}^{r} {a_i}^2$$

Corollary 6.18. *A representation V is irreducible if and only if* $\langle \psi_v, \psi_v \rangle = 1$.

Proof. If V is irreducible, then by theorem 6.11, $\langle \psi_v, \psi_v \rangle = 1$. Now suppose we are given any representation V with the property $\langle \psi_v, \psi_v \rangle = 1$. V can be decomposed into nonisomorphic irreducible representations so that

$$\langle \psi_v, \psi_v \rangle = \sum_{i=1}^{r} {a_i}^2 = 1$$

where the a_i are the multiplicities of the irreducible representations U_i. Then this implies that exactly one of the a_i equals 1 and the rest zero. Hence V is equal to an irreducible representation.

□

We now revisit the topic of class functions. Suppose G is a group with l conjugacy classes. Once again, let $\mathbb{C}_{\text{class}}(G)$ be the set of all class functions. Then with the proper definition of addition and scalar multiplication, $\mathbb{C}_{\text{class}}(G)$ is a vector space over $\mathbb{C}$. A basis for $\mathbb{C}_{\text{class}}$ can be simply a set of functions that take the value 1 along a conjugacy class and zero elsewhere. Then dim $\mathbb{C}_{\text{class}} = l$.

Theorem 6.19. *Let $\chi_1, \dots, \chi_k$ be the irreducible characters of a group G. Then $\chi_1, \dots, \chi_k$ form a basis for $\mathbb{C}_{class}$.*

Proof. We first seek to show that $\chi_1, \dots, \chi_k$ are linearly independent in $\mathbb{C}_{\text{class}}$. For any $\alpha_1, \dots, \alpha_k \in \mathbb{C}$, let

$$\alpha_1\chi_1 + \ldots + \alpha_k\chi_k = 0$$

But we know that for all χ_i, $\langle \alpha_1\chi_1 + \ldots + \alpha_k\chi_k, \chi_i\rangle = \alpha_i = 0$. Thus, $\chi_1, \dots, \chi_k$ are linearly independent. If dim $\mathbb{C}_{\text{class}} = l$, then $k \leq l$. Now we need to show that these vectors span $\mathbb{C}_{\text{class}}$. Consider the regular representation $\mathbb{C}G$. Let $V_1, \dots, V_k$ be the complete set of nonisomorphic irreducible representations. By corollary 3.4, we can decompose $\mathbb{C}G$ into the form

$$\mathbb{C}G = W_1 \oplus ... \oplus W_k$$

where for each i, W_i is isomorphic to the direct sum of a_i copies of V_i. Take now the center of $\mathbb{C}G$. Suppose $z \in Z(\mathbb{C}G)$. Then for each i, there exists $\lambda_i \in \mathbb{C}$ such that $\lambda_i(zI_{w_i}) = I_{w_i}$, where I_{w_i} is the identity vector in W_i. This directly implies that for all $w \in W_i$, $zw = \lambda_i w$. Then if we take $1_v \in \mathbb{C}G$,

$$z = z \cdot 1_v = z(w_1 + \ldots + w_k)$$

where $w_1, \dots, w_k$ are chosen such that $w_1 + \ldots + w_k = 1_v$. Hence,

$$z = \lambda_1 w_1 + \ldots + \lambda_k w_k$$

This reveals that $Z(\mathbb{C}G) \subseteq sp\{w_1, \dots, w_k\}$. We only have left to show that dim $Z(\mathbb{C}G) =$ dim $\mathbb{C}_{\text{class}} = l$, the number of conjugacy classes of G. Take any $z \in Z(\mathbb{C}G)$. We can rewrite it as

$$z = \sum_{g \in G} a_g g$$

For all $h \in G$,

$$hzh^{-1} = \sum_{g \in G} a_{hgh^{-1}} g$$

which implies that $a_g = a_{hgh^{-1}}$. Hence the a_g coefficients are constant along conjugacy classes. Then if $C_1, \dots, C_l$ are the conjugacy classes, elements $b_i = \sum_{g \in C_i} g$ form a basis $b_1, \dots, b_l$ for $Z(\mathbb{C}G)$.

Hence $k = l$, and $\chi_1, \dots, \chi_k$ are a maximal set of linearly independent vectors. ∎

Corollary 6.20. *The number of irreducible representations of a group is equal to the number of conjugacy classes for a group.*

We arrive at last at a few important consequences from our study of characters.

Theorem 6.21. *Every representation is determined up to isomorphism by its character.*

Proof. Suppose two representations V and W are isomorphic. Then they share a decomposition such that $V \cong W \cong {U_1}^{\oplus a_1} \oplus ... \oplus {U_r}^{\oplus a_r}$, where the U_i are distinct irreducible representations and the a_i are the multiplicities. Thus the characters for V and W are both given by $a_1\chi_{u_i} + \ldots + a_r\chi_{u_r}$. Conversely, suppose we are given that $\chi_v = \chi_w$. We can decompose V and W into $V \cong {U_1}^{\oplus a_1} \oplus ... \oplus {U_r}^{\oplus a_r}$ and $W \cong {U_1}^{\oplus b_1} \oplus ... \oplus {U_r}^{\oplus b_r}$. Since $\chi_v = \chi_w$, it follows that $a_1\chi_{u_i} + \ldots + a_r\chi_{u_r} = b_1\chi_{u_i} + \ldots + b_r\chi_{u_r}$. Now because the χ_i are linearly independent, we observe that $a_i = b_i$ for every i. Hence, $V \cong W$.

□

Theorem 6.22. *(Column Orthogonality Relation) Suppose $\chi_1, \ldots, \chi_k$ are all the irreducible characters of a group G. Let $g_1, \ldots, g_k$ be representatives of the conjugacy classes of G. Then for all g_i, g_j, and any χ_l,*

$$\sum_{l=1}^{k} \chi_l(g_i)\overline{\chi_l(g_j)} = \delta_{ij}\frac{|G|}{c(g_j)} \tag{6.23}$$

Proof. For $1 \leq j \leq k$, define ψ_j to be the class function

$$\psi_j(g_i) = \delta_{ij}$$

where $1 \leq i \leq k$. By theorem 5.19, ψ_j is a linear combination of $\chi_1, \ldots, \chi_k$ so that

$$\psi_j = \sum_{l=1}^{k} \alpha_l \chi_l \qquad (\alpha_l \in \mathbb{C})$$

To obtain each α_l, we simply take the inner product $\langle \psi_j, \chi_l \rangle$ so that

$$\alpha_l = \langle \psi_j, \chi_l \rangle = \frac{1}{|G|}\sum_{g \in G} \psi_j(g)\overline{\chi_l(g)}$$

$\psi_j(g) = 1$ if g is conjugate to g_j and zero otherwise. $c(g_j)$ denotes the number of elements in G conjugate to g_j. Thus, we have

$$\alpha_l = \frac{1}{|G|}\sum_{g \in G} \psi_j(g)\overline{\chi_l(g)} = \frac{1}{|G|}c(g_j)\overline{\chi_l(g_j)},$$

from which we conclude

$$\delta_{ij} = \psi_j(g_i) = \sum_{l=1}^{k} \alpha_l \chi_l(g_i) = \frac{1}{|G|}\sum_{l=1}^{k} c(g_j)\chi_l(g_i)\overline{\chi_l(g_j)}.$$

Rearrange the equation and the result follows. □

7. Character Tables for S_4 and $\mathbb{Z}_3$

We will now build the character tables for the two elementary groups S_4 and $\mathbb{Z}_3$. Character tables provide character values along each conjugacy class for each irreducible character. They are used because they offer useful information about the irreducible representations of a group. This section unfortunately must use a few facts not accounted for earlier in the paper. We largely assume that the reader is already familiar with S_n and $\mathbb{Z}_n$, understands how to determine their conjugacy classes, and has seen their basic representations.

Let us begin with the symmetric group. The conjugacy classes of S_n are determined by their cycle type. There are thus five conjugacy classes for S_4 and we can give them representatives 1, (12), (123), (1234), (12)(34). One can also determine that there are 1, 6, 8, 6, 3 conjugates in each conjugacy class, respectively.

For S_4, we know we can find the trivial, alternating, and standard representations. The trivial representation takes every element in S_n to the identity. The alternating representation takes each $g \in S_4$ to the sign of g times the identity, i.e. $gv = sgn(g)v$. Now consider a 4-dimensional vector space V with basis vectors v_1, v_2, v_3, and v_4. The standard representation takes each $g \in S_4$ to a linear transformation in V such that $gv_i = v_{gi}$. There is another irreducible representation obtained by taking the tensor product of the standard representation with the alternating representation. This already yields the table:

g_i	1	(12)	(123)	(1234)	(12)(34)
c(g)	1	6	8	6	3
trivial χ_1	1	1	1	1	1
alternating χ_2	1	-1	1	-1	1
standard χ_3	3	1	0	-1	-1
$\chi_3 \otimes \chi_2$	3	-1	0	1	-1

To find the last character χ_5, we simply need to apply the column orthogonality relation to each column. For instance, for the conjugacy class (12), we know that

$$\sum_{l=1}^{k} \chi_l(g)\overline{\chi_l(g)} = \frac{4!}{6} = 4$$
$$1^2 + (-1)^2 + 1^2 + (-1)^2 + (\chi_5(1\ 2))^2 = 4$$
$$\chi_5(1\ 2) = 0$$

This can also be done for the other columns. The final character table looks like:

g_i	1	(12)	(123)	(1234)	(12)(34)
c(g)	1	6	8	6	3
trivial χ_1	1	1	1	1	1
alternating χ_2	1	-1	1	-1	1
standard χ_3	3	1	0	-1	-1
$\chi_3 \otimes \chi_2$	3	-1	0	1	-1
χ_5	2	0	-1	0	2

We will now do the same for $\mathbb{Z}_3$. $\mathbb{Z}_3$ has three conjugacy classes which each hold one element (this follows immediately from the fact that $\mathbb{Z}_3$ is abelian). As always, we can find a trivial representation for $\mathbb{Z}_3$. Moreover, one can easily find another representation of $\mathbb{Z}_3$ for which each element maps to a 1 x 1 matrix whose entry is a third root of unity. Then the last character can be obtained from the orthogonality relations. The completed character table for $\mathbb{Z}_3$ is thus:

g_i	0	1	2
c(g)	1	1	1
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

(ω is the primitive third root of unity).

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