The Symmetric Group and Young Diagrams
Proseminar: Algebra, Topology and Group Theory in Physics, organised by Prof. Matthias Gaberdiel

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1 Introduction

The symmetric group plays an important role in some aspects of Lie theory. Therefore we want to understand its irreducible representations. In this paper we will give a complete description of the irreducible representations of the symmetric group, that is, to provide a construction of the irreducible representations via Young symmetrizers and a formula for the characters of the irreducible representations, the Frobenius formula. Also, a physical application of the theory is of interest. We will study a quantum mechanical system of identical particles and examine the physical meaning of the irreducible representations of the symmetric group.

2 The symmetric group

The symmetric group S_n is the set of invertible transformations of the set of natural numbers $\{1, \ldots, n\}$ with the group operation being the composition of transformations. The composition of transformations is associative, there is an identity element being the trivial transformation and for every transformation there is an inverse transformation such that the composition of the two is equal to the trivial transformation. In the case of $n \geq 3$ the group is non-abelian. The order of the group, i.e. the number of its elements, is given by

$$|S_n| = n! \tag{2.1}$$

There are several ways of describing an element in S_n. As an example a permutation $\sigma \in S_7$ can be written as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 2 & 3 & 6 & 7 & 4 & 1 \end{pmatrix}$$

$$\longleftrightarrow \sigma = (1\,5\,7)\,(2)\,(3)\,(4\,6) = (1\,5\,7)\,(4\,6)$$

The latter notation is the notation in terms of *cycles*, i.e. there is cyclic permutation of the elements inside each brackets. The cycles of length one, that is the cycles containing the numbers on which the permutation acts trivially, are left out.

2.1 Conjugacy classes

Claim: Two permutations are conjugate if and only if they have the same cyclic decomposition.

Proof. Assumption: σ, $\tau \in S_n$ conjugate, i.e. $\exists \rho \in S_n$ s.t. $\sigma = \rho\tau\rho^{-1}$
Write τ in its cyclic notation, where τ for now consists of only one cycle: $\tau = (a_1 a_2 \cdots a_k)$, i.e. τ permutes the elements of the set $T = \{a_1, \ldots, a_k\}$
For $j \in \{1, \ldots, n\}$: $\sigma(j) = \rho\tau\rho^{-1}(j) = \begin{cases} \rho(a_{i+1}) & \text{if } \rho^{-1}(j) \equiv a_i \in T \\ j & \text{if } \rho^{-1}(j) \notin T \end{cases}$,
i.e. $\sigma = (\rho(a_1)\rho(a_2)\cdots\rho(a_k))$ and indeed, the length of the cycle is preserved under conjugation.
In the general case, where τ consists of more than one cycle: $\tau = \tau_1 \cdots \tau_r \longrightarrow \sigma = \rho\tau_1 \cdots \tau_r\rho^{-1} = \rho\tau_1\rho^{-1} \cdot \rho\tau_2\rho^{-1} \cdots \rho\tau_r\rho^{-1}$ and with the above argument the length of each cycle is preserved.

Assumption: σ, $\tau \in S_n$ have the same cyclic decomposition $\sigma = \sigma_1 \cdots \sigma_k$, $\tau = \tau_1 \cdots \tau_k$ where σ_i, τ_i are cycles of length l_i.
Denote the cycle j by $\sigma_j = \left(\sigma_{j1} \cdots \sigma_{jl_j}\right)$. Since the numbers σ_{jl} are pairwise disjoint: $\cup_{jl}\sigma_{jl} = \{1, \ldots, n\}$, the same holds for the τ_{jl}.
Define another permutation $\lambda \in S_n$ by $\lambda(\sigma_{jl}) = \tau_{jl}$. Then $\lambda\sigma\lambda^{-1}(\tau_{jl}) = \lambda\sigma(\sigma_{jl}) = \lambda(\sigma_{jl+1}) = \tau_{jl+1} = \tau(\tau_{jl})$ and indeed, σ and τ are conjugate. □

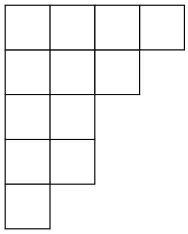
Since two permutations are conjugate if and only if they have the same cyclic decomposition a conjugacy class of the symmetric group S_n is labelled by the cyclic decomposition of its elements. Denote a conjugacy class of the symmetric group S_n by $C_{\mathbf{i}}$, where $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ is a multi-index such that i_j equals the number of cycles of length j.
For the previous example $\sigma = (157)\,(46) \in S_7$ the corresponding multi-index is $\mathbf{i} = (2, 1, 1, 0, 0, 0, 0)$.
From this notation one finds the partition of n into positive integers

$$n = \sum_{j=0}^{n} j \cdot i_j \tag{2.2}$$

e.g. for the above example $7 = 2 \cdot 1 + 2 + 3 = 1 + 1 + 2 + 3$.

3 Partitions and Young diagrams

A partition can be described in terms of a *Young diagram*. Therefore a Young diagram also stands for a conjugacy class of the symmetric group. As an example we want to find the Young diagram corresponding to the conjugacy class containing $\sigma = (1\,10\,11)\,(2\,4\,5\,7)\,(3\,8)\,(9\,12) \in S_{12}$. From the cyclic notation of the permutation σ we obtain the partition $\lambda = (4,3,2,2,1) = 4+3+2+2+1$ of $n = 12$. The corresponding Young diagram takes the form



The Young diagram is obtained by ordering the summands of the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ in decreasing order and then draw λ_i boxes in the i-th row of the diagram. The rows of boxes are lined up in the left.

3.1 Motivation

As seen above, each conjugacy class of the symmetric group S_n defines a partition of n into positive integers. This partition then can be described in terms of a Young diagram, i.e. each conjugacy class of S_n can be described by its corresponding Young diagram. Our aim is to understand the irreducible representations of S_n. Those can be obtained from the conjugacy classes of the symmetric group due to a direct correspondence between an irreducible representation and a conjugacy class in the symmetric group. This correspondence is not available for general groups.

3.2 Young tableaux

A *filling* of a Young diagram is any way of putting a positive integer in each box. A *Young tableau* is a filling such that the entries are

1. weakly increasing across each row from left to right, i.e. equalities are allowed.
2. strictly increasing down each column, i.e. equalities are not allowed.

A *standard* Young tableau is a Young tableau in which the numbers $\{1, \ldots, n\}$ each occur once. This leads to the entries being strictly increasing across each row from left to right.
For the partition $\lambda = (4,3,2,2,1)$ of $n = 12$ for example one finds the Young tableaux

1	4	5	6
2	7	8	
3	10		
9	12		
11			

and

1	2	3	4
5	6	7	
8	9		
10	11		
12			

The second Young tableau is called *canonical*.
From Young tableaux there can be found projection operators for the regular representation of the symmetric group which then give rise to the irreducible representations of S_n.

3.3 Interlude: regular representation and group algebra

The *regular representation* of a finite group G is the left-action of G on itself. The underlying vector space V has a basis $\{e_a\}_{a\in G}$, where each basis-vector is labelled by a group element. The action of $b \in G$ on $e_a \in V$ is given by

$$b(e_a) = e_{ba} \in V \tag{3.1}$$

The *group algebra* $\mathbb{C}G$ associated to a finite group G is the underlying vector space of the regular representation enlarged with the algebra structure

$$e_a \cdot e_b = e_{ab} \tag{3.2}$$

3.4 Young symmetrizer

Now the projection operators for the regular representation can be introduced and the irreducible representations for the symmetric group S_n will be obtained. We provide an example for $n = 3$ including all the calculations in subsection 3.5.

For a given standard Young tableau, define two subgroups of the symmetric group

$$P_\lambda = \{\sigma \in S_n : \sigma \text{ preserves each row}\} \tag{3.3}$$

and

$$Q_\lambda = \{\sigma \in S_n : \sigma \text{ preserves each column}\} \tag{3.4}$$

The condition for a permutation $\sigma \in S_n$ to preserve each row can be understood in the following sense: from the rows of a given Young tableau there can be defined subsets of the set $\{1, \dots, n\}$, i.e. the numbers in the first row define a subset and so do the numbers of the second row and so on. The permutation $\sigma \in S_n$ preserves each row if and only if σ permutes only the numbers inside each subset. The same of course holds for the columns.

Introduce two elements in the group algebra $\mathbb{C}S_n$ corresponding to these subgroups by defining

$$a_\lambda = \sum_{\sigma \in P_\lambda} e_\sigma \tag{3.5}$$

and

$$b_\lambda = \sum_{\sigma \in Q_\lambda} \operatorname{sgn}(\sigma) e_\sigma \ , \tag{3.6}$$

where $\operatorname{sgn}(\sigma)$ is the sign of the permutation $\sigma \in S_n$. Define then the *Young symmetrizer* associated to the partition λ by

$$c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}S_n \tag{3.7}$$

The images on the group algebra of the Young symmetrizers corresponding to the different partitions of n provide all the irreducible representations of S_n. Concretely, one has the following theorem:

Theorem Some scalar multiple of c_λ is idempotent, i.e. $c_\lambda^2 = n_\lambda c_\lambda$, where $n_\lambda \in \mathbb{R}$. The image of c_λ by right-multiplication on the group algebra $\mathbb{C}S_n$ is an irreducible representation V_λ of S_n. The representations corresponding to the different diagrams λ are inequivalent, hence all irreducible representations can be obtained in this manner.

A proof of the theorem can be found, for example, in section 4.2 of [FH].

3.5 Example: S_3

In this section the irreducible representations of S_3 will be obtained according to the theory introduced above. Since S_3 contains three conjugacy classes there will be found three irreducible representations. First identify the conjugacy classes of S_3 which is to identify the partitions of $n = 3$ into positive integers:

$$3 = 2 + 1 = 1 + 1 + 1$$

Then draw the corresponding Young diagrams.

- $\lambda = (3)$:

$$\begin{array}{|c|c|c|}\hline & & \\ \hline\end{array}$$

- $\lambda = (2, 1)$:

$$\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$$

- $\lambda = (1, 1, 1)$:

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}$$

Now perform fillings in order to find the standard Young tableaux for each conjugacy class.

- $\lambda = (3)$:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$$

- $\lambda = (2, 1)$:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{or} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

- $\lambda = (1, 1, 1)$:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

For the partition $\lambda = (2, 1)$ two different standard Young tableaux are found. In fact, the irreducible representation obtained from a certain Young diagram does not depend on the filling. This will become clear when calculating the character of the irreducible representation, which will only depend on the conjugacy class of the symmetric group, i.e. on the Young diagram, but not on the filling.

The next step is to define two subgroups of S_3 for each Young tableau and introduce the corresponding elements in the group algebra.

- $\lambda = (3)$:

$$P_{(3)} = S_3 \quad , \quad Q_{(3)} = \{1\}$$
$$a_{(3)} = \sum_{\sigma \in S_3} e_\sigma \quad , \quad b_{(3)} = e_1$$

- $\lambda = (2,1)$, for the choice of the canonical Young tableau:

$$P_{(2,1)} = \{1, (12)\} \quad , \quad Q_{(2,1)} = \{1, (13)\}$$
$$a_{(2,1)} = e_1 + e_{(12)} \quad , \quad b_{(2,1)} = e_1 - e_{(13)}$$

- $\lambda = (1,1,1)$:

$$P_{(1,1,1)} = \{1\} \quad , \quad Q_{(1,1,1)} = S_3$$
$$a_{(1,1,1)} = e_1 \quad , \quad b_{(1,1,1)} = \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) e_\sigma$$

Define the Young symmetrizers and obtain the irreducible representation V_λ for each partition λ according to the theorem in the previous section.

- $\lambda = (3)$:

  $$c_{(3)} = a_{(3)} \cdot b_{(3)} = \sum_{\sigma \in S_3} e_\sigma$$
  $$\longrightarrow V_{(3)} = \mathbb{C}S_3 \cdot c_{(3)} = \mathbb{C}\left\{e_\tau : \tau \in S_3\right\} \cdot \sum_{\sigma \in S_3} e_\sigma = \mathbb{C}\left\{\sum_{\sigma \in S_3} e_{\tau\sigma} : \tau \in S_3\right\} = \mathbb{C}\left\{\sum_{\sigma' \in S_3} e_{\sigma'} : \tau \in S_3\right\}$$
  $$= \mathbb{C} \sum_{\sigma' \in S_3} e_{\sigma'}$$

  This is the trivial representation, since any $\tau \in S_3$ acts trivially on $V_{(3)}$:

  $$\tau \cdot \sum_{\sigma \in S_3} e_\sigma = \sum_{\sigma \in S_3} e_{\tau\sigma} = \sum_{\sigma' \in S_3} e_{\sigma'}$$

- $\lambda = (2,1)$:

  $$c_{(2,1)} = (e_1 + e_{(12)}) \cdot (e_1 - e_{(13)}) = e_1 + e_{(12)} - e_{(13)} - e_{(132)}$$
  $$\longrightarrow V_{(2,1)} = \mathbb{C}S_3 \cdot c_{(2,1)}$$

  The Young symmetrizer acts on each basis-vector of the group algebra by right-multiplication:

  $$\begin{aligned}
  e_1 \cdot c_{(2,1)} &= c_{(2,1)} \\
  e_{(12)} \cdot c_{(2,1)} &= e_{(12)1} + e_{(12)(12)} - e_{(12)(13)} - e_{(12)(132)} = e_{(12)} + e_1 - e_{(132)} - e_{(13)} = c_{(2,1)} \\
  e_{(13)} \cdot c_{(2,1)} &= e_{(13)} + e_{(123)} - e_1 - e_{(12)} \\
  e_{(23)} \cdot c_{(2,1)} &= e_{(23)} + e_{(132)} - e_{(123)} - e_{(12)} \\
  e_{(123)} \cdot c_{(2,1)} &= e_{(123)} + e_{(13)} - e_{(23)} - e_1 = e_{(13)} \cdot c_{(2,1)} \\
  e_{(132)} \cdot c_{(2,1)} &= e_{(132)} + e_{(23)} - e_{(12)} - e_{(123)} = e_{(23)} \cdot c_{(2,1)}
  \end{aligned}$$
  $$\longrightarrow V_{(2,1)} = \mathbb{C}\left\{c_{(2,1)}, e_{(13)} \cdot c_{(2,1)}, e_{(23)} \cdot c_{(2,1)}\right\} = \mathbb{C}\left\{c_{(2,1)}, e_{(23)} \cdot c_{(2,1)}\right\}$$
  $$\text{since } c_{(2,1)} + e_{(23)} \cdot c_{(2,1)} = -e_{(13)} \cdot c_{(2,1)},$$

  i.e. $V_{(2,1)}$ is a two-dimensional irreducible representation of S_3. This irreducible representation is often referred to as the *standard* representation of S_3.

- $\lambda = (1,1,1)$:

$$c_{(1,1,1)} = \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) e_\sigma$$

$$\longrightarrow V_{(1,1,1)} = \mathbb{C}S_3 \cdot c_{(1,1,1)} = \mathbb{C}\left\{\sum_{\sigma\in S_3} \mathrm{sgn}(\sigma)e_{\tau\sigma} : \tau \in S_3\right\} = \mathbb{C}\left\{\mathrm{sgn}(\tau)\sum_{\sigma'\in S_3} \mathrm{sgn}(\sigma')e_{\sigma'} : \tau \in S_3\right\}$$
$$= \mathbb{C}\sum_{\sigma'\in S_3} \mathrm{sgn}(\sigma')e_{\sigma'}$$

We obtained the alternating representation of S_3, since any $\tau \in S_3$ acts on $V_{(1,1,1)}$ by multiplication of its sign:

$$\tau \cdot \sum_{\sigma\in S_3} \mathrm{sgn}(\sigma)e_\sigma = \sum_{\sigma\in S_3} \mathrm{sgn}(\sigma)e_{\tau\sigma} = \mathrm{sgn}(\tau)\sum_{\sigma'\in S_3} \mathrm{sgn}(\sigma')e_{\sigma'}$$

Indeed, all of the obtained Young symmetrizers are idempotent:

$$c_{(3)}^2 = \sum_{\sigma\in S_3} e_\sigma \cdot \sum_{\tau\in S_3} e_\tau = \sum_{\sigma\in S_3}\sum_{\tau\in S_3} e_{\sigma\tau} = \sum_{\sigma\in S_3}\sum_{\sigma'\in S_3} e_{\sigma'} = 6 \cdot c_{(3)}$$
$$c_{(2,1)}^2 = (e_1 + e_{(12)} - e_{(13)} - e_{(132)}) \cdot (e_1 + e_{(12)} - e_{(13)} - e_{(132)}) = 3 \cdot c_{(2,1)}$$
$$c_{(1,1,1)}^2 = \left(\sum_{\sigma\in S_3} \mathrm{sgn}(\sigma)e_\sigma\right) \cdot \left(\sum_{\tau\in S_3} \mathrm{sgn}(\tau)e_\tau\right) = \sum_{\sigma\in S_3}\sum_{\tau\in S_3} \mathrm{sgn}(\sigma\tau)e_{\sigma\tau} = \sum_{\sigma\in S_3}\sum_{\sigma'\in S_3} \mathrm{sgn}(\sigma')e_{\sigma'} = 6 \cdot c_{(1,1,1)}$$

As denoted in the theorem by $c_\lambda^2 = n_\lambda c_\lambda$, the scalar n_λ depends on the Young diagram λ. In fact for λ standing for a conjugacy class of S_n, n_λ is given by

$$n_\lambda = \frac{n!}{\mathrm{dim} V_\lambda} \tag{3.8}$$

4 Frobenius formula

For understanding a representation of a finite group G it is sufficient to know the character of the representation. For the irreducible representations of the symmetric group there is an elegant formula to obtain their characters, the Frobenius formula. This formula will be introduced in this section.

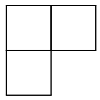
- introduce independent variables $\mathbf{x} = x_1, \ldots, x_k$, where k is at least as large as the number of rows of the Young diagram λ
- evaluate the character on the conjugacy class $C_\mathbf{i}$, $\mathbf{i} = (i_1, i_2, \ldots, i_n)$, where i_j equals the number of cycles of length j
- define power sums $P_j(\mathbf{x}) = x_1^j + x_2^j + \cdots + x_k^j$ for $1 \leq j \leq n$
- define the discriminant $\Delta(\mathbf{x}) = \prod_{i<j}(x_i - x_j)$
- if $f(\mathbf{x}) = f(x_1, \ldots, x_k)$ is a formal power series and $(l_1, \ldots, l_k)$ a k-tuple of non-negative integers, define
$$[f(\mathbf{x})]_{(l_1,\ldots,l_k)} = \text{coefficient of } x_1^{l_1} \cdots x_k^{l_k}$$
- given $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition of n such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$, define a strictly increasing sequence of k non-negative integers
$$l_j = \lambda_j + k - j$$

The character of the irreducible representation of S_n associated to λ evaluated on the conjugacy class $C_{\mathbf{i}}$ is then given by the **Frobenius formula**:

$$\chi_\lambda(C_{\mathbf{i}}) = \left[\Delta(\mathbf{x}) \cdot \prod_{j=1}^{n} P_j(\mathbf{x})^{i_j} \right]_{(l_1,\dots,l_k)} \tag{4.1}$$

A proof of this formula can, for example, be found in section 4.3 of [FH].

As an illustration the character of the two-dimensional irreducible representation of S_3, i.e. the irreducible representation corresponding to $\lambda = (2,1)$, will be calculated.
Define k, $\mathbf{x} = x_1, \dots, x_k$ and the sequence of non-negative integers according to the Young diagram



by $k = 2$, $\mathbf{x} = x_1, x_2$ and $l_1 = 2 + 2 - 1 = 3$, $l_2 = 1 + 2 - 2 = 1$.
First we evaluate the character on the conjugacy class containing the identity, i.e. on $C_{(3,0,0)}$.

$$\chi_{(2,1)}(C_{(3,0,0)}) = \left[(x_1 - x_2)(x_1 + x_2)^3\right]_{(3,1)} = 2$$

This equals the dimension of the irreducible representation $V_{(2,1)}$.
To obtain the character on the remaining two conjugacy classes of S_3, i.e. on $C_{(1,1,0)}$ and on $C_{(0,0,1)}$, the discriminant $\Delta(\mathbf{x}) = \prod_{i<j}(x_i - x_j)$ and the pair of non-negative integers (l_1, l_2) have not to be calculated again. Both of them only depend on the Young diagram, i.e. on the irreducible representation, but not on the conjugacy class on which the character is evaluated.
Evaluating the character on the conjugacy class containing the transpositions then leads to:

$$\chi_{(2,1)}(C_{(1,1,0)}) = \left[(x_1 - x_2)(x_1 + x_2)(x_1^2 + x_2^2)\right]_{(3,1)} = \left[x_1^4 - x_2^4\right]_{(3,1)} = 0$$

Where evaluating the character on the conjugacy class of the cyclic permutation (1 2 3) leads to

$$\chi_{(2,1)}(C_{(0,0,1)}) = \left[(x_1 - x_2)(x_1^3 + x_2^3)\right]_{(3,1)} = -1.$$

Indeed, the character does not depend on the filling but only on the Young diagram. This confirms the idea that the filling is needed for the construction of the projection operator, where the irreducible representation which then is obtained only depends on the Young diagram. In particular, the irreducible representation only depends on the conjugacy class of the symmetric group S_n due to the direct correspondence between a conjugacy class and an irreducible representation in the symmetric group.

4.1 Dimension of an irreducible representation of the symmetric group

The Frobenius formula can be used to calculate the dimension of the irreducible representation V_λ associated to λ. The character evaluated on the conjugacy class containing the identity is equal to the dimension of the representation. Inserting the conjugacy class of the identity inside the Frobenius formula gives the expression

$$\dim(V_\lambda) = \chi_\lambda(C_{(n,0,\dots,0)}) = \left[\Delta(\mathbf{x}) \cdot (x_1 + \cdots + x_k)^n\right]_{(l_1,\dots,l_k)} \tag{4.2}$$

This can be simplified to the formula

$$\dim(V_\lambda) = \frac{n!}{l_1! \cdots l_k!} \prod_{i<j} (l_i - l_j) \tag{4.3}$$

A derivation of equation (4.3) starting from equation (4.2) can be found, for example, in section 2.2 of [G]. However, there is another particularly simple formula to obtain the dimension of the irreducible representation V_λ associated to λ, the so-called **hook formula**. For a given Young diagram λ define the *hook length* of a box in the diagram to be the number of boxes directly to the right of the box or directly below the box, including the box once. If r_i and c_j denote the row lengths and the column lengths respectively, the hook length of the box at position (i, j) is given by

$$h_\lambda(i, j) = r_i + c_j - (i + j - 1) \tag{4.4}$$

Example of a Young diagram with each box labelled by its hook length:

8	6	5	3	2
7	5	4	2	1
4	2	1		
1				

The above Young diagram denotes the partition $\lambda = (5, 5, 3, 1)$ of $n = 12$.
The dimension of the irreducible representation V_λ of S_n corresponding to a given Young diagram λ is then given by the **hook formula**:

$$\dim(V_\lambda) = n! \prod_{(i,j)\in\lambda} \frac{1}{h_\lambda(i, j)} \tag{4.5}$$

Evaluating the formula for the irreducible representation corresponding to the above Young diagram one finds the dimension of $V_{(5,5,3,1)}$:

$$\dim(V_{(5,5,3,1)}) = \frac{14!}{8 \cdot 6 \cdot 5 \cdot 3 \cdot 2 \cdot 7 \cdot 5 \cdot 4 \cdot 2 \cdot 4 \cdot 2} = 7 \cdot 13 \cdot 3 \cdot 11 \cdot 9 = 27'027$$

5 Application: identical particles

As an application of the theory consider a quantum mechanical system of N *identical particles*. Assume the wave-function $\psi(x_1, \ldots, x_N)$ is an eigenfunction of the Hamiltonian for a certain energy eigenvalue. The term identical particles means that we are dealing with particles that cannot be distinguished, not even in principle. In particular the Hamiltonian of the system is invariant under exchanging the particles and therefore any of the $N!$ permutations applied on the coordinates inside the wave-function $\psi(x_1, \ldots, x_N)$ yields another eigenfunction of the Hamiltonian for the same energy eigenvalue. In such a case one says that the system possesses *exchange degeneracy*. Unlike in the one particle case specifying the energy eigenvalue does not determine the wave-function of the system completely.
When considering indistinguishable fermions or indistinguishable bosons the relevant physical wave-functions are

$$\Psi^\epsilon(x_1, \ldots, x_N) = \sum_{\sigma\in S_N} (\mathrm{sgn}(\sigma))^\epsilon \psi(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) \tag{5.1}$$

where

- $\epsilon = 0$: bosons
- $\epsilon = 1$: fermions

Comparing this with the two one dimensional representations of the symmetric group S_N one obtains that bosons transform according to the trivial representation and fermions transform according to the alternating representation of S_N.
Particles that transform in another irreducible representation of the symmetric group are said to obey *parastatistics*. Parastatistics have never been observed for fundamental particles in $3 + 1$ space-time dimensions. Still in the next section we provide an example of how to find a suitable basis of wave-functions for a system of three identical particles that obey parastatistics.

5.1 Example: three identical particles that obey parastatistics

Let $\psi(x_1, x_2, x_3)$ be an eigenfunction of the Hamiltonian for a certain energy eigenvalue. For the three particles being bosons or fermions, the relevant physical wave-function of the system can be obtained from equation (5.1) for $\epsilon = 0$ or $\epsilon = 1$. In the case of $N = 3$ there is only one other irreducible representation of the symmetric group, i.e. the two-dimensional irreducible representation $V_{(2,1)}$. To obtain a basis if the particles were to obey parastatistics we apply the Young symmetrizer $c_{(2,1)}$ on $\psi(x_1, x_2, x_3)$ and find

$$\Psi^{(1)} = \psi(x_1, x_2, x_3) + \psi(x_2, x_1, x_3) - \psi(x_3, x_2, x_1) - \psi(x_3, x_1, x_2)$$

For the second basis-vector we recall that $V_{(2,1)} = \mathbb{C}\left\{c_{(2,1)}, e_{(23)} \cdot c_{(2,1)}\right\}$. Therefore the second basis-vector is obtained by applying the transposition $e_{(23)}$ on $\Psi^{(1)}(x_1, x_2, x_3)$:

$$\Psi^{(1)'} = \psi(x_1, x_3, x_2) + \psi(x_3, x_1, x_2) - \psi(x_2, x_3, x_1) - \psi(x_2, x_1, x_3)$$

As mentioned before the space of all possible wave-functions for a certain energy eigenvalue is $N! = 6$ dimensional. The two missing basis-vectors can be obtained by considering the other possible filling of the Young diagram

$$\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$$

i.e. the standard Young tableau

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

This then yields the two wave-functions

$$\Psi^{(2)} = \psi(x_1, x_2, x_3) + \psi(x_3, x_2, x_1) - \psi(x_2, x_1, x_3) - \psi(x_2, x_3, x_1)$$

and

$$e_{(23)}\Psi^{(2)} \equiv \Psi^{(2)'} = \psi(x_1, x_3, x_2) + \psi(x_2, x_3, x_1) - \psi(x_3, x_1, x_2) - \psi(x_3, x_2, x_1)$$

to complete the basis of six possible wave-functions.

6 Conclusions

Based on the direct correspondence between a conjugacy class and an irreducible representation in the symmetric group we introduced a mathematical framework to construct the irreducible representations explicitly via Young symmetrizers. The Frobenius formula was introduced to determine the character of an irreducible representation and in particular to calculate its dimension. Also, the even simpler Hook formula was introduced to compute the dimension of an irreducible representation of the symmetric group. Throughout the whole paper only elementary techniques were used, for example in section 4 in order to calculate the polynomials. This emphasises the elegance of the theory.

The physical application of the theory contemplates the idea of the behaviour under exchange operations of a system containing N indistinguishable bosons, or N indistinguishable fermions respectively, from the mathematical point of view. Since bosons and fermions transform according to an irreducible representation of the symmetric group each permutation acting on the coordinates of the wave-function of the system of N bosons, or N fermions respectively, will lead to the same wave-function of the system. This is to say that each irreducible representation of a group is invariant under the action of the group itself.

7 References

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