On Wigner's D-matrix and Angular Momentum

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Abstract

Using the Analytic Oscillator space (Fock Bargmann space) and the generating function of Wigner's D-matrix we will determine the generating functions of spherical harmonics and the 3-j symbols of SU(2). We will find also a relation between the Gegenbauer Polynomials and Wigner's d-matrix. We drive the 3-j symbols and we find simply the expression of 3-j symbols with magnetic momentum nulls. We will find the generating function the $3 - lm$ symbols of SO(3) and a new relation between the 3-j symbols. We will derive also the generating function of 6-j symbols. We will apply our method for the determination of the generating function and the spherical basis of four dimensional R^4 Harmonic oscillators.

1 Introduction

The Harmonic oscillator and the Angular Momentum are parts of the course of quantum mechanics and has been studied since time ago by many scientists using different methods [1, 7]. Racah has made an important contribution to the study of coefficients of 3-j and 6-j symbols. Furthermore Wigner fined the representation matrix of rotation. Then Schwinger studied the rotation groups starting from oscillator basis and found the generating functions of 3-j and 6-j. By analogy with the Schwinger work, Bargmann used the Fock space to study the angular momentum.

We observe that many spaces of the oscillators that have wave functions are function of complex numbers and it is conjugated. We will use this property to make a new simple study of angular momentum. For example, the two-dimensional isotropic harmonic oscillator with plane polar coordinates and the R^4 wave function in terms of Euler-angles, and the angular part is the Wigner D-matrix [7].

This work starts with the determination of the generating function of Wigner D-matrix and the exploitation of its properties which are very useful for the study of the angular momentum. Then we deduce the generating function of spherical harmonics and we find a relationship between the Gegenbauer Polynomials and Wigner's d-matrix.

Starting from the integration of the product of three Wigner D-matrix and three spherical harmonics we will find the 3-j symbols and the 3-lm symbols of SO (3) [8, 17]. We will simply calculate 3-j symbols with magnetic momentum nulls [6, 10].

We find a new relation between the 3-j symbols and we derive also the generating function of 6-j symbols [6, 16]. Moreover, we will apply our method for the determination of the generating function and the spherical basis for four dimension R4 Harmonic oscillators [7, 17].

It is important to emphasize that our method is simple and elementary.

This paper is organized as follows: in section 2 the derivation of the spherical harmonics and the representation of SU(2) is illustrated. In sections 3 and 4 we will derive The generating function of Wigner's D-matrix of SU(2), the generating functions of spherical harmonics and Legendre polynomials. In section 5 the relation between Gegenbauer polynomial and Wigner's D-matrix will be presented. The well known Coupling of two angular momenta and the generating function of 3-j symbols are exposed in parts six and seven. In section 8 we expose the new derivation of the 3-j with magnetic moments $m_i = 0$. We will derive the generating functions of $3-lm$ in section 9. Then we present the derivation of the new relation between the 3-j symbols in section 10. In section 11 we expose the derivation of the generating function of Racah coefficients. In section 12 we will find the generating function and the spherical basis of four dimensional of harmonic oscillator.

2 The spherical harmonics and the representation of SU(2)

We make a summary of the determination of spherical harmonics and the representation of SU(2) using the analytic Hilbert space [11, 16].

2.1 The spherical harmonics

The rotation group SO(3) leaves invariant the quadratic form: $x^2 + y^2 + z^2$. And it is well known in the theory of angular momentum [1]:

$$\overrightarrow{L} = \overrightarrow{r} \times \overrightarrow{p} = L_x \overrightarrow{i} + L_y \overrightarrow{j} + L_z \overrightarrow{k} \tag{2.1}$$

And

$$L_x = yp_x - zp_y, L_y = zp_x - xp_z, L_z = xp_y - yp_x \tag{2.2}$$

Where $\overrightarrow{r}$ the position is vector of the particle and $\overrightarrow{p}$ is the momentum. The commutators of the generators of SO(3) are:

$$\lfloor L_x, L_y \rfloor = i\eta L_z, \lfloor L_y, L_z \rfloor = i\eta L_x, \lfloor L_z, L_x \rfloor = i\eta L_y \tag{2.3}$$

The Casimir operator $\overrightarrow{L}^2 = L_x^2 + L_y^2 + L_z^2$ commutes with L_x, L_y, L_z.

The spherical harmonics are the eigenfunctions of angular momentum operators $\overrightarrow{L}^2$

and L_z. We found that

$$L_z Y_{lm}(\theta\varphi) = \eta m Y_{lm}(\theta\varphi), \overrightarrow{L}^2 Y_{lm}(\theta\varphi) = \eta^2 l(l+1) Y_{lm}(\theta\varphi) \tag{2.4}$$

In Dirac notation [1] we write:

$$Y_{lm}(\theta\varphi) = \langle\theta\varphi\|lm\rangle \tag{2.5}$$

2.2 The Analytic Hilbert space

The Analytic Oscillator space (or Fock space) defined by:

$$u_n(z) = \frac{z^n}{\sqrt{n!}} \tag{2.6}$$

With scalar product is:

$$(u_{n'}, u_n) = \int \frac{\bar{z}_{n'}}{\sqrt{n'!}} \frac{z_n}{\sqrt{n!}} d\mu(z) = \delta_{n,n'} \tag{2.7}$$

And $d\mu(z)$ is the measure of integration:

$$d\mu(z) = (\frac{1}{\pi}) exp[-\bar{z}z] dx dy_i, z = x_i + iy_i \tag{2.8}$$

Furthermore, it is easy to check the useful relationship:

$$\int exp[\alpha z + \beta\bar{z}] d\mu(z) = e^{\alpha\beta} \tag{2.9}$$

2.3 The representation of SU(2) in the analytic Hilbert space

By analogy with the angular momentum $\overrightarrow{L}$ we define the generators of SU(2) in the analytic Hilbert space, u_i, $i = 1, 2$, by:

$$J_1 = \left(u_1 \frac{1}{\partial u_2} - u_2 \frac{1}{\partial u_1}\right), J_2 = \frac{1}{i}\left(u_1 \frac{1}{\partial u_2} - u_2 \frac{1}{\partial u_1}\right), J_3 = \frac{1}{2}\left(u_1 \frac{1}{\partial u_1} - u_2 \frac{1}{\partial u_2}\right) \tag{2.10}$$

And

$$\overrightarrow{J}^2 = N(N+1), N = \frac{1}{2}\left(u_1 \frac{1}{\partial u_1} - u_2 \frac{1}{\partial u_2}\right) \tag{2.11}$$

We find that:

$$\overrightarrow{J}^2 \varphi_{jm}(u) = j(j+1)\varphi_{jm}(u), J_3 \varphi_{jm}(u) = m\varphi_{jm}(u) \tag{2.12}$$

With $\varphi_{jm}(u)$ is the representation of SU(2) in the analytic Hilbert space with:

$$\varphi_{jm}(u) = \frac{u_1^{(j+m)} u_2^{(j-m)}}{\sqrt{(j+m)!(j-m)!}} \tag{2.13}$$

The functions $\varphi_{jm}(u)$ are orthonormal basis isomorphs to the Schwinger realization of SU(2) in terms of boson operators [6] :

$$|jm\rangle = \frac{(a_1^+)^{j+m}(a_2^+)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|00\rangle \tag{2.14}$$

It is important to note that the functions $\left\{u_n(z)e^{-z\overline{z}/2}\right\}$ are the wave functions of subspaces of the cylindrical basis of R2 of the Harmonics oscillators [18, 19, 22].

3 The generating function of Wigner's D-matrix of SU(2)

We will determine the Wigner's D-matrix and its generating function. We Show that the orthogonality of D-matrices is performed simply [11].

3.1 The Wigner's D-matrix

The Wigner's D-matrix of the rotation $R(\Omega) = e^{i\psi J_z}e^{i\theta J_y}e^{i\varphi J_z}$ is given in Dirac notations by:

$$D^j_{(m',m\}}(\Omega) = (jm|R(\Omega)|jm) = e^{im'\psi}d^j_{(m',m)}(\theta)e^{im\varphi} \tag{3.1}$$

We have [11] also

$$\rho^{2j}R(\Omega)\varphi_{jm}(u) = \frac{(z_1u_1 + z_2u_2)^{j+m}(-\overline{z}_2u_1 + z_1u_2)^{j-m}}{\sqrt{(j+m)!(j-m)!}} \tag{3.2}$$

with

$$z_1 = x_1 + iy_1 = \rho cos\frac{\theta}{2}e^{i(\psi+\varphi)/2}, z_2 = x_2 + iy_2 = \rho sin\frac{\theta}{2}e^{i(\psi-\varphi)/2} \tag{3.3}$$

and

$$D^j_{(m',m\}}(z) = \rho^{2j}D^j_{(m',m\}}(\Omega), \Omega = (\psi\theta\varphi) \tag{3.4}$$

with (ψ, θ, φ) are the angles of Euler and $\left\{D^j_{(m',m\}}(z)\right\}$ is a subspace of spherical basis of R^4 Harmonic oscillators [7, 22].
For m = j we find that: $\varphi_{jm'}(u) = \rho^{2j}D^j_{(m',0\}}(\Omega)$

3.2 The generating function of Wigner's D-matrix

Multiplying (2.10) by $\varphi_{jm'}(v)$ and summing with respect to j, m, m' we find the generating function of Wigner's D-matrix

$$G(v, z, u) = exp\left(\begin{pmatrix} v_1 & v_2 \end{pmatrix}\begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = \sum_{j,m,m'} \varphi_{jm'}(v)D^j_{(m',m\}}(z)\varphi_{jm}(u) \tag{3.5}$$

3.3 Orthogonality of D-Matrices

The Orthogonality of D-Matrices is:

$$\int exp\left(\begin{pmatrix} v'_1 & v'_2 \end{pmatrix}\begin{pmatrix} \overline{z}_1 & z_2 \\ -\overline{z}_2 & z_1 \end{pmatrix}\begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} + \begin{pmatrix} v_1 & v_2 \end{pmatrix}\begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) d\mu(z) =$$

$$\sum_{jm}\prod_{i=}^{2}\varphi^{ji}_{(mi)}(v^i)\varphi^{ji}_{(m'i)}(v^i)\left(\varphi^{ji}_{(mi)}(v^i)\varphi^{ji}_{m'i}(v^i)\right)\int \overline{D}^{j_1}_{(m',m\}}(z)D^{j_2}_{(m',m\}}(z)d\mu(z) \quad (3.6)$$

with

$$d\mu(z) = 2exp(-\rho^2)\rho^3 d\rho d\mu(\Omega)$$

and

$$d\mu(\Omega) = \frac{1}{8\pi^2}d\psi sin\theta d\theta d\varphi \quad (3.7)$$

In addition the integration of the first member of (3.6) is:

$$exp\left[(u_2v_2 + v_1u_1)(v'_1u'_1 + u'_2v'_2)\right] \quad (3.8)$$

And, the integration of the second member on the variable ρ of (3.6), using the formula:

$$\int_0^\infty \rho^{2n+1}e^{-\rho^2}d\rho = n!/2, n = j_1 + j_2 + 1 \quad (3.9)$$

After development of (3.8) and identification with (3.6) we find that:

$$\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\overline{D}^l_{(m_1,m'_1)}(\psi\theta\varphi)D^l_{(m_2,m'_2)}(\psi\theta\varphi)d\Omega = \frac{1}{2j_1+1}\delta_{m_1,m_2}\delta_{m'_1,m'_2}\delta_{j_1,j_2} \quad (3.10)$$

4 The generating functions of spherical harmonics And Legendre polynomials

We will determine the generating function of spherical harmonics and the generating function of Legendre polynomials [20, 21].

4.1 The generating function of spherical harmonics

The relationship between $D^l_{m',m}(\psi\theta\varphi)$ and $Y_{lm}(\theta\varphi)$ is well known [7, 10, 11]:

$$D^l_{(0,m)}(\psi\theta\varphi) = \left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}},$$

and

$$D^l_{(0,0)}(\Omega) = P_l(cos\theta) \quad (4.1)$$

$Y_{lm}(\theta\varphi)$ is the spherical harmonics and is Laguerre polynomial.
By putting $m' = 0$ in (3.1) we find a very useful expression for the rest of this work:

$$\langle\varphi_{jm}(v_1,\overline{v}_1)|R(\Omega)|\varphi_{jm}(t_1,t_2)\rangle/j! = D^l_{(0,m)}(\psi\theta\varphi) \tag{4.2}$$

Then we find

$$\int exp\left[\begin{pmatrix} v & \overline{v}\end{pmatrix}\begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1\end{pmatrix}\begin{pmatrix} u_1 \\ u_2\end{pmatrix}\right] d\mu(v) = exp\left[\frac{(\overrightarrow{a}.\overrightarrow{r})}{2}\right] \tag{4.3}$$

After integration we find the generating function of spherical harmonics.

$$exp\left[\frac{(\overrightarrow{a}.\overrightarrow{r})}{2}\right] = \sum_{lm}\left[\frac{4\pi}{2l+1}\right]^{\frac{1}{2}}\varphi_{lm}(u)Y_{lm}(\overrightarrow{r}) \tag{4.4}$$

$\overrightarrow{a} = \overrightarrow{a}(a)$ is a vector of length zero, $\overrightarrow{a}.\overrightarrow{a} = 0$ and has the components

$$a_x = -u_1^2 + u_2^2, a_y = -i(u_1^2 + u_2^2), a_z = 2u_1u_2 \tag{4.5}$$

and

$$\overrightarrow{r} = (x,y,z), x = z_1\overline{z}_2 + z_2\overline{z}_1, y = i(-z_1\overline{z}_2 + z_2\overline{z}_1), \tag{4.6}$$

$$z = z_1\overline{z}_1 - z_2\overline{z}_2 = rcos\theta, \text{ with } r = \rho^2.$$

4.2 The generating function of Legendre polynomials

Put $u_2 = \overline{u}_1$ in the expression (4.4) and $u_1 = v_1 + iv_2$ we find that:

$$\int exp\frac{1}{2}\left[x(-u_1^2 + \overline{u}_1^2) - iy(u_1^2 + \overline{u}_1^2) + 2z(u_1 \times \overline{u}_1)\right] d\mu(u) = \sum_l r^l P_l(cos\theta) \tag{4.7}$$

A- We write the first member of (4.7) in matrix form:

$$\left(-2ixv_1v_2 + (-iy+z)v_1^2 + (iy+z)v_1^2\right) = \begin{pmatrix} v_1 & v_2\end{pmatrix}\begin{pmatrix} -i(y+iz) & -ix \\ -ix & i(y-iz)\end{pmatrix}\begin{pmatrix} v_1 & v_2\end{pmatrix} \tag{4.8}$$

Let $r = \lambda$, we find:

$$\int exp\left[\lambda\begin{pmatrix} v_1 & v_2\end{pmatrix}\begin{pmatrix} -i(y+iz) & -ix \\ -ix & i(y-iz)\end{pmatrix}\begin{pmatrix} v_1 & v_2\end{pmatrix}\right] d\ \mu(u) = \frac{1}{\sqrt{1 - 2\lambda cos\theta + \lambda^2}} \tag{4.9}$$

B- We find the generating function of Legendre polynomials [20, 21] after comparing the two sides of (4.7):

$$(1 - 2\lambda cos\theta + \lambda^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(cos\theta)\lambda^n \tag{4.10}$$

5 The relation between Gegenbauer Polynomial and the Wigner's D-matrix

We will determine a relationship between Gegenbauer Polynomials [20, 21] and the Wigner's D-matrix.
We suppose that in (3.1) and we replace in the generating function (3.5), v_1 by $\overline{u}_1$, v_2 by $\overline{u}_2$ and $r = \lambda$, we thus find:

$$exp\left[\begin{pmatrix}\overline{u}_1 & \overline{u}_2\end{pmatrix}\begin{pmatrix}z_1 & z_2\\ -\overline{z}_2 & \overline{z}_1\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}\right] = \sum_{j,m,m'} \varphi_{jm'}(\overline{u}) D^j_{(m',m\}}(z)\varphi_{jm}(u) \tag{5.1}$$

After the integration of the expression

$$\int exp\left[(\overline{u})\begin{pmatrix}z_1 & z_2\\ -\overline{z}_2 & \overline{z}_1\end{pmatrix}(u)\right] d\mu(u) = \sum_{j,m,m'} \lambda^{2j} D^j_{(m',m\}}(\Omega)\int \varphi_{jm'}(\overline{u})\varphi_{jm}(u)d\mu(u) \tag{5.2}$$

Using the Gauss integral:

$$\left(\frac{1}{\pi}\right)^n \int \prod_{i=1}^{n} du_i dv_i exp(-\overline{V}^t XV) = (det(X))^{-1} \tag{5.3}$$

With: $V = (v_1, v_2, \ldots, v_n)$, $v_i = x_i + iy_i$.
We find that:

$$\frac{1}{(1 - 2\lambda cos\frac{\theta}{2} cos(\frac{\psi+\varphi}{2}) + \lambda^2)} = \sum_{jm} \lambda^{2j} D^j_{(m,m)}(\Omega) \tag{5.4}$$

The development of the first member using generating function formula of the Gegenbauer Polynomials $C_n^\beta(t)$:

$$(1 - 2t\lambda + \lambda^2)^{-\beta} = \sum_{n=0}^{\infty} \lambda^n C_n^\beta(t) \tag{5.5}$$

We find the following relation:

$$C^1_{2j}\left(cos\frac{\theta}{2} cos\frac{(\psi+\varphi)}{2}\right) = \sum_m D^j_{(m,m)}(\Omega) \tag{5.6}$$

If we put $(\psi + \varphi) = 0$ we find also the new relation:

$$C^1_{2j}(cos\frac{\theta}{2}) = \sum_m d^j_{(m,m)}(\theta) \tag{5.7}$$

6 The Coupling of two angular momenta

We make a simple revision of the couplings of two angular moments [10, 11].
We deduce from the orthogonality of the D-Matrices the integration formulas of the product of three D-matrices and the product of three spherical harmonics.

6.1 The coupled function of two angular moments

The function of two coupled angular momentum, $\overrightarrow{J} = \overrightarrow{j}_1 + \overrightarrow{j}_2$, in the Hilbert space of the analytic functions is:

$$\Psi_{(j_1 j_2) j_3 m}(u^1, u^2) = \sum_{m_1 m_2} \langle j_1 m_1, j_2 m_2 \| (j_1 j_2) j_3 m_3 \rangle \varphi_{j_1 m_1}(u^1) \varphi_{j_2 m_2}(u^2) \tag{6.1}$$

with

$$D^{j_3}_{(m_3, m_3)}(\Omega) = \left(\Psi_{(j_1 j_2) j_3 m}(u^1, u^2), R(\Omega) \Psi_{(j_1 j_2) j_3 m}(u^1, u^2) \right) \tag{6.2}$$

And $(u^i) = (u^i_1, u^i_2)$

The coefficient

$$\langle j_1 m_1, j_2 m_2 \| (j_1 j_2) j_3 m_3 \rangle = \langle (j_1 j_2) j_3 m_3 \| j_1 m_1, j_2 m_2 \rangle$$

is the Clebsh-Gordan [6].

Conversely of (6.1) we can write:

$$\varphi_{j_1 m_1}(u^1) \varphi_{j_2 m_2}(u^2) = \sum_{m_1 m_2} \langle j_1 m_1, j_2 m_2 \| (j_1 j_2) j_3 m_3 \rangle \Psi_{(j_1 j_2) j_3 m}(u^1, u^2) \tag{6.3}$$

6.2 The relationships of Wigner 3-j symbols and the Clebsh-Gordan coefficients

The relationships of 3-j symbols are related to the Clebsh-Gordan coefficients by [10, 11]:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1 - j_2 - m_2} (2j_3 + 1)^{-1/2} \langle j_1 m_1, j_2 m_2 \| (j_1 j_2) j_3 - m_3 \rangle \tag{6.4}$$

Wigner introduced the 3-j symbols for symmetries purpose.

6.3 The integral of the product of three D-matrices and three spherical harmonics

Using the relation (6.3) and (6.4) we find that:

$$D^{j_1}_{(m'_1, m_1)}(\Omega) D^{j_2}_{(m'_2, m_2)}(\Omega) = \sum_{jmm'} (j_3 + 1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix} \overline{D}^{j_3}_{(m'_3, m_3)}(\Omega) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{6.5}$$

We deduce from the orthogonality of matrices D (.) the relations which are very important for the calculation of the symbols of 3-j and $3 - lm$.

$$\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} D^{j_1}_{(m_1',m_1)}(\Omega)D^{j_2}_{(m_2',m_2)}(\Omega)D^{j_3}_{(m_3',m_3)}(\Omega)d\Omega = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad (6.6)$$

In posing $m_1' = m_{,2} = m_3' = 0$ and using (6.6) we find the second relation that interests us for the determination of $3 - lm$.

$$\int_0^{2\pi}\int_0^{\pi} [Y_{l_1m_1}(\theta,\varphi)Y_{l_2m_2}(\theta,\varphi)Y_{l_3m_3}(\theta,\varphi)]\, sin\theta d\theta d\varphi$$
$$= \left[\frac{1}{4\pi}(2l_1+1)(2l_2+1)(2l_3+1)\right]^{1/2}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad (6.7)$$

With l_1, l_2, and l_3 are integers, and $J = l_1 + l_2 + l_3$ is even.

7 The generating function of 3-j symbols

We will determine the generating function of the 3-j symbols [6, 16]. In addition we will give the generating function, G.F, of four coupling as an example of the generalization.

7.1 The generating function of 3-j symbols

We will deduct simply the G.F of 3-j symbols by using the integration of the product of three G.F of Wigner's D-matrix (6.6) and the formulas (3.9) and (3.10).
We multiply the two members of (6.6) by

$$r^{2(J+1)+1}dr, J = j_1 + j_2 + j_3 \quad (7.1)$$

And we carry out the integration and then we identify the two members.

A- The integral of the first member or the product of three generating functions is:

$$G^3 = \int\sum_{i=1}^{3}\left[G(u^i, z^i, v^i)\right] d\mu(z_1)d\mu(z_2) =$$
$$exp\left[\left[v^3, v^2\right]\left[u^3, u^2\right] + \left[v^3, v^1\right]\left[u^3, u^1\right] + \left[v^1, v^2\right]\left[u^1, u^2\right]\right] \quad (7.2)$$

B-The integration of the second member with the help of (2.9) and (3.9) we find:

$$\int exp[det(\bar{t}, v)] \times exp[det(t, u)]d\mu(t) = \int G_{3j}(\bar{t}, v) \times G_{3j}(t, u)d\mu(t) \quad (7.3)$$

C- The function $G_3(t, u)$ is the generating function of the 3-j symbols [6].

$$G_{3j}(t, u) = exp[det(\lambda, u)] = exp\left[t_1[u^3, u^3] + t_2[u^3, u^1] + t_3[u^1, u^2]\right] =$$
$$\sum_{jm}\varphi_{j_1m_1}(u^1)\varphi_{j_2m_2}(u^2)\varphi_{j_3m_3}(u^3)\Phi(t_1t_2t_3)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad (7.4)$$

And

$$\Phi(t_1t_2t_3) = [(J+1)!]^{1/2}\,\frac{t_1^{J-2j_1}t_2^{J-2j_2}t_3^{J-2j_3}}{[(J-2j_1)!(J-2j_2)!(J-2j_3)!]^{1/2}} \tag{7.5}$$

With $\lfloor u^i, u^j\rfloor = u_1^i u_2^j - u_2^i u_1^j$
And

$$(u^i.u^j) = u_1^i u_2^j + u_1^i u_2^j \tag{7.6}$$

We can write also:

$$det(t,u) = \begin{vmatrix} t_1 & t_2 & t_3 \\ u_1^1 & u_1^2 & u_1^3 \\ u_2^1 & u_2^2 & u_2^3 \end{vmatrix} \tag{7.7}$$

Note that Regge has studied the symmetries of the 3-j symbols starting from the invariance of the determinant $det(t,u)$ [13].
Moreover, if we put in (7.4) $u_1^3 = u_2^3 = 0$ or $j_3 = 0$, $j_1 = j_2$ we find that:

$$\begin{pmatrix} j_1 & j_1 & 0 \\ m_1 & -m_1 & 0 \end{pmatrix} = (-1)^{j_1-m_1}(2j_1+1)^{-\frac{1}{2}} \tag{7.8}$$

7.2 The relation between the generating functions of the coupling and the uncoupling

Using the expression (7.1) we deduce the generating function of the couplings:

$$exp\lfloor t_3[u^1,u^2] + t_2(u^3.u^1) + t_1(u^3.u^2)\rfloor = \sum_{jm}(2j_3+1)^{-1/2}\varphi_{j_3m_3}(u^3)\Phi(t_1t_2t_3)\Psi_{(j_1j_2)j_3m}(u^1,u^2) \tag{7.9}$$

And it is very important to observe that [6]:

$$exp\lfloor t_3[u^1,u^2] + t_2(u^3.u^1) + t_1(u^3.u^2)\rfloor = exp\left[t_3[\frac{\partial}{v^1},\frac{\partial}{v^2}] + t_2\left(u^3.\frac{\partial}{v^1}\right) + t_1\left(u^3.\frac{\partial}{v^2}\right)\right] exp\left[(v^1.u^1) + (v^2.u^2)\right] \tag{7.10}$$

So the function (7.9) is simply deduced from the generating function of the uncoupling functions. Moreover the expression (7.10) is very interesting for the couplings of the several angular moments and for the calculations of the 6-j symbols and Racah coefficients.

7.3 The generating function of four coupling

The generalization of (7.2) is simple and we are going to give the generating function of the couplings of four angular moments.
Indeed

$$G^5 = \int \sum_{i=1}^{5}[G(u^i,z^i,v^i)]d\mu(z_1)d\mu(z_2) = \int G_{5j}(\overrightarrow{\lambda},v)\times G_{5j}(\lambda,v)d\mu(\lambda) \tag{7.11}$$

We follow the same method of calculations of (7.2) and we find:

$$G_{5j}(t,u) = exp\left[\sum_{i=1,j<i}^{4} \lambda_{ij}\left[u^i, u^j\right]\right] \tag{7.12}$$

8 A new derivation of the 3j with magnetic moments $m_i = 0$

The coefficients 3-j with $m_i = 0$ are frequently found in kinetic momentum calculations but the proposed methods are difficult [6, 11]. In following section, I will determine these coefficients by a very simple method of calculation.
By posing $u_2^1 = \overline{u}_1^1$, $u_2^2 = \overline{u}_1^2$, and $u_2^3 = \overline{u}_1^3$ in (7.4), we find that:

$$\begin{pmatrix} t_1 & t_2 & t_3 \\ u_1^1 & u_1^2 & u_1^3 \\ \overline{u}_1^1 & \overline{u}_1^2 & \overline{u}_1^3 \end{pmatrix} = \begin{pmatrix} \overline{u}_1^1 & \overline{u}_1^2 & \overline{u}_1^3 \end{pmatrix} \begin{pmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{pmatrix} \begin{pmatrix} u_1^1 \\ u_1^2 \\ u_1^3 \end{pmatrix} =^t (\overline{X})A(X) \tag{8.1}$$

We find after integration that

$$\int exp^t(\overline{X})A(X)\mathbf{X}_{i=1}^3 d\mu(u^i) = \frac{1}{1+\lambda^2(t_1^2+t_2^2+t_3^2)} \tag{8.2}$$

And consequently:

$$\frac{1}{1+\lambda^2(t_1^2+t_2^2+t_3^2)} = \sum_j \Phi(t_1 t_2 t_3) \begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} \tag{8.3}$$

After development and identification of two members of (8.3) we find the 3-j symbols with magnetic moments $m_i = 0$, $i = 1,2,3$:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{J/2}\left[\frac{\prod_{i=1}^3 (J-2j_i)!}{(J+1)!}\right]^{\frac{1}{2}} \times \frac{(\frac{J}{2})!}{\prod_{i=1}^3(\frac{J}{2}-j_i)!} \tag{8.4}$$

If J is even.

9 The generating function of the $3-lm$ symbols of SO(3)

We will determine for the first time the generating function of the $3-lm$ symbols.
The generating function of Spherical Harmonics is:

$$exp\left[\frac{\overrightarrow{a}(u).\overrightarrow{r}}{2}\right] = \sum_{lm}\left[\frac{4\pi}{2l+1}\right]^{\frac{1}{2}} \varphi_{lm}(u)(r)^l Y_{lm}(\theta\varphi) \tag{9.1}$$

Therefore we write:

$$
exp\left[\frac{(\overrightarrow{a}(u^1)+\overrightarrow{a}(u^2)+\overrightarrow{a}(u^3))\,.\,\overrightarrow{r}}{2}\right]=
$$
$$
\sum_{lm}\left\{\left[\prod_{i=1}^{3}\left[\frac{4\pi}{2l_i+1}\right]^{\frac{1}{2}}\right](r)^{l_1+l_2+l_3}\times\left[\prod_{i=1}^{3}(Y_{\lim i}(\theta,\varphi))\right]\varphi_{l_1m_1}(u^1)\varphi_{l_1m_1}(u^2)\varphi_{l_1m_1}(u^3)\right\} \tag{9.2}
$$

Multiplying the two members by:

$$
\frac{1}{\pi^{3/2}}exp(-r^2)dxdydz \tag{9.3}
$$

We find after integration of the first member:

$$
\frac{1}{\pi^{3/2}}\int_{-\infty}^{\infty}exp\left[\frac{(\overrightarrow{a}(u^1)+\overrightarrow{a}(u^2)+\overrightarrow{a}(u^3))\,.\,\overrightarrow{r}}{2}\right]exp(-\overrightarrow{r}^2)dxdydz=
$$
$$
exp\left[-\frac{1}{4}\left([u^1,u^2]^2+[u^1,u^3]^2+[u^2,u^3]^2\right)\right] \tag{9.4}
$$

And with the help of the integral:

$$
\int_0^{\infty}x^ne^{-x^2}dx=\Gamma\left(\frac{n+1}{2}\right)/2 \tag{9.5}
$$

Moreover, after the integration of the second member (9.2), using (6.7) and (9.5), we deduce the generating function of the $3-lm$ symbols of $SO(3)$:

$$
exp\left[-\frac{1}{4}\left([u^1,u^2]^2+[u^1,u^3]^2+[u^2,u^3]^2\right)\right]=
$$
$$
\sum_{lm}\frac{4}{\sqrt{\pi}}\Gamma\left(\frac{J+3}{2}\right)\begin{pmatrix}l_1 & l_2 & l_3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{pmatrix}\varphi_{l_1m_1}(u)\varphi_{l_1m_1}(v)\varphi_{l_1m_1}(w) \tag{9.6}
$$

With the help of (8.4) and Gamma duplication formula:

$$
\Gamma\left(z+\frac{1}{2}\right)\Gamma(z)=2^{1-2z}\sqrt{\pi}(2z) \tag{9.7}
$$

And the developments of the fist side of (9.6) we find the expression of $3-lm$ symbols for SO (3).

10 The new relationship between 3-j symbols

We will determine a new relation between the symbols 3-j starting from the expression (9.6) and using the generating function of the symbols 3-j, (7.4).

We note that the generating function of the $3-lm$ symbols (9.6) could be written as:

$$exp\left[-\frac{1}{4}\left([u^1,u^2]^2+[u^1,u^3]^2+[u^2,u^3]^2\right)\right]=$$
$$\int\left\{exp\left[-\frac{1}{4}\left(\bar{t}_3[u^1,u^2]+\bar{t}_2[u^1,u^3]+\bar{t}_1[u^2,u^3]\right)\right]exp\left[\left(t_3[u^1,u^2]+t_2[u^1,u^3]+t_1[u^2,u^3]\right)\right]\right\}d\mu(t) \tag{10.1}$$

After the development of (10.1), using (7.4) and the identification of the second member of (9.6) we find the new relation:

$$\left[\left(\frac{J}{2}+1\right)!\right]\sum_{m_1,m_2'}\left[\prod_{i=1}^{3}\frac{[(j_i+m_i)!(j_i-m_i)!]}{\prod_{i=1}^{3}(j_i'+m_i')!(j_i'-m_i')![(j_i'+m_i'')!(j_i'-m_i'')!]}\right]^{1/2}\times$$
$$\begin{pmatrix}\frac{l_1}{2} & \frac{l_2}{2} & \frac{l_3}{2}\\ m_1' & m_2' & m_3'\end{pmatrix}\begin{pmatrix}\frac{l_1}{2} & \frac{l_2}{2} & \frac{l_3}{2}\\ m_1'' & m_2'' & m_3''\end{pmatrix}=\left[(J+1)!\prod_{i=1}^{3}(J-2j_i)!\right]^{\frac{1}{2}}\begin{pmatrix}l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{pmatrix} \tag{10.2}$$

With:

$$J'=J/2, j_i'=j_i/2, m_i'+m_i''=m_i, i=1,2,3 \tag{10.3}$$

11 The generating function of 6-j symbols

We followed at the beginning the simple method proposed by Schwinger [6]. This method starts with the coupling of four angular moments instead of three. But we propose a new simpler method for the calculation of the G.F of 6-j symbols.
We note that it is important to introduce the 6-j symbols instead of the Racah coefficients for symmetry reasons. The couplings of the four moments are:

$$\vec{J}=\left(\vec{J}_1+\vec{J}_2\right)+\left(\vec{J}_3+\vec{J}_4\right)=\left(\vec{J}_1+\vec{J}_3\right)+\left(\vec{J}_2+\vec{J}_4\right) \tag{11.1}$$

Taking into account of the two coupling states and the relation (7.8), the overlap of the states is written then:

$$\langle((j_1j_2)j'(j_3j_4)j')\,j=0,m=0\|\,((j_1j_3)j''(j_2j_4)j'')\,j=0,m=0\rangle=$$
$$(-1)^{j'+j''-j_1-j_2}\left[(2j'+1)(2j''+1)\right]^{1/2}W(j_1j_2j_3j_4;j'j'') \tag{11.2}$$

With

$$j_{12}=j_{34}=j', j_{13}=j_{24}=j'' \tag{11.3}$$

But our method that we used in part three makes the calculation of the overlap elementary much easier than the Schwinger method that we will present later.

11.1 The coupling of the four angular moments

The uncoupled functions are:

$$\Psi(t,u) = exp\left((t^1u^1) + (t^2u^2) + (t^3u^3) + (t^4u^4)\right) \tag{11.4}$$

We will make the couplings $\overrightarrow{J}_{12} = \overrightarrow{J}_1 + \overrightarrow{J}_2$, $\overrightarrow{J}_34 = \overrightarrow{J}_3 + \overrightarrow{J}_4$ with the help of (7.10)

$$\begin{aligned} \Psi' = &\left\{ exp\left[\alpha_3[\frac{\partial}{t^1}, \frac{\partial}{t^2}] + \alpha_2\left(x.\frac{\partial}{t^1}\right) + \alpha_1\left(x.\frac{\partial}{t^2}\right)\right] \times \right. \\ &\left. exp\left[\beta_3[\frac{\partial}{t^3}, \frac{\partial}{t^4}] + \beta_2\left(y.\frac{\partial}{t^3}\right) + \beta_1\left(y.\frac{\partial}{t^3}\right)\right]\right\} \Psi(t,u) \end{aligned} \tag{11.5}$$

We do the coupling again $\overrightarrow{J} = \overrightarrow{J}_{12} + \overrightarrow{J}_{34}$:

$$exp\left[\gamma_3[\frac{\partial}{x}, \frac{\partial}{y}] + \gamma_2\left(z.\frac{\partial}{x}\right) + \gamma_1\left(z.\frac{\partial}{y}\right)\right]\Psi' \tag{11.6}$$

11.2 The expression of the coupling of the four angular moments

Then we put $z_1 = \gamma_1 = z_2 = \gamma_2 = 0$, $\gamma_3 = 1$ and so we get the generating function of the function $\Psi^L_{()}$ of the coupling $((j_1j_2)j'(j_3j_4)j')\, j = 0, m = 0$ and are of the form (7.12):

$$G^L_s(t,u) = exp\left[\sum_{i=1,j>i}^{4} \lambda_{ij}[u^i, u^j]\right] \tag{11.7}$$

with

$$\begin{array}{ccc} \lambda_{12} = \alpha_3 & \lambda_{13} = \alpha_2\beta_2 & \lambda_{14} = \alpha_2\beta_1 \\ \lambda_{23} = \alpha_1\beta_2 & \lambda_{24} = \alpha_1\beta_1 & \lambda_{34} = \beta_4 \end{array} \tag{11.8}$$

Similarly, we get $\Psi^R_{()}$ the generating function of the coupling $((j_1j_3)j''(j_2j_4)j'')\, j = 0, m = 0$, which has the form (7.12):

$$G^R_s(t,u) = exp\left[\sum_{i=1,j>i}^{4} \mu_{ij}[u^i, u^j]\right] \tag{11.9}$$

With

$$\begin{array}{ccc} \mu_{12} = \alpha'_2\beta'_2 & \mu_{13} = \alpha'_1\beta'_1 & \mu_{14} = \beta'_3 \\ \mu_{23} = -\alpha'_1\beta'_2 & \mu_{24} = \alpha'_2\beta'_1 & \mu_{34} = \beta'_3 \end{array} \tag{11.10}$$

We will find the generating function of 6-j symbols by calculating the expression:

$$\begin{aligned} I = \int exp\left[\sum_{i=1,j>i}^{4} \mu_{ij}\left[u^i, u^j\right]\right] exp\left[\sum_{i=1,j>i}^{4} \lambda_{ij}\left[\overline{u}^i, \overline{u}^j\right]\right] \prod_{i=1}^{4} d\mu(u^I) = \\ \sum_{j_1 \dots j''} (-1)^{j'+j''-j_1-j_2} \varphi_{j_1j_2j'}(\alpha)\varphi_{j_3j_4j'}(\beta)\varphi_{j_1j_3j''}(\alpha)\varphi_{j_2j_4j''}(\beta) W(j_1j_2j_3j_4; j'j'') \end{aligned} \tag{11.11}$$

We will propose a simple method for calculating the first member of (11.11).

11.3 The generating function of 6-j symbols

In the first member of (11.11) we make the change of the variables u_2^i by $\overline{u}_2^i$ $j = 1\dots8$ and we denote $\begin{pmatrix} u_1^1 & u_1^2 & u_1^3 & u_1^4 & u_2^5 & u_2^6 & u_2^7 & u_2^8 \end{pmatrix}$ by (X) we write then:

$$(A) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ 0 & 0 & 0 & 0 & -\lambda_{12} & 0 & \lambda_{23} & \lambda_{24} \\ 0 & 0 & 0 & 0 & -\lambda_{13} & -\lambda_{23} & 0 & \lambda_{34} \\ 0 & 0 & 0 & 0 & -\lambda_{14} & -\lambda_{24} & -\lambda_{34} & 0 \\ 0 & -\mu_{12} & -\mu_{13} & -\mu_{14} & 0 & 0 & 0 & 0 \\ \mu_{12} & 0 & -\mu_{23} & -\mu_{24} & 0 & 0 & 0 & 0 \\ \mu_{13} & \mu_{23} & 0 & -\mu_{34} & 0 & 0 & 0 & 0 \\ \mu_{14} & \mu_{24} & \mu_{34} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{11.12}$$

As a result we write (11.11) in the form:

$$I = \int exp\left[{}^t\left(\overline{X}\right)(A)(X)\right]\prod_{i=1}^{4} d\mu(u^I) \tag{11.13}$$

The integration of this expression is carried out using Gauss's formula (5.3) so we find:

$$I = \left(1 - \left(\sum_{i=1, j<i}^{4} \lambda_{ij}\mu_{ij}\right) + (\mu_{12}\mu_{34} + \mu_{13}\mu_{24} + \mu_{14}\mu_{23}) \times (\lambda_{12}\lambda_{34} + \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23})\right)^{-2} \tag{11.14}$$

By replacing λ_{ij}, μ_{ij} by their expressions (11.7), (11.9) and by changing the signs of α_1 and β_1', We find the generating function of 6-j symbols [6]:

$$\sum_{j_1..j''} \varphi_{j_1j_2j'}(\alpha)\varphi_{j_3j_4j'}(\beta)\varphi_{j_1j_3j''}(\alpha)\varphi_{j_2j_4j''}(\beta)W(j_1j_2j_3j_4;j'j'') = \tag{11.15}$$

$$(1-\alpha_3\alpha_2'\beta_2'-\beta_3\alpha_1'\beta_1'-\alpha_3'\alpha_2\beta_2-\beta_3'\alpha_1\beta_1-\alpha_1\beta_2\alpha_1'\beta_2'-\alpha_2\beta_1\alpha_2'\beta_1'+\alpha_3\beta_3\alpha_3'\beta_3')^{-2} \tag{11.16}$$

The development of (11.16) and its comparisons with (11.15) give the expression of the Racah coefficients [6, 16].

12 The generating function and the basis of the four dimensional spherical basis of the Harmonic oscillators

The resolution of the Schrdinger equation for an isotropic harmonic oscillator in R^4 has been performed by many others [7, 8, 18, 22] using the method of separations of variables. However as a simple application of angular momentum couplings we will propose a new method for the determination of the wave function of R^4 in terms of Euler-angles $(r, \theta, \varphi, \psi)$. We give the orthogonal transformation of the operators of the creations of Dirac which allows us to determine the eigenfunctions of the kinetic moment L_z . Then using the generating method we will find the wave functions of R^4.

12.1 The basis of two dimensions Harmonic oscillator

The basis of two dimensions Harmonic oscillator in Dirac notations [1, 22] is:

$$|n_1, n_2\rangle = (a_x^+)^{n_1}(a_y^+)^{n_1}|0, 0\rangle \tag{12.1}$$

This basis is not eigenfunctions of L_z. So to obtain the basis which has this property we must take the transformation [1]:

$$\begin{aligned}
&A_1^+ = \frac{\sqrt{2}}{2}(a_x^+ - ia_y^+), \qquad A_2^+ = \frac{\sqrt{2}}{2}(a_x^+ + ia_y^+),\\
&N_1 = A_1^+ A_1, \quad N_2 = A_2^+ A_2\\
&L_z = (N_1 - N_2), \qquad N = N_1 + N_2
\end{aligned} \tag{12.2}$$

The new basis $|N_1, N_2\rangle$ can be written in the form:

$$|jm\rangle = |j + m, j - m\rangle = \frac{A_1^{+j+m} A_2^{+j-m}}{\sqrt{(j + m)!(j - m)!}}|0, 0\rangle \tag{12.3}$$

This basis is eigenfunctions of L_z and N with the values $2m$ and $2j$.

12.2 The generating function of the two dimensions oscillator

The generating function of the two dimensions oscillator may be written in the form:

$$\begin{aligned}
|F_{jm}(t_1, t_2)\rangle = exp[t_1 A_1^+ + t_2 A_2^+]|0, 0\rangle &= \sum_{jm} \frac{t_1^{j+m} t_2^{+j-m}}{\sqrt{(j + m)!(j - m)!}}|j, m\rangle\\
&= exp[a_x^+\sqrt{2}(t_1 + t_2)/2 + ia_y^+\sqrt{2}(-t_1 + t_2)/2]|0, 0\rangle
\end{aligned} \tag{12.4}$$

But the generating function of one dimension of harmonic oscillator is:

$$\pi^{-1/4} exp\left(-\frac{x^2}{2} + \sqrt{2}x - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{n!}}\langle x\|n\rangle, \langle x\|n\rangle = u_n(x) \tag{12.5}$$

Put z in term of Cartesian coordinates $z = x + iy$, $x = \rho cos\varphi$, $y = \rho sin\varphi$ and using (12.5) we find:

$$F_{jm}(t_1, t_2, z) = \frac{1}{\sqrt{\pi}} exp\left[-\frac{x^2 + y^2}{2} + t_1(z) + t_2(\overline{z}) - t_1 t_2\right] = \sum_{jm} \varphi_{jm}(t)\varphi_{jm}(\rho, \varphi) \tag{12.6}$$

12.3 The cylindrical basis of Harmonic oscillator and the Analytic Hilbert space

The wave function of the cylindrical base in term of Cartesian coordinates is very well known [18, 19], so we put $\lambda = \sqrt{m\omega/\eta} = 1$ and we write:

$$\Phi_{nm}(r, \varphi) = \sqrt{\frac{2p!}{(p + |m|)!}} e^{-r^2/2} L_p^{|m|}(r) r^{|m|} e^{im\varphi} \tag{12.7}$$

With

$$n = n_1 + n_2, \quad p = (n - |m|)/2 \tag{12.8}$$

$L_p^{|m|}(r)$ Are the Associates Laguerre polynomial:
If we put p = 0 in (12.7) we find the relation of $u_n(z)$, (2.6), with the wave function of the cylindrical basis:

$$\Phi_{nm}(r,\varphi) = u_n(z)e^{-z\overline{z}/2} = \frac{z^n}{\sqrt{n!}}e^{-z\overline{z}/2} \tag{12.9}$$

12.4 The generating function of the spherical base of R^4

A- We consider the product of the two uncoupled basis (7.10) by:

$$\frac{t_1^{j_1+m_1}t_4^{j_2-m_2}}{\sqrt{(j_1+m_1)!(j_2-m_2)!}} \times \frac{t_3^{j_2+m_2}t_2^{j_1-m_1}}{\sqrt{(j_2+m_2)!(j_1-m_1)!}} \tag{12.10}$$

And the generating function of 3-j of this basis is:

$$G_s((\alpha,\xi),t) = exp\left[\gamma(t_1t_2 - t_3t_4) + \xi_1(t_1u_1 + t_4u_2) + \xi_2(t_3u_1 + t_2u_2)\right] \tag{12.11}$$

B- We will derive the generating function of the spherical base of R^4.

12.4.1 The generating function of R^4

To determine the generating function of the spherical base of R^4 we can perform the integration of the below expression using Gauss formula

$$\begin{aligned} G_c(s,\xi u,\lambda p) &= \int \overline{G_S((\gamma\xi),t)}F_{2c}((s,t),\lambda\rho_1,\lambda\rho_2)\prod_{i=1}^{4}d\mu(t_i) \\ \text{and} \quad F_{2c}(t_1,t_2,\lambda z) &= F_{j_1m_1}(t_1,t_2,\lambda z_1)F_{j_2m_2}(t_3,t_4,\lambda z_2) \end{aligned} \tag{12.12}$$

The Gauss formula is:

$$\left(\frac{1}{\pi}\right)^n \int \prod_{i=1}^{n} du_idv_iexp(-\overline{V}^tXV + A^tV + \overline{V}^t\overline{B}) = (det(X))^{-1}exp(A^tX^{-1}\overline{B}) \tag{12.13}$$

With $V = (v_1, v_2, \ldots, v_n,\ v_i = x_i + iy - i$
After integration and putting $\gamma = -1$ we find the generating function of the basis $\Psi_{(j_1j_2)j(m_1m_2)m}(\lambda\rho,\psi\theta\varphi)$ of R^4:

$$\begin{aligned} G_c(s,\xi u,\lambda p) = \left(\frac{\lambda}{\pi}\right)\frac{1}{(1-s^2)^2}Exp\left\{-\lambda\rho^2 - \frac{s^2}{(1-s^2)}2\lambda\rho^2 + \right. \\ \left. \frac{s}{1-s^2}\sqrt{\lambda}\begin{pmatrix}\xi_1 & \xi_2\end{pmatrix}\begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & -\overline{z}_1\end{pmatrix}\begin{pmatrix}u_1 & u_2\end{pmatrix}\right\} \end{aligned} \tag{12.14}$$

12.4.2 The spherical basis of R^4

We find the spherical base R^4 after development (12.14) and using the generating function of the Associates Laguerre polynomial [20, 21]:

$$\begin{aligned} &\sum_{p=0}^{\infty} t^p L_p^{(\alpha)}(x) = \frac{1}{(1-t)^{\beta+1}} e^{-\frac{t}{1-t}x}, x = \lambda r \\ \text{With} \qquad &L_p^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \Gamma(n+\alpha+1)}{k!(n-k)!\Gamma(k+\alpha+1)} x^k \end{aligned} \tag{12.15}$$

We write the development of (12.14) in the form:

$$\sum \frac{s^{n+j}}{(1-s)^{n+2+2j}} \varphi_{j(j_1-j_2)}(\xi) \varphi_{jm}(u) L_{j_1+j_2-j}^{2j+1}(2\lambda\rho^2) D^j_{((j_1-j_2),m)}(z) \tag{12.16}$$

The spherical base of R^4 is therefore

$$\Psi_{njm,m'}(\rho, \psi\theta\varphi) = N_{njm,m'} e^{-\rho^2/2} R_{nk}(\rho) D^j_{(j_1-j_2),m)}(\psi\theta\varphi) \tag{12.17}$$

The radial part is:

$$R_{nk}(\rho) = N_{nk} \rho^k L_n^{(k+(d-d)2)}(\rho^2) e^{-\rho^2/2} \tag{12.18}$$

With $n = j_1 + j_2 - j_3$, and $k = 2j_3$, $d = 4$, $2\lambda = 1$. The normalization factor is:

$$N_{nk} = \left[\frac{2\Gamma(n+1)}{\Gamma(n+k+d/2)} \right] \tag{12.19}$$

12.5 Generalization of the transformation of R^2 basis to R^4 and R^8 basis.

We observe that (12.2) analogue to the polar transformation [22] written in the matrix form by:

$$\begin{pmatrix} A_1^+ \\ A_2^+ \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} a_x^+ \\ ia_y^+ \end{pmatrix}, \qquad H_2(1) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{12.20}$$

With $H_2(1)$ is an orthogonal matrix.

A- The transformation for R^4 basis With Hurwitz orthogonal matrices [22] we can easily generalize transformations (12.2) to oscillator R^4 using the polar cylindrical transformation [22]. So we write:

$$\begin{pmatrix} A_1^+ \\ A_2^+ \\ A_3^+ \\ A_4^+ \end{pmatrix} = \frac{\sqrt{2}}{4} H_4(1) \begin{pmatrix} a_{x_1}^+ \\ ia_{y_1}^+ \\ ia_{y_2}^+ \\ a_{x_2}^+ \end{pmatrix}, \quad H_4(1) = \frac{\sqrt{2}}{4} \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \tag{12.21}$$

B- The transformation for R^8 basis The generalization of the transformation generalization for $n = 8$ is:

$$\begin{pmatrix} A_1^+ \\ A_2^+ \\ A_3^+ \\ A_4^+ \\ A_5^+ \\ A_6^+ \\ A_7^+ \\ A_8^+ \end{pmatrix} = \frac{1}{4} H_8(1) \begin{pmatrix} a_1^+ \\ ia_2^+ \\ ia_3^+ \\ ia_4^+ \\ ia_5^+ \\ a_6^+ \\ a_7^+ \\ a_8^+ \end{pmatrix}, \quad H_8(1) = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 \end{pmatrix} \tag{12.22}$$

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