Quantum Mechanics - II
Angular Momentum - II : Addition of Angular Momentum - Clebsch-Gordan Coefficients

Dipan Kumar Ghosh
UM-DAE Centre for Excellence in Basic Sciences
Kalina, Mumbai 400098

September 1, 2019

1 Introduction

Consider a spin-half particle. The state space of the particle is spanned by position kets $\{| \vec{x}\rangle\}$ and two dimensional spin space spanned by $| \uparrow\rangle$ and $| \downarrow\rangle$. If the spin-orbit coupling is weak, the Hilbert space is the product of the position space and the spin space, and is spanned by

$$| \vec{x}, \pm\rangle = | \vec{x}\rangle \otimes | \uparrow, \downarrow\rangle$$

The rotation operator is still given by $\exp\left(-i\vec{J}\cdot\hat{n}\theta/\hbar\right)$, where $\vec{J}$ is given by

$$\vec{J} = \vec{L} \otimes \mathbb{I} \oplus \mathbb{I} \otimes \vec{S}$$

which is usually abbreviated as $\vec{J} = \vec{L} + \vec{S}$. The rotation operator in the product space is given by the product of the operators for the orbital and spin parts :

$$U_R(\hat{n}, \theta) = \exp\left(-i\vec{L}\cdot\hat{n}\theta/\hbar\right)\exp\left(-i\vec{S}\cdot\hat{n}\theta/\hbar\right)$$

The total wavefunction is a a product of the space part and a spin part

$$\psi_\alpha(\vec{x}) = \psi(\vec{x})\chi_\alpha$$

where $\alpha =\uparrow$ or $\downarrow$. The wavefunction is a two component tensor

$$\begin{pmatrix} \psi_\uparrow(\vec{x}) \\ \psi_\downarrow(\vec{x}) \end{pmatrix}$$

We have seen that the orbital angular momentum is described by the operator L^2 and L_z, while the spin operators by S^2 and S_z. The set of operators required to describe the composite system is $\{L^2, L_z, S^2, S_z\}$. Since the state space is a product of independent states, we can write

$$\mid l, s, m_l, m_s\rangle =\mid l, m_m\rangle \otimes \mid s, m_s\rangle$$

where

$$\begin{aligned}
L^2 \mid l, m_l\rangle &= l(l+1)\hbar^2 \mid l, m_l\rangle \\
L_z \mid l, m_l\rangle &= m_l\hbar \mid l, m_l\rangle \\
S^2 \mid s, m_s\rangle &= s(s+1)\hbar^2 \mid s, m_s\rangle \\
S_z \mid s, m_s\rangle &= m_s\hbar \mid s, m_s\rangle
\end{aligned}$$

Instead of considering spin and orbital angular momenta of a single particle, we could consider more complex system consisting of two or more particles. We could, for instance, talk about orbital momenta of two spinless particles or angular momenta of more complex systems with many particles with different angular momenta. However, the basic formulation remains the same as for adding two angular momenta. Let us consider two angular momenta J_1 and J_2. We define the total angular momenta of the system by

$$\vec{J} = \vec{J_1} \otimes \mathbb{I} \oplus \mathbb{I} \otimes \vec{J_2}$$

abbreviated as

$$\hat{J} = \hat{J}_1 + \hat{J}_2$$

[The hat or the vector signs (which is more commonly used) indicating the operator nature of angular momentum will be frequently omitted. The commutation relations satisfied by

J_1 and J_2 are

$$\begin{aligned}
[J_{1i}, J_{1j}] &= i\hbar\epsilon_{ijk}J_{1k} \\
[J_{2i}, J_{2j}] &= i\hbar\epsilon_{ijk}J_{2k} \\
[J_{1i}, J_{2j}] &= 0
\end{aligned} \tag{1}$$

Since the components of J_1 commute with those of J_2, they can have common eigenstates. We can choose, J_1^2, J_{1z}, J_2^2 and J_{2z} to be a set of operators which have common eigenstates. Let us denote the common eigenstates by $\mid j_1, j_2; m_1, m_2\rangle \equiv \mid j_1, m_1\rangle \mid j_2, m_2\rangle$ [Note : This notation is not standard, several books will denote this as $\mid j_1, m_1, j_2, m_2\rangle$. We follow Sakurai here.]. We then have

$$\begin{aligned}
J_1^2 \mid j_1, j_2; m_1, m_2\rangle &= j_1(j_1+1)\hbar^2 \mid j_1, j_2; m_1, m_2\rangle \\
J_1^z \mid j_1, j_2; m_1, m_2\rangle &= m_1\hbar \mid j_1, j_2; m_1, m_2\rangle \\
J_2^2 \mid j_1, j_2; m_1, m_2\rangle &= j_2(j_2+1)\hbar^2 \mid j_1, j_2; m_1, m_2\rangle \\
J_2^z \mid j_1, j_2; m_1, m_2\rangle &= m_2\hbar \mid j_1, j_2; m_1, m_2\rangle
\end{aligned}$$

The dimension of the space to which J_1 and J_2 belong is $(2j_1+1)(2j_2+1)$. The set of states $\mid j_1, j_2; m_1, m_2\rangle$ form a complete and orthonormal set

$$\sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} \mid j_1, j_2; m_1, m_2\rangle\langle j_1, j_2; m_1, m_2 \mid = 1 \tag{2}$$

$$\langle j_1, j_2; m_1, m_2 \mid j_1', j_2'; m_1', m_2'\rangle = \delta_{j_1,j_1'}\delta_{j_2,j_2'}\delta_{m_1,m_1'}\delta_{m_2,m_2'} \tag{3}$$

As mentioned for the case of the orbital and spin angular momenta, the rotation operator in this case is given by

$$U_{1R}(\hat{n},\theta) \otimes U_{2R}(\hat{n},\theta) = \exp\left(-i\frac{\vec{J}_1\cdot\hat{n}\theta}{\hbar}\right)\exp\left(-i\frac{\vec{J}_2\cdot\hat{n}\theta}{\hbar}\right)$$

As a consequence of (1), the components of the *total angular momentum* $\hat{J}$ satisfy

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k \tag{4}$$

which shows that $\hat{J}$ is an angular momentum operator. It is easy to check that J^2 commutes with J_z, J_1^2 and J_2^2. However, J_z does not commute with either J_{1z} or J_{2z}. As a result, it is possible to consider an alternate set of commuting operators, viz., $\{J^2,_z, J_1^2, J_2^2\}$

which would describe the same space as did our old set $\{J_1^2, J_2^2, J_{1z}, J_{2z}\}$. We denote the simultaneous eigenstates represented by this new set by $\mid j_1, j_2; j, m\rangle$. In terms of this new set, we have

$$J^2 \mid j_1, j_2; j, m\rangle = j(j+1)\hbar^2 \mid j_1, j_2; j, m\rangle \tag{5}$$

$$J_z \mid j_1, j_2; j, m\rangle = m\hbar \mid j_1, j_2; j, m\rangle \tag{6}$$

These kets are also eigenkets of J_1^2 and J_2^2, though not of J_{1z} and J_{2z}. They satisfy the completeness relation

$$\sum_{m=-j}^{j} \mid j_1, j_2; j, m\rangle\langle j_1, j_2; j, m \mid = 1 \tag{7}$$

They can be taken to be orthonormalized

$$\langle j_1, j_2; , j, m \mid j_1', j_2'; j', m'\rangle = \delta_{m,m'}\delta_{j,j'} \tag{8}$$

It is tacitly assumed that j_1 and j_2 in a given problem are given and fixed . We have not yet found out what values j can take. However, the sum over j in (7) must be over all values of j consistent with given values of j_1 and j_2. Using the completeness property(2) of the old set $\{\mid j_1, j_2; m_1, m_2\rangle\}$, we may express a member of the new set $\{\mid j_1, j_2; j, m\rangle\}$ as

$$\mid j_1, j_2; j, m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \mid j_1, j_2; m_1, m_2\rangle\langle j_1, j_2, m_1, m_2 \mid j_1, j_2; j, m\rangle \tag{9}$$

The coefficients $\langle j_1, j_2, m_1, m_2 \mid j_1, j_2; j, m\rangle$ defined in (9) are called **Clebsch-Gordan (C-G)Coefficients.**. The basis defined here as represented by the kets on the left hand side of (9) will be also referred to as the *Coupled Representation.*

2 Properties of C-G Coefficients

1. The C-G coefficients $\langle j_1, j_2, m_1, m_2 \mid j_1, j_2; j, m\rangle$ are zero unless $m = m_1 + m_2$. To show this, note that $J_z - J_{1z} - J_{2z}$ is a null operator,

$$\langle j_1, j_2, m_1, m_2 \mid J_z - J_{1z} - J_{2z} \mid j_1, j_2; j, m\rangle = 0$$

because J_z acts to the ket on its right giving m times the ket, while $J_{1z} - J_{2z}$ acts to the bra on the left giving $m_1 - m_2$ times the bra,

$$(m - m_1 - m_2)\langle j_1, j_2, m_1, m_2 \mid j_1, j_2; j, m\rangle = 0$$

which shows that of $m \neq m_1 + m_2$, the C-G coefficient would be zero.

2. By convention, the C-G coefficients are taken to be real, so that

$$\langle j_1, j_2, m_1, m_2 \mid j_1, j_2; j, m\rangle = \langle j_1, j_2; jm \mid j_1, j_2; m_1, m_2\rangle$$

3. Orthonormality of C-G Coefficients : In the expression (8) given above, if we insert a complete set given by (2) of old states, we get

$$\langle j_1, j_2; , j, m \mid j_1', j_2'; j', m'\rangle = \delta_{m,m'}\delta_{j,j'}$$

$$\sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} \langle j_1, j_2; j, m \mid j_1, j_2; m_1, m_2\rangle\langle j_1, j_2; m_1, m_2 \mid j_1', j_2'; j', m'\rangle = \delta_{m,m'}\delta_{j,j'} \tag{10}$$

Substituting $j = j'$ and $m = m'$ in the above, we get

$$\boxed{\sum_{m_1,m_2} \mid \langle j_1, j_2; m_1, m_2 \mid j_1, j_2; j, m\rangle \mid^2 = 1} \tag{11}$$

In a similar way may start with the orthonormality condition on the old basis set (3) and insert the completeness condition (7) of the new basis set to obtain

$$\boxed{\sum_j \sum_{m=-j}^{+j} \mid \langle j_1, j_2; m_1, m_2 \mid j_1, j_2; j, m\rangle \mid^2 = 1} \tag{12}$$

4. The C-G coefficients vanish unless

$$\boxed{\mid j_1 - j_2 \mid \leq j \leq j_1 + j_2} \tag{13}$$

The inequality on the right is obvious because $m_1^{max} = j_1$ and $m_2^{max} = j_2$. Hence $m^{max} = j_1 + j_2$. Since the maximum value of m is j, the right inequality follows

and $j_{max} = j_1 + j_2$.
The inequality on the left requires a bit of working out. We know that there are $(2j_1 + 1)(2j_2 + 1)$ number of states. For each value of j, there are $2j + 1$ states. Thus we must have

$$\sum_{j=j_{min}}^{j_{max}} (2j + 1) = (2j_1 = 1)(2j_2 + 1)$$

Left hand side of above is an arithmetic series of finite number of terms. The sum works out as follows :

$$\begin{aligned} l.h.s. &= \frac{j_{max} - j_{min} + 1}{2}(2j_{max} + 1 + 2j_{min} + 1) \\ &= (j_{max} - j_{min} + 1)(j_{max} + J_{min} + 1) \\ &= (j_{max} + 1)^2 - j_{min}^2 \\ &= (j_1 + j_2 + 1)^2 - j_{min}^2 \end{aligned}$$

Equating the above to $(2j_1 + 1)(2j_2 + 1)$, we get $j_{min}^2 = (j_1 - j_2)^2$, which gives

$$j_{min} =\mid j_1 - j_2 \mid$$

Thus the values of j satisfies

$$\mid j_1 - j_2 \mid \leq j \leq j_1 + j_2 \tag{14}$$

5. For a given j, the possible m values are $-j \leq m \leq +j$.

A notation, primarily used by the nuclear physicists is known as Wigner's 3-j symbol and is related to C-G coefficients as follows. While the C-G coefficients are for adding two angular moments, the 3-j symbols are for addition of three angular momenta such that their sum gives zero angular momentum state.

$$\langle j_1, j_2; m_1, m_2 \mid j_1, j_2; j, m\rangle = (-1)^{j_1 - j_2 + m}\sqrt{2j + 1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \tag{15}$$

3 Calculation of Clebsch-Gordan Coefficients

In the two limiting cases, i.e., for $m_1 = j_1, m_2 = j_2; m = j_1 + j_2$ and $m_1 = -j_1, m_2 = -j_2; m = -j_1 - j_2$, the CG coefficients are 1. For the other cases we need to obtain recursion relations. Note that

$$\begin{aligned} J_\pm \mid j_1, j_2; j, m\rangle &= (J_{1,\pm} + J_{2,\pm}) \mid j_1, j_2; j, m\rangle \\ &= \sum_{m_1', m_2'} \langle j_1, j_2; m_1', m_2' \mid j_1, j_2; j, m\rangle (J_{1,\pm} + J_{2,\pm}) \mid j_1, j_2; m_1', m_2'\rangle \end{aligned}$$

Applying the ladder operator on both sides

$$\begin{aligned} \sqrt{(j \mp m)(j \pm m + 1)} \mid j_1, j_2; j, m \pm 1\rangle \ &= \ \sum_{m_1', m_2'} \langle j_1, j_2; m_1', m_2' \mid j_1, j_2; j, m\rangle \\ & \Big(\sqrt{(j_1 \mp m_1')(j_1 \pm m_1' + 1)} \mid j_1, j_2; m_1' \pm 1, m_2'\rangle \\ & \sqrt{(j_2 \mp m_2')(j_2 \pm m_2' + 1)} \mid j_1, j_2; m_1', m_2' \pm 1\rangle \Big) \end{aligned} \tag{16}$$

Multiply both sides of the above equation by $\langle j_1, j_2; m_1, m_2 \mid$ and use orthogonality condition. The term on the left gives a CG coefficient. The first term on the right is non-zero only if $m_1 = m_1' \pm 1$ and $m_2 = m_2'$ while the second term is non-zero if $m_1 = m_1'$ and $m_2 = m_2' \pm 1$. Using these on the right, we get a relationship between three CG coefficients.

$$\begin{aligned} \sqrt{(j \mp m)(j \pm m_1 + 1)} \langle j_1, j_2; m_1, m_2 \mid j_1, j_2; j, m \pm 1\rangle \ &= \\ \sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)} \langle j_1, j_2; m_1 \mp 1, m_2 \mid j_1, j_2; j, m\rangle & \\ + \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)} \langle j_1, j_2; m_1, m_2 \mp 1 \mid j_1, j_2; j, m\rangle & \end{aligned} \tag{17}$$

Note that the left there are CG coefficients which require $m_1 + m_2 = m \pm 1$.

3.1 Examples of CG Coefficient Calculation

Example 1 - Adding two spin 1/2 angular momenta :
We will denote the spins by $\hat{s}_1$ and $\hat{s}_2$ and total spin by $\hat{S}$. This could, for instance, represent the case of the spin states of hydrogen atom wherein the electron and the proton each have spin 1/2. Clearly, the spin angular moments of two different particles commute. The total spin S can take values from $s_1 - s_2$ to $s_1 + s_2$, i.e. 0 and 1. Corresponding to $s = 0$, there is only one state with $m = 0$ while for $s = 1$, there are three states with $m = -1, 0$ and $+1$. The former is called a singlet while the latter trio are known as triplets. ($s_1 = 1/2$ has two states and $s_2 = 1/2$ also has two states, so that there are

$2 \times 2 = 4$ states). We note that the state $\mid \frac{1}{2}, \frac{1}{2}; 1, 1\rangle$ in the representation $\mid s_1, s_2, s, m\rangle$ is uniquely obtained from the state $\mid \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\rangle$ of $\mid s_1, s_2, m_1, m_2\rangle$ representation.

$$\mid \frac{1}{2}, \frac{1}{2}; 1, 1\rangle = \mid \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\rangle$$

Apply J_- on the left and $(J_{1-} + J_{2-})$ on the right and use the formula $J_|j, m\rangle = \sqrt{(j+m)(j-m+1)}$

$$\begin{aligned} J_- \mid \frac{1}{2}, \frac{1}{2}; 1, 1\rangle &= (J_{1-} + J_{2-}) \mid \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\rangle \\ \sqrt{2} \mid \frac{1}{2}, \frac{1}{2}; 1, 0\rangle &= \mid \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, \frac{1}{2}\rangle + \mid \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\rangle \end{aligned}$$

Thus the Clebsch-Gordon coefficients are

$$\begin{aligned} \langle 1/2, 1/2; 1/2, 1/2 \mid 1/2, 1/2, 1, 1\rangle &= 1 \\ \langle 1/2, 1/2; 1/2, -1/2 \mid 1/2, 1/2, 1, 0\rangle &= \frac{1}{\sqrt{2}} \\ \langle 1/2, 1/2; -1/2, 1/2 \mid 1/2, 1/2, 1, 0\rangle &= \frac{1}{\sqrt{2}} \end{aligned}$$

The coefficient for $s = 1.m = -1$ need not be calculated as it is obtained by symmetry to be also equal to 1. For $s = 0$, there is only state with $m = 0$. This state is orthogonal to the state $\mid 1/2, 1/2, 1, 0\rangle$ and is given by

$$\mid 1/2, 1/2, 0, 0\rangle = \frac{1}{\sqrt{2}} \left(\mid 1/2, 1/2, 1/2, -1/2\rangle - \mid 1/2, 1/2, -1/2, 1/2\rangle\right)$$

Thus

$$\begin{aligned} \langle 1/2, 1/2; 1/2, -1/2 \mid 1/2, 1/2, 1, 0\rangle &= \frac{1}{\sqrt{2}} \\ \langle 1/2, 1/2; -1/2, 1/2 \mid 1/2, 1/2, 1, 0\rangle &= -\frac{1}{\sqrt{2}} \end{aligned}$$

It is conventional to indicate the state with $m = 1/2$ as $|\uparrow\rangle$ or as α. The state with $m = -1/2$ is then indicated as $|\downarrow\rangle$ or as β. The triplet states are then $s = 1$:

$$m = +1 : |\uparrow\uparrow\rangle$$
$$m = 0 : \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$
$$m = -1 : |\downarrow\downarrow\rangle$$

It may be noted that the triplets are symmetric with respect to the two spins. The singlet state $s = 0$ is given by

$$\frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

which is antisymmetric under exchange of spins.
When the magnitudes of the two angular momenta are different, it is conventional to choose the one with larger magnitude as the first spin and the other one as the second. This convention has some repercussion in fixing the signs of Clebsch-Gordan coefficients, which we will talk about in the next example.
Example 2: A Hydrogen atom is known to be in the state $n = 3, l = 2$. Find all the values of the total angular momentum j and for each j, the eigenvalue of J^2 operator.
We have $l = 2, s = 1/2$. Thus allowed values of j are from $2 - \frac{1}{2}$ to $2 + \frac{1}{2}$, i.e. $\frac{5}{2}, \frac{3}{2}$. The eigenvalue of J^2 being $j(j+1)\hbar^2$, we have, for $j = 5/2$ the eigenvalue is $(35/4)\hbar^2$ and $(15/4)\hbar^2$ respectively.
Example 3: For two spins $s = 1/2$ each, show that $\frac{3}{4}I + \frac{1}{\hbar^2}\hat{S}_1 \cdot \hat{S}_2$ is a projection operator for $S = 1$ and $\frac{1}{4}I - \frac{1}{\hbar^2}\hat{S}_1 \cdot \hat{S}_2$ is a projection operator for $S = 0$
Solution:
Note

$$\frac{3}{4}I + \frac{1}{\hbar^2}\hat{S}_1 \cdot \hat{S}_2 = \frac{3}{4}I + \frac{1}{2\hbar^2}[S^2 - S_1^2 - S_2^2]$$

For $S = 1$, S^2 has eigenvalue $1(1+1)\hbar^2 = 2\hbar^2$ and S_1^2 and S_2^2 have eigenvalue $\frac{3}{4}\hbar^2$ each. Thus we get for $S = 1$ this expression to have value 1 and for $S = 0$ it has value 0. Thus the operator projects $S = 1$ state.

Example 4: Adding $j_1 = 1$ with $j_2 = 1/2$

In this case, in the $j_1, j_2; m_1, m_2$ representation there are $3 \times 2 = 6$ states. The resultant angular momentum which satisfies $| j_1 - j_2 | \leq j \leq j_1 + j_2$ can take values $\frac{3}{2}$ and $\frac{1}{2}$, the former has 4 states and the latter 2, making a total of 6 states as before. We proceed as follows. Start with the state having maximum j value and maximum possible m value corresponding to this j. In this case it is $j = \frac{3}{2}$ and $m = \frac{3}{2}$. This state is obtained uniquely from $j_1 = 1, m_1 = 1$ and $j_2 = 1/2, m_2 = 1/2$. Thus

$$| 1, 1/2; 3/2, 3/2\rangle = | 1, 1/2; 1, 1/2\rangle$$

We apply J_- on the ket on the left and $J_{1-} + J_{2-}$ to the ket on the right, using the formula $J_- \mid j, m\rangle = \sqrt{(j+m)(j-m+1)} \mid j, m-1\rangle$. We then get

$$\sqrt{3} \mid 1, 1/2; 3/2, 1/2\rangle = \sqrt{2} \mid 1, 1/2; 0, 1/2\rangle + \mid 1, 1/2; 1, -1/2\rangle \tag{18}$$

From this it follows that

$$\langle 1, 1/2; 3/2, 1/2 \mid 1, 1/2; 0, 1/2\rangle = \sqrt{\frac{2}{3}}$$
$$\langle 1, 1/2; 3/2, 1/2 \mid 1, 1/2; 1, -1/2\rangle = \sqrt{\frac{1}{3}}$$

The remaining coefficients belonging to the negative values of m are obtained by symmetry. We now consider the $j = 1/2$ state with $= \pm 1/2$. Once again, we will only compute $m = 1/2$ term. Note that the state $| 1, 1/2; 1/2, 1/2\rangle$ is orthogonal to the state with the same m but $j = 3/2$. Thus this state is orthogonal to

$$| 1, 1/2; 3/2, 1/2\rangle = \sqrt{\frac{2}{3}} \mid 1, 1/2; 0, 1/2\rangle + \sqrt{\frac{1}{3}} \mid 1, 1/2; 1, -1/2\rangle$$

The state which is orthogonal to the state on the right is obviously

$$| 1, 1/2; 1/2, 1/2\rangle = \pm \left(\sqrt{\frac{1}{3}} \mid 1, 1/2; 0, 1/2\rangle - \sqrt{\frac{2}{3}} \mid 1, 1/2; 1, -1/2\rangle \right)$$

Thus, there is an ambiguity on the overall phase. In order to fix the sign, we use what is known as **Condon-Shortley sign convention** according which the state with the highest m of the larger component of the angular momentum that is being added is *assigned* a positive sign. In case where $j_1 = j_2$, as was the case in the previous example, the first mentioned one, i.e. j_1 will be taken to be the larger component for application of this convention. Thus in this case the state with $j_1 = 1, m_1 = 1$ will be taken to be positive. Thus we have

$$| 1, 1/2; 1/2, 1/2\rangle = \sqrt{\frac{2}{3}}\, | 1, 1/2; 1, -1/2\rangle - \sqrt{\frac{1}{3}}\, | 1, 1/2; 0, 1/2\rangle$$

This gives two Clebsch-Gordan coefficients as follows

$$\langle 1, 1/2; 1/2, 1/2 \mid 1, 1/2; 1, -1/2\rangle = \sqrt{\frac{2}{3}}$$
$$\langle 1, 1/2; 1/2, 1/2 \mid 1, 1/2; 0, 1/2\rangle = -\sqrt{\frac{1}{3}}$$

Example 5: Adding $j_1 = 1$ with $j_2 = 1$

Here in the j_1, j_2, m_1, m_2 basis, there are $3 \times 3 = 9$ members. In the coupled representation the possible angular momenta are $j = 2, 1, 0$ with respectively 5, 3 and 1 states, making it 9, as expected. In this case we start with

$$| 1, 1; 2, 2\rangle =| 1, 1; 1, 1\rangle$$

where the coupled representation is on the left. we apply J_- on the left and $J_{1-} + J_{2-}$ on the right. we get

$$\sqrt{(2+2)(2-2+1)} \mid 1, 1; 2, 1\rangle = \sqrt{(1+1)(1-1+1)} \mid 1, 1; 0, 1\rangle + \sqrt{(1+1)(1-1+1)} \mid 1, 1; 1, 0\rangle$$

Thus

$$| 1, 1; 2, 1\rangle = \sqrt{\frac{1}{2}} \mid 1, 1; 0, 1\rangle + \sqrt{\frac{1}{2}} \mid 1, 1; 1, 0\rangle \tag{19}$$

Further application of $J_- = J_{1-} + J_{2-}$ gives

$$\sqrt{6} \mid 1, 1, 2, 0\rangle = \sqrt{\frac{1}{2}}(\sqrt{2} \mid 1, 1; -1, 1\rangle + \sqrt{2} \mid 1, 1, 0, 0\rangle) + \sqrt{\frac{1}{2}}(\sqrt{2} \mid 1, 1; 0, 0\rangle + \sqrt{2} \mid 1, 1, 1, -1\rangle)$$

$$| 1, 1, 2, 0\rangle = \sqrt{\frac{1}{6}}[| 1, 1; 1, -1\rangle + 2 \mid 1, 1, 0, 0\rangle) + \sqrt{2} \mid 1, 1; -1, 1\rangle] \tag{20}$$

The state with $j = 1, m = 1$ is orthogonal to the state (19). Using the sign convention discussed above

$$| 1, 1; 1, 1\rangle = \sqrt{\frac{1}{2}} \mid 1, 1; 1, 0\rangle - \sqrt{\frac{1}{2}} \mid 1, 1; 0, 1\rangle \tag{21}$$

An application of $J_- = J_{1-} + J_{2-}$ on (21) gives

$$| 1, 1; 1, 0\rangle = \sqrt{\frac{1}{2}} \mid 1, 1; 1, -1\rangle - \sqrt{\frac{1}{2}} \mid 1, 1; -1, 1\rangle \tag{22}$$

It may be noted that in the above state there is no contribution from $m_1 = m_2 = 0$ state to $m = 1$ state.

Now the state $| 1, 1; 0, 0\rangle$ is orthogonal to both (20) and (22). Let the state be written as

$$| 1, 1; 0, 0\rangle = a \mid 1, 1; 1, -1\rangle + b \mid 1, 1; 0, 0\rangle + c \mid 1, 1; -1, 1\rangle$$

where a, b and c are constants to be determined. Since the state is orthogonal tio (22), we have $a = c$. Orthogonality to (20) gives $a + 2b + c = 0$. Solving, we get $b = -c$. Thus the state is

$$| \, 1, 1; 0, 0\rangle = a[| \, 1, 1; 1, -1\rangle - | \, 1, 1; 0, 0\rangle + | \, 1, 1; -1, 1\rangle]$$

The constant a is determined to be $\frac{1}{\sqrt{3}}$ by normalization of the state. Thus we get

$$| \, 1, 1; 0, 0\rangle = \frac{1}{\sqrt{3}}[| \, 1, 1; 1, -1\rangle - | \, 1, 1; 0, 0\rangle + | \, 1, 1; -1, 1\rangle]$$

Collecting the above results together, we get the Clebsch-Gordan coefficients to be given by

$$m = 2, 1$$

$$\begin{aligned}
\langle 1, 1; 2, 2 \mid 1, 1; 1, 1\rangle &= 1 \\
\langle 1, 1; 2, 1 \mid 1, 1; 1, 0\rangle &= \frac{1}{\sqrt{2}} \\
\langle 1, 1; 2, 1 \mid 1, 1; 0, 1\rangle &= \frac{1}{\sqrt{2}} \\
\langle 1, 1; 1, 1 \mid 1, 1; 1, 0\rangle &= \frac{1}{\sqrt{2}} \\
\langle 1, 1; 1, 1 \mid 1, 1; 0, 1\rangle &= -\frac{1}{\sqrt{2}}
\end{aligned}$$

$m = 0$:

$$\langle 1,1;2,0 \mid 1,1;1,-1\rangle = \frac{1}{\sqrt{6}}$$
$$\langle 1,1;2,0 \mid 1,1;0,0\rangle = \sqrt{\frac{2}{3}}$$
$$\langle 1,1;2,0 \mid 1,1;-1,1\rangle = \frac{1}{\sqrt{6}}$$
$$\langle 1,1;1,0 \mid 1,1;1,-1\rangle = \frac{1}{\sqrt{2}}$$
$$\langle 1,1;1,0 \mid 1,1;-1,1\rangle = -\frac{1}{\sqrt{2}}$$
$$\langle 1,1;0,0 \mid 1,1;1,-1\rangle = \frac{1}{\sqrt{3}}$$
$$\langle 1,1;0,0 \mid 1,1;0,0\rangle = -\frac{1}{\sqrt{3}}$$
$$\langle 1,1;0,0 \mid 1,1;-1,1\rangle = \frac{1}{\sqrt{3}}$$