On the angular integrals

The following expansion for the potential is proposed:

$$\hat{V}(R,\Omega,\omega)=\sum_{F}^{\infty}\sum_{m_F}\sum_{L}\sum_{J} G^{F,m_F}_{L,J}(R)\, T^{F,m_F}_{LJ}(\Omega,\omega)$$

where

$$T^{F,m_F}_{L,J}(\Omega,\omega)=\sum_{m_L m_J}(-1)^{L-J+m_F}\sqrt{2F+1}\begin{pmatrix} L & J & F\\ m_L & m_J & -m_F\end{pmatrix} Y^{L}_{m_L}(\Omega)\left[\frac{2J+1}{8\pi^2}\right]^{1/2} D^{J*}_{m_J,K}(\omega)$$

We also propose the basis set for constructing the matrix elements

$$|\Psi\rangle = |n\, l\, m_l\, j\, K\, m_j\rangle \longrightarrow |n\, l\, j\, K\, \lambda\, m_\lambda\rangle$$

where

$$|n\, l\, j\, k\, \lambda\, m_\lambda\rangle = F_{n_l}(R)\, Y^{l}_{m_l}(\Omega)\left[\frac{2j+1}{8\pi^2}\right]^{1/2} D^{j*}_{K,m_j}(\omega)$$

which can be also written as (depending parametrically on K)

$$|n\, l\, j\, K\, \lambda\, m_\lambda\rangle_K = F_{n_l}(R)\left[\sum_{m_l}\sum_{m_j}(-1)^{l-j+m_\lambda}\sqrt{2\lambda+1}\begin{pmatrix} l & j & \lambda\\ m_l & m_j & -m_\lambda\end{pmatrix} Y^{l}_{m_l}(\Omega)\right]\left[\frac{2j+1}{8\pi^2}\right]^{1/2} D^{j*}_{m_j,K}(\omega)$$

Thus, if we compute the eigenvalues we will have

$$\langle n', l', j', K', \lambda', m'_\lambda |\, G^{F,m_F}_{LJ}(R)\, T^{F,m_F}_{L,J}(\Omega,\omega)\,| n, l, j, K, \lambda, m_\lambda\rangle = \langle n' |\, G^{F,m_F}_{L,J}(R)\,| n\rangle$$

$$\times\underbrace{\langle l', j', K', \lambda', m'_\lambda |\, T^{F,m_F}_{L,J}(\Omega,\omega)\,| l, j, K, \lambda, m_\lambda\rangle}_{\xi}$$

$$\xi=\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{\pi}\left\{\sum_{m'_l}\sum_{m'_j}(-1)^{l'-j'+m'_\lambda}\sqrt{2\lambda'+1}\begin{pmatrix} l' & j' & \lambda'\\ m'_l & m'_j & -m'_\lambda\end{pmatrix} Y^{l'*}_{m'_l}(\Omega)\left[\frac{2j'+1}{8\pi^2}\right]^{1/2} D^{j'}_{m'_j,K}(\omega)\right\}$$

$$\times\left\{\sum_{m_L}\sum_{m_J}(-1)^{L-J+m_F}\sqrt{2F+1}\begin{pmatrix} L & J & F\\ m_L & m_J & -m_F\end{pmatrix} Y^{L}_{m_L}(\Omega)\left[\frac{2J+1}{8\pi^2}\right]^{1/2} D^{J}_{M,K}(\omega)\right\}$$

$$\times\left\{\sum_{m_l}\sum_{m_j}(-1)^{l-j+m_\lambda}\sqrt{2\lambda+1}\begin{pmatrix} l & j & \lambda\\ m_l & m_j & -m_\lambda\end{pmatrix} Y^{l}_{m_l}(\Omega)\left[\frac{2j+1}{8\pi^2}\right]^{1/2} D^{j*}_{m_j,K}(\omega)\right\} d\omega\, d\Omega$$

Let us now compute term by term as separation of ω from Ω terms is possible:

$$\int_{\Omega} Y^{l'*}_{m'_l}(\Omega)\, Y^{L}_{m_L}(\Omega)\, Y^{l}_{m_l}(\Omega)\, d\Omega$$

First let us remember that

$$\int d\Omega\, Y^*_{L_3M_3}(\theta,\phi)Y_{L_2M_2}(\theta,\phi)Y_{L_1M_1}(\theta,\phi)$$

$$= \left[\frac{(2L_1+1)(2L_2+1)}{4\pi(2L_3+1)}\right]^{\frac{1}{2}} \langle L_1M_1, L_2M_2|L_3M_3\rangle\langle L_10, L_20|L_30\rangle$$

$$\int_\Omega Y^{l'*}_{m_l'}(\Omega)\, Y^{L}_{m_L}(\Omega)\, Y^{l}_{m_l}(\Omega)\, d\omega = \left[\frac{(2l+1)(2L+1)}{4\pi(2l'+1)}\right]^{1/2} \langle l\, m_l, L\, m_L | l'\, m_l'\rangle\langle l\,0, L\,0|l'\,0\rangle$$

$$\langle j_1m_1, j_2m_2|j_3m_3\rangle \equiv (-1)^{j_1-j_2+m_3}(2j_3+1)^{\frac{1}{2}}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3\end{pmatrix}$$

$$\langle l0, L0|l'0\rangle \equiv (-1)^{l-L+0}(2l'+1)^{1/2}\begin{pmatrix} l & L & l' \\ 0 & 0 & 0\end{pmatrix} = (-1)^{l-L}(2l'+1)^{1/2}\begin{pmatrix} l & L & l' \\ 0 & 0 & 0\end{pmatrix}$$

$$\langle l\, m_l, L\, m_L|l'\, m_l'\rangle \equiv (-1)^{l-L+m_l'}(2l'+1)^{1/2}\begin{pmatrix} l & L & l' \\ m_l & m_L & -m_l'\end{pmatrix}$$

So we will have that

$$\int_\Omega Y^{l'}_{m_l'}(\Omega)\, Y^{L}_{m_L}\, Y^{l}_{m_l}(\Omega)\, d\Omega = \left[\frac{(2l+1)(2L+1)}{4\pi(2l'+1)}\right]^{1/2}(-1)^{l-L+m_l'}(2l'+1)^{1/2}\begin{pmatrix} l & L & l' \\ m_l & m_L & -m_l'\end{pmatrix}$$
$$\times (-1)^{l-L}(2l'+1)^{1/2}\begin{pmatrix} l & L & l' \\ 0 & 0 & 0\end{pmatrix}$$

$$= (-1)^{2(l-L)+m_l'}\left[\frac{(2l+1)(2L+1)(2l'+1)}{4\pi}\right]^{1/2}\begin{pmatrix} l & L & l' \\ m_l & m_L & -m_l'\end{pmatrix}\begin{pmatrix} l & L & l' \\ 0 & 0 & 0\end{pmatrix}$$

$$= (-1)^{m_l'}\left[\frac{(2l+1)(2L+1)(2l'+1)}{4\pi}\right]^{1/2}\begin{pmatrix} l & L & l' \\ m_l & m_L & -m_l'\end{pmatrix}\begin{pmatrix} l & L & l' \\ 0 & 0 & 0\end{pmatrix}$$

- Now we have to work with the integrals of the Wigner D-functions. We have that:

$$\int_\omega D^{j'}_{m_j',K'}(\omega)\, D^{J}_{m_J,K}(\omega)\, D^{j*}_{m_j,K} = \ldots$$

Let us remember that

$$\int d\Omega\, D^{J_3^*}_{M_3' M_3}(R)\, D^{J_2}_{M_2' M_2}(R)\, D^{J_1}_{M_1' M_1}(R)$$

$$= \sum_J \langle J_1 M_1, J_2 M_2 | J, M_1 + M_2 \rangle \langle J_1 M_1', J_2 M_2' | J, M_1' + M_2' \rangle$$

$$\times \int d\Omega\, D^{J_3^*}_{M_3' M_3}(R)\, D^{J}_{M_1'+M_2', M_1+M_2}(R)$$

$$= \sum_J \langle J_1 M_1, J_2 M_2 | J, M_1 + M_2 \rangle \langle J_1 M_1', J_2 M_2' | J, M_1' + M_2' \rangle$$

$$\times \frac{8\pi^2}{2J_3+1} \delta_{M_1+M_2, M_3} \delta_{M_1'+M_2', M_3'} \delta_{J_3, J}$$

$$= \frac{8\pi^2}{2J_3+1} \langle J_1 M_1, J_2 M_2 | J_3 M_3 \rangle \langle J_1 M_1', J_2 M_2' | J_3 M_3' \rangle \qquad (3.114)$$

So we will have

$$\int_\omega D^{j'}_{m_j', K'}(\omega)\, D^{J}_{m_J, K}(\omega)\, D^{j*}_{m_j' K} \doteq \int_\omega D^{j*}_{m_j' K}(\omega)\, D^{J}_{m_J, K}(\omega)\, D^{j'}_{m_j', K'}(\omega)\, d\omega$$

$$= \frac{8\pi^2}{2j+1} \langle j' m_j', J m_J | j\, m_j \rangle \langle j' K', J \bar{K} | j\, K \rangle$$

and similarly to the previous case we will have

$$\langle j' m_j', J m_J | j\, m_j \rangle = (-1)^{j'-J+m_j} (2j+1)^{1/2} \begin{pmatrix} j' & J & j \\ m_j' & m_J & -m_j \end{pmatrix}$$

$$\langle j' K', J \bar{K} | j\, K \rangle = (-1)^{j'-J+K} (2j+1)^{1/2} \begin{pmatrix} j' & J & j \\ K' & \bar{K} & -K \end{pmatrix}$$

this means that

$$\int_\omega D^{j*}_{m_j' K}(\omega)\, D^{J}_{m_J, K}(\omega)\, D^{j'}_{m_j', K'}(\omega)\, d\omega = (-1)^{2(j'-J)+m_j+m_K} \times 8\pi^2 \begin{pmatrix} j' & J & j \\ m_j' & m_J & -m_j \end{pmatrix} \begin{pmatrix} j' & J & j \\ K' & \bar{K} & -K \end{pmatrix}$$

$$= (-1)^{m_j+m_K}\, 8\pi^2 \begin{pmatrix} j' & J & j \\ m_j' & m_J & -m_j \end{pmatrix} \begin{pmatrix} j' & J & j \\ K' & \bar{K} & -K \end{pmatrix}$$

So in the end the full expression will have the form

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{\pi} \left\{ \sum_{m_\ell'} \sum_{m_j'} (-1)^{\ell'-j'+m_\lambda'} \sqrt{2\lambda'+1} \begin{pmatrix} \ell' & j' & \lambda' \\ m_\ell' & m_j' & -m_\lambda' \end{pmatrix} Y^{\ell'*}_{m_\ell'}(\Omega) \left[\frac{2j'+1}{8\pi^2}\right]^{1/2} D^{j'}_{m_j', K'}(\omega) \right\}$$

$$\times \left\{ \sum_{m_L} \sum_{m_J} (-1)^{L-J+m_F} \sqrt{2F+1} \begin{pmatrix} L & J & F \\ m_L & m_J & -m_F \end{pmatrix} Y^{L}_{m_L}(\Omega) \left[\frac{2J+1}{8\pi^2}\right]^{1/2} D^{J}_{M, K}(\omega) \right\}$$

$$\times \left\{ \sum_{m_\ell} \sum_{m_j} (-1)^{\ell-j+m_\lambda} \sqrt{2\lambda+1} \begin{pmatrix} \ell & j & \lambda \\ m_\ell & m_j & -m_\lambda \end{pmatrix} Y^{\ell}_{m_\ell}(\Omega) \left[\frac{2j+1}{8\pi^2}\right]^{1/2} D^{j*}_{m_j, K}(\omega) \right\} d\omega\, d\Omega$$

$$\sum_{m_\ell'}\sum_{m_j'}\sum_{m_L}\sum_{m_J}\sum_{m_\ell}\sum_{m_j} (-1)^{\ell'-j'+m_\lambda' + L - J + m_F + \ell - j + m_\lambda + m_\ell' + m_j + K}$$

$$\times (2\lambda'+1)^{1/2} (2F+1)^{1/2} (2\lambda+1)^{1/2} (2j'+1)^{1/2} (2J+1)^{1/2} (2j+1)^{1/2} (2\ell'+1)^{1/2} (2L+1)^{1/2} (2\ell+1)^{1/2} \cdot \frac{\cancel{8\pi^2}}{\cancel{8\pi^2}\sqrt{8\pi^2}} \cdot \frac{1}{\sqrt{4\pi}} \qquad \frac{1}{\sqrt{32\pi^3}}$$

$$\times \begin{pmatrix} \ell' & j' & \lambda' \\ m_\ell' & m_j' & -m_\lambda' \end{pmatrix} \begin{pmatrix} L & J & F \\ m_L & m_J & -m_F \end{pmatrix} \begin{pmatrix} \ell & j & \lambda \\ m_\ell & m_j & -m_\lambda \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ m_\ell & m_L & -m_\ell' \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} j' & J & j \\ m_j' & m_J & -m_j \end{pmatrix} \begin{pmatrix} j' & J & j \\ K' & \bar{K} & -K \end{pmatrix}$$

This can be rewritten into a more elegant form if we see that

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix} = \sum_{\text{all } m} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_4 & j_5 & j_6 \\ m_4 & m_5 & m_6 \end{pmatrix}\begin{pmatrix} j_7 & j_8 & j_9 \\ m_7 & m_8 & m_9 \end{pmatrix}$$
$$\times \begin{pmatrix} j_1 & j_4 & j_7 \\ m_1 & m_4 & m_7 \end{pmatrix}\begin{pmatrix} j_2 & j_5 & j_8 \\ m_2 & m_5 & m_8 \end{pmatrix}\begin{pmatrix} j_3 & j_6 & j_9 \\ m_3 & m_6 & m_9 \end{pmatrix} \quad (4.23)$$

So we would need

$$\begin{Bmatrix} l' & i' & \lambda' \\ L & J & F \\ l & i & \lambda \end{Bmatrix} = \sum_{\text{all } m} \begin{pmatrix} l' & i' & \lambda' \\ m_l' & m_i' & m_\lambda' \end{pmatrix}\begin{pmatrix} L & J & F \\ m_L & m_J & m_F \end{pmatrix}\begin{pmatrix} l & i & \lambda \\ m_l & m_i & m_\lambda \end{pmatrix}$$
$$\times \begin{pmatrix} l' & L & l \\ m_l' & m_L & m_l \end{pmatrix}\begin{pmatrix} i' & J & i \\ m_i' & m_J & m_i \end{pmatrix}\begin{pmatrix} \lambda' & F & \lambda \\ m_\lambda' & m_F & m_\lambda \end{pmatrix} \leftarrow \text{Missing}$$

comparing to what we have

must be permuted

$$\sum_{\text{all } m} \begin{pmatrix} l' & i' & \lambda' \\ m_l' & m_i' & -m_\lambda' \end{pmatrix}\begin{pmatrix} L & J & F \\ m_L & m_J & -m_F \end{pmatrix}\begin{pmatrix} l & i & \lambda \\ m_l & m_i & -m_\lambda \end{pmatrix}\begin{pmatrix} l & L & l' \\ m_l & m_L & -m_l' \end{pmatrix}\begin{pmatrix} l & L & l' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} i' & J & i \\ m_i' & m_J & -m_i \end{pmatrix}\begin{pmatrix} i' & J & i \\ k' & K & -k \end{pmatrix}$$

Let us focus on the noted terms, we will have that

$$\sum_{m_1, m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = (2j_3+1)^{-1}\delta_{j_3,j_3'}\delta_{m_3,m_3'} \quad (2.32)$$

thus

$$(2\lambda+1)\sum_{m_\lambda' m_F} \begin{pmatrix} \lambda' & F & \lambda \\ m_\lambda' & m_F & m_\lambda \end{pmatrix}\begin{pmatrix} \lambda' & F & \lambda'' \\ m_\lambda' & m_F & m_\lambda'' \end{pmatrix} = \delta_{\lambda',\lambda''}\delta_{m_\lambda',m_\lambda''}$$

On the other hand, let us notice that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix}$$
$$= (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}$$
$$= (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix}$$

$$\begin{pmatrix} l & L & l' \\ m_l & m_L & m_l' \end{pmatrix} = (-1)^{l+L+l'}\begin{pmatrix} l' & L & l \\ m_l' & m_L & m_l \end{pmatrix}$$

So getting everything together we will have

$$\sum_{\text{all } m} \begin{pmatrix} l' & i' & \lambda' \\ m_l' & m_i' & -m_\lambda' \end{pmatrix}\begin{pmatrix} L & J & F \\ m_L & m_J & -m_F \end{pmatrix}\begin{pmatrix} l & i & \lambda \\ m_l & m_i & -m_\lambda \end{pmatrix}\begin{pmatrix} l & L & l' \\ m_l & m_L & -m_l' \end{pmatrix}\begin{pmatrix} l & L & l' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} i' & J & i \\ m_i' & m_J & -m_i \end{pmatrix}\begin{pmatrix} i' & J & i \\ k' & K & -k \end{pmatrix}$$

$$= \sum_{\text{all } m} \begin{pmatrix} l & i' & \lambda' \\ m_l & m_i' & m_\lambda' \end{pmatrix}\begin{pmatrix} L & J & F \\ m_L & m_J & m_F \end{pmatrix}\begin{pmatrix} l & i & \lambda \\ m_l & m_i & m_\lambda \end{pmatrix}(-1)^{l+L+l'}\begin{pmatrix} l' & L & l \\ m_l' & m_L & m_l \end{pmatrix}\begin{pmatrix} i' & J & i \\ m_i' & m_J & m_i \end{pmatrix}\delta_{\lambda',\lambda''}\delta_{m_\lambda',m_\lambda''}\begin{pmatrix} l & L & l' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} i' & J & i \\ k' & K & -k \end{pmatrix}$$

$$= (-1)^{l+L+l'}(2\lambda+1)\sum_{\text{all } m}\begin{pmatrix} l' & j' & \lambda' \\ m_l' & m_j' & m_\lambda' \end{pmatrix}\begin{pmatrix} L & J & F \\ m_L & m_J & m_F \end{pmatrix}\begin{pmatrix} l & j & \lambda \\ m_l & m_j & m_\lambda \end{pmatrix}\begin{pmatrix} l' & L & l \\ m_l' & m_L & m_l \end{pmatrix}\begin{pmatrix} j' & J & j \\ m_j' & J & m_j \end{pmatrix}$$

$$\times \sum_{\substack{m_\lambda' m_F \\ m\ (\lambda'=\lambda'')}}\begin{pmatrix} \lambda' & F & \lambda \\ m_\lambda' & m_F & m_\lambda \end{pmatrix}\begin{pmatrix} \lambda' & F & \lambda'' \\ m_\lambda' & m_F & m_\lambda'' \end{pmatrix} \times \begin{pmatrix} l & L & l' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} j' & J & j \\ K' & K_\Sigma & -K \end{pmatrix}$$

$$= \sum_{m_l'}\sum_{m_j'}\sum_{m_L}\sum_{m_J}\sum_{m_l}\sum_{m_j}\sum_{m_\lambda}\sum_{m_\lambda'}\sum_{m_F}\begin{pmatrix} l' & j' & \lambda' \\ m_l' & m_j' & m_\lambda' \end{pmatrix}\begin{pmatrix} L & J & F \\ m_L & m_J & m_F \end{pmatrix}\begin{pmatrix} l & j & \lambda \\ m_l & m_j & m_\lambda \end{pmatrix}\begin{pmatrix} l' & L & l \\ m_l' & m_L & m_l \end{pmatrix}\begin{pmatrix} j' & J & j \\ m_j' & J & m_j \end{pmatrix}\begin{pmatrix} \lambda' & F & \lambda \\ m_\lambda' & m_F & m_\lambda \end{pmatrix} \times (-1)^{l+L+l'}(2\lambda+1)\begin{pmatrix} l & L & l' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} j' & J & j \\ K' & K_\Sigma & K \end{pmatrix}$$

$$= \begin{Bmatrix} l' & j' & \lambda' \\ L & J & F \\ l & j & \lambda \end{Bmatrix} \times (-1)^{l+L+l'}(2\lambda+1)\begin{pmatrix} l & L & l' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} j' & J & j \\ K' & K_\Sigma & K \end{pmatrix}$$

NOTE: Notice that I have made the change $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \longmapsto \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ this can be justified by reversing the change done in (Wolfram's page for the 3j-Wigner symbol and by abusing the notation.

The parameters of the $3\,j$ symbol $\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix}$ (where m_3 has been written as $-M$)

Therefore, we come to the final result

$$\frac{1}{\sqrt{32\pi^3}}$$

$$\to \sum_{m_l'}\sum_{m_j'}\sum_{m_L}\sum_{m_J}\sum_{m_l}\sum_{m_j}(-1)^{l'-j'+m_\lambda'+L-J+m_F+l-j+m_\lambda+m_l'+m_j+K}$$

$$\times (2\lambda'+1)^{1/2}(2F+1)^{1/2}(2\lambda+1)^{1/2}(2j'+1)^{1/2}(2J+1)^{1/2}(2j+1)^{1/2}(2l'+1)^{1/2}(2L+1)^{1/2}(2l+1)^{1/2}\cdot\frac{\cancel{8\pi^2}}{\cancel{8\pi^2}\sqrt{8\pi^2}}\cdot\frac{1}{\sqrt{4\pi}}$$

$$\times \begin{pmatrix} l' & j' & \lambda' \\ m_l' & m_j' & -m_\lambda' \end{pmatrix}\begin{pmatrix} L & J & F \\ m_L & m_J & -m_F \end{pmatrix}\begin{pmatrix} l & j & \lambda \\ m_l & m_j & -m_\lambda \end{pmatrix}\begin{pmatrix} l & L & l' \\ m_l & m_L & -m_l' \end{pmatrix}\begin{pmatrix} l & L & l' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} j' & J & j \\ m_j' & m_J & -m_j \end{pmatrix}\begin{pmatrix} j' & J & j \\ K' & K_\Sigma & -K \end{pmatrix}$$

$$= (-1)^{2l'-j'+m_\lambda'+2L-J+m_F+2l-j+m_\lambda+m_l'+m_j+K}(2\lambda+1)\left[(2\lambda'+1)(2F+1)(2\lambda+1)\cancel{(2\lambda+1)}(2j+1)(2J+1)(2j+1)\right.$$

$$\left.\times (2l'+1)(2L+1)(2l+1)\right]^{1/2}\frac{1}{\sqrt{32\pi^3}}\begin{Bmatrix} l' & j' & \lambda' \\ L & J & F \\ l & j & \lambda \end{Bmatrix}\begin{pmatrix} l & L & l' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} j' & J & j \\ K' & K_\Sigma & K \end{pmatrix}$$

This is

$$\boxed{\begin{aligned} \langle l' j' K' \lambda' m_\lambda' | \hat{V}(\Omega,\omega) | l j K \lambda m_\lambda \rangle &= (-1)^{-J-j-j'+m_\lambda'+m_\lambda+m_F+m_l'+m_j+K} \\ &\times (2\lambda+1)\left[(2\lambda'+1)(2F+1)(2\lambda+1)(2j+1)(2J+1)(2j+1)(2l'+1)(2L+1)(2l+1)\right]^{1/2} \times \sqrt{\frac{1}{32\pi^3}} \\ &\times \begin{Bmatrix} l' & j' & \lambda' \\ L & J & F \\ l & j & \lambda \end{Bmatrix}\begin{pmatrix} l & L & l' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} j' & J & j \\ K' & K_\Sigma & K \end{pmatrix} \end{aligned}}$$