

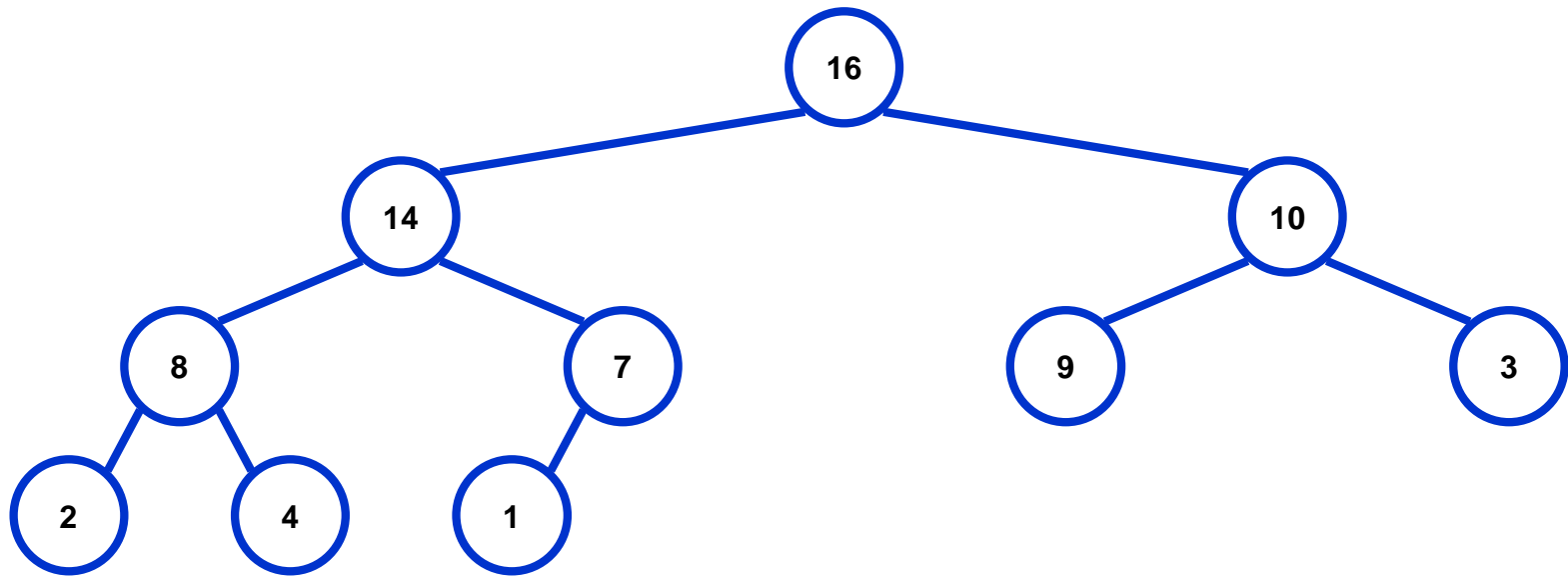


CS 435: Algorithms

Heap sort

Heaps

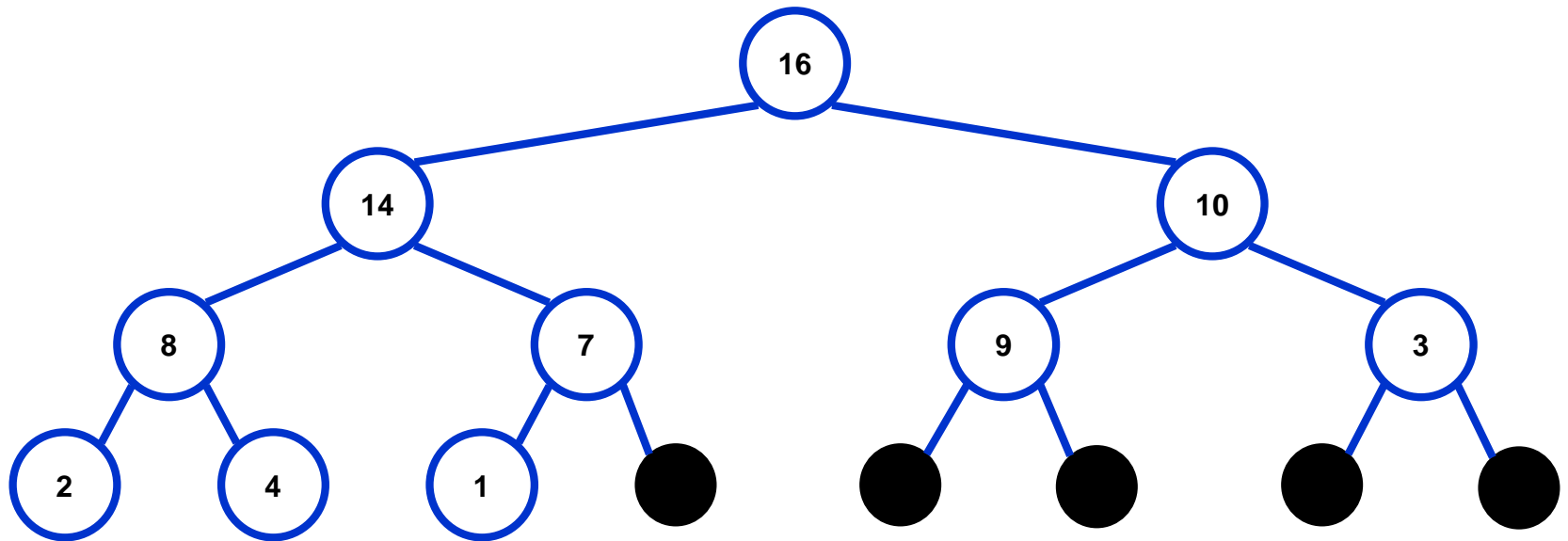
- A *heap* can be seen as a complete binary tree:



- *What makes a binary tree complete?*
- *Is the example above complete?*

Heaps

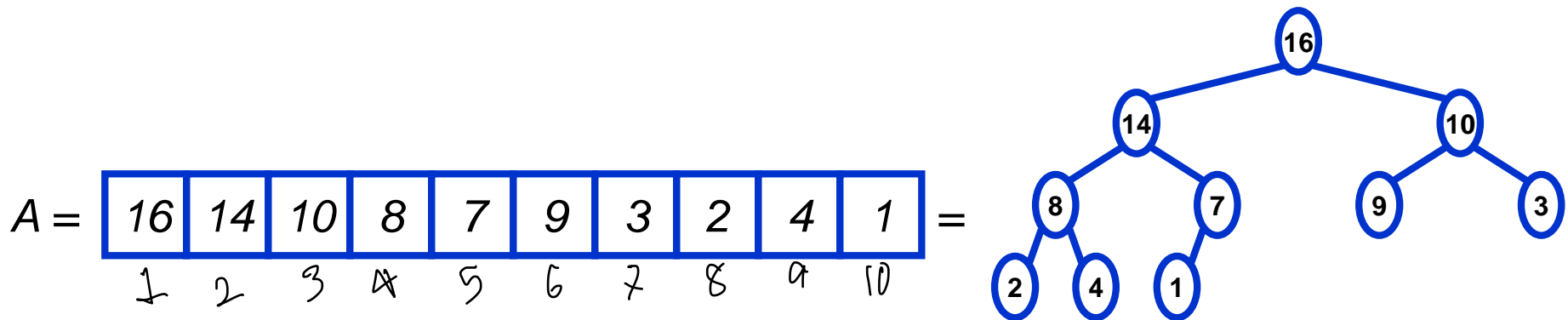
- A *heap* can be seen as a complete binary tree:



- The book calls them “nearly complete” binary trees; can think of unfilled slots as null pointers

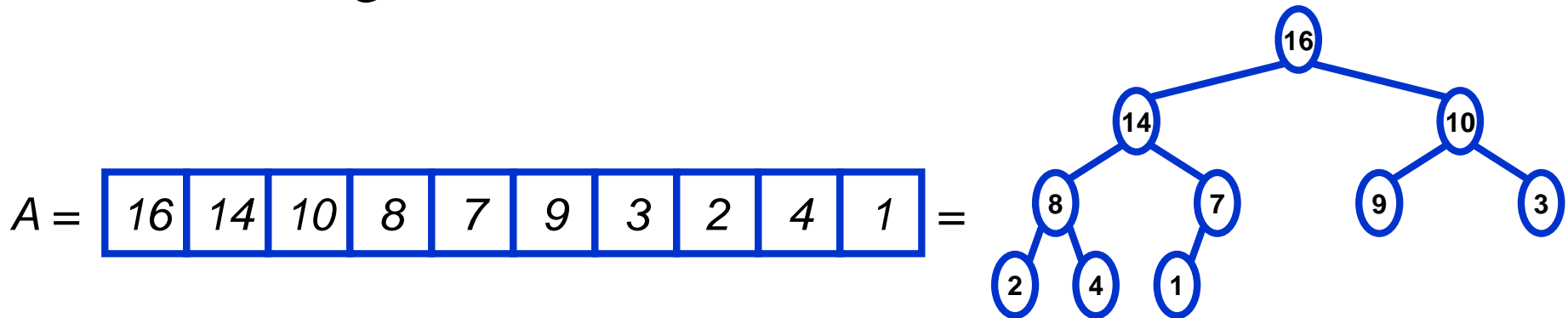
Heaps

- In practice, heaps are usually implemented as arrays:



Heaps

- To represent a complete binary tree as an array:
 - The root node is $A[1]$
 - Node i is $A[i]$
 - The parent of node i is $A[i/2]$ (note: integer divide)
 - The left child of node i is $A[2i]$
 - The right child of node i is $A[2i + 1]$



Referencing Heap Elements

- So...
`Parent(i) { return $\lfloor i/2 \rfloor$; }`
`Left(i) { return $2*i$; }`
`right(i) { return $2*i + 1$; }`
- An aside: *How would you implement this most efficiently?*
- Another aside: *Really?*

The Heap Property

- Heaps also satisfy the *heap property*:

$$A[\textit{Parent}(i)] \geq A[i] \quad \text{for all nodes } i > 1$$

- In other words, the value of a node is at most the value of its parent

- *Where is the largest element in a heap stored?*

- Definitions:

- The *height* of a node in the tree = the number of edges on the longest downward path to a leaf
- The height of a tree = the height of its root

which
enumerates
BST's
 $O(n)$
worst
case
as it
would
never
be
linear

Heap Height

$\log n$?

- *What is the height of an n -element heap? Why?*
- This is nice: basic heap operations take at most time proportional to the height of the heap

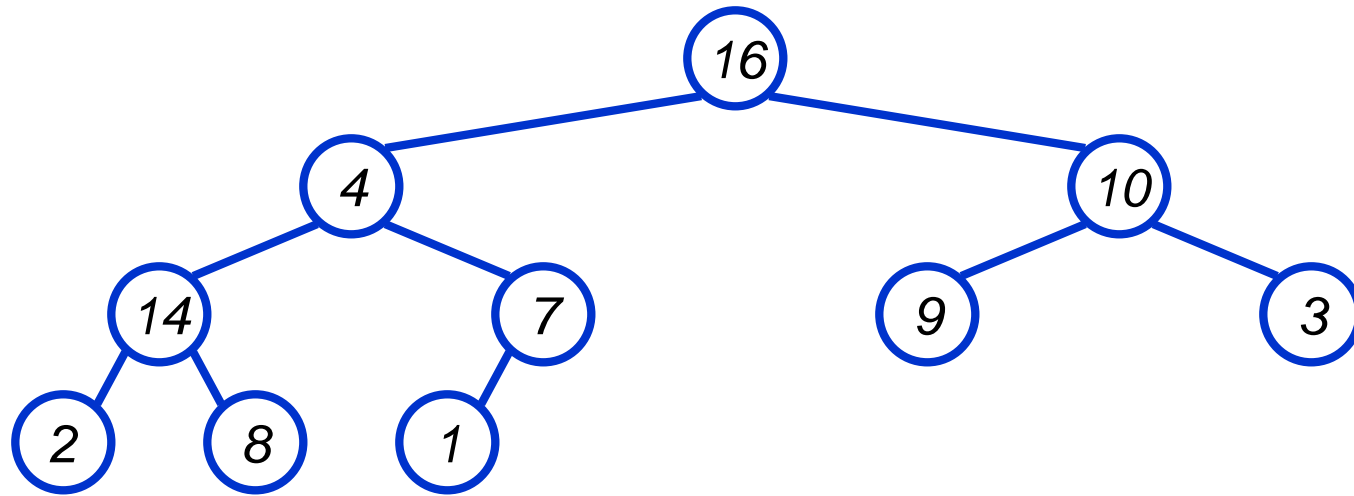
Heap Operations: Heapify()

- **Heapify()** : maintain the heap property
 - Given: a node i in the heap with children l and r
 - Given: two subtrees rooted at l and r , assumed to be heaps
 - Problem: The subtree rooted at i may violate the heap property (*How?*)
 - Action: let the value of the parent node “float down” so subtree at i satisfies the heap property
 - *What do you suppose will be the basic operation between i , l , and r ?*

Heap Operations: Heapify()

```
Heapify(A, i)
{
    l = Left(i); r = Right(i);
    if (l <= heap_size(A) && A[l] > A[i])
        largest = l;
    else
        largest = i;
    if (r <= heap_size(A) && A[r] > A[largest])
        largest = r;
    if (largest != i)
        Swap(A, i, largest);
    Heapify(A, largest);
}
```

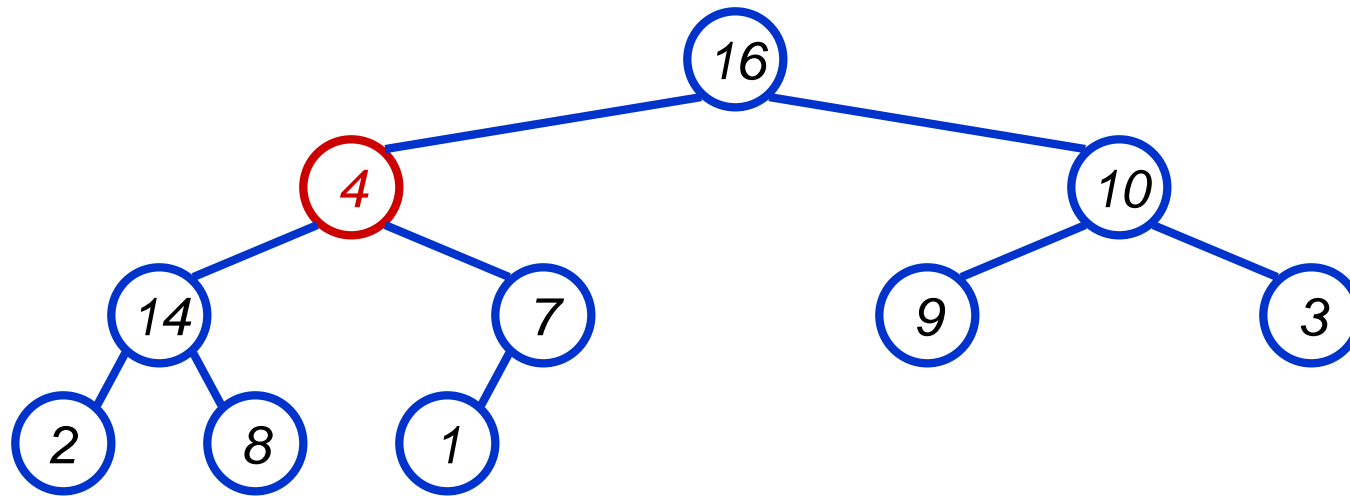
Heapify() Example



$A =$

16	4	10	14	7	9	3	2	8	1
----	---	----	----	---	---	---	---	---	---

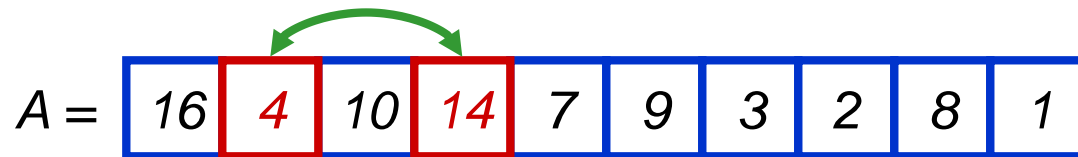
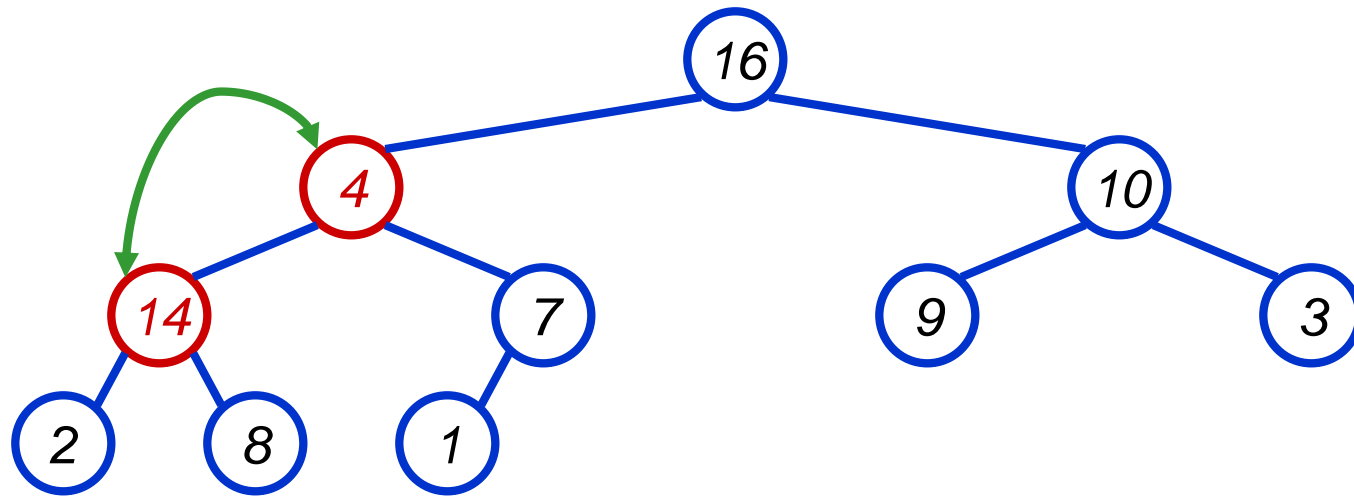
Heapify() Example



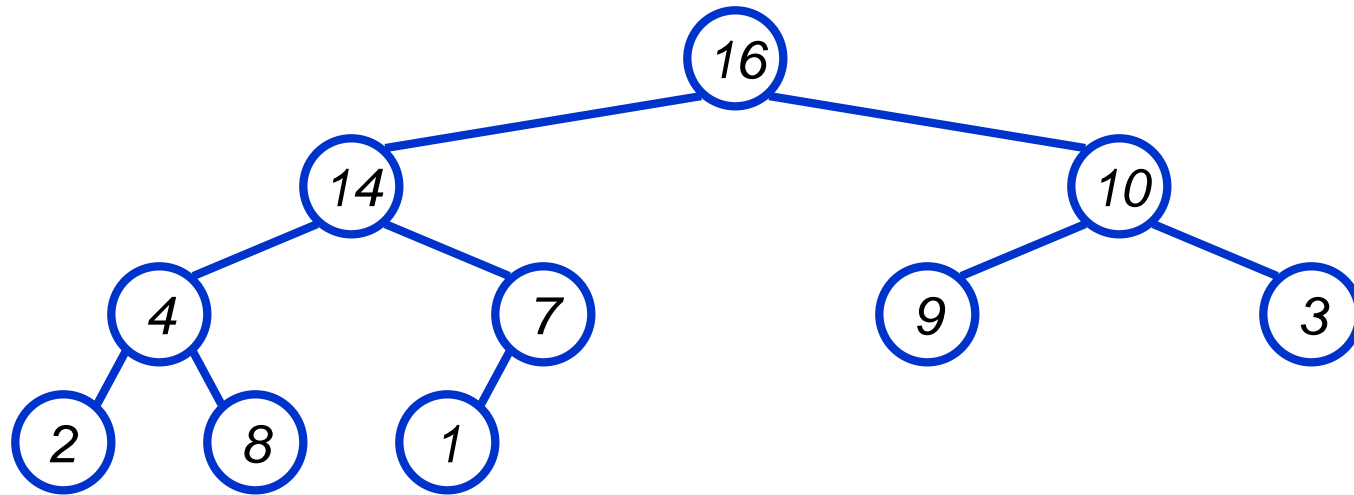
$A =$

16	4	10	14	7	9	3	2	8	1
----	---	----	----	---	---	---	---	---	---

Heapify() Example



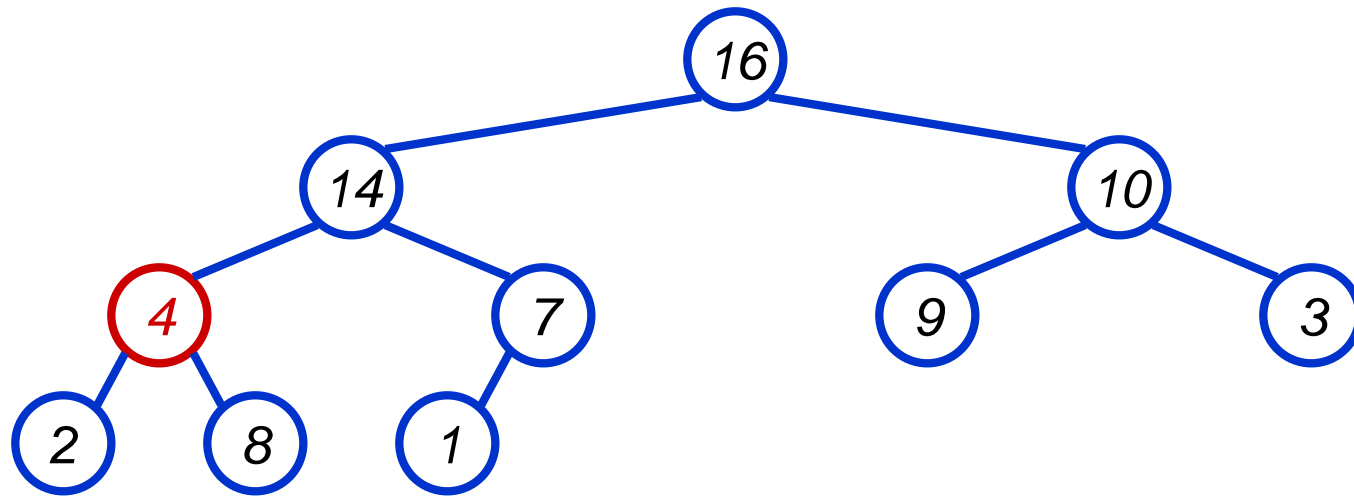
Heapify() Example



$A =$

16	14	10	4	7	9	3	2	8	1
----	----	----	---	---	---	---	---	---	---

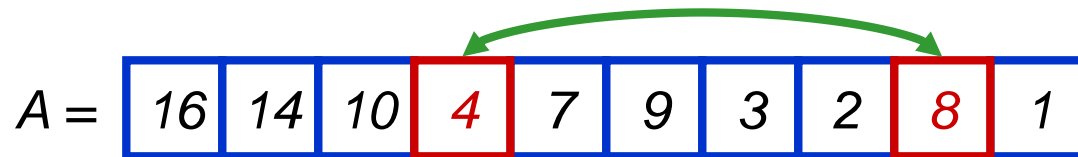
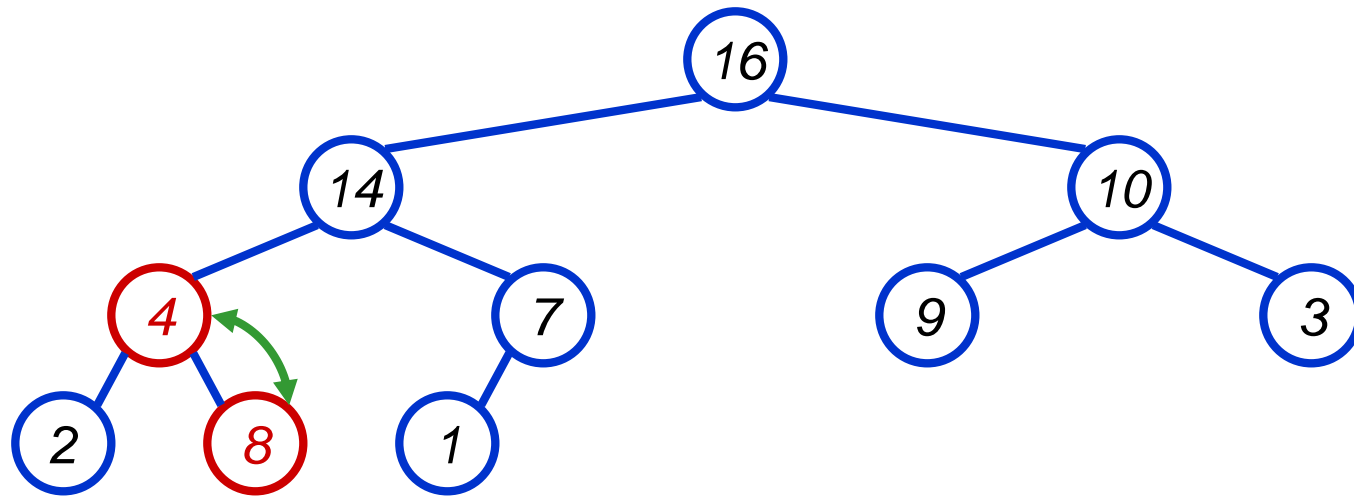
Heapify() Example



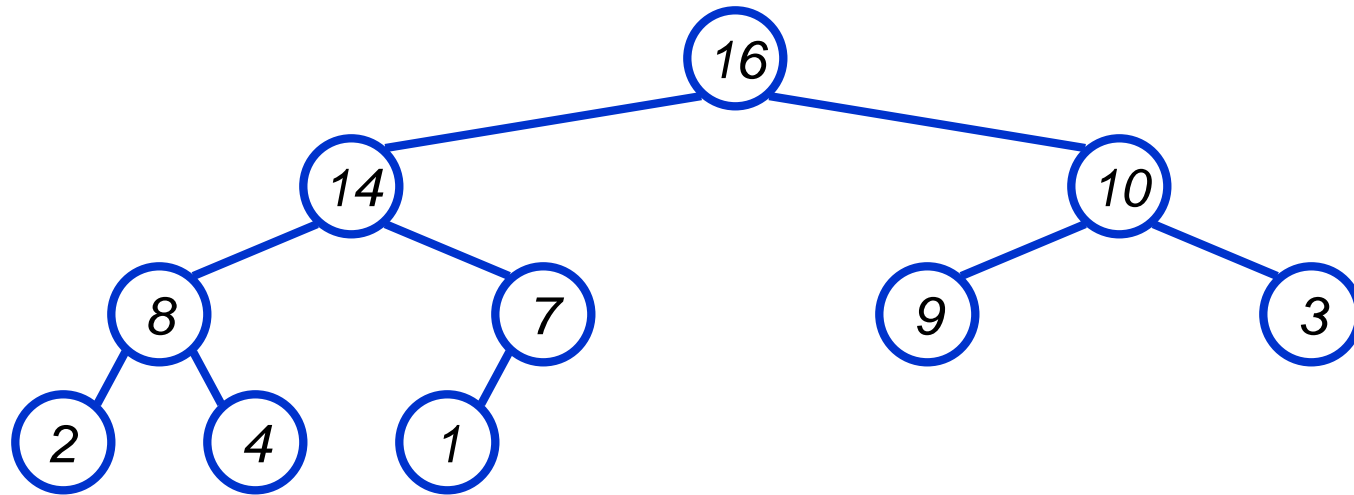
$A =$

16	14	10	4	7	9	3	2	8	1
----	----	----	---	---	---	---	---	---	---

Heapify() Example



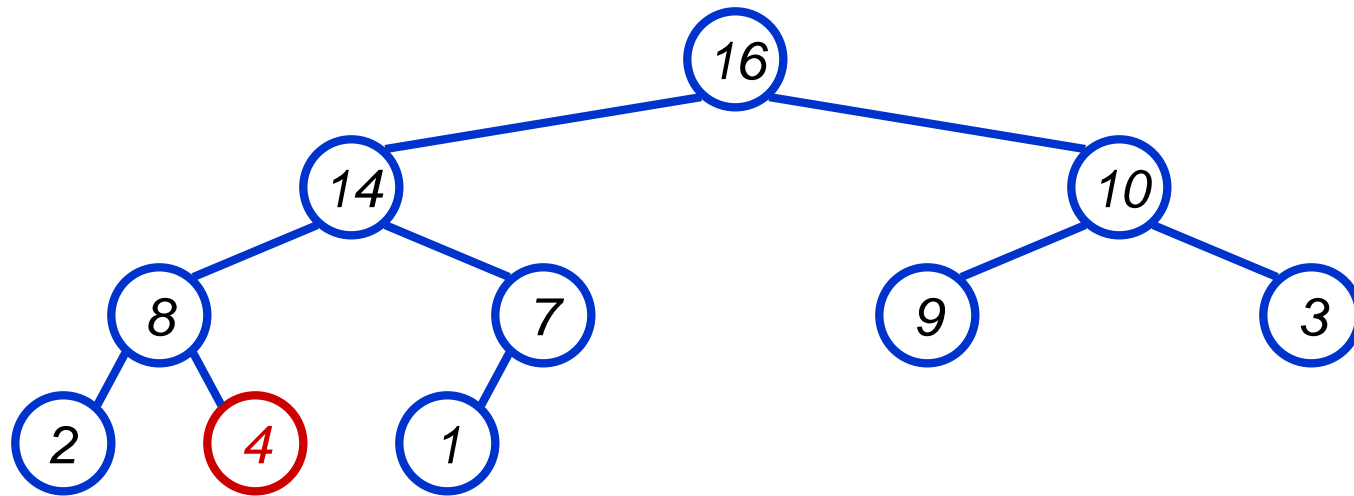
Heapify() Example



$A =$

16	14	10	8	7	9	3	2	4	1
----	----	----	---	---	---	---	---	---	---

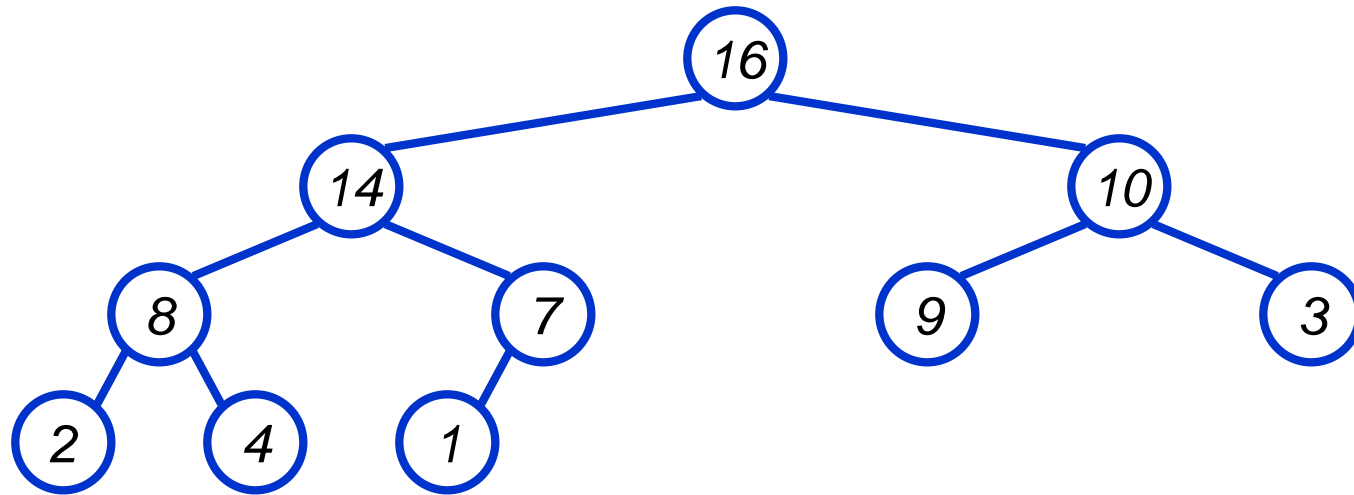
Heapify() Example



$A =$

16	14	10	8	7	9	3	2	4	1
----	----	----	---	---	---	---	---	---	---

Heapify() Example



$A =$

16	14	10	8	7	9	3	2	4	1
----	----	----	---	---	---	---	---	---	---

Analyzing Heapify(): Informal

- *Aside from the recursive call, what is the running time of **Heapify()**?* $\rightarrow O(1)$
- *How many times can **Heapify()** recursively call itself?* $\rightarrow \log n$
- *What is the worst-case running time of **Heapify()** on a heap of size n ?*

$\rightarrow \log n$

Analyzing Heapify(): Formal

- Fixing up relationships between i , l , and r takes $\Theta(1)$ time
- *If the heap at i has n elements, how many elements can the subtrees at l or r have?*
 - Draw it $\rightarrow \left(\frac{n-1}{2}\right)$ is more suitable
- Answer: $2n/3$ (worst case: bottom row 1/2 full)
- So time taken by **Heapify()** is given by
$$T(n) \leq T(2n/3) + \Theta(1)$$

Analyzing Heapify(): Formal

- So we have

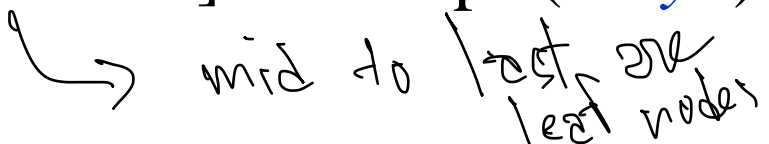
$$T(n) \leq T(2n/3) + \Theta(1)$$

- By case 2 of the Master Theorem,

$$T(n) = O(\lg n)$$

- Thus, **Heapify()** takes linear time

Heap Operations: BuildHeap()

- We can build a heap in a bottom-up manner by running **Heapify()** on successive subarrays
 - Fact: for array of length n , all elements in range $A[\lfloor n/2 \rfloor + 1 .. n]$ are heaps (*Why?*)
 - $n=14$
7 to 14 are heap
 - So:  mid to last or leaf nodes
 - Walk backwards through the array from $n/2$ to 1, calling **Heapify()** on each node.
 - Order of processing guarantees that the children of node i are heaps when i is processed

BuildHeap()

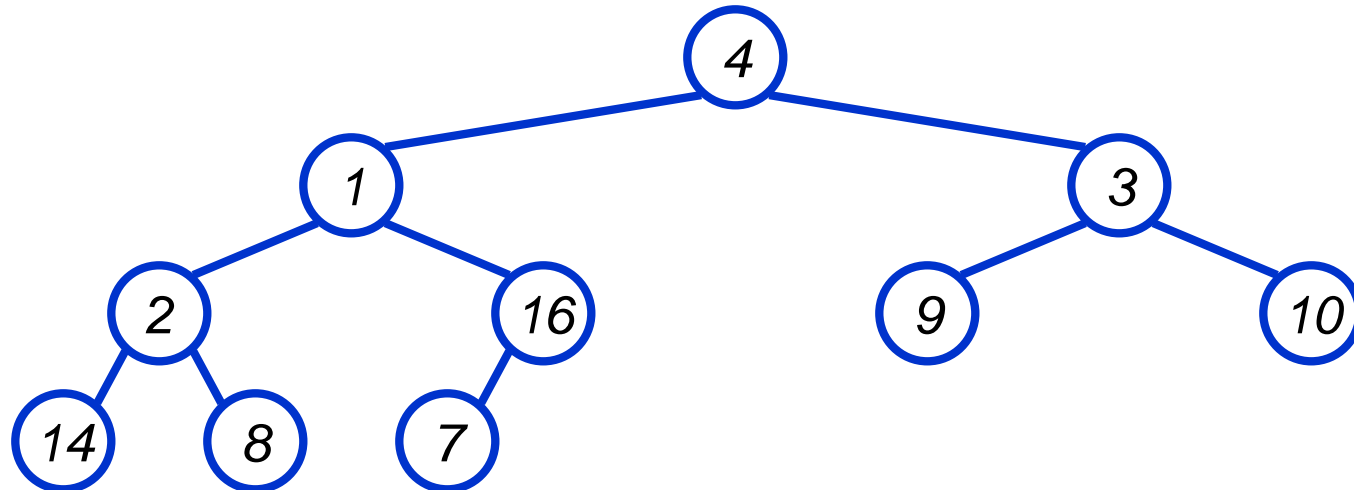
```
// given an unsorted array A, make A a heap
BuildHeap(A)
{
    heap_size(A) = length(A);
    for (i = ⌊length[A]/2⌋ downto 1)
        Heapify(A, i);
}
```


BuildHeap() Example

- Work through example

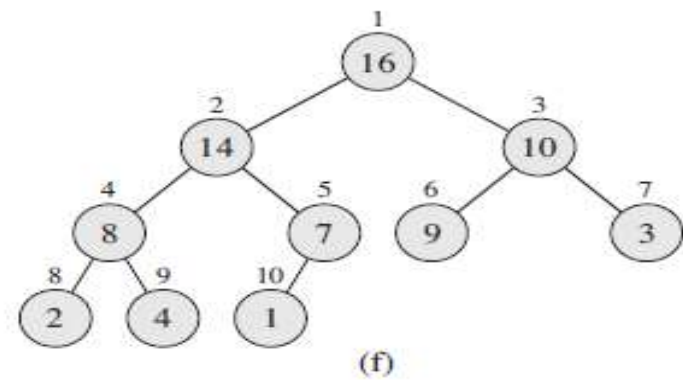
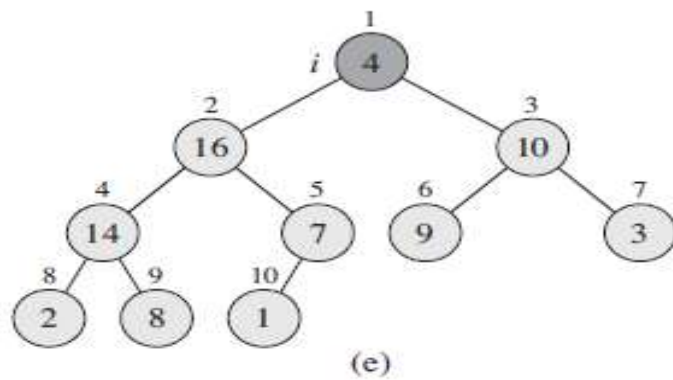
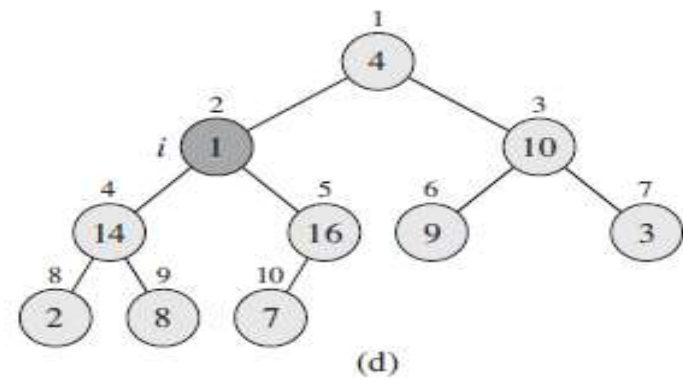
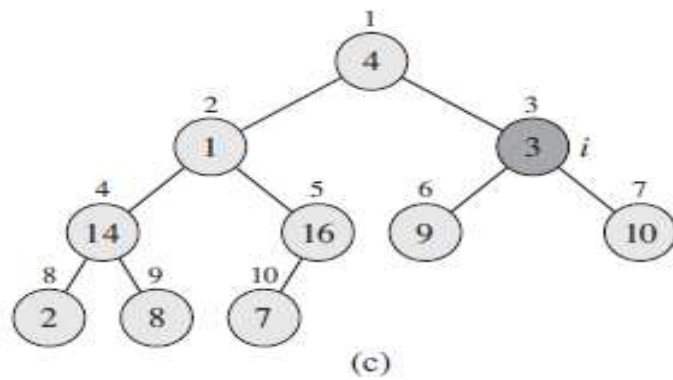
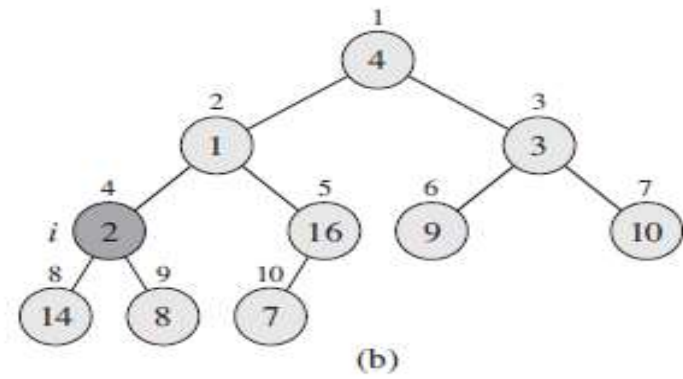
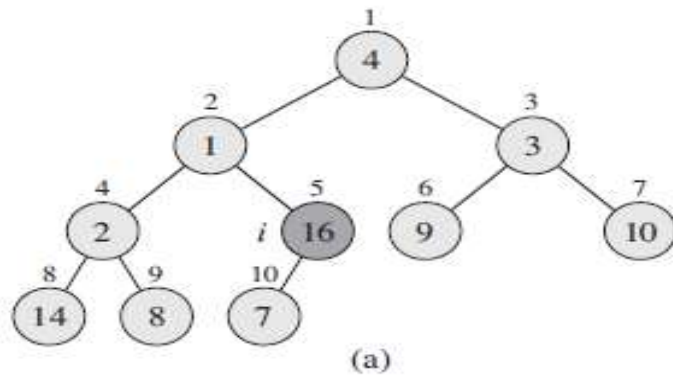
$A = \{4, 1, 3, 2, 16, 9, 10, 14, 8, 7\}$

1 2 3 4 5 6 7 8 9 10



A

4	1	3	2	16	9	10	14	8	7
---	---	---	---	----	---	----	----	---	---



Analyzing BuildHeap()

- Each call to **Heapify()** takes $O(\lg n)$ time
- There are $O(n)$ such calls (specifically, $\lfloor n/2 \rfloor$)
- Thus the running time is $O(n \lg n)$

- *Is this a correct asymptotic upper bound?*

- *Is this an asymptotically tight bound?*

- A tighter bound is $O(n)$

- *How can this be? Is there a flaw in the above reasoning?*

no. of
nodes at
height h
 $h =$
 $\left\lceil \frac{\lg n}{2} \right\rceil$

Analyzing BuildHeap(): Tight

- To **Heapify** () a subtree takes $O(h)$ time where h is the height of the subtree
 - $h = O(\lg m)$, $m = \#$ nodes in subtree
 - The height of most subtrees is small
- Fact: an n -element heap has at most $\lceil n/2^{h+1} \rceil$ nodes of height h
- CLR 7.3 uses this fact to prove that **BuildHeap** () takes $O(n)$ time

Heapsort

- Given **BuildHeap()**, an in-place sorting algorithm is easily constructed:
 - Maximum element is at $A[1]$
 - Discard by swapping with element at $A[n]$
 - Decrement $\text{heap_size}[A]$
 - $A[n]$ now contains correct value
 - Restore heap property at $A[1]$ by calling **Heapify()**
 - Repeat, always swapping $A[1]$ for $A[\text{heap_size}(A)]$

Heapsort

Heapsort (A)

```
{  
    BuildHeap(A);  
    for (i = length(A) downto 2)  
    {  
        Swap(A[1], A[i]);  
        heap_size(A) -= 1;  
        Heapify(A, 1);  
    }  
}
```

$O(n)$

$\rightarrow O(n)$



last 2 elements
are already
sorted

$O(\log n)$

hence $O(n \log n)$

Heapsort - Example

replace root with highest index

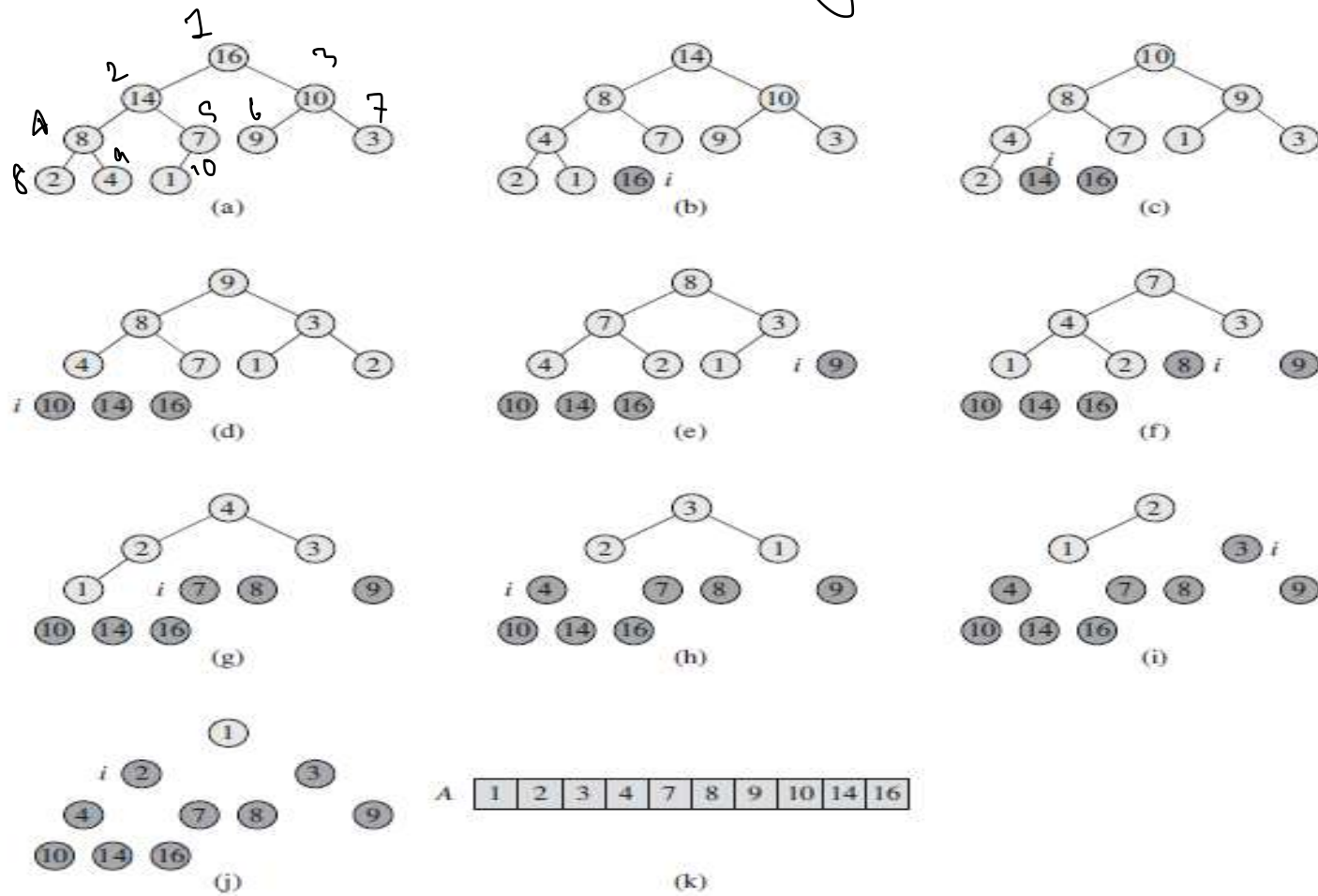


Figure 6.4 The operation of HEAPSORT. (a) The max-heap data structure just after BUILD-MAX-HEAP has built it in line 1. (b)–(j) The max-heap just after each call of MAX-HEAPIFY in line 5, showing the value of i at that time. Only lightly shaded nodes remain in the heap. (k) The resulting sorted array A .

Analyzing Heapsort

- The call to **BuildHeap** () takes $O(n)$ time
- Each of the $n - 1$ calls to **Heapify** () takes $O(\lg n)$ time
- Thus the total time taken by **HeapSort** ()
$$= O(n) + (n - 1) O(\lg n)$$
$$= O(n) + O(n \lg n)$$
$$= O(n \lg n)$$

Priority Queues

- Heapsort is a nice algorithm, but in practice Quicksort usually wins
- But the heap data structure is incredibly useful for implementing *priority queues*
 - A data structure for maintaining a set S of elements, each with an associated value or *key*
 - Supports the operations **Insert()**, **Maximum()**, and **ExtractMax()**
 - *What might a priority queue be useful for?*

Priority Queue Operations

- **Insert(S, x)** inserts the element x into set S
- **Maximum(S)** returns the element of S with the maximum key
- **ExtractMax(S)** removes and returns the element of S with the maximum key
- *How could we implement these operations using a heap?*
 - *E.g. scheduling jobs on a shared computer*

Priority Queue Operations – Implementation of Extract-Max

HEAP-EXTRACT-MAX(A)

```
1  if  $A.heap-size < 1$ 
2      error "heap underflow"
3   $max = A[1]$ 
4   $A[1] = A[A.heap-size]$ 
5   $A.heap-size = A.heap-size - 1$ 
6  MAX-HEAPIFY( $A, 1$ )
7  return  $max$ 
```

EXTRACT-MAX(S) removes and returns the element of S with the largest key.