Part-1

ⓐ What is the relationship between the number of vertices and the number of edges of a graph.

$\mathcal{E} \leq \frac{V(V-1)}{2}$ where $\mathcal{E} \Rightarrow$ Edges (number of edges

$u \rightarrow$ number of vertices

ⓑ Prove the result you stated in part (a)

In a complete graph every pair of vertices is connected by an edge. So number of edges is just the number of pairs of vertices

Max number of edges is ${}^{V}C_2$

$${}^{V}C_2 = \frac{V(V-1)}{1 \times 2} = \frac{V(V-1)}{2}$$

$$\boxed{\mathcal{E} \leq \frac{V(V-1)}{2}}$$

ⓒ What is the relationship between odd cycles and bipartite graph

* The graph is bipartite in and only if it contains no Odd (Sim cycle

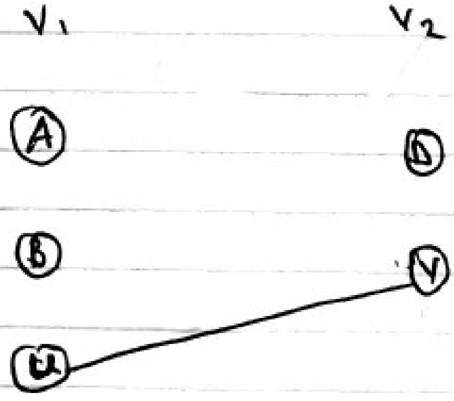
~~ⓓ Prove that if G is disconnected then G' is connected~~

ⓓ Prove the result you stated in part C

$V = V_1 \cup V_2$ where $V_1 \cap V_2 = \emptyset$

Let $u \in V_1$. Any path of odd length say (u, v),

V is in V_2



(e) Prove that if G is disconnected then G' is connected

Let G be a disconnected graph

We can partition $V(G)$ in to V_1, V_2 such that for all $x \in V_1$, $y \in V_2$ there is no path from x to y. (There is no edge from x to y)

Let x, y be any two vertices of $V(G^c)$

Case 1. $x \in V_1$, $y \in V_2$. The edge $(x,y) \notin G$.

Hence $(x,y) \in G^c$. There is a path from x to y.

Case 2 $x, y \in V_1$

Let $z \in V_2$. Hence the path $(x,z),(z,y)$ is in G^c

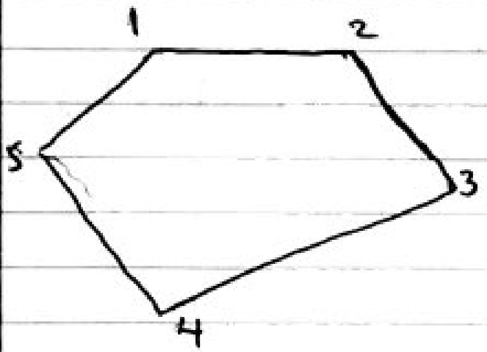
Hence there is a path from x to y in G^c

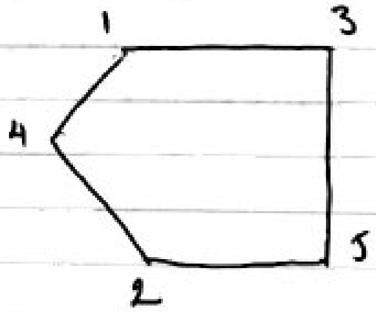
Case 3 $x, y \in V_2$

Similar to Case 2

(f). Draw a graph G with 5 more vertices such that both G and G' are connected. If such graph doesn't exist please write "G doesn't exist".

G





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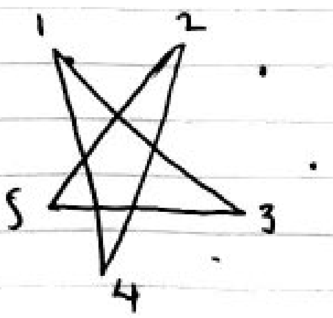
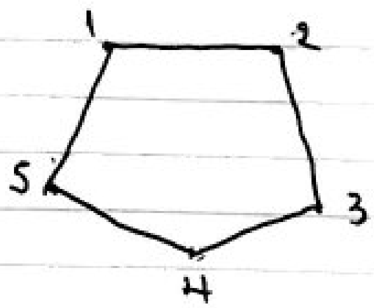




Part II

(a) What are the properties of Red-Black tree.

1- Every node is either red or black
2- Every leaf (null pointer) is black
3- If a node is red, both children are black
4- Every path from a node to descendent leaf contains the same number of black nodes.
5- The root is always black.

(b) True or false: Number of red nodes <= number of Black nodes.

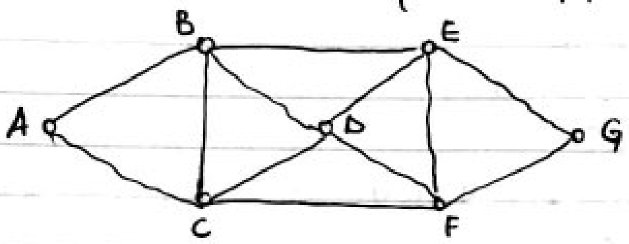
False - Minimum number of red node is 0 is all the tree is black nodes
- Maximum number of red node is 2 : 1 black node.

(c) True or false: The time complexity to build n node Red-black is $O(n)$

False : It is $O(\log n)$.

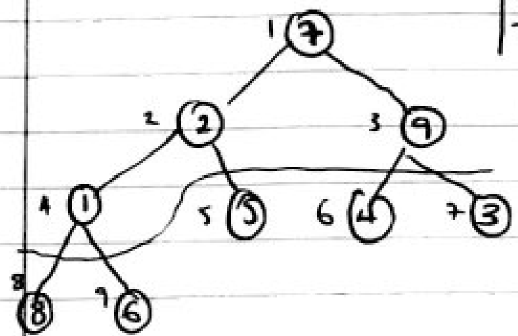
(d) Write a nondeterministic algorithm to search an "item" in an integer array. What is its time complexity.

(e) Illustrate the proof that the Hamiltonian cycle problem is polynomial reducable to TSP. By considering the following Hamiltonian graph - an instance of HamiltonianCycle - and transforming it to a TSP instance in polynomial time so that a solution to the HC problem is a solution to the TSP problem, and conversely.



Part III

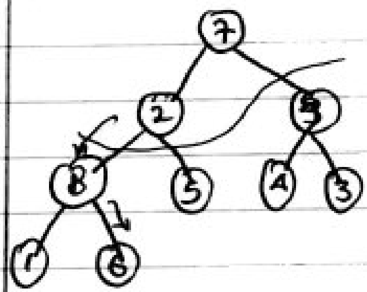
① (to be illustrated step by step) Heap sort [7, 2, 9, 1, 5, 4, 3, 8, 6] in ascending order using in-place bottom-up iterative method. In this case, heap is maintained in an array as explained in your class notes



$\lceil \frac{n}{2} \rceil$ = 5 leaves

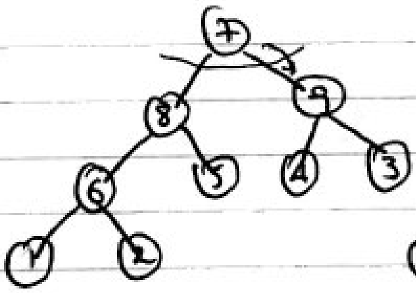
$\lfloor \frac{n}{2} \rfloor$ = 4 internal nods.

i=4

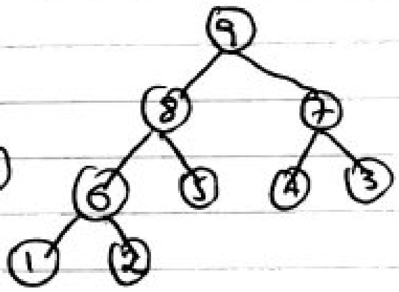


i=3 - no change

i=2

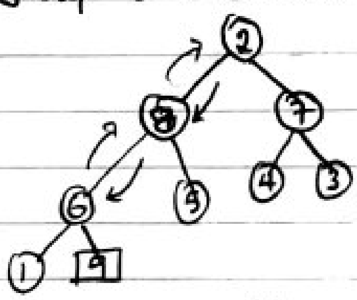


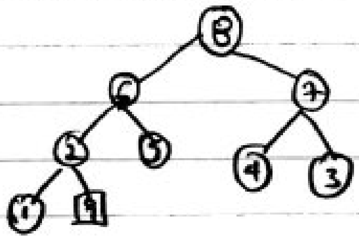
i=1



	9	8	7	6	5	4	3	1	2

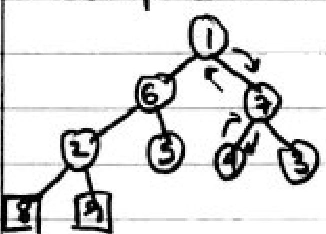
i- Swap i-1 with i=9 and rebuild.

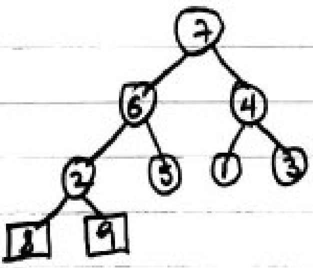




	8	6	7	2	5	4	3	1	9

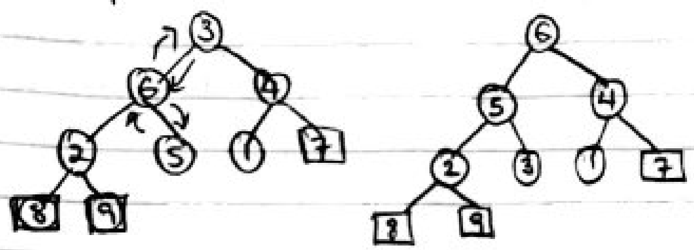
ii- Swap i-1 with i=8 and re-build





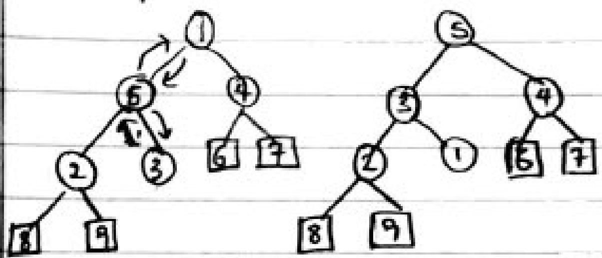
	7	6	4	2	5	1	3	8	9

iii- Swap i=1 with i=7 and rebuild



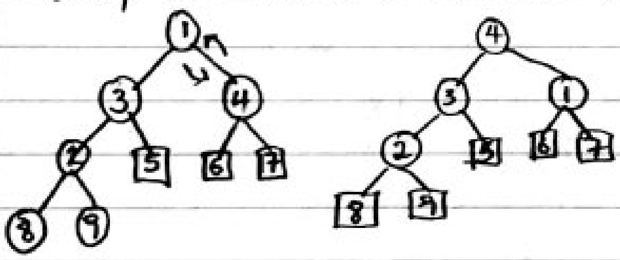
	6	5	4	2	3	1	7	8	9

iv- Swap i=1 with i=6 and rebuild



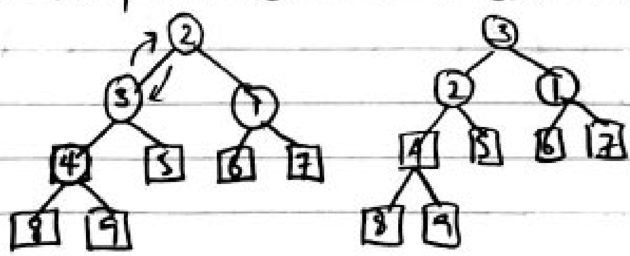
	5	3	4	2	1	6	7	8	9

v- Swap i=1 with i=5 and rebuild



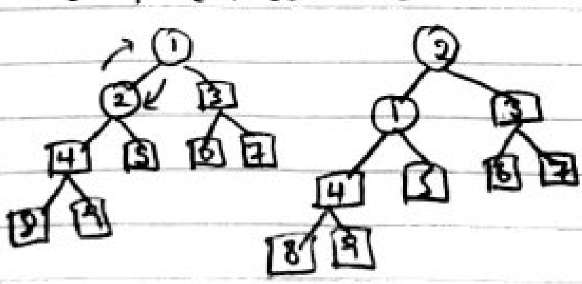
	4	3	1	2	5	6	7	8	9

vi- Swap i=1 with i=4 and rebuild



	3	2	1	4	5	6	7	8	9

vii- Swap i=1 with i=3 and rebuild



	2	1	3	4	5	6	7	8	9

Swap i=1 with i=2

	1	2	3	4	5	6	7	8	9

(b) What is the time complexity to build a heap using in-place bottom-up iterative method.

Ans. $O(n)$

(c) Prove the result in (b)

Assume $n = 2^{(h+1)} - 1$ where $n \rightarrow$ Number of nodes.

$h \rightarrow$ Height of tree.

Thus maximum number of operations (in the worst case) is

$$\sum_{j=0}^{h} j2^{h-j} = 2^h \sum_{j=0}^{h} j2^{-j}$$

Since $$\sum_{j=0}^{h} j2^{-j} < \sum_{j=0}^{\infty} j2^{-j} = 2$$

Thus maximum number of operations (in the worst case) is

$$\sum_{j=0}^{h} j2^{h-j} < 2^{h+1}$$

Since $n = 2^{(h+1)} - 1$, $\sum_{j=0}^{h} j2^{h-j} < n+1 = O(n)$.

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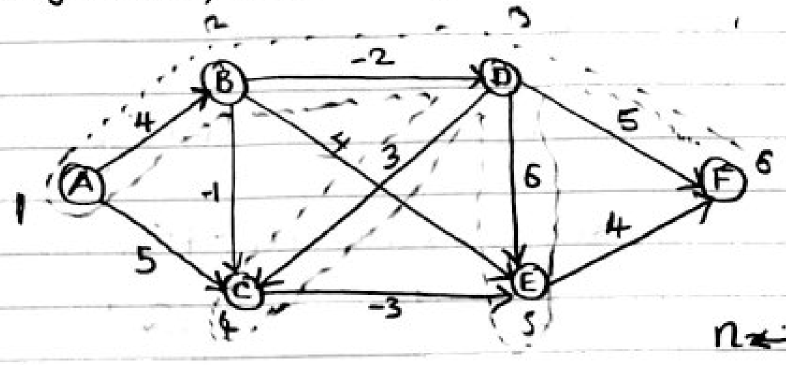
Part IV:

(a) (to be illustrated step by step). Compute shortest path from A to F based on the adjacency matrix given below. Show all steps.

	A	B	C	D	E	F
A	0	4	5	0	0	0
B	0	0	-1	-2	4	0
C	0	0	0	0	-3	0
D	0	0	3	0	6	5
E	0	0	0	0	0	4
F	0	0	0	0	0	0



n = 6

Topological order - ABDCEF

Dist (A) = 0

Dist (B) = min { dist (A) + wt (A, B) } = 0 + 4 = 4

Dist (D) = min { dist (B) + wt (B, D) } = 4 - 2 = 2

Dist (C) = min { dist (A) + wt (A, C) = 0 + 5 = 5 ; dist (B) + wt (B, C) = 4 - 1 = 3 ; dist (D) + wt (D, C) = 2 + 3 = 5 } = 3

Dist (E) = min { dist (C) + wt (C, E) = 3 - 3 = 0 ; dist (B) + wt (B, E) = 4 + 4 = 8 ; dist (D) + wt (D, E) = 2 + 6 = 8 } = 0

Dist (F) = min { dist (D) + wt (D, F) = 2 + 5 = 7 ; dist (E) + wt (E, F) = 0 + 4 = 4 } = 4

∴ The shortest path A to F = 4 { ABCEF }

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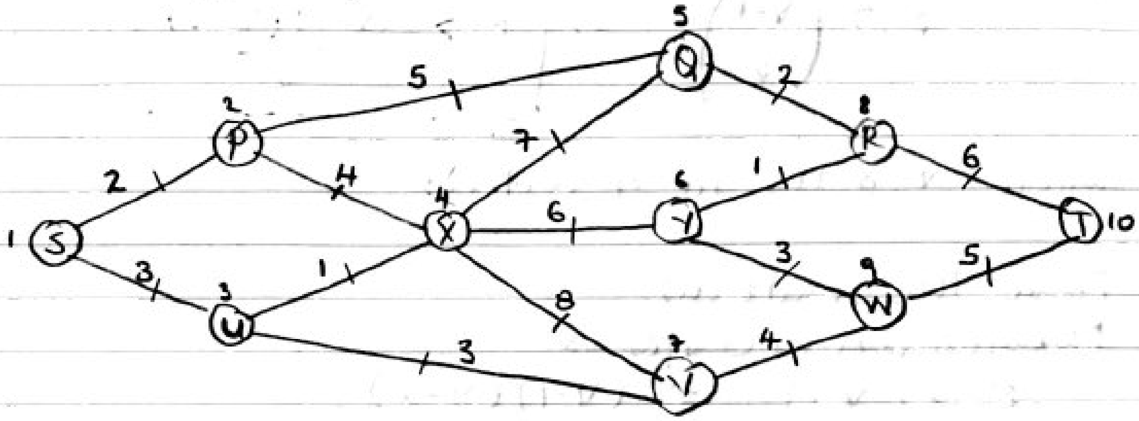
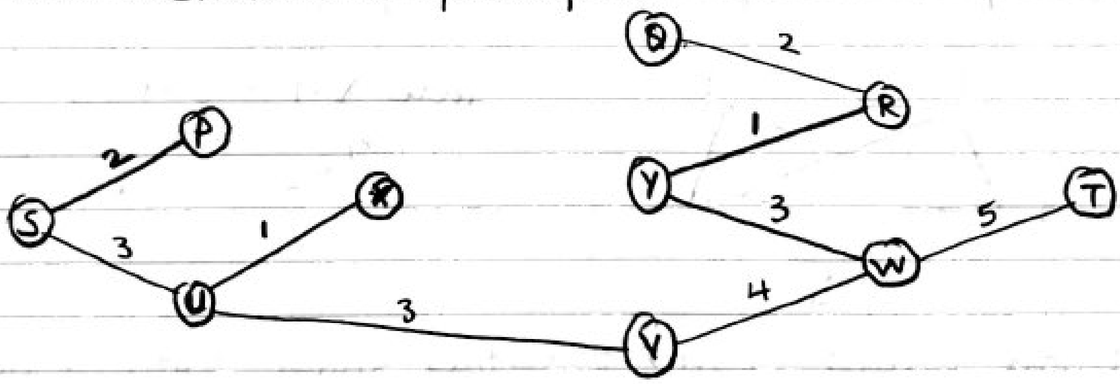
(b) (To be illustrated step by step) Apply Kruskal's algorithm to the following graph to compute minimum spanning tree. Please show the minimum spanning tree. What is the length minimum spanning tree



Step-1 Sort based on the length of edges

UX	YR	SP	QR	SU	UV	YW	VW	PX	PQ	WT	XY	RT	XQ	XV
1	1	2	2	3	3	3	4	4	5	5	6	6	7	8

Step-2. Draw the minmum spanning tree



Step-3 Calculate the length of the minmum spanning tree

count the total weight of the edges to get the minmum spanning tree length.

Hence total length = 2+3+1+3+4+3+1+2+5 = 24 //

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Part V:

(a) What is the minmum number of edges required to guarantee that a graph on n vertices is connected (irrespective of which edges are present).

$$\varepsilon > \binom{V-1}{2} \Rightarrow \boxed{\varepsilon > \frac{(V-1)(V-2)}{2}}$$

Where V is the no. of vertices.

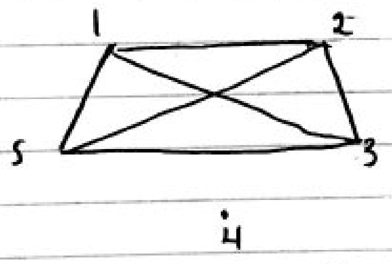
(b) Prove the result (a).

~~Maximum number of edges is~~

$$\varepsilon > \binom{V-1}{2} = \frac{(V-1)(V-2)}{1 \times 2}$$

$$\varepsilon > \frac{(V-1)(V-2)}{2}$$

• Let say for example the following graph of V = 5



Here $V-1 = 5-1 = 4$

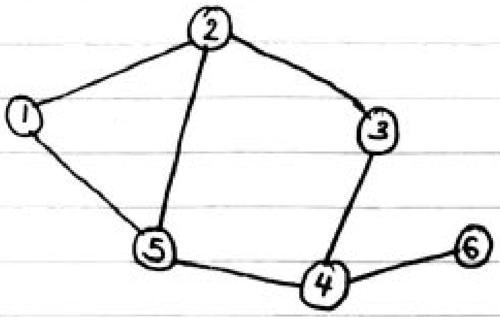
$${}^4C_2 = \frac{4 \times 3}{1 \times 2} = 6$$

* We have 6 edges in this 5 vertices graph but it is not connected because of the node 4.

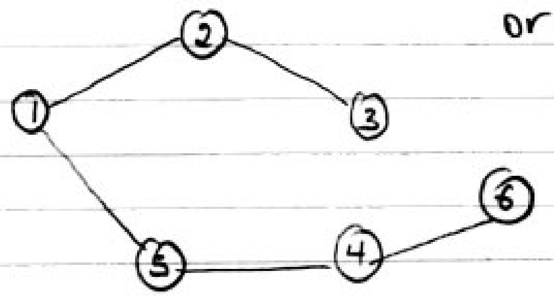
∴ The no of edges should be greater than 6. Here if we add one more edge to 4 it will be connected.

- You can connect the graph with < 6 edges but to be guarented > 6 edges is needed

Part VI: Consider the graph G = (V, E)



ⓐ Give a spanning tree



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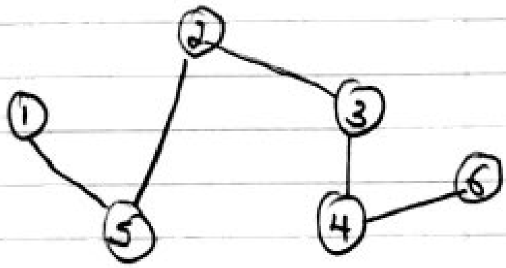
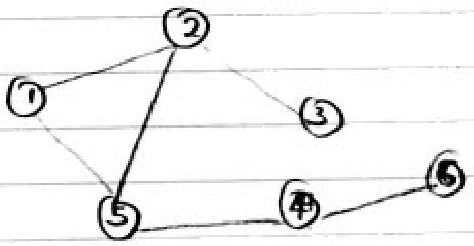


ⓑ Give a spanning subgraph that is not a tree.



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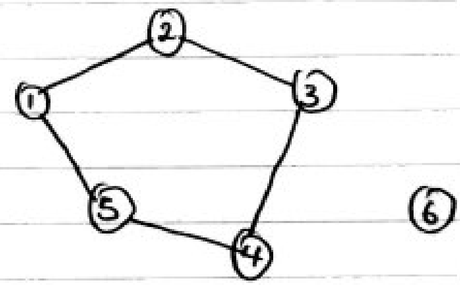

ⓒ Let W = {2, 3, 5, 6}. What is G[W] (G[W] is the subgraph induce
by Graph G

Note: G(W) is induced subgraph, which
is a subset of the vertices graph
G together with any edges where
end points are both in this subs