We consider a pendulum shown in Fig. 1, where the length is l, the mass is m. The gravitational acceleration is denoted by g. The coordinates of the sphere is $[p_x \, p_y]^\top = [-l \sin\theta \; l\cos\theta]$.

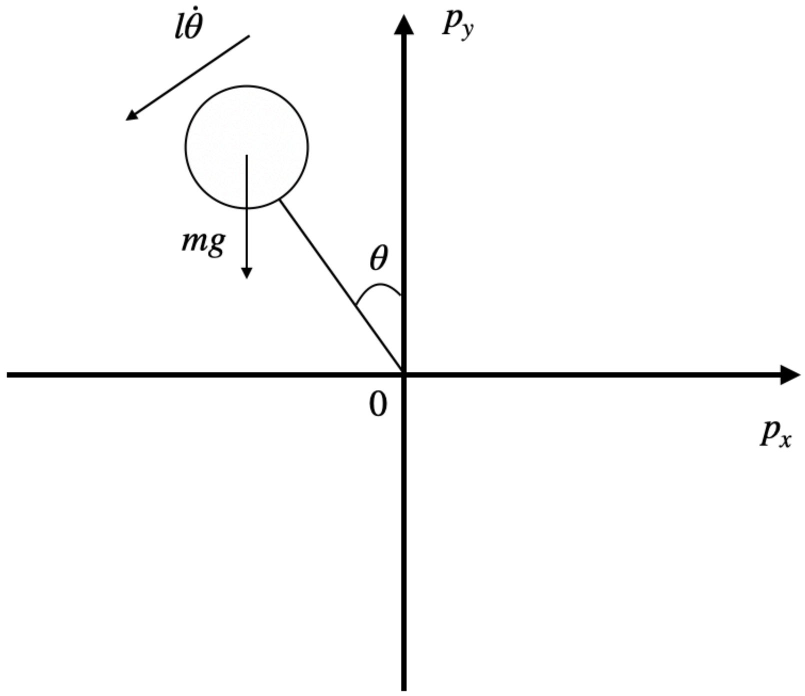

Fig. 1 Illustration of a pendulum.

Then,

$$v = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} -l\dot{\theta}\cos\theta \\ -l\dot{\theta}\sin\theta \end{bmatrix}. \tag{1}$$

The kinetic energy T is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2, \tag{2}$$

and the potential energy U is

$$U = mgl\cos\theta. \tag{3}$$

The Lagrangian $L = T - U$ is

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta. \tag{4}$$

By the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0, \tag{5}$$

the equation of the pendulum is

$$ml^2\ddot{\theta} = mgl\sin\theta. \tag{6}$$

Taking the friction into consideration, the equation is

$$ml^2\ddot{\theta} = mgl\sin\theta - \kappa l\dot{\theta}, \tag{7}$$

where $\kappa > 0$ is frictional coefficient. Additionally, in the case with a torque control u,

$$ml^2\ddot{\theta} = mgl\sin\theta - \kappa l\dot{\theta} + u. \tag{8}$$

Thus, the dynamics of the controlled pendulum is modeled by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta} \\ \frac{g}{l}\sin\theta - \frac{\kappa}{ml}\dot{\theta} + \frac{1}{ml^2}u\end{bmatrix}. \tag{9}$$

In this study, we use the discrete-time model based on (9) using the Euler method as follows;

$$\begin{bmatrix}\theta_{k+1} \\ \omega_{k+1}\end{bmatrix} = \begin{bmatrix}\theta_k + d\omega_k \\ \omega_k + d(g\sin\theta_k - \xi_1\omega_k + \xi_2 u_k)\end{bmatrix}, \tag{10}$$

where $l = 1$ and $d > 0$ is a step size.