QR FACTORIZATION METHOD FOR FINDING EIGENVALUES

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Abstract. The QR factorization method is a powerful and important method for finding approximations for all eigenvalues of a given matrix. Eigenvalues are an important concept that is used extensively in physics and engineering for analyzing rotary motion, oscillations, and stability. There are many methods for finding eigenvalues; the QR factorization method is efficient when it is necessary to find all eigenvalues of a given matrix. It is commonly implemented in various algorithms that use matrix factorizations.

The QR factorization method is a very powerful and stable numerical method for finding approximations of eigenvalues. This method is able to find the set of all eigenvalues of a given matrix, as opposed to methods such as the Rayleigh Quotient, the Power Method, or the Inverse Power Method, all of which find only a single eigenvalue. Consequently, the QR factorization method is inefficient when a specific eigenvalue or eigenvector needs to be found, because it requires extensive computations. However, unlike many other methods for finding eigenvalues, such as those which use determinants, the QR method can be used to find eigenvalues of relatively large matrices, which would otherwise be impossible to compute accurately.

To better understand the importance of this method, a short explanation of eigenvalues is needed. For any square matrix A, consider the equation $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$, where λ is some scalar. The roots of this equation, represented by λ, are known as the eigenvalues of A and they exist only if there exists a nontrivial solution $\vec{\mathbf{x}}$ for the equation $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$; such an $\vec{\mathbf{x}}$ is called an eigenvector corresponding to λ. Finding eigenvalues is essential for many applications in physics, engineering, and computer science: for instance, they are used for analyzing rotational motion and in graph theory.

The main idea behind the method of QR factorization is that any $m \times n$ matrix can be represented in the form

$$A = QR$$

where Q is an orthogonal[1] $m \times n$ matrix such that its columns constitute an orthonormal[2] basis for $Col\, A$, and R is an $n \times n$ invertible upper triangular matrix with positive entries on its diagonal. Q can be found via the Gram-Schmidt process (described in detail in [2]). R can be easily derived from Q by noticing that

$$Q^T A = Q^T(QR) = (Q^T Q)R = IR = R.$$

Knowing how to obtain a Q and an R for a given matrix, we can use the iterative algorithm described in [1] to find the eigenvalues. The algorithm is as follows:

Let $A_0 = A = Q_0 R_0$.

While A_i is not a diagonal matrix:

- Find $Q_i R_i = A_{i-1}$ (using the method above, for instance).
- Let $A_i = R_i Q_i$.

Assuming the algorithm converges, the resulting diagonal matrix A_i will contain the eigenvalues on the main diagonal.

Of course, the algorithm is not faultless – it may converge very slowly, or it may not converge at all. Convergence of this algorithm is a very extensive topic, and will not be analyzed in this paper. Various techniques are generally used in numeric implementations to speed up convergence and to guarantee it for a larger set of matrices.

[1] If for a matrix M the property $M \cdot M^T = I$ is true, then M is an orthogonal matrix.
[2] An orthonormal basis is constructed from an orthogonal basis by normalizing each column vector.

This algorithm would be quite tedious to perform by hand even for a small matrix, there are many computer-based implementations that can accurately perform the QR factorizations. In fact, variations of this method are used in practically every numeric algorithm that computes eigenvalues. An example of this algorithm can be found in [1].

The advantages of this method become readily apparent for large problem sizes. Other methods may have time complexity that increases dramatically with increasing problem sizes, while the time complexity of the QR factorization method scales much more slowly. This makes it useful for solving large numeric problems which would be completely impossible otherwise.

References

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