San Jose State University
Department of Mechanical Engineering

ME 160 Introduction to Finite Element Method

Chapter 6

Thermal Stress Analysis of Solid Structures Using Finite Element Method

Instructor
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References for FE formulation:

"A First Course in the Finite element Method," by Daryl L. Logan, 6th Edition, Cengage Learning, 2017, Chapter 15, P. 727
"The Finite Element Method in Thermomechanics," by T.R. Hsu, Allen and Unwin, 1986, Chapter 3.

Fundamentals of Thermomechanics Analysis

It is a law of nature that "matter change shapes by temperature changes."

For most materials, "Increase temperature will make its shape larger, and the reverse is true too."

Almost all metals expand with temperature increase, and contract with decrease of temperature.

In general temperature, or temperature changes in a solid may induce the following effects:

1) Temperature increase will change material properties: Such as decrease the Young's modulus (E) and yield strength of materials (σ_y).

2) Induce thermal stresses that will be added to mechanically induced stresses in solid structures.

3) Induce creep of the material, and thereby make materials vulnerable for failure at high temperature

Causes of thermal stress:

There are two causes of thermal stress in solid structures:

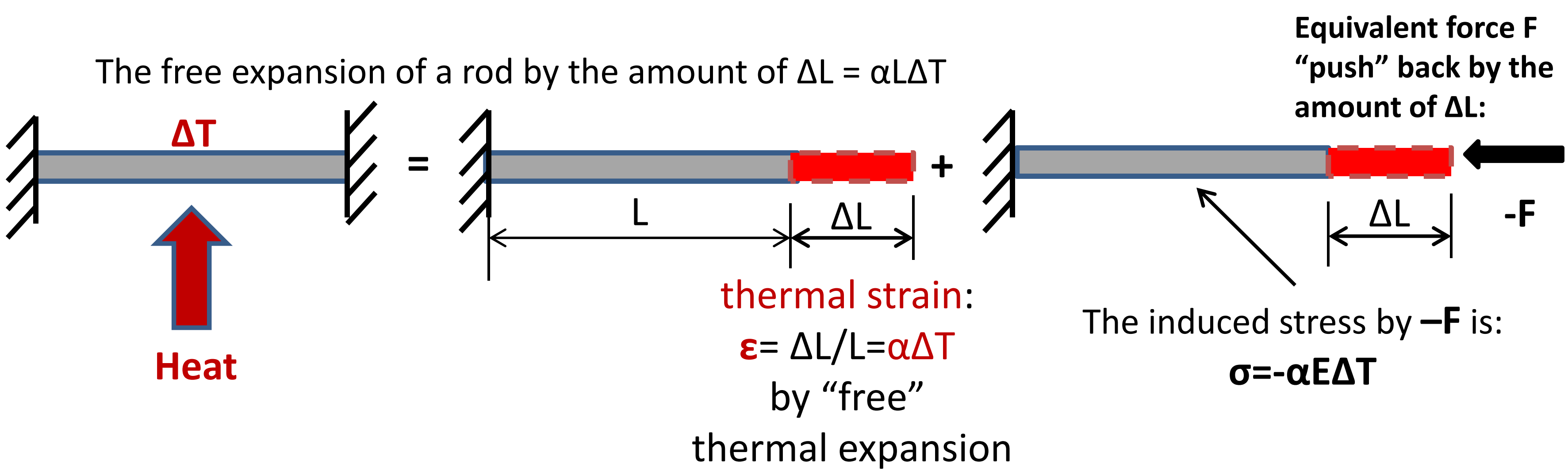
1) Uniform temperature rise in solids with physical constraints-:

A uniform temperature increase in a sold rod with both ends fixed will induce compressive stress in the rod with amount equal to:

$$\sigma = - \alpha \Delta T$$

where α = coefficient of linear thermal expansion of the material with a unit of /^{o}C
ΔT = temperature rise from a reference temperature

The above expression is derived by the following superposition of the situations:



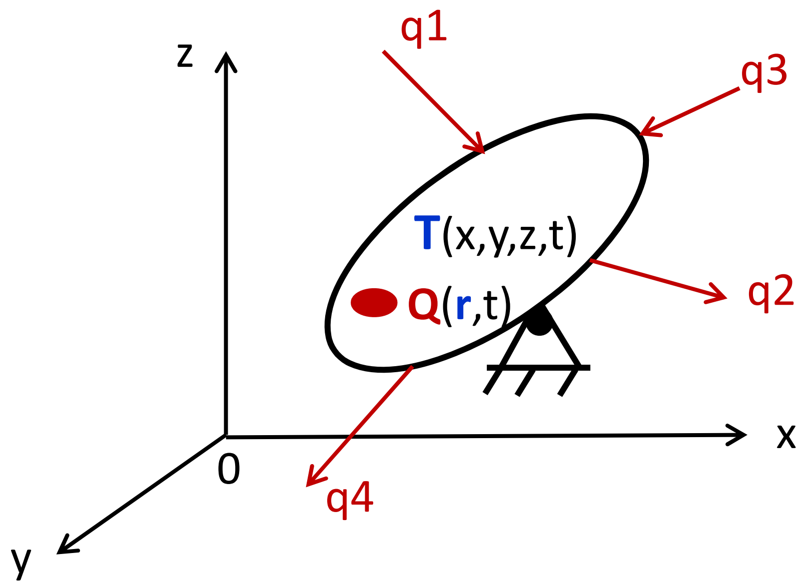
Causes of thermal stress – cont'd:

2) Solid with non-uniform temperature distributions:

Stress induced by non-uniform temperature distribution in solids cause "internal restraints" for thermal expansion or contraction.

3) Solids with partial mechanical constraints coupled by non-uniform temperature distributions. A common feature of structures subject to combine thermomechanical loading:

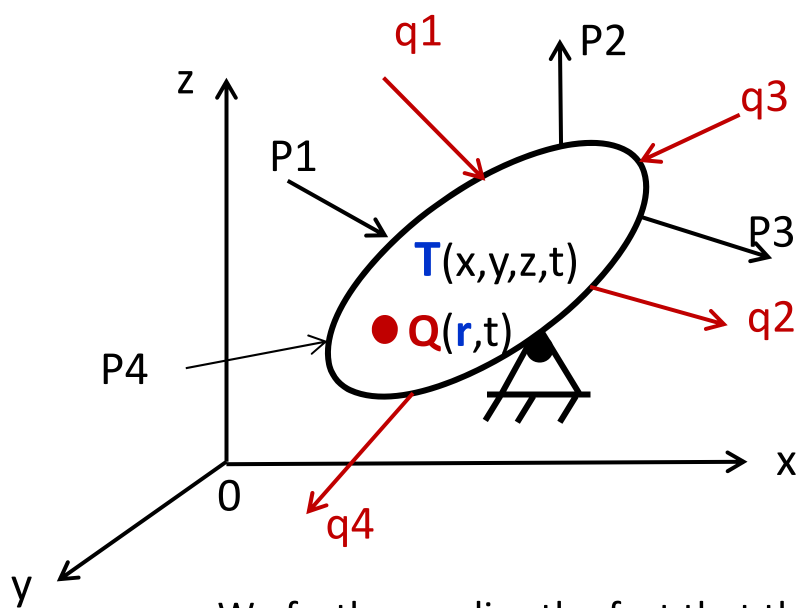
$$\frac{\partial}{\partial x}\left(k\frac{\partial T(x,y,z,t)}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T(x,y,z,t)}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T(x,y,z,t)}{\partial z}\right)+Q(x,y,z,t)=\rho c\frac{\partial T(x,y,z,t)}{\partial t}$$



Continuum solid

The solution T(x,y,z,t) of the above equation is usually a necessary step proceeding to the thermal stress analysis of the solid in the figure.

FE Formulation of Induced Thermal Stress in in Solid Structures



The figure shows a solid with a temperature field (distribution) T(x,y,z,t) produced by the thermal forces: q1, q2, q3, q4,………, and self-heat generation Q. The solid is also subjected to mechanical forces, P1, P2, P3, P4,………

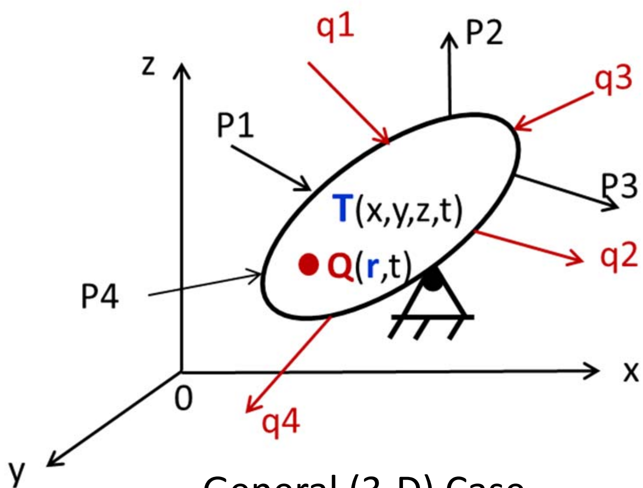
We expect the solid responds to the simultaneous applications of thermal and mechanical load by producing both **combine mechanical and thermal strains and stresses** to reach a new state of equilibrium

We further realize the fact that the solid will return to its original shape at "free-stress" state once ALL the thermal and mechanical forces are removed. (we term this as "unloading.")

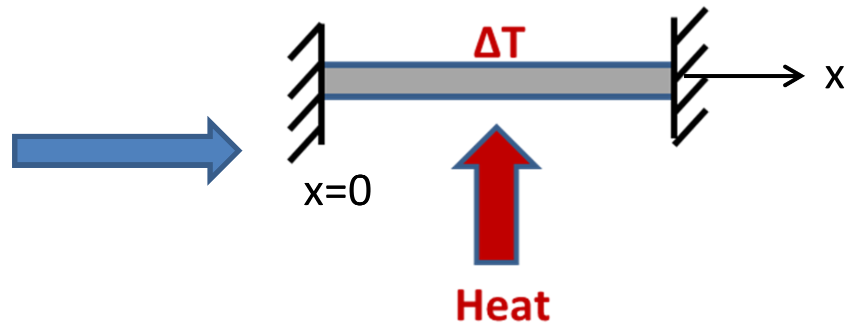
The energy that prompts the restoration of the solid to its original shape after unloading is the "**Strain energy**" that was stored in the deformed state by the applied loads (combined thermal and mechanical loads in this case).

FE Formulation of Induced Thermal Stress in in Solid Structures – cont'd

We begin our FE formulation by considering a simple 1-D case as illustrated below:



General (3-D) Case



Special (1-D) Case

Let the thermal strain ε_T induced in the 1-D rod by a temperature rise ΔT to be expressed as:

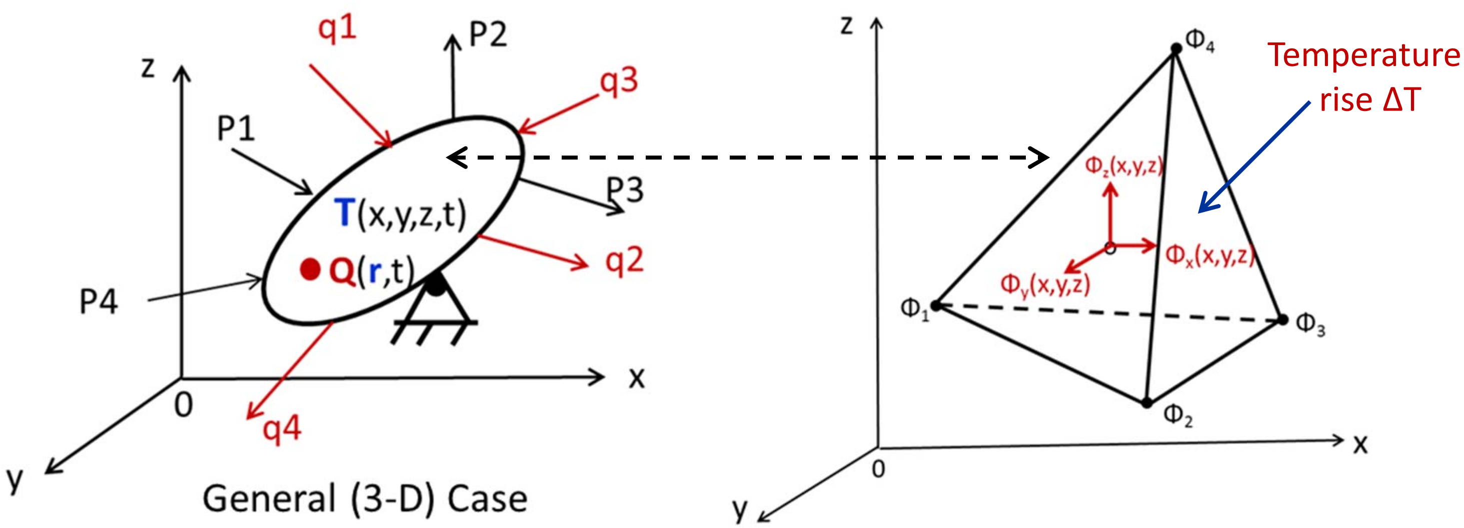
$$\varepsilon_T = \alpha \Delta T \tag{6.1}$$

Due to the fact that temperature is a "scalar" quantity, and thermal strain in the solid structure such as in a special case of 1-D bar, the induced thermal strain by the temperature rise or field may be added on to the mechanical strains in the following way with Hooke's law:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} + \varepsilon_T \tag{6.2}$$

where E is the Young's modulus of the rod material, and ε_T is the induced thermal strain by temperature rise ΔT, and is equal to $\varepsilon_T = \alpha\Delta T$. One should note that Thermal expansion of matter due to temperature rise is **UNIFORM in ALL directions. Consequently, only NORMAL strains are produced.**

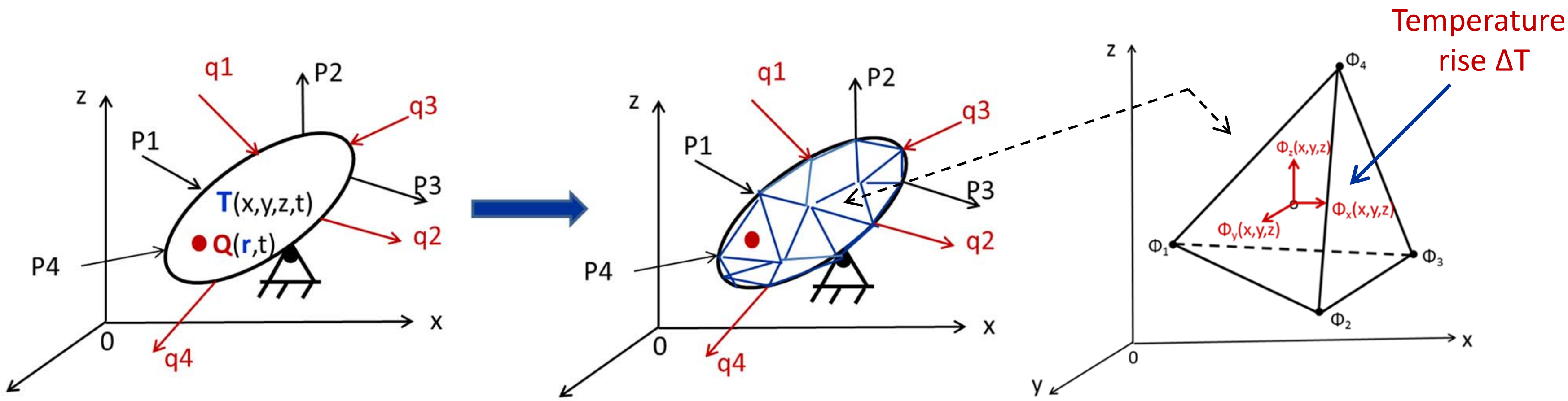
FE Formulation of Induced Thermal Stress in in Solid Structures – cont'd



We will derive the FE formulas for thermal stress analysis of general 3-D solid structures with discretized FE models involving tetrahedral elements as illustrated in the above figure. The element is subjected to both mechanical loading and a temperature rise ΔT from a reference state.

By following the derivations in Chapter 4, we will have the FE formulas for the present case of combined mechanical and thermal forces acting on the solid simultaneously.

FE Formulation of Induced Thermal Stress in in Solid Structures – cont'd



1) Interpolation function:

The element displacement is: {Φ(x,y,z)} with three components:
$\Phi_x(x,y,z)$ = the element displacement component along the x-direction
$\Phi_y(x,y,z)$ = the element displacement component along the y-direction, and
$\Phi_z(x,y,z)$ = the element displacement component along the z-direction

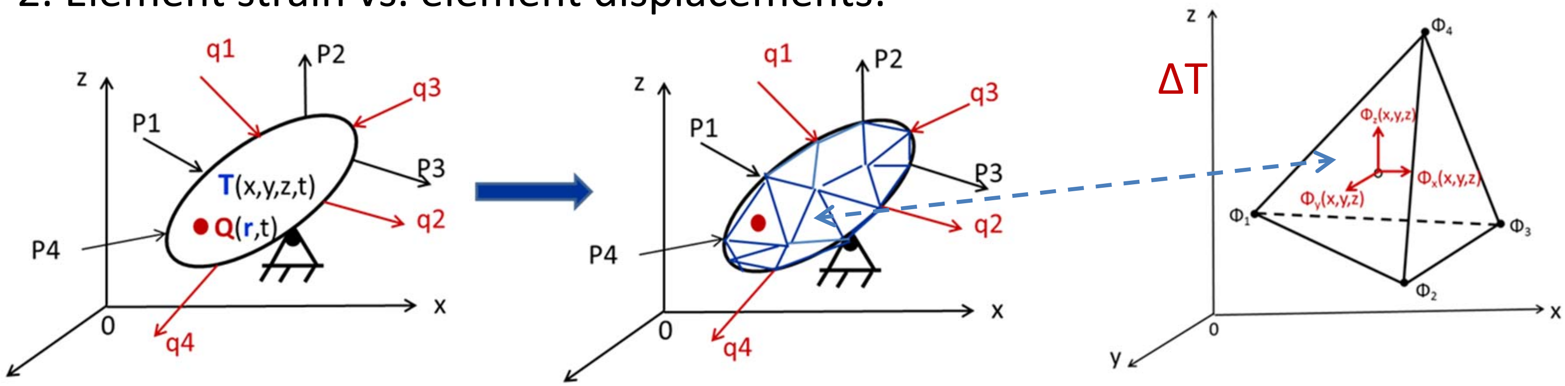
Nodal displacement s:

$$\{\phi\}^T = \{\phi_{1x} \quad \phi_{1y} \quad \phi_{1z} \quad \phi_{2x} \quad \phi_{2y} \quad \phi_{2z} \quad \phi_{3x} \quad \phi_{3y} \quad \phi_{3z} \quad \phi_{4x} \quad \phi_{4y} \quad \phi_{4z}\}^T$$

$$\{\Phi(x,y,z)\} = [N(x,y,z)]\,\{\phi\} \qquad (6.3)$$

where [N(x,y,z)] = interpolation function

2. Element strain vs. element displacements:



Let the total strain in the element to be:

$$\{\varepsilon\} = \{\varepsilon_M\} + \{\varepsilon_T\} \qquad (6.4)$$

where $\{\varepsilon_M\}$ = induced strains by mechanical means, e.g., P1, P2, P3 and P4.

$\{\varepsilon_T\}$ = induced strains by temperature change ΔT produced by q1, q2, q3, and q4 and Q

$$\{\varepsilon_M\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} \Phi_x(x,y,z) \\ \Phi_y(x,y,z) \\ \Phi_z(x,y,z) \end{Bmatrix}$$

The induced thermal strain in the element with UNIFORM temperature change ΔT has CONSTANT values as:

$$\{\varepsilon_T\} = \begin{Bmatrix} \varepsilon_{T,xx} \\ \varepsilon_{T,yy} \\ \varepsilon_{T,zz} \end{Bmatrix} = \begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ \alpha\Delta T \end{Bmatrix} \qquad (6.5)$$

$$\{\varepsilon(x,y,z)\} = [D]\{\Phi(x,y,z)\}+[D_T]\{\Phi_T(x,y,z)\} \tag{6.6}$$

where

$$[D] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \tag{4.4}$$

$$[D_T] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix} \tag{6.7}$$

From Equation (6.5), we will have $[D_T]\{\Phi(x,y,z\} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix} \begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ \alpha\Delta T \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

We will thus use the truncated Equation (6.6), i.e. $\{\varepsilon(x,y,z)\} = [D]\{\Phi(x,y,z)\}$ for further FE formulations

3. Element strains vs. Nodal displacements:

Hence $\{\varepsilon\} = [D][N(x,y,z)]\{\Phi\} = [B(x,y,z)]\{\Phi\}$ (4.12)

with $[B(x,y,z)] = [D][N(x,y,z)]$ (4.13)

4. Element stresses vs. nodal displacements:

$$\{\sigma\} = [C]\{\varepsilon\} \quad (4.6)$$

in which the elasticity matrix [C] in Equation (4.7)

Hence $\{\sigma\} = [C]\,[B(x,y,z)]\{\Phi\}$ (4.14)

5. Strain energy with nodal displacements:

$$U = \frac{1}{2}\int_v \{\varepsilon\}^T \{\sigma\} dv \quad (4.9)$$

Hence

$$U = \frac{1}{2}\int_v ([B(x,y,z)]\{\phi\})^T [C]([B(x,y,z)\{\phi\}])dv \quad (4.15)$$

or

$$U = \frac{1}{2}\int_v \{\phi\}^T [B(x,y,z)]^T [C][B(x,y,z)]\{\phi\}dv \quad \mathbf{(4.16)}$$

Derivation of Element Equations-Cont'd

Potential energy in a deformed solid subjected mechanical forces:

Strain energy:

$$U = \frac{1}{2}\int_v \{\varepsilon\}^T \{\sigma\} dv = \frac{1}{2}\int_v \left(\{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \varepsilon_{xy} \quad \varepsilon_{yz} \quad \varepsilon_{xz}\} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} \right) dv$$

$$= \frac{1}{2}\int_v \left(\varepsilon_{xx}\sigma_{xx} + \varepsilon_{yy}\sigma_{yy} + \varepsilon_{zz}\sigma_{zz} + \varepsilon_{xy}\sigma_{xy} + \varepsilon_{yz}\sigma_{yz} + \varepsilon_{xz}\sigma_{xz} \right) dv \qquad (4.9)$$

Both the strain and stress components are function of (x,y,z), and dv = (dx)(dy)(dz) = the volume of given points in the deformed solid by mechanical forces.

We did not include the thermal strain in the derivation of the strain energy because they are constants as indicated in Equation (6.6). Thermal strain induced by thermal expansion (or contraction) at the nodes Will be accounted for as nodal forces.

Strain energy is a scalar quantity.

Derivation of Element Equations – cont'd

Potential energy in a deformed solid subjected to external mechanical forces:

Work done to deform the solid: Definition of "work": **Work** (W) = **Force** x **Displacement** (deformatio

Two kinds of forces: (1) **body forces** (uniformly distributed throughout the volume of the solid (**v**)), e.g., the weight. Equivalent force of thermal strain by heat is a body force

(2) **surface tractions**, e.g., the pressure or concentrated forces acting on the boundary surface (**s**)

Mathematical expression of work: $W = \int_v \{\Phi(x,y,z)\}^T \{f\} dv + \int_s \{\Phi(x,y,z)\}^T \{t\} ds$

$$\boxed{\begin{aligned} &= \int_v \{\Phi_x(x,y,z) \quad \Phi_y(x,y,z) \quad \Phi_z(x,y,z)\} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} dv \\ &+ \int_s \{\Phi_x(x,y,z) \quad \Phi_y(x,y,z) \quad \Phi_z(x,y,z)\} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} ds \end{aligned}} \qquad (4.10)$$

where {Φ(x,y,z)} = the element displacement of the solid at (x,y,z), {f} = body forces, and {t} = the surface tractions , and ds = the part of the surface boundary on which the surface tractions apply

Derivation of Element Equations – cont'd

Potential energy in a deformed solid subjected to external mechanical forces:

So, the potential energy stored in a deformed solid is: **P = U – W**, or:

$$P = U - W = \frac{1}{2}\int_v \{\varepsilon(x,y,z)\}^T \{\sigma(x,y,z)\} dv - \left(\int_v \{\phi(x,y,z)\}^T \{f\} dv + \int_s \{\phi(x,y,z)\}^T \{t\} ds\right) \quad (4.11)$$

Following the Rayleigh-Ritz Variational principle, the equilibrium condition for the deformed solid should satisfy the following conditions:

$$\frac{\partial P(\phi)}{\partial\{\phi\}} = 0$$

From which, equations for each element may be derived from:

$$\frac{\partial P(\phi)}{\partial \phi_1} = 0, \quad \frac{\partial P(\phi)}{\partial \phi_2} = 0, \quad \frac{\partial P(\phi)}{\partial \phi_3} = 0, \ldots\ldots\ldots\ldots$$

Derivation of Element Equations – cont'd

Element equation by Variational process

The above variation results in:

$$\left(\int_v [B(x,y,z)]^T [C][B(x,y,z)]dv\right)\{\phi\} - \left(\int_v [N(x,y,z)]^T \{f\}dv\right) - \left(\int_s [N(x,y,z)]^T \{t\}ds\right) = 0$$

Upon moving the last two items to the right-hand side:

$$\left(\int_v [B(x,y,z)]^T [C][B(x,y,z)]dv\right)\{\phi\} = \left(\int_v [N(x,y,z)]^T \{f\}dv\right) + \left(\int_s [N(x,y,z)]^T \{t\}ds\right) \tag{4.20}$$

We may represent Equation by the following **element equation**:

$$[K_e]\{\Phi\} = \{q\} \tag{4.21}$$

where $[K_e] = \int_v [B(x,y,z)]^T [C][B(x,y,z)]dv$ = Element stiffness matrix (4.22)

$\{\phi\} = \textit{Nodal displacement copmponents}$

$\{q\} = \int_v [N(x,y,z)]^T \{f\}dv + \int_s [N(x,y,z)]^T \{t\}ds = \textit{Nodal force matrix}$ (4.23)

In Equation (4.23): {f} = Body force and {t} = the surface traction, including the pressure loading on specific boundary face of the element.

Thermal strain as nodal forces in thermal stress analysis

Due to the fact that we have excluded thermal strains from the computation of strain energy "U" in computing the potential energy in the deformed solids, the thermal strain is treated as A component of the nodal forces called "thermal forces", acting at the nodes in the following element equation:

$$[K_e]\{\Phi\} = \{q\} \tag{4.21}$$

where
$$[K_e] = \int_v [B(x,y,z)]^T [C][B(x,y,z)]dv \quad \text{= Element stiffness matrix} \tag{4.22}$$

$$\{\phi\} = \textit{Nodal displacement copmponents}$$

$$\{q\} = \int_v [N(x,y,z)]^T \{f\}dv + \int_v [\boldsymbol{B}]^T [\boldsymbol{C}]\{\boldsymbol{\varepsilon_T}\}\boldsymbol{dv} + \int_s [N(x,y,z)]^T \{t\}ds = \textit{Nodal force matrix} \tag{6.8}$$

The thermal forces at the nodes: $\{f_T\} = \int_v [B]^T [C]\{\varepsilon_T\}dv$ obtained from the general expression shown in the integral in bold face in Equation (6.8) has expressions for the following special cases:

We may express the element equation with lumped thermomechanical node forces in the following expression from Equations (4.21) and (6.8):

$$[K_e]\{\phi\} = \{f_M\} + \{f_T\} \tag{6.9}$$

Element Stiffness Matrix

Nodal Displacement Matrix

Mechanical Nodal Forces

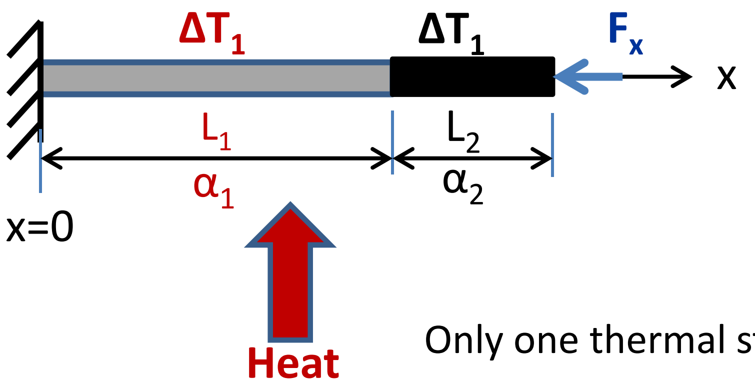
Thermal Forces

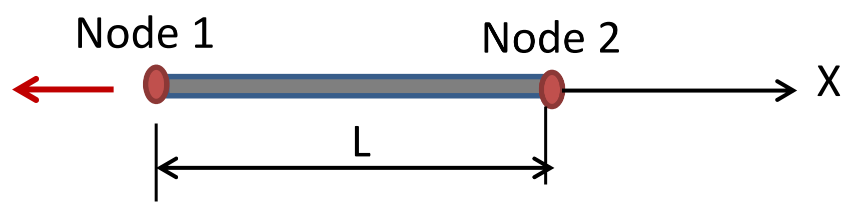
Nodal Forces Contributed by Thermal Strains

General expression of nodal forces contributed by thermal strains:

$$\{q_T\} = \int_v [B(x,y,z)]^T [C]\{\varepsilon_T\} dv \qquad (6.10)$$

For 1-D bar elements:





Only one thermal strain component along the x-coordinate:

$$\{\varepsilon_T\} = \{\varepsilon_{Tx}\} = \{\alpha T\} \qquad (6.11)$$

$$[B] = [D]\{N(x)\} = \frac{d\{N(x)\}}{dx} = \left\{-\frac{1}{L} \quad \frac{1}{L}\right\} \qquad \text{and} \qquad [C] = E$$

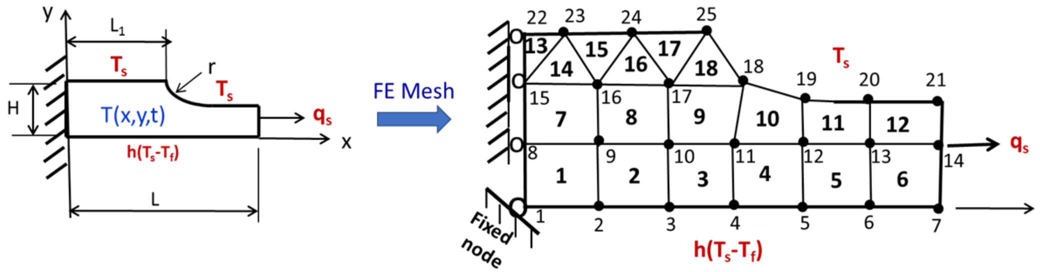
Hence the nodal forces at the two nodes of the 1-D bar element becomes:

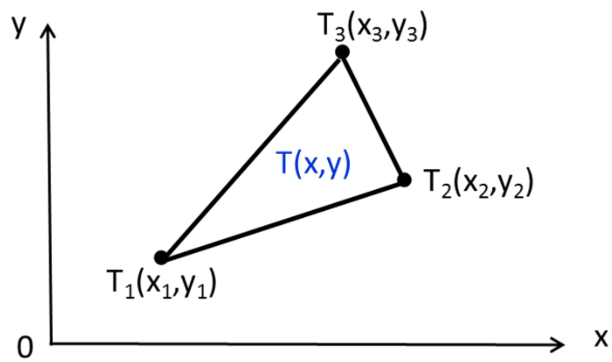
$$\{f_T\} = \begin{Bmatrix} f_{T1} \\ f_{T2} \end{Bmatrix} = \begin{Bmatrix} -E\alpha\Delta T \\ E\alpha\Delta T \end{Bmatrix} A \qquad (6.12)$$

where A = cross-sectional area of the bar element

Nodal Forces Contributed by Thermal Strains - Cont'd

For Plate Elements:





Because temperature change in the element does not can only produce normal strains, but not shear strain, so the induced thermal strain components in the element by temperature change ΔT care:

$$\{\varepsilon_T\} = \begin{Bmatrix} \varepsilon_{xx,T} \\ \varepsilon_{yy,T} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{Bmatrix} \qquad (6.13)$$

The equivalent nodal forces are:

$$\{f_T\} = \int_v [B(x,y,z)]^T [C]\{\varepsilon_T\}dv \approx [B]^T [C]\{\varepsilon_T\}\Delta v = \frac{\alpha E \Delta T}{2(1-\nu)} \begin{Bmatrix} b_1 \\ c_1 \\ b_2 \\ c_2 \\ b_3 \\ c_3 \end{Bmatrix} (At) \qquad (6.14)$$

where

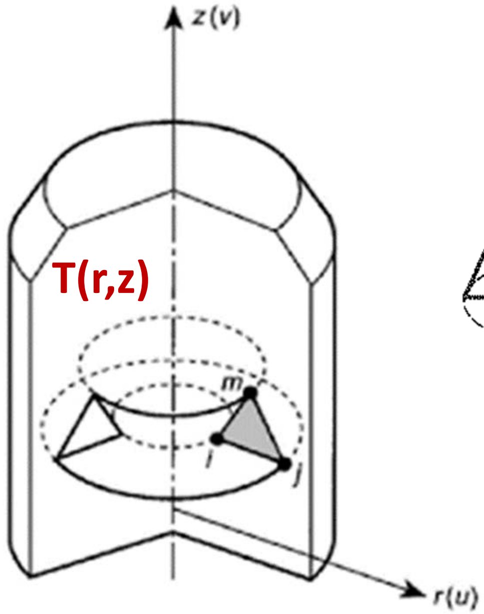
$$a_1 = (x_2 y_3 - x_3 y_2) \quad b_1 = (y_2 - y_3) \quad c_1 = (x_3 - x_2)$$
$$a_2 = (x_3 y_1 - x_1 y_3) \quad b_2 = (y_3 - y_1) \quad c_2 = (x_1 - x_3)$$
$$a_3 = (x_1 y_2 - x_2 y_1) \quad b_3 = (y_1 - y_2) \quad c_3 = (x_2 - x_1)$$

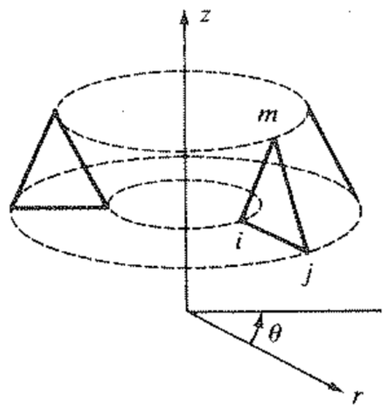
A = plan area of the element with $2A = (x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3)$ and t = thickness

Nodal Forces Contributed by Thermal Strains - Cont'd

For Axisymmetric Elements:

Triangular torus element:





Strains induced in the torus elements induced by temperature change ΔT are those normal components of: ε_{rr}, $\varepsilon_{\theta\theta}$ and ε_{zz} only, or in a matrix form:

$$\{\varepsilon_T\} = \begin{Bmatrix} \varepsilon_{rr,T} \\ \varepsilon_{zz,T} \\ \varepsilon_{\theta\theta,T} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{Bmatrix} \qquad (6.15)$$

The equivalent nodal forces ae:

$$\{f_T\} = \int_v [B(x,y,z)]^T [C]\{\varepsilon_T\} dv = 2\pi \int_A [B]^T [C]\{\varepsilon_T\} r dA \approx 2\pi \bar{r} A [B]^T [C]\{\varepsilon_T\} \qquad (6.16)$$

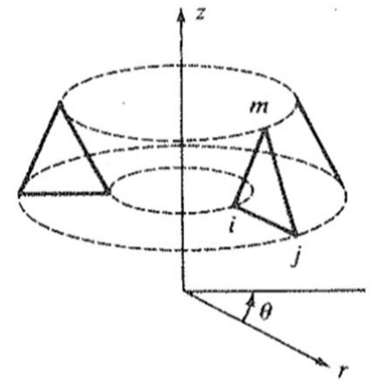
where the average radius: $\bar{r} = \dfrac{r_i + r_j + r_m}{3}$

and the cross-sectional area of the triangular torus element is:

$$A = \frac{1}{2}(x_i y_j - x_j y_i) + (x_j y_m - x_m y_j) + (x_m y_i - x_i y_m)$$

Nodal Forces Contributed by Thermal Strains - Cont'd

For Axisymmetric Elements: Triangular torus element:



$$\{f_T\} = \int_V [B(x,y,z)]^T [C]\{\varepsilon_T\} dv = 2\pi \int_A [B]^T [C]\{\varepsilon_T\} r dA \approx 2\pi \bar{r} A [B]^T |$$

where

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{r} + \beta_i + \frac{\gamma_i z}{r} & 0 & \frac{\alpha_j}{r} + \beta_j + \frac{\gamma_j z}{r} & 0 & \frac{\alpha_m}{r} + \beta_m + \frac{\gamma_m z}{r} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

with

$$\alpha_i = r_j z_m - z_j r_m \quad \alpha_j = r_m z_i - z_m r_i \quad \alpha_m = r_i z_j - z_i r_j$$

$$\beta_i = z_j - z_m \quad \beta_j = z_m - z_i \quad \beta_m = z_i - z_j$$

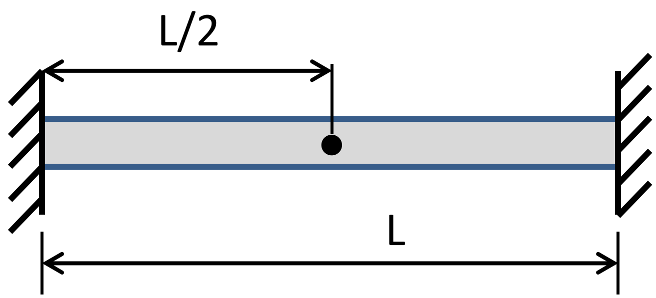
$$\gamma_i = r_m - r_j \quad \gamma_j = r_i - r_m \quad \gamma_m = r_j - r_i$$

and

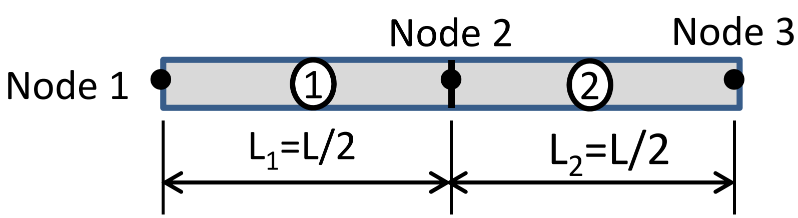
$$[C] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Example 6.1 (Example 15.1, Textbook by Logan):

For the one-dimensional bar fixed at both ends and subjected to a uniform temperature rise of $\Delta T = 50^{o}F$ as shown in the figure below. Determine the reaction at both ends and the axial stress in the bar. Let the Young's modulus $E = 30\text{x}10^6$ psi, cross-sectional area $A = 4\ in^2$, L=4 ft., and linear thermal expansion coefficient $\alpha = 7\text{x}10^{-6}$ in/in-^{o}F.



Bar subjected to temperature rise



FE model

Solution:

We recall that the [B] matrix for 1-D elements is:

$$[B] = \left\{ -\frac{1}{L/2} \quad \frac{1}{L/2} \right\}$$ for both Element 1 and 2

and with [C] = E, we may express the element equation for both elements to be:

$$\begin{array}{cc} & \begin{array}{ccc} \text{Node} & 1 & 2 \end{array} \\ \left[K_e^1\right] = \dfrac{AE}{L/2} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$ for Element 1

$$\begin{array}{cc} & \begin{array}{ccc} \text{Node} & 2 & 3 \end{array} \\ \left[K_e^2\right] = \dfrac{AE}{L/2} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$ for element 2

Example 6.1 (Example 15.1, Textbook by Logan)- cont'd

The thermal force matrix for each element is given by:

$$\left\{f_T^1\right\}=\begin{Bmatrix}-\alpha E\Delta T\\ \alpha E\Delta T\end{Bmatrix} \quad \text{for Element 1} \qquad \text{(a)}$$

$$\left\{f_T^2\right\}=\begin{Bmatrix}-\alpha E\Delta T\\ \alpha E\Delta T\end{Bmatrix} \quad \text{for Element 2} \qquad \text{(b)}$$

The overall stiffness matrix [K] and the thermal force matrix for the structure can be obtained by assembling the element stiffness matrices and the thermal force matrices as follows:

$$[K]=\left[K_e^1\right]+\left[K_e^2\right]=\frac{AE}{L/2}\begin{bmatrix}1 & -1 & 0\\ -1 & 1+1 & -1\\ 0 & -1 & 1\end{bmatrix} \quad \text{and} \quad \{f_T\}=\left\{f_T^1\right\}+\left\{f_T^2\right\}=\begin{Bmatrix}-\alpha E\Delta T\\ \alpha E\Delta T-\alpha E\Delta T\\ \alpha E\Delta T\end{Bmatrix}=\begin{Bmatrix}-\alpha E\Delta T\\ 0\\ \alpha E\Delta T\end{Bmatrix}$$

We thus get the overall stiffness structure equation for the applied thermal forces at the 3 nodes to be:

$$\frac{AE}{L/2}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}-\alpha E\Delta T\\ 0\\ \alpha E\Delta T\end{Bmatrix} \qquad \text{(c)}$$

Since we have the given boundary conditions of: $u_1 = u_3 = 0$, the solution of Equation (c) leads to $u_2 = 0$.

Example 6.1 (Example 15.1, Textbook by Logan)- cont'd

Equation (6.9): $[K_e]\{\phi\} = \{f_M\} + \{f_T\} = 0$ for the present example, leads to the following situation that **$\{f_M\} = -\{f_T\}$, or**:

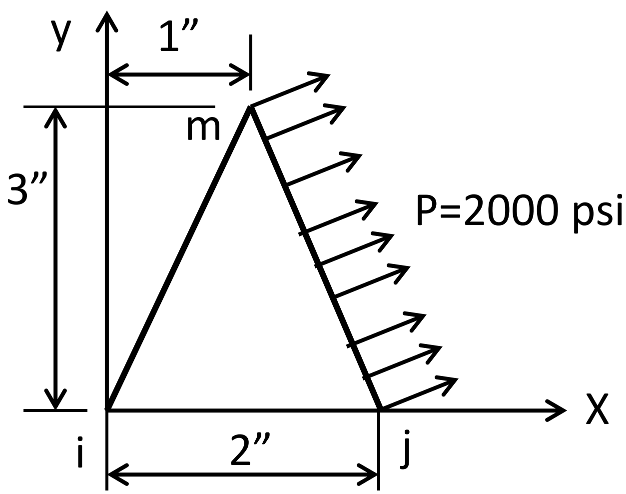
$$\{f_M\} = \begin{Bmatrix} f_{1x} \\ f_{2x} \\ f_{3x} \end{Bmatrix} = \left(\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -\alpha E \Delta T \\ 0 \\ \alpha E \Delta T \end{Bmatrix} \right) \qquad \text{(d)}$$

We solve for: f_{1x} = 42,000 lb, f_{2x} = 0, and f_{3x} =-42000 lb with the given material properties of α and E and ΔT = 50 ^oF.

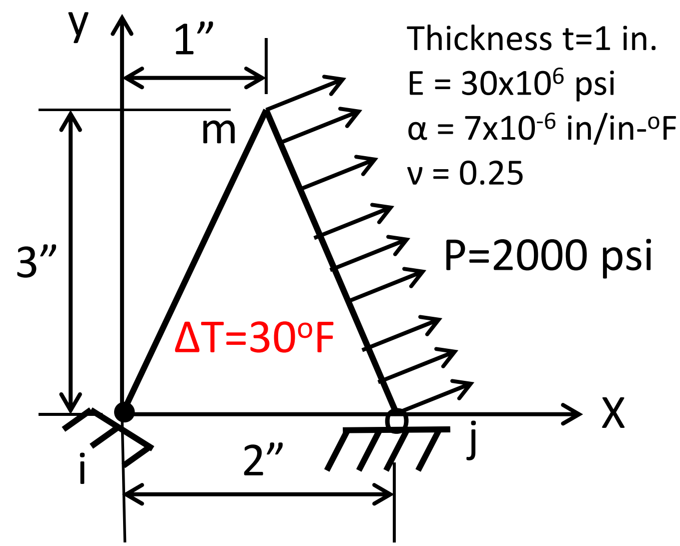
The induced stress in the bar by the forces f1x and f3x at repective nodes 1 and 3 is:

$\sigma_{xx} = f_{1x}/A = f_{3x}/A$ = 42000/4 = 10500 psi (compressive)

Example 6.2 (Example 15.4, Textbook by Logan): Derive the element equation for a thin plate subjected to a 2000 psi pressure acting perpendicular to the side j-m and is subjected to a 30°F temperature rise. (Modified example: to find nodal displacements and the induced stress in the element, with Node i being fixed and Node j is allowed to move along the x-axis)



The element and the structure



FE Model

Solution:

Given nodal coordinates: x_i=0, yi=0; x_j=2, y_j =0; x_m=1, y_m=3, we may compute the following constant coefficients required to determine the [B] matrix in Equation (6.14):

$$b_i = y_j - y_m = -3 \qquad c_i = x_m - x_j = -1$$

$$b_j = y_m - y_i = 3 \qquad c_j = x_i - x_m = -1$$

$$b_m = y_i - y_j = 0 \qquad c_m = x_j - x_i = 2$$

$$and \quad A = \frac{(2)(3)}{2} = 3 \ in^2$$

Example 6.2 - Cont'd

We may compute the [B] matrix by Equation (4.37) to be:

$$[B]=\frac{1}{6}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix} \qquad \text{(a)}$$

The element stiffness matrix may be obtained by using Equation (4.22):

$$[K_e]\approx[B]^T[C][B](At) \qquad \text{(b)}$$

with [B] in Equation (a), and the [C] matrix in the following expression:

$$[C]=\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}=\frac{30\times10^6}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}=(4\times10^6)\begin{bmatrix} 8 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \text{(c)}$$

From Equations (a) and (c), we have the following:

$$[B]^T[C]=\frac{1}{6}\begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}(4\times10^6)\begin{bmatrix} 8 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix}=\frac{4\times10^6}{6}\begin{bmatrix} -24 & -6 & -3 \\ -2 & -8 & -9 \\ 24 & 6 & -3 \\ -2 & -8 & 9 \\ 0 & 0 & 6 \\ 4 & 16 & 0 \end{bmatrix} \qquad \text{(d)}$$

Example 6.2 - Cont'd

We may thus compute the element stiffness matrix as follows:

$$[K_e] = (1in)\frac{(3in^2)}{6}\frac{4\times10^6}{6}\begin{bmatrix} -24 & -6 & -3 \\ -2 & -8 & -9 \\ 24 & 6 & -3 \\ -2 & -8 & 9 \\ 0 & 0 & 6 \\ 4 & 16 & 0 \end{bmatrix}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$= \frac{10^6}{3}\begin{bmatrix} 75 & 15 & -69 & -3 & -6 & -12 \\ 15 & 35 & 3 & -19 & -18 & -16 \\ -69 & 3 & 75 & -15 & -6 & 12 \\ -3 & -19 & -15 & 35 & 18 & -16 \\ -6 & -18 & -6 & 18 & 12 & 0 \\ -12 & -16 & 12 & -16 & 0 & 32 \end{bmatrix} \qquad \text{(e)}$$

Example 6.2 - Cont'd

We are now ready to compute the nodal force matrices that include both the thermal force matrix and the mechanical force matrix.

Thermal force matrix:

The thermal force matrix at the nodes can be obtained by following Equation (6.14) as follows:

$$\{f_T\} = \frac{\alpha E \Delta T}{2(1-\nu)} \begin{Bmatrix} b_i \\ c_i \\ b_j \\ c_j \\ b_m \\ c_m \end{Bmatrix} = \frac{(7\times10^{-6})(30\times10^{6})(1)(30)}{2\times(1-0.25)} \begin{Bmatrix} -3 \\ -1 \\ 3 \\ -1 \\ 0 \\ 2 \end{Bmatrix} = 4200 \begin{Bmatrix} -3 \\ -1 \\ 3 \\ -1 \\ 0 \\ 2 \end{Bmatrix} = \begin{Bmatrix} -12600 \\ -4200 \\ 12600 \\ -4200 \\ 0 \\ 8400 \end{Bmatrix} \qquad \text{(f)}$$

Mechanical force matrix:

We realize that uniform pressure is applied along the edge j-m, which leads to the following derivations:

The length of the edge j-m: $L_{j\text{-}m} = [(2-1)^2+(3-0)^2]^{1/2} = 3.163$ in.
Its components along the x- and y-coordinates are:
$P_x = P\cos\theta = 2000(3/3.163) = 1896$ psi, and
$p_y = P\sin\theta = 2000(1/3.163) = 632$ psi .

Example 6.2 - Cont'd

The equivalent forces applied to Node j and m can be obtained by the following expression:

$$\{f_M\} = \int_S [N_s]^T \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} dS = \int_S \begin{bmatrix} N_i & 0 \\ 0 & N_i \\ N_j & 0 \\ 0 & N_j \\ N_m & 0 \\ 0 & N_m \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} dS \approx \frac{tL_{j-m}}{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} \tag{g}$$

From which, we get:

$$\{f_M\} = \frac{(1in)(3.163\,in)}{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1896 \\ 632 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 3000 \\ 1000 \\ 3000 \\ 1000 \end{Bmatrix} \tag{h}$$

Example 6.2 - Cont’d

The element equation following Equation (6.9) takes the following form:

$$\frac{1\times10^{6}}{3}\begin{bmatrix} 75 & 15 & -69 & -3 & -6 & -12 \\ 15 & 35 & 3 & -19 & -18 & -16 \\ -69 & 3 & 75 & -15 & -6 & 12 \\ -3 & -19 & -15 & 35 & 18 & -16 \\ -6 & -18 & -6 & 18 & 12 & 0 \\ -12 & -16 & 12 & -16 & 0 & 32 \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}=\begin{Bmatrix} -12600 \\ -4200 \\ 15600 \\ -3200 \\ 3000 \\ 9400 \end{Bmatrix} \tag{j}$$

We realize that the elements in the matrix at right-hand-side of Equation (j) are the summation of Those elements in Equations (f) and (h).

Example 6.2 - Cont'd

Solve for nodal displacements

Since there is only one element in the structure, we have the overall stiffness equation
Of the structure to be identical to that for the element, we thus have:

$$\frac{1\times10^6}{3}\begin{bmatrix} 75 & 15 & -69 & -3 & -6 & -12 \\ 15 & 35 & 3 & -19 & -18 & -16 \\ -69 & 3 & 75 & -15 & -6 & 12 \\ -3 & -19 & -15 & 35 & 18 & -16 \\ -6 & -18 & -6 & 18 & 12 & 0 \\ -12 & -16 & 12 & -16 & 0 & 32 \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} = \begin{Bmatrix} -12600 \\ -4200 \\ 15600 \\ -3200 \\ 3000 \\ 9400 \end{Bmatrix} \qquad \text{(j)}$$

We will impose the boundary conditions with: $u_i=v_i= 0$, and $v_j = 0$ into the above equations.

Example 6.2 - Cont'd

Solve for nodal displacements

$$\frac{1\times 10^6}{3}\begin{bmatrix} 75 & 15 & -69 & -3 & -6 & -12 \\ 15 & 35 & 3 & -19 & -18 & -16 \\ -69 & 3 & 75 & -15 & -6 & 12 \\ -3 & -19 & -15 & 35 & 18 & -16 \\ -6 & -18 & -6 & 18 & 12 & 0 \\ -12 & -16 & 12 & -16 & 0 & 32 \end{bmatrix}\begin{Bmatrix} u_i = 0 \\ v_i = 0 \\ u_j \\ v_j = 0 \\ u_m \\ v_m \end{Bmatrix} = \begin{Bmatrix} -12600 \\ -4200 \\ 15600 \\ -3200 \\ 3000 \\ 9400 \end{Bmatrix} \qquad \text{(k)}$$

By shoveling rows and columns of the matrices in Equation (k) with unknown displacement components Following the rules stipulated in Step 6 in Chapter 3, we will interchange Row 3 and 4, followed by Similar change of column 3 and 4 in the stiffness matrix to give:

$$\frac{10^6}{3}\left[\begin{array}{ccc|ccc} 75 & 15 & -3 & -69 & -6 & -12 \\ 15 & 35 & -19 & 3 & -18 & -16 \\ -3 & -19 & 35 & -15 & 18 & -16 \\ \hline -69 & 3 & -15 & 75 & -6 & 12 \\ -6 & -18 & 18 & -6 & 12 & 0 \\ -12 & -16 & -16 & 12 & 0 & 32 \end{array}\right]\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline u_j \\ u_m \\ v_m \end{array}\right\} = \left\{\begin{array}{c} -12600 \\ -4200 \\ -3200 \\ \hline 15600 \\ 3000 \\ 9400 \end{array}\right\} \qquad \text{(m)}$$

Example 6.2 - Cont'd

Solve for nodal displacements

The partitioned matrix equation in Equation (m) provides the solution of the three unknown displacements $\{q_b\} = \{u_j \quad u_m \quad v_m\}^T$ obtainable by the expression:

$$\{q_b\} = [K_{bb}]^{-1} \left(\{R_b\} - [K_{ba}]\{q_a\}\right)$$

$$[K_{bb}] = \begin{bmatrix} 75 & -6 & 12 \\ -6 & 12 & 0 \\ 12 & 0 & 32 \end{bmatrix} \qquad \{R_b\} = \begin{Bmatrix} 15600 \\ 3000 \\ 9400 \end{Bmatrix} \qquad [K_{ba}] = \begin{bmatrix} -69 & 3 & -15 \\ -6 & -18 & 18 \\ -12 & -16 & -16 \end{bmatrix} \qquad \{q_a\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} u_j \\ u_m \\ v_m \end{Bmatrix} = \frac{10^{-6}}{3} \begin{bmatrix} 75 & -6 & 12 \\ -6 & 12 & 0 \\ 12 & 0 & 32 \end{bmatrix}^{-1} \begin{Bmatrix} 15600 \\ 3000 \\ 9400 \end{Bmatrix} = \frac{10^{-6}}{3} \begin{bmatrix} 0.0148 & 0.0074 & -0.0056 \\ 0.0074 & 0.08703 & -0.0028 \\ -0.0056 & -0.0028 & 0.03333 \end{bmatrix} \begin{Bmatrix} 15600 \\ 3000 \\ 9400 \end{Bmatrix}$$

$$= \frac{10^{-6}}{3} \begin{Bmatrix} 200.44 \\ 350.21 \\ 217.54 \end{Bmatrix} = \begin{Bmatrix} 66.8 \times 10^{-6} \\ 116.67 \times 10^{-6} \\ 72.33 \times 10^{-6} \end{Bmatrix} \text{ in.}$$

So, the displacements of the three nodes are:

$u_j = 66.8\text{x}10^{-6}$ inch, $u_m = 116.67\ \text{x}10^{-6}$ inch and $v_m = 72.33\text{x}10^{-6}$ inch.

Example 6.2 - Cont'd

Solve for element stresses

Equation (4.14) relates the element stress with nodal displacements:

$$\{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [C][B]\{\phi\} = \left(4\times10^{6}\right)\begin{bmatrix} 8 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix}\frac{1}{6}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 66.67 \\ 0 \\ 116.67 \\ 72.51 \end{Bmatrix}\times10^{-6}$$

$$= \frac{4}{6}\begin{bmatrix} 8 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 66.67 \\ 0 \\ 116.67 \\ 72.51 \end{Bmatrix}$$

$$= \frac{4}{6}\begin{bmatrix} -24 & -2 & 24 & -2 & 0 & 4 \\ -6 & -8 & 6 & -8 & 0 & 16 \\ -3 & -9 & -3 & 9 & 6 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 66.67 \\ 0 \\ 116.67 \\ 72.51 \end{Bmatrix} = \frac{4}{6}\begin{Bmatrix} 1890.12 \\ 1560.18 \\ 500 \end{Bmatrix} = \begin{Bmatrix} 1260 \\ 1040.12 \\ 333.33 \end{Bmatrix} \; psi$$

Summary on Thermal Stress analysis by FE Method

1. Temperature gradients will be induced in a solid whenever there is heat flow within the solid.

2. Solids will change their shape and dimensions whenever there is a temperature change: A solid will expand with a temperature rise and contract with a temperature drop.

3. Sources of heat flow include: heat generation by certain parts or entire solid; heat fluxes entering or leaving the solid across the solid boundaries, or parts of the solid in contact with surrounding fluids with heat exchange between the solid and fluid through convective heat transfer.

4. Thermal stresses may be introduced to the solid if: (a) the induced solid is physically constrained with a uniform temperature change in the solid, (b) unconstrained solid but with temperature gradients in the solid. Unconstrained solids with uniform temperature change does not generate thermal stress.

5. Thermal stresses induced in solids by temperature change are treated as a form of "Body Force." As such, the induced thermal stress in the solids is treated as a form of additional nodal forces in a FE stress analysis.

6. The FE formulation in this chapter is based on "uniform" temperature in individual elements. thermal stress analysis normally follows a "heat transfer analysis" from which the temperature distributions in sloid structures are made available.

7. Thermal stress analysis is an important part of ME and AE structure analyses. The induced thermal stresses should be considered as a part of total induced stresses in all stress analysis, in addition to the accompanied material deteriorations with high temperature environment..