

# MATH 421 Lecture Notes

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# Properties of Real Number

**Definition 1.** Given any  $a \in \mathbb{R}$ , we define its absolute value to be

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

**Theorem 2** (Triangular Inequality). Given  $a, b \in \mathbb{R}$ , there holds

$$|a + b| \leq |a| + |b|$$

# Method of Proof

## Direct proof

some statements can be shown to be true through a direct argument e.g. our proof of Theorem 1

**Theorem 3.** hello

## Proof by induction

the aim is to proof that a statement is true for all rational number

- (i) Show the statement is true for  $n = 1$
- (ii) Assume the statement is true for general  $n \in \mathbb{N}$
- (iii) Using assumption (ii), prove the statement is true for  $n + 1$
- (iv) Conclude your proof with a sentence like "by mathematical information, the result holds for all  $n \in \mathbb{N}$ "

**Example 4.** Show that  $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$

**Theorem 5.** Let  $x \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Then, there holds the formula

$$(1+x)^n = \sum_{j=0}^n \binom{n}{j} x^j$$

# 1 Real Intervals

$\forall a, b \in \mathbb{R}$  such that  $a < b$ , we denote  $[a, b]$ , the set of all  $\mathbb{R}$  between  $a$  and  $b$  (inclusive)

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

Similarly, we have

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

by convention,  $(a, a) = \emptyset$ , the empty set

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

Subset of this form are call **intervals**. We also adopt the notation

$$(\infty, a] = \{x \in \mathbb{R} : x \leq a\}$$

$$(b, \infty) = \{x \in \mathbb{R} : x > b\}$$

We'll never write  $[\infty, a]$ , since  $\pm\infty$  are **not** real numbers.

$[a, b], (a, b], [a, b), (a, b)$ , they are **bounded**

**Definition 6.** A set  $B \subseteq \mathbb{R}$  is bounded below (respectively bounded above) if  $\exists b \in \mathbb{R}$  such that  $x \geq b \forall x \in B$  (respectively  $x \leq b$  for all  $x \in B$ )

e.g.  $\{0, 1, 50^{72}, -350\pi\}$  and  $\left[-\frac{1}{\sqrt{10}}, 3\right)$  are bounded while  $\mathbb{R}$  and  $\mathbb{N}$  are not bounded

e.g.  $[-357, \infty)$  is bounded below but not above

**Definition 7.** Let  $B \subseteq \mathbb{R}$  be a subset that is bounded. We say that  $b \in \mathbb{R}$  is the least upper bound of  $B$  (also call the supremum of  $B$ ) if

- (i)  $b$  is an upper bound for  $B$
- (ii) if  $b'$  is also an upper bound for  $B$ , then we have  $b \leq b'$

We denote this least upper bound by  $\sup B$

**Remark 8.** It is easy to see that for a set  $B$  bounded above,  $\sup B$  is unique. To see this, suppose that both  $\beta_1$  and  $\beta_2$  are least upper bound for  $B$ . Then since  $\beta_2$  is least upper bound and  $\beta_1$  is an upper bound. We have  $\beta_2 \leq \beta_1$ . But also since  $\beta_1$  is least upper bound and  $\beta_2$  is a lower bound, we have  $\beta_1 \leq \beta_2$ . Hence  $\beta_1 = \beta_2$

We have the corresponding notation for lower bounds

**Definition 9.** Let  $A \subseteq \mathbb{R}$  be a subset bounded below. We say that  $a \in \mathbb{R}$  is the greatest lower bound for  $A$  (also called the infimum of  $A$ ) if

- (i)  $a$  is a lower bound for  $A$
- (ii) if  $a'$  is also a lower bound for  $A$ , then  $a' \leq a$

For  $B = (-1, \infty)$ ,  $\inf B = -1$ .

For  $B = [-1, \infty)$ ,  $\inf B = -1$ .

For  $A = [2, 10) \cup (510, 511] \cup \{520\}$ ,  $\inf A = 2$ ,  $\sup A = 520$

Note that some sets contain their infimum/supremum while others do not. We note down a property of the real-numbers which we state but do not prove

**Example.** Prove that if  $a = (0, 1)$ ,  $\sup A = 1$

*Proof.* Notice that if  $x \in A$  then  $x < 1$ , so 1 is an upper bound for  $A$ . Suppose for contradiction that  $\sup A \neq 1$ . Then we must have  $\sup A < 1$  but  $m = \frac{1}{2}(\sup A + 1) \in A$  but  $m > \sup A$ . So  $\sup A$  is not an upper bound for  $A$   $\square$

## 2 Functions & Their Representation

A function is a “thing” that assigns a number to another number

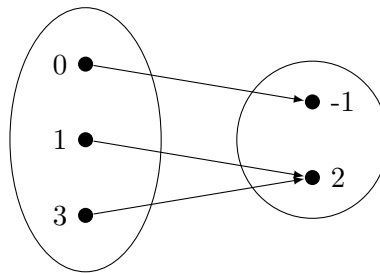
**Example.** the square function  $x \mapsto x^2$

The way we represent this is by writing that  $f$ , the function such that  $f(x) = x^2$ , also written  $f : x \mapsto x^2$

**Example.** We could also define a function, say  $g$ , that acts on  $\{0, 1, 3\}$  and maps from elements of this set to  $\{-1, 2\}$ , for instance

$$g(0) = 1, \quad g(1) = 2, \quad g(3) = 2$$

One way of representing this is with the diagram



When defining a function  $f$ , we write  $f : A \rightarrow B$ , where  $A$  is domain and  $B$  is range

**Example.** Define the function  $r : [-17, -\frac{\pi}{3}] \rightarrow \mathbb{R}$  by the explicit formula

$$r(x) = x^3, r : [-17, -\frac{\pi}{3}] \rightarrow [-17^3, -(\frac{\pi}{3})^3] \subseteq \mathbb{R}$$

### 2.1 Operation between functions

Suppose  $f_1, f_2$  have the same domain  $A$ , then we can define a new function, say  $g$ , to take the values of the sum of  $f_1$  and  $f_2$  i.e., for  $f_1 : A \rightarrow B$  and  $f_2 : A \rightarrow B$  we define  $g : A \rightarrow B'$  to be

$$g(x) = f_1(x) + f_2(x) \quad \forall x \in A$$

Note that  $B'$  might not be equal to  $B$

**Example.**  $f_1, f_2 : [0, 1] \rightarrow [0, 1]$ ,  $f_1(x) = x$ ,  $f_2(x) = \frac{1}{2}x$ ,  $g(x) = \frac{3}{2}x$  and  $g : [0, 1] \rightarrow [0, \frac{3}{2}]$

For ease of notation, we write  $g$  as  $(f_1 + f_2)$

Similarly, we define the product function  $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x) \forall x \in A$

**Example.**  $f(x) = \log x$  for  $x \geq 1$ ,  $g(x) = 10x^2 \forall x \in \mathbb{R}$  To define  $f + g$  and  $f \cdot g$ , we must to the smaller domain  $\{x \in \mathbb{R} : x \geq 1\}$

## 2.2 Some examples of functions

### Polynomials

**Definition 10.**  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a polynomial function, if  $\exists N \in \mathbb{N}$  and  $\exists \{a_0, \dots, a_N\} \in \mathbb{R}^{N+1}$

$$f(x) = a_0 + a_1x + \dots + a_Nx^N \forall x \in \mathbb{R}$$

### Rational function

**Definition 11.** We say that  $f$  is a rational function if for some polynomial functions  $p : \mathbb{R} \rightarrow \mathbb{R}$  and  $q : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x) = \frac{p(x)}{q(x)} \forall x \in \mathbb{R} \setminus R_q$$

where  $R_q = \{x \in \mathbb{R} : q(x) = 0\}$  is the set of roots of  $q$

### Construct functions

**Definition 12.**  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a constant function if  $\exists c \in \mathbb{R}$  such that  $f(x) = c \forall x \in \mathbb{R}$

### The identity

**Definition 13.** If  $f(x) = x \forall x \in \mathbb{R}$  then we say that  $f$  is the identity map.

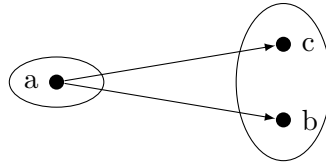
## 2.3 Composition

**Definition 14.** Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be functions. We define the composition  $g \circ f : A \rightarrow C$  by  $g \circ f(x) = g(f(x)) \forall x \in A$



## 2.4 Formal definition

**Definition 15.** A function is a collection of pairs of points with the property if  $(a, b)$  and  $(a, c)$  belong to the collection, the  $b = c$ . The pairs of points are of the form  $(a, f(a))$ . The property in **Definition 15** ensure that we stay clear of a confusion of the sort  $f(2) = 2$  and  $f(2) = 3$ , which would using the diagram representation.



**NOT** a function

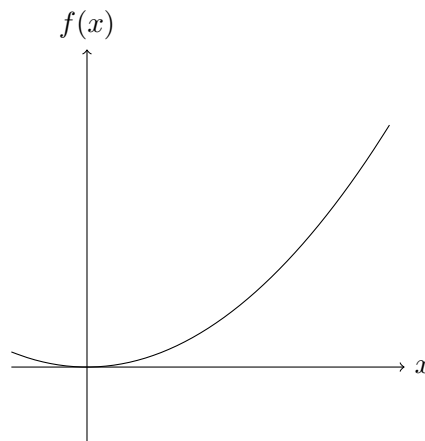
**Definition 16.** Let  $f$  be a function and denote by  $\mathcal{F}$  its collection of points. The domain of  $f$ , written  $\text{dom}(f)$ , is the set of all points  $a$  such that there exists some  $b$  for which  $(a, b) \in \mathcal{F}$ .

i.e.,  $\text{dom}(f) = \{a : \exists b \text{ for which } (a, b) \in \mathcal{F}\}$

Moreover, by **Definition 15** for each  $a \in \text{dom}(f)$  there exists a unique  $b$  such that  $(a, b) \in \mathcal{F}$

## 2.5 Graphs of functions

An intimate way to represent a function is by writing its coordinate pair on curves, i.e., drawing its graph



This diagram is representation of  $\{(x, f(x)), x \in A\}$

**Definition 17.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. We say  $f$  is **linear** if  $\exists a \in \mathbb{R}$  such that

$$f(x) = ax, \forall x \in \mathbb{R}$$

**Definition 18.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. We say  $f$  is **affine** if  $\exists a \in \mathbb{R}$  such that

$$f(x) = ax + b, \forall x \in \mathbb{R}$$

**Definition 19.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. We say  $f$  is **even** if  $\exists a \in \mathbb{R}$  such that

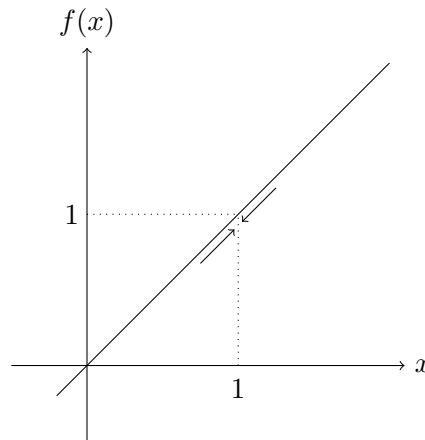
$$f(x) = f(-x), \forall x \in \mathbb{R}$$

**Definition 20.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. We say  $f$  is **odd** if  $\exists a \in \mathbb{R}$  such that

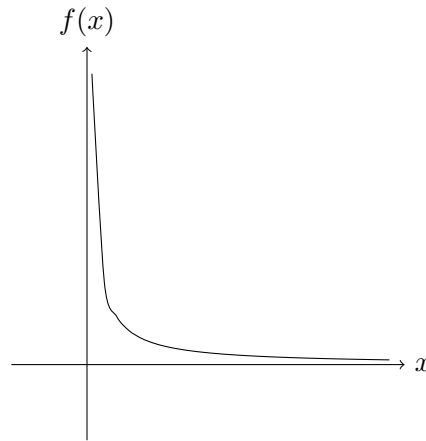
$$f(x) = -f(-x), \forall x \in \mathbb{R}$$

## 2.6 What is limit

What is a limit? Intuitively, a function has a limit at a point  $x_*$  if the function values  $f(x)$  “approach” this limit number as  $x$  gets closer to  $x_*$



if  $f(x) = x \forall x \in \mathbb{R}$  that as  $x$  increases to 1



as  $x \rightarrow \infty$ ,  $f(x)$  goes arbitrary close to 0, as  $x \rightarrow 0$ ,  $f(x)$  “explodes” and has not limit

This idea of a function having a limit is also preserve for more basic objects, e.g., sequence  
e.g., the sequence of points  $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$  where the  $n^{th}$  element of the sequence may be written as  $a_n = 1 - \frac{1}{n}$ , converge to 1 as  $n \rightarrow \infty$

### definition of limit

**Definition 21.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function and let  $a, l \in \mathbb{R}$ . We say that  $f$  approach the limit  $l$  near  $a$  if for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$0 < |x - a| < \delta \implies |f(x) - l| < \varepsilon$$

We write  $\lim_{x \rightarrow a} f(x) = l$

Some comments on **Definition 21**

- (i)  $\delta$  is allowed to depend on  $\varepsilon, a, l$
- (ii) “for all  $\varepsilon > 0$ ” can be read as “given any  $\varepsilon > 0$ ”

**Example.** Let  $f(x) = cx$  for some  $c \in \mathbb{R}$  we show that  $\lim_{x \rightarrow 1} f(x) = c$

*Proof.* let  $\varepsilon > 0$  be given. Then

$$\begin{aligned} |f(x) - c| &= |cx - c| \\ &= |c| \cdot |1 - x| \end{aligned}$$

So, letting  $\delta = \delta(\varepsilon) = |c|^{-1} \cdot \varepsilon$ , we get that

$$0 < |1 - x| < \delta \implies |f(x) - c| < \varepsilon$$

Since this hold for all  $\varepsilon > 0$ , we define  $\lim_{x \rightarrow 1} f(x) = c$

□

**Example.** Let  $g(x) = x \sin(\frac{1}{x})$  for some  $x \in (0, \infty)$ . Then  $\lim_{x \rightarrow 0} g(x) = 0$

*Proof.* Indeed, let  $\varepsilon > 0$  be given. Notice that  $|g(x)| = |x| \cdot |\sin(\frac{1}{x})| \leq |x|$

, thus, letting  $\delta = \delta(\varepsilon) = \varepsilon$ , we see that

$$0 < |x| < \delta \implies |g(x)| < \varepsilon$$

□

**Definition 22.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and let  $l \in \mathbb{R}$ . We say that  $f$  approaches the limit  $l$  as  $x$  tends to infinity if: for all  $\varepsilon > 0$ , there exists  $R > 0$  such that

$$x > R \implies |f(x) - l| < \varepsilon$$

We write  $\lim_{x \rightarrow \infty} f(x) = l$  ( $R$  is allowed to depend on  $\varepsilon, l$ )

**Example.** let  $f(x) = \frac{1}{x}$  for  $x > 0$ . We show that  $\lim_{x \rightarrow \infty} f(x) = 0$

letting  $R(\varepsilon) = \varepsilon^{-1}$ , we see that  $x > R \implies |f(x) - 0| < \varepsilon$

**Definition 23.** Let  $l \in \mathbb{R}$  and  $\{a_n\}_{n \in \mathbb{N}}$  be a sequence of real numbers. We say that  $a_n$  approaches the limit  $l$  as  $n$  tends to infinity if for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$n > N \implies |a_n - l| < \varepsilon$$

Write  $\lim_{n \rightarrow \infty} a_n = l$

**Example.** For the sequence  $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$  where  $a_n = 1 - \frac{1}{n} \forall n \in \mathbb{N}$  we see that  $\lim_{n \rightarrow \infty} a_n = 1$

*Proof.* Indeed, let  $\varepsilon > 0$  be given. Observe that  $|a_n - 1| < \frac{1}{n}$ , letting  $N(\varepsilon) = \lceil \varepsilon^{-1} \rceil$ , we see that, whenever  $n > N$ ,  $n > \varepsilon^{-1} \implies \frac{1}{n} < \varepsilon$  and  $|a_n - 1| < \varepsilon$  for such  $n$  □

What does it mean to not have a limit?

### what is no limit

**Corollary 24.**  $f : \mathbb{R} \rightarrow \mathbb{R}$  does not approach the limit  $l \in \mathbb{R}$  at the point  $a \in \mathbb{R}$  if there exists some  $\varepsilon_0 > 0$  such that for all  $\delta > 0$  there exists  $x_\delta \in \mathbb{R}$  for which there holds

$$|x_\delta - a| < \delta \text{ and } |f(x_\delta) - l| \geq \varepsilon_0$$

**Example.** We show that  $f: (0,1) \rightarrow (0,\infty)$  has no limit at  $x = 0$

*Proof.* We show that  $\forall p \geq 0$ ,  $f$  does not approach the limit  $p$  at  $x = 0$ . Let  $p \geq 0$  be given. We'll show that Corollary 24 holds with  $\varepsilon_0 = 1$ . Note that  $|f(x) - p| = |\frac{1}{x} - p| = \frac{1}{x} - p$  provided  $0 < x \leq \frac{1}{p}$ . Also observe that  $0 < x \leq \frac{1}{p+1} \implies \frac{1}{x} - p \geq p+1 - p = 1$ . This given any  $\delta > 0$ , choosing  $x_\delta = \min\{\frac{\delta}{2}, \frac{1}{p+1}\}$  we get  $0 < x_\delta < \delta$  and by  $|f(x_\delta) - p| \geq 1$ .  $\square$

**Example.** Let  $f: (0,\infty) \rightarrow \mathbb{R}$   $x \mapsto \sin(\frac{1}{x})$ . We show  $f$  does not approach the value 0 as  $x \rightarrow 0$ .

*Proof.* Indeed, for this case set  $\varepsilon_0 = \frac{1}{2}$  and for every  $\delta > 0$ , set  $x_\delta = \frac{1}{\frac{\pi}{2} + 2\pi n_\delta}$  where  $n_\delta \in \mathbb{N}$  chosen sufficiently large such that  $0 < x_\delta < \delta$ . For instance,  $n_\delta = \lceil \frac{\delta^{-1}}{2\pi} \rceil$  clarify that  $x_\delta = \frac{1}{\frac{\pi}{2} + 2\pi n_\delta} < \frac{1}{2\pi n_\delta}$  and

$$\begin{aligned} n_\delta &\geq \frac{\delta^{-1}}{2\pi} \\ 2\pi n_\delta &\geq \delta^{-1} \\ \frac{1}{2\pi n_\delta} &\leq \delta \end{aligned}$$

Then,  $0 < x_\delta < \delta$ , and

$$\begin{aligned} f(x) &= \sin\left(\frac{1}{x_\delta}\right) \\ &= \sin\left(\frac{\pi}{2} + \frac{1}{x_\delta}\right) \\ &= \sin\left(\frac{\pi}{2}\right) = 1 \end{aligned}$$

So,  $|x_\delta - 0| < \delta$  and  $|f(x_\delta) - 0| = 1 > \frac{1}{2} = \varepsilon_0$  (So,  $\lim_{x \rightarrow 0} f(x) \neq 0$ )  $\square$

**Example 25.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

$\lim_{x \rightarrow 0} f(x) = 0$  but  $f$  has no limit at any other point  $a \neq 0$

**Fact** Given  $s < t$  real numbers:

- (i)  $\exists q \in \mathbb{Q}$  such that  $s < q < t$
- (ii)  $\exists r \in \mathbb{R} \setminus \mathbb{Q}$  such that  $s < r < t$

*Proof.* Fix  $a > 0$  and let  $l \in \mathbb{R}$  be arbitrary. There are 2 cases

1. Suppose  $l = 0$  set  $\varepsilon_0 = a$ . Then, given  $\delta > 0$  by Fact(i),  $\exists x_\delta \in \mathbb{Q}$  such that  $a < x_\delta < a + \delta$  and thus  $|x_\delta - a| < \delta$  and  $|f(x_\delta) - l| = x_\delta > a = \varepsilon_0$  so  $f(x) \not\rightarrow 0$  as  $x \rightarrow a$
2. Suppose  $l \neq 0$  set  $\varepsilon_0 = \frac{|l|}{2}$  then given any  $\delta > 0$  by Fact(ii),  $\exists x_\delta \in \mathbb{R} \setminus \mathbb{Q}$  such that  $a < x_\delta < a + \delta$ ,  $|x_\delta - a| < \delta$  and  $|f(x_\delta) - l| = |l| > \frac{|l|}{2} = \varepsilon_0$  repeating the same strategy for  $a < 0$  concludes the proof.

□

## 2.7 Identity of Limit

**Theorem 26.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $a \in \mathbb{R}$ . Suppose that for  $\mu, \nu \in \mathbb{R}$  we have  $\lim_{x \rightarrow a} f(x) = \mu$  and  $\lim_{x \rightarrow a} f(x) = \nu$  then  $\mu = \nu$  (i.e., the limit is unique)

*Proof.* Let  $\varepsilon > 0$  be given. By the definition of the limit  $\exists \delta_1 = \delta_1(\varepsilon, a, \mu) > 0$  such that  $0 < |x - a| < \delta_1 \implies |f(x) - \mu| < \frac{\varepsilon}{2}$  also  $\exists \delta_2 = \delta_2(\varepsilon, a, \nu) > 0$  such that  $0 < |x - a| < \delta_2 \implies |f(x) - \nu| < \frac{\varepsilon}{2}$ . Letting  $\delta = \min\{\delta_1, \delta_2\} > 0$ , we see that  $|\mu - \nu| \leq |\mu - f(x)| + |f(x) - \nu|$ , which provided  $|x - a| < \delta$ . Hence,  $|\mu - \nu| < \varepsilon$  whenever  $|x - a| < \delta$

We will show that  $\mu - \nu = 0$ . Suppose  $\mu - \nu \neq 0$  then  $|\mu - \nu| \geq 0$  but then, choosing  $\varepsilon = \frac{1}{2}|\mu - \nu|$  we get  $|\mu - \nu| < \frac{1}{2}|\mu - \nu|$

□

**Theorem 27.** Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  and  $a \in \mathbb{R}$ . Suppose that for  $\mu, \nu \in \mathbb{R}$ ,  $\lim_{x \rightarrow a} f(x) = \mu$  and  $\lim_{x \rightarrow a} g(x) = \nu$  then

- (a)  $\lim_{x \rightarrow a} (f + g)(x) = \mu + \nu$
- (b)  $\lim_{x \rightarrow a} (f \cdot g)(x) = \mu \cdot \nu$

*Proof.* We will prove each separately

- (a) Let  $\varepsilon > 0$  be given. by the definition of limit,  $\exists \delta_1 = \delta_1(\varepsilon, a, \mu) > 0$  such that  $0 < |x - a| < \delta_1 \implies |f(x) - \mu| < \frac{\varepsilon}{2}$  and  $\exists \delta_2 = \delta_2(\varepsilon, a, \nu) > 0$  such that  $0 < |x - a| < \delta_2 \implies |g(x) - \nu| < \frac{\varepsilon}{2}$ . Let  $\delta = \min\{\delta_1, \delta_2\}$ , provided  $0 < |x - a| < \delta$ ,

and observe that

$$\begin{aligned} |(f+g)(x) - (\mu + \nu)| &= |(f(x) - \mu) + (g(x) - \nu)| \\ &\leq |f(x) - \mu| + |g(x) - \nu| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and  $0 < |x - a| < \delta \implies |(f+g)(x) - (\mu + \nu)| < \varepsilon$

(b) Let  $\varepsilon > 0$  be given, and observe that

$$\begin{aligned} |(f \cdot g)(x) - (\mu\nu)| &= |(f(x)g(x) - \mu g(x)) + (\mu g(x) - \mu\nu)| \\ &\leq |g(x)| \cdot |f(x) - \mu| + |\mu| \cdot |g(x) - \nu| \end{aligned}$$

By the definition of limit  $\exists \delta_g = \delta_g(\varepsilon, a, \nu) > 0$  such that  $|g(x) - \nu| < \min\{\frac{\varepsilon}{2(1+|\mu|)}, 1\}$ , whenever  $0 < |x - a| < \delta_g$ .

Note: whenever  $0 < |x - a| < \delta_g$ , we have

$$(i) \quad |g(x) - \nu| < \frac{\varepsilon}{2(1+|\mu|)} \quad \text{and} \quad |\mu| \cdot |g(x) - \nu| < \frac{\varepsilon}{2}$$

$$(ii) \quad |g(x) - \nu| < 1 \quad \text{and} \quad |g(x)| \leq |g(x) - \nu| + |\nu| < 1 + |\nu|$$

Again, by the definition of limit,  $\exists \delta_f = \delta_f(\varepsilon, a, \mu, \nu) > 0$  such that

$$|x - a| < \delta_f \implies |f(x) - \mu| < \frac{\varepsilon}{2(1+|\nu|)}$$

then, we see that, for  $\delta = \min\{\delta_f, \delta_g\}$  we have

$$|(f \cdot g)(x) - (\mu\nu)| < (1 + |\nu|) \frac{\varepsilon}{2(1+|\nu|)} + \frac{\varepsilon}{2} = \varepsilon$$

□

## 2.8 Infimum / Supremum

Our objective is to give a sense of infimum/supremum as limits. For example, consider  $[1, 2]$ . This set has the property that for every  $x \in [1, 2]$ , there exists a sequence of points  $(x_n)_{n \in \mathbb{N}}$  belonging to  $[1, 2]$  such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . Indeed,  $x \in (1, 2)$ , then for  $M_x > 0$  sufficiently large.  $x_n = x + \frac{1}{n \cdot M_x}$  is such that  $x_n \in (1, 2)$  and  $x_n \rightarrow x$ . And for when  $x \in \{1, 2\}$ , we can build the sequences  $x_n = \frac{1}{100n}$  or  $x_n = 2 - \frac{1}{100n}$ . This property also holds for  $(1, 2)$ , but also even though  $1, 2 \notin (1, 2)$ , there exists sequences  $(y_n)_{n \in \mathbb{N}}$  and  $(z_n)_{n \in \mathbb{N}}$  such that  $y_n, z_n \notin (1, 2) \forall n \in \mathbb{N}$  and  $y_n \rightarrow 1$  as  $n \rightarrow \infty$ ,  $z_n \rightarrow 2$  as  $n \rightarrow \infty$ .

It turns out that the property of “having a sequence inside the set converging to this point” is a property that holds true for the inf and sup of any bounded set.

To this end, we prove the following lemma

**Lemma 28.** Let  $B \subseteq \mathbb{R}$  be a nonempty set bounded above. Then, given any  $\varepsilon > 0$ , there exists some  $b_\varepsilon \in B$  such that

$$\sup B - \varepsilon < b_\varepsilon (\leq \sup B)$$

*Proof.* Let  $\varepsilon > 0$  be given. Denote  $\sup B$  by  $\beta$ . Suppose for contradiction that no such  $b_\varepsilon$  exists. Then for all  $b \in B$ , we must have  $b \leq \beta - \varepsilon$  but then  $\beta - \varepsilon$  is the least upper bound for  $B$   $\square$

An analogous argument prove

**Lemma 29.** Let  $A \subseteq \mathbb{R}$  be a nonempty set bounded below. Then, given any  $\varepsilon > 0$ , there exists some  $a_\varepsilon \in A$  such that

$$(\inf A \leq) a_\varepsilon < \inf A + \varepsilon$$

**Corollary 30.** Let  $A \subseteq \mathbb{R}$  be nonempty and bounded, then,  $\exists (x_n)_{n \in \mathbb{N}}$  and  $\exists (y_n)_{n \in \mathbb{N}}$  for which  $x_n, y_n \in A$  for all  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} x_n = \inf A$ ,  $\lim_{n \rightarrow \infty} y_n = \sup A$

*Proof.* By Lemma 28 for each  $n \in \mathbb{N}$ ,  $\exists y_n \in A$  such that  $\sup A - \frac{1}{n} < y_n \leq \sup A$  and  $|y_n - \sup A| < \frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$ . So,  $\lim_{n \rightarrow \infty} y_n = \sup A$ . Also, for each  $n \in \mathbb{N}$ , by Lemma 29,  $\exists x_n \in A$  such that  $\inf A \leq x_n < \inf A + \frac{1}{n}$ . i.e.,  $|x_n - \inf A| < \frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$ . So,  $\lim_{n \rightarrow \infty} x_n = \inf A$ .  $\square$

**Lemma 31.** Suppose  $A$  is non-empty and bounded below. Let  $B$  be the set of all lower bounds of  $A$ . Then  $\inf A = \sup B$

*Proof.* There are 3 steps

**Step 1** [ $B$  is nonempty] Since  $A$  is bounded below, there exists at least one lower bound, which belongs to  $B$ , so  $B \neq \emptyset$

**Step 2** [ $B$  is bounded above] Suppose for contradiction that  $B$  is not bounded above. Then given any  $n \in \mathbb{N}$ ,  $\exists x_n \in B$  such that  $x_n \geq n$ . Then by the definition of  $B$ ,  $x_n$  is a lower bound for  $A$  for each  $n \in \mathbb{N}$ . Thus given any  $a \in A$ , we have  $a \geq x_n \geq n \forall n \in \mathbb{N}$ . Here  $B$  is bounded above.

**Step 3** [showing the equality]

( $\leq$ ) Let  $\nu = \inf A$  and  $\mu = \sup B$ . Since  $\nu$  is the infimum of  $A$ ,  $\nu$  is a lower bound for  $A$ . So  $\nu \in B \implies \nu \leq \sup B = \mu$



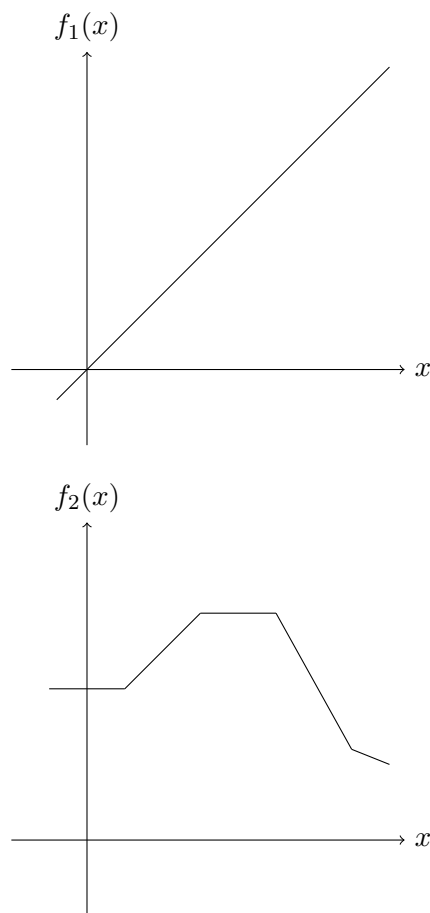
( $\geq$ ) Let  $\varepsilon > 0$  be arbitrary. Then by **Lemma 28**  $\exists b_\varepsilon \in B$  such that  $\mu - \varepsilon < b_\varepsilon \leq \mu$ . Hence,  $\mu < \varepsilon + b_\varepsilon$ . Now, let  $a \in A$  be any point of  $A$  and observe that since  $b_\varepsilon \in B$ ,  $b_\varepsilon \leq a \implies \mu < \varepsilon + b_\varepsilon \leq \varepsilon + a$ . i.e.,  $\mu < \varepsilon + a$  for all  $a \in A$ . i.e.,  $\mu - \varepsilon < a \forall a \in A$ . So,  $\mu - \varepsilon$  is a lower bound for  $A \implies \mu - \varepsilon < \inf A = \nu$  i.e.,  $\mu < \nu + \varepsilon$ , but  $\varepsilon > 0$  was arbitrary  $\implies \mu \leq \nu$

□

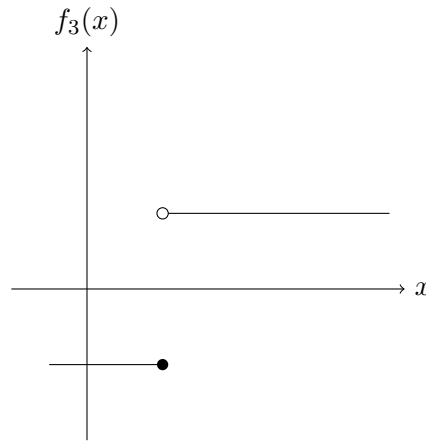
### 3 Continuous Function

What does it mean for a function to be continuous?

Infinitely, this is some smoothness to the function i.g.,



But, on the other hand



is not continuous

### 3.1 Definition of Continuous Function

**Definition 32.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . We say  $f$  is continuous at the point  $x_0 \in \mathbb{R}$  if there holds  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

**Remark.** For  $f$  to be continuous at  $x_0 \in \mathbb{R}$ , we require

- (i)  $\lim_{x \rightarrow x_0} f(x)$  exists
- (ii)  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

Another way of writing Definition 32 is

**Definition (32).**  $f$  is continuous at  $x_0$  if for all  $\varepsilon > 0$ ,  $\exists \delta = \delta(\varepsilon, x_0, f(x_0)) > 0$  such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$$

**Example.**  $f_3$  is not continuous at the point  $x = 1$ .

*Proof.* Indeed, setting  $\varepsilon_0 = 1$ , we see that, given any  $\delta > 0$ , the point  $x_\delta = 1 + \frac{\delta}{2}$  is such that  $|x_\delta - 1| < \delta$  and  $|f(x_\delta) - f(1)| = |1 - (-1)| = 2 > \varepsilon_0$   $\square$

**Example.**  $f(x) = x^2$  is continuous.

*Proof.* Indeed, let  $x_0 \in \mathbb{R}$  be any point and observe that

$$\begin{aligned} |f(x) - f(x_0)| &= |x^2 - x_0^2| \\ &= |(x + x_0)(x - x_0)| \\ &= |x + x_0| \cdot |x - x_0| \end{aligned}$$

Let  $\varepsilon > 0$  be given. Now let  $\delta = \min \left\{ 1, \frac{\varepsilon}{2(1+|x_0|)} \right\}$ , then

$$\begin{aligned} |x + x_0| &= |x - x_0 + 2x_0| \\ &\leq |x - x_0| + 2|x_0| \\ &\leq 1 + 2|x_0| \end{aligned}$$

Then provided  $|x - x_0| < \delta$  we get

$$|f(x) - f(x_0)| \leq (1 + 2|x_0|) \cdot \frac{\varepsilon}{2(1 + |x_0|)} < \varepsilon$$

□

**Example.**

$$f(x) = \begin{cases} 0 & x = 0 \\ x \sin\left(\frac{1}{x}\right) & x \neq 0 \end{cases}$$

$f$  is continuous at  $x = 0$

*Proof.* Indeed, let  $\varepsilon > 0$  be given and observe that

$$\begin{aligned} |f(x) - f(0)| &= |x| \cdot \left| \sin\left(\frac{1}{x}\right) \right| \text{ for } x \neq 0 \\ &\leq |x| \end{aligned}$$

So, letting  $\delta(\varepsilon) = \frac{\varepsilon}{2}$ , we see that

$$|x - 0| < \delta \implies |f(x) - f(0)| \leq \frac{\varepsilon}{2} < \varepsilon$$

□

## 3.2 Identity of Continuous Function

**Lemma 33.** Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be continuous at  $a \in \mathbb{R}$ . Then

- (i)  $f + g$  is continuous at  $a$
- (ii)  $f \cdot g$  is continuous at  $a$

*Proof.* We will prove each separately

(i) let  $\varepsilon > 0$  be given. By the definition of continuous,  $\exists \delta_f = \delta_f(\varepsilon, a) > 0$  such that

$$|x - a| < \delta_f \implies |f(x) - f(a)| < \frac{\varepsilon}{2}$$

and,  $\exists \delta_g = \delta_g(\varepsilon, a) > 0$  such that

$$|x - a| < \delta_g \implies |g(x) - g(a)| < \frac{\varepsilon}{2}$$

So, letting  $\delta = \min\{\delta_f, \delta_g\}$ , suppose  $|x - a| < \delta$ , we see that

$$\begin{aligned} |f(x) + g(x) - (f(a) + g(a))| &\leq |f(x) - f(a)| + |g(x) - g(a)| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

(ii) let  $\varepsilon$  be given. Note that

$$|f(x)g(x) - f(a)g(a)| \leq |g(x)| \cdot |f(x) - f(a)| + |f(a)| \cdot |g(x) - g(a)|$$

Since  $g$  is continuous at  $a$ ,  $\exists \delta_g = \delta_g(\varepsilon, a) > 0$  such that

$$|x - a| < \delta_g \implies |g(x) - g(a)| < \min \left\{ 1, \frac{\varepsilon}{2(1 + |f(a)|)} \right\}$$

Then, provided  $|x - a| < \delta_g$ , we get

$$|g(x)| \leq \overbrace{|g(x) - g(a)|}^{<1} + |g(a)| < 1 + |g(a)|$$

Also, since  $f$  is continuous at  $a$ ,  $\exists \delta_f = \delta_f(\varepsilon, a) > 0$  such that

$$|x - a| < \delta_f \implies |f(x) - f(a)| < \frac{\varepsilon}{2(1 + |g(a)|)}$$

Then, letting  $\delta = \min\{\delta_f, \delta_g\}$ , we see that whenever  $|x - a| < \delta$ , we have from

$$|f(x)g(x) - f(a)g(a)| < (1 + |g(a)|) \left( \frac{\varepsilon}{2(1 + |g(a)|)} \right) + |f(a)| \cdot \frac{\varepsilon}{2(1 + |f(a)|)} < \varepsilon$$

□

**Lemma 34.** Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be continuous at  $a \in \mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous at  $g(a)$ . Then  $f \circ g$  is continuous at  $a$

*Proof.* Let  $\varepsilon > 0$  be given. Since  $f$  is continuous at  $g(a)$ ,  $\exists \delta_f = \delta_f(\varepsilon, a) > 0$  such that

$$|y - g(a)| < \delta_f \implies |f(y) - f(g(a))| < \varepsilon$$

Meanwhile,  $g$  is continuous at  $a$ , so  $\exists \delta_g = \delta_g(\delta_f(\varepsilon, a), a) > 0$  such that

$$|x - a| < \delta_g \implies |g(x) - g(a)| < \delta_f$$

So, letting  $\delta = \delta_g$ , we see that

$$\begin{aligned} |x - a| < \delta &\implies |g(x) - g(a)| < \delta_f \\ &\implies |f(g(x)) - f(g(a))| < \varepsilon \end{aligned}$$

□

**Lemma 35.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous at  $a$ , and suppose  $f(a) > 0$ . Then  $\exists \delta > 0$  such that  $f(x) > 0 \forall x \in [a - \delta, a + \delta]$

*Proof.* Since  $f$  is continuous at  $a$ ,  $\exists \delta_f = \delta_f(a, \overbrace{f(a)}^{\varepsilon}) > 0$  such that

$$|x - a| < \delta_f \implies |f(x) - f(a)| < \overbrace{\frac{1}{2}f(a)}^{\varepsilon}$$

It follows that, for  $x \in (a - \delta_f, a + \delta_f)$ , we have

$$\begin{aligned} f(x) &= (f(x) - f(a)) + f(a) \\ &\geq f(a) - |f(x) - f(a)| \\ &> f(a) - \frac{1}{2}f(a) \\ &= \frac{1}{2}f(a) > 0 \end{aligned}$$

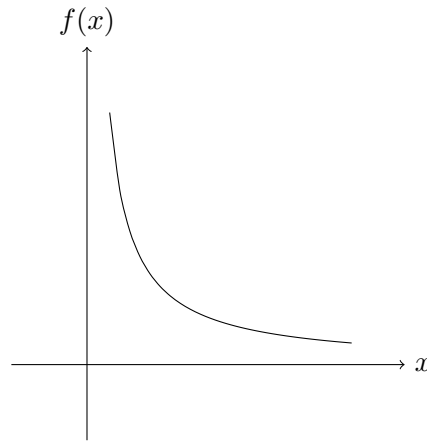
In turn, letting  $\delta = \frac{1}{2}\delta_f$ , we see that  $f(x) > 0 \forall x \in [a - \delta, a + \delta]$

□

### 3.3 Definition of Left/Right Continuity

$f$  continuous on  $(a, b)$  if  $f$  is continuous at  $x$ , for all  $x \in (a, b)$ . What does it mean for  $f$  to be continuous at on  $[a, b]$ ? Should there be a difference between “continuous on  $(a, b)$ ” and “continuous on  $[a, b]$ ”.

To gather intuition, let's look at  $f(x) = \frac{1}{x}$  on  $(0, 1)$  and  $[0, 1]$ .



It's clear that  $f$  is continuous at every point  $a \in (0, 1)$  but  $\lim_{x \rightarrow 0} f(x)$  is not defined. So, it ought to not be continuous on  $[0, 1]$ . We make the following definition

**Definition (32).** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $a < b$  be real numbers.

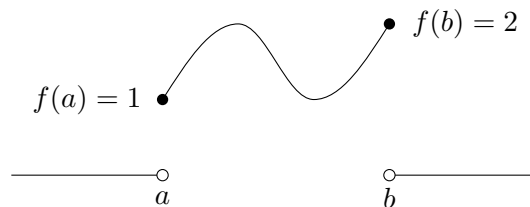
- (i) We say  $f$  is continuous on  $(a, b)$  if  $f$  is continuous at  $x$  for every  $x \in (a, b)$
- (ii) We say  $f$  is continuous on  $[a, b]$  if  $f$  is continuous on  $(a, b)$  and  $\lim_{x \rightarrow a^+} f(x) = f(a)$  and  $\lim_{x \rightarrow b^-} f(x) = f(b)$

We write  $\lim_{x \rightarrow a^+} f(x)$  to mean “The limit  $f$  as  $x$  tends to  $a$  from above” also written  $\lim_{x \searrow a} f(x)$  and  $\lim_{x \rightarrow b^-} f(x)$  to mean “The limit  $f$  as  $x$  tends to  $b$  from below” also written  $\lim_{x \nearrow b} f(x)$

**Definition (32).** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $a \in \mathbb{R}$

- (i) We write  $\mu = \lim_{x \searrow a} f(x)$  if for all  $\varepsilon > 0$ ,  $\exists \delta > 0$  such that whenever  $a < x < a + \delta$  we have  $|\mu - f(x)| < \varepsilon$
- (ii) We write  $\nu = \lim_{x \nearrow a} f(x)$  if for all  $\varepsilon > 0$ ,  $\exists \delta > 0$  such that whenever  $a - \delta < x < a$  we have  $|\nu - f(x)| < \varepsilon$

**Example.** Considered this graph



then,  $\lim_{x \searrow a} f(x) = 1$  and  $\lim_{x \nearrow b} f(x) = 2$  on the other hand  $\lim_{x \nearrow a} f(x) = 0$  and  $\lim_{x \searrow b} f(x) = 0$

**Example.**  $\lim_{x \rightarrow x_0} f(x)$  exists  $\iff \lim_{x \nearrow x_0} f(x)$  and  $\lim_{x \searrow x_0} f(x)$  exists and are equal.

### 3.4 3 Hard Theorems

**Theorem 36** (Intermediate Value Theorem). Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  for  $a < b$ . Suppose  $f(a) < 0 < f(b)$  Then  $\exists \xi \in (a, b)$  such that  $f(\xi) = 0$

**Theorem 37.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  for  $a < b$ . Then  $f$  is bounded above on  $[a, b]$ , i.e.,  $\exists M \in \mathbb{R}$  such that  $f(x) \leq M$   $x \in [a, b]$

**Theorem 38.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous on  $[a, b]$ . Then  $\exists \xi \in [a, b]$  such that  $f(x) \leq f(\xi) \forall x \in [a, b]$  i.e.,  $f(\xi) = \sup\{f(x) : x \in [a, b]\}$  (we say that  $f$  achieves its supremum on  $[a, b]$ )

**Lemma (35').** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $b \in \mathbb{R}$ . Suppose  $\lim_{x \nearrow b} f(x) = f(b) > 0$  Then  $\exists \delta > 0$  such that  $f(x) > 0$  for all  $x \in (b - \delta, b)$

*Proof.* Directly from Definition 32(ii) (definition of  $\lim_{x \nearrow b} f(x)$ ) such that

$$x \in (b - \delta, b) \implies |f(x) - f(b)| < \frac{1}{2}f(b)$$

Then for such  $x \in (b - \delta, b)$  we have

$$\begin{aligned} f(x) &= (f(x) - f(b)) + f(b) \\ &\geq f(b) - \overbrace{|f(x) - f(b)|}^{< \frac{1}{2}f(b)} \\ &> \frac{1}{2}f(b) > 0 \end{aligned}$$

Hence, for  $x \in (b - \frac{\delta}{2}, b)$  we have  $f(x) > 0$  □

**Lemma (35'').** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $b \in \mathbb{R}$ . Suppose  $\lim_{x \searrow a} f(x) = f(a) > 0$  Then  $\exists \delta > 0$  such that  $f(x) > 0$  for all  $x \in (a, a + \delta)$

*Proof Theorem 36.* Define the set  $A = \{x \in [a, b] : f(y) < 0 \forall y \in [a, x]\}$  Since  $f(a) < 0$ , so  $a \in A$ , so  $A \neq \emptyset$  Also, using Lemma 35''  $\exists \delta_1 > 0$  such that  $f(y) < 0 \forall y \in [a, a + \delta_1]$  so  $a + \delta_1 \in A$ , and by Lemma 35'  $\exists \delta_2 > 0$  such that  $f(y) > 0 \forall y \in [b - \delta_2, b]$  where



$b - \delta_2$  is an upper bound for  $A$ . So  $A$  is bounded above and  $\sup A$  is well-defined. Let  $\alpha = \sup A$ . We already know that  $\alpha \in (a, b)$  our aim is to show that  $f(\alpha) \neq 0$ . We proceed by contradiction:

Suppose for contradiction that  $f(\alpha) \neq 0$ . There are 2 possibilities

(i)  $f(\alpha) < 0$

(ii)  $f(\alpha) > 0$

Suppose (i) holds. Since  $\alpha \in (a, b)$  and  $f(\alpha) < 0$  by **Lemma 35**,  $\exists \delta_3 > 0$  such that  $f(y) < 0 \forall y \in [\alpha - \delta_3, \alpha + \delta_3]$ . But then  $\alpha + \delta_3 \in A$  and  $\alpha + \delta_3 > \alpha$ .

Suppose (ii) holds. Then since  $\alpha \in (a, b)$ ,  $f(\alpha) > 0$  and  $f$  is continuous. By **Lemma 35**,  $\exists \delta_4 > 0$  such that  $f(x) > 0 \forall x \in [\alpha - \delta_4, \alpha + \delta_4]$ . But then  $\alpha = \sup A$  by **Lemma 28**  $\exists x_0 \in A$  such that  $\alpha - \frac{\delta_4}{2} < x_0$ . Thus  $x_0 \in (\alpha - \frac{\delta_4}{2}, \alpha) \subseteq [\alpha - \delta_4, \alpha + \delta_4] \implies f(x_0) > 0$ . But  $x_0 \in A$  so  $f(x_0) < 0$ .  $\square$

**Corollary 39.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and let  $c \in \mathbb{R}$ . Suppose  $f(a) < c < f(b)$ . Then  $\exists \xi \in (a, b)$  such that  $f(\xi) = c$ .

*Proof.* Define  $g(x) = f(x) - c$  and apply **Theorem 36** to  $g$ .  $\square$

**Example 40.** Let  $f(x) = x^4 + x - 3 \forall x \in \mathbb{R}$ . **Fact:** all polynomials are continuous  $\forall x \in \mathbb{R}$ . A nice application of the Intermediate Value Theorem is to find roots of continuous functions. We can see by plugging in that

$$f(1) = 1 + (-1) - 3 = -3$$

$$f(2) = 16 + 2 - 3 = 15$$

IVT  $\implies \exists x_0 \in (1, 2)$  such that  $f(x_0) = 0$ . This at least lets us estimate where roots are.

**Example 41.** Let  $f(x) = x^4 + x - 3 + \tan\left(\frac{x}{2}\right)$  (continuous on  $(-\pi, \pi)$ )

$$f(-1) = -3 - \tan\left(\frac{1}{2}\right) < 0$$

$$f(2) = 15 - \tan\left(\frac{1}{2}\right) > 0$$

IVT  $\implies \exists x_0 \in (-1, 2)$  such that  $f(x_0) = 0$ .

What is it useful for? If we look at the set  $f([a, b]) = \{f(x) : x \in [a, b]\}$  and **Theorem 37** tell us that set is bounded. Since the set is bounded, it has a supremum. You can think of this as “local max” of  $f$  on the interval  $[a, b]$ .

Before proving **Theorem 37**, let's look at one of its consequences.

**Corollary 42.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous on  $[a, b]$ . Then  $f$  is bounded below on  $[a, b]$ , i.e.,  $\exists m \in \mathbb{R}$  such that  $m \leq f(x) \forall x \in [a, b]$

*Proof.* Since  $f$  is continuous, so is  $(-f)$ . Now apply Theorem 37 to  $-f$ .  $\exists M \in \mathbb{R}$  such that  $-f(x) \leq M \forall x \in [a, b]$  then,  $f(x) \leq -M \forall x \in [a, b]$   $\square$

**Takeaway:** If  $f$  is continuous on  $[a, b]$ , then  $f$  is bounded above + below on  $[a, b]$

To prove Theorem 37, we'll need a few Lemmas.

**Lemma 43.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous at  $a \in \mathbb{R}$ , then  $\exists \delta > 0$  such that  $f$  is bounded above on the interval  $[a - \delta, a + \delta]$

*Proof.* Since  $f$  is continuous at  $a$ ,  $\exists \delta = \delta(a, \overbrace{1}^{\varepsilon})$  such that  $|x - a| < \delta \implies |f(x) - f(a)| < 1$  This for such  $x$  we have

$$\begin{aligned} f(x) &= f(x) - f(a) + f(a) \\ &\leq |f(x) - f(a)| + |f(a)| \\ &< 1 + |f(a)| \end{aligned}$$

For  $x$  satisfying  $|x - a| < \delta$ , we have  $f(x) < 1 + f(a)$ .

In particular,  $f(x) < 1 + f(a) \forall x \in [a - \frac{\delta}{2}, a + \frac{\delta}{2}]$   $\square$

**Lemma.** (43') Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function and  $b \in \mathbb{R}$ . Suppose  $\lim_{x \nearrow b} f(x) = f(b)$ . Then  $\exists \delta > 0$  such that  $f$  is bounded above on  $[b - \delta, b]$

*Proof.* By Definition 32'',  $\exists \delta = \delta(b, 1)$  such that

$$0 < |x - b| < \delta \implies |f(x) - f(b)| < 1$$

Therefore, for such  $x$ ,

$$\begin{aligned} f(x) &= f(x) - f(b) + f(b) \\ &\leq |f(x) - f(b)| + |f(b)| \\ &< 1 + |f(b)| \end{aligned}$$

$f(x) < f(b) + 1 \forall x \in [b - \frac{\delta}{2}, b]$   $\square$

**Lemma.** (43'') Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function and  $a \in \mathbb{R}$ . Suppose  $\lim_{x \searrow a} f(x) = f(a)$ . Then  $\exists \delta > 0$  such that  $f$  is bounded above on  $[a, a + \delta]$

*Proof Theorem 37.* As in the proof of Theorem 36, consider the set

$$A = \{x \in [a, b] : f \text{ is bounded above on } [a, x]\}$$

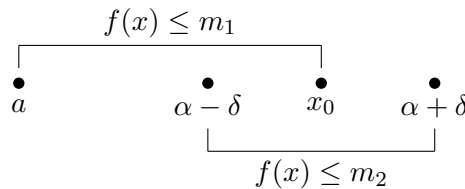
Since  $a \in A$ , we know  $a \neq \emptyset$ . Moreover, the point  $b$  is an upper bound for  $A$ , so  $\sup A = \alpha$  exists.

Our objective is to show that  $\alpha = b$ .

Suppose for contradiction that  $\alpha < b$ . (Note that we must have  $a < \alpha$ . We can't have  $a > \alpha$  since  $a \in A$ . and  $\sup A \geq a$ . If  $\alpha = a$ , then  $A = \{a\}$ , but we know from Lemma 43'' that  $\exists \delta > 0$  such that  $[a, a + \delta] \subseteq A$ )

By assumption  $a < \alpha < b$  and so Lemma 43  $\implies \exists \delta > 0$  such that  $f$  is bounded on  $[\alpha - \delta, \alpha + \delta]$ . Let's say  $f(x) \leq m_2$  on this interval  $[\alpha - \delta, \alpha + \delta]$ .

By Lemma 28 (Alternate definition of supremum)  $\exists x_0 \in A$  such that  $\alpha - \delta < x_0 \leq \alpha$ .  $f$  is bounded above on  $[a, x_0]$  (by the definition of  $A$ ). say  $f(x) \leq m_1$  on  $[a, x_0]$



Thus,  $f(x) \leq \max\{m_1, m_2\} \forall x \in [a, \alpha + \delta]$  We deduce that  $\alpha + \delta \in A$  and  $\alpha + \delta > \alpha = \sup A$ . Hence,

$$\begin{aligned} \alpha = b &\iff \sup A = b \\ &\implies f \text{ is bounded above on } [a, b] \text{ for every } x < b \quad \textcircled{1} \end{aligned}$$

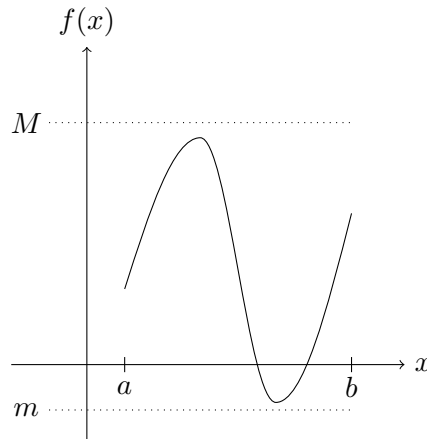
Finally, using continuity at the point  $b$  by Lemma 43'  $\exists \delta' > 0$  such that  $f$  is bounded on  $[b - \delta', b]$   $\textcircled{2}$ .

Hence, choosing  $x = b - \delta'$  in  $\textcircled{1}$ ,  $\exists M$  such that  $f(x) \leq M, \forall x \in [a, b - \delta']$ . and by  $\textcircled{2}$ ,  $\exists M_2$  such that  $f(x) \leq M_2, \forall x \in [b - \delta', b]$ . So,  $f(x) \leq \max\{M, M_2\} \forall x \in [a, b]$ .  $\square$

Summarize steps:

- (i) define a good set  $A$
- (ii) show  $b = \sup A$
- (iii) show  $b \in A$

The picture is



Whenever  $f$  is continuous on  $[a, b]$ ,  $\exists M > m$  such that  $m \leq f(x) \leq M \forall x \in [a, b]$

**Note:** We must be careful about being continuous on  $[a, b]$ , and not just  $(a, b)$ . Indeed,  $f: (0, 1) \rightarrow (0, \infty)$ ,  $f(x) = \frac{1}{x}$ ,  $f$  is continuous on  $(\tilde{x}, \infty)$  for every  $\tilde{x} > 0$ , but it is not continuous on  $[0, \infty)$ .

**Question:** does there exist  $\xi_1, \xi_2 \in [a, b]$  such that

$$f(\xi_1) = \inf_{[a,b]} f \text{ and } f(\xi_2) = \sup_{[a,b]} f$$

**Answer:** Yes

Later on, when we discuss differentiability, if sup/inf is achieved in  $(a, b)$ , then  $f' = 0$  at such points. This we will prove later.

*Proof of Theorem 38.* We already know from Theorem 37 that  $f$  is bounded on  $[a, b]$ , i.e., the set  $B = f([a, b]) = \{f(x) : x \in [a, b]\}$  is bounded. This set is nonempty and so  $\beta = \sup B$  is well-defined; Since  $\beta \geq f(x) \forall x \in [a, b]$  it suffices to show that  $\exists \xi \in [a, b]$  such that  $f(\xi) = \beta$ .

Suppose for contradiction that this is not the case, i.e.,  $\beta \neq f(y) \forall y \in [a, b]$ . Then the function  $g : [a, b] \rightarrow \mathbb{R}$ , defined by  $g(x) = \frac{1}{\beta - f(x)} \forall x \in [a, b]$ , is well-defined and  $g$  is continuous on  $[a, b]$  by virtue of Lemma 33

Since  $g$  is continuous, by Theorem 37  $\implies g$  is bounded above on  $[a, b]$ . However, by Lemma 28, given any  $n \in \mathbb{N}$ ,  $\exists x_n \in [a, b]$  such that

$$\beta - \frac{1}{n} < f(x_n) \leq \beta \implies g(x_n) \geq \frac{1}{\beta - (\beta - \frac{1}{n})} = n$$

Hence given any  $n \in \mathbb{N}$ ,  $\exists x_n \in [a, b]$  such that  $g(x_n) \geq n$  and therefore  $g$  is unbounded on  $[a, b]$ .  $\square$

We've actually proved

**Corollary 44.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous on  $[a, b]$ . Then  $\exists \xi \in [a, b]$  such that  $f(\xi) = \sup\{f(x) : x \in [a, b]\}$  (we often write with the shorthand  $\sup_{[a,b]} f$ )

**Corollary 45.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous on  $[a, b]$ . Then  $\exists \xi \in [a, b]$  such that  $f(\xi) = \inf\{f(x) : x \in [a, b]\}$

*Proof.* Apply Corollary 44 to the function  $-f$  and use the result  $\inf B = -\sup(-B)$ .  $\square$

### 3.5 Usage of 3 Hard Theorem

**Example 46.** Suppose  $f, g$  are continuous on  $[a, b]$  and  $f(a) < g(a)$  and  $f(b) > g(b)$ . Then  $\exists x \in [a, b]$  such that  $f(x) = g(x)$  (in actual fact,  $x \in (a, b)$ )

*Proof.* define  $h(x) = f(x) - g(x)$ . Then  $h$  is continuous on  $[a, b]$ ,  $h(a) < 0 < h(b)$  so from Theorem 36,  $\exists \xi \in (a, b)$  such that  $h(\xi) = 0 \implies f(\xi) = g(\xi)$   $\square$

**Example 47.** Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous on  $[0, 1]$  and suppose  $0 \leq f(x) \leq 1 \forall x \in [0, 1]$ . Then  $\exists x_0 \in [0, 1]$  such that  $f(x_0) = x_0$  (we can imagine that  $f$  cross  $y = x$ )

*Proof.* Note that if  $f(0) = 0$  or if  $f(1) = 1$ , then we are done. Suppose that  $f(0) \neq 0$  and  $f(1) \neq 1$  then  $0 < f(0)$  and  $f(1) < 1$ . Let  $g(x) = x - f(x)$ . Then,  $g(0) = 0 - f(0) < 0$  and  $g(1) = 1 - f(1) > 0$ . So,  $g$  is continuous and  $g(0) < 0 < g(1)$ , where Theorem 36  $\exists x_0 \in [0, 1]$  such that  $g(x_0) = 0$  and hence  $x_0 = f(x_0)$   $\square$

**Example 48.** There are 3 sub-examples here:

- (a) Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $|f(x)| \leq |x|$  for all  $x \in \mathbb{R}$ . Then  $f$  is continuous at 0
- (b) There exists a function which satisfies the assumption of a.) but is not continuous at any other points other than  $x = 0$
- (c) Suppose  $g$  is continuous at 0 and  $g(0) = 0$  and suppose  $|f(x)| \leq |g(x)| \forall x \in \mathbb{R}$ . Then  $f$  is continuous at 0.

*Proof.* We will prove each separately:

- (a) The inequality implies  $f(0) = 0$ . Let  $\varepsilon > 0$  be given, then the inequality show that

$$|f(x) - f(0)| = |f(x)| \leq |x - 0|$$

so letting  $\delta = \varepsilon$ , we see that

$$|x - 0| < \delta \implies |f(x) - f(0)| < \varepsilon$$

so  $f$  is continuous at 0

- (b)

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then  $|f(x)| \leq |x| \forall x$  but  $f$  is not continuous at any points other than 0

- (c) Since  $g(0) = 0$ , we immediately get  $f(0) = 0$ . Let  $\varepsilon > 0$  be given. Since  $g$  is continuous at 0,  $\exists \delta = \delta(\varepsilon, 0) > 0$  such that

$$|x - 0| < \delta \implies |g(x) - g(0)| \leq \varepsilon$$

but then, in view of the bound  $|f(x)| \leq |g(x)| \forall x$ , we see that

$$|x - 0| < \delta \implies |f(x) - f(0)| = |f(x)| \leq |g(x)| = |g(x) - g(0)| < \varepsilon$$

□

**Example 49.** This exercise is here to help us gain more familiarity with limits– it's not concern with continuous functions per se.

- (i) Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  and suppose  $f(x) \leq g(x) \forall x \in \mathbb{R}$  and suppose  $\mu := \lim_{x \rightarrow a} f(x), \nu := \lim_{x \rightarrow a} g(x)$  Show that  $\mu \leq \nu$
- (ii) Now suppose  $f(x) < g(x) \forall x \in \mathbb{R}$ . Does this guarantee  $\mu < \nu$ ?

*Proof.* We will prove each separately:

- (i) Let  $\varepsilon > 0$  be given. Then  $\exists \delta_1 = \delta_1(\varepsilon, a, \mu) > 0$  and  $\exists \delta_2 = \delta_2(\varepsilon, a, \nu) > 0$  such that

$$|x - a| < \delta_1 \implies |f(x) - \mu| < \frac{\varepsilon}{2},$$

$$|x - a| < \delta_2 \implies |g(x) - \nu| < \frac{\varepsilon}{2}$$

Set  $\delta := \min(\delta_1, \delta_2)$  Then, provided  $|x - a| < \delta$ , we have

$$\begin{aligned} \nu - \mu &= (\nu - g(x)) + (g(x) - f(x)) + (f(x) - \mu) \\ &\geq \underbrace{g(x) - f(x)}_{\geq 0} - \underbrace{|\nu - g(x)|}_{< \frac{\varepsilon}{2}} - \underbrace{|\mu - f(x)|}_{< \frac{\varepsilon}{2}} \\ &> -\varepsilon \end{aligned}$$

So,  $\nu - \mu > -\varepsilon$  for all  $\varepsilon > 0 \implies \nu - \mu \geq 0$

- (ii) NO: Suppose  $f(x) = 0$  and  $g(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ \frac{1}{x} & \text{if } |x| \geq 1 \end{cases}$

Then  $\lim_{x \rightarrow \infty} f(x) = 0$  and  $\lim_{x \rightarrow \infty} g(x) = 0$

□

**Example 50.** Let  $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & x = 0 \end{cases}$

- (a) Show that  $f$  is not continuous on  $[-1, 1]$   
 (b) Show that  $f$  satisfies the conclusion of Theorem 36 (IVT)

*Proof.*

- (a) for every  $\delta > 0$ ,  $n_\delta := \max\left(\left\lceil \frac{1}{2\pi}\delta^{-1} \right\rceil, 1\right) \in \mathbb{N}$  such that

$$\frac{1}{\frac{\pi}{2} + 2\pi n_\delta} < \delta \text{ and } x_\delta := \frac{1}{\frac{\pi}{2} + 2\pi n_\delta}$$

we get  $0 < x_\delta < \delta$  and

$$|f(x_\delta) - f(0)| = \left| \sin\left(\frac{\pi}{2} + 2\pi n_\delta\right) \right| = 1$$

so, for all  $\delta > 0$ ,  $\exists x_\delta$  such that  $0 < x_\delta < \delta$  and  $|f(x_\delta) - f(0)| = 1$ , so  $f$  is not continuous at 0.

- (b)  $f$  is not continuous at 0, however  $f$  is continuous on  $(-1, 0)$  and on  $(0, 1]$  and so Theorem 36 holds on any interval of the form  $[-1, y]$  and  $[x, 1]$  for  $y < 0$  and  $x > 0$

It remains to check that

\*Suppose  $a > 0$  and  $f(a) > 0$ . Then, for every  $c \in [0, f(a)]$ ,  $\exists \xi_c \in [0, a]$  such that  $f(\xi_c) = c$

Note that  $f(a) \leq 1$ , Indeed  $\xi = \frac{1}{\arcsin(c)}$  is such that

$$\begin{aligned} f(\xi) &= c \\ \sin\left(\frac{1}{\xi}\right) &= \sin(\arcsin(c)) \end{aligned}$$

So the only remaining issue is that we do not necessarily have  $\xi \in [0, a]$ .

To this end, notice that, for every  $N \in \mathbb{N}$ ,  $\xi = \frac{1}{2\pi N + \arcsin(c)}$  also satisfies  $f(\xi) = c$  and hence, choosing  $N$  sufficiently large such that  $\frac{1}{2\pi N + \arcsin(c)} \leq a$ , we have that  $\xi = \frac{1}{2\pi N + \arcsin(c)}$  is a point that verifies \*

□

**Example 51.** Suppose  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  are continuous, and  $f(x)^2 = g(x)^2 \forall x \in \mathbb{R}$  and  $f(x) \neq 0$ . Then either

$$(i) \ f(x) = g(x) \ \forall x \in \mathbb{R}$$

$$(ii) \ f(x) = -g(x) \ \forall x \in \mathbb{R}$$

i.e.,  $f$  cannot ‘jump’ between  $\pm g$ .

*Proof.* Suppose for contradiction that  $\exists a, b \in \mathbb{R}$  such that  $f(a) = g(a)$  and  $f(b) = -g(b)$   $\circledast$  and wlog (without loss of generality), assume  $a < b$ . Since  $f(x) \neq 0 \forall x$ , we also assume wlog  $f(a) < 0$ . Then it can’t be the case that  $f(b) > 0$ . Indeed, if this were the case, then by Theorem 36,  $\exists \xi \in (a, b)$  such that  $f(\xi) = 0$ , which contradicts  $f(x) \neq 0 \forall x$ .

Hence  $f(a) < 0$  and  $f(b) < 0$ .

Then,  $\circledast \implies g(a) < 0$  and  $g(b) > 0$ , so Theorem 36  $\implies \exists \zeta \in (a, b)$  such that  $g(\zeta) = 0$ . But then  $f(\zeta) = 0$ , which is again a contradiction. □

**Example 52.** Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and such that  $f(x)^2 = x^2 \forall x \in \mathbb{R}$ . Then, either  $f(x) = x \forall x \in \mathbb{R}$ , or  $f(x) = -x \forall x \in \mathbb{R}$ , or  $f(x) = |x| \forall x \in \mathbb{R}$ .

*Proof.* It suffices to show that

$$(A) \text{ for } x < 0, \text{ either: } f(x) = x \ \forall x < 0, \text{ or } f(x) = -x \ \forall x < 0$$

$$(B) \text{ for } x > 0, \text{ either: } f(x) = x \ \forall x > 0, \text{ or } f(x) = -x \ \forall x > 0$$

We only prove (B), as the proof for (A) is identical.

Suppose for contradiction  $\exists 0 < a < b$  such that (wlog)  $f(a) = -a$  and  $f(b) = b$ . Then, observe that  $f(a) < 0$ , while  $f(b) > 0$ .

Thus, Theorem 36  $\implies \exists \xi \in (a, b)$  such that  $f(\xi) = 0$ . But,  $(f(\xi))^2 = \xi^2 > a^2 > 0$  □

**Example 53.** Suppose  $f$  is continuous on  $[a, b]$  and  $f(x) \in \mathbb{Q} \forall x \in [a, b]$ . Then,  $f$  is a constant function, i.e.,  $\exists q \in \mathbb{Q}$  such that  $f(x) = q \forall x \in [a, b]$ .

*Proof.* Suppose for contradiction that  $f$  is not constant, i.e.,  $\exists a, b \in \mathbb{R}$  such that  $f(a) < f(b)$  and wlog  $a < b$ . Since between any 2 real numbers, there exists an irrational number, it follows that there exists  $c \in \mathbb{R} \setminus \mathbb{Q}$  such that  $f(a) < c < f(b)$ .

Then, from IVT,  $\exists \xi_c \in (a, b)$  such that  $f(\xi_c) = c \in \mathbb{R} \setminus \mathbb{Q}$ . □



**Example 54.** Suppose  $f$  is continuous on  $[0, 1]$  and  $f(0) = f(1)$ . Let  $n \in \mathbb{N}$  be arbitrary. Then,  $\exists x_* \in [0, 1]$  such that  $f(x_*) = f(x_* + \frac{1}{n})$ .

*Proof.* Define  $g : [0, 1 - \frac{1}{n}] \rightarrow \mathbb{R}$  by  $g(x) := f(x) - f(x + \frac{1}{n})$ .

Suppose for contradiction that  $g(x) \neq 0 \forall x \in [0, 1 - \frac{1}{n}]$ . By cty (using Theorem 36), we must have either  $g(x) > 0$  or  $g(x) < 0 \forall x \in [0, 1 - \frac{1}{n}]$ .

Wlog, assume  $g(x) > 0 \forall x \in [0, 1 - \frac{1}{n}]$ . Then,  $f(x) > f(x + \frac{1}{n}) \forall x \in [0, 1 - \frac{1}{n}]$ . It follows that, by setting  $x = 0$ ,  $f(0) > f(\frac{1}{n})$ , but also by setting  $x = \frac{1}{n}$ ,

$$\begin{aligned} f\left(\frac{1}{n}\right) &> f\left(\frac{2}{n}\right), \dots, f\left(\frac{m}{n}\right) > f\left(\frac{m+1}{n}\right) \quad \forall m \in \left\{0, \dots, \frac{n-1}{n}\right\} \\ \implies f(0) &> f\left(\frac{1}{n}\right) > f\left(\frac{2}{n}\right), \dots, f\left(\frac{n-1}{n}\right) > f(1) \\ &\implies f(0) > f(1) \end{aligned}$$

but we assumed  $f(0) = f(1)$ , which is a contradiction.  $\square$

**Example 55.** Suppose  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $n \in \mathbb{N}$ , and  $\lim_{x \rightarrow \infty} \frac{\phi(x)}{x^n} = 0 = \lim_{x \rightarrow -\infty} \frac{\phi(x)}{x^n}$ . Then,

- (a) if  $n$  is odd,  $\exists x_* \in \mathbb{R}$  such that  $(x_*)^n + \phi(x_*) = 0$
- (b) if  $n$  is even,  $\exists y \in \mathbb{R}$  such that  $(y)^n + \phi(y) \leq x^n + \phi(x) \forall x \in \mathbb{R}$

*Proof.* Define  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  by  $\psi(x) := x^n + \phi(x) \forall x \in \mathbb{R}$  and note that  $\psi$  is also continuous on  $\mathbb{R}$ .

- (a) Since  $n$  is odd,  $\lim_{x \rightarrow -\infty} \frac{\psi(x)}{|x|^n} = -1 + \underbrace{\lim_{x \rightarrow -\infty} \frac{\phi(x)}{|x|^n}}_{=0}$  and similarly  $\lim_{x \rightarrow \infty} \frac{\psi(x)}{|x|^n} = 1$ .

Note that  $x \mapsto \frac{\psi(x)}{|x|^n}$  is continuous on any interval excluding 0.

Then, since  $\frac{\psi(x)}{|x|^n}$  is continuous on  $(-\infty, 0)$ ,  $\exists R_1 = R_1(\frac{1}{2}) > 0$  such that

$$x < -R_1 \implies \left| \frac{\psi(x)}{|x|^n} - (-1) \right| < \frac{1}{2}$$

i.e., for  $x < -R_1$ , we have  $\frac{\psi(x)}{|x|^n} < (-1) + \frac{1}{2} = -\frac{1}{2}$ .

$$\implies \psi(x) < -\frac{1}{2}|x|^n \quad \forall x \in \mathbb{R}$$

i.e., for all  $x < -R_1$ , we have  $\psi(x) < 0$   $\circledast$ .

Similarly,  $\exists R_2 = R_2(\frac{1}{2}) > 0$  such that

$$\begin{aligned} x > R_2 &\implies \left| \frac{\psi(x)}{|x|^n} - 1 \right| < \frac{1}{2} \\ &\implies \psi(x) > \frac{1}{2}|x|^n \quad \forall x > R_2 \end{aligned}$$

Therefore,  $\psi(x) > 0$  for all  $x > R_2$   $\circledast$   $\circledast$ .

By  $\circledast$  and  $\circledast\circledast, \exists a, b \in \mathbb{R}$  ( $a < b$ ) such that

$$\psi(a) < 0 < \psi(b)$$

Then since  $\psi$  is continuous, by Theorem 36  $\implies \exists x_* \in (a, b)$  such that  $\psi(x_*) = 0$ , i.e.,  $x_*^n + \phi(x_*) = 0$ .

□

**Example 56.**

**Example 57.**

**Example 58.**

**Example 59.** Suppose  $f$  is continuous and  $\circledast \lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow \infty} f(x) = 0$ , and  $f(x) > 0 \quad \forall x \in \mathbb{R}$ . Then,  $\exists x_* \in \mathbb{R}$  such that  $f(x) \leq f(x_*) \quad \forall x \in \mathbb{R}$ .

*Proof.* Let  $\mu := \max_{y \in [-1, 1]} f(y)$ , by  $\circledast$ ,  $\exists R_1, R_2 > 0$  such that

$$x < -R_1 \implies 0 < f(x) < \frac{1}{2}\mu$$

$$x > R_2 \implies 0 < f(x) < \frac{1}{2}\mu$$

Hence  $0 < f(x) < \frac{1}{2}\mu$  for all  $|x| \in \mathbb{R} := \max\{R_1, R_2\}$ . and meanwhile  $\sup_{x \in \mathbb{R}} f(x) \geq$

$$\sup_{x \in [-1, 1]} f(x) = \mu.$$

$\sup_{x \in \mathbb{R}} f(x)$  is well-defined Since  $\sup_{[-R, -R]} f$  is well-defined and achieved by Theorem and  $|f(x)| < \frac{1}{2}\mu$  for  $|x| > R$ .

$$+\infty > \sup_{x \in \mathbb{R}} f(x) \geq \max_{x \in [-R, R]} f(x) \geq \mu > \sup_{|x| > R} f(x)$$

It follows that  $\sup_{x \in \mathbb{R}} f(x) = \sup_{x \in [-R, R]} f(x)$  ( $\mathbb{R} = \underbrace{\{x : |x| \leq R\}}_{=[-R, R]} \cup \{x : |x| > R\}$ )

Since  $f$  is continuous, it achieves its bounds by Theorem 38  $\implies \exists x_* \in [-R, R]$  such that  $f(x_*) = \sup_{[-R, R]} f = \sup_{\mathbb{R}} f$ .  $\square$

**Example 60.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = (\sin x)^2 + (\sin(x + (\cos x)^7))^2$$

Then,  $\exists c > 0$  such that  $f(x) \geq c \forall x \in \mathbb{R}$ .

*Proof.* Observe that  $f(x) \geq 0$  for all  $x$  and  $A := \{f(x) : x \in \mathbb{R}\}$  is bounded below by 0.

Define  $c := \inf A$  is well-defined.

$$\begin{aligned} f(x + 2\pi) &= (\sin(x + 2\pi))^2 + \sin(\overbrace{(x + 2\pi) + (\cos(x + 2\pi))^7}^{=x+2\pi+(\cos x)^7})^2 \\ &= (\sin x)^2 + \sin(x + (\cos x)^7)^2 \\ &= f(x) \end{aligned}$$

$f$  is  $2\pi$ -periodic,  $\implies c = \inf A = \inf\{f(x) : x \in [0, 2\pi]\}$

Since  $f$  is continuous, Theorem 38  $\implies \exists x_* \in [0, 2\pi]$  such that  $f(x_*) = c$ .

Suppose for contradiction that  $c = 0$

$$\begin{aligned} &\implies f(x_*) = 0 \\ &\implies \underbrace{(\sin x_*)^2}_{=0} + \underbrace{(\sin(x_* + (\cos x_*)^7))^2}_{=0} = 0 \\ &\implies x_* \in \{0, \pi, 2\pi\} \text{ but then } \cos x_* \in \{1, -1\} \\ &\implies x_* + (\cos x_*)^7 \in \{1, \pi - 1, 2\pi + 1\} \\ &\implies \sin(x_* + (\cos x_*)^7) \in \{\sin(1), \sin(\pi - 1)\} \text{ neither of which are } 0 \end{aligned}$$

$\square$

### 3.6 Uniform Continuity

Finally, we look at uniform continuity

**Definition 61.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . We say  $f$  is uniformly continuous on an interval  $A$  if for all  $\varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$  such that

$$|x - y| < \delta \text{ and } x, y \in A \implies |f(x) - f(y)| < \varepsilon$$

**KEY:**  $\delta$  is not depend on a specific point.

**Example.**  $f(x) = x$  is uniformly continuous on  $\mathbb{R}$ . Let  $\varepsilon > 0$  be given then letting  $\delta = \varepsilon$ , we see that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

**Example.**  $f(x) = x^2$  is not uniformly continuous on  $\mathbb{R}$ .

Fix  $\varepsilon > 0$  and recall from Lecture 10 that

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0||x + x_0|$$

and so we need  $\delta = \min\left(1, \frac{\varepsilon}{1+2|x_0|}\right)$  to have  $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$ .

We see that  $\delta$  depends on specific point  $x_0$ .

This is only an indication that  $f$  is not uniformly continuous – not a proof yet.

The negation of Definition 61

**Definition (61').**  $\exists \varepsilon_0 > 0$  such that for all  $\delta > 0$  there exist corresponding  $x_\delta, y_\delta \in A$  such that

$$|x_\delta - y_\delta| < \delta \text{ and } |f(x_\delta) - f(y_\delta)| \geq \varepsilon_0$$

*Proof of Example .* Let  $\varepsilon_0 = 1$ . Observe that for  $x > y > 0$ ,

$$|f(x) - f(y)| = x^2 - y^2 = (x + y)(x - y)$$

For each  $\delta > 0$  choose  $y_\delta = \delta^{-1}$  and  $x_\delta = \delta^{-1} + \frac{\delta}{2}$

Then,  $x_\delta + y_\delta = 2\delta^{-1} + \frac{\delta}{2} > 2\delta^{-1}$  and  $|x_\delta - y_\delta| = \frac{\delta}{2} < \delta$ .

Hence,  $|x_\delta - y_\delta| < \delta$  and also

$$\begin{aligned} |f(x_\delta) - f(y_\delta)| &= (x_\delta + y_\delta)(x_\delta - y_\delta) \\ &= \left(2\delta^{-1} + \frac{\delta}{2}\right) \cdot \frac{\delta}{2} \\ &= 1 + \frac{\delta^2}{4} \\ &> 1 = \varepsilon_0 \end{aligned}$$

□

**Remark.**  $x \mapsto x^2$  is uniformly continuous on  $[-1, 1]$ , even though it is not uniformly continuous on  $\mathbb{R}$ .

**Example 62.** Let  $f : [0, \infty) \rightarrow [0, \infty)$ ,  $x \mapsto x^{\frac{1}{2}}$ . Then  $f$  is uniform continuous on  $[0, \infty)$ .

*Proof.* Let  $x, y \in [0, \infty)$  and wlog assume  $x > y$ . Notice that

$$\oplus |f(x) - f(y)| = \sqrt{x} - \sqrt{y} \stackrel{\oplus}{\leq} \sqrt{x - y}$$

Hence, given any  $\varepsilon > 0$ ,  $|x - y| < \varepsilon^2 \stackrel{\oplus}{\implies} |f(x) - f(y)| < \varepsilon$ .

proof of  $\oplus$ : let  $a > b \geq 0$

$$\begin{aligned} (\sqrt{a} - \sqrt{b})^2 &= a + b \underbrace{-2\sqrt{a}\sqrt{b}}_{< -2\sqrt{b}\sqrt{b} = -2b} \\ &\leq a - b \\ \implies \sqrt{a} - \sqrt{b} &\leq \sqrt{a - b} \end{aligned}$$

□

**Theorem 63.** If  $f$  is continuous on  $[a, b]$ , then  $f$  is uniformly continuous on  $[a, b]$ .

The choice of the interval  $A$  matters on the Definition 61.

*Proof.* We first make the following definition

For  $\varepsilon > 0$ , we say that  $g$  is  $\varepsilon$ -good on  $[a, b]$  if  $\exists \delta = \delta(\varepsilon)$  such that for all  $y, z \in [a, b]$ ,

$$|y - z| < \delta \implies |g(y) - g(z)| < \varepsilon$$

We want to prove that  $f$  is  $\varepsilon$ -good on  $[a, b]$  for every  $\varepsilon > 0$ .

For each  $\varepsilon > 0$ , define

$$A_\varepsilon := \{x \in [a, b] : f \text{ is } \varepsilon\text{-good on } [a, x]\}$$

Then,  $A_\varepsilon \neq \emptyset$  since  $a \in A_\varepsilon$ , and  $A_\varepsilon$  is certainly bounded above by  $b$ . Hence,  $\sup A_\varepsilon$  is well-defined and we set  $\alpha_\varepsilon := \sup A_\varepsilon$ .

Fix  $\varepsilon > 0$ . Our aim is to prove that  $\alpha_\varepsilon = b$ . Suppose for contradiction  $\alpha_\varepsilon < b$ . Since  $f$  is continuous at  $\alpha_\varepsilon$ ,  $\exists \delta_0 = \delta_0(\varepsilon, \alpha_\varepsilon)$  such that

$$|y - \alpha_\varepsilon| < \delta_0 \implies |f(y) - f(\alpha_\varepsilon)| < \frac{\varepsilon}{2}$$

Hence if both  $y, z \in [\alpha_\varepsilon - \delta_0, \alpha_\varepsilon + \delta_0]$  there holds

$$|y - \alpha_\varepsilon| < \delta_0 \implies |f(y) - f(\alpha_\varepsilon)| < \frac{\varepsilon}{2}$$

$$|z - \alpha_\varepsilon| < \delta_0 \implies |f(z) - f(\alpha_\varepsilon)| < \frac{\varepsilon}{2}$$

So, triangle inequality gives  $|f(y) - f(z)| < \varepsilon$ .

This,  $f$  is  $\varepsilon$ -good on  $[\alpha_\varepsilon - \delta_0, \alpha_\varepsilon + \delta_0]$ . Also since  $\alpha_\varepsilon = \sup A_\varepsilon$ , it is also clear (from Lemma 28) that  $f$  is  $\varepsilon$ -good on  $[a, \alpha_\varepsilon - \delta_0]$ .

Claim:  $f$  is  $\varepsilon$ -good on  $[a, \alpha_\varepsilon + \delta_0]$ .

We will prove this claim later. Assuming it holds, we get that  $f$  is  $\varepsilon$ -good on  $[a, \alpha_\varepsilon + \delta_0] \implies \alpha_\varepsilon + \delta_0 \in A_\varepsilon$  but  $\alpha_\varepsilon + \delta_0 > \alpha_\varepsilon = \sup A_\varepsilon$ .

Hence,  $\alpha_\varepsilon = b$ . We now show that  $b \in A$ . Since  $f$  is continuous at  $b$ ,  $\exists \delta_1 = \delta_1(\varepsilon, b)$  such that

$$b - \delta_1 < y \leq b \implies |f(y) - f(b)| < \frac{\varepsilon}{2}$$

So we again see that  $f$  is  $\varepsilon$ -good on  $[b - \delta_1, b]$ . But  $f$  is also  $\varepsilon$ -good on  $[a, b - \delta_1]$ . Since  $b - \delta_1 \in A$  by Lemma 28. So, using the claim again we get that  $b \in A_\varepsilon$ .  $\square$

*proof of Claim.* Since  $f$  is continuous at  $\alpha_\varepsilon - \delta_0$ ,  $\exists \delta_2 = \delta_2(\varepsilon, \alpha_\varepsilon - \delta_0)$  such that

$$(\dagger \dagger \dagger) |x - (\alpha_\varepsilon - \delta_0)| < \delta_2 \implies |f(x) - f(\alpha_\varepsilon - \delta_0)| < \frac{\varepsilon}{2}$$

Meanwhile,  $f$  is  $\varepsilon$ -good on  $[a, \alpha_\varepsilon - \delta_0]$ , so  $\exists \delta_3 = \delta_3(\varepsilon)$  such that

$$x, y \in [a, \alpha_\varepsilon - \delta_0], |x - y| < \delta_3 \implies |f(x) - f(y)| < \frac{\varepsilon}{2}(\dagger)$$

and similarly,  $\exists \delta_4 = \delta_4(\varepsilon)$  such that

$$x, y \in [\alpha_\varepsilon - \delta_0, \alpha_\varepsilon + \delta_0], |x - y| < \delta_4 \implies |f(x) - f(y)| < \frac{\varepsilon}{2}(\dagger \dagger)$$

Now, choose any  $x, y \in [a, \alpha_\varepsilon + \delta_0]$ . If  $x, y$  both belong either to  $[a, \alpha_\varepsilon - \delta_0]$  or to  $[\alpha_\varepsilon - \delta_0, \alpha_\varepsilon + \delta_0]$ , then there is nothing to show (by  $\dagger, \dagger \dagger$ ). The final possibility is  $x \in [a, \alpha_\varepsilon - \delta_0]$  and  $y \in [\alpha_\varepsilon - \delta_0, \alpha_\varepsilon + \delta_0]$ .

In this case, let  $\delta := \min(\delta_2, \delta_3, \delta_4)$  and observe that

$$\begin{aligned} |x - y| < \delta & \xRightarrow{\text{since } y > x} 0 \leq y - x < \delta \\ & \implies 0 \leq (y - (\alpha_\varepsilon - \delta_0)) + ((\alpha_\varepsilon - \delta_0) - x) < \delta \\ & \implies |y - (\alpha_\varepsilon - \delta_0)| < \delta \\ & \implies |f(y) - f(\alpha_\varepsilon - \delta_0)| < \frac{\varepsilon}{2}(\dagger \dagger \dagger) \text{ and } |f(x) - f(\alpha_\varepsilon - \delta_0)| < \frac{\varepsilon}{2} \\ & \implies |f(y) - f(x)| < \varepsilon \end{aligned}$$

Note that  $\delta = \min(\delta_2(\varepsilon, \alpha_\varepsilon - \delta_0(\varepsilon, \alpha_\varepsilon)), \delta_3(\varepsilon), \delta_4(\varepsilon))$ .

$\delta$  only depends on  $\varepsilon, \alpha_\varepsilon$ , and since  $\alpha_\varepsilon$  only depends on  $\varepsilon$ , we define that  $\delta$  only depends on  $\varepsilon$ , as required.  $\square$

**Example 64.**

- (i)  $f(x) = \sin\left(\frac{1}{x}\right)$  is continuous and bounded on  $(0, 1]$  however it is not uniformly continuous on  $(0, 1]$ .
- (ii)  $f(x) = \sin(e^x)$  is continuous and bounded on  $[0, \infty)$  however it is not uniformly continuous on  $[0, \infty)$ .

*Proof.*

- (i) Fix any  $\delta > 0$  and let  $x_\delta = \frac{1}{2\pi n_\delta}$  and  $y_\delta = \frac{1}{\frac{\pi}{2} + 2\pi n_\delta}$ , where  $n_\delta \in \mathbb{N}$  is to be chosen. Notice that

$$0 < x_\delta - y_\delta = \frac{\frac{\pi}{2} + 2\pi n_\delta - 2\pi n_\delta}{2\pi n_\delta \left(\frac{\pi}{2} + 2\pi n_\delta\right)} = \frac{1}{4n_\delta \left(\frac{\pi}{2} + 2\pi n_\delta\right)}$$

thus, by choosing  $n_\delta$  large enough,

$$\frac{1}{4n_\delta \left(\frac{\pi}{2} + 2\pi n_\delta\right)} < \delta$$

and thus  $|x_\delta - y_\delta| < \delta$ , and yet  $|f(x_\delta) - f(y_\delta)| = 1$

So,  $f$  is not uniformly continuous on  $(0, 1]$ .

- (ii) Fix any  $\delta > 0$  and let  $x_\delta = \log\left(2\pi n_\delta + \frac{\pi}{2}\right)$ ,  $y_\delta = \log(2\pi n_\delta)$  where  $n_\delta$  is to be chosen. Observe that

$$0 < x_\delta - y_\delta = \log\left(1 + \frac{1}{4n_\delta}\right)$$

Since  $\log : [1, \infty) \rightarrow [0, \infty)$  is continuous at 1, and  $\log(1) = 0$ ,  $\exists n_\delta \in \mathbb{N}$  sufficiently large such that

$$0 < \log\left(1 + \frac{1}{4n_\delta}\right) < \delta$$

Thus,  $|x_\delta - y_\delta| < \delta$  and yet  $\left| \underbrace{f(x_\delta)}_{\sin\left(2\pi n_\delta + \frac{\pi}{2}\right)=1} - \underbrace{f(y_\delta)}_{\sin(2\pi n_\delta)=0} \right| = 1$ .

So,  $f$  is not uniformly continuous on  $[0, \infty)$ .  $\square$

This concludes our section on continuity. We are now ready to look at differentiation.

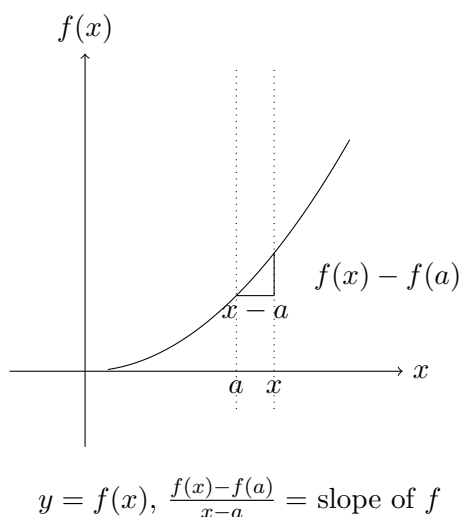
## 4 Differentiation

Office hours on Monday

1. Office hour 6.pm to 7.pm on Monday
2. can meet before 8:50 am Monday in my office Van Vleck 613 (send an email on sunday)

Consider a function defined on an interval  $I$ , with real values.  $f : I \rightarrow \mathbb{R}$

**Definition.**  $f$  is differentiable at the point  $a \in I$  if the limit  $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$  exists, then we call this limit the derivative  $f'(a)$



Computation of some derivatives

**Example.**

- (i)  $f(x) = c$  ( $c$  is some fixed point) we get  $f'(a) = 0$  for all  $a$ ,

$f(x) = f(a) = 0$  for all  $x$ ,  $\frac{f(x) - f(a)}{x - a} = 0 \implies f$  is differentiable and  $f'(a) = 0$  for all  $a$

$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(x)$  is equivalent with saying  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a)$



(ii)  $f(x) = x$ , then

$$\frac{f(a+h) - f(a)}{h} = \frac{a+h-a}{h} = 1$$

(written  $f'(x) = 1$ )

(iii)  $f(x) = x^2$ , then fix  $a$ ,

$$\frac{f(a+h) - f(a)}{h} = \frac{(a+h)^2 - a^2}{h} = \frac{a^2 + 2ah + h^2 - a^2}{h} = 2a + h$$

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} 2a + h = 2a$$

(iv)  $f(x) = |x|$ , We should examine the differentiability of  $f$  at  $\underline{a=0}$

$$\frac{f(0+h) - \overbrace{f(0)}^{=0}}{h} = \frac{|h|}{h} = \begin{cases} 1 & \text{if } h > 0 \\ -1 & \text{if } h < 0 \end{cases}$$

The limit does not exist, and thus  $f$  is not differentiable at 0.

(v)  $f(x) = \sqrt{|x|}$ ,  $f$  is not differentiable at 0 because  $f(0) = 0$  and  $\frac{f(0+h)-f(0)}{h} = \frac{\sqrt{|h|}}{h}$ , this limit also does not exist

Examine differentiability and derivative of  $f(x) = \sqrt{|x|}$  at  $x = a, a > 0$

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{\sqrt{|a+h|} - \sqrt{|a|}}{h} \\ &= \frac{\sqrt{a+h} - \sqrt{a}}{h} \\ &= \frac{a+h-a}{\sqrt{a+h} + \sqrt{a}} \cdot \frac{1}{h} \\ &= \frac{1}{\sqrt{a+h} + \sqrt{a}} \rightarrow \frac{1}{2\sqrt{a}} \end{aligned}$$

(vi)  $f(x) = x^n$

$$\frac{f(a+h) - f(a)}{h} = \frac{(a+h)^n - a^n}{h} = n \cdot a^{n-1}$$

## 4.1 Basic fact about differentiation

Continuity is necessary (but not sufficient) for differentiation

**Theorem.** If  $f : I \rightarrow \mathbb{R}$  is differentiable at  $a$  the  $f$  is continuous at  $a$ .

**Reminder** If  $\lim_{x \rightarrow a} F(x) = l$  and  $\lim_{x \rightarrow a} G(x) = m$ , then  $\lim_{x \rightarrow a} F(x)G(x) = lm$

If  $\lim_{x \rightarrow a} F(x) = l$  and  $\lim_{x \rightarrow a} G(x) = m$ , then  $\lim_{x \rightarrow a} \frac{F(x)}{G(x)} = \frac{l}{m}$  or not? Yes if  $m \neq 0$

*Proof.* We know that  $\lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h} = f'(a)$

$$\begin{aligned} f(a+h) - f(a) &= \frac{f(a+h) - f(a)}{h} \cdot h \\ \implies \lim_{h \rightarrow 0} f(a+h) - f(a) &= f'(a) \cdot 0 = 0 \\ \lim_{h \rightarrow 0} f(a+h) &= f(a) \end{aligned}$$

this is continuity of  $f$  at  $a$

Another argument : for sufficiently small  $h$ ,  $|f(a+h) - f(a)| \leq C|h|$  □

## 4.2 Sum Rule

**Theorem.** Let  $f : I \rightarrow \mathbb{R}$  and  $g : I \rightarrow \mathbb{R}$ ,  $a \in I$  assume that  $f$  and  $g$  are differentiable at  $a$ . Then  $f+g$ ,  $(f+g)(x) = f(x) + g(x)|_{x=a}$  is differentiable and its derivative  $f'(a) + g'(a)$  (The derivative of the sum is the sum of the derivatiives)

*Proof.*

$$\begin{aligned} \frac{(f+g)(a+h) - (f+g)(a)}{h} &= \frac{f(a+h) + g(a+h) - (f(a) + g(a))}{h} \\ &= \frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h} \end{aligned}$$

As  $h \rightarrow 0$  this has limit  $f'(a) + g'(a)$  □

## 4.3 Product Rule

**Theorem.** Let  $f : I \rightarrow \mathbb{R}$  and  $g : I \rightarrow \mathbb{R}$ ,  $a \in I$  assume that  $f$  and  $g$  are differentiable at  $a$ . the  $f \cdot g$  is differentiable at  $a$

$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$$

*Proof.*

$$\begin{aligned} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} &= \frac{(f(a+h) - f(a))g(a+h) + f(a)g(a+h) - f(a)g(a)}{h} \\ &= \underbrace{\frac{f(a+h) - f(a)}{h}}_{f'(a)} \cdot \underbrace{g(a+h)}_{\rightarrow g(a)} + \underbrace{\frac{g(a+h) - g(a)}{h}}_{\rightarrow g'(a)} \cdot \underbrace{f(a)}_{f(a)} \end{aligned}$$

By theorem about products and of limits, and the continuity of  $g$  at  $a$ , we get

$$\lim_{h \rightarrow 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} = f'(a)g(a) + g'(a)f(a)$$

□

**Theorem.** Let  $f : I \rightarrow \mathbb{R}$  and  $g : I \rightarrow \mathbb{R}$ ,  $a \in I$  is differentiable at  $a$ , and if  $g(a) \neq 0$  then  $\frac{1}{g}$  is differentiable at  $a$  and

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{(g(a))^2}$$

*Proof.*

$$\begin{aligned} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} &= \frac{g(a) - g(a+h)}{g(a+h)g(a)} \cdot \frac{1}{h} \\ &= \frac{1}{g(a+h)g(a)} \cdot (-1) \frac{g(a+h) - g(a)}{h} \\ &\rightarrow \frac{1}{(g(a))^2} \cdot (-1)g'(a) \end{aligned}$$

□

## 4.4 Quotient Rule

**Theorem.** Let  $f : I \rightarrow \mathbb{R}$  and  $g : I \rightarrow \mathbb{R}$ ,  $a \in I$  assume  $f$  and  $g$  are differentiable at  $a$ , and if  $g(a) \neq 0$  then  $\frac{f}{g}$  is differentiable at  $a$  and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

*Proof.* Combine the theorems about products and reciprocals of differentiable function

$$\begin{aligned}
\left(\frac{f}{g}\right)'(a) &= \left(f \cdot \frac{1}{g}\right)'(a) \\
&= f'(a) \cdot \frac{1}{g(a)} + f(a) \cdot \left(\frac{1}{g}\right)'(a) \\
&= \frac{f'(a)}{g(a)} + f(a) \left(-\frac{g'(a)}{g(a)^2}\right) \\
&= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}
\end{aligned}$$

□

**Example.**

$$\begin{aligned}
\left(\frac{\sin x}{\cos x}\right)' &= \frac{\sin'(x) \cos(x) - \cos' x \sin x}{(\cos x)^2} \\
&= \frac{(\cos x)^2 - (-\sin^2 x)}{(\cos x)^2} \\
&= \frac{1}{(\cos x)^2}
\end{aligned}$$

**Example.**  $f(x) = x^n$  then  $f'(x) = nx^{n-1}$ *Proof.*

$$\begin{aligned}
f_0(x) &= 1, f'_0(x) = 0 \\
f_1(x) &= x, f'_1(x) = 1 \\
f_1(x) &= x^2, f'_2(x) = 2x
\end{aligned}$$

We want to show this formula for a given  $n$ , assuming that we already know if for  $n = 1$ , In other words, the formula  $f'_{n-1}(x) = (n-1)x^{n-2}$ ,  $n \geq 2$ , implies the formula for  $f_n$

Induction step:

$$f_n(x) = x^n = \underbrace{x^{n-1}}_{f_{n-1}} \cdot \underbrace{x}_{f_1}$$

By using Product Rule, we get

$$\begin{aligned}
f'_n(x) &= f'_{n-1}(x)f_1(x) + f_{n-1}(x)f'_1(x) \\
&= (n-1)x^{n-2} \cdot x + x^{n-1} \cdot 1 \\
&= nx^{n-1}
\end{aligned}$$

□

**Example.**

$$\begin{aligned}
 (fg)'' &= (f'g + fg')' \\
 &= (f'g)' + (fg')' \\
 &= f''g + f'g' + f'g' + fg'' \\
 &= f''g + 2f'g' + fg''
 \end{aligned}$$

$(fg)''' = f'''g + 3f''g' + 3f'g'' + fg'''$  and can be written as  $(fg)^{(3)}$

$$(fg)^{(n)}(x) = \sum_{k=0}^n \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)$$

As the analogy

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

We will not talk much about higher derivative in this class

## 4.5 Chain Rule

Let  $\zeta(x) = f(g(x))$ . Assume that  $g$  is defined on an interval containing  $a$ , and  $g$  is differentiable at  $a$ . Let  $f$  be defined in an interval that contain the range (image) of  $g$ , and let  $f$  be differentiable at  $g(a)$ . then,  $\zeta = f \circ g$  is differentiable at  $a$  and

$$\zeta'(a) = f'(g(a))g'(a)$$

**Example.**

$$F(x) = (x^3 + 7x^2 + 1)^8$$

Fix a point  $a$ , what  $F'(a)$

Let  $F(x) = f(g(x))$ ,  $g(x) = x^3 + 7x^2 + 1$  and  $f(w) = w^8$

First, calculate  $f'$  and  $g'$

$$f'(w) = 8w^7$$

$$g'(x) = 3x^2 + 14x$$

Then calculate  $F'(x)$

$$\begin{aligned}
 F'(x) &= f'(g(x))g'(x) \\
 &= 8(g(x))^7 \cdot (3x^2 + 14x) \\
 &= 8(x^3 + 7x^2 + 1)^7 \cdot (3x^2 + 14x)
 \end{aligned}$$

Attempt to prove the chain rule

*Proof.*

$$\begin{aligned}\frac{\zeta(a+h) - \zeta(a)}{h} &= \frac{f(g(a+h)) - f(g(a))}{h} \\ &= \underbrace{\frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)}}_{\rightarrow f'(g(a))} \cdot \underbrace{\frac{g(a+h) - g(a)}{h}}_{\rightarrow g'(a)}\end{aligned}$$

But  $g(a+h) - g(a)$  might be equal to 0, So, we can't use this method to prove the chain rule.  $\square$

**Theorem** (Decomposition theorem for differentiation). The function  $f$  is differentiable at  $a$  (with derivative  $f'(a)$ ) if and only if there is another function  $u$  with the same domain as  $f$ , so that  $u$  is continuous at  $a$  and

$$f(x) = f(a) + (x - a)u(x)$$

Then

$$u(a) = f'(a)$$

*Proof.* Assume that  $f$  is differentiable at  $a$ ,  $f'(a)$  is the derivative

$$u(x) = \begin{cases} \frac{f(x) - f(a)}{x - a} & \text{if } x \neq a \\ f'(a) & \text{if } x = a \end{cases}$$

( $u$  depends on  $a$  but  $a$  is fixed)

$u$  is continuous at  $a$  because  $\lim_{x \rightarrow a} \frac{f(a+h) - f(a)}{h} = f'(a) = u(a)$

Suppose that

$$\zeta(x) = f(g(x)) \implies \zeta'(a) = f'(g(a))g'(a)$$

**Assumption**

- (1)  $g$  is differentiable at  $a$
- (2)  $f$  is differentiable at  $g(a)$

we can write

$$g(x) = g(a) + (x - a)u(x) *$$

where  $u$  is continuous at  $a$ ,  $g'(a) = u(a)$ , and

$$f(y) = f(g(a)) + (y - g(a))v(y) **$$

where  $v$  is continuous at  $g(a)$ ,  $v(g(a)) = f'(g(a))$

Goal is to find a function  $w$  continuous at  $a$  such that

$$\zeta(x) = \zeta(a) + (x - a)w(x)$$

with  $w(a) = f'(g(a))g'(a)$

from \*\*,

$$f(g(x)) = f(g(a)) + (g(x) - g(a)) \underbrace{v(g(x))}_{\text{cts at } a}$$

from \*,

$$f(g(x)) = f(g(a)) + (x - a) \underbrace{u(x)v(g(x))}_{\text{cts at } a}$$

Then, we get

$$w(x) := u(x)v(g(x))$$

and

$$w(a) = u(a)v(g(a)) = g'(a)f'(g(a))$$

□

## 4.6 Geometric meaning of Differentiation

**Theorem.** Let  $f$  be defined on an interval  $I$  and let  $a$  be a point in the interior of this interval.

Assume:

1.  $f$  has a maximum at  $a$
2.  $f$  is differentiable at  $a$

Then,  $f'(a) = 0$

formally  $f$  has a maximum in  $I$  at  $a$ , means  $f(x) \leq f(a)$  for all  $x \in I$  (Also works for min in place of max)

*Proof.* We know by the assumption  $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(x)$  exists.

1. If  $x > a$  then  $f(x) \leq f(a) \implies \frac{f(x) - f(a)}{x - a} \leq 0$  (slope of right side  $\leq 0$ )
2. If  $x < a$  then  $f(x) \leq f(a)$  but now  $x - a < 0$ ,  $\frac{f(x) - f(a)}{x - a} \geq 0$  (slope of left side  $\geq 0$ )

So,  $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$  has to be  $\geq 0$  and  $\leq 0$ , so it must be 0.

□

## 4.7 Mean-Value Theorem

**Theorem** (Mean-value theorem). Let  $f$  be defined on  $[a, b]$  and  $f$  continuous in  $[a, b]$  and differentiable in  $(a, b)$ . Then there is a  $\xi \in (a, b)$  such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

*Proof.* First step in the proof is a special case where  $f(a) = f(b)$  (then there is a  $\xi \in (a, b)$  such that  $f'(\xi) = 0$ )

1. if  $f$  has a max and a min at the endpoint,  $f$  is constant and therefore  $f'(\xi) = 0$  for all  $\xi \in (a, b)$
2. if  $f$  has a maximum and a minimum in  $(a, b)$ , then we know already, at such a point, the derivative is 0, so at that point  $\xi \implies f'(\xi) = 0$

This particular case is called “*Rolle’s theorem*”

Consider

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then,  $g(a) = 0$  and  $g(b) = 0$  and  $g$  is continuous in  $(a, b)$

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

Apply “*Rolle’s theorem*” to  $g$  on  $(a, b)$ , we get a  $\xi \in (a, b)$  such that  $g'(\xi) = 0$  □

Aforementioned theorem can be written as

$$f(b) - f(a) = f'(\xi)(b - a)$$

## 4.8 Application of the Mean-Value Theorem

**Example.** Prove  $|\sin x| \leq |x|$

*Proof.* We know that  $\sin 0 = 0$  and  $\sin' x = \cos x$

$$\sin x = \sin x - \underbrace{\sin 0}_{=0} = \sin'(\xi)(x - 0)$$

where  $\xi$  is between 0 and  $x$

$$\begin{cases} 0 < \xi < x & \text{if } x > 0 \\ x < \xi < 0 & \text{if } x < 0 \end{cases}$$



So,  $\sin x = (\cos \xi)x$ , where  $-1 \leq \cos \xi \leq 1 \implies |\cos \xi| \leq 1$

Therefore,

$$|\sin x| = |\cos \xi| \cdot |x| \leq |x|$$

□

**Example.** Can we get an estimate for  $\cos x - 1$  where  $x$  is small?

$$\cos x - 1 = \cos x - \cos 0 = \cos'(\xi)(x - 0) = (-\sin \xi)x$$

We get  $|\cos x - 1| \leq |x|$

Can do better

$$\begin{aligned} |\cos x - 1| &\leq |(\sin \xi)| \cdot |x| \text{ for } \xi \text{ between } 0 \text{ and } x \\ &\leq |\xi| \cdot |x| \leq |x|^2 \end{aligned}$$

for  $|x| < 1$  this is a better estimate than the previous one

**Theorem.** If  $f$  is differentiable on  $(a, b)$  and if  $f'(x) = 0$  for all  $x \in (a, b)$  then  $f$  is constant.

*Proof.* take  $x_1 < x_2$ , both in the interval and apply the MVT

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1), \quad x_1 < \xi < x_2$$

we know  $f'(\xi) = 0 \implies f(x_2) - f(x_1) = 0 \implies f(x_2) = f(x_1)$

So,  $f$  is constant function

□

**Example.** for a differential equation

Q: Find  $f(x)$  (differentiable, for  $x > 0$ ) such that

$$xf'(x) = f(x)$$

*Proof.* By guessing  $f(x) = x$  is a solution because  $f'(x) = 1, x \cdot 1 = x$

In fact, for any constant  $C$ ,  $f(x) = Cx$  is a solution.

Show that for an arbitrary solution  $g$ ,  $yg'(y) = g(y)$  for  $y > 0$  try to show that  $\frac{g(y)}{y}$  is constant

To do this, show that the derivative of  $\frac{g(x)}{x}$  is zero

$$\frac{g(x)}{x} = \frac{g'(x)x - g(x) \cdot 1}{x^2} = 0, \text{ since } g \text{ satisfies the differential equation}$$

So,  $\frac{g(x)}{x}$  is constant

□

**Example.**

$$xf'(x) = af(x)$$

$Cx^a$  is a solution

*Proof.* Conjecture: All solutions are of the form  $f(x) = Cx^a$

Let  $g$  be a solution of the equation, we have  $xf'(x) = ag(x)$  Consider

$$\begin{aligned} \left( \frac{g(x)}{x^a} \right)' &= \frac{g'(x)x^a - g(x)ax^{a-1}}{x^{2a}} \\ &= \frac{x^{a-1}}{x^{2a}} \cdot \underbrace{(g'(x)x - ag(x))}_{=0} \end{aligned}$$

So,  $\frac{g(x)}{x^a}$  is constant, so  $g(x) = Cx^a$  for some  $C$  □

**Theorem.** If  $f, f'$  are differentiable on  $(a, b)$  if  $f'(c) = 0$  and  $f''(x) > 0$  for all  $x$  in  $(a, b)$  then  $f$  has a minimum at  $c$

*Proof.* To do this, we want to check that  $f$  is strictly increasing for  $x > c$  and strictly decreasing for  $x < c$

We do this by checking  $f'(x) < 0, x < c$  and  $f'(x) > 0, x > c$

$$f''(x) > 0 \implies f' \text{ is increasing (strictly) on } (a, b)$$

$$f'(c) = 0 \implies f'(x) > 0, x > c \text{ and } f'(x) < 0, x < c$$

□

## 4.9 Inverse Function

**one-to-one (injective)**

$f$  is one-to-one if  $x_1 \neq x_2$  implies that  $f(x_1) \neq f(x_2)$

**Example.**

$$\begin{aligned} f : (1, 2) &\rightarrow \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

Show that  $f$  is one-to-one

*Proof.* proof by contradiction

$$x_1^2 = x_2^2 \text{ and } x_1, x_2 \in (1, 2)$$

$$\sqrt{x_1^2} = x_1, \sqrt{x_2^2} = x_2 \implies x_1 = x_2 \quad \square$$

**Example.**

$$\begin{aligned} f : (-5, 5) &\rightarrow \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

Show that  $f$  is not one-to-one

$$\textit{Proof. } 2^2 = (-2)^2, x_1 = 2, x_2 = -2 \implies x_1^2 = x_2^2 \quad \square$$

### onto (surjective)

$f$  is “onto” means that every element in  $B$  is a value  $f(x)$  for some  $x \in A$

Every function is onto if the target space is equal to the range of  $f$

### one-to-one and onto (bijective)

If a function is both one-to-one and onto (injective and surjective)

$f : A \rightarrow B$  bijective mean that for ever  $x \in A$  there is exactly one  $y \in B$  such that  $y = f(x)$  and for every  $y \in B$  there is exactly one  $x$ , such that  $y = f(x)$  we say

$$x = f^{-1}(y) \iff y = f(x)$$

We pronounce  $f^{-1}$  as “f inverse”

**Theorem.** If  $f$  is strictly increasing on  $[a, b]$  and continuous then  $f[a, b] \rightarrow [f(a), f(b)]$  is bijective and  $f$  has an inverse function

$$f^{-1}[f(a), f(b)] \rightarrow [a, b]$$

$$y \mapsto x$$

As the result, we get  $f(f^{-1}(y)) = y$  and  $f^{-1}(f(x)) = x$

If  $f$  and  $f^{-1}$

$$f : [a, b] \rightarrow [f(a), f(b)]$$

$$f^{-1} : [f(a), f(b)] \rightarrow [a, b]$$

are both differentiable, what is the relation between the derivatives

Apply Chain rule on  $f^{-1}(f(x)) = x$  then  $(f^{-1})'(f(x))f'(x) = 1$

Apply Chain rule on  $f(f^{-1}(y)) = y$  then  $f'(f^{-1}(y))(f^{-1})'(y) = 1$

if and only if  $y = f(x)$  and  $x = f^{-1}(y)$

**Theorem.** If  $f$  is increasing or decreasing on some interval then it has an inverse function  $f^{-1}$

*Proof.* If  $f$  and  $f^{-1}$  are differentiable then we may get a formula for  $(f^{-1})'$  from the chain rule applied to  $f^{-1}(f(x)) = x$

The chain rule given us

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

Derivative of  $f^{-1}$ , evaluated at  $f(x)$

$$\implies (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

□

**Theorem.** Let  $f$  be (strictly) increasing on  $[a, b]$  and  $f'(x_0)$  exists for  $x_0 \in (a, b)$  and  $f'(x_0) \neq 0$  then  $f^{-1}$  is differentiable at  $f(x_0)$  and  $(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$

*Proof.* Recall

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

If  $f(x) = y$  then

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{y - f(x_0)}{f^{-1}(y) - x_0} = \frac{1}{\frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)}}$$

$$\lim_{y \rightarrow f(x_0)} \frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} = (f^{-1})'(f(x_0))$$

if that limit exists

We know that  $f'(x_0) = c > 0$  and  $\frac{f(x) - f(x_0)}{x - x_0} \rightarrow C$  There exists  $\delta > 0$  such that

$$\frac{f(x) - f(x_0)}{x - x_0} > \frac{C}{2} \text{ for } |x - x_0| < \delta$$

$$\frac{f(x) - f(x_0)}{x - x_0} < 2C \text{ for } |x - x_0| < \delta$$

$$f(x) - f(x_0) \text{ is between } \frac{C}{2}(x - x_0) \text{ and } 2C(x - x_0)$$

$$x - x_0 \text{ is between } \frac{f(x) - f(x_0)}{2C} \text{ and } \frac{f(x) - f(x_0)}{\frac{C}{2}}$$

Then we get

$$\begin{aligned} \lim_{y \rightarrow f(x_0)} \frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} &= \lim_{y \rightarrow f(x_0)} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} \\ &= \frac{1}{f'(x_0)} \end{aligned}$$

□

**Example.** For  $x > 0$ ,  $f(x) = x^n$ ,  $f : (0, \infty) \rightarrow (0, \infty)$  and  $f^{-1} : (0, \infty) \rightarrow (0, \infty)$  The inverse function  $f^{-1}$  is called nth to of

$$f'(x) = nx^{n-1}$$

$$\begin{aligned} (f^{-1})'(y) &= \frac{1}{f'(x)} \Big|_{x=f^{-1}(y)} \\ &= \frac{1}{nx^{n-1}} \Big|_{x=f^{-1}(y)} \\ &= \frac{1}{n(\sqrt[n]{y^{n-1}})} \\ &= \frac{1}{n} \frac{1}{\sqrt[n]{y}} \sqrt[n]{y} \\ &= \frac{1}{n} \frac{1}{y} \sqrt[n]{y} \end{aligned}$$

**Example.**  $f(x) = \frac{\sin x}{\cos x} = \tan x$  where  $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$f(x)$  is well-defined whenever  $x \neq \frac{\pi}{2} + k\pi$  for  $k \in \mathbb{Z}$

$$f'(x) = \frac{(\cos x) \cos x - \sin x(-\sin x)}{(\cos x)^2} = \frac{1}{(\cos x)^2}$$

$\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \mapsto (-\infty, \infty)$  and we found that  $\tan$  is increasing  $\tan' x = \frac{1}{(\cos x)^2} > 0$

What is  $(\tan^{-1})'(y)$

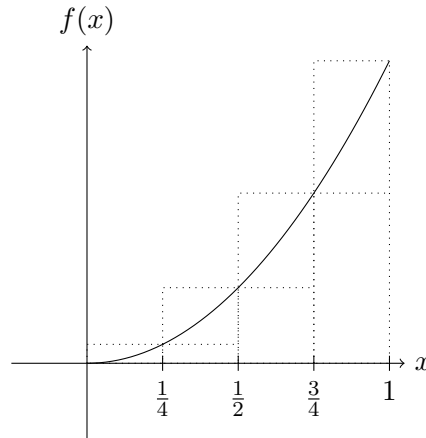
$$\begin{aligned} (f^{-1})'(y) &= \frac{1}{f'(f^{-1}(y))} \\ &= \frac{1}{\frac{1}{\cos^2(f^{-1}(y))}} \\ &= (\cos(f^{-1}(y)))^2 \\ &= (\cos(\tan^{-1}(y)))^2 \\ &= (\cos(\arctan(y)))^2 \end{aligned}$$

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1 - (\cos x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2} - (\cos x)^2 - 1$$

$$\frac{1}{(\cos x)^2} = 1 + (\tan x)^2 \implies (\cos x)^2 = \frac{1}{1 + (\tan x)^2}$$

$$(\cos(\arctan(y)))^2 = \frac{1}{1 + (\tan(\arctan y))^2} = \frac{1}{1 + y^2}$$

## 5 Integration



$$f(x) = x^2 \text{ defined on } [0, 1]$$

1. Given an interval  $[a, b]$  and a function  $f$  defined by  $f : [a, b] \rightarrow \mathbb{R}$ , assume that  $f$  is bounded (there is an  $M$  such that  $|f(x)| \leq M$  for all  $x \in [a, b]$ )

**Definition.** A partition of  $[a, b]$  is a finite collection of distinct numbers in  $[a, b]$ . which contains the end point  $a$  and  $b$ .

We can order these

$$a = x_0 < x_1 < x_2 < \dots < x_N = b$$

Are finement  $\tilde{P}$  of partition  $P$ , is a partition which contains  $P$

**Definition** (Lower Riemann Sum).

$$L(f, P) = \sum_{j=1}^N (x_j - x_{j-1}) \cdot m_j$$

$$m_j = \inf_{x_{j-1} \leq t \leq x_j} f(t)$$

Lower Riemann sum, given  $f$ , and  $P = \{x_0 < x_1 < \dots < x_N\}$ ,  $\inf$  is the greatest lower bound for  $f$  on  $[x_{j-1}, x_j]$ ,  $m_j$  is the greatest lower bound for  $f$  on the partition  $[x_{j-1}, x_j]$

**Definition** (Uower Riemann Sum).

$$U(f, P) = \sum_{j=1}^N (x_j - x_{j-1}) \cdot M_j$$

$$M_j = \sup_{x_{j-1} \leq t \leq x_j} f(t)$$

where sup is the least upper bound,  $M_j$  is the least upper bound

the example  $[a, b] = [0, 1]$

$$P = \left\{ 0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N}{N} = 1 \right\}$$

where  $x_j = \frac{j}{N}, 0 \leq j \leq N$

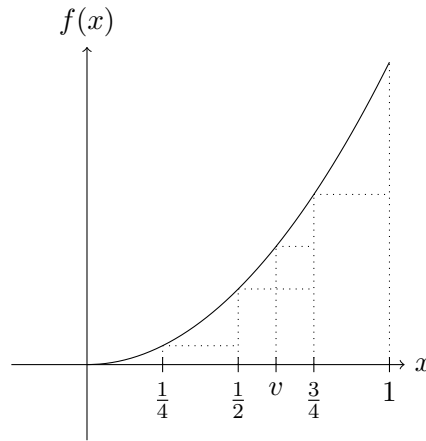
$$\begin{aligned} L(f, P) &= \sum_{j=1}^N \frac{1}{N} \left( \frac{j-1}{N} \right)^2 \\ &= \frac{1}{N^3} \sum_{j=1}^N (j-1)^2 \\ &= \frac{1}{N^3} (0^2 + 1^2 + 2^2 + \dots + (N-1)^2) \\ U(f, P) &= \sum_{j=1}^N \frac{1}{N} \left( \frac{j}{N} \right)^2 \\ &= \frac{1}{N^3} \sum_{j=1}^N (j)^2 \\ &= \frac{1}{N^3} (1^2 + 2^2 + \dots + N^2) \end{aligned}$$

What happens with  $L(f, P)$  if we refine the partition?

What happens with  $U(f, P)$  if we refine the partition?

**Proposition:** if  $P, \tilde{P}$  are partitions  $P \subset \tilde{P}$  then  $L(f, P) \leq L(f, \tilde{P})$  and  $U(f, P) \geq U(f, \tilde{P})$





$$f(x) = x^2 \text{ defined on } [0, 1]$$

If we refine the partition by adding a point  $\tilde{P} = P \cup \{v\}$

The key for the proof  $x_{j-1}, x_j \in P$  take a new partition point  $v$  between  $x_{j-1}$  and  $x_j$

$$\begin{aligned} (x_j - x_{j-1}) \inf_{x_{j-1} \leq t \leq x_j} f(t) &= (x_j - v) \inf_{x_{j-1} \leq t \leq x_j} f(t) + (v - x_{j-1}) \inf_{x_{j-1} \leq t \leq x_j} f(t) \\ &\leq (x_j - v) \inf_{v \leq t \leq x_j} f(t) + (v - x_{j-1}) \inf_{x_{j-1} \leq t \leq v} f(t) \end{aligned}$$

**Theorem.** Let  $f$  be a bounded function on  $[a, b]$ . Then

$$\sup_P L(f, P) \leq \inf_P U(f, P)$$

where  $\sup_P$  is supremum over all partition and  $\inf_P$  is infimum over all partitions.

**Definition.** Given the theorem, we say that  $f$  is integrable (or Riemann integrable) if

$$\sup_P L(f, P) = \inf_P U(f, P)$$

*Proof.*

(i) If  $P_1, P_2$  are two partitions then

$$L(f, P_1) \leq U(f, P_2)$$

Key: Take a refinement  $P$  of both  $P_1, P_2$  where  $P \supset P_1 \cup P_2$  then

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2)$$

- (ii) Conjecture:  $U(f, P_2)$  is an upper bound for all  $L(f, P)$  where  $P$  is any partition so the least upper bound for the  $L(f, P)$  cannot exceed  $U(f, P)$  that means

$$\sup_P L(f, P) \leq U(f, P_2)$$

for any fixed partition

For all partition  $P_2$ , the number  $\sup_P L(f, P)$  is a lower bound for  $U(f, P_2)$  The greatest lower bound for the  $U(f, P_2)$  cannot be smaller than  $\sup_P L(f, P)$

$$\implies \sup_P L(f, P) \leq \inf_P U(f, P)$$

□

**Example.**

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational on } [0, 1] \\ 0 & \text{if } x \text{ is rational on } [0, 1] \end{cases}$$

Take any partition  $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$  on  $[x_{j-1}, x_j]$

$$\inf_{[x_{j-1}, x_j]} f = 0 \text{ and } \sup_{[x_{j-1}, x_j]} f = 1$$

$$L(f, P) = \sum_{j=1}^N (x_j - x_{j-1}) \cdot 0 = 0$$

$$U(f, P) = \sum_{j=1}^N (x_j - x_{j-1}) \cdot 1 = 1$$

So this function is not integrable

**Example.**

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$P = \left\{ 0, \frac{1}{2}, 1 \right\}, \inf_{[0, \frac{1}{2}]} f = 0, \sup_{[0, \frac{1}{2}]} f = 1$$

$$L(f, P) = 0 \cdot \frac{1}{2} + \frac{1}{2} = \frac{1}{2}$$

$$U(f, P) = 1 \cdot \frac{1}{2} + \frac{1}{2} = 1$$

We pick new points

$$x_0 = 0, x_1 = \frac{1}{2} - \frac{1}{N}, x_2 = \frac{1}{2}, x_3 = 1$$

$$\begin{aligned}
L(f, P) &= \left(\frac{1}{2} - \frac{1}{N}\right) \cdot 0 + \frac{1}{N} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2} \\
U(f, P) &= \left(\frac{1}{2} - \frac{1}{N}\right) \cdot 0 + \frac{1}{N} \cdot 1 + \frac{1}{2} \cdot 1 = \frac{1}{2} + \frac{1}{N} \\
\sup L(f, P) &\geq \frac{1}{2} \text{ and } \inf U(f, P) \leq \frac{1}{2} + \frac{1}{N} \\
\implies \sup L(f, P) &= \inf U(f, P) = \frac{1}{2}
\end{aligned}$$

**Example.** Define on  $[0, 1]$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x \text{ is rational, } x = \frac{p}{q} \text{ in lowest term} \end{cases}$$

Question: Is  $f$  integrable on  $[0, 1]$ ?

Answer: Yes, and the integral is 0

## 5.1 Simple criteria for Riemann integrability

**Theorem.** Let  $f$  be a bounded function on  $[a, b]$ . Then  $f$  is integrable if and only if for every  $\varepsilon > 0$  there is a partition  $P$ , such that

$$U(f, P) - L(f, P) < \varepsilon$$

Definition of Riemann integrable by if

$$\sup_{P_1} L(f, P_1) = \inf_{P_2} U(f, P_2)$$

then  $f$  is Riemann integrable, and the (common) value is

$$\int_a^b f$$

If  $\tilde{P}$  is a refinement of  $P$  then  $L(f, \tilde{P}) \geq L(f, P)$  and  $U(f, \tilde{P}) \leq U(f, P)$

*Proof.*

1. Assume that  $f$  is Riemann integrable. Let  $\varepsilon > 0$  there is a partition  $P_1$  such that

$$\int_a^b f - \frac{\varepsilon}{10} < L(f, P_1)$$

There is a partition  $P_2$  such that

$$U(f, P_2) < \int_a^b f + \frac{\varepsilon}{10}$$

Then we get

$$\begin{aligned} \int_a^b f - \frac{\varepsilon}{10} < L(f, P_1) &\leq L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2) \leq U(f, P_2) < \int_a^b f + \frac{\varepsilon}{10} \\ \implies U(f, P_1 \cup P_2) - L(f, P_1 \cup P_2) &< \frac{\varepsilon}{5} \end{aligned}$$

2. Assume that for every  $\varepsilon > 0$ , there is a partition  $P$  such that

$$U(f, P) - L(f, P) < \varepsilon$$

We want to show that  $\inf_P U(f, P) - \sup_P L(f, P)$  differs by no more than  $\varepsilon$ . If I succeed, then we get (since  $\varepsilon > 0$  is arbitrary) that  $\inf_P U(f, P) = \sup_P L(f, P)$  (i.e. by definition  $f$  is Riemann integrable)

We here shown  $\inf U(f, P) \geq \sup L(f, P)$

$$U(f, P_\varepsilon) \geq \inf_{\text{all } P} U(f, P) \geq \sup_{\text{all } P} L(f, P) \geq L(f, P_\varepsilon)$$

□

## 5.2 Continuous functions

**Theorem.** A continuous function  $f$  on  $[a, b]$  is Riemann integrable

Recall the definition of continuity. A function is continuous at  $x_0 \in [a, b]$

$$\forall \varepsilon > 0, \exists \delta > 0, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon (x \in [a, b])$$

Theorem: A continuous function on  $[a, b]$  is uniformly continuous.

*Proof.*  $f$  is uniformly continuous, we want to check for arbitrary  $\varepsilon > 0$  that there is a partition  $P$  such that  $U(f, P) - L(f, P) < \varepsilon$

Know: There is a  $\delta > 0$  such that  $|f(x_1) - f(x_2)| < \frac{\varepsilon}{b-a}$  provided That  $|x_1 - x_2| < \delta$

$$P = \left\{ x_j = a + j \frac{\varepsilon}{b-a}, \right\}, x_j - x_{j-1} = \frac{\varepsilon}{b-a} = \delta$$

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{j=1}^N M_j \cdot (x_j - x_{j-1}) - \sum_{j=1}^N m_j \cdot (x_j - x_{j-1}) \\ &= \sum_{j=1}^N \underbrace{(M_j - m_j)}_{\leq \frac{\varepsilon}{b-a} \cdot \frac{1}{100}} \cdot (x_j - x_{j-1}) \\ &\leq \sum_{j=1}^N \frac{\varepsilon}{b-a} \cdot \frac{1}{100} \cdot (x_j - x_{j-1}) \\ &\leq \frac{\varepsilon}{b-a} \cdot \frac{1}{100} \cdot (b-a) \\ &= \frac{\varepsilon}{100} \end{aligned}$$

□