# MATH 421 Lecture Notes

Pongsaphol Pongsawakul

Fall 2022

## Contents

# Properties of Real Number

**Definition 1.** Given any $a \in \mathbb{R}$, we define its absolute value to be

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ a & \text{if } a < 0 \end{cases}$$

**Theorem 2** (Triangular Inequality)**.** Given $a, b \in \mathbb{R}$, there holds

$$|a + b| \leq |a| + |b|$$

# Method of Proof

## Direct proof

some statements can be shown to be true through a direct arguement e.g. our proof of Theorem 1

**Theorem 3.** hello

## Proof by induction

the aim is to proof that a statement is true for all rational number

(i) Show the statement is true for $n = 1$

(ii) Assume the statement is true for general $n \in \mathbb{N}$

(iii) Using assumption (ii), prove the statement is true for $n + 1$

(iv) Conclude your proof with a sentence like "by mathematical information, the result holds for all $n \in \mathbb{N}$"

**Example 4.** Show that $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$

**Theorem 5.** Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, there holds the formula

$$(1 + x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j$$

# 1 Real Intervals

$\forall a, b \in \mathbb{R}$ such that $a < b$, we denote $[a, b]$, the set of all $\mathbb{R}$ between $a$ and $b$ (inclusive)

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

Similarly, we have

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

by convention, $(a, a) = \emptyset$, the empty set

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

Subset of this form are call **intervals**. We also adopt the notation

$$(\infty, a] = \{x \in \mathbb{R} : x \leq a\}$$

$$(b, \infty] = \{x \in \mathbb{R} : x > a\}$$

We'll never write $[\infty, a]$, since $\pm\infty$ are **not** real numbers.

$[a, b]\,, (a, b]\,, [a, b)\,, (a, b)$, they are **bounded**

**Definition 6.** A set $B \subseteq \mathbb{R}$ is bounded below (respectively bounded above) if $\exists b \in \mathbb{R}$ such that $x \geq b \; \forall x \in B$ (respectively $x \leq b$ for all $x \in B$)

e.g. $\{0, 1, 50^{72}, -350\pi\}$ and $\left[-\frac{1}{\sqrt{10}}, 3\right)$ are bounded while $\mathbb{R}$ and $\mathbb{N}$ are not bounded

e.g. $[-357, \infty)$ is bounded below but not above

**Definition 7.** Let $B \subseteq \mathbb{R}$ be a subset that is bounded. We say that $b \in \mathbb{R}$ is the least upper bound of $B$ (also call the supremum of $B$) if

(i) $b$ is an upper bound for $B$

(ii) if $b'$ is also an upper bound for $B$, then we have $b \leq b'$

We denote this least upper bound by $\sup B$

**Remark 8.** It is easy to see that for a set $B$ bounded above. $\sup B$ is unique. To see this, suppose that both $\beta_1$ and $\beta_2$ are least upper bound for $B$. Then since $\beta_2$ is least upper bound and $\beta_1$ is an upper bound. We have $\beta_2 \leq \beta_1$. But also since $\beta_1$ is least upper bound and $\beta_2$ is a lower bound, we have $\beta_1 \leq \beta_2$. Hence $\beta_1 = \beta_2$

We have the corresponding notation for lower bounds

**Definition 9.** Let $A \subseteq \mathbb{R}$ be a subset bounded below. We say that $a \in \mathbb{R}$ is the greatest lower bound for $A$ (also called the infimum of $A$) if

(i) $a$ is an lower bound for $A$

(ii) if $a'$ is also an lower bound for $A$, then $a' \leq a$

For $B = (-1, \infty)\,, \inf B = -1$.

For $B = [-1, \infty)\,, \inf B = -1$.

For $A = [2, 10) \cup (510, 511] \cup \{520\}, \inf A = 2, \sup A = 520$

Note that some sets contain their infimum/supremum while others do not. We note down a property of the real-numbers which we state but do not prove

**Example.** Prove that if $a = (0, 1)\,, \sup A = 1$

*Proof.* Notice that if $x \in A$ then $x < 1$, so 1 is an upper bound for $A$. Suppose for contradiction that $\sup A \neq 1$. Then we must have $\sup A < 1$ but $m = \frac{1}{2}(\sup A + 1) \in A$ but $m > \sup A$. So $\sup A$ is not an upper bound for $A$ $\square$

# 2 Functions & Their Representation

A function is a "thing" that assigns a number to another number

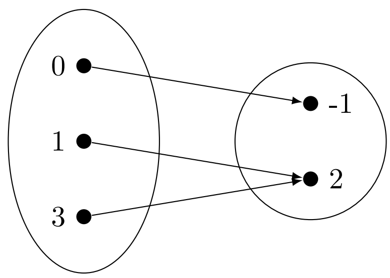
**Example.** the square function $x \mapsto x^2$

The way we represent this is by writing that $f$, the function such that $f(x) = x^2$, also written $f : x \mapsto x^2$

**Example.** We could also define a function, say $g$, that acts on $\{0, 1, 3\}$ and maps from elements of this set to $\{-1, 2\}$, for instance

$$g(0) = 1, \; g(1) = 2, \; g(3) = 2$$

One way of representing this is with the diagram




When defining a function $f$, we write $f : A \to B$, where $A$ is domain and $B$ is range

**Example.** Define the function $r : \left[-17, -\frac{\pi}{3}\right] \to \mathbb{R}$ by the explicit formula

$$r(x) = x^3, r : \left[-17, -\frac{\pi}{3}\right] \to \left[-17^3, -\left(\frac{\pi}{3}\right)^3\right] \subseteq \mathbb{R}$$

## 2.1 Operation between functions

Suppose $f_1, f_2$ have the same domain $A$, then we can define a new function, say $g$, to take the values of the sum of $f_1$ and $f_2$ i.e., for $f_1 : A \to B$ and $f_2 : A \to B$ we define $g : A \to B'$ bo be

$$g(x) = f_1(x) + f_2(x) \; \forall x \in A$$

Note that $B'$ might not be equal to $B$

**Example.** $f_1, f_2 : [0,1] \to [0,1]\,,\ f_1(x) = x,\ f_2(x) = \frac{1}{2}x,\ g(x) = \frac{3}{2}x$ and $g : [0,1] \to \left[0, \frac{3}{2}\right]$

For ease of notation, we write $g$ as $(f_1 + f_2)$

Similarly, we define the product function $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)\ \forall x \in A$

**Example.** $f(x) = \log x$ for $x \geq 1$, $g(x) = 10x^2\ \forall x \in \mathbb{R}$ To define $f + g$ and $f \cdot g$, we must to the smaller domain $\{x \in \mathbb{R} : x \geq 1\}$

## 2.2 Some examples of functions

### Polynomials

**Definition 10.** $f : \mathbb{R} \to \mathbb{R}$ is a polynomial function, if $\exists N \in \mathbb{N}$ and $\exists\{a_0, \ldots, a_N\} \in \mathbb{R}^{N+1}$

$$f(x) = a_0 + a_1 x + \ldots a_N x^N\ \forall x \in \mathbb{R}$$

### Rational function

**Definition 11.** We say that $f$ is a rational function if for some polynomial functions $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \frac{p(x)}{q(x)}\ \forall x \in \mathbb{R} \setminus R_q$$

where $R_q = \{x \in \mathbb{R} : q(x) = 0\}$ is the set of roots of $q$

### Construct functions

**Definition 12.** $f : \mathbb{R} \to \mathbb{R}$ is a constant function if $\exists c \in \mathbb{R}$ such that $f(x) = c\ \forall x \in \mathbb{R}$

### The identity

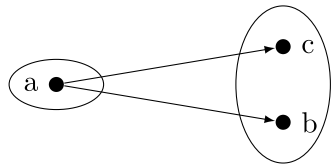
**Definition 13.** If $f(x) = x\ \forall x \in \mathbb{R}$ then we say that $f$ is the identity map.

## 2.3 Composition

**Definition 14.** Let $f : A \to B$ and $g : B \to C$ be functions. We define the composition $g \circ f : A \to C$ by $g \circ f(x) = g(f(x))\ \forall x \in A$

## 2.4 Formal definition

**Definition 15.** A function is a collection of pairs of points with the property if $(a, b)$ and $(a, c)$ belong to the collection, the $b = c$. The pairs of points are of the form $(a, f(a)$. The property in **Definition 15** ensure that we stay clear of a confusion of the sort $f(2) = 2$ and $f(2) = 3$, which would using the diagram representation.




**NOT** a function

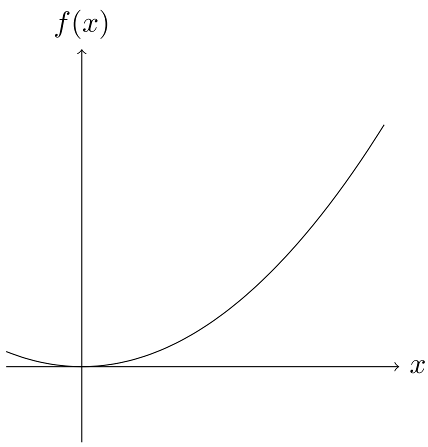
**Definition 16.** Let $f$ be a function and denote by $\mathcal{F}$ its collection of points. The domain of $f$, written $\mathrm{dom}(f)$, is the set of all points $a$ such that there exists some $b$ for which $(a, b) \in \mathcal{F}$.

i.e., $\mathrm{dom}(f) = \{a : \exists b \text{ for which } (a, b) \in \mathcal{F}\}$

Moreover, by **Definition 15** for each $a \in \mathrm{dom}(f)$ there exists $a$ unique $b$ such that $(a, b) \in \mathbf{F}$

## 2.5 Graphs of functions

An intimidate way to represent a function is by writing its coordinate pair on curves, i.e., drawing its graph




This diagram is representation of $\{(x, f(x)\}, x \in A$

**Definition 17.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say $f$ is **linear** if $\exists a \in \mathbb{R}$ such that

$$f(x) = ax, \ \forall x \in \mathbb{R}$$

**Definition 18.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say $f$ is **affine** if $\exists a \in \mathbb{R}$ such that

$$f(x) = ax + b, \ \forall x \in \mathbb{R}$$

**Definition 19.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say $f$ is **even** if $\exists a \in \mathbb{R}$ such that

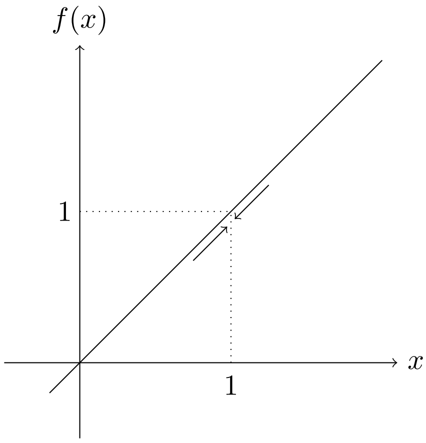
$$f(x) = f(-x), \ \forall x \in \mathbb{R}$$

**Definition 20.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say $f$ is **odd** if $\exists a \in \mathbb{R}$ such that

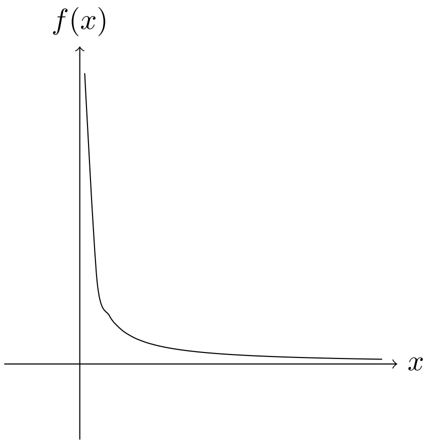
$$f(x) = -f(-x), \ \forall x \in \mathbb{R}$$

## 2.6 What is limit

What is a limit? Intutively, a function has a limit at a point $x_*$ if the function values $f(x)$ "approach" this limit number as $x$ gets closer to $x_*$




if $f(x) = x$ $\forall x \in \mathbb{R}$ that as $x$ increases to 1




as $x \to \infty, f(x)$ goes arbitrary close to 0, as $x \to 0, f(x)$ "explodes" and has not limit

This idea of a function having a limit is also preserve for more basic objects, e.g., sequence

e.g., the sequence of points $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$ where the $n^{th}$ element of the sequnec may be written as $a_n = 1 - \frac{1}{n}$, converge to 1 as $n \to \infty$

## definition of limit

**Definition 21.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $a, l \in \mathbb{R}$. We say that $f$ approach the limit $l$ near $a$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - l| < \varepsilon$$

We write $\lim_{x \to a} f(x) = l$

Some comments on **Definition** 21

(i) $\delta$ is allowed to depend on $\varepsilon, a, l$

(ii) "for all $\varepsilon > 0$" can be read as "given any $\varepsilon > 0$"

**Example.** Let $f(x) = cx$ for some $c \in \mathbb{R}$ we show that $\lim_{x \to 1} f(x) = c$

*Proof.* let $\varepsilon > 0$ be given. Then

$$\begin{aligned} |f(x) - c| &= |cx - c| \\ &= |c| \cdot |1 - x| \end{aligned}$$

So, letting $\delta = \delta(\varepsilon) = |c|^{-1} \cdot \varepsilon$, we get that

$$0 < |1 - x| < \delta \implies |f(x) - c| < \varepsilon$$

Since this hold for all $\varepsilon > 0$, we define $\lim_{x \to 1} f(x) = c$ $\square$

**Example.** Let $g(x) = x\sin(\frac{1}{x})$ for some $x \in (0, \infty)$. Then $\lim_{x\to 0} g(x) = 0$

*Proof.* Indeed, let $\varepsilon > 0$ be given. Notice that $|g(x)| = |x| \cdot |\sin(\frac{1}{x})| \leq |x|$

, thus, letting $\delta = \delta(\varepsilon) = \varepsilon$, we see that

$$0 < |x| < \delta \implies |g(x)| < \varepsilon$$

□

**Definition 22.** Let $f : \mathbb{R} \to \mathbb{R}$ and let $l \in \mathbb{R}$. We say that $f$ apporaches the limit $l$ as $x$ tends to infinity if: for all $\varepsilon > 0$, there exists $R > 0$ such that

$$x > R \implies |f(x) - l| < \varepsilon$$

We write $\lim_{x\to\infty} f(x) = l$ ($R$ is allowed to depend on $\varepsilon, l$)

**Example.** let $f(x) = \frac{1}{x}$ for $x > 0$. We show that $\lim_{x\to\infty} f(x) = 0$

letting $R(\varepsilon) = \varepsilon^{-1}$, wee see that $x > R \implies |f(x) - 0| < \varepsilon$

**Definition 23.** Let $l \in \mathbb{R}$ and $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that $a_n$ approaches the limit $l$ as $n$ tends to infinty if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n > N \implies |a_n - l| < \varepsilon$$

Write $\lim_{x\to\infty a_n} = l$

**Example.** For the sequence $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$ where $a_n = 1 - \frac{1}{n}$ $\forall n \in \mathbb{N}$ we see that $\lim_{x\to\infty a_n} = 1$

*Proof.* Indeed, let $\varepsilon > 0$ be given. Observe that $|a_n - 1| < \frac{1}{n}$ ,letting $N(\varepsilon) = \lceil \varepsilon^{-1} \rceil$, we see that, whenever $n > N$, $n > \varepsilon^{-1} \implies \frac{1}{n} < \varepsilon$ and $|a_n - 1| < \varepsilon$ for such $n$ □

What does it mean to not have a limit?

### what is no limit

**Corollary 24.** $f : \mathbb{R} \to \mathbb{R}$ does not approach the limit $l \in \mathbb{R}$ at the point $a \in \mathbb{R}$ if there exists some $\varepsilon_0 > 0$ such that for all $\delta > 0$ there exists $x_\delta \in \mathbb{R}$ for which there holds

$$|x_\delta - a| < \delta \text{ and } |f(x_\delta) - l| \geq \varepsilon_0$$

**Example.** We show that $\underset{x\mapsto\frac{1}{x}}{f:(0,1)\to(0,\infty)}$ has no limit at $x=0$

*Proof.* We show that $\forall p \geq 0$, $f$ does not approach the limit $p$ at $x=0$ Let $p\geq 0$ be given. We'll show that Corollary 24 holds with $\varepsilon_0 = 1$ Note that $|f(x)-p| = |\frac{1}{x}-p| = \frac{1}{x}-p$ provided $0 < x \leq \frac{1}{p}$. Also observe that $0 < x \leq \frac{1}{p+1} \implies \frac{1}{x} - p \geq p+1-p = 1$ This given any $\delta > 0$, choosing $x_\delta = \min\{\frac{\delta}{2}, \frac{1}{p+1}\}$ we get $0 < x_\delta < \delta$ and by $|f(x_\delta - p) \geq 1$ □

**Example.** Let $\underset{x\mapsto\sin(\frac{1}{x})}{f:(0,\infty)\to\mathbb{R}}$. We show $f$ does not approach the value 0 as $x\to 0$.

*Proof.* Indeed, for this case set $\varepsilon_0 = \frac{1}{2}$ and for every $\delta > 0$, set $x_\delta = \frac{1}{\frac{\pi}{2}+2\pi n_\delta}$ where $n_\delta \in \mathbb{N}$ chosen sufficiently large such that $0 < x_\delta < \delta$. For instance, $n_\delta = \lceil\frac{\delta^{-1}}{2\pi}\rceil$ clearify that $x_\delta = \frac{1}{\frac{\pi}{2}+2\pi n_\delta} < \frac{1}{2\pi n_\delta}$ and

$$\begin{aligned} n_\delta &\geq \frac{\delta^{-1}}{2\pi} \\ 2\pi n_\delta &\geq \delta^{-1} \\ \frac{1}{2\pi n_\delta} &\leq \delta \end{aligned}$$

Then, $0 < x_\delta < \delta$, and

$$\begin{aligned} f(x) &= \sin\left(\frac{1}{x_\delta}\right) \\ &= \sin\left(\frac{\pi}{2} + \frac{1}{x_\delta}\right) \\ &= \sin\left(\frac{\pi}{2}\right) = 1 \end{aligned}$$

So, $|x_\delta - 0| < \delta$ and $f(x_\delta) - 0| = 1 > \frac{1}{2} = \varepsilon_0 \left(\text{So, } \lim_{x\to 0} f(x) \neq 0\right)$ □

**Example 25.** Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x\in\mathbb{Q} \\ 0 & \text{if } x\notin\mathbb{Q} \end{cases}$$

$\lim_{x\to 0} f(x) = 0$ but $f$ has no limit at any other point $a \neq 0$

**Fact** Given $s < t$ real numbers:

(i) $\exists q \in \mathbb{Q}$ such that $s < q < t$

(ii) $\exists r \in \mathbb{R}\setminus\mathbb{Q}$ such that $s < r < t$

*Proof.* Fix $a > 0$ and let $l \in \mathbb{R}$ be arbitrary. There are 2 cases

1. Suppose $l = 0$ set $\varepsilon_0 = a$ Then, given $\delta > 0$ by Fact(i), $\exists x_\delta \in \mathbb{Q}$ such that $a < x_\delta < a + \delta$ and thus $|x_\delta - a| < \delta$ and $|f(x_\delta) - l| = x_\delta > a = \varepsilon_0$ so $f(x) \nrightarrow 0$ as $x \to a$

2. Suppose $l \neq 0$ set $\varepsilon_0 = \frac{|l|}{2}$ then given any $\delta > 0$ by Fact(ii), $\exists x_\delta \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x_\delta < a + \delta$, $|x_\delta - a| < \delta$ and $|f(x_\delta) - l| = |l| > \frac{|l|}{2} = \varepsilon_0$ repeating the same strategy for $a < 0$ concludes the proof.

□

## 2.7 Identity of Limit

**Theorem 26.** Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Suppose that for $\mu, \nu \in \mathbb{R}$ we have $\lim_{x \to a} f(x) = \mu$ and $\lim_{x \to a} f(x) = \nu$ then $\mu = \nu$ (i.e., the limit is unique)

*Proof.* Let $\varepsilon > 0$ be given. By the definition of the limit $\exists \delta_1 = \delta_1(\varepsilon, a, \mu) > 0$ such that $0 < |x - a| < \delta_1 \implies |f(x) - \mu| < \frac{\varepsilon}{2}$ also $\exists \delta_2 = \delta_2(\varepsilon, a, \nu) > 0$ such that $0 < |x - a| < \delta_2 \implies |f(x) - \nu| < \frac{\varepsilon}{2}$ Letting $\delta = \min\{\delta_1, \delta_2\} > 0$, we see that $|\mu - \nu| \leq |\mu - f(x)| + |f(x) - \nu|$, which provided $|x - a| < \delta$. Hence, $|\mu - \nu| < \varepsilon$ whenever $|x - a| < \delta$

We will show that $\mu - \nu = 0$. Suppose $\mu - \nu \neq 0$ then $|\mu - \nu| \geq 0$ but then, choosing $\varepsilon = \frac{1}{2}|\mu - \nu|$ we get $|\mu - \nu| < \frac{1}{2}|\mu - \nu|$

□

**Theorem 27.** Let $f, g : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Suppose that for $\mu, \nu \in \mathbb{R}$, $\lim_{x \to a} f(x) = \mu$ and $\lim_{x \to a} g(x) = \nu$ then

(a) $\lim_{x \to a} (f + g)(x) = \mu + \nu$

(b) $\lim_{x \to a} (f \cdot g)(x) = \mu \cdot \nu$

*Proof.* We will prove each separately

(a) Let $\varepsilon > 0$ be given. by the definition of limit, $\exists \delta_1 = \delta_1(\varepsilon, a, \mu) > 0$ such that $0 < |x - a| < \delta_1 \implies |f(x) - \mu| < \frac{\varepsilon}{2}$ and $\exists \delta_2 = \delta_2(\varepsilon, a, \nu) > 0$ such that $0 < |x - a| < \delta_2 \implies |g(x) - \nu| < \frac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$, provided $0 < |x - a| < \delta$, and observe that

$$\begin{aligned} |(f+g)(x) - (\mu+\nu)| &= |(f(x)-\mu) + (g(x)-\nu)| \\ &\le |f(x)-\mu| + |g(x)-\nu| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and $0 < |x-a| < \delta \implies |(f+g)(x) - (\mu+\nu)| < \varepsilon$

(b) Let $\varepsilon > 0$ be given, and observe that

$$\begin{aligned} |(f\cdot g)(x) - (\mu\nu)| &= |(f(x)g(x) - \mu g(x)) + (\mu g(x) - \mu\nu)| \\ &\le |g(x)| \cdot |f(x)-\mu| + |\mu| \cdot |g(x)-\nu| \end{aligned}$$

By the definition of limit $\exists \delta_g = \delta_g(\varepsilon, a, \nu) > 0$ such that $|g(x)-\nu| < \min\{\frac{\varepsilon}{2(1+|\mu|)}, 1\}$, whenever $0 < |x-a| < \delta_g$.

Note: whenever $0 < |x-a| < \delta_g$, we have

(i) $|g(x)-\nu| < \frac{\varepsilon}{2(1+|\mu|)}$ and $|\mu| \cdot |g(x)-\nu| < \frac{\varepsilon}{2}$

(ii) $|g(x)-\nu| < 1$ and $g(x) \le |g(x)-\nu| + |\nu| < 1 + |\nu|$

Again, by the definition of limit, $\exists \delta_f = \delta_f(\varepsilon, a, \mu, \nu) > 0$ such that

$$|x-a| < \delta_f \implies |f(x)-\mu| < \frac{\varepsilon}{2(1+|\nu|)}$$

then, we see that, for $\delta = \min\{\delta_f, \delta_g\}$ we have

$$|(f\cdot g)(x) - (\mu\nu)| < (1+|\nu|)\frac{\varepsilon}{2(1+|\nu|)} + \frac{\varepsilon}{2} = \varepsilon$$

$\square$

## 2.8 Infremum / Supremum

Our objective is to give a sense of infremum/supremum as limits. For example, consider $[1,2]$. This set has the property that for every $x \in [1,2]$, there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ belonging to $[1,2]$ such that $x_n \to x$ as $n \to \infty$. Indeed, $x \in (1,2)$, then for $M_x > 0$ sufficiently large. $x_n = x + \frac{1}{n\cdot M_x}$ is such that $x_n \in (1,2)$ and $x_n \to x$. And for when $x \in \{1,2\}$, we can build the sequences $x_n = \frac{1}{100n}$ or $x_n = 2 - \frac{1}{100n}$ This property also holds for $(1,2)$, but also even though $1,2 \notin (1,2)$, there exists sequences $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ such that $y_n, z_n \notin (1,2)$ $\forall n \in \mathbb{N}$ and $y_n \to 1$ as $n \to \infty$, $z_n \to 2$ as $n \to \infty$

It turns out that the property of "having a sequence inside the set converging to this point" is a property that holds true for the inf and sup of any bounded set.

To this end, we prove the following lemma

**Lemma 28.** Let $B \subseteq \mathbb{R}$ be a nonempty set bounded above. Then, given any $\varepsilon > 0$, there exists some $b_\varepsilon \in B$ such that

$$\sup B - \varepsilon < b_\varepsilon \ (\leq \sup B)$$

*Proof.* Let $\varepsilon > 0$ be given. Denote $\sup B$ by $\beta$. Suppose for contradiction that no such $b_\varepsilon$ exists, Then for all $b \in B$, we must have $b \leq \beta - \varepsilon$ but then $\beta - \varepsilon$ is the least upper bound for $B$ □

An analogous argument prove

**Lemma 29.** Let $A \subseteq \mathbb{R}$ be a nonempty set bounded below. Then, given any $\varepsilon > 0$, there exists some $a_\varepsilon \in B$ such that

$$(\inf A \leq)\ a_\varepsilon < \inf A + \varepsilon$$

**Corollary 30.** Let $A \subseteq \mathbb{R}$ be nonempty and bounded, then, $\exists (x_n)_{n\in\mathbb{N}}$ and $\exists (y_n)_{n\in\mathbb{N}}$ for which $x_n, y_n \in A$ for all $n \in \mathbb{N}$ and $\lim\limits_{x\to\infty} x_n = \inf A$, $\lim\limits_{x\to\infty} y_n = \sup A$

*Proof.* By Lemma 28 for each $n \in \mathbb{N}$, $\exists y_n \in A$ such that $\sup A - \frac{1}{n} < y_n \leq \sup A$ and $|y_n - \sup A| < \frac{1}{n} \to 0$ as $n \to \infty$ So, $\lim\limits_{x\to\infty} y_n = \sup A$. Also, for each $n \in \mathbb{N}$, by Lemma 29, $\exists x_n \in A$ such that $\inf A \leq x_n < \inf A + \frac{1}{n}$. i.e., $|x_n - \inf A| < \frac{1}{n} \to 0$ as $n \to \infty$. So, $\lim\limits_{x\to\infty} x_n = \inf A$. □

**Lemma 31.** Suppose $A$ is non-empty and bounded below. Let $B$ be the set of all lower bounds of $A$. Then $\inf A = \sup B$

*Proof.* There are 3 steps

**Step 1** [$B$ is nonempty] Since $A$ is bounded below, there exists at least one lower bound, which belongs to $B$, so $B \neq \emptyset$

**Step 2** [$B$ is bounded above] Suppose for contradiction that $B$ is not bounded above. Then given any $n \in \mathbb{N}, \exists x_n \in B$ such that $x_n \geq n$. Then by the definition of $B$, $x_n$ is a lower bound for $A$ for each $n \in \mathbb{N}$. Thus given any $a \in A$, we have $a \geq x_n \geq n\ \forall n \in \mathbb{N}$. Here $B$ is bounded above.

**Step 3** [showing the equality]

($\leq$) Let $\nu = \inf A$ nad $\mu = \sup B$. Since $\nu$ is the infimum of $A$, $\nu$ is a lower bound for $A$. So $\nu \in B \implies \nu \leq \sup B = \mu$

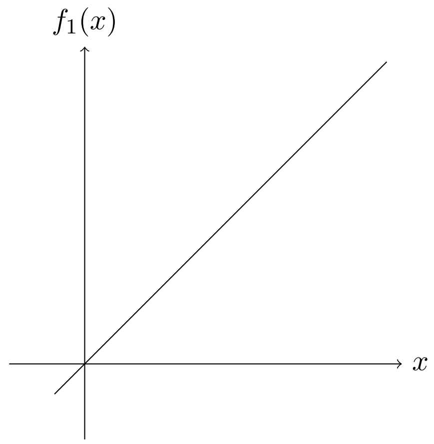
($\geq$) Let $\varepsilon > 0$ be arbitrary. Then by **Lemma** 28 $\exists b_\varepsilon \in B$ such that $\mu - \varepsilon < b_\varepsilon \leq \mu$. Hence, $\mu < \varepsilon + b_\varepsilon$. Now, let $a \in A$ be any point of $A$ and observe that since $b_\varepsilon \in B$, $b_\varepsilon \leq a \implies \mu < \varepsilon + b_\varepsilon \leq \varepsilon + a$. i.e., $\mu < \varepsilon + a$ for all $a \in A$. i.e., $\mu - \varepsilon < a \ \forall a \in A$. So, $\mu - \varepsilon$ is a lower bound for $A \implies \mu - \varepsilon < \inf A = \nu$ i.e., $\mu < \nu + \varepsilon$, but $\varepsilon > 0$ was arbitrary $\implies \mu \leq \nu$

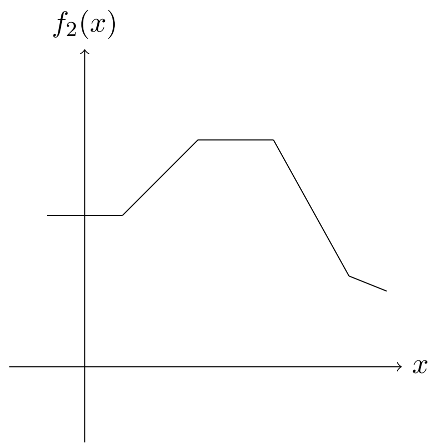
$\square$

# 3 Continuous Function

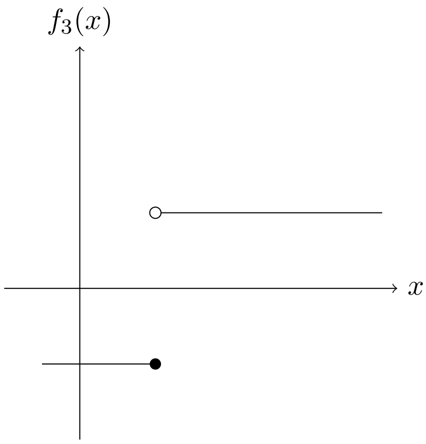
What does it mean for a function to be continuous?

Infinitely, this is some smoothness to the function i.g.,




But, on the other hand




is not continuous

## 3.1 Definition of Continuous Function

**Definition 32.** Let $f : \mathbb{R} \to \mathbb{R}$. We say $f$ is continuous at the point $x_0 \in \mathbb{R}$ if there holds $\lim_{x \to x_0} f(x) = f(x_0)$

**Remark.** For $f$ to be continuous at $x_0 \in \mathbb{R}$, we require

(i) $\lim_{x \to 0} f(x)$ exists

(ii) $\lim_{x \to 0} f(x) = f(x_0)$

Another way of writing Definition 32 is

**Definition** (32)**.** $f$ is continuous at $x_0$ if for all $\varepsilon > 0$, $\exists \delta = \delta(\varepsilon, x_0, f(x_0)) > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$$

**Example.** $f_3$ is not continuous at the point $x = 1$.

*Proof.* Indeed, setting $\varepsilon_0 = 1$, we see that, given any $\delta > 0$, the point $x_\delta = 1 + \frac{\delta}{2}$ is such that $|x_\delta - 1| < \delta$ and $|f(x_\delta) - f(1)| = |1 - (-1)| = 2 > \varepsilon_0$ $\square$

**Example.** $f(x) = x^2$ is continuous.

*Proof.* Indeed, let $x_0 \in \mathbb{R}$ be any point and observe that

$$\begin{aligned} |f(x) - f(x_0)| &= |x^2 - x_0^2| \\ &= |(x + x_0)(x - x_0)| \\ &= |x + x_0| \cdot |x - x_0| \end{aligned}$$

Let $\varepsilon > 0$ be given. Now let $\delta = \min\left\{1, \frac{\varepsilon}{2(1+|x_0|)}\right\}$, then

$$\begin{aligned} |x + x_0| &= |x - x_0 + 2x_0| \\ &\leq |x - x_0| + 2|x_0| \\ &\leq 1 + 2|x_0| \end{aligned}$$

Then provided $|x - x_0| < \delta$ we get

$$|f(x) - f(x_0)| \leq (1 + 2|x_0|) \cdot \frac{\varepsilon}{2(1 + |x_0|)} < \varepsilon$$

$\square$

**Example.**

$$f(x) = \begin{cases} 0 & x = 0 \\ x \sin\left(\frac{1}{x}\right) & x \neq 0 \end{cases}$$

$f$ is continuous at $x = 0$

*Proof.* Indeed, let $\varepsilon > 0$ be given and observe that

$$\begin{aligned} |f(x) - f(0)| &= |x| \cdot \left|\sin\left(\frac{1}{x}\right)\right| \quad \text{for } x \neq 0 \\ &\leq |x| \end{aligned}$$

So, letting $\delta(\varepsilon) = \frac{\varepsilon}{2}$, we see that

$$|x - 0| < \delta \implies |f(x) - f(0)| \leq \frac{\varepsilon}{2} < \varepsilon$$

$\square$

## 3.2 Identity of Continuous Function

**Lemma 33.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous at $a \in \mathbb{R}$. Then

(i) $f + g$ is continuous at $a$

(ii) $f \cdot g$ is continuous at $a$

*Proof.* We will prove each separately

(i) let $\varepsilon > 0$ be given. By the definition of continuous, $\exists \delta_f = \delta_f(\varepsilon, a) > 0$ such that

$$|x - a| < \delta_f \implies |f(x) - f(a)| < \frac{\varepsilon}{2}$$

and, $\exists \delta_g = \delta_g(\varepsilon, a) > 0$ such that

$$|x - a| < \delta_g \implies |g(x) - g(a)| < \frac{\varepsilon}{2}$$

So, letting $\delta = \min\{\delta_f, \delta_g\}$, suppose $|x - a| < \delta$, we see that

$$\begin{aligned} |f(x) + g(x) - (f(a) + g(a))| &\le |f(x) - f(a)| + |g(x) - g(a)| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

(ii) let $\varepsilon$ be given. Note that

$$|f(x)g(x) - f(a)g(a)| \le |g(x)| \cdot |f(x) - f(a)| + |f(a)| \cdot |g(x) - g(a)|$$

Since $g$ is continuous at $a$, $\exists \delta_g = \delta_g(\varepsilon, a) > 0$ such that

$$|x - a| < \delta_g \implies |g(x) - g(a)| < \min\left\{1, \frac{\varepsilon}{2(1 + |f(a)|)}\right\}$$

Then, provided $|x - a| < \delta_g$, we get

$$|g(x)| \le \overbrace{|g(x) - g(a)|}^{<1} + |g(a)| < 1 + |g(a)|$$

Also, since $f$ is continuous at $a$, $\exists \delta_f = \delta_f(\varepsilon, a) > 0$ such that

$$|x - a| < \delta_f \implies |f(x) - f(a)| < \frac{\varepsilon}{2(1 + |g(a)|)}$$

Then, letting $\delta = \min\{\delta_f, \delta_g\}$, we see that whenever $|x - a| < \delta$, we have form

$$|f(x)g(x) - f(a)g(a)| < (1 + |g(a)|)\left(\frac{\varepsilon}{2(1 + |g(a)|)}\right) + |f(a)| \cdot \frac{\varepsilon}{2(1 + |f(a)|)} < \varepsilon$$

$\square$

**Lemma 34.** Let $g : \mathbb{R} \to \mathbb{R}$ be continuous at $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be continuous at $g(a)$. Then $f \circ g$ is continuous at $a$

*Proof.* Let $\varepsilon > 0$ be given. Since $f$ is continuous at $g(a)$, $\exists \delta_f = \delta_f(\varepsilon, a) > 0$ such that

$$|y - g(a)| < \delta_f \implies |f(y) - f(g(a))| < \varepsilon$$

Meanwhile, $g$ is continuous at $a$, so $\exists \delta_g = \delta_g(\delta_f(\varepsilon, a), a) > 0$ such that

$$|x - a| < \delta_g \implies |g(x) - g(a)| < \delta_f$$

So, letting $\delta = \delta_g$, we see that

$$\begin{aligned} |x - a| < \delta &\implies |g(x) - g(a)| < \delta_f \\ &\implies |f(g(x)) - f(g(a))| < \varepsilon \end{aligned}$$

□

**Lemma 35.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at $a$, and suppose $f(a) > 0$. Then $\exists \delta > 0$ such that $f(x) > 0 \ \forall x \in [a - \delta, a + \delta]$

*Proof.* Since $f$ is continuous at $a$, $\exists \delta_f = \delta_f(a, \overbrace{f(a)}^{\varepsilon}) > 0$ such that

$$|x - a| < \delta_f \implies |f(x) - f(a)| < \overbrace{\frac{1}{2} f(a)}^{\varepsilon}$$

It follows that, for $x \in (a - \delta_f, a + \delta_f)$, we have

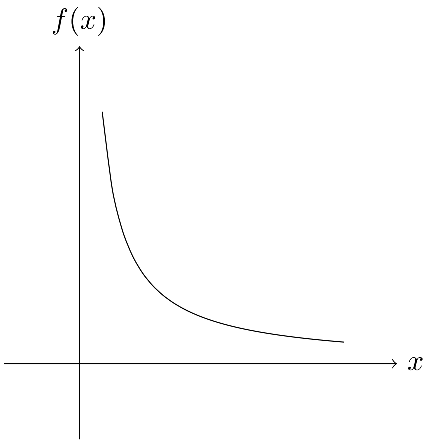
$$\begin{aligned} f(x) &= (f(x) - f(a)) + f(a) \\ &\geq f(a) - |f(x) - f(a)| \\ &> f(a) - \frac{1}{2} f(a) \\ &= \frac{1}{2} f(a) > 0 \end{aligned}$$

In turn, letting $\delta = \frac{1}{2}\delta_f$, we see that $f(x) > 0 \ \forall x \in [a - \delta, a + \delta]$ □

## 3.3 Definition of Left/Right Continuity

$f$ continuous on $(a, b)$ if $f$ is continuous at $x$, for all $x \in (a, b)$. What does it mean for $f$ to be continuous at on $[a, b]$? Should there be a difference between "continuous on $(a, b)$" and "continuous on $[a, b]$".

To gather intution, let's look at $f(x) = \frac{1}{x}$ on $(0, 1)$ and $[0, 1]$.




It's clar that $f$ is continuous at every point $a \in (0,1)$ but $\lim_{x \to 0} f(x)$ is not defined. So, it ought to not be continuous on $[0,1]$ We make the following define

**Definition** (32)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $a < b$ be real numbers.

(i) We say f is continuous on $(a,b)$ if $f$ is continuous at $x$ for every $x \in (a,b)$

(ii) We say $f$ is continuous on $[a,b]$ if $f$ is continuous on $(a,b)$ and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$

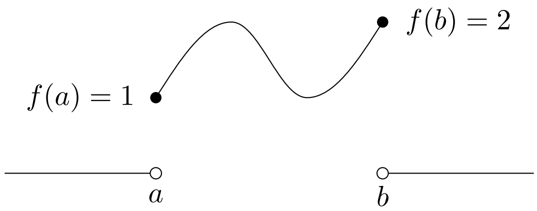
We write $\lim_{x \to a^+} f(x)$ to mean "The limit $f$ as $x$ tends to $a$ from above" also written $\lim_{x \searrow a} f(x)$ and $\lim_{x \to b^-} f(x)$ to mean "The limit $f$ as $x$ tends to $b$ from below" also written $\lim_{x \nearrow a} f(x)$

**Definition** (32)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$

(i) We write $\mu = \lim_{x \searrow a} f(x)$ if for all $\varepsilon > 0$, $\exists \delta > 0$ such that whenever $a < x < a + \delta$ we have $|\mu - f(x)| < \varepsilon$

(ii) We write $\nu = \lim_{x \nearrow a} f(x)$ if for all $\varepsilon > 0$, $\exists \delta > 0$ such that whenever $a - \delta < x < a$ we have $|\nu - f(x)| < \varepsilon$

**Example.** Considered this graph



then, $\lim_{x \searrow a} f(x) = 1$ and $\lim_{x \nearrow b} f(x) = 2$ on the other hand $\lim_{x \nearrow a} f(x) = 0$ and $\lim_{x \searrow b} f(x) = 0$

**Example.** $\lim_{x \to x_0} f(x)$ exists $\iff$ $\lim_{x \nearrow x_0} f(x)$ and $\lim_{x \searrow x_0} f(x)$ exists and are equal.

## 3.4 3 Hard Theorems

**Theorem 36** (Intermediate Value Theorem)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ for $a < b$. Suppose $f(a) < 0 < f(b)$ Then $\exists \xi \in (a, b)$ such that $f(\xi) = 0$

**Theorem 37.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ for $a < b$. Then $f$ is bounded above on $[a, b]$, i.e., $\exists M \in \mathbb{R}$ such that $f(x) \le M$ $x \in [a, b]$

**Theorem 38.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$. Then $\exists \xi \in [a, b]$ such that $f(x) \le f(\xi)$ $\forall x \in [a, b]$ i.e., $f(\xi) = \sup\{f(x) : x \in [a, b]\}$ (we say that $f$ achieves its supremum on $[a, b]$)

**Lemma** ($35'$)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $b \in \mathbb{R}$. Suppose $\lim_{x \nearrow b} f(x) = f(b) > 0$ Then $\exists \delta > 0$ such that $f(x) > 0$ for all $x \in (b - \delta, b)$

*Proof.* Directly from Definition 32(ii) (definition of $\lim_{x \nearrow b} f(x)$) such that

$$x \in (b - \delta, b) \implies |f(x) - f(b)| < \frac{1}{2} f(b)$$

Then for such $x \in (b - \delta, b)$ we have

$$\begin{aligned} f(x) &= (f(x) - f(b)) + f(b) \\ &\ge f(b) - \overbrace{|f(x) - f(b)|}^{< \frac{1}{2} f(b)} \\ &> \frac{1}{2} f(b) > 0 \end{aligned}$$

Hence, for $x \in \left(b - \frac{\delta}{2}, b\right)$ we have $f(x) > 0$ $\square$

**Lemma** ($35''$)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $b \in \mathbb{R}$. Suppose $\lim_{x \searrow a} f(x) = f(a) > 0$ Then $\exists \delta > 0$ such that $f(x) > 0$ for all $x \in (a, a + \delta)$

*Proof Theorem 36.* Define the set $A = \{x \in [a, b] : f(y) < 0\ \forall y \in [a, x]\}$ Since $f(a) < 0$, so $a \in A$, so $A \neq \emptyset$ Also, using Lemma $35''$ $\exists \delta_1 > 0$ such that $f(y) < 0$ $\forall y \in [a, a + \delta_1]$ so $a + \delta_1 \in A$, and by Lemma $35'$ $\exists \delta_2 > 0$ such that $f(y) > 0$ $\forall y \in [b - \delta_2, b]$ where

$b - \delta_2$ is an upper bound for $A$. So $A$ is bounded above and $\sup A$ is well-defined. Let $\alpha = \sup A$. We already know that $\alpha \in (a, b)$ our aim is to show that $f(\alpha) \neq 0$ We proceed by contradiction:

Suppose for contradiction that $f(\alpha) \neq 0$ There are 2 possibilities

(i) $f(\alpha) < 0$

(ii) $f(\alpha) > 0$

Suppose (i) holds, Since $\alpha \in (a, b)$ and $f(\alpha) < 0$ by **Lemma 35**, $\exists \delta_3 > 0$ such that $f(y) < 0 \; \forall y \in [\alpha - \delta_3, \alpha + \delta_3]$ But then $\alpha + \delta_3 \in A$ and $\alpha + \delta_3 > \alpha$

Suppose (ii) holds. Then since $\alpha \in (a, b)$, $f(\alpha) > 0$ and $f$ is continuous. By **Lemma 35**, $\exists \delta_4 > 0$ such that $f(x) > 0 \; \forall x \in [\alpha - \delta_4, \alpha + \delta_4]$ But then $\alpha = \sup A$ by **Lemma 28** $\exists x_0 \in A$ such that $\alpha - \frac{\delta_4}{2} < x_0$ Thus $x_0 \in (\alpha - \frac{\delta_4}{2}, \alpha) \subseteq [\alpha - \delta_4, \alpha + \delta_4] \implies f(x_0) > 0$ But $x_0 \in A$ so $(f_x) < 0$ $\square$

**Corollary 39.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and let $c \in \mathbb{R}$. Suppose $f(a) < c < f(b)$. Then $\exists \xi \in (a, b)$ such that $f(\xi) = c$

*Proof.* Define $g(x) = f(x) - c$ and apply **Theorem** 36 to $g$ $\square$

**Example 40.** Let $f(x) = x^4 + x - 3 \; \forall x \in \mathbb{R}$ **Fact**: all polynomials are continuous $\forall x \in \mathbb{R}$ A nice application of the Intermidiate Value Theorem is to find roots of continuous functions We can see by plugging in that

$$f(1) = 1 + (-1) - 3 = -3$$

$$f(2) = 16 + 2 - 3 = 15$$

IVT $\implies \exists x_0 \in (1, 2)$ such that $f(x_0) = 0$ This at least lets us estimate where roots are

**Example 41.** Let $f(x) = x^4 + x - 3 + \tan\left(\frac{x}{2}\right)$ (continuous on $(-\pi, \pi)$)

$$f(-1) = -3 - \tan\left(\frac{1}{2}\right) < 0$$

$$f(2) = 15 - \tan\left(\frac{1}{2}\right) > 0$$

IVT $\implies \exists x_0 \in (-1, 2)$ such that $f(x_0) = 0$

What is it useful for? If we look at the set $f([a, b]) = \{f(x) : x \in [a, b]\}$ and Theorem 37 tell us that set is bounded. Since the set is bounded, it has a supremum. You can think of this as "local max" of $f$ on the interval $[a, b]$

Before proving Theorem 37, let's look at one of its consequences.

**Corollary 42.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$. Then $f$ is bounded below on $[a, b]$, i.e., $\exists m \in \mathbb{R}$ such that $m \leq f(x)\ \forall x \in [a, b]$

*Proof.* Since $f$ is continuous, so is $(-f)$. Now apply Theorem 37 to $-f$. $\exists M \in \mathbb{R}$ such that $-f(x) \leq M\ \forall x \in [a, b]$ the, $f(x) \leq -M\ \forall x \in [a, b]$ $\square$

**Takeaway**: If $f$ is continuous on $[a, b]$, then $f$ is bounded above + below on $[a, b]$

To prove Theorem 37, we'll need a few Lemmas.

**Lemma 43.** Let $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$, then $\exists \delta > 0$ such that $f$ is bounded above on the interval $[a - \delta, a + \delta]$

*Proof.* Since $f$ is continuous at $a$, $\exists \delta = \delta(a, \overbrace{1}^{\varepsilon})$ such that $|x - a| < \delta \implies |f(x) - f(a)| < 1$ This for such $x$ we have

$$\begin{aligned} f(x) &= f(x) - f(a) + f(a) \\ &\leq |f(x) - f(a)| + |f(a)| \\ &< 1 + |f(a)| \end{aligned}$$

For $x$ satisfying $|x - a| < \delta$, we have $f(x) < 1 + f(a)$.

In particular, $f(x) < 1 + f(a)\ \forall x \in \left[a - \frac{\delta}{2}, a + \frac{\delta}{2}\right]$ $\square$

**Lemma.** (43′) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $b \in \mathbb{R}$. Suppose $\lim\limits_{x \nearrow b} f(x) = f(b)$. Then $\exists \delta > 0$ such that $f$ is bounded above on $[b - \delta, b]$

*Proof.* By Definition 32″, $\exists \delta = \delta(b, 1)$ such that

$$0 < |x - b| < \delta \implies |f(x) - f(b)| < 1$$

Therefore, for such $x$,

$$\begin{aligned} f(x) &= f(x) - f(b) + f(b) \\ &\leq |f(x) - f(b)| + |f(b)| \\ &< 1 + |f(b)| \end{aligned}$$

$f(x) < f(b) + 1\ \forall x \in \left[b - \frac{\delta}{2}, b\right]$ $\square$

**Lemma.** (43″) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $a \in \mathbb{R}$. Suppose $\lim\limits_{x \searrow a} f(x) = f(a)$. Then $\exists \delta > 0$ such that $f$ is bounded above on $[a, a + \delta]$

*Proof Theorem 37.* As in the proof of Theorem 36, consider the set

$$A = \{x \in [a, b] : f \text{ is bounded above on } [a, x]\}$$

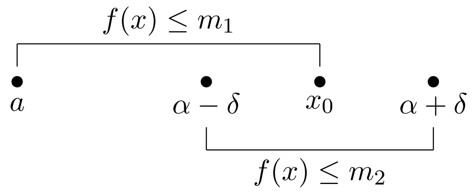
Since $a \in A$, we know $a \neq \emptyset$. Moreover, the point $b$ is an upper bound for $A$, so $\sup A = \alpha$ exists.

Our objective is to show that $\alpha = b$.

Suppose for contradiction that $\alpha < b$. (Note that we must have $a < \alpha$. We can't have $a > \alpha$ since $a \in A$. and $\sup A \geq a$. If $\alpha = a$, then $A = \{a\}$, but we know from Lemma $43''$ that $\exists \delta > 0$ such that $[a, a + \delta] \subseteq A$)

By assumption $a < \alpha < b$ and so Lemma 43 $\implies$ $\exists \delta > 0$ such that $f$ isbounded on $[\alpha - \delta, \alpha + \delta]$. Let's say $f(x) \leq m_2$ on this interval $[\alpha - \delta, \alpha + \delta]$.

By Lemma 28 (Alternate definition of supremum) $\exists x_0 \in A$ such that $\alpha - \delta < x_0 \leq \alpha$. $f$ is bounded above on $[a, x_0]$ (by the definition of $A$). say $f(x) \leq m_1$ on $[a, x_0]$




Thus, $f(x) \leq \max\{m_1, m_2\}$ $\forall x \in [a, \alpha + \delta]$ We deduce that $\alpha + \delta \in A$ and $\alpha + \delta > \alpha = \sup A$. Hence,

$$\begin{aligned} \alpha = b &\iff \sup A = b \\ &\implies f \text{ is bounded above on } [a, b] \text{ for every } x < b \text{ } 1 \end{aligned}$$

Finally, using continuity at the point $b$ by Lemma $43'$ $\exists \delta'$ such that $f$ is bounded on $[b - \delta', b]$ 2.

Hence, choosing $x = b - \delta'$ in 1, $\exists M$ such that $f(x) \leq M$, $\forall x \in [a, b - \delta']$. and by 2, $\exists M_2$ such that $f(x) \leq M_2$, $\forall x \in [b - \delta', b]$. So, $f(x) \leq \max\{M, M_2\}$ $\forall x \in [a, b]$. ☐

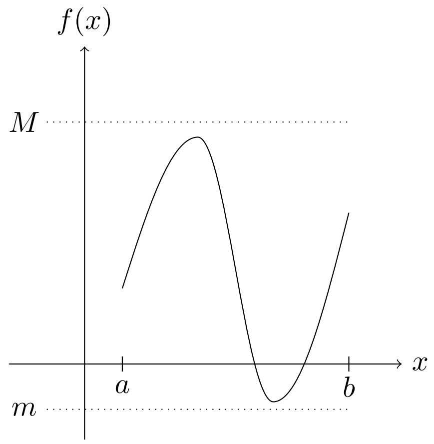
Summarize steps:

(i) define a good set $A$

(ii) show $b = \sup A$

(iii) show $b \in A$

The picture is




Whenever $f$ is continuous on $[a, b]$, $\exists M > m$ such that $m \leq f(x) \leq M\ \forall x \in [a, b]$

**Note:** We must be careful aboue being continuous on $[a, b]$, and mot just $(a, b)$. Indeed, $\underset{x \mapsto \frac{1}{x}}{f:(0,1) \to (0,\infty)}$, $f$ is continuous on $[\tilde{x}, \infty)$ for every $\tilde{x} > 0$, but it is <u>not</u> continuous on $[0, \infty)$.

**Question:** does these exists $\xi_1, \xi_2 \in [a, b]$ such that

$$f(\xi_1) = \inf_{[a,b]} f \text{ and } f(\xi_2) = \sup_{[a,b]} f$$

**Anwer:** Yes

Later on, when we discuss differentiability, if sup/inf is achieved in $(a, b)$, then $f' = 0$ at such points. This we will prove later.

*Proof of Theorm 38.* We already know from Theorem 37 that $f$ is bounded on $[a, b]$, i.e., the set $B = f([a, b]) = \{f(x) : x \in [a, b]\}$ is bounded. This set is nonempty and so $\beta = \sup B$ is well-defined; Since $\beta \geq f(x)\ \forall x \in [a, b]$ it suffies to show that $\exists \xi \in [a, b]$ such that $f(\xi) = \beta$.

Suppose for contradiction that this is not the case, i.e., $\beta \neq f(y)\ \forall y \in [a, b]$ Then the function $g : [a, b] \to \mathbb{R}$, defined by $g(x) = \frac{1}{\beta - f(x)} \forall x \in [a, b]$, is well-defined and $g$ is continuous on $[a, b]$ by virtue of Lemma 33

Since $g$ is continuous, by Theorem 37 $\implies$ $g$ is bounded above on $[a, b]$ However, by Lemma 28, given any $n \in \mathbb{N}, \exists x_n \in [a, b]$ such that

$$\beta - \frac{1}{n} < f(x_n) \leq \beta \implies g(x_n) \geq \frac{1}{\beta - \left(\beta - \frac{1}{n}\right)} = n$$

Hence given any $n \in \mathbb{N}, \exists x_n \in [a, b]$ such that $g(x_n) \geq n$ and therefore $g$ is unbounded on $[a, b]$. $\square$

We've actually proved

**Corollary 44.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$. Then $\exists \xi \in [a, b]$ such that $f(\xi) = \sup\{f(x) : x \in [a, b]\}$ (we often write with the shorthand $\sup_{[a,b]} f$)

**Corollary 45.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$. Then $\exists \xi \in [a, b]$ such that $f(\xi) = \inf\{f(x) : x \in [a, b]\}$

*Proof.* Apploy Corollary 44 to the function $-f$ and use the result $\inf B = -\sup(-B)$. $\square$

**Example 46.** Suppose $f, g$ are continuous on $[a, b]$ and $f(a) < g(a)$ and $f(b) > g(b)$. Then $\exists x \in [a, b]$ such that $f(x) = g(x)$ (in actual fact, $x \in (a, b)$)

*Proof.* define $h(x) = f(x) - g(x)$. Then $h$ is continuous on $[a, b]$, $h(a) < 0 < h(b)$ so from Theorem 36, $\exists \xi \in (a, b)$ such that $h(\xi) = 0 \implies f(\xi) = g(\xi)$ $\square$

**Example 47.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[0, 1]$ and suppose $0 \leq f(x) \leq 1$ $\forall x \in [0, 1]$. Then $\exists x_0 \in [0, 1]$ such that $f(x_0) = x_0$ (we can imagine that $f$ cross $y = x$)

*Proof.* Note that if $f(0) = 0$ on if $f(1) = 1$, then we are done. Suppose that $f(0) \neq 0$ and $f(1) \neq 1$ then $0 < f(0)$ and $f(1) < 1$ Let $g(x) = x - f(x)$. Then, $g(0) = 0 - f(0) < 0$ and $g(1) = 1 - f(1) > 0$. So, $g$ is continuous and $g(0) < 0 < g(1)$, where Theorem 36 $\exists x_0 \in [0, 1]$ such that $g(x_0) = 0$ and hence $x_0 = f(x_0)$ $\square$