MATH 421 Lecture Notes

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Fall 2022

Contents

Properties of Real Number

Definition 1. Given any $a \in \mathbb{R}$, we define its absolute value to be

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ a & \text{if } a < 0 \end{cases}$$

Theorem 2 (Triangular Inequality)**.** Given $a, b \in \mathbb{R}$, there holds

$$|a + b| \leq |a| + |b|$$

Method of Proof

Direct proof

some statements can be shown to be true through a direct arguement e.g. our proof of Theorem 1

Theorem 3. hello

Proof by induction

the aim is to proof that a statement is true for all rational number

(i) Show the statement is true for $n = 1$

(ii) Assume the statement is true for general $n \in \mathbb{N}$

(iii) Using assumption (ii), prove the statement is true for $n + 1$

(iv) Conclude your proof with a sentence like "by mathematical information, the result holds for all $n \in \mathbb{N}$"

Example 4. Show that $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$

Theorem 5. Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, there holds the formula

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j$$

1. Real Intervals

$\forall a, b \in \mathbb{R}$ such that $a < b$, we denote $[a, b]$, the set of all $\mathbb{R}$ between a and b (inclusive)

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

Similarly, we have

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

by convention, $(a, a) = \emptyset$, the empty set

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

Subset of this form are call **intervals**. We also adopt the notation

$$(\infty, a] = \{x \in \mathbb{R} : x \leq a\}$$

$$(b, \infty] = \{x \in \mathbb{R} : x > a\}$$

We'll never write $[\infty, a]$, since $\pm\infty$ are **not** real numbers.

$[a, b]\, , (a, b]\, , [a, b)\, , (a, b)$, they are **bounded**

Definition 6. A set $B \subseteq \mathbb{R}$ is bounded below (respectively bounded above) if $\exists b \in \mathbb{R}$ such that $x \geq b \; \forall x \in B$ (respectively $x \leq b$ for all $x \in B$)

e.g. $\{0, 1, 50^{72}, -350\pi\}$ and $\left[-\frac{1}{\sqrt{10}}, 3\right)$ are bounded while $\mathbb{R}$ and $\mathbb{N}$ are not bounded

e.g. $[-357, \infty)$ is bounded below but not above

Definition 7. Let $B \subseteq \mathbb{R}$ be a subset that is bounded. We say that $b \in \mathbb{R}$ is the least upper bound of B (also call the supremum of B) if

(i) b is an upper bound for B

(ii) if b' is also an upper bound for B, then we have $b \leq b'$

We denote this least upper bound by $\sup B$

Remark 8. It is easy to see that for a set B bounded above. $\sup B$ is unique. To see this, suppose that both β_1 and β_2 are least upper bound for B. Then since β_2 is least upper bound and β_1 is an upper bound. We have $\beta_2 \leq \beta_1$. But also since β_1 is least upper bound and β_2 is a lower bound, we have $\beta_1 \leq \beta_2$. Hence $\beta_1 = \beta_2$

We have the corresponding notation for lower bounds

Definition 9. Let $A \subseteq \mathbb{R}$ be a subset bounded below. We say that $a \in \mathbb{R}$ is the greatest lower bound for A (also called the infimum of A) if

(i) a is an lower bound for A

(ii) if a' is also an lower bound for A, then $a' \leq a$

For $B = (-1, \infty)\,, \inf B = -1$.

For $B = [-1, \infty)\,, \inf B = -1$.

For $A = [2, 10) \cup (510, 511] \cup \{520\}, \inf A = 2, \sup A = 520$

Note that some sets contain their infimum/supremum while others do not. We note down a property of the real-numbers which we state but do not prove

Example. Prove that if $a = (0, 1)\,, \sup A = 1$

Proof. Notice that if $x \in A$ then $x < 1$, so 1 is an upper bound for A. Suppose for contradiction that $\sup A \neq 1$. Then we must have $\sup A < 1$ but $m = \frac{1}{2}(\sup A + 1) \in A$ but $m > \sup A$. So $\sup A$ is not an upper bound for A $\square$

2. Functions & Their Representation

A function is a “thing” that assigns a number to another number

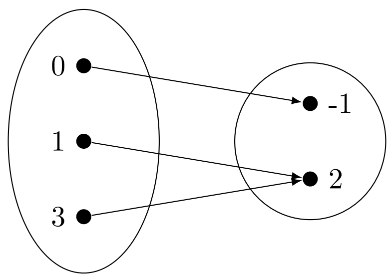
Example. the square function $x \mapsto x^2$

The way we represent this is by writing that f, the function such that $f(x) = x^2$, also written $f : x \mapsto x^2$

Example. We could also define a function, say g, that acts on $\{0, 1, 3\}$ and maps from elements of this set to $\{-1, 2\}$, for instance

$$g(0) = 1,\ g(1) = 2,\ g(3) = 2$$

One way of representing this is with the diagram



When defining a function f, we write $f : A \to B$, where A is domain and B is range

Example. Define the function $r : \left[-17, -\frac{\pi}{3}\right] \to \mathbb{R}$ by the explicit formula

$$r(x) = x^3, r : \left[-17, -\frac{\pi}{3}\right] \to \left[-17^3, -\left(\frac{\pi}{3}\right)^3\right] \subseteq \mathbb{R}$$

2.1. Operation between functions

Suppose f_1, f_2 have the same domain A, then we can define a new function, say g, to take the values of the sum of f_1 and f_2 i.e., for $f_1 : A \to B$ and $f_2 : A \to B$ we define $g : A \to B'$ bo be

$$g(x) = f_1(x) + f_2(x)\ \forall x \in A$$

Note that B' might not be equal to B

Example. $f_1, f_2 : [0,1] \to [0,1]$, $f_1(x) = x$, $f_2(x) = \frac{1}{2}x$, $g(x) = \frac{3}{2}x$ and $g : [0,1] \to \left[0, \frac{3}{2}\right]$

For ease of notation, we write g as $(f_1 + f_2)$

Similarly, we define the product function $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)\ \forall x \in A$

Example. $f(x) = \log x$ for $x \geq 1$, $g(x) = 10x^2\ \forall x \in \mathbb{R}$ To define $f + g$ and $f \cdot g$, we must to the smaller domain $\{x \in \mathbb{R} : x \geq 1\}$

2.2. Some examples of functions

Polynomials

Definition 10. $f : \mathbb{R} \to \mathbb{R}$ is a polynomial function, if $\exists N \in \mathbb{N}$ and $\exists \{a_0, \ldots, a_N\} \in \mathbb{R}^{N+1}$

$$f(x) = a_0 + a_1 x + \ldots a_N x^N\ \forall x \in \mathbb{R}$$

Rational function

Definition 11. We say that f is a rational function if for some polynomial functions $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \frac{p(x)}{q(x)}\ \forall x \in \mathbb{R} \setminus R_q$$

where $R_q = \{x \in \mathbb{R} : q(x) = 0\}$ is the set of roots of q

Construct functions

Definition 12. $f : \mathbb{R} \to \mathbb{R}$ is a constant function if $\exists c \in \mathbb{R}$ such that $f(x) = c\ \forall x \in \mathbb{R}$

The identity

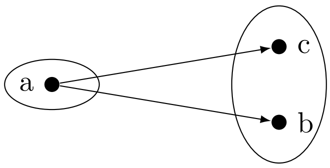
Definition 13. If $f(x) = x\ \forall x \in \mathbb{R}$ then we say that f is the identity map.

2.3. Composition

Definition 14. Let $f : A \to B$ and $g : B \to C$ be functions. We define the composition $g \circ f : A \to C$ by $g \circ f(x) = g(f(x))\ \forall x \in A$

2.4. Formal definition

Definition 15. A function is a collection of pairs of points with the property if (a, b) and (a, c) belong to the collection, the $b = c$. The pairs of points are of the form $(a, f(a)$. The property in **Definition 15** ensure that we stay clear of a confusion of the sort $f(2) = 2$ and $f(2) = 3$, which would using the diagram representation.



NOT a function

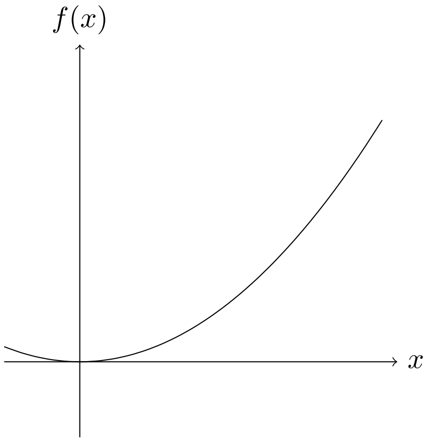
Definition 16. Let f be a function and denote by $\mathcal{F}$ its collection of points. The domain of f, written $\mathrm{dom}(f)$, is the set of all points a such that there exists some b for which $(a, b) \in \mathcal{F}$.

i.e., $\mathrm{dom}(f) = \{a : \exists b \text{ for which } (a, b) \in \mathcal{F}\}$

Moreover, by **Definition 15** for each $a \in \mathrm{dom}(f)$ there exists a unique b such that $(a, b) \in \mathbf{F}$

2.5. Graphs of functions

An intimidate way to represent a function is by writing its coordinate pair on curves, i.e., drawing its graph



This diagram is representation of $\{(x, f(x)\}, x \in A$

Definition 17. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say f is **linear** if $\exists a \in \mathbb{R}$ such that

$$f(x) = ax, \ \forall x \in \mathbb{R}$$

Definition 18. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say f is **affine** if $\exists a \in \mathbb{R}$ such that

$$f(x) = ax + b, \ \forall x \in \mathbb{R}$$

Definition 19. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say f is **even** if $\exists a \in \mathbb{R}$ such that

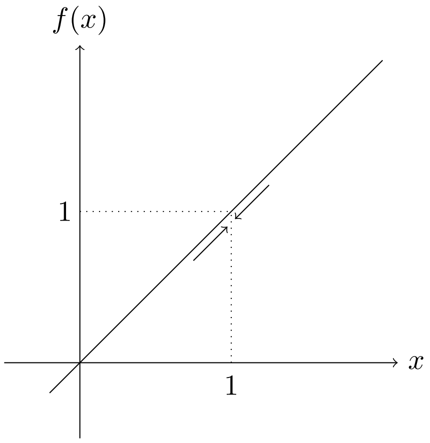
$$f(x) = f(-x), \ \forall x \in \mathbb{R}$$

Definition 20. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say f is **odd** if $\exists a \in \mathbb{R}$ such that

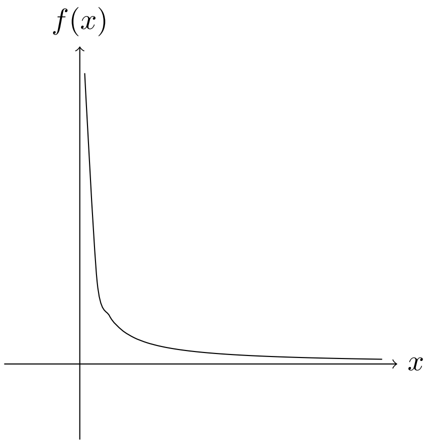
$$f(x) = -f(-x), \ \forall x \in \mathbb{R}$$

2.6. What is limit

What is a limit? Intutively, a function has a limit at a point x_* if the function values $f(x)$ "approach" this limit number as x gets closer to x_*



if $f(x) = x \ \forall x \in \mathbb{R}$ that as x increases to 1



as $x \to \infty, f(x)$ goes arbitrary close to 0, as $x \to 0, f(x)$ "explodes" and has not limit

This idea of a function having a limit is also preserve for more basic objects, e.g., sequence

e.g., the sequence of points $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$ where the n^{th} element of the sequnec may be written as $a_n = 1 - \frac{1}{n}$, converge to 1 as $n \to \infty$

definition of limit

Definition 21. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $a, l \in \mathbb{R}$. We say that f approach the limit l near a if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - l| < \varepsilon$$

We write $\lim\limits_{x \to a} f(x) = l$

Some comments on **Definition** 21

(i) δ is allowed to depend on ε, a, l

(ii) "for all $\varepsilon > 0$" can be read as "given any $\varepsilon > 0$"

Example. Let $f(x) = cx$ for some $c \in \mathbb{R}$ we show that $\lim\limits_{x \to 1} f(x) = c$

Proof. let $\varepsilon > 0$ be given. Then

$$\begin{aligned} |f(x) - c| &= |cx - c| \\ &= |c| \cdot |1 - x| \end{aligned}$$

So, letting $\delta = \delta(\varepsilon) = |c|^{-1} \cdot \varepsilon$, we get that

$$0 < |1 - x| < \delta \implies |f(x) - c| < \varepsilon$$

Since this hold for all $\varepsilon > 0$, we define $\lim\limits_{x \to 1} f(x) = c$ ☐

Example. Let $g(x) = x\sin(\frac{1}{x})$ for some $x \in (0, \infty)$. Then $\lim\limits_{x\to 0} g(x) = 0$

Proof. Indeed, let $\varepsilon > 0$ be given. Notice that $|g(x)| = |x| \cdot |\sin(\frac{1}{x})| \leq |x|$

, thus, letting $\delta = \delta(\varepsilon) = \varepsilon$, we see that

$$0 < |x| < \delta \implies |g(x)| < \varepsilon$$

□

Definition 22. Let $f : \mathbb{R} \to \mathbb{R}$ and let $l \in \mathbb{R}$. We say that f apporaches the limit l as x tends to infinity if: for all $\varepsilon > 0$, there exists $R > 0$ such that

$$x > R \implies |f(x) - l| < \varepsilon$$

We write $\lim\limits_{x\to\infty} f(x) = l$ (R is allowed to depend on ε, l)

Example. let $f(x) = \frac{1}{x}$ for $x > 0$. We show that $\lim\limits_{x\to\infty} f(x) = 0$

letting $R(\varepsilon) = \varepsilon^{-1}$, wee see that $x > R \implies |f(x) - 0| < \varepsilon$

Definition 23. Let $l \in \mathbb{R}$ and $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that a_n approaches the limit l as n tends to infinty if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n > N \implies |a_n - l| < \varepsilon$$

Write $\lim\limits_{x\to\infty} a_n = l$

Example. For the sequence $\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$ where $a_n = 1 - \frac{1}{n}$ $\forall n \in \mathbb{N}$ we see that $\lim\limits_{x\to\infty} a_n = 1$

Proof. Indeed, let $\varepsilon > 0$ be given. Observe that $|a_n - 1| < \frac{1}{n}$, letting $N(\varepsilon) = \lceil \varepsilon^{-1} \rceil$, we see that, whenever $n > N$, $n > \varepsilon^{-1} \implies \frac{1}{n} < \varepsilon$ and $|a_n - 1| < \varepsilon$ for such n □

What does it mean to not have a limit?

what is no limit

Corollary 24. $f : \mathbb{R} \to \mathbb{R}$ does not approach the limit $l \in \mathbb{R}$ at the point $a \in \mathbb{R}$ if there exists some $\varepsilon_0 > 0$ such that for all $\delta > 0$ there exists $x_\delta \in \mathbb{R}$ for which there holds

$$|x_\delta - a| < \delta \text{ and } |f(x_\delta) - l| \geq \varepsilon_0$$

Example. We show that $\underset{x\mapsto\frac{1}{x}}{f:(0,1)\to(0,\infty)}$ has no limit at $x=0$

Proof. We show that $\forall p \geq 0$, f does not approach the limit p at $x=0$ Let $p\geq 0$ be given. We'll show that Corollary 24 holds with $\varepsilon_0 = 1$ Note that $|f(x)-p| = |\frac{1}{x}-p| = \frac{1}{x}-p$ provided $0<x\leq\frac{1}{p}$. Also observe that $0<x\leq\frac{1}{p+1} \implies \frac{1}{x}-p\geq p+1-p=1$ This given any $\delta>0$, choosing $x_\delta=\min\{\frac{\delta}{2},\frac{1}{p+1}\}$ we get $0<x_\delta<\delta$ and by $|f(x_\delta-p)\geq 1$ $\square$

Example. Let $\underset{x\mapsto\sin(\frac{1}{x})}{f:(0,\infty)\to\mathbb{R}}$. We show f does not approach the value 0 as $x\to 0$.

Proof. Indeed, for this case set $\varepsilon_0=\frac{1}{2}$ and for every $\delta>0$, set $x_\delta=\frac{1}{\frac{\pi}{2}+2\pi n_\delta}$ where $n_\delta\in\mathbb{N}$ chosen sufficiently large such that $0<x_\delta<\delta$. For instance, $n_\delta=\lceil\frac{\delta^{-1}}{2\pi}\rceil$ clearify that $x_\delta=\frac{1}{\frac{\pi}{2}+2\pi n_\delta}<\frac{1}{2\pi n_\delta}$ and

$$\begin{aligned} n_\delta &\geq \frac{\delta^{-1}}{2\pi} \\ 2\pi n_\delta &\geq \delta^{-1} \\ \frac{1}{2\pi n_\delta} &\leq \delta \end{aligned}$$

Then, $0<x_\delta<\delta$, and

$$\begin{aligned} f(x) &= \sin\left(\frac{1}{x_\delta}\right) \\ &= \sin\left(\frac{\pi}{2}+\frac{1}{x_\delta}\right) \\ &= \sin\left(\frac{\pi}{2}\right)=1 \end{aligned}$$

So, $|x_\delta-0|<\delta$ and $f(x_\delta)-0|=1>\frac{1}{2}=\varepsilon_0$ $\left(\text{So, } \lim_{x\to 0} f(x)\neq 0\right)$ $\square$

Example 25. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases} x & \text{if } x\in\mathbb{Q} \\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$$

$\lim_{x\to 0} f(x)=0$ but f has no limit at any other point $a\neq 0$

Fact Given $s<t$ real numbers:

(i) $\exists q\in\mathbb{Q}$ such that $s<q<t$

(ii) $\exists r\in\mathbb{R}\setminus\mathbb{Q}$ such that $s<r<t$

Proof. Fix $a > 0$ and let $l \in \mathbb{R}$ be arbitrary. There are 2 cases

1. Suppose $l = 0$ set $\varepsilon_0 = a$ Then, given $\delta > 0$ by Fact(i), $\exists x_\delta \in \mathbb{Q}$ such that $a < x_\delta < a + \delta$ and thus $|x_\delta - a| < \delta$ and $|f(x_\delta) - l| = x_\delta > a = \varepsilon_0$ so $f(x) \nrightarrow 0$ as $x \to a$

2. Suppose $l \neq 0$ set $\varepsilon_0 = \frac{|l|}{2}$ then given any $\delta > 0$ by Fact(ii), $\exists x_\delta \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x_\delta < a + \delta$, $|x_\delta - a| < \delta$ and $|f(x_\delta) - l| = |l| > \frac{|l|}{2} = \varepsilon_0$ repeating the same strategy for $a < 0$ concludes the proof.

□

2.7. Identity of Limit

Theorem 26. Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Suppose that for $\mu, \nu \in \mathbb{R}$ we have $\lim_{x \to a} f(x) = \mu$ and $\lim_{x \to a} f(x) = \nu$ then $\mu = \nu$ (i.e., the limit is unique)

Proof. Let $\varepsilon > 0$ be given. By the definition of the limit $\exists \delta_1 = \delta_1(\varepsilon, a, \mu) > 0$ such that $0 < |x - a| < \delta_1 \implies |f(x) - \mu| < \frac{\varepsilon}{2}$ also $\exists \delta_2 = \delta_2(\varepsilon, a, \nu) > 0$ such that $0 < |x - a| < \delta_2 \implies |f(x) - \nu| < \frac{\varepsilon}{2}$ Letting $\delta = \min\{\delta_1, \delta_2\} > 0$, we see that $|\mu - \nu| \leq |\mu - f(x)| + |f(x) - \nu|$, which provided $|x - a| < \delta$. Hence, $|\mu - \nu| < \varepsilon$ whenever $|x - a| < \delta$

We will show that $\mu - \nu = 0$. Suppose $\mu - \nu \neq 0$ then $|\mu - \nu| \geq 0$ but then, choosing $\varepsilon = \frac{1}{2}|\mu - \nu|$ we get $|\mu - \nu| < \frac{1}{2}|\mu - \nu|$

□

Theorem 27. Let $f, g : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Suppose that for $\mu, \nu \in \mathbb{R}$, $\lim_{x \to a} f(x) = \mu$ and $\lim_{x \to a} g(x) = \nu$ then

(a) $\lim_{x \to a} (f + g)(x) = \mu + \nu$

(b) $\lim_{x \to a} (f \cdot g)(x) = \mu \cdot \nu$

Proof. We will prove each separately

(a) Let $\varepsilon > 0$ be given. by the definition of limit, $\exists \delta_1 = \delta_1(\varepsilon, a, \mu) > 0$ such that $0 < |x - a| < \delta_1 \implies |f(x) - \mu| < \frac{\varepsilon}{2}$ and $\exists \delta_2 = \delta_2(\varepsilon, a, \nu) > 0$ such that $0 < |x - a| < \delta_2 \implies |g(x) - \nu| < \frac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$, provided $0 < |x - a| < \delta$, and observe that

$$\begin{aligned}|(f+g)(x)-(\mu+\nu)| &= |(f(x)-\mu)+(g(x)-\nu)| \\ &\leq |f(x)-\mu|+|g(x)-\nu| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

and $0<|x-a|<\delta \implies |(f+g)(x)-(\mu+\nu)|<\varepsilon$

(b) Let $\varepsilon>0$ be given, and observe that

$$\begin{aligned}|(f\cdot g)(x)-(\mu\nu)| &= |(f(x)g(x)-\mu g(x))+(\mu g(x)-\mu\nu)| \\ &\leq |g(x)|\cdot|f(x)-\mu|+|\mu|\cdot|g(x)-\nu|\end{aligned}$$

By the definition of limit $\exists\delta_g=\delta_g(\varepsilon,a,\nu)>0$ such that $|g(x)-\nu|<\min\{\frac{\varepsilon}{2(1+|\mu|)},1\}$, whenever $0<|x-a|<\delta_g$.

Note: whenever $0<|x-a|<\delta_g$, we have

(i) $|g(x)-\nu|<\frac{\varepsilon}{2(1+|\mu|)}$ and $|\mu|\cdot|g(x)-\nu|<\frac{\varepsilon}{2}$

(ii) $|g(x)-\nu|<1$ and $g(x)\leq|g(x)-\nu|+|\nu|<1+|\nu|$

Again, by the definition of limit, $\exists\delta_f=\delta_f(\varepsilon,a,\mu,\nu)>0$ such that

$$|x-a|<\delta_f \implies |f(x)-\mu|<\frac{\varepsilon}{2(1+|\nu|)}$$

then, we see that, for $\delta=\min\{\delta_f,\delta_g\}$ we have

$$|(f\cdot g)(x)-(\mu\nu)|<(1+|\nu|)\frac{\varepsilon}{2(1+|\nu|)}+\frac{\varepsilon}{2}=\varepsilon$$

□

2.8. Infremum / Supremum

Our objective is to give a sense of infremum/supremum as limits. For example, consider $[1,2]$. This set has the property that for every $x\in[1,2]$, there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ belonging to $[1,2]$ such that $x_n\to x$ as $n\to\infty$. Indeed, $x\in(1,2)$, then for $M_x>0$ sufficiently large. $x_n=x+\frac{1}{n\cdot M_x}$ is such that $x_n\in(1,2)$ and $x_n\to x$. And for when $x\in\{1,2\}$, we can build the sequences $x_n=\frac{1}{100n}$ or $x_n=2-\frac{1}{100n}$ This property also holds for $(1,2)$, but also even though $1,2\notin(1,2)$, there exists sequences $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ such that $y_n,z_n\notin(1,2)$ $\forall n\in\mathbb{N}$ and $y_n\to 1$ as $n\to\infty$, $z_n\to 2$ as $n\to\infty$

It turns out that the property of “having a sequence inside the set converging to this point” is a property that holds true for the inf and sup of any bounded set.

To this end, we prove the following lemma

Lemma 28. Let $B \subseteq \mathbb{R}$ be a nonempty set bounded above. Then, given any $\varepsilon > 0$, there exists some $b_\varepsilon \in B$ such that

$$\sup B - \varepsilon < b_\varepsilon \ (\leq \sup B)$$

Proof. Let $\varepsilon > 0$ be given. Denote $\sup B$ by β. Suppose for contradiction that no such b_ε exists, Then for all $b \in B$, we must have $b \leq \beta - \varepsilon$ but then $\beta - \varepsilon$ is the least upper bound for B □

An analogous argument prove

Lemma 29. Let $A \subseteq \mathbb{R}$ be a nonempty set bounded below. Then, given any $\varepsilon > 0$, there exists some $a_\varepsilon \in B$ such that

$$(\inf A \leq) \ a_\varepsilon < \inf A + \varepsilon$$

Corollary 30. Let $A \subseteq \mathbb{R}$ be nonempty and bounded, then, $\exists (x_n)_{n \in \mathbb{N}}$ and $\exists (y_n)_{n \in \mathbb{N}}$ for which $x_n, y_n \in A$ for all $n \in \mathbb{N}$ and $\lim\limits_{x \to \infty} x_n = \inf A$, $\lim\limits_{x \to \infty} y_n = \sup A$

Proof. By Lemma 28 for each $n \in \mathbb{N}$, $\exists y_n \in A$ such that $\sup A - \frac{1}{n} < y_n \leq \sup A$ and $|y_n - \sup A| < \frac{1}{n} \to 0$ as $n \to \infty$ So, $\lim\limits_{x \to \infty} y_n = \sup A$. Also, for each $n \in \mathbb{N}$, by Lemma 29, $\exists x_n \in A$ such that $\inf A \leq x_n < \inf A + \frac{1}{n}$. i.e., $|x_n - \inf A| < \frac{1}{n} \to 0$ as $n \to \infty$. So, $\lim\limits_{x \to \infty} x_n = \inf A$. □

Lemma 31. Suppose A is non-empty and bounded below. Let B be the set of all lower bounds of A. Then $\inf A = \sup B$

Proof. There are 3 steps

Step 1 [B is nonempty] Since A is bounded below, there exists at least one lower bound, which belongs to B, so $B \neq \emptyset$

Step 2 [B is bounded above] Suppose for contradiction that B is not bounded above. Then given any $n \in \mathbb{N}, \exists x_n \in B$ such that $x_n \geq n$. Then by the definition of B, x_n is a lower bound for A for each $n \in \mathbb{N}$. Thus given any $a \in A$, we have $a \geq x_n \geq n \ \forall n \in \mathbb{N}$. Here B is bounded above.

Step 3 [showing the equality]

($\leq$) Let $\nu = \inf A$ nad $\mu = \sup B$. Since ν is the infimum of A, ν is a lower bound for A. So $\nu \in B \implies \nu \leq \sup B = \mu$

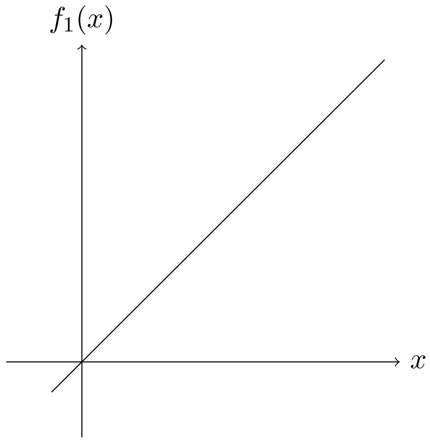
($\geq$) Let $\varepsilon > 0$ be arbitrary. Then by **Lemma** 28 $\exists b_\varepsilon \in B$ such that $\mu - \varepsilon < b_\varepsilon \leq \mu$. Hence, $\mu < \varepsilon + b_\varepsilon$. Now, let $a \in A$ be any point of A and observe that since $b_\varepsilon \in B$, $b_\varepsilon \leq a \implies \mu < \varepsilon + b_\varepsilon \leq \varepsilon + a$. i.e., $\mu < \varepsilon + a$ for all $a \in A$. i.e., $\mu - \varepsilon < a \ \forall a \in A$. So, $\mu - \varepsilon$ is a lower bound for $A \implies \mu - \varepsilon < \inf A = \nu$ i.e., $\mu < \nu + \varepsilon$, but $\varepsilon > 0$ was arbitrary $\implies \mu \leq \nu$

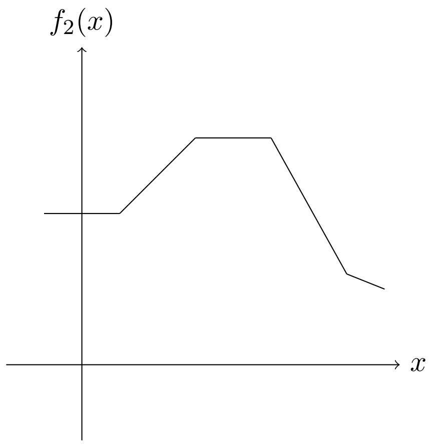
$\square$

3. Continuous Function

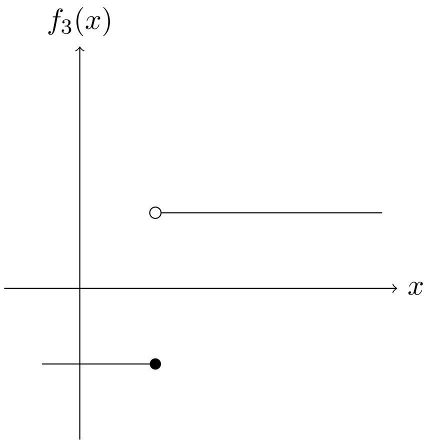
What does it mean for a function to be continuous?

Infinitely, this is some smoothness to the function i.g.,





But, on the other hand



is not continuous

3.1. Definition of Continuous Function

Definition 32. Let $f : \mathbb{R} \to \mathbb{R}$. We say f is continuous at the point $x_0 \in \mathbb{R}$ if there holds $\lim_{x \to x_0} f(x) = f(x_0)$

Remark. For f to be continuous at $x_0 \in \mathbb{R}$, we require

(i) $\lim_{x \to 0} f(x)$ exists

(ii) $\lim_{x \to 0} f(x) = f(x_0)$

Another way of writing Definition 32 is

Definition (32)**.** f is continuous at x_0 if for all $\varepsilon > 0$, $\exists \delta = \delta(\varepsilon, x_0, f(x_0)) > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$$

Example. f_3 is not continuous at the point $x = 1$.

Proof. Indeed, setting $\varepsilon_0 = 1$, we see that, given any $\delta > 0$, the point $x_\delta = 1 + \frac{\delta}{2}$ is such that $|x_\delta - 1| < \delta$ and $|f(x_\delta) - f(1)| = |1 - (-1)| = 2 > \varepsilon_0$ □

Example. $f(x) = x^2$ is continuous.

Proof. Indeed, let $x_0 \in \mathbb{R}$ be any point and observe that

$$\begin{aligned} |f(x) - f(x_0)| &= |x^2 - x_0^2| \\ &= |(x + x_0)(x - x_0)| \\ &= |x + x_0| \cdot |x - x_0| \end{aligned}$$

Let $\varepsilon > 0$ be given. Now let $\delta = \min\left\{1, \frac{\varepsilon}{2(1+|x_0|)}\right\}$, then

$$\begin{aligned} |x + x_0| &= |x - x_0 + 2x_0| \\ &\leq |x - x_0| + 2|x_0| \\ &\leq 1 + 2|x_0| \end{aligned}$$

Then provided $|x - x_0| < \delta$ we get

$$|f(x) - f(x_0)| \leq (1 + 2|x_0|) \cdot \frac{\varepsilon}{2(1 + |x_0|)} < \varepsilon$$

□

Example.

$$f(x) = \begin{cases} 0 & x = 0 \\ x \sin\left(\frac{1}{x}\right) & x \neq 0 \end{cases}$$

f is continuous at $x = 0$

Proof. Indeed, let $\varepsilon > 0$ be given and observe that

$$\begin{aligned} |f(x) - f(0)| &= |x| \cdot \left|\sin\left(\frac{1}{x}\right)\right| \quad \text{for } x \neq 0 \\ &\leq |x| \end{aligned}$$

So, letting $\delta(\varepsilon) = \frac{\varepsilon}{2}$, we see that

$$|x - 0| < \delta \implies |f(x) - f(0)| \leq \frac{\varepsilon}{2} < \varepsilon$$

□

3.2. Identity of Continuous Function

Lemma 33. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous at $a \in \mathbb{R}$. Then

(i) $f + g$ is continuous at a

(ii) $f \cdot g$ is continuous at a

Proof. We will prove each separately

(i) let $\varepsilon > 0$ be given. By the definition of continuous, $\exists \delta_f = \delta_f(\varepsilon, a) > 0$ such that

$$|x-a| < \delta_f \implies |f(x) - f(a)| < \frac{\varepsilon}{2}$$

and, $\exists \delta_g = \delta_g(\varepsilon, a) > 0$ such that

$$|x-a| < \delta_g \implies |g(x) - g(a)| < \frac{\varepsilon}{2}$$

So, letting $\delta = \min\{\delta_f, \delta_g\}$, suppose $|x-a| < \delta$, we see that

$$\begin{aligned} |f(x) + g(x) - (f(a) + g(a))| &\leq |f(x) - f(a)| + |g(x) - g(a)| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

(ii) let ε be given. Note that

$$|f(x)g(x) - f(a)g(a)| \leq |g(x)| \cdot |f(x) - f(a)| + |f(a)| \cdot |g(x) - g(a)|$$

Since g is continuous at a, $\exists \delta_g = \delta_g(\varepsilon, a) > 0$ such that

$$|x-a| < \delta_g \implies |g(x) - g(a)| < \min\left\{1, \frac{\varepsilon}{2(1 + |f(a)|)}\right\}$$

Then, provided $|x - a| < \delta_g$, we get

$$|g(x)| \leq \overbrace{|g(x) - g(a)|}^{<1} + |g(a)| < 1 + |g(a)|$$

Also, since f is continuous at a, $\exists \delta_f = \delta_f(\varepsilon, a) > 0$ such that

$$|x-a| < \delta_f \implies |f(x) - f(a)| < \frac{\varepsilon}{2(1 + |g(a)|)}$$

Then, letting $\delta = \min\{\delta_f, \delta_g\}$, we see that whenever $|x - a| < \delta$, we have form

$$|f(x)g(x) - f(a)g(a)| < (1 + |g(a)|)\left(\frac{\varepsilon}{2(1 + |g(a)|)}\right) + |f(a)| \cdot \frac{\varepsilon}{2(1 + |f(a)|)} < \varepsilon$$

$\square$

Lemma 34. Let $g : \mathbb{R} \to \mathbb{R}$ be continuous at $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be continuous at $g(a)$. Then $f \circ g$ is continuous at a

Proof. Let $\varepsilon > 0$ be given. Since f is continuous at $g(a)$, $\exists \delta_f = \delta_f(\varepsilon, a) > 0$ such that

$$|y - g(a)| < \delta_f \implies |f(y) - f(g(a))| < \varepsilon$$

Meanwhile, g is continuous at a, so $\exists \delta_g = \delta_g(\delta_f(\varepsilon, a), a) > 0$ such that

$$|x - a| < \delta_g \implies |g(x) - g(a)| < \delta_f$$

So, letting $\delta = \delta_g$, we see that

$$\begin{aligned} |x - a| < \delta &\implies |g(x) - g(a)| < \delta_f \\ &\implies |f(g(x)) - f(g(a))| < \varepsilon \end{aligned}$$

□

Lemma 35. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at a, and suppose $f(a) > 0$. Then $\exists \delta > 0$ such that $f(x) > 0 \ \forall x \in [a - \delta, a + \delta]$

Proof. Since f is continuous at a, $\exists \delta_f = \delta_f(a, \overbrace{f(a)}^{\varepsilon}) > 0$ such that

$$|x - a| < \delta_f \implies |f(x) - f(a)| < \overbrace{\frac{1}{2} f(a)}^{\varepsilon}$$

It follows that, for $x \in (a - \delta_f, a + \delta_f)$, we have

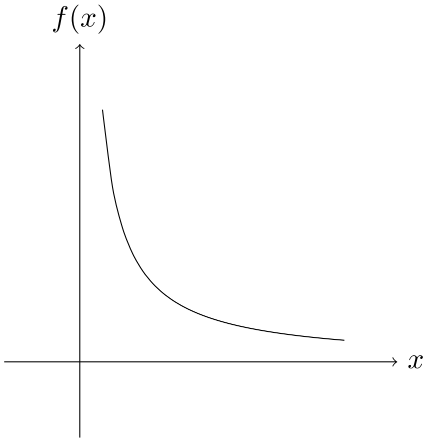
$$\begin{aligned} f(x) &= (f(x) - f(a)) + f(a) \\ &\geq f(a) - |f(x) - f(a)| \\ &> f(a) - \frac{1}{2} f(a) \\ &= \frac{1}{2} f(a) > 0 \end{aligned}$$

In turn, letting $\delta = \frac{1}{2}\delta_f$, we see that $f(x) > 0 \ \forall x \in [a - \delta, a + \delta]$ □

3.3. Definition of Left/Right Continuity

f continuous on (a, b) if f is continuous at x, for all $x \in (a, b)$. What does it mean for f to be continuous at on $[a, b]$? Should there be a difference between "continuous on (a, b)" and "continuous on $[a, b]$".

To gather intution, let's look at $f(x) = \frac{1}{x}$ on $(0, 1)$ and $[0, 1]$.



It's clar that f is continuous at every point $a \in (0,1)$ but $\lim_{x\to 0} f(x)$ is not defined. So, it ought to not be continuous on $[0,1]$ We make the following define

Definition (32)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $a < b$ be real numbers.

(i) We say f is continuous on (a,b) if f is continuous at x for every $x \in (a,b)$

(ii) We say f is continuous on $[a,b]$ if f is continuous on (a,b) and $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$

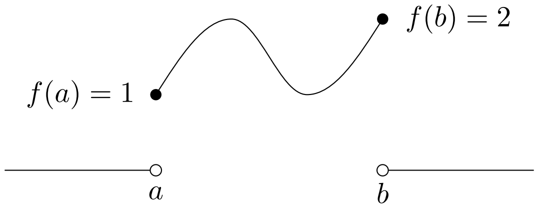
We write $\lim_{x\to a^+} f(x)$ to mean "The limit f as x tends to a from above" also written $\lim_{x\searrow a} f(x)$ and $\lim_{x\to b^-} f(x)$ to mean "The limit f as x tends to b from below" also written $\lim_{x\nearrow a} f(x)$

Definition (32)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$

(i) We write $\mu = \lim_{x\searrow a} f(x)$ if for all $\varepsilon > 0$, $\exists \delta > 0$ such that whenever $a < x < a + \delta$ we have $|\mu - f(x)| < \varepsilon$

(ii) We write $\nu = \lim_{x\nearrow a} f(x)$ if for all $\varepsilon > 0$, $\exists \delta > 0$ such that whenever $a - \delta < x < a$ we have $|\nu - f(x)| < \varepsilon$

Example. Considered this graph



then, $\lim_{x \searrow a} f(x) = 1$ and $\lim_{x \nearrow b} f(x) = 2$ on the other hand $\lim_{x \nearrow a} f(x) = 0$ and $\lim_{x \searrow b} f(x) = 0$

Example. $\lim_{x \to x_0} f(x)$ exists $\iff$ $\lim_{x \nearrow x_0} f(x)$ and $\lim_{x \searrow x_0} f(x)$ exists and are equal.

3.4. 3 Hard Theorems

Theorem 36 (Intermediate Value Theorem)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ for $a < b$. Suppose $f(a) < 0 < f(b)$ Then $\exists \xi \in (a, b)$ such that $f(\xi) = 0$

Theorem 37. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ for $a < b$. Then f is bounded above on $[a, b]$, i.e., $\exists M \in \mathbb{R}$ such that $f(x) \leq M$ $x \in [a, b]$

Theorem 38. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$. Then $\exists \xi \in [a, b]$ such that $f(x) \leq f(\xi)$ $\forall x \in [a, b]$ i.e., $f(\xi) = \sup\{f(x) : x \in [a, b]\}$ (we say that f achieves its supremum on $[a, b]$)

Lemma ($35'$)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $b \in \mathbb{R}$. Suppose $\lim_{x \nearrow b} f(x) = f(b) > 0$ Then $\exists \delta > 0$ such that $f(x) > 0$ for all $x \in (b - \delta, b)$

Proof. Directly from Definition 32(ii) (definition of $\lim_{x \nearrow b} f(x)$) such that

$$x \in (b - \delta, b) \implies |f(x) - f(b)| < \frac{1}{2} f(b)$$

Then for such $x \in (b - \delta, b)$ we have

$$\begin{aligned} f(x) &= (f(x) - f(b)) + f(b) \\ &\geq f(b) - \overbrace{|f(x) - f(b)|}^{< \frac{1}{2} f(b)} \\ &> \frac{1}{2} f(b) > 0 \end{aligned}$$

Hence, for $x \in \left(b - \frac{\delta}{2}, b\right)$ we have $f(x) > 0$ $\square$

Lemma ($35''$)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $b \in \mathbb{R}$. Suppose $\lim_{x \searrow a} f(x) = f(a) > 0$ Then $\exists \delta > 0$ such that $f(x) > 0$ for all $x \in (a, a + \delta)$

Proof Theorem 36. Define the set $A = \{x \in [a, b] : f(y) < 0 \ \forall y \in [a, x]\}$ Since $f(a) < 0$, so $a \in A$, so $A \neq \emptyset$ Also, using Lemma $35''$ $\exists \delta_1 > 0$ such that $f(y) < 0$ $\forall y \in [a, a + \delta_1]$ so $a + \delta_1 \in A$, and by Lemma $35'$ $\exists \delta_2 > 0$ such that $f(y) > 0$ $\forall y \in [b - \delta_2, b]$ where

$b - \delta_2$ is an upper bound for A. So A is bounded above and $\sup A$ is well-defined. Let $\alpha = \sup A$. We already know that $\alpha \in (a, b)$ our aim is to show that $f(\alpha) \neq 0$ We proceed by contradiction:

Suppose for contradiction that $f(\alpha) \neq 0$ There are 2 possibilities

(i) $f(\alpha) < 0$

(ii) $f(\alpha) > 0$

Suppose (i) holds, Since $\alpha \in (a, b)$ and $f(\alpha) < 0$ by **Lemma 35**, $\exists \delta_3 > 0$ such that $f(y) < 0\ \forall y \in [\alpha - \delta_3, \alpha + \delta_3]$ But then $\alpha + \delta_3 \in A$ and $\alpha + \delta_3 > \alpha$

Suppose (ii) holds. Then since $\alpha \in (a, b)$, $f(\alpha) > 0$ and f is continuous. By **Lemma 35**, $\exists \delta_4 > 0$ such that $f(x) > 0\ \forall x \in [\alpha - \delta_4, \alpha + \delta_4]$ But then $\alpha = \sup A$ by **Lemma 28** $\exists x_0 \in A$ such that $\alpha - \frac{\delta_4}{2} < x_0$ Thus $x_0 \in (\alpha - \frac{\delta_4}{2}, \alpha) \subseteq [\alpha - \delta_4, \alpha + \delta_4] \implies f(x_0) > 0$ But $x_0 \in A$ so $(f_x) < 0$ □

Corollary 39. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and let $c \in \mathbb{R}$. Suppose $f(a) < c < f(b)$. Then $\exists \xi \in (a, b)$ such that $f(\xi) = c$

Proof. Define $g(x) = f(x) - c$ and apply **Theorem** 36 to g □

Example 40. Let $f(x) = x^4 + x - 3\ \forall x \in \mathbb{R}$ **Fact**: all polynomials are continuous $\forall x \in \mathbb{R}$ A nice application of the Intermidiate Value Theorem is to find roots of continuous functions We can see by plugging in that

$$f(1) = 1 + (-1) - 3 = -3$$

$$f(2) = 16 + 2 - 3 = 15$$

IVT $\implies \exists x_0 \in (1, 2)$ such that $f(x_0) = 0$ This at least lets us estimate where roots are

Example 41. Let $f(x) = x^4 + x - 3 + \tan\left(\frac{x}{2}\right)$ (continuous on $(-\pi, \pi)$)

$$f(-1) = -3 - \tan\left(\frac{1}{2}\right) < 0$$

$$f(2) = 15 - \tan\left(\frac{1}{2}\right) > 0$$

IVT $\implies \exists x_0 \in (-1, 2)$ such that $f(x_0) = 0$

What is it useful for? If we look at the set $f([a, b]) = \{f(x) : x \in [a, b]\}$ and Theorem 37 tell us that set is bounded. Since the set is bounded, it has a supremum. You can think of this as "local max" of f on the interval $[a, b]$

Before proving Theorem 37, let's look at one of its consequences.

Corollary 42. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$. Then f is bounded below on $[a, b]$, i.e., $\exists m \in \mathbb{R}$ such that $m \leq f(x)\ \forall x \in [a, b]$

Proof. Since f is continuous, so is $(-f)$. Now apply Theorem 37 to $-f$. $\exists M \in \mathbb{R}$ such that $-f(x) \leq M\ \forall x \in [a, b]$ the, $f(x) \leq -M\ \forall x \in [a, b]$ ☐

Takeaway: If f is continuous on $[a, b]$, then f is bounded above + below on $[a, b]$

To prove Theorem 37, we'll need a few Lemmas.

Lemma 43. Let $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$, then $\exists \delta > 0$ such that f is bounded above on the interval $[a - \delta, a + \delta]$

Proof. Since f is continuous at a, $\exists \delta = \delta(a, \overbrace{1}^{\varepsilon})$ such that $|x-a| < \delta \implies |f(x)-f(a)| < 1$ This for such x we have

$$\begin{aligned} f(x) &= f(x) - f(a) + f(a) \\ &\leq |f(x) - f(a)| + |f(a)| \\ &< 1 + |f(a)| \end{aligned}$$

For x satisfying $|x - a| < \delta$, we have $f(x) < 1 + f(a)$.

In particular, $f(x) < 1 + f(a)\ \forall x \in \left[a - \frac{\delta}{2}, a + \frac{\delta}{2}\right]$ ☐

Lemma. (43′) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $b \in \mathbb{R}$. Suppose $\lim_{x \nearrow b} f(x) = f(b)$. Then $\exists \delta > 0$ such that f is bounded above on $[b - \delta, b]$

Proof. By Definition 32″, $\exists \delta = \delta(b, 1)$ such that

$$0 < |x - b| < \delta \implies |f(x) - f(b)| < 1$$

Therefore, for such x,

$$\begin{aligned} f(x) &= f(x) - f(b) + f(b) \\ &\leq |f(x) - f(b)| + |f(b)| \\ &< 1 + |f(b)| \end{aligned}$$

$f(x) < f(b) + 1\ \forall x \in \left[b - \frac{\delta}{2}, b\right]$ ☐

Lemma. (43″) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $a \in \mathbb{R}$. Suppose $\lim_{x \searrow a} f(x) = f(a)$. Then $\exists \delta > 0$ such that f is bounded above on $[a, a + \delta]$

Proof Theorem 37. As in the proof of Theorem 36, consider the set

$$A = \{x \in [a, b] : f \text{ is bounded above on } [a, x]\}$$

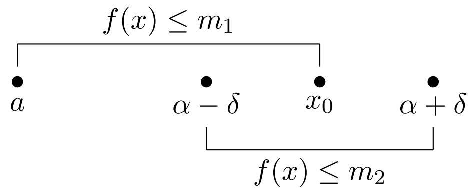
Since $a \in A$, we know $a \neq \emptyset$. Moreover, the point b is an upper bound for A, so $\sup A = \alpha$ exists.

Our objective is to show that $\alpha = b$.

Suppose for contradiction that $\alpha < b$. (Note that we must have $a < \alpha$. We can't have $a > \alpha$ since $a \in A$. and $\sup A \geq a$. If $\alpha = a$, then $A = \{a\}$, but we know from Lemma 43″ that $\exists \delta > 0$ such that $[a, a + \delta] \subseteq A$)

By assumption $a < \alpha < b$ and so Lemma 43 $\implies \exists \delta > 0$ such that f is bounded on $[\alpha - \delta, \alpha + \delta]$. Let's say $f(x) \leq m_2$ on this interval $[\alpha - \delta, \alpha + \delta]$.

By Lemma 28 (Alternate definition of supremum) $\exists x_0 \in A$ such that $\alpha - \delta < x_0 \leq \alpha$. f is bounded above on $[a, x_0]$ (by the definition of A). say $f(x) \leq m_1$ on $[a, x_0]$



Thus, $f(x) \leq \max\{m_1, m_2\}$ $\forall x \in [a, \alpha + \delta]$ We deduce that $\alpha + \delta \in A$ and $\alpha + \delta > \alpha = \sup A$. Hence,

$$\begin{aligned} \alpha = b &\iff \sup A = b \\ &\implies f \text{ is bounded above on } [a, b] \text{ for every } x < b \text{ (1)} \end{aligned}$$

Finally, using continuity at the point b by Lemma 43′ $\exists \delta'$ such that f is bounded on $[b - \delta', b]$ (2).

Hence, choosing $x = b - \delta'$ in (1), $\exists M$ such that $f(x) \leq M$, $\forall x \in [a, b - \delta']$. and by (2), $\exists M_2$ such that $f(x) \leq M_2$, $\forall x \in [b - \delta', b]$. So, $f(x) \leq \max\{M, M_2\}$ $\forall x \in [a, b]$. $\square$

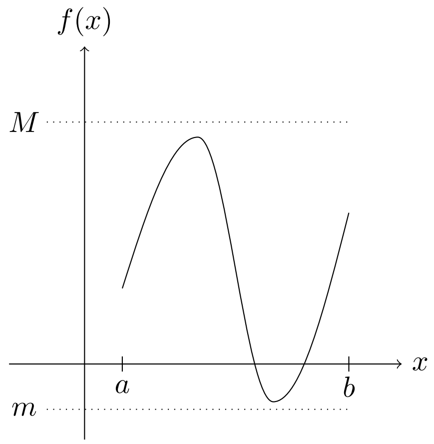
Summarize steps:

(i) define a good set A

(ii) show $b = \sup A$

(iii) show $b \in A$

The picture is



Whenever f is continuous on $[a, b]$, $\exists M > m$ such that $m \leq f(x) \leq M$ $\forall x \in [a, b]$

Note: We must be careful aboue being continuous on $[a, b]$, and mot just (a, b). Indeed, $\underset{x \mapsto \frac{1}{x}}{f:(0,1)\to(0,\infty)}$, f is continuous on $[\tilde{x}, \infty)$ for every $\tilde{x} > 0$, but it is <u>not</u> continuous on $[0, \infty)$.

Question: does these exists $\xi_1, \xi_2 \in [a, b]$ such that

$$f(\xi_1) = \inf_{[a,b]} f \text{ and } f(\xi_2) = \sup_{[a,b]} f$$

Anwer: Yes

Later on, when we discuss differentiability, if sup/inf is achieved in (a, b), then $f' = 0$ at such points. This we will prove later.

Proof of Theorm 38. We already know from Theorem 37 that f is bounded on $[a, b]$, i.e., the set $B = f([a, b]) = \{f(x) : x \in [a, b]\}$ is bounded. This set is nonempty and so $\beta = \sup B$ is well-defined; Since $\beta \geq f(x)$ $\forall x \in [a, b]$ it suffies to show that $\exists \xi \in [a, b]$ such that $f(\xi) = \beta$.

Suppose for contradiction that this is not the case, i.e., $\beta \neq f(y)$ $\forall y \in [a, b]$ Then the function $g : [a, b] \to \mathbb{R}$, defined by $g(x) = \frac{1}{\beta - f(x)} \forall x \in [a, b]$, is well-defined and g is continuous on $[a, b]$ by virtue of Lemma 33

Since g is continuous, by Theorem 37 $\implies$ g is bounded above on $[a, b]$ However, by Lemma 28, given any $n \in \mathbb{N}, \exists x_n \in [a, b]$ such that

$$\beta - \frac{1}{n} < f(x_n) \leq \beta \implies g(x_n) \geq \frac{1}{\beta - \left(\beta - \frac{1}{n}\right)} = n$$

Hence given any $n \in \mathbb{N}, \exists x_n \in [a, b]$ such that $g(x_n) \geq n$ and therefore g is unbounded on $[a, b]$. $\square$

We've actually proved

Corollary 44. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$. Then $\exists \xi \in [a, b]$ such that $f(\xi) = \sup\{f(x) : x \in [a, b]\}$ (we often write with the shorthand $\sup_{[a,b]} f$)

Corollary 45. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$. Then $\exists \xi \in [a, b]$ such that $f(\xi) = \inf\{f(x) : x \in [a, b]\}$

Proof. Apploy Corollary 44 to the function $-f$ and use the result $\inf B = -\sup(-B)$. $\square$

3.5. Usage of 3 Hard Theorem

Example 46. Suppose f, g are continuous on $[a, b]$ and $f(a) < g(a)$ and $f(b) > g(b)$. Then $\exists x \in [a, b]$ such that $f(x) = g(x)$ (in actual fact, $x \in (a, b)$)

Proof. define $h(x) = f(x) - g(x)$. Then h is continuous on $[a, b]$, $h(a) < 0 < h(b)$ so from Theorem 36, $\exists \xi \in (a, b)$ such that $h(\xi) = 0 \implies f(\xi) = g(\xi)$ $\square$

Example 47. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[0, 1]$ and suppose $0 \le f(x) \le 1$ $\forall x \in [0, 1]$. Then $\exists x_0 \in [0, 1]$ such that $f(x_0) = x_0$ (we can imagine that f cross $y = x$)

Proof. Note that if $f(0) = 0$ on if $f(1) = 1$, then we are done. Suppose that $f(0) \neq 0$ and $f(1) \neq 1$ then $0 < f(0)$ and $f(1) < 1$ Let $g(x) = x - f(x)$. Then, $g(0) = 0 - f(0) < 0$ and $g(1) = 1 - f(1) > 0$. So, g is continuous and $g(0) < 0 < g(1)$, where Theorem 36 $\exists x_0 \in [0, 1]$ such that $g(x_0) = 0$ and hence $x_0 = f(x_0)$ $\square$

Example 48. There are 3 sub-examples here:

(a) Suppose $f : \mathbb{R} \to \mathbb{R}$ satsfies $|f(x)| \le |x|$ for all $x \in \mathbb{R}$. Then f is continuous at 0

(b) There exists a function which satisfies the assumption of a.) but is not continuous at any other points other than $x = 0$

(c) Suppose g is continuous at 0 and $g(0) = 0$ and suppose $|f(x)| \le |g(x)|$ $\forall x \in \mathbb{R}$. Then f is continuous at 0.

Proof. We will prove each separately:

(a) The inequallity implies $f(0) = 0$. Let $\varepsilon > 0$ be given, then the inequallity show that

$$|f(x) - f(0)| = |f(x)| \leq |x - 0|$$

so letting $\delta = \varepsilon$, we see that

$$|x - 0| < \delta \implies |f(x) - f(0)| < \varepsilon$$

so f is continuous at 0

(b)

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then $|f(x)| \leq |x| \ \forall x$ but f is not continuous at any points other than 0

(c) Since $g(0) = 0$, we immediately get $f(0) = 0$. Let $\varepsilon > 0$ be given. Since g is continuous at 0, $\exists \delta = \delta(\varepsilon, 0) > 0$ such that

$$|x - 0| < \delta \implies |g(x) - g(0)| \leq \varepsilon$$

but then, in view of the bound $|f(x)| \leq |g(x)| \ \forall x$, we see that

$$|x - 0| < \delta \implies |f(x) - f(0)| = |f(x)| \leq |g(x)| = |g(x) - g(0)| < \varepsilon$$

□

Example 49. This exercisc is here to help us gain more familiarity with limits– it's not concern with continuous functions per se.

(i) Let $f, g : \mathbb{R} \to \mathbb{R}$ and suppose $f(x) \leq g(x) \ \forall x \in \mathbb{R}$ and suppose $\mu := \lim_{x \to a} f(x), \nu := \lim_{x \to a} g(x)$ Show that $\mu \leq \nu$

(ii) Now suppose $f(x) < g(x) \ \forall x \in \mathbb{R}$. Does this guarantee $\mu < \nu$?

Proof. We will prove each separately:

(i) Let $\varepsilon > 0$ be given. Then $\exists \delta_1 = \delta_1(\varepsilon, a, \mu) > 0$ and $\exists \delta_2 = \delta_2(\varepsilon, a, \nu) > 0$ such that

$$|x - a| < \delta_1 \implies |f(x) - \mu| < \frac{\varepsilon}{2},$$
$$|x - a| < \delta_2 \implies |g(x) - \nu| < \frac{\varepsilon}{2}$$

Set $\delta := \min(\delta_1, \delta_2)$ Then, provided $|x - a| < \delta$, we have

$$\begin{aligned} \nu - \mu &= (\nu - g(x)) + (g(x) - f(x)) + (f(x) - \mu) \\ &\geq \underbrace{g(x) - f(x)}_{\geq 0} - \underbrace{|\nu - g(x)|}_{< \frac{\varepsilon}{2}} - \underbrace{|\mu - f(x)|}_{< \frac{\varepsilon}{2}} \\ &> -\varepsilon \end{aligned}$$

So, $\nu - \mu > -\varepsilon$ for all $\varepsilon > 0 \implies \nu - \mu \geq 0$

(ii) NO: Suppose $f(x) = 0$ and $g(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ \frac{1}{x} & \text{if } |x| \geq 1 \end{cases}$

Then $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$

□

Example 50. Let $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & x = 0 \end{cases}$

(a) Show that f is not continuous on $[-1, 1]$

(b) Show that f satisfies the conclusion of Theorem 36 (IVT)

Proof.

(a) for every $\delta > 0$, $n_\delta := \max\left(\left\lceil \frac{1}{2\pi}\delta^{-1} \right\rceil, 1\right) \in \mathbb{N}$ such that

$$\frac{1}{\frac{\pi}{2} + 2\pi n_\delta} < \delta \text{ and } x_\delta := \frac{1}{\frac{\pi}{2} + 2\pi n_\delta}$$

we get $0 < x_\delta < \delta$ and

$$|f(x_\delta) - f(0)| = \left|\sin\left(\frac{\pi}{2} + 2\pi n_\delta\right)\right| = 1$$

so, for all $\delta > 0, \exists x_\delta$ such that $0 < x_\delta < \delta$ and $|f(x_\delta) - f(0)| = 1$, so f is not continuous at 0.

(b) f is not continuous at 0, however f is continuous on $(-1, 0)$ and on $(0, 1]$ and so Theorem 36 holds on any interval of the form $[-1, y]$ and $[x, 1]$ for $y < 0$ and $x > 0$

It remains to check that

*Suppose $a > 0$ and $f(a) > 0$. Then, for every $c \in [0, f(a)], \exists \xi_c \in [0, a]$ such that $f(\xi_c) = c$

Note that $f(a) \leq 1$, Indeed $\xi = \frac{1}{\arcsin(c)}$ is such that

$$\begin{aligned} f(\xi) &= c \\ \sin\left(\frac{1}{\xi}\right) &= \sin(\arcsin(c)) \end{aligned}$$

So the only remaining issue is that we do not necessarily have $\xi \in [0, a]$.

To this end, notice that, for every $N \in \mathbb{N}$, $\xi = \frac{1}{2\pi N + \arcsin(c)}$ also satisfies $f(\xi) = c$ and hence, choosing N sufficiently large such that $\frac{1}{2\pi N + \arcsin(c)} \leq a$, we have that $\xi = \frac{1}{2\pi N + \arcsin(c)}$ is a point that verifies *

$\square$

Example 51. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, and $f(x)^2 = g(x)^2 \ \forall x \in \mathbb{R}$ and $f(x) \neq 0$. Then either

(i) $f(x) = g(x) \ \forall x \in \mathbb{R}$

(ii) $f(x) = -g(x) \ \forall x \in \mathbb{R}$

i.e., f cannot 'jump' between $\pm g$.

Proof. Suppose for contradiction that $\exists a, b \in \mathbb{R}$ such that $f(a) = g(a)$ and $f(b) = -g(b)\circledast$ and wlog(without loss of generality), assume $a < b$. Since $f(x) \neq 0 \ \forall x$, we also assume wlog $f(a) < 0$ Then it can't be the case that $f(b) > 0$. Indeed, if this were the case, then by Theorem 36, $\exists \xi \in (a, b)$ such that $f(\xi) = 0$, which contradicts $f(x) \neq 0 \ \forall x$.

Hence $f(a) < 0$ and $f(b) < 0$.

Then, $\circledast \implies g(a) < 0$ and $g(b) > 0$, so Theorem 36 $\implies \exists \zeta \in (a, b)$ such that $g(\zeta) = 0$. But then $f(\zeta) = 0$, which is again a contradiction. $\square$

Example 52. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and such that $f(x)^2 = x^2 \ \forall x \in \mathbb{R}$. Then, either $f(x) = x \ \forall x \in \mathbb{R}$, or $f(x) = -x \ \forall x \in \mathbb{R}$, or $f(x) = |x| \ \forall x \in \mathbb{R}$.

Proof. It sufficies to show that

(A) for $x < 0$, either: $f(x) = x \ \forall x < 0$, or $f(x) = -x \ \forall x < 0$

(B) for $x > 0$, either: $f(x) = x \ \forall x > 0$, or $f(x) = -x \ \forall x > 0$

We only prove (B), as the proof for (A) is identical.

Suppose for contradiction $\exists 0 < a < b$ such that (wlog) $f(a) = -a$ and $f(b) = b$. Then, observe that $f(a) < 0$, while $f(b) > 0$.

Thus, Theorem 36 $\implies \exists \xi \in (a, b)$ such that $f(\xi) = 0$. But, $(f(\xi))^2 = \xi^2 > a^2 > 0$ $\square$

Example 53. Suppose f is continuous on $[a, b]$ and $f(x) \in \mathbb{Q} \ \forall x \in [a, b]$. Then, f is a constant function, i.e., $\exists q \in \mathbb{Q}$ such that $f(x) = q \ \forall x \in [a, b]$.

Proof. Suppose for contradiction that f is not constant, i.e., $\exists a, b \in \mathbb{R}$ such that $f(a) < f(b)$ and wlog $a < b$. Since between any 2 real numbers, there exists an innational number, it follows that there exists $c \in \mathbb{R} \setminus \mathbb{Q}$ such that $f(a) < c < f(b)$.

Then, from IVT, $\exists \xi_c \in (a, b)$ such that $f(\xi_c) = c \in \mathbb{R} \setminus \mathbb{Q}$. $\square$

Example 54. Suppose f is continuous on $[0,1]$ and $f(0) = f(1)$. Let $n \in \mathbb{N}$ be arbitrary. Then, $\exists x_* \in [0,1)$ such that $f(x_*) = f\left(x_* + \frac{1}{n}\right)$.

Proof. Define $g : \left[0, 1 - \frac{1}{n}\right] \to \mathbb{R}$ by $g(x) := f(x) - f\left(x + \frac{1}{n}\right)$.

Suppose for contradiction that $g(x) \neq 0 \ \forall x \in \left[0, 1 - \frac{1}{n}\right]$. By cty (using Theorm 36), we must have either $g(x) > 0$ or $g(x) < 0 \ \forall x \in \left[0, 1 - \frac{1}{n}\right]$.

Wlog, assume $g(x) > 0 \ \forall x \in \left[0, 1 - \frac{1}{n}\right]$. Then, $f(x) > f\left(x + \frac{1}{n}\right) \ \forall x \in \left[0, 1 - \frac{1}{n}\right]$. It follows that, by setting $x = 0$, $f(0) > f\left(\frac{1}{n}\right)$, but also by setting $x = \frac{1}{n}$,

$$f\left(\frac{1}{n}\right) > f\left(\frac{2}{n}\right), \ldots, f\left(\frac{m}{n}\right) > f\left(\frac{m+1}{n}\right) \ \forall m \in \left\{0, \ldots, \frac{n-1}{n}\right\}$$

$$\implies f(0) > f\left(\frac{1}{n}\right) > f\left(\frac{2}{n}\right), \ldots, f\left(\frac{n-1}{n}\right) > f(1)$$

$$\implies f(0) > f(1)$$

but we assumed $f(0) = f(1)$, which is a contradiction. □

Example 55. Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is continuous and $n \in \mathbb{N}$, and $\lim_{x \to \infty} \frac{\phi(x)}{x^n} = 0 = \lim_{x \to -\infty} \frac{\phi(x)}{x^n}$. Then,

(a) if n is odd, $\exists x_* \in \mathbb{R}$ such that $(x_*)^n + \phi(x_*) = 0$

(b) if n is even, $\exists y \in \mathbb{R}$ such that $(y)^n + \phi(y) \leq x^n + \phi(x) \ \forall x \in \mathbb{R}$

Proof. Define $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(x) := x^n + \phi(x) \ \forall x \in \mathbb{R}$ and note that ψ is also continuous on $\mathbb{R}$.

(a) Since n is odd, $\lim_{x \to -\infty} \frac{\psi(x)}{|x|^n} = -1 + \underbrace{\lim_{x \to -\infty} \frac{\phi(x)}{|x|^n}}_{=0}$ and similarly $\lim_{x \to \infty} \frac{\psi(x)}{|x|^n} = 1$.

Note that $x \mapsto \frac{\psi(x)}{|x|^n}$ is continuous on any internal excluding 0.

Then, since $\frac{\psi(x)}{|x|^n}$ is continuous on $(-\infty, 0)$, $\exists R_1 = R_1(\frac{1}{2}) > 0$ such that

$$x < -R_1 \implies \left|\frac{\psi(x)}{|x|^n} - (-1)\right| < \frac{1}{2}$$

i.e., for $x < -R_1$, we have $\frac{\psi(x)}{|x|^n} < (-1) + \frac{1}{2} = -\frac{1}{2}$.

$$\implies \psi(x) < -\frac{1}{2}|x|^n \ \forall x \in \mathbb{R}$$

i.e., for all $x < -R_1$, we have $\psi(x) < 0$ ⊛.

Similarly, $\exists R_2 = R_2(\frac{1}{2}) > 0$ such that

$$\begin{aligned} x > R_2 &\implies \left|\frac{\psi(x)}{|x|^n} - 1\right| < \frac{1}{2} \\ &\implies \psi(x) > \frac{1}{2}|x|^n \ \forall x > R_2 \end{aligned}$$

Therefore, $\psi(x) > 0$ for all $x > R_2$ ⊛⊛.

By ⊛ and ⊛⊛,$\exists a, b \in \mathbb{R}$ $(a < b)$ such that

$$\psi(a) < 0 < \psi(b)$$

Then since ψ is continuous, by Theorem 36 $\implies \exists x_* \in (a, b)$ such that $\phi(x_*) = 0$, i.e., $x_*^n + \phi(x_*) = 0$.

□

Example 56.

Example 57.

Example 58.

Example 59. Suppose f is continuous and ⊛ $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0$, and $f(x) > 0\ \forall x \in \mathbb{R}$. Then, $\exists x_* \in \mathbb{R}$ such that $f(x) \le f(x_*)\ \forall x \in \mathbb{R}$.

Proof. Let $\mu := \max_{y\in[-1,1]} f(y)$, by ⊛, $\exists R_1, R_2 > 0$ such that

$$x < -R_1 \implies 0 < f(x) < \frac{1}{2}\mu$$

$$x > R_2 \implies 0 < f(x) < \frac{1}{2}\mu$$

Hence $0 < f(x) < \frac{1}{2}\mu$ for all $|x| \in \mathbb{R} := \max\{R_1, R_2\}$. and meanwhile $\sup_{x\in\mathbb{R}} f(x) \ge \sup_{x\in[-1,1]} f(x) = \mu$.

$\sup_{x\in\mathbb{R}} f(x)$ is well-defined Since $\sup_{[-R,-R]} f$ is well-defined and achieved by Theorem and $|f(x)| < \frac{1}{2}\mu$ for $|x| > R$.

$$+\infty > \sup_{x\in\mathbb{R}} f(x) \geq \max_{x\in[-R,R]} f(x) \geq \mu > \sup_{|x|>R} f(x)$$

It follows that $\sup\limits_{x\in\mathbb{R}} f(x) = \sup\limits_{x\in[-R,R]} f(x)$ ($\mathbb{R} = \underbrace{\{x : |x| \leq R\}}_{=[-R,R]} \cup \{x : |x| > R\}$)

Since f is continuous, it achieves its boundes by Theorem 38 $\implies \exists x_* \in [-R, R]$ such that $f(x_*) = \sup\limits_{[-R,R]} f = \sup\limits_{\mathbb{R}} f$. $\square$

Example 60. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = (\sin x)^2 + (\sin(x + (\cos x)^7))^2$$

Then, $\exists c > 0$ such that $f(x) \geq c \ \forall x \in \mathbb{R}$.

Proof. Observe that $f(x) \geq 0 \ forall x$ and $A := \{f(x) : x \in \mathbb{R}\}$ is bounded below by 0.

Define $c := \inf A$ is well-defined.

$$\begin{aligned} f(x + 2\pi) &= (\sin(x + 2\pi))^2 + \sin(\overbrace{(x + 2\pi) + (\underbrace{\cos(x + 2\pi)}_{=\cos x})^7}^{=x+2\pi+(\cos x)^7})^2 \\ &= (\sin x)^2 + \sin(x + (\cos x)^7)^2 \\ &= f(x) \end{aligned}$$

f is 2π-periodic, $\implies c = \inf A = \inf\{f(x) : x \in [0, 2\pi]\}$

Since f is continuous, Theorem 38 $\implies \exists x_* \in [0, 2\pi]$ such that $f(x_*) = c$.

Suppose for contradiction that $c = 0$

$$\begin{aligned} &\implies f(x_*) = 0 \\ &\implies (\underbrace{\sin x_*}_{=0})^2 + (\underbrace{\sin(x_* + (\cos x_*)^7)}_{=0})^2 = 0 \\ &\implies x_* \in \{0, \pi, 2\pi\} \text{ but then } \cos x_* \in \{1, -1\} \\ &\implies x_* + (\cos x_*)^7 \in \{1, \pi - 1, 2\pi + 1\} \\ &\implies \sin(x_* + (\cos x_*)^7) \in \{\sin(1), \sin(\pi - 1)\} \text{ neither of which are } 0 \end{aligned}$$

$\square$

3.6. Uniform Continuity

Finally, we look at uniform continuity

Definition 61. Let $f : \mathbb{R} \to \mathbb{R}$. We say f is uniformly continuous on an interval A if for all $\varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ such that

$$|x - y| < \delta \text{ and } x, y \in A \implies |f(x) - f(y)| < \varepsilon$$

KEY: δ is not depend on a specific point.

Example. $f(x) = x$ is uniformly continuous on $\mathbb{R}$. Let $\varepsilon > 0$ be given then letting $\delta = \varepsilon$, we see that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Example. $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Fix $\varepsilon > 0$ and recall from Lecture 10 that

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0||x + x_0|$$

and so we need $\delta = \min\left(1, \frac{\varepsilon}{1+2|x_0|}\right)$ to have $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$.

We see that δ depends on specific point x_0.

This is only an indication that f is not uniformly continuous – not a proof yet.

The negation of Definition 61

Definition (61′)**.** $\exists \varepsilon_0 > 0$ such that for all $\delta > 0$ there exist corresponding $x_\delta, y\delta \in A$ such that

$$|x_\delta - y_\delta| < \delta \text{ and } |f(x_\delta) - f(y_\delta)| \geq \varepsilon_0$$

Proof of Example . Let $\varepsilon_0 = 1$. Observe that for $x > y > 0$,

$$|f(x) - f(y)| = x^2 - y^2 = (x + y)(x - y)$$

For each $\delta > 0$ choose $y_\delta = \delta^{-1}$ and $x_\delta = \delta^{-1} + \frac{\delta}{2}$

Then, $x_\delta + y_\delta = 2\delta^{-1} + \frac{\delta}{2} > 2\delta^{-1}$ and $|x_\delta - y_\delta| = \frac{\delta}{2} < \delta$.

Hence, $|x_\delta - y_\delta| < \delta$ and also

$$\begin{aligned}
|f(x_\delta) - f(y_\delta)| &= (x_\delta + y_\delta)(x_\delta - y_\delta) \\
&= (2\delta^{-1} + \frac{\delta}{2}) \cdot \frac{\delta}{2} \\
&= 1 + \frac{\delta^2}{4} \\
&> 1 = \varepsilon_0
\end{aligned}$$

□

Remark. $x \mapsto x^2$ is uniformly continuous on $[-1,1]$, even though it is not uniformly continuous on $\mathbb{R}$.

Example 62. Let $f:[0,\infty)\to[0,\infty), x\mapsto x^{\frac{1}{2}}$ Then f is uniform continuous on $[0,\infty)$.

Proof. Let $x,y\in[0,\infty)$ and wlog assume $x>y$. Notice that

$$\oplus|f(x)-f(y)| = \sqrt{x}-\sqrt{y} \overbrace{\leq}^{\circledast} \sqrt{x-y}$$

Hence, given any $\varepsilon>0, |x-y|<\varepsilon^2 \underbrace{\implies}_{\oplus} |f(x)-f(y)|<\varepsilon$.

<u>proof of $\circledast$</u>: let $a>b\geq 0$

$$\begin{aligned}(\sqrt{a}-\sqrt{b})^2 &= a+b \quad \overbrace{-2\sqrt{a}\sqrt{b}}^{<-2\sqrt{b}\sqrt{b}=-2b} \\ &\leq a-b \\ \implies \sqrt{a}-\sqrt{b} &\leq \sqrt{a-b}\end{aligned}$$

□

Theorem 63. If f is continuous on $[a,b]$, then f is uniformly continuous on $[a,b]$.

The choice of the interval A matters on the Definition 61.

Proof. We first make the following definition

For $\varepsilon>0$, we say that g is <u>ε-good on $[a,b]$</u> if $\exists\delta=\delta(\varepsilon)$ such that for all $y,z\in[a,b]$,

$$|y-z|<\delta \implies |g(y)-g(z)|<\varepsilon$$

We want to prove that f is ε-good on $[a,b]$ for every $\varepsilon>0$.

For each $\varepsilon>0$, define

$$A_\varepsilon := \{x\in[a,b] : f \text{ is } \varepsilon\text{-good on } [a,x]\}$$

Then, $A_\varepsilon\neq\emptyset$ since $a\in A_\varepsilon$, and A_ε is certainly bounded above by b. Hence, $\sup A_\varepsilon$ is well-defined and we set $\alpha_\varepsilon := \sup A_\varepsilon$

Fix $\varepsilon>0$. Our aim is to prove that $\alpha_\varepsilon=b$. Suppose for contradiction $\alpha_\varepsilon<b$. Since f is continuous at $\alpha_\varepsilon, \exists\delta_0=\delta_0(\varepsilon,\alpha_\varepsilon)$ such that

$$|y-\alpha_\varepsilon|<\delta_0 \implies |f(y)-f(\alpha_\varepsilon)|<\frac{\varepsilon}{2}$$

Hence if both $y, z \in [\alpha_\varepsilon - \delta_0, \alpha_\varepsilon + \delta_0]$ there holds

$$|y - \alpha_\varepsilon| < \delta_0 \implies |f(y) - f(\alpha_\varepsilon)| < \frac{\varepsilon}{2}$$

$$|z - \alpha_\varepsilon| < \delta_0 \implies |f(z) - f(\alpha_\varepsilon)| < \frac{\varepsilon}{2}$$

So, triangle inequality gives $|f(y) - f(z)| < \varepsilon$.

This, f is ε-good on $[\alpha_\varepsilon - \delta_0, \alpha_\varepsilon + \delta_0]$. Also since $\alpha_\varepsilon = \sup A_\varepsilon$, it is also clear (from Lemma 28) that f is ε-good on $[a, \alpha_\varepsilon - \delta_0]$.

Claim: f is ε-good on $[a, \alpha_\varepsilon + \delta_0]$.

We will prove this claim later. Assuming it holds, we get that f is ε-good on $[a, \alpha_\varepsilon + \delta_0] \implies \alpha_\varepsilon + \delta_0 \in A_\varepsilon$ but $\alpha_\varepsilon + \delta_0 > \alpha_\varepsilon = \sup A_\varepsilon$.

Hence, $\alpha_\varepsilon = b$. We now show that $b \in A$. Since f is continuous at b, $\exists \delta_1 = \delta_1(\varepsilon, b)$ such that

$$b - \delta_1 < y \leq b \implies |f(y) - f(b)| < \frac{\varepsilon}{2}$$

So we again see that f is ε-good on $[b - \delta_1, b]$. But f is also ε-good on $[a, b - \delta_1]$. Since $b - \delta_1 \in A$ by Lemma 28. So, using the claim again we get that $b \in A_\varepsilon$. ☐

proof of Claim. Since f is continuous at $\alpha_\varepsilon - \delta_0$, $\exists \delta_2 = \delta_2(\varepsilon, \alpha_\varepsilon - \delta_0)$ such that

$$(\dagger\dagger\dagger)|x - (\alpha_\varepsilon - \delta_0)| < \delta_2 \implies |f(x) - f(\alpha_\varepsilon - \delta_0)| < \frac{\varepsilon}{2}$$

Meanwhile, f is ε-good on $[a, \alpha_\varepsilon - \delta_0]$, so $\exists \delta_3 = \delta_3(\varepsilon)$ such that

$$x, y \in [a, \alpha_\varepsilon - \delta_0], |x - y| < \delta_3 \implies |f(x) - f(y)| < \frac{\varepsilon}{2}(\dagger)$$

and similarly, $\exists \delta_4 = \delta_4(\varepsilon)$ such that

$$x, y \in [\alpha_\varepsilon - \delta_0, \alpha_\varepsilon + \delta_0], |x - y| < \delta_4 \implies |f(x) - f(y)| < \frac{\varepsilon}{2}(\dagger\dagger)$$

Now, choose any $x, y \in [a, \alpha_\varepsilon + \delta_0]$. If x, y both belong either to $[a, \alpha_\varepsilon - \delta_0]$ or to$[\alpha_\varepsilon - \delta_0, \alpha_\varepsilon + \delta_0]$, then there is nothing to show (by †, ††). The final possibility is $x \in [a, \alpha_\varepsilon - \delta_0]$ and $y \in [\alpha_\varepsilon - \delta_0, \alpha_\varepsilon + \delta_0]$.

In this case, let $\delta := \min(\delta_2, \delta_3, \delta_4)$ and observe that

$$\begin{aligned}
|x - y| < \delta &\overbrace{\implies}^{\text{since } y > x} 0 \leq y - x < \delta \\
&\implies 0 \leq (y - (\alpha_\varepsilon - \delta_0)) + ((\alpha_\varepsilon - \delta_0) - x) < \delta \\
&\implies |y - (\alpha_\varepsilon - \delta_0)| < \delta \\
&\implies |f(y) - f(\alpha_\varepsilon - \delta_0)| < \frac{\varepsilon}{2}(\dagger\dagger\dagger) \text{ and } |f(z) - f(\alpha_\varepsilon - \delta_0)| < \frac{\varepsilon}{2} \\
&\implies |f(y) - f(z)| < \varepsilon
\end{aligned}$$

Note that $\delta = \min(\delta_2(\varepsilon, \alpha_\varepsilon - \delta_0(\varepsilon, \alpha_\varepsilon)), \delta_3(\varepsilon), \delta_4(\varepsilon))$.

δ only depends on $\varepsilon, \alpha_\varepsilon$, and since α_ε only depends on ε, we define that $\underline{\delta \text{ only depends on } \varepsilon}$, as required. □

Example 64.

(i) $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous and bounded on $(0, 1]$ however it it not uniformly continuous on $(0, 1]$.

(ii) $f(x) = \sin(e^x)$ is continuous and bounded on $[0, \infty)$ however it is not uniformly continuous on $[0, \infty)$.

Proof.

(i) Fix any $\delta > 0$ and let $x_\delta = \frac{1}{2\pi n_\delta}$ and $y_\delta = \frac{1}{\frac{\pi}{2} + 2\pi n_\delta}$, where $n_\delta \in \mathbb{N}$ is to be chosen. Notice that

$$0 < x_\delta - y_\delta = \frac{\frac{\pi}{2} + 2\pi n_\delta - 2\pi n_\delta}{2\pi n_\delta \left(\frac{\pi}{2} + 2\pi n_\delta\right)} = \frac{1}{4n_\delta \left(\frac{\pi}{2} + 2\pi n_\delta\right)}$$

thus, by choosing n_δ large enough,

$$\frac{1}{4n_\delta \left(\frac{\pi}{2} + 2\pi n_\delta\right)} < \delta$$

and thus $|x_\delta - y_\delta| < \delta$, and yet $|f(x_\delta) - f(y_\delta)| = 1$

So, f is not uniformly continuous on $(0, 1]$.

(ii) Fix any $\delta > 0$ and let $x_\delta = \log\left(2\pi n_\delta + \frac{\pi}{2}\right), y_\delta = \log(2\pi n_\delta)$ where n_δ is to be chosen. Observe that

$$0 < x_\delta - y_\delta = \log\left(1 + \frac{1}{4n_\delta}\right)$$

Since $\log : [1, \infty) \to [0, \infty)$ is continuous at 1, and $\log(1) = 0$, $\exists n_\delta \in \mathbb{N}$ sufficiently large such that

$$0 < \log\left(1 + \frac{1}{4n_\delta}\right) < \delta$$

Thus, $|x_\delta - y_\delta| < \delta$ and yet $|\underbrace{f(x_\delta)}_{\sin\left(2\pi n_\delta + \frac{\pi}{2}\right) = 1} - \underbrace{f(y_\delta)}_{\sin(2\pi n_\delta) = 0}| = 1$.

So, f is not uniformly continuous on $[0, \infty)$.

□

This concludes our section on continuity. We are now ready to look at differentation.

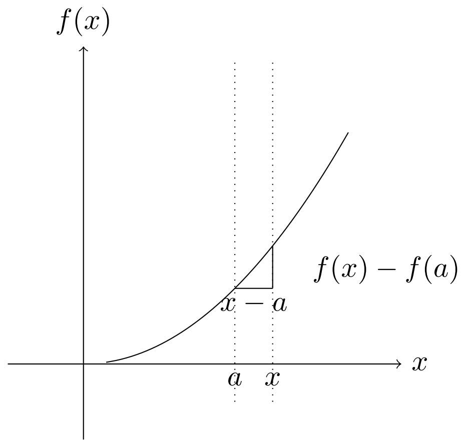
4. Differenitiation

Office hours on Monday

1. Office hour 6.pm to 7.pm on Monday
2. can meet before 8:50 am Monday in my office Van Vleck 613 (send an email on sunday)

Consider a function defined on on interval I, with real values. $f : I \to \mathbb{R}$

Definition. f is differentiable at the point $a \in I$ if the limit $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists, then we call this limit the deriviative $f'(a)$



$$y = f(x),\ \frac{f(x)-f(a)}{x-a} = \text{slope of } f$$

Computation of some derivatives

Example.

(i) $f(x) = c$ (c is some fixed point) we get $f'(a) = 0$ for all a,

$f(x) = f(a) = 0$ for all x, $\frac{f(x)-f(a)}{x-a} = 0 \implies f$ is differentiable and $f'(a) = 0$ for all a

$\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(x)$ is equivalent with saying $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$

(ii) $f(x) = x$, then
$$\frac{f(a+h)-f(a)}{h} = \frac{a+h-a}{h} = 1$$
(written $f'(x) = 1$)

(iii) $f(x) = x^2$, then fix a,
$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^2-a^2}{h} = \frac{a^2+2ah+h^2-a^2}{h} = 2a+h$$
$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} 2a+h = 2a$$

(iv) $f(x) = |x|$, We should examine the differentiability of f at $\underline{a=0}$
$$\frac{f(0+h)-\overbrace{f(0)}^{=0}}{h} = \frac{|h|}{h} = \begin{cases} 1 & \text{if } h > 0 \\ -1 & \text{if } h < 0 \end{cases}$$
The limit does not exist, and thus f is not differentiable at 0.

(v) $f(x) = \sqrt{|x|}$, f is not differentiable at 0 because $f(0) = 0$ and $\frac{f(0+h)-f(0)}{h} = \frac{\sqrt{|h|}}{h}$, this limit also does not exist

Examine differentiability and derivative of $f(x) = \sqrt{|x|}$ at $x = a, a > 0$
$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{\sqrt{|a+h|}-\sqrt{|a|}}{h} \\ &= \frac{\sqrt{a+h}-\sqrt{a}}{h} \\ &= \frac{a+h-a}{\sqrt{a+h}+\sqrt{a}}\cdot\frac{1}{h} \\ &= \frac{1}{\sqrt{a+h}+\sqrt{a}} \to \frac{1}{2\sqrt{a}}\end{aligned}$$

(vi) $f(x) = x^n$
$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^n-a^n}{h} = n\cdot a^{n-1}$$

4.1. Basic fact about differentiation

Continuity is necessary (but not sufficient) for differentiation

Theorem. If $f : I \to \mathbb{R}$ is differentiable at a the f is continuous at a.

Reminder If $\lim_{x\to a} F(x) = l$ and $\lim_{x\to a} G(x) = m$, then $\lim_{x\to a} F(x)G(x) = lm$

If $\lim_{x\to a} F(x) = l$ and $\lim_{x\to a} G(x) = m$, then $\lim_{x\to a} \frac{F(x)}{G(x)} = \frac{l}{m}$ or not? Yes if $m \neq 0$

Proof. We know that $\lim_{x\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$

$$f(a+h) - f(a) = \frac{f(a+h)-f(a)}{h} \cdot h$$

$$\implies \lim_{h\to 0} f(a+h) - f(a) = f'(a) \cdot 0 = 0$$

$$\lim_{h\to 0} f(a+h) = f(a)$$

this is continuity of f at a

Another argument : for sufficiently small h, $|f(a+h) - f(a)| \leq C|h|$ □

4.2. Sum Rule

Theorem. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$, $a \in I$ assume that f and g are differentiable at a. Then $f+g$, $(f+g)(x) = f(x)+g(x)|_{x=a}$ is differentiable and its derivative $f'(a)+g'(a)$ (The derivative of the sum is the sum of the derivatiives)

Proof.

$$\begin{aligned}\frac{(f+g)(a+h)-(f+g)(a)}{h} &= \frac{f(a+h)+g(a+h)-(f(a)+g(a))}{h} \\ &= \frac{f(a+h)-f(a)}{h} + \frac{g(a+h)-g(a)}{h}\end{aligned}$$

As $h \to 0$ this has limit $f'(a) + g'(a)$ □

4.3. Product Rule

Theorem. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$, $a \in I$ assume that f and g are differentiable at a. the $f \cdot g$ is differentiable at a

$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$$

Proof.

$$\begin{aligned}\frac{f(a+h)g(a+h)-f(a)g(a)}{h} &= \frac{(f(a+h)-f(a))g(a+h)+f(a)g(a+h)-f(a)g(a)}{h} \\ &= \underbrace{\frac{f(a+h)-f(a)}{h}}_{f'(a)} \cdot \underbrace{g(a+h)}_{\to g(a)} + \underbrace{\frac{g(a+h)-g(a)}{h}}_{\to g'(a)} \cdot \underbrace{f(a)}_{f(a)}\end{aligned}$$

By theorem about products and of limits, and the continuity of g at a, we get

$$\lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a)g(a)}{h} = f'(a)g(a)+g'(a)f(a)$$

□

Theorem. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$, $a \in I$ is differentiable at a, and if $g(a) \neq 0$ then $\frac{1}{g}$ is differentiable at a and

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{(g(a))^2}$$

Proof.

$$\begin{aligned}\frac{\frac{1}{g(a+h)}-\frac{1}{g(a)}}{h} &= \frac{g(a)-g(a+h)}{g(a+h)g(a)}\cdot\frac{1}{h} \\ &= \frac{1}{g(a+h)g(a)}\cdot(-1)\frac{g(a+h)-g(a)}{h} \\ &\to \frac{1}{(g(a))^2}\cdot(-1)g'(a)\end{aligned}$$

□

4.4. Quotient Rule

Theorem. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$, $a \in I$ assume f and g are differentiable at a, and if $g(a) \neq 0$ then $\frac{f}{g}$ is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a)-f(a)g'(a)}{g(a)^2}$$

Proof. Combine the theorems about products and reciprocals of differentiable function

$$\begin{aligned}
\left(\frac{f}{g}\right)'(a) &= \left(f \cdot \frac{1}{g}\right)'(a) \\
&= f'(a) \cdot \frac{1}{g(a)} + f(a) \cdot \left(\frac{1}{g}\right)'(a) \\
&= \frac{f'(a)}{g(a)} + f(a)\left(-\frac{g'(a)}{g(a)^2}\right) \\
&= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}
\end{aligned}$$

□

Example.

$$\begin{aligned}
\left(\frac{\sin x}{\cos x}\right)' &= \frac{\sin'(x)\cos(x) - \cos' x \sin x}{(\cos x)^2} \\
&= \frac{(\cos x)^2 - (-\sin^2 x)}{(\cos x)^2} \\
&= \frac{1}{(\cos x)^2}
\end{aligned}$$

Example. $f(x) = x^n$ then $f'(x) = nx^{n-1}$

Proof.

$$f_0(x) = 1, f_0'(x) = 0$$

$$f_1(x) = x, f_1'(x) = 1$$

$$f_1(x) = x^2, f_2'(x) = 2x$$

We want to show this formula for a given n, assuming that we already know if for $n = 1$, In other words, the formula $f_{n-1}'(x) = (n-1)x^{n-1}, n \geq 2$, implies the formula for f_n

Induction step:

$$f_n(x) = x^n = \underbrace{x^{n-1}}_{f_{n-1}} \cdot \underbrace{x}_{f_1}$$

By using Product Rule, we get

$$\begin{aligned}
f_n'(x) &= f_{n-1}'(x)f_1(x) + f_{n-1}(x)f_1'(x) \\
&= (n-1)x^{n-1} \cdot x + x^{n-1} \cdot 1 \\
&= nx^{n-1}
\end{aligned}$$

□

Example.

$$
\begin{aligned}
(fg)'' &= (f'g + fg')' \\
&= (f'g)' + (fg')' \\
&= f''g + f'g' + f'g' + fg'' \\
&= f''g + 2f'g' + fg''
\end{aligned}
$$

$(fg)''' = f'''g + 3f''g' + 3f'g'' + fg'''$ and can be written as $(fg)^{(3)}$

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)$$

As the analogy

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

We will not talk much about higher derivative in this class

4.5. Chain Rule

Let $\zeta(x) = f(g(x))$ Assume that g is defined on an interval containing a, and g is differentiable at $\underline{a}$. Let f be defined in an interval that contain the range (image) of g, and let f be differentiable at $g(a)$. then, $\zeta = f \circ g$ is differentiable at a and

$$\zeta'(a) = f'(g(a))g'(a)$$

Example.

$$F(x) = (x^3 + 7x^2 + 1)^8$$

Fix a point a, what $F'(a)$

Let $F(x) = f(g(x))$, $g(x) = x^3 + 7x^2 + 1$ and $f(w) = w^8$

First, calculate f' and g'

$$f'(w) = 8w^7$$

$$g'(x) = 3x^2 + 14x$$

Then cancluate $F'(x)$

$$
\begin{aligned}
F'(x) &= f'(g(x))g'(x) \\
&= 8(g(x))^7 \cdot (3x^2 + 14x) \\
&= 8(x^3 + 7x^2 + 1)^7 \cdot (3x^2 + 14x)
\end{aligned}
$$

Attempt to prove the chain rule

Proof.

$$\begin{aligned}\frac{\zeta(a+h)-\zeta(a)}{h} &= \frac{f(g(a+h))-f(g(a))}{h} \\ &= \underbrace{\frac{f(g(a+h))-f(g(a))}{g(a+h)-g(a)}}_{\to f'(g(a))} \cdot \underbrace{\frac{g(a+h)-g(a)}{h}}_{\to g'(a)}\end{aligned}$$

But $g(a+h)-g(a)$ might be equal to 0, So, we can't use this method to prove the chain rule. □

Theorem (Decomposition theorem for differentation)**.** The function f is differentiable at a (with derivative $f'(a)$) <u>if and only if</u> there is another function u with the same domain as f, so that u is continuous at a and

$$f(x) = f(a) + (x-a)u(x)$$

Then

$$u(a) = f'(a)$$

Proof. Assume that f is differentiable at a, $f'(a)$ is the derivative

$$u(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} & \text{if } x \neq a \\ f'(a) & \text{if } x = a \end{cases}$$

(u depends on a but a is fixed)

u is continuous at a because $\lim\limits_{x\to a} \frac{f(a+h)-f(a)}{h} = f'(a) = u(a)$

Suppose that

$$\zeta(x) = f(g(x)) \implies \zeta'(a) = f'(g(a))g'(a)$$

Assumption

(1) g is differentiable at a

(2) f is differentiable at $g(a)$

we can write

$$g(x) = g(a) + (x-a)u(x) \; *$$

where u is continuous at a, $g'(a) = u(a)$, and

$$f(y) = f(g(a)) + (y-g(a))v(y) \;\; **$$

where v is continuous at $g(a)$, $v(g(a)) = f'(g(a))$

Goal is to find a function w continuous at a such that

$$\zeta(x) = \zeta(a) + (x-a)w(x)$$

with $w(a) = f'(g(a))g'(a)$

from $**$,

$$f(g(x)) = f(g(a)) + (g(x) - g(a)) \underbrace{v(g(x))}_{\text{cts at } a}$$

from $*$,

$$f(g(x)) = f(g(a)) + (x-a) \underbrace{u(x)v(g(x))}_{\text{cts at } a}$$

Then, we get

$$w(x) := u(x)v(g(x))$$

and

$$w(a) = u(a)v(g(a)) = g'(a)f'(g(a))$$

□

4.6. Geometric meaning of Differentiation

Theorem. Let f be defined on an interval I and let a be a point in the interior of this interval.

Assume:

1. f has a maximum at a
2. f is differentiable at a

Then, $f'(a) = 0$

formally f has a maximum in I at a, means $f(x) \leq f(a)$ for all $x \in I$ (Also works for min in place of max)

Proof. We know by the assumption $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(x)$ exists.

1. If $x > a$ then $f(x) \leq f(a) \implies \frac{f(x)-f(a)}{x-a} \leq 0$ (slope of right side ≤ 0)
2. If $x < a$ then $f(x) \leq f(a)$ but now $x - a < 0, \frac{f(x)-f(a)}{x-a} \geq 0$ (slope of left side ≥ 0)

So, $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$ has to be ≥ 0 and ≤ 0, so it must be 0.

□

4.7. Mean-Value Theorem

Theorem (Mean-value theorem)**.** Let f be defined on $[a, b]$ and f continuous in $[a, b]$ and differentiable in (a, b). Then there is a $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

Proof. First step in the proof is a special case where $f(a) = f(b)$ (then there is a $\xi \in (a, b)$ such that $f'(\xi) = 0$)

1. if f has a max and a min at the endpoint, f is contant and therefore $f'(\xi) = 0$ for all $\xi \in (a, b)$
2. if f has a maximum and a minimum in (a, b), then we know already, at such a point, the derivative is 0, so at that point $\xi \implies f'(\xi) = 0$

This particular case is called "*Rolle's theorem*"

Consider

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then, $g(a) = 0$ and $g(b) = 0$ and g is continuous in (a, b)

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

Apply "*Rolle's theorem*" to g on (a, b), we get a $\xi \in (a, b)$ such that $g'(\xi) = 0$ □

Aforementioned theorem can be written as

$$f(b) - f(a) = f'(\xi)(b - a)$$

4.8. Application of the Mean-Value Theorem

Example. Prove $|\sin x| \leq |x|$

Proof. We know that $\sin 0 = 0$ and $\sin' x = \cos x$

$$\sin x = \sin x - \underbrace{\sin 0}_{=0} = \sin'(\xi)(x - 0)$$

where ξ is between 0 and x

$$\begin{cases} 0 < \xi < x & \text{if } x > 0 \\ x < \xi < 0 & \text{if } x < 0 \end{cases}$$

So, $\sin x = (\cos \xi)x$, where $-1 \le \cos \xi \le 1 \implies |\cos \xi| \le 1$

Therefore,

$$|\sin x| = |\cos \xi| \cdot |x| \le |x|$$

□

Example. Can we get an estimate for $\cos x - 1$ where x is small?

$$\cos x - 1 = \cos x - \cos 0 = \cos'(\xi)(x - 0) = (-\sin \xi)x$$

We get $|\cos x - 1| \le |x|$

Can do better

$$\begin{aligned} |\cos x - 1| &\le |(\sin \xi)| \cdot |x| \quad \text{for } \xi \text{ between } 0 \text{ and } x \\ &\le |\xi| \cdot |x| \le |x|^2 \end{aligned}$$

for $|x| < 1$ this is a better estimate than the previous one

Theorem. If f is differentiable on (a, b) and if $f'(x) = 0$ for all $x \in (a, b)$ then f is constant.

Proof. take $x_1 < x_2$, both in the interval and apply the MVT

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1), \; x_1 < \xi < x_2$$

we know $f'(\xi) = 0 \implies f(x_2) - f(x_1) = 0 \implies f(x_2) = f(x_1)$

So, f is constant function □

Example. for a differential equation

Q: Find $f(x)$ (differentiable, for $x > 0$) such that

$$xf'(x) = f(x)$$

Proof. By guessing $f(x) = x$ is a solution because $f'(x) = 1, x \cdot 1 = x$

In fact, for any constant C, $f(x) = Cx$ is a solution.

Show that for an arbitrary solution g, $xg'(x) = g(x)$ for $x > 0$ try to show that $\frac{g(x)}{x}$ is constant

To do this, show that the derivative of $\frac{g(x)}{x}$ is zero

$$\frac{g(x)}{x} = \frac{g'(x)x - g(x) \cdot 1}{x^2} = 0, \text{ since } g \text{ satisfies the differential equation}$$

So, $\frac{g(x)}{x}$ is constant □

Example.

$$xf'(x) = af(x)$$

Cx^a is a solution

Proof. Conjecture: All solutions are of the form $f(x) = Cx^a$

Let g be a solution of the equation, we have $xg'(x) = ag(x)$ Consider

$$\begin{aligned}\left(\frac{g(x)}{x^a}\right)' &= \frac{g'(x)x^a - g(x)ax^{a-1}}{x^{2a}} \\ &= \frac{x^{a-1}}{x^{2a}} \cdot \underbrace{(g'(x)x - ag(x))}_{=0}\end{aligned}$$

So, $\frac{g(x)}{x^a}$ is constant, so $g(x) = Cx^a$ for some C ☐

Theorem. If f, f' are differentiable on (a, b) if $f'(c) = 0$ and $f''(x) > 0$ for all x in (a, b) then f has a minimum at c

Proof. To do this, we want to check that f is strictly increasing for $x > c$ and strictly decreasing for $x < 0$

We do this by checking $f'(x) < 0, x < c$ and $f'(x) > 0, x > c$

$$f''(x) > 0 \implies f' \text{ is increasing (strictly) on } (a, b)$$

$$f'(c) = 0 \implies f'(x) > 0, x > c \text{ and } f'(x) < 0, x < c$$

☐

4.9. Inverse Function

one-to-one (injective)

f is one-to-one if $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$

Example.

$$f : (1, 2) \to \mathbb{R}$$
$$x \mapsto x^2$$

Show that f is one-to-one

Proof. proof by contradiction

$x_1^2 = x_2^2$ and $x_1, x_2 \in (1, 2)$

$\sqrt{x_1^2} = x_1, \sqrt{x_2^2} = x_2 \implies x_1 = x_2$ □

Example.

$$f : (-5, 5) \to \mathbb{R}$$
$$x \mapsto x^2$$

Show that f is not one-to-one

Proof. $2^2 = -2^2, x_1 = 2, x_2 = -2 \implies x_1^2 = x_2^2$ □

onto (surjective)

f is "onto" means that every element in B is a value $f(x)$ for some $x \in A$

Every function is onto if the target space is equal to the range of f

one-to-one and onto (bijective)

If a function is both one-to-one and onto (injective and surjective)

$f : A \to B$ bijective mean that for ever $x \in A$ there is exactly one $y \in B$ such that $y = f(x)$ and for every $y \in B$ there is exactly one x, such that $y = f(x)$ we say

$$x = f^{-1}(y) \iff y = f(x)$$

We pronounce f^{-1} as "f inverse"

Theorem. If f is strictly increasing on $[a, b]$ and continuous then $f[a, b] \to [f(a), f(b)]$ is bijective and f has an inverse function

$$f^{-1}[f(a), f(b)] \to [a, b]$$

$$y \mapsto x$$

As the result, we get $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$

If f and f^{-1}

$$f : [a, b] \to [f(a), f(b)]$$
$$f^{-1} : [f(a), f(b)] \to [a, b]$$

are both differentiable, what is the relation between the derivatives

Apply Chain rule on $f^{-1}(f(x)) = x$ then $(f^{-1})'(f(x))f'(x) = 1$

Apply Chain rule on $f(f^{-1}(y)) = y$ then $f'(f^{-1}(y))(f^{-1})'(y) = 1$

if and only if $y = f(x)$ and $x = f^{-1}(y)$

Theorem. If f is increasing or decreasing on some interval then it has an inverse function f^{-1}

Proof. If f and f^{-1} are differentiable then we may get a formula for $(f^{-1})'$ from the chain rule applied to $f^{-1}(f(x)) = x$

The chain rule given us

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

Derivative of f^{-1}, evaluated at $f(x)$

$$\implies (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

□

Theorem. Let f be (strictly) increasing on $[a, b]$ and $f'(x_0)$ exists for $x_0 \in (a, b)$ and $f'(x_0) \neq 0$ then f^{-1} is differentiable at $f(x_0)$ and $(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$

Proof. Precall

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

If $f(x) = y$ then

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{y - f(x_0)}{f^{-1}(y) - x_0} = \frac{1}{\frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)}}$$

$$\lim_{y \to f(x_0)} \frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} = (f^{-1})'(f(x_0))$$

if that limit exists

We know that $f'(x_0) = c > 0$ and $\frac{f(x) - f(x_0)}{x - x_0} \to C$ There exists $\delta > 0$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} > \frac{C}{2} \text{ for } |x - x_0| < \delta$$

$$\frac{f(x) - f(x_0)}{x - x_0} < 2C \text{ for } |x - x_0| < \delta$$

$$f(x) - f(x_0) \text{ is between } \frac{C}{2}(x - x_0) \text{ and } 2C(x - x_0)$$

$$x - x_0 \text{ is between } \frac{f(x)-f(x_0)}{2C} \text{ and } \frac{f(x)-f(x_0)}{\frac{C}{2}}$$

Then we get

$$\begin{aligned}\lim_{y\to f(x_0)} \frac{f^{-1}(y)-f^{-1}(f(x_0))}{y-f(x_0)} &= \lim_{y\to f(x_0)} \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}} \\ &= \frac{1}{f'(x_0)}\end{aligned}$$

□

Example. For $x > 0, f(x) = x^n, f : (0,\infty) \to (0,\infty)$ and $f^{-1} : (0,\infty) \to (0,\infty)$ The inverse function f^{-1} is called nth to of

$$f'(x) = nx^{n-1}$$

$$\begin{aligned}(f^{-1})'(y) &= \left.\frac{1}{f'(x)}\right|_{x=f^{-1}(y)} \\ &= \left.\frac{1}{nx^{n-1}}\right|_{x=f^{-1}(y)} \\ &= \frac{1}{n(\sqrt[n]{y}^{n-1})} \\ &= \frac{1}{n}\frac{1}{\sqrt[n]{y}}\sqrt[n]{y} \\ &= \frac{1}{n}\frac{1}{y}\sqrt[n]{y}\end{aligned}$$

Example. $f(x) = \frac{\sin x}{\cos x} = \tan x$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$f(x)$ is well-defined whenever $x \neq \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$

$$f'(x) = \frac{(\cos x)\cos x - \sin x(-\sin x)}{(\cos x)^2} = \frac{1}{(\cos x)^2}$$

$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \mapsto (-\infty,\infty)$ and we found that tan is increasing $\tan' x = \frac{1}{(\cos x)^2} > 0$

What is $(\tan^{-1})'(y)$

$$\begin{aligned}(f^{-1})'(y) &= \frac{1}{f'(f^{-1}(y))} \\ &= \frac{1}{\frac{1}{\cos^2(f^{-1}(y))}} \\ &= (\cos(f^{-1}(y)))^2 \\ &= (\cos(\tan^{-1}(y)))^2 \\ &= (\cos(\arctan(y)))^2\end{aligned}$$

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-(\cos x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2} - (\cos x)^2 - 1$$

$$\frac{1}{(\cos x)^2} = 1 + (\tan x)^2 \implies (\cos x)^2 = \frac{1}{1+(\tan x)^2}$$

$$(\cos(\arctan(y)))^2 = \frac{1}{1+(\tan(\arctan y))^2} = \frac{1}{1+y^2}$$

4.10. Practice Problems

Example (Q8). $f(t) = t^n - 1 - nt + n$ for $0 < t < 1$ what sign does f have?

$$f(1) = 1^n - 1 - n + n = 0$$

If $f'(t) > 0$ on $(0,1)$ then f is increasing on $(0,1)$

If $f'(t) < 0$ on $(0,1)$ then f is decreasing on $(0,1)$

For $t \in (0,1)$, $f'(t) = nt^{n-1} - n$, $0 < t^{n-1} < 1$ on $(0,1)$. So, $f'(t) < 0$ on $(0,1)$ and $f(0) = n-1$, f is decreasing on $(-1,0)$. Then $f(t) > n-1$ on $(-1,0)$

Example (Q2). f differentiable at a, $f'(a) > 0$

$$f(x) > f(a) \text{ for } a < x < a+\beta$$

$$f(x) < f(a) \text{ for } a-\beta < x < a$$

$$0 < f'(x) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(x)}{h}$$

Choose in the definition of limit, choose $\varepsilon = \frac{f'(a)}{2}$ There is a δ such that for $0 < |h| < \delta$ (i.e. $h \in (-\delta, \delta)$ and $h \neq 0$)

$$\left| \frac{f(a+h)-f(a)}{h} - f'(a) \right| < \varepsilon = \frac{f'(a)}{2}$$

We get that

$$\frac{f(a+h)-f(a)}{h} \text{ is in the interval } \left(\frac{f'(a)}{2}, \frac{3f'(a)}{2} \right)$$

In the partition

$$\frac{f(a+h)-f(a)}{h} > \frac{f'(a)}{2}$$

First case: $h > 0$ we get

$$f(a+h) - f(a) > \frac{f'(a)}{2} h > 0$$

i.e., $f(a+h) > f(a)$ for $0 < h < \delta$

Second case: $h < 0$ we get

$$\frac{f(a+h)-f(a)}{h} = \frac{f(a)-f(a+h)}{-h} > \frac{f'(a)}{2}$$

$$\implies f(a)-f(a+h) > \frac{f'(a)}{2}(-h) > 0$$

i.e., $f(a) > f(a+h)$ for $-\delta < h < 0$

Example (Q9)**.** $f : [a,b] \to [c,d]$ (strictly) decreasing $\implies f$ is one-to-one, $f(c) = d, f(b) = c$, f is onto So, $f^{-1} : [c,d] \to [a,b]$ exists

$$f^{-1}(y) = \left.\frac{1}{f'(x)}\right|_{x=f^{-1}(y)} = \frac{1}{f'(f^{-1}(y))} \text{ defined on } [c,d]$$

From the Quotient Rule,

$$\left(\frac{1}{g(y)}\right)' = -\frac{g'(y)}{g(y)^2}$$

apply this with $g(y) = f'(f^{-1}(y))$ then using Chain Rule

$$g'(y) = f''(f^{-1}(y)) \cdot (f^{-1})'(y) = f''(f^{-1}(y)) \cdot \frac{1}{f'(f^{-1}(y))}$$

$$(f^{-1})''(y) = -\frac{f''(f^{-1}(y)) \cdot \frac{1}{f'(f^{-1}(y))}}{f'(f^{-1}(y))^2} = -\frac{f''(f^{-1}(y))}{(f'(f^{-1}(y)))^3}$$

There are formula for the second and higher derivatives of composite functions

Example (Q6)**.** Given a function $f'(x) = x^2 + x + 1$ and $f(3) = 5$ One such function satisfying $f'(x) = x^2 + x + 1$ is

$$f(x) = \frac{x^3}{3} + \frac{x^2}{2} + x + c$$

For this function $f(3) = \frac{27}{3} + \frac{9}{2} + 3 + c = 5$ Can compute c so that this true

If on an interval ($\mathbb{R}$ here) a differentiable function has $F'(x) = 0$ everywhere then $F(x)$ is constant. In particular $F(x) = F(3)$.

Now let f, g be function such that their derivatiive is $x^2 + x + 1$ and their value at 3 is 5

$$F(x) := f(x) - g(x) \implies F'(x) = 0$$

$$F(3) = f(3) - g(3) = 5 - 5 = 0$$

$\implies F$ is constant, and equal to $F(3) = 0 \implies f(x) = g(x)$ everywhere

Example (Q10)**.** $f : \tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$

$$f'(x) = \frac{1}{(\cos x)^2} > 0$$

So, f is increasing and f is one-to-one

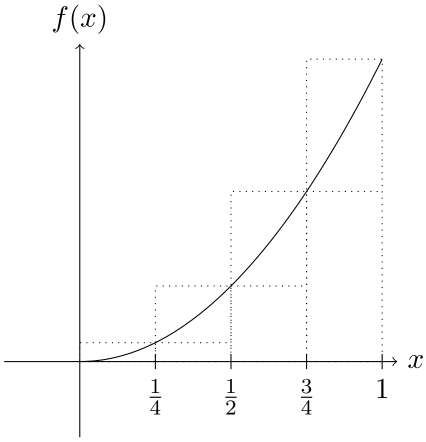
Pick $y \in \mathbb{R}$ and show there is x such that $f(x) = y, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\lim_{x \to \frac{\pi}{2}} \tan x = \infty \ \triangle$$

$$\lim_{x \to -\frac{\pi}{2}} \tan x = -\infty \blacktriangle$$

$\triangle$ For every R there is an interval $\left(\frac{\pi}{2} - \delta, \frac{\pi}{2}\right)$ such that $\tan x > R$ in this interval

5. Integration



$f(x) = x^2$ defined on $[0, 1]$

1. Given an interval $[a, b]$ and a function f defined by $f : [a, b] \to \mathbb{R}$, assume that f is bounded (there is an M such that $|f(x)| \leq M$ for all $x \in [a, b]$)

Definition. A partition of $[a, b]$ is a finite collection of distinct numbers in $[a, b]$. which contains the end point a and b.

We can order these

$$a = x_0 < x_1 < x_2 < \ldots < x_N = b$$

Are finement $\tilde{P}$ of partition P, is a partition which contains P

Definition (Lower Riemann Sum)**.**

$$L(f, P) = \sum_{j=1}^{N} (x_j - x_{j-1}) \cdot m_j$$

$$m_j = \inf_{x_{j-1} \leq t \leq x_j} f(t)$$

Lower Riemann sum, given f, and $P = \{x_0 < x_1 < \ldots < x_N\}$, inf is the greatest lower bound for f on $[x_{j-1}, x_j]$, m_j is the greatest lower bound for f on the partition$[x_{j-1}, x_j]$

Definition (Uower Riemann Sum)**.**

$$U(f, P) = \sum_{j=1}^{N} (x_j - x_{j-1}) \cdot M_j$$

$$M_j = \sup_{x_{j-1} \le t \le x_j} f(t)$$

where sup is the least upper bound, M_j is the least upper bound

the example $[a, b] = [0, 1]$

$$P = \left\{0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N}{N} = 1\right\}$$

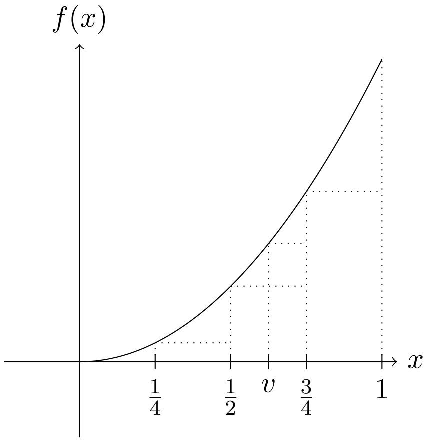
where $x_j = \frac{j}{N}, 0 \le j \le N$

$$\begin{aligned} L(f, P) &= \sum_{j=1}^{N} \frac{1}{N} \left(\frac{j-1}{N}\right)^2 \\ &= \frac{1}{N^3} \sum_{j=1}^{N} (j-1)^2 \\ &= \frac{1}{N^3} (0^2 + 1^2 + 2^2 + \dots + (N-1)^2) \\ U(f, P) &= \sum_{j=1}^{N} \frac{1}{N} \left(\frac{j}{N}\right)^2 \\ &= \frac{1}{N^3} \sum_{j=1}^{N} (j)^2 \\ &= \frac{1}{N^3} (1^2 + 2^2 + \dots + N^2) \end{aligned}$$

What happens with $L(f, P)$ if we refine the partition?

What happens with $U(f, P)$ if we refine the partition?

Proposition:if $P, \tilde{P}$ are partitions $P \subset \tilde{P}$ then $L(f, P) \le L(f, \tilde{P})$ and $U(f, P) \ge U(f, \tilde{P})$



$f(x) = x^2$ defined on $[0, 1]$

If we refine the partition by adding a point $\tilde{P} = P \cup \{v\}$

The key for the proof $x_{j-1}, x_j \in P$ take a new partition point v between x_{j-1} and x_j

$$\begin{aligned}(x_j - x_{j-1}) \inf_{x_{j-1} \le t \le x_j} f(t) &= (x_j - v) \inf_{x_{j-1} \le t \le x_j} f(t) + (v - x_{j-1}) \inf_{x_{j-1} \le t \le x_j} f(t) \\ &\le (x_j - v) \inf_{v \le t \le x_j} f(t) + (v - x_{j-1}) \inf_{x_{j-1} \le t \le v} f(t)\end{aligned}$$

Theorem. Let f be a bounded function on $[a, b]$. Then

$$\sup_P L(f, P) \le \inf_P U(f, P)$$

where $\sup_P$ is supremum over all partition and $\inf_P$ is infimum over all partitions.

Definition. Given the theorem, we say that f is integrable (or Riemann integrable) if

$$\sup_P L(f, P) = \inf_P U(f, P)$$

Proof.

(i) If P_1, P_2 are two partitions then

$$L(f, P_1) \le U(f, P_2)$$

Key: Take a refinement P of both P_1, P_2 where $P \supset P_1 \cup P_2$ then

$$L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2)$$

(ii) Conjecture: $U(f, P_2)$ is an upper bound for all $L(f, P)$ where P is any partition so the least upper bound for the $L(f, P)$ cannot exceed $U(f, P)$ that menas

$$\sup_P L(f, P) \leq U(f, P_2)$$

for any fixed partition

For all partition P_2, the number $\sup\limits_P L(f, P)$ is a lower bound for $U(f, P_2)$ The greatest lower bound for the $U(f, P_2)$ cannot be smaller than $\sup\limits_P L(f, P)$

$$\implies \sup_P L(f, P) \leq \inf_P U(f, P)$$

□

Example.

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational on } [0, 1] \\ 0 & \text{if } x \text{ is rational on } [0, 1] \end{cases}$$

Take any partition $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ on $[x_{j-1}, x_j]$

$$\inf_{[x_{j-1}, x_j]} f = 0 \text{ and } \sup_{[x_{j-1}, x_j]} f = 1$$

$$L(f, P) = \sum_{j=1}^{N} (x_j - x_{j-1}) \cdot 0 = 0$$

$$U(f, P) = \sum_{j=1}^{N} (x_j - x_{j-1}) \cdot 1 = 1$$

So this function is not integrable

Example.

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$P = \left\{0, \frac{1}{2}, 1\right\}, \inf_{[0, \frac{1}{2}]} = 0, \sup_{[0, \frac{1}{2}]} f = 1$$

$$L(f, P) = 0 \cdot \frac{1}{2} + \frac{1}{2} = \frac{1}{2}$$

$$U(f, P) = 1 \cdot \frac{1}{2} + \frac{1}{2} = 1$$

We pick new points

$$x_0 = 0, x_1 = \frac{1}{2} - \frac{1}{N}, x_2 = \frac{1}{2}, x_3 = 1$$

$$
\begin{aligned}
L(f,P) &= \left(\frac{1}{2}-\frac{1}{N}\right)\cdot 0 + \frac{1}{N}\cdot 0 + \frac{1}{2}\cdot 1 = \frac{1}{2}\\
U(f,P) &= \left(\frac{1}{2}-\frac{1}{N}\right)\cdot 0 + \frac{1}{N}\cdot 1 + \frac{1}{2}\cdot 1 = \frac{1}{2}+\frac{1}{N}\\
\sup L(f,P) &\geq \frac{1}{2} \text{ and } \inf U(f,P) \leq \frac{1}{2}+\frac{1}{N}\\
&\implies \sup L(f,P) = \inf U(f,P) = \frac{1}{2}
\end{aligned}
$$

Example. Define on $[0,1]$

$$
f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational}\\ \frac{1}{q} & \text{if } x \text{ is rational}, x = \frac{p}{q} \text{ in lowest term}\end{cases}
$$

Question: Is f integrable on $[0,1]$?

Answer: Yes, and the integral is 0

Proof. <u>Claim</u> f is integrable on $[0,1]$, and $\int_0^1 f = 0$

$L(f,P) = 0$ for all partitions P. Given $\varepsilon > 0$ we have to find a partition P_ε such $U(f,P_\varepsilon) < \varepsilon$.

Choose N large, $\frac{1}{N} \ll \varepsilon$ Form a partition which contain the fractions $\frac{p}{q}$ (in lowest terms) such that $1 \leq q \leq N$

Observe, there are no more than N^2 such numbers whenever

$$
\left|\frac{p}{q}-\frac{p'}{q'}\right| = \left|\frac{pq'-qp'}{qq'}\right| \geq \frac{1}{qq'} \geq \frac{1}{N^2}
$$

We choose the partition P_ε by including all $\frac{p}{q} \in [0,1], 1 \leq q \leq N$ and then for each of those, $v = \frac{p}{q}$, add $v - \frac{1}{N^3}, v + \frac{1}{N^3}$

Then number of partition point is $\leq 3N(N+1)$

Estimate

$$
U(f,P) = \sum_{j=1}^{M} (x_j - x_{j-1})M_j,\ M_j = \sup_{[x_{j-1},x_j} f
$$

I have the estimate

$$
\begin{aligned}
(x_j - x_{j-1})M_j &\leq \max\left\{\frac{1}{N^3}, (x_j - x_{j-1})\frac{1}{N+1}\right\}\\
&\leq \frac{1}{N^3} + (x_j - x_{j-1})\frac{1}{N+1}
\end{aligned}
$$

$$\begin{aligned}
U(f,P) &\le \sum_{j=1}^{M\le 3N(N+1)} \frac{1}{N^3} + (x_j - x_{j-1})\frac{1}{N+1} \\
&= \sum_{j=1}^{M} \frac{1}{N^3} + \sum_{j=1}^{M}(x_j - x_{j-1})\frac{1}{N+1} \\
&= \frac{M}{N^3} + \frac{1}{N+1}\underbrace{\sum_{j=1}^{M}(x_j - x_{j-1})}_{=1} \\
&\le \frac{3N(N+1)}{N^3} + \frac{1}{N+1}
\end{aligned}$$

□

5.1. Simple criteria for Riemann integrability

Theorem. Let f be a bounded function on $[a,b]$. Then f is integrable if and only if for every $\varepsilon > 0$ there is a partition P, such that

$$U(f,P) - L(f,P) < \varepsilon$$

Definition of Riemann integrable by if

$$\sup_{P_1} L(f_1, P_1) = \inf_{P_2} U(f_1, P_2)$$

then f is Riemann integrable, and the (common) value is

$$\int_a^b f$$

If $\tilde{P}$ is a refinement of P then $L(f,\tilde{P}) \ge L(f,P)$ and $U(f,\tilde{P}) \le U(f,P)$

Proof.

1. Assume that f is Riemann integrable. Let $\varepsilon > 0$ there is a partition P_1 such that

$$\int_a^b f - \frac{\varepsilon}{10} < L(f, P_1)$$

There is a partition P_2 such that

$$U(f, P_2) < \int_a^b f + \frac{\varepsilon}{10}$$

Then we get

$$\int_a^b f - \frac{\varepsilon}{10} < L(f, P_1) \leq L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2) \leq U(f, P_2) < \int_a^b f + \frac{\varepsilon}{10}$$

$$\implies U(f, P_1 \cup P_2) - L(f, P_1 \cup P_2) < \frac{\varepsilon}{5}$$

2. Assume that for every $\varepsilon > 0$, there is a partition P such that

$$U(f, P) - L(f, P) < \varepsilon$$

We want to show that $\inf_P U(f, P) - \sup_P L(f, P)$ differs by no more than ε If I succeed, then we get (since $\varepsilon > 0$ is arbitrary) that $\inf_P U(f, P) = \sup_P L(f, P)$ (i.e. by definition f is Riemann integrable)

We here shown $\inf U(f, P) \geq \sup L(f, P)$

$$U(f, P_\varepsilon) \geq \inf_{\text{all } P} U(f, P) \geq \sup_{\text{all } P} L(f, P) \geq L(f, P_\varepsilon)$$

□

5.2. Continuous functions

Theorem. A continuous function f on $[a, b]$ is Riemann integrable

Recall the defnition of continuity. A function is continuous at $x_0 \in [a, b]$

$$\forall \varepsilon > 0, \exists \delta > 0, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon \text{ (for } x \in [a, b])$$

Theorem: A continuous function on $[a, b]$ is uniformly continuous.

Proof. f is uniformly continuous, we <u>want to check</u> for arbitrary $\varepsilon > 0$ that there is a partition P such that $U(f, P) - L(f, P) < \varepsilon$

<u>Know:</u> There is a $\delta > 0$ such that $|f(x_1) - f(x_2)| < \frac{\varepsilon}{b-a}$ provided That $|x_1 - x_2| < \delta$

$$P = \left\{ x_j = a + j\frac{\varepsilon}{b-a}, \right\}, x_j - x_{j-1} = \frac{\varepsilon}{b-a} = \delta$$

$$
\begin{aligned}
U(f,P) - L(f,P) &= \sum_{j=1}^{N} M_j \cdot (x_j - x_{j-1}) - \sum_{j=1}^{N} m_j \cdot (x_j - x_{j-1}) \\
&= \sum_{j=1}^{N} \underbrace{(M_j - m_j)}_{\le \frac{\varepsilon}{b-a} \cdot \frac{1}{100}} \cdot (x_j - x_{j-1}) \\
&\le \sum_{j=1}^{N} \frac{\varepsilon}{b-a} \cdot \frac{1}{100} \cdot (x_j - x_{j-1}) \\
&\le \frac{\varepsilon}{b-a} \cdot \frac{1}{100} \cdot (b-a) \\
&= \frac{\varepsilon}{100}
\end{aligned}
$$

□

5.3. Estimation of integrals

Theorem. If f and g are integrable on $[a,b]$ and if $f(x) \le g(x)$ for all $x \in [a,b]$ then

$$\int_a^b f \le \int_a^b g$$

Proof. For any partition P show that $L(f,P) \le F(g,P)$ and $U(f,P) \le U(g,P)$

$$L(f,P) = \sum (x_j - x_{j-1}) \inf_{[x_{j-1},x_j]} f$$

$$U(f,P) = \sum (x_j - x_{j-1}) \sup_{[x_{j-1},x_j]} f$$

use that if $f(x) \le g(x)$ then

$$
\begin{aligned}
\inf_{[x_{j-1},x_j]} f &\le \inf_{[x_{j-1},x_j]} g \\
\sup_{[x_{j-1},x_j} f &\le \sup_{[x_{j-1},x_j]} g
\end{aligned}
$$

□

5.4. Class of Riemann Integrable functions

$$\mathcal{I} : f \mapsto \int_a^b f$$

$\Re(a,b) =$ class of Riemann integrable function

- If $f, g \in \mathfrak{R}(a,b)$ then $f+g \in \mathfrak{R}(a,b)$

  If $f \in \mathfrak{R}(a,b)$ and $c \in \mathbb{R}$ then $cf \in \mathfrak{R}(a,b)$

- $\displaystyle\int_a^b f+g = \int_a^b f + \int_a^b g$

  $\displaystyle\int_a^b cf = c\int_a^b f$

The cast two properties say that

$\mathcal{I} : \mathfrak{R}(a,b) \to \mathbb{R}$ is <u>linear</u>

$$\mathcal{I}(f,g) = \mathcal{I}(f) + \mathcal{I}(g)$$
$$\mathcal{I}(cf) = c\mathcal{I}(f)$$

Given afined partition

It is usually not true that $L(f+g,P) = L(f,P) + L(g,P)$

Suppose that $M_j(f) = \sup_{[x_{j-1},x_j]} f(x)$ and $m_j(f) = \inf_{[x_{j-1},x_j]} f(x)$

$$M_j(f+g) \le M_j(f) + M_j(g)$$

$$m_j(f+g) \ge m_j(f) + m_j(g)$$

$$L(f,P) + L(g,P) \le L(f+g,P) \le U(f+g,P) \le U(f,P) + U(g,P)$$

We get $M_j(cf) = cM_j(f)$ and $m_j(cf) = cm_j(f)$ if $c \ge 0$ and $M_j(-f) = -m_j(f)$

We recognized that f, g are integrable and if $f(x) \le g(x)$ for all $x \in [a,b]$ then

$$\int_a^b f(x) \le \int_a^b g(x)$$

This is because $L(f,P) \le L(g,P)$ and $U(f,P) \le U(g,P)$

Theorem. If f is integrable, so is $|f|$

Proof. Prove this by showing

$$0 \le U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P)$$

Check this and apply our integrability criteria

Check that $M_j(|f|) - m_j(|f|) \le M_j(f) - m_j(f)$

Given $\varepsilon > 0$

- There is a w such that $M_j \geq f(w) \geq M_j - \varepsilon$
- There is a z such that $m_j \geq f(z) \geq m_j - \varepsilon$

We know that $||f(w)| - |f(z)|| \leq |f(w) - f(z)|$

$$\begin{aligned} M_j(|f|) - m_j(|f|) &\leq |f(w)| - |f(z)| + 2\varepsilon \\ &\leq |f(w) - f(z)| \\ &\leq M_j(f) - m_j(f) \end{aligned}$$

□

As the result if $-|f(x)| \leq f(x) \leq |f(x)|$ for all x in $[a, b]$ then

$$\int_a^b -|f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

So $\int_a^b f(x)\,dx$ has absolute value $\leq \int_a^b |f(x)|\,dx$

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

$$|f(x)| \leq \underbrace{\sup_{w\in[a,b]} |f(w)|}_{\text{constant function which is larger than all } f(x)}$$

$$\int_a^b |f(x)|\,dx \leq \int_a^b \sup_{[a,b]} |f|\,dx = (b-a)\sup_{[a,b]} |f|$$

We can use this to bound the integral of f by the integral of $|f|$ and the supremum of $|f|$ over $[a, b]$

Example.

$$\int_0^{\frac{1}{2}} \frac{\sin x}{x}\,dx \leq \int_0^{\frac{1}{2}} 1\,dx = \frac{1}{2}$$

$$\int_0^{\frac{1}{2}} \frac{\sin x}{x + \frac{1}{10^6}e^{-x}}\,dx \leq \frac{1}{2}$$

<u>Note</u>: prefer to write $\int_{[a,b]}$

Theorem (Exercise)**.** Let f be integrable. Let g be such that $f(x) = g(x)$ except at a finite number of points. Then g is integrable and

$$\int_a^b f = \int_a^b g$$

Theorem (Lebesgue)**.** A subset E of $[0, 1]$ is called a null set if for every $\varepsilon > 0$ there is a sequence of intervals I_j such $\sum length(I_j) < \varepsilon$

Lebesque states f is Riemann integrable if and only if the set of discontinuities is a null sets.

5.5. Fundamental Theorem of Calculus

Theorem. Given an interval $[a, b]$ and and integrable function on $[a, b]$, given $a < c < b$ then f is integrable on $[a, c]$ and integrable on $[c, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof. We work with partition P of $[a, b]$, $c \in P$.

$$P = \{x_0 = a < x_1 < x_2 < \cdots < x_{M-1} < x_M = c < x_{M+1} < x_N = b\}$$

$P = P' \cup P''$ where P' is the partition of $[a, c]$ and P'' is the partition of $[c, b]$

Then we get, $L(f, P) = L(f, P') + L(f, P'')$ and $U(f, P) = U(f, P') + U(f, P'')$ □

$$x \mapsto \int_a^x f = \int_a^x f(t)\,dt$$

Example. $[-1, 1]$

$$f(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 \le t \le 1 \end{cases}$$

$$\int_{-1}^x f(t)\,dt = \begin{cases} -1(x - (-1)) = -(x + 1) & -1 \le x < 0 \\ -1 + x & 0 \le x \le 1 \end{cases}$$

$$F(x) = \begin{cases} -x - 1 & -1 \le x < 0 \\ x - 1 & 0 \le x \le 1 \end{cases} = -1 + |x|$$

Claim: If f is integrable on $[a, b]$ the function $F(x) = \displaystyle\int_a^x f(t)\,dt$ is continuous on $[a, b]$

We show that $F(x + h) - F(x) \to 0$ as $h \to 0$.

$F(x + h) = \displaystyle\int_a^{x+h} f$ and $F(x) = \displaystyle\int_a^x f$

If $h > 0$,

$$F(x + h) - F(x) = \int_a^x f + \int_x^{x+h} f - \int_a^x f = \int_x^{x+h} f$$

If $h < 0$,

$$F(x + h) - F(x) = \int_a^{x+h} f - \left(\int_a^{x+h} f + \int_{x+h}^x f\right) = \int_{x+h}^x f$$

Precall: If $|f(x)| \leq M$ then

$$\left|\int_{a_1}^{a_2} f\right| \leq M(a_2 - a_1)$$

apply this to our f which is bounded, There is some M such that $|f(t)| \leq M$ for all t

$$\left|\int_x^{x+h} f\right| \leq M|h|$$

$$\left|\int_{x+h}^x f\right| \leq M|h|$$

Goes to 0 as $h \to 0$

Theorem (Fundamental theorem of calculus)**.** If f is integrable on $[a, b]$ and continuous at some point $c \in [a, b]$ then the function F, defined by $F(x) = \displaystyle\int_a^x f$ is differentiable at c, and $F'(c) = f(c)$

Proof. For $h > 0$,

$$\begin{aligned}
\frac{F(c + h) - F(c)}{h} - f(c) &= \frac{1}{h}\int_c^{c+h} f - f(c) \\
&= \frac{1}{h}\int_c^{c+h} f(t)\,dt - \frac{1}{h}\int_c^{c+h} f(c)\,dt \\
&= \frac{1}{h}\int_c^{c+h} [f(t) - f(c)]\,dt
\end{aligned}$$

Show that $\left|\dfrac{F(c + h) - F(c)}{h} - f(c)\right| < \varepsilon$ if $h > 0$ is small enough

Know: There is a δ such $|f(t) - f(c)| < \varepsilon$ provided $|t - c| < \delta$

If $0 < h < \delta$ then

$$\begin{aligned} \frac{1}{h}\int_c^{c+h} |f(t) - f(c)| &\leq \frac{1}{h}\int_c^{c+h} \underbrace{|f(t) - f(c)|}_{\text{Integrate is } \leq \varepsilon \text{ if } 0<h<\delta} \, dt \\ &\leq \frac{1}{h}\int_c^{c+h} \varepsilon \, dt \\ &= \varepsilon \end{aligned}$$

□

5.6. Logarithm function

Definition. $\exp(x)$ is the unique differentiable function on $(-\infty, \infty)$ where $\exp'(x) = \exp(x)$ and $\exp(0) = 1$

Example. $x \in (0, \infty)$

$$x \mapsto \log x = \int_1^x \frac{1}{t}\, dt$$

Goal To show this log is "ln" with preserves the properties $\ln(ab) = \ln a + \ln b$

Theorem. $\log : (0, \infty) \to \mathbb{R}$ satisfies $\log(xy) = \log x + \log y$ for all $x, y > 0$

Proof. FTC give us $\log'(x) = \frac{1}{x}$. Fix $y > 0$ then $F_1(x) = \log(xy)$, $F_2(x) = \log x + \log y$

First check that $F_1'(x) = F_2'(x)$

$$\begin{aligned} F_2'(x) &= \frac{1}{x} \\ F_1'(x) = \log'(xy)\frac{d}{dx}(xy) &= \frac{1}{xy}y = \frac{1}{x} \\ F_1(x) &= F_2(x) + C(y) \end{aligned}$$

know $\log 1 = 0$, by the definition

$$\begin{aligned} F_1(x) - F_2(x) = C(y) &= F_1(1) - F_2(1) \\ &= \log(1 - y) - (\log 1 + \log y) = 0 \end{aligned}$$

so we have $C(y) = 0$, so $F_1 = F_2$ □

Range of log

- know: log is increasing
- Want $\lim_{x\to\infty} \log x = \infty$, $\lim_{x\to 0^+} \log x = -\infty$

$$\begin{aligned}\log(2^N) &= \int_1^{2^N} \frac{1}{t}\,dt \\ &= \int_1^2 + \int_2^4 + \int_{2^2}^{2^3} + \cdots + \int_{2^{N-1}}^{2^N} \frac{1}{t}\,dt \\ &= \sum_{K=1}^{N} \int_{2^{K-1}}^{2^K} \frac{1}{t}\,dt, \quad \int_{2^{K-1}}^{2^K} \frac{1}{t}\,dt \geq \int_{2^{K-1}}^{2^K} \frac{1}{2^K}\,dt = \frac{1}{2} \\ \implies \int_1^{2^N} \frac{1}{t}\,dt &\geq \frac{1}{2}N\end{aligned}$$

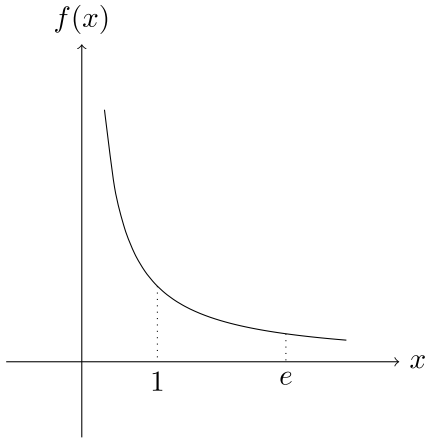
$$\log(2^{-N}) = \int_1^{2^{-N}} \frac{1}{t}\,dt = -\int_{2^{-N}}^1 \frac{1}{t}\,dt$$

want to show $\int_{2^{-N}}^1 \frac{1}{t}\,dt \to \infty$ if $N \to \infty$

$$\int_{2^{-N}}^1 \frac{1}{t}\,dt = \sum_{K=1}^{N} \int_{2^{-K}}^{2^{1-K}} \frac{1}{t}\,dt, \quad \int_{2^{-K}}^{2^{1-K}} \frac{1}{t}\,dt \geq \int_{2^{-K}}^{2^{1-K}} \frac{1}{2^{1-K}}\,dt = \frac{1}{2}$$

$$\implies \log(2^{-N}) \leq -\frac{1}{2}N$$

We define e as the unique x for which $\log x = 1$



$\log^{-1} : (-\infty, \infty) \to (0, \infty)$ is exp

$\exp(0) = 1$,

$$\exp'(y) = \left.\frac{1}{\log(x)}\right|_{x=\exp(y)} = \left.\frac{1}{\frac{1}{x}}\right|_{x=\exp(y)} = x|_{x=\exp y} = \exp(y)$$

<u>claim</u> $\exp(a+b) = \exp(a) \cdot \exp(b)$

verify the claim by applying the log to both sides

$$\log(\exp(a+b)) = a+b$$
$$\log(\exp a \cdot \exp b) = \log(\exp a) + \log(\exp b) = a+b$$

$\exp(n) = \exp(1+1+\ldots+1) = (\exp(1))^n = e^n$

5.7. Review version of FTC

f is integrable on $[a, b]$

$$A(x) = \int_a^x f(t)\,dt \implies A \text{ is a continuous function}$$

<u>FTC</u> If $c \in [a, b]$ and f is continuous at c then A is differentiable at c and $A'(c) = f(c)$

Compute

$$\frac{A(c+h) - A(c)}{h} = \frac{1}{h}\left[\int_a^{c+h} f - \int_a^c f\right] = \frac{1}{h}\int_c^{c+h} f$$
$$\frac{A(c+h) - A(c)}{h} - f(c) = \frac{1}{h}\int_c^{c+h} [f(t) - f(c)]\,dt$$

Compute for $h < 0$ ($-h = |h|$ if $h < 0$)

$$\frac{A(c+h) - A(c)}{h} = \frac{1}{h}\left(-\int_{c+h}^{c} f(t)\,dt\right) = \frac{1}{|h|}\int_{c-|h|}^{c} f(t)\,dt$$

Let $G(x) = \displaystyle\int_x^b f(t)\,dt$, Assume f is continuous

if $h > 0$

$$\frac{G(x+h) - G(x)}{h} = \frac{1}{h}\left[\int_{x+h}^{b} f - \int_x^h f\right] = \frac{1}{h}\left(-\int_x^{x+h} f\right) = -\frac{1}{h}\int_x^{x+h} f$$

if $h < 0$

$$\frac{G(x+h) - G(x)}{h} = \frac{1}{h}\left[\int_{x+h}^{b} f - \int_{x}^{h} f\right] = \frac{1}{h}\int_{x+h}^{x} f = -\frac{1}{|h|}\int_{x-|h|}^{x} f$$

$$\frac{d}{dx}\int_{x}^{b} f(t)dt\bigg|_{x=c} = -f(c)$$

$a < b$

$$\int_{b}^{a} f(t)\,dt = -\int_{a}^{b} f(t)\,dt$$

Theorem (Second version of FTC)**.** Given a function g which is differentiable on $[a, b]$ and assume g' is continuous function Then

$$\int_{a}^{b} g'(t)\,dt = g(b) - g(a)$$

Proof. Goal $F(x) = \displaystyle\int_{a}^{x} g'(t)\,dt$ and $G(x) = g(x) - g(a)$ is the same function

Apply FTC (main version) we get $F'(x) = g'(x)$ and $G'(x) = g'(x)$

$F(x) - G(x)$ differ only by a constant $F(x) = G(x) = C \implies C = F(a) - G(a) = 0$ $\square$

Theorem (More sophisticated version of the second FTC)**.** let g be a function on $[a, b]$ which is differentiable such that g' is integrable then

$$\int_{a}^{b} g'(t)\,dt = g(b) - g(a)$$

Proof. Idea is to examine lower and upper sums for the g', and use the mean value theorem for derivatives

Suppose that $P = [a = t_0 < t_1 < \ldots < t_N = b]$

$$L(g', P) = \sum_{j=1}^{N}(t_j - t_{j-1}m_j(g'))$$

$$U(g', P) = \sum_{j=1}^{N}(t_j - t_{j-1}M_j(g'))$$

where $m_j(g') = \inf\limits_{t\in[t_{j-1},t_j]} g'(t)$ and $M_j(g') = \sup\limits_{t\in[t_{j-1},t_j]} g'(t)$

Mean value theorem applied in $[t_{j-1}, t_j]$ gives

$$g(t_j) - g(t_{j-1}) = g'(\xi_j)(t_j - t_{j-1})$$

where ξ_j is between t_{j-1} and t_j

$$\begin{aligned}
L(g', P) &= \sum_{j=1}^{N} (t_j - t_{j-1}) m_j(g') \\
&\leq \sum_{j=1}^{N} (t_j - t_{j-1}) g'(\xi_j) \\
&= \sum_{j=1}^{N} g(t_j) - g(t_{j-1}) \\
U(g', P) &= \sum_{j=1}^{N} (t_j - t_{j-1}) M_j(g') \\
&\geq \sum_{j=1}^{N} (t_j - t_{j-1}) g'(\xi_j) \\
&= \sum_{j=1}^{N} g(t_j) - g(t_{j-1})
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^{N} (g(t_j) - g(t_{j-1})) &= g(t_1) - g(t_0) + g(t_2) - g(t_1) + \ldots + g(t_N) - g(t_{N-1}) \\
&= g(t_N) - g(t_0) \\
&= g(b) - g(a)
\end{aligned}$$

$$L(g', P) \leq \sum_{j=1}^{N} (g(t_j) - g(t_{j-1})) \leq U(g', P)$$

$$L(g', P) \leq g(b) - g(a) \leq U(g', P)$$

□

Example. Assume f is continuous

$$\frac{d}{dx} \int_0^{x^2} f(t)dt = ?$$

let $F(w) = \int_0^w f(t)\,dt$ then $F(x^2) = \int_0^{x^2} f(t)\,dt$

$$\begin{aligned}\frac{d}{dx}F(x^2) &= F'(x^2)\frac{d}{dx}x^2 \\ &= f(x^2)2x\end{aligned}$$

Therefore

$$\frac{d}{dx}\int_0^{g(x)} f(t)dt = f(g(x))g'(x)$$

5.8. Mean Value Theorem for Integrals

Theorem. if f is integrable on $[a,b]$ and if $m = f(x) = M$ for all $x \in [a,b]$ then

$$\int_a^b f(t)\,dt = (b-a)\mu$$

where $m \le \mu \le M$ (in other words $\frac{1}{b-a}\int_a^b f(t)\,dt = \mu$)

Proof. Another way of expression

$$m(b-a) \le \int_a^b f(t)\,dt \le M(b-a)$$

Given any partition $P = \{a = t_0 < t_1 < \ldots < t_N = b\}$

$$L(P,f) = \sum_{j=1}^{N}(t_j - t_{j-1})\overbrace{\inf_{t_{j-1}\le t\le t_j} f(t)}^{m_j(f)}$$

$$U(P,f) = \sum_{j=1}^{N}(t_j - t_{j-1})\overbrace{\sup_{t_{j-1}\le t\le t_j} f(t)}^{M_j(f)}$$

We have that for all j, $m_j(f) \ge m$ and $M_j(f) \le M$

We get

$$L(P,f) \ge \sum_{j=1}^{N}(t_j - t_{j-1})m = m(b-a)$$

$$U(P,f) \le \sum_{j=1}^{N}(t_j - t_{j-1})M = M(b-a)$$

$$m(b-a) \leq L(P,f) \leq \int_a^b f(t)\,dt \leq U(P,f) \leq M(b-a)$$

□

Theorem. If f is continuous on $[a,b]$ (thus certainly integrable) then there is a point $\xi \in [a,b]$ such that

$$\int_a^b f(x)\,dx = (b-a)f(\xi)$$

Proof. Use that a continuous function on a closed and bounded interval $[a,b]$ has a minimum and a maximum Means there is a point X_{max} so that $\sup_{x\in[a,b]} f = f(X_{max})$ and a point X_{min} so that $\inf_{x\in[a,b]} f = f(X_{min})$. We call $M = \max_{[a,b]} f(x)$ and $m = \min_{[a,b]} f(x)$ then M and m are in the range of f.

We have

$$\underbrace{m}_{\text{in the range of } f} \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \underbrace{M}_{\text{in the range of } f}$$

Use the intermediate value theorem to conclude that $\frac{1}{b-a}\int_a^b f(t)\,dt$ is in the range of f. i.e. there is $\xi \in [a,b]$ such that $f(\xi) = \frac{1}{b-a}\int_a^b f(t)\,dt$ □

Counter example if f is not continuous

Example. $f(x) = \lfloor x \rfloor$ on $[0,2]$

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x < 2 \\ 2 & \text{if } x = 2 \end{cases}$$

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = 1 = \frac{1}{2}\underbrace{(b-a)}_{=2}$$

Theorem. Given a function g that is integrable nad non-negative. Given a continuous function f on $[a,b]$

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$$

for some $\xi \in [a,b]$, in other words

$$f(\xi) = \frac{1}{\int_a^b g(x)\,dx}\int_a^b f(x)g(x)\,dx$$

Proof. If $g(x) = 1$ for all x we get the previous theorem.

Let $m = \min_{[a,b]} f$ and $M = \max_{[a,b]} f$

$$\int_a^b f(x)g(x)\,dx \geq \int_a^b mg(x)\,dx = m\int_a^b g(x)\,dx$$

$$\int_a^b f(x)g(x)\,dx \leq \int_a^b Mg(x)\,dx = M\int_a^b g(x)\,dx$$

Assume $\displaystyle\int_a^b g(x)\,dx > 0$

$$m \leq \underbrace{\frac{\displaystyle\int_a^b f(x)g(x)\,dx}{\displaystyle\int_a^b g(x)\,dx}}_{\text{this is a value } f(\xi)} \leq M$$

□

5.9. Example on Function

If f is integrable and $f \geq 0$ everywhere. Suppose I know that $\int_a^b f(x)\,dx = 0$ what can we say about f? (If f is continuous implies that f is a zero function)

Example. f defined on $(-1, 1)$

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x^a \sin\left(\frac{1}{x^c}\right) & x > 0 \end{cases}$$

Questions

1. Is f bounded?

   No, if a is negative, i.e., $a < 0, c > 0$

   Bounded, there is M such that $|f(x)| \leq M$, consider the choices of x_k [1]

$$\frac{1}{x_k^c} = 2k\pi + \frac{\pi}{2} \iff x_k \left(\frac{1}{2k\pi + \frac{\pi}{2}}\right)^{\frac{1}{c}} \ (k \geq 1)$$

[1] $\sin 2k\pi = 0$ and $\sin\left(2k\pi + \frac{\pi}{2}\right) = 1$

$$f(x_k) = x_k^a (a < 0)$$
$$= (2k\pi + \frac{\pi}{2})^{\frac{\overbrace{(-a)}^{a<0}}{c}}$$

Given M, I claim that for largn enough k we have $|f(x_k)| > M$

$$f(x_k) = \left(2k\pi + \frac{\pi}{2}\right)^b > M, 2k\pi + \frac{\pi}{2} > M^{\frac{1}{b}}$$

$b = \frac{(-a)}{c} > 0$

No, if $a < 0, c = 0$

Yes, if $a = 0, c \in \mathbb{R}$

$a < 0, c < 0, c = -b, b > 0$

$|x^a \sin(x^b)| \leq x^{a+b}$ for $x > 0$ [2]

If $a + b \geq 0$ (i.e., $a - c \geq 0$)

Yes, if $a < 0, c \leq a$

No, if $a < 0, c > a$

2. Is f continuous at $x = 0$?

   $|x^a \sin(x^{-c})| \leq x^{a-c}$, continuity, $\lim_{x\to 0} f(x) = 0$ if $a - c > 0$ $\lim_{x\to 0} f(x) = 0$ if $a > 0$

   If $a = 0, c > 0$, $f(x) = \sin\left(\frac{1}{x^c}\right)$ for $x > 0$

   If $x_k = \left(\frac{1}{2k\pi + \frac{\pi}{2}}\right)^{\frac{1}{c}}$ then $f(x_k) = 1$

   If $x_k = \left(\frac{1}{2k\pi}\right)^{\frac{1}{c}}$ then $f(x_k) = 0$

   f does not have the limit 0 as $x \to 0$ means: there exists an $\varepsilon > 0$ such that there is no δ with $|f(x) - \underbrace{f(0)}_{=0}| < \varepsilon$

   No if $a \leq 0, c \geq 0$

3. Is f differentiable at 0?

$$\frac{f(h) - \underbrace{\overset{=0}{f(0)}}}{h} = \begin{cases} 0 & h < 0 \\ h^{a-1} \sin\left(\frac{1}{h^c}\right) & h > 0 \end{cases}$$

[2] $|\sin w| \leq |w|$

$\lim\limits_{h\to 0} h^{a-1}\sin\left(\frac{1}{h^c}\right) = 0$ if $a > 1$, does not exists if $a = 1$

Yes If $c \geq 0, a - 1 > 0$

4. Is f' bounded?

   If $a > 1$, then f is differentiable and

$$f'(x) = \begin{cases} 0 & x \leq 0 \\ ax^{a-1}\sin(x^{-c}) + x^a\cos(x^{-c})(-c)x^{-c-1} & x > 0 \end{cases}$$

   $a > 1$, unbounded $a - c - 1 \geq 0$

   Yes If $a > 1$ and $a - c - 1 \leq 0$

5. Is f' continuous at $x = 0$?

$$f'(x) = \begin{cases} 0 & x \leq 0 \\ \underbrace{ax^{a-1}\sin(x^{-c})}_{|\ |\leq a|x|^{a-1}} + x^a\cos(x^{-c})(-c)x^{-c-1} & x > 0 \end{cases}$$

   Yes If $a > 1$, $a - c - 1 > 0$

5.10. Substitution Rule

Rule on how to compute integrals directly reflected to chain rule

Theorem. Given

- $g : [a, b]$, assuming that g is differentiable and g' is continuous
- f which is defined on an interval containing the range of g, f is continuous

Then

$$\int_a^b f(g(t))g'(t)\,dt = \int_{g(a)}^{g(b)} f(u)\,du$$

Proof. Let $F_1(x) = \displaystyle\int_a^x f(g(t))g'(t)\,dt$ and $F_2(x) = \displaystyle\int_{g(a)}^{g(x)} f(u)\,du$

Since F_1, F_2 is continuous, so, we can compute the derivatiive

$$F_1'(x) = f(g(x))g'(x) \text{ by FTC}$$

Define $A(w) \int_{g(a)}^{w} f(u)\,du$. Then $F_2(x) = A(g(x))$

By the cain rule $F_2'(x) = A'(g(x))g'(x) = f(g(x))g'(x)$

$A(w) = f(w)$ by again FTC

We see that $F_1(x) - F_2(x)$ is constant, inparticular equal to $F_1(a) - F_2(a) = 0$ $\square$

Example.

$$\int_0^3 x\sin(x^2)\,dx$$

Let $g(x) = x^2$, $g'(x) = 2x$ and $g(x) = x^2$

$$\begin{aligned}\int_0^3 x\sin(x^2)\,dx &= \frac{1}{2}\int_0^3 2x\sin(x^2)\,dx \\ &= \frac{1}{2}\int_{0^2}^{3^2} \sin(u)\,du \\ &= \frac{1}{2}(-\cos(3^2) + \cos(0^2))\end{aligned}$$

5.11. Integration by Parts

is related to the product rule

Theorem. Given f, g both differentialbe, with f', g' continuous in $[a, b]$ Then

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx$$

Proof.

$$\begin{aligned}\int_a^b f(x)g'(x)\,dx + \int_a^b f(x)g'(x)\,dx &= \int_a^b f(x)g'(x) + f'(x)g(x)\,dx \\ &= \int_a^b (f(x)g(x))'\,dx \\ &= f(b)g(b) - f(a)g(a)\end{aligned}$$

$\square$

Example. Given a nice function $f(x)$ on an interval $[a, b]$ (f, f' are continuous on $[a, b]$)

$$\int_a^b f(x)\cos(Nx)\,dx$$

what can we say about the size of this integral?

Standard estimate: assume $a < b$, then

$$\left|\int_a^b f(x)\cos(Nx)\,dx\right| \le \int_a^b |f(x)|\,dx \le (b-a)\cdot\max|f|$$

We know $N\cos Nx = (\sin Nx)'$

$$\begin{aligned}\int_a^b f(x)\cos Nx\,dx &= \frac{1}{N}\int_a^b f(x)(\sin(Nx))'\,dx \\ &= \frac{1}{N}\left[f(b)(\sin(Nb)) - f(a)(\sin Na) - \int_a^b f'(x)\sin Nx\,dx\right] \\ \left|\int_a^b f'(x)\sin Nx\,dx\right| &\le (b-a)\cdot\max|f'|\end{aligned}$$

Example. Try to estimate

$$\int_A^B f(x)\sin(x^2)\,dx = \int_A^B \frac{f(x)}{2x}\underbrace{2x\sin(x^2)}_{\frac{d}{dx}(\sin x^2)}\,dx$$

6. Taylor's Theorem

Example. Our fundamental theorem of calculus

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

Meanwhile theorem for integrals

$$f(x) = f(a) = (x-a)f'(\xi)$$

(also we get this from the mean value theorem for derivatives)

Do an integration by parts, in the integral (If f is differentiable twice)

$$\begin{aligned}
\int_a^x 1f'(t)\,dt &= \int_a^x \underbrace{\frac{d}{dx}(t-x)}_{u'}\,\underbrace{f'(t)}_{v}\;dt \\
&= \underbrace{(x-x)f'(x)}_{=0} - (a-x)f'(a) - \int_a^x (t-x)f''(t)\,dt \\
&= (x-a)f'(a) + \int_a^x (x-t)f''(t)\,dt
\end{aligned}$$

$$f(x) - f(a) = f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt$$

View this formula as approximating $f(x)$ for x near a by $f(a)+f'(a)(x-a)+$ Error term

Error term $= \displaystyle\int_a^x (x-t)f''(t)\,dt$

A. Definition and Theorem

Infremum and Supremum

Definition. A set $B \subseteq \mathbb{R}$ is bounded below if there exists $b \in \mathbb{R}$ such that $x \geq b$ for all $x \in B$.

Definition. A set $A \subseteq \mathbb{R}$ is bounded above if there exists $a \in \mathbb{R}$ such that $x \leq a$ for all $x \in A$.

Definition. Let $B \subseteq \mathbb{R}$ be a bounded set. We say that $b \in \mathbb{R}$ is the least upper bound of B ($\sup B$) if

1. b is an upper bound of B.
2. if b' is an upper bound of B, then $b \leq b'$.

Definition. Let $A \subseteq \mathbb{R}$ be a bounded set. We say that $a \in \mathbb{R}$ is the greatest lower bound of A ($\inf A$) if

1. a is an lower bound of A.
2. if a' is an lower bound of A, then $a' \leq a$.

Limit

Definition. $\lim\limits_{x \to a} f(x) = l$ means: for any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta$, then $|f(x) - l| < \varepsilon$.

Definition. $\lim\limits_{x \to a^+} f(x) = l$ means: for any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < x - a < \delta$, then $|f(x) - l| < \varepsilon$.

Definition. $\lim\limits_{x \to a^-} f(x) = l$ means: for any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < a - x < \delta$, then $|f(x) - l| < \varepsilon$.

Continuous

Definition. Let $f : \mathbb{R} \to \mathbb{R}$. f is continuous at a if $\lim\limits_{x\to a} f(x) = f(a)$.

Definition. Let $f : \mathbb{R} \to \mathbb{R}$, and $a <$ be real numbers.

1. We say f is continuous on (a, b) if f is continuous at x for every $x \in (a, b)$
2. We say f is continuous on $[a, b]$ if f is continuous on (a, b) and $\lim\limits_{x\to a^+} f(x) = f(a)$ and $\lim\limits_{x\to b^-} f(x) = f(b)$.

3 Hard theorems

Theorem (Intermediate Value Theorem)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ for $a < b$. Suppose $f(a) < 0 < f(b)$ then there exists $\xi \in (a, b)$ such that $f(\xi) = 0$.

Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ for $a < b$. Then f is bounded above on $[a, b]$. i.e, there exists $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in [a, b]$.

Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ for $a < b$. Then there exists $\xi \in [a, b]$ such that $f(x) \leq f(\xi)$ for all $x \in [a, b]$. i.e., $f(\xi) = \sup\{f(x) : x \in [a, b]\}$

Uniform Continuity

Definition. Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is uniformly continuous on an interval A if for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|x - y| < \delta$ and $x, y \in A$ implies $|f(x) - f(y)| < \varepsilon$.

Theorem. If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.

Differentiation

Definition. f is differentiable at the point a if $\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. We call this limit $f'(a)$.

Theorem. Let f be differentiable at a and f has a maximum at a. Then $f'(a) = 0$.

Theorem (Rolle's Theorem)**.** If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there is a number x in (a, b) such that $f'(x) = 0$.

Theorem (Mean Value Theorem)**.** If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number x in (a, b) such that $f(b) - f(a) = f'(x)(b - a)$

Inverse function

Theorem. If f is increasing on some interval the it has an inverse function f^{-1}.

Theorem. If f be (strictly) increasing on $[a, b]$ and $f'(x_0)$ exists for $x_0 \in (a, b)$ and $f'(x_0) \neq 0$ then f^{-1} is differentiable at $f(x_0)$ and $f^{-1}(f(x_0)) = \frac{1}{f'(x_0)}$.

Integration

Definition (Partition)**.** A partition of $[a, b]$ (called P) is a finite collection of distinct numbers in $[a, b]$. which contains the end point a and b;

Definition (Lower Riemann Sum)**.** Let $P = \{a = x_0 < x_1 < \ldots < x_N = b\}$ be a partition of $[a, b]$. The lower Riemann sum of f on P is defined as

$$L(f, P) = \sum_{j=1}^{N} (x_j - x_{j-1}) \cdot m_j$$

where $m_j = \inf\limits_{t \in [x_{j-1}, x_j]} f(t)$.

Definition (Upper Riemann Sum)**.** Let $P = \{a = x_0 < x_1 < \ldots < x_N = b\}$ be a partition of $[a, b]$. The upper Riemann sum of f on P is defined as

$$U(f, P) = \sum_{j=1}^{N} (x_j - x_{j-1}) \cdot M_j$$

where $M_j = \sup\limits_{t \in [x_{j-1}, x_j]} f(t)$.

Theorem. If $\tilde{P} \supseteq P$ is a partition of $[a, b]$ then

$$L(f, P) \leq L(f, \tilde{P})$$

$$U(f, P) \geq U(f, \tilde{P})$$

Theorem. Let P_1 and P_2 be partitions of $[a, b]$. Let f be a function which is bounded on $[a, b]$ then

$$L(f, P_1) \leq U(f, P_2)$$

Definition. A function f which is bounded on $[a, b]$ is **integrable** on $[a, b]$ if

$$\sup_P L(f, P) = \inf_P U(f, P)$$

In this case, this common number is called the **integral** of f on $[a, b]$ and is denoted by

$$\int_a^b f$$

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Theorem. If f is bounded on $[a, b]$ then f is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon$$

Theorem. If f is continuous on $[a, b]$ then f is integrable on $[a, b]$.