MATH 421 Lecture Notes

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Fall 2022

Contents

Properties of Real Number

Definition 1. Given any $a \in \mathbb{R}$, we define its absolute value to be

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ a & \text{if } a < 0 \end{cases}$$

Theorem 2 (Triangular Inequality)**.** Given $a, b \in \mathbb{R}$, there holds

$$|a + b| \leq |a| + |b|$$

Method of Proof

Direct proof

some statements can be shown to be true through a direct arguement e.g. our proof of Theorem 1

Theorem 3. hello

Proof by induction

the aim is to proof that a statement is true for all rational number

(i) Show the statement is true for $n = 1$

(ii) Assume the statement is true for general $n \in \mathbb{N}$

(iii) Using assumption (ii), prove the statement is true for $n + 1$

(iv) Conclude your proof with a sentence like "by mathematical information, the result holds for all $n \in \mathbb{N}$"

Example 4. Show that $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$

Theorem 5. Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, there holds the formula

$$(1 + x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j$$

1 Real Intervals

$\forall a, b \in \mathbb{R}$ such that $a < b$, we denote $[a, b]$, the set of all $\mathbb{R}$ between a and b (inclusive)

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

Similarly, we have

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

by convention, $(a, a) = \emptyset$, the empty set

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

Subset of this form are call **intervals**. We also adopt the notation

$$(\infty, a] = \{x \in \mathbb{R} : x \leq a\}$$

$$(b, \infty] = \{x \in \mathbb{R} : x > a\}$$

We'll never write $[\infty, a]$, since $\pm\infty$ are **not** real numbers.

$[a, b], (a, b], [a, b), (a, b)$, they are **bounded**

Definition 6. A set $B \subseteq \mathbb{R}$ is bounded below (respectively bounded above) if $\exists b \in \mathbb{R}$ such that $x \geq b \; \forall x \in B$ (respectively $x \leq b$ for all $x \in B$)

e.g. $\{0, 1, 50^{72}, -350\pi\}$ and $\left[-\frac{1}{\sqrt{10}}, 3\right)$ are bounded while $\mathbb{R}$ and $\mathbb{N}$ are not bounded

e.g. $[-357, \infty)$ is bounded below but not above

Definition 7. Let $B \subseteq \mathbb{R}$ be a subset that is bounded. We say that $b \in \mathbb{R}$ is the least upper bound of B (also call the supremum of B) if

(i) b is an upper bound for B

(ii) if b' is also an upper bound for B, then we have $b \leq b'$

We denote this least upper bound by $\sup B$

Remark 8. It is easy to see that for a set B bounded above. $\sup B$ is unique. To see this, suppose that both β_1 and β_2 are least upper bound for B. Then since β_2 is least upper bound and β_1 is an upper bound. We have $\beta_2 \leq \beta_1$. But also since β_1 is least upper bound and β_2 is a lower bound, we have $\beta_1 \leq \beta_2$. Hence $\beta_1 = \beta_2$

We have the corresponding notation for lower bounds

Definition 9. Let $A \subseteq \mathbb{R}$ be a subset bounded below. We say that $a \in \mathbb{R}$ is the greatest lower bound for A (also called the infimum of A) if

(i) a is an lower bound for A

(ii) if a' is also an lower bound for A, then $a' \leq a$

For $B = (-1, \infty)\,, \inf B = -1$.

For $B = [-1, \infty)\,, \inf B = -1$.

For $A = [2, 10) \cup (510, 511] \cup \{520\}, \inf A = 2, \sup A = 520$

Note that some sets contain their infimum/supremum while others do not. We note down a property of the real-numbers which we state but do not prove

Example. Prove that if $a = (0, 1)\,, \sup A = 1$

Proof. Notice that if $x \in A$ then $x < 1$, so 1 is an upper bound for A. Suppose for contradiction that $\sup A \neq 1$. Then we must have $\sup A < 1$ but $m = \frac{1}{2}(\sup A + 1) \in A$ but $m > \sup A$. So $\sup A$ is not an upper bound for A $\square$

2 Functions & Their Representation

A function is a "thing" that assigns a number to another number

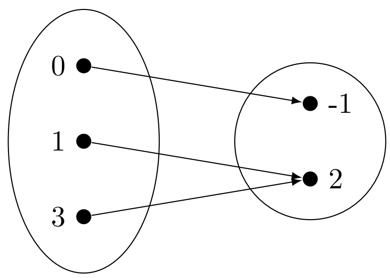
Example. the square function $x \mapsto x^2$

The way we represent this is by writing that f, the function such that $f(x) = x^2$, also written $f : x \mapsto x^2$

Example. We could also define a function, say g, that acts on $\{0, 1, 3\}$ and maps from elements of this set to $\{-1, 2\}$, for instance

$$g(0) = 1,\ g(1) = 2,\ g(3) = 2$$

One way of representing this is with the diagram



When defining a function f, we write $f : A \to B$, where A is domain and B is range

Example. Define the function $r : \left[-17, -\frac{\pi}{3}\right] \to \mathbb{R}$ by the explicit formula

$$r(x) = x^3, r : \left[-17, -\frac{\pi}{3}\right] \to \left[-17^3, -\left(\frac{\pi}{3}\right)^3\right] \subseteq \mathbb{R}$$

2.1 Operation between functions

Suppose f_1, f_2 have the same domain A, then we can define a new function, say g, to take the values of the sum of f_1 and f_2 i.e., for $f_1 : A \to B$ and $f_2 : A \to B$ we define $g : A \to B'$ bo be

$$g(x) = f_1(x) + f_2(x)\ \forall x \in A$$

Note that B' might not be equal to B

Example. $f_1, f_2 : [0,1] \to [0,1]$, $f_1(x) = x$, $f_2(x) = \frac{1}{2}x$, $g(x) = \frac{3}{2}x$ and $g : [0,1] \to \left[0, \frac{3}{2}\right]$

For ease of notation, we write g as $(f_1 + f_2)$

Similarly, we define the product function $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)\ \forall x \in A$

Example. $f(x) = \log x$ for $x \geq 1$, $g(x) = 10x^2\ \forall x \in \mathbb{R}$ To define $f + g$ and $f \cdot g$, we must to the smaller domain $\{x \in \mathbb{R} : x \geq 1\}$

2.2 Some examples of functions

Polynomials

Definition 10. $f : \mathbb{R} \to \mathbb{R}$ is a polynomial function, if $\exists N \in \mathbb{N}$ and $\exists \{a_0, \ldots, a_N\} \in \mathbb{R}^{N+1}$

$$f(x) = a_0 + a_1 x + \ldots a_N x^N\ \forall x \in \mathbb{R}$$

Rational function

Definition 11. We say that f is a rational function if for some polynomial functions $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \frac{p(x)}{q(x)}\ \forall x \in \mathbb{R} \setminus R_q$$

where $R_q = \{x \in \mathbb{R} : q(x) = 0\}$ is the set of roots of q

Construct functions

Definition 12. $f : \mathbb{R} \to \mathbb{R}$ is a constant function if $\exists c \in \mathbb{R}$ such that $f(x) = c\ \forall x \in \mathbb{R}$

The identity

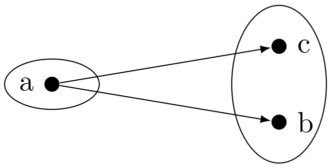
Definition 13. If $f(x) = x\ \forall x \in \mathbb{R}$ then we say that f is the identity map.

2.3 Composition

Definition 14. Let $f : A \to B$ and $g : B \to C$ be functions. We define the composition $g \circ f : A \to C$ by $g \circ f(x) = g(f(x))\ \forall x \in A$

2.4 Formal definition

Definition 15. A function is a collection of pairs of points with the property if (a, b) and (a, c) belong to the collection, the $b = c$. The pairs of points are of the form $(a, f(a)$. The property in **Definition 15** ensure that we stay clear of a confusion of the sort $f(2) = 2$ and $f(2) = 3$, which would using the diagram representation.



NOT a function

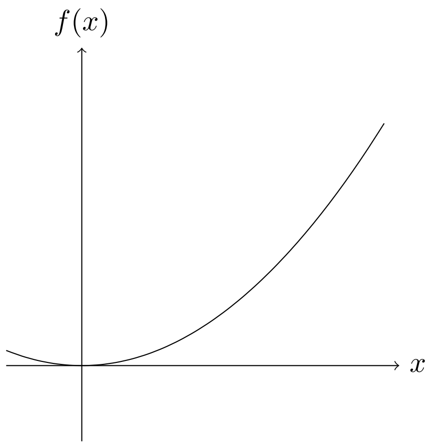
Definition 16. Let f be a function and denote by $\mathcal{F}$ its collection of points. The domain of f, written $\text{dom}(f)$, is the set of all points a such that there exists some b for which $(a, b) \in \mathcal{F}$.

i.e., $\text{dom}(f) = \{a : \exists b \text{ for which } (a, b) \in \mathcal{F}\}$

Moreover, by **Definition 15** for each $a \in \text{dom}(f)$ there exists a unique b such that $(a, b) \in \mathbf{F}$

2.5 Graphs of functions

An intimidate way to represent a function is by writing its coordinate pair on curves, i.e., drawing its graph



This diagram is representation of $\{(x, f(x)\}, x \in A$

Definition 17. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say f is **linear** if $\exists a \in \mathbb{R}$ such that

$$f(x) = ax, \ \forall x \in \mathbb{R}$$

Definition 18. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say f is **affine** if $\exists a \in \mathbb{R}$ such that

$$f(x) = ax + b, \ \forall x \in \mathbb{R}$$

Definition 19. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say f is **even** if $\exists a \in \mathbb{R}$ such that

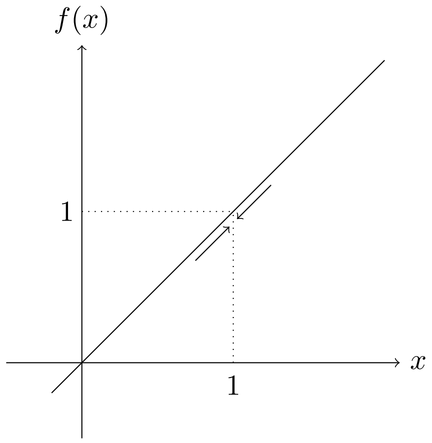
$$f(x) = f(-x), \ \forall x \in \mathbb{R}$$

Definition 20. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say f is **odd** if $\exists a \in \mathbb{R}$ such that

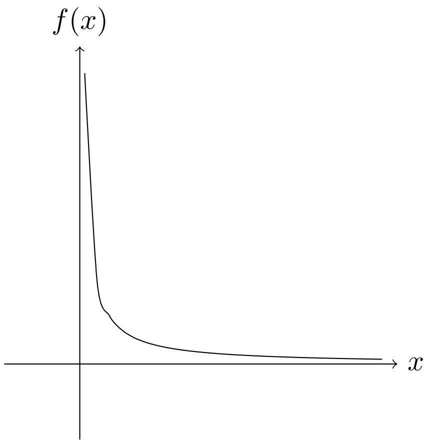
$$f(x) = -f(-x), \ \forall x \in \mathbb{R}$$

2.6 What is limit

What is a limit? Intutively, a function has a limit at a point x_* if the function values $f(x)$ "approach" this limit number as x gets closer to x_*



if $f(x) = x \ \forall x \in \mathbb{R}$ that as x increases to 1



as $x \to \infty, f(x)$ goes arbitrary close to 0, as $x \to 0, f(x)$ "explodes" and has not limit

This idea of a function having a limit is also preserve for more basic objects, e.g., sequence

e.g., the sequence of points $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$ where the n^{th} element of the sequnec may be written as $a_n = 1 - \frac{1}{n}$, converge to 1 as $n \to \infty$

definition of limit

Definition 21. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $a, l \in \mathbb{R}$. We say that f approach the limit l near a if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - l| < \varepsilon$$

We write $\lim\limits_{x \to a} f(x) = l$

Some comments on **Definition** 21

(i) δ is allowed to depend on ε, a, l

(ii) "for all $\varepsilon > 0$" can be read as "given any $\varepsilon > 0$"

Example. Let $f(x) = cx$ for some $c \in \mathbb{R}$ we show that $\lim\limits_{x \to 1} f(x) = c$

Proof. let $\varepsilon > 0$ be given. Then

$$\begin{aligned} |f(x) - c| &= |cx - c| \\ &= |c| \cdot |1 - x| \end{aligned}$$

So, letting $\delta = \delta(\varepsilon) = |c|^{-1} \cdot \varepsilon$, we get that

$$0 < |1 - x| < \delta \implies |f(x) - c| < \varepsilon$$

Since this hold for all $\varepsilon > 0$, we define $\lim\limits_{x \to 1} f(x) = c$ $\square$

Example. Let $g(x) = x\sin(\frac{1}{x})$ for some $x \in (0, \infty)$. Then $\lim_{x\to 0} g(x) = 0$

Proof. Indeed, let $\varepsilon > 0$ be given. Notice that $|g(x)| = |x| \cdot |\sin(\frac{1}{x})| \leq |x|$

, thus, letting $\delta = \delta(\varepsilon) = \varepsilon$, we see that

$$0 < |x| < \delta \implies |g(x)| < \varepsilon$$

□

Definition 22. Let $f : \mathbb{R} \to \mathbb{R}$ and let $l \in \mathbb{R}$. We say that f apporaches the limit l as x tends to infinity if: for all $\varepsilon > 0$, there exists $R > 0$ such that

$$x > R \implies |f(x) - l| < \varepsilon$$

We write $\lim_{x\to\infty} f(x) = l$ (R is allowed to depend on ε, l)

Example. let $f(x) = \frac{1}{x}$ for $x > 0$. We show that $\lim_{x\to\infty} f(x) = 0$

letting $R(\varepsilon) = \varepsilon^{-1}$, wee see that $x > R \implies |f(x) - 0| < \varepsilon$

Definition 23. Let $l \in \mathbb{R}$ and $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that a_n approaches the limit l as n tends to infinty if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n > N \implies |a_n - l| < \varepsilon$$

Write $\lim_{x\to\infty a_n} = l$

Example. For the sequence $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$ where $a_n = 1 - \frac{1}{n}$ $\forall n \in \mathbb{N}$ we see that $\lim_{x\to\infty a_n} = 1$

Proof. Indeed, let $\varepsilon > 0$ be given. Observe that $|a_n - 1| < \frac{1}{n}$,letting $N(\varepsilon) = \lceil \varepsilon^{-1} \rceil$, we see that, whenever $n > N$, $n > \varepsilon^{-1} \implies \frac{1}{n} < \varepsilon$ and $|a_n - 1| < \varepsilon$ for such n □

What does it mean to not have a limit?

what is no limit

Corollary 24. $f : \mathbb{R} \to \mathbb{R}$ does not approach the limit $l \in \mathbb{R}$ at the point $a \in \mathbb{R}$ if there exists some $\varepsilon_0 > 0$ such that for all $\delta > 0$ there exists $x_\delta \in \mathbb{R}$ for which there holds

$$|x_\delta - a| < \delta \text{ and } |f(x_\delta) - l| \geq \varepsilon_0$$

Example. We show that $\underset{x\mapsto\frac{1}{x}}{f:(0,1)\to(0,\infty)}$ has no limit at $x = 0$

Proof. We show that $\forall p \geq 0$, f does not approach the limit p at $x = 0$ Let $p \geq 0$ be given. We'll show that Corollary 24 holds with $\varepsilon_0 = 1$ Note that $|f(x) - p| = |\frac{1}{x} - p| = \frac{1}{x} - p$ provided $0 < x \leq \frac{1}{p}$. Also observe that $0 < x \leq \frac{1}{p+1} \implies \frac{1}{x} - p \geq p + 1 - p = 1$ This given any $\delta > 0$, choosing $x_\delta = \min\{\frac{\delta}{2}, \frac{1}{p+1}\}$ we get $0 < x_\delta < \delta$ and by $|f(x_\delta - p) \geq 1$ ☐

Example. Let $\underset{x\mapsto\sin(\frac{1}{x})}{f:(0,\infty)\to\mathbb{R}}$. We show f does not approach the value 0 as $x \to 0$.

Proof. Indeed, for this case set $\varepsilon_0 = \frac{1}{2}$ and for every $\delta > 0$, set $x_\delta = \frac{1}{\frac{\pi}{2}+2\pi n_\delta}$ where $n_\delta \in \mathbb{N}$ chosen sufficiently large such that $0 < x_\delta < \delta$. For instance, $n_\delta = \lceil \frac{\delta^{-1}}{2\pi} \rceil$ clearify that $x_\delta = \frac{1}{\frac{\pi}{2}+2\pi n_\delta} < \frac{1}{2\pi n_\delta}$ and

$$\begin{aligned} n_\delta &\geq \frac{\delta^{-1}}{2\pi} \\ 2\pi n_\delta &\geq \delta^{-1} \\ \frac{1}{2\pi n_\delta} &\leq \delta \end{aligned}$$

Then, $0 < x_\delta < \delta$, and

$$\begin{aligned} f(x) &= \sin\left(\frac{1}{x_\delta}\right) \\ &= \sin\left(\frac{\pi}{2} + \frac{1}{x_\delta}\right) \\ &= \sin\left(\frac{\pi}{2}\right) = 1 \end{aligned}$$

So, $|x_\delta - 0| < \delta$ and $f(x_\delta) - 0| = 1 > \frac{1}{2} = \varepsilon_0$ $\left(\text{So, } \lim_{x\to 0} f(x) \neq 0\right)$ ☐

Example 25. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

$\lim_{x\to 0} f(x) = 0$ but f has no limit at any other point $a \neq 0$

Fact Given $s < t$ real numbers:

(i) $\exists q \in \mathbb{Q}$ such that $s < q < t$

(ii) $\exists r \in \mathbb{R} \setminus \mathbb{Q}$ such that $s < r < t$

Proof. Fix $a > 0$ and let $l \in \mathbb{R}$ be arbitrary. There are 2 cases

1. Suppose $l = 0$ set $\varepsilon_0 = a$ Then, given $\delta > 0$ by Fact(i), $\exists x_\delta \in \mathbb{Q}$ such that $a < x_\delta < a + \delta$ and thus $|x_\delta - a| < \delta$ and $|f(x_\delta) - l| = x_\delta > a = \varepsilon_0$ so $f(x) \nrightarrow 0$ as $x \to a$

2. Suppose $l \neq 0$ set $\varepsilon_0 = \frac{|l|}{2}$ then given any $\delta > 0$ by Fact(ii), $\exists x_\delta \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x_\delta < a + \delta$, $|x_\delta - a| < \delta$ and $|f(x_\delta) - l| = |l| > \frac{|l|}{2} = \varepsilon_0$ repeating the same strategy for $a < 0$ concludes the proof.

□

2.7 Identity of Limit

Theorem 26. Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Suppose that for $\mu, \nu \in \mathbb{R}$ we have $\lim\limits_{x\to a} f(x) = \mu$ and $\lim\limits_{x\to a} f(x) = \nu$ then $\mu = \nu$ (i.e., the limit is unique)

Proof. Let $\varepsilon > 0$ be given. By the definition of the limit $\exists \delta_1 = \delta_1(\varepsilon, a, \mu) > 0$ such that $0 < |x - a| < \delta_1 \implies |f(x) - \mu| < \frac{\varepsilon}{2}$ also $\exists \delta_2 = \delta_2(\varepsilon, a, \nu) > 0$ such that $0 < |x - a| < \delta_2 \implies |f(x) - \nu| < \frac{\varepsilon}{2}$ Letting $\delta = \min\{\delta_1, \delta_2\} > 0$, we see that $|\mu - \nu| \leq |\mu - f(x)| + |f(x) - \nu|$, which provided $|x - a| < \delta$. Hence, $|\mu - \nu| < \varepsilon$ whenever $|x - a| < \delta$

We will show that $\mu - \nu = 0$. Suppose $\mu - \nu \neq 0$ then $|\mu - \nu| \geq 0$ but then, choosing $\varepsilon = \frac{1}{2}|\mu - \nu|$ we get $|\mu - \nu| < \frac{1}{2}|\mu - \nu|$

□

Theorem 27. Let $f, g : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Suppose that for $\mu, \nu \in \mathbb{R}$, $\lim\limits_{x\to a} f(x) = \mu$ and $\lim\limits_{x\to a} g(x) = \nu$ then

(a) $\lim\limits_{x\to a}(f + g)(x) = \mu + \nu$

(b) $\lim\limits_{x\to a}(f \cdot g)(x) = \mu \cdot \nu$

Proof. We will prove each separately

(a) Let $\varepsilon > 0$ be given. by the definition of limit, $\exists \delta_1 = \delta_1(\varepsilon, a, \mu) > 0$ such that $0 < |x - a| < \delta_1 \implies |f(x) - \mu| < \frac{\varepsilon}{2}$ and $\exists \delta_2 = \delta_2(\varepsilon, a, \nu) > 0$ such that $0 < |x - a| < \delta_2 \implies |g(x) - \nu| < \frac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$, provided $0 < |x - a| < \delta$, and observe that

$$\begin{aligned}|(f+g)(x)-(\mu+\nu)| &= |(f(x)-\mu)+(g(x)-\nu)| \\ &\leq |f(x)-\mu|+|g(x)-\nu| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

and $0<|x-a|<\delta \implies |(f+g)(x)-(\mu+\nu)|<\varepsilon$

(b) Let $\varepsilon>0$ be given, and observe that

$$\begin{aligned}|(f\cdot g)(x)-(\mu\nu)| &= |(f(x)g(x)-\mu g(x))+(\mu g(x)-\mu\nu)| \\ &\leq |g(x)|\cdot|f(x)-\mu|+|\mu|\cdot|g(x)-\nu|\end{aligned}$$

By the definition of limit $\exists\delta_g=\delta_g(\varepsilon,a,\nu)>0$ such that $|g(x)-\nu|<\min\{\frac{\varepsilon}{2(1+|\mu|)},1\}$, whenever $0<|x-a|<\delta_g$.

Note: whenever $0<|x-a|<\delta_g$, we have

(i) $|g(x)-\nu|<\frac{\varepsilon}{2(1+|\mu|)}$ and $|\mu|\cdot|g(x)-\nu|<\frac{\varepsilon}{2}$

(ii) $|g(x)-\nu|<1$ and $g(x)\leq|g(x)-\nu|+|\nu|<1+|\nu|$

Again, by the definition of limit, $\exists\delta_f=\delta_f(\varepsilon,a,\mu,\nu)>0$ such that

$$|x-a|<\delta_f \implies |f(x)-\mu|<\frac{\varepsilon}{2(1+|\nu|)}$$

then, we see that, for $\delta=\min\{\delta_f,\delta_g\}$ we have

$$|(f\cdot g)(x)-(\mu\nu)|<(1+|\nu|)\frac{\varepsilon}{2(1+|\nu|)}+\frac{\varepsilon}{2}=\varepsilon$$

□

2.8 Infremum / Supremum

Our objective is to give a sense of infremum/supremum as limits. For example, consider $[1,2]$. This set has the property that for every $x\in[1,2]$, there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ belonging to $[1,2]$ such that $x_n\to x$ as $n\to\infty$. Indeed, $x\in(1,2)$, then for $M_x>0$ sufficiently large. $x_n=x+\frac{1}{n\cdot M_x}$ is such that $x_n\in(1,2)$ and $x_n\to x$. And for when $x\in\{1,2\}$, we can build the sequences $x_n=\frac{1}{100n}$ or $x_n=2-\frac{1}{100n}$ This property also holds for $(1,2)$, but also even though $1,2\notin(1,2)$, there exists sequences $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ such that $y_n,z_n\notin(1,2)$ $\forall n\in\mathbb{N}$ and $y_n\to 1$ as $n\to\infty$, $z_n\to 2$ as $n\to\infty$

It turns out that the property of "having a sequence inside the set converging to this point" is a property that holds true for the inf and sup of any bounded set.

To this end, we prove the following lemma

Lemma 28. Let $B \subseteq \mathbb{R}$ be a nonempty set bounded above. Then, given any $\varepsilon > 0$, there exists some $b_\varepsilon \in B$ such that

$$\sup B - \varepsilon < b_\varepsilon \ (\leq \sup B)$$

Proof. Let $\varepsilon > 0$ be given. Denote $\sup B$ by β. Suppose for contradiction that no such b_ε exists, Then for all $b \in B$, we must have $b \leq \beta - \varepsilon$ but then $\beta - \varepsilon$ is the least upper bound for B $\square$

An analogous argument prove

Lemma 29. Let $A \subseteq \mathbb{R}$ be a nonempty set bounded below. Then, given any $\varepsilon > 0$, there exists some $a_\varepsilon \in B$ such that

$$(\inf A \leq)\ a_\varepsilon < \inf A + \varepsilon$$

Corollary 30. Let $A \subseteq \mathbb{R}$ be nonempty and bounded, then, $\exists (x_n)_{n\in\mathbb{N}}$ and $\exists (y_n)_{n\in\mathbb{N}}$ for which $x_n, y_n \in A$ for all $n \in \mathbb{N}$ and $\lim\limits_{x\to\infty} x_n = \inf A$, $\lim\limits_{x\to\infty} y_n = \sup A$

Proof. By Lemma 28 for each $n \in \mathbb{N}$, $\exists y_n \in A$ such that $\sup A - \frac{1}{n} < y_n \leq \sup A$ and $|y_n - \sup A| < \frac{1}{n} \to 0$ as $n \to \infty$ So, $\lim\limits_{x\to\infty} y_n = \sup A$. Also, for each $n \in \mathbb{N}$, by Lemma 29, $\exists x_n \in A$ such that $\inf A \leq x_n < \inf A + \frac{1}{n}$. i.e., $|x_n - \inf A| < \frac{1}{n} \to 0$ as $n \to \infty$. So, $\lim\limits_{x\to\infty} x_n = \inf A$. $\square$

Lemma 31. Suppose A is non-empty and bounded below. Let B be the set of all lower bounds of A. Then $\inf A = \sup B$

Proof. There are 3 steps

Step 1 [B is nonempty] Since A is bounded below, there exists at least one lower bound, which belongs to B, so $B \neq \emptyset$

Step 2 [B is bounded above] Suppose for contradiction that B is not bounded above. Then given any $n \in \mathbb{N}, \exists x_n \in B$ such that $x_n \geq n$. Then by the definition of B, x_n is a lower bound for A for each $n \in \mathbb{N}$. Thus given any $a \in A$, we have $a \geq x_n \geq n\ \forall n \in \mathbb{N}$. Here B is bounded above.

Step 3 [showing the equality]

($\leq$) Let $\nu = \inf A$ nad $\mu = \sup B$. Since ν is the infimum of A, ν is a lower bound for A. So $\nu \in B \implies \nu \leq \sup B = \mu$

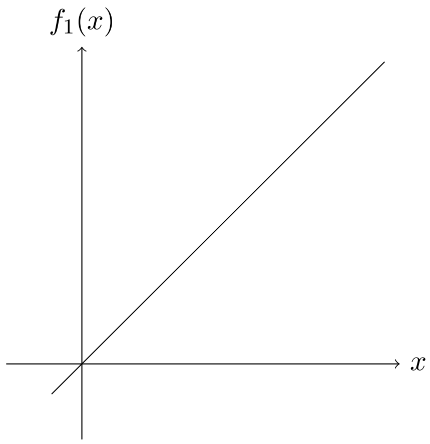
($\geq$) Let $\varepsilon > 0$ be arbitrary. Then by **Lemma** 28 $\exists b_\varepsilon \in B$ such that $\mu - \varepsilon < b_\varepsilon \leq \mu$. Hence, $\mu < \varepsilon + b_\varepsilon$. Now, let $a \in A$ be any point of A and observe that since $b_\varepsilon \in B$, $b_\varepsilon \leq a \implies \mu < \varepsilon + b_\varepsilon \leq \varepsilon + a$. i.e., $\mu < \varepsilon + a$ for all $a \in A$. i.e., $\mu - \varepsilon < a \; \forall a \in A$. So, $\mu - \varepsilon$ is a lower bound for $A \implies \mu - \varepsilon < \inf A = \nu$ i.e., $\mu < \nu + \varepsilon$, but $\varepsilon > 0$ was arbitrary $\implies \mu \leq \nu$

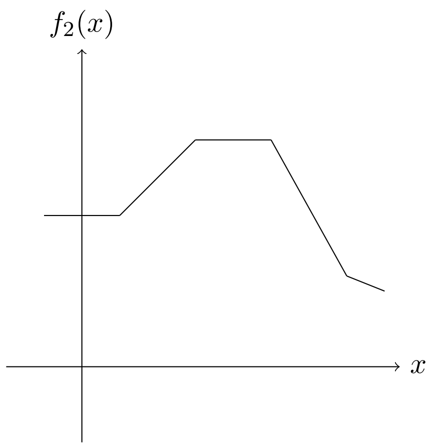
$\square$

3 Continuous Function

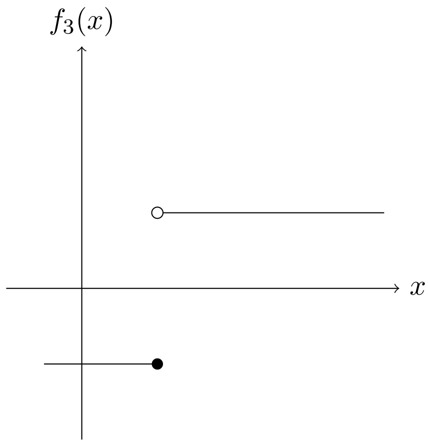
What does it mean for a function to be continuous?

Infinitely, this is some smoothness to the function i.g.,





But, on the other hand



is not continuous

3.1 Definition of Continuous Function

Definition 32. Let $f : \mathbb{R} \to \mathbb{R}$. We say f is continuous at the point $x_0 \in \mathbb{R}$ if there holds $\lim\limits_{x \to x_0} f(x) = f(x_0)$

Remark. For f to be continuous at $x_0 \in \mathbb{R}$, we require

(i) $\lim\limits_{x \to 0} f(x)$ exists

(ii) $\lim\limits_{x \to 0} f(x) = f(x_0)$

Another way of writing Definition 32 is

Definition (32)**.** f is continuous at x_0 if for all $\varepsilon > 0$, $\exists \delta = \delta(\varepsilon, x_0, f(x_0)) > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$$

Example. f_3 is not continuous at the point $x = 1$.

Proof. Indeed, setting $\varepsilon_0 = 1$, we see that, given any $\delta > 0$, the point $x_\delta = 1 + \frac{\delta}{2}$ is such that $|x_\delta - 1| < \delta$ and $|f(x_\delta) - f(1)| = |1 - (-1)| = 2 > \varepsilon_0$ □

Example. $f(x) = x^2$ is continuous.

Proof. Indeed, let $x_0 \in \mathbb{R}$ be any point and observe that

$$\begin{aligned}|f(x) - f(x_0)| &= |x^2 - x_0^2| \\ &= |(x + x_0)(x - x_0)| \\ &= |x + x_0| \cdot |x - x_0|\end{aligned}$$

Let $\varepsilon > 0$ be given. Now let $\delta = \min\left\{1, \frac{\varepsilon}{2(1+|x_0|)}\right\}$, then

$$\begin{aligned}|x + x_0| &= |x - x_0 + 2x_0| \\ &\leq |x - x_0| + 2|x_0| \\ &\leq 1 + 2|x_0|\end{aligned}$$

Then provided $|x - x_0| < \delta$ we get

$$|f(x) - f(x_0)| \leq (1 + 2|x_0|) \cdot \frac{\varepsilon}{2(1 + |x_0|)} < \varepsilon$$

$\square$

Example.

$$f(x) = \begin{cases} 0 & x = 0 \\ x \sin\left(\frac{1}{x}\right) & x \neq 0 \end{cases}$$

f is continuous at $x = 0$

Proof. Indeed, let $\varepsilon > 0$ be given and observe that

$$\begin{aligned}|f(x) - f(0)| &= |x| \cdot \left|\sin\left(\frac{1}{x}\right)\right| \quad \text{for } x \neq 0 \\ &\leq |x|\end{aligned}$$

So, letting $\delta(\varepsilon) = \frac{\varepsilon}{2}$, we see that

$$|x - 0| < \delta \implies |f(x) - f(0)| \leq \frac{\varepsilon}{2} < \varepsilon$$

$\square$

3.2 Identity of Continuous Function

Lemma 33. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous at $a \in \mathbb{R}$. Then

(i) $f + g$ is continuous at a

(ii) $f \cdot g$ is continuous at a

Proof. We will prove each separately

(i) let $\varepsilon > 0$ be given. By the definition of continuous, $\exists \delta_f = \delta_f(\varepsilon, a) > 0$ such that

$$|x - a| < \delta_f \implies |f(x) - f(a)| < \frac{\varepsilon}{2}$$

and, $\exists \delta_g = \delta_g(\varepsilon, a) > 0$ such that

$$|x - a| < \delta_g \implies |g(x) - g(a)| < \frac{\varepsilon}{2}$$

So, letting $\delta = \min\{\delta_f, \delta_g\}$, suppose $|x - a| < \delta$, we see that

$$\begin{aligned} |f(x) + g(x) - (f(a) + g(a))| &\leq |f(x) - f(a)| + |g(x) - g(a)| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

(ii) let ε be given. Note that

$$|f(x)g(x) - f(a)g(a)| \leq |g(x)| \cdot |f(x) - f(a)| + |f(a)| \cdot |g(x) - g(a)|$$

Since g is continuous at a, $\exists \delta_g = \delta_g(\varepsilon, a) > 0$ such that

$$|x - a| < \delta_g \implies |g(x) - g(a)| < \min\left\{1, \frac{\varepsilon}{2(1 + |f(a)|)}\right\}$$

Then, provided $|x - a| < \delta_g$, we get

$$|g(x)| \leq \overbrace{|g(x) - g(a)|}^{<1} + |g(a)| < 1 + |g(a)|$$

Also, since f is continuous at a, $\exists \delta_f = \delta_f(\varepsilon, a) > 0$ such that

$$|x - a| < \delta_f \implies |f(x) - f(a)| < \frac{\varepsilon}{2(1 + |g(a)|)}$$

Then, letting $\delta = \min\{\delta_f, \delta_g\}$, we see that whenever $|x - a| < \delta$, we have form

$$|f(x)g(x) - f(a)g(a)| < (1 + |g(a)|)\left(\frac{\varepsilon}{2(1 + |g(a)|)}\right) + |f(a)| \cdot \frac{\varepsilon}{2(1 + |f(a)|)} < \varepsilon$$

$\square$

Lemma 34. Let $g : \mathbb{R} \to \mathbb{R}$ be continuous at $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be continuous at $g(a)$. Then $f \circ g$ is continuous at a

Proof. Let $\varepsilon > 0$ be given. Since f is continuous at $g(a)$, $\exists \delta_f = \delta_f(\varepsilon, a) > 0$ such that

$$|y - g(a)| < \delta_f \implies |f(y) - f(g(a))| < \varepsilon$$

Meanwhile, g is continuous at a, so $\exists \delta_g = \delta_g(\delta_f(\varepsilon, a), a) > 0$ such that

$$|x - a| < \delta_g \implies |g(x) - g(a)| < \delta_f$$

So, letting $\delta = \delta_g$, we see that

$$\begin{aligned} |x - a| < \delta &\implies |g(x) - g(a)| < \delta_f \\ &\implies |f(g(x)) - f(g(a))| < \varepsilon \end{aligned}$$

□

Lemma 35. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at a, and suppose $f(a) > 0$. Then $\exists \delta > 0$ such that $f(x) > 0$ $\forall x \in [a - \delta, a + \delta]$

Proof. Since f is continuous at a, $\exists \delta_f = \delta_f(a, \overbrace{f(a)}^{\varepsilon}) > 0$ such that

$$|x - a| < \delta_f \implies |f(x) - f(a)| < \overbrace{\frac{1}{2} f(a)}^{\varepsilon}$$

It follows that, for $x \in (a - \delta_f, a + \delta_f)$, we have

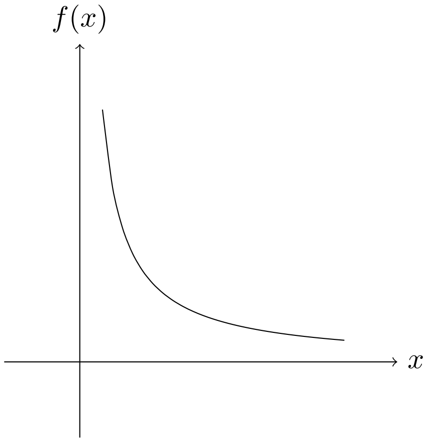
$$\begin{aligned} f(x) &= (f(x) - f(a)) + f(a) \\ &\geq f(a) - |f(x) - f(a)| \\ &> f(a) - \frac{1}{2} f(a) \\ &= \frac{1}{2} f(a) > 0 \end{aligned}$$

In turn, letting $\delta = \frac{1}{2}\delta_f$, we see that $f(x) > 0$ $\forall x \in [a - \delta, a + \delta]$ □

3.3 Definition of Left/Right Continuity

f continuous on (a, b) if f is continuous at x, for all $x \in (a, b)$. What does it mean for f to be continuous at on $[a, b]$? Should there be a difference between "continuous on (a, b)" and "continuous on $[a, b]$".

To gather intution, let's look at $f(x) = \frac{1}{x}$ on $(0, 1)$ and $[0, 1]$.



It's clar that f is continuous at every point $a \in (0,1)$ but $\lim_{x\to 0} f(x)$ is not defined. So, it ought to not be continuous on $[0,1]$ We make the following define

Definition (32)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $a < b$ be real numbers.

(i) We say f is continuous on (a,b) if f is continuous at x for every $x \in (a,b)$

(ii) We say f is continuous on $[a,b]$ if f is continuous on (a,b) and $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$

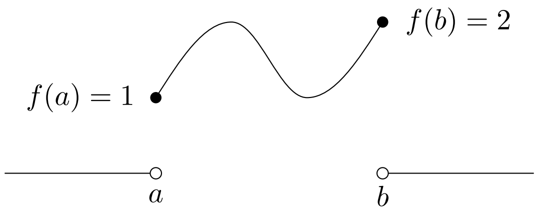
We write $\lim_{x\to a^+} f(x)$ to mean "The limit f as x tends to a from above" also written $\lim_{x\searrow a} f(x)$ and $\lim_{x\to b^-} f(x)$ to mean "The limit f as x tends to b from below" also written $\lim_{x\nearrow a} f(x)$

Definition (32)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$

(i) We write $\mu = \lim_{x\searrow a} f(x)$ if for all $\varepsilon > 0$, $\exists \delta > 0$ such that whenever $a < x < a + \delta$ we have $|\mu - f(x)| < \varepsilon$

(ii) We write $\nu = \lim_{x\nearrow a} f(x)$ if for all $\varepsilon > 0$, $\exists \delta > 0$ such that whenever $a - \delta < x < a$ we have $|\nu - f(x)| < \varepsilon$

Example. Considered this graph



then, $\lim_{x \searrow a} f(x) = 1$ and $\lim_{x \nearrow b} f(x) = 2$ on the other hand $\lim_{x \nearrow a} f(x) = 0$ and $\lim_{x \searrow b} f(x) = 0$

Example. $\lim_{x \to x_0} f(x)$ exists $\iff$ $\lim_{x \nearrow x_0} f(x)$ and $\lim_{x \searrow x_0} f(x)$ exists and are equal.

3.4 3 Hard Theorems

Theorem 36 (Intermediate Value Theorem)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ for $a < b$. Suppose $f(a) < 0 < f(b)$ Then $\exists \xi \in (a, b)$ such that $f(\xi) = 0$

Theorem 37. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ for $a < b$. Then f is bounded above on $[a, b]$, i.e., $\exists M \in \mathbb{R}$ such that $f(x) \leq M$ $x \in [a, b]$

Theorem 38. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$. Then $\exists \xi \in [a, b]$ such that $f(x) \leq f(\xi)$ $\forall x \in [a, b]$ i.e., $f(\xi) = \sup\{f(x) : x \in [a, b]\}$ (we say that f achieves its supremum on $[a, b]$)

Lemma ($35'$)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $b \in \mathbb{R}$. Suppose $\lim_{x \nearrow b} f(x) = f(b) > 0$ Then $\exists \delta > 0$ such that $f(x) > 0$ for all $x \in (b - \delta, b)$

Proof. Directly from Definition 32(ii) (definition of $\lim_{x \nearrow b} f(x)$) such that

$$x \in (b - \delta, b) \implies |f(x) - f(b)| < \frac{1}{2} f(b)$$

Then for such $x \in (b - \delta, b)$ we have

$$\begin{aligned} f(x) &= (f(x) - f(b)) + f(b) \\ &\geq f(b) - \overbrace{|f(x) - f(b)|}^{< \frac{1}{2} f(b)} \\ &> \frac{1}{2} f(b) > 0 \end{aligned}$$

Hence, for $x \in \left(b - \frac{\delta}{2}, b\right)$ we have $f(x) > 0$ $\square$

Lemma ($35''$)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $b \in \mathbb{R}$. Suppose $\lim_{x \searrow a} f(x) = f(a) > 0$ Then $\exists \delta > 0$ such that $f(x) > 0$ for all $x \in (a, a + \delta)$

Proof Theorem 36. Define the set $A = \{x \in [a, b] : f(y) < 0 \ \forall y \in [a, x]\}$ Since $f(a) < 0$, so $a \in A$, so $A \neq \emptyset$ Also, using Lemma $35''$ $\exists \delta_1 > 0$ such that $f(y) < 0$ $\forall y \in [a, a + \delta_1]$ so $a + \delta_1 \in A$, and by Lemma $35'$ $\exists \delta_2 > 0$ such that $f(y) > 0$ $\forall y \in [b - \delta_2, b]$ where

$b - \delta_2$ is an upper bound for A. So A is bounded above and $\sup A$ is well-defined. Let $\alpha = \sup A$. We already know that $\alpha \in (a, b)$ our aim is to show that $f(\alpha) \neq 0$ We proceed by contradiction:

Suppose for contradiction that $f(\alpha) \neq 0$ There are 2 possibilities

(i) $f(\alpha) < 0$

(ii) $f(\alpha) > 0$

Suppose (i) holds, Since $\alpha \in (a, b)$ and $f(\alpha) < 0$ by **Lemma 35**, $\exists \delta_3 > 0$ such that $f(y) < 0 \ \forall y \in [\alpha - \delta_3, \alpha + \delta_3]$ But then $\alpha + \delta_3 \in A$ and $\alpha + \delta_3 > \alpha$

Suppose (ii) holds. Then since $\alpha \in (a, b)$, $f(\alpha) > 0$ and f is continuous. By **Lemma 35**, $\exists \delta_4 > 0$ such that $f(x) > 0 \ \forall x \in [\alpha - \delta_4, \alpha + \delta_4]$ But then $\alpha = \sup A$ by **Lemma 28** $\exists x_0 \in A$ such that $\alpha - \frac{\delta_4}{2} < x_0$ Thus $x_0 \in (\alpha - \frac{\delta_4}{2}, \alpha) \subseteq [\alpha - \delta_4, \alpha + \delta_4] \implies f(x_0) > 0$ But $x_0 \in A$ so $(f_x) < 0$ $\square$

Corollary 39. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and let $c \in \mathbb{R}$. Suppose $f(a) < c < f(b)$. Then $\exists \xi \in (a, b)$ such that $f(\xi) = c$

Proof. Define $g(x) = f(x) - c$ and apply **Theorem** 36 to g $\square$

Example 40. Let $f(x) = x^4 + x - 3 \ \forall x \in \mathbb{R}$ **Fact**: all polynomials are continuous $\forall x \in \mathbb{R}$ A nice application of the Intermidiate Value Theorem is to find roots of continuous functions We can see by plugging in that

$$f(1) = 1 + (-1) - 3 = -3$$

$$f(2) = 16 + 2 - 3 = 15$$

IVT $\implies \exists x_0 \in (1, 2)$ such that $f(x_0) = 0$ This at least lets us estimate where roots are

Example 41. Let $f(x) = x^4 + x - 3 + \tan\left(\frac{x}{2}\right)$ (continuous on $(-\pi, \pi)$)

$$f(-1) = -3 - \tan\left(\frac{1}{2}\right) < 0$$

$$f(2) = 15 - \tan\left(\frac{1}{2}\right) > 0$$

IVT $\implies \exists x_0 \in (-1, 2)$ such that $f(x_0) = 0$

What is it useful for? If we look at the set $f([a, b]) = \{f(x) : x \in [a, b]\}$ and Theorem 37 tell us that set is bounded. Since the set is bounded, it has a supremum. You can think of this as "local max" of f on the interval $[a, b]$

Before proving Theorem 37, let's look at one of its consequences.

Corollary 42. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$. Then f is bounded below on $[a, b]$, i.e., $\exists m \in \mathbb{R}$ such that $m \leq f(x)\ \forall x \in [a, b]$

Proof. Since f is continuous, so is $(-f)$. Now apply Theorem 37 to $-f$. $\exists M \in \mathbb{R}$ such that $-f(x) \leq M\ \forall x \in [a, b]$ the, $f(x) \leq -M\ \forall x \in [a, b]$ $\square$

Takeaway: If f is continuous on $[a, b]$, then f is bounded above + below on $[a, b]$

To prove Theorem 37, we'll need a few Lemmas.

Lemma 43. Let $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$, then $\exists \delta > 0$ such that f is bounded above on the interval $[a - \delta, a + \delta]$

Proof. Since f is continuous at a, $\exists \delta = \delta(a, \overbrace{1}^{\varepsilon})$ such that $|x-a| < \delta \implies |f(x)-f(a)| < 1$ This for such x we have

$$\begin{aligned} f(x) &= f(x) - f(a) + f(a) \\ &\leq |f(x) - f(a)| + |f(a)| \\ &< 1 + |f(a)| \end{aligned}$$

For x satisfying $|x - a| < \delta$, we have $f(x) < 1 + f(a)$.

In particular, $f(x) < 1 + f(a)\ \forall x \in \left[a - \frac{\delta}{2}, a + \frac{\delta}{2}\right]$ $\square$

Lemma. (43′) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $b \in \mathbb{R}$. Suppose $\lim_{x \nearrow b} f(x) = f(b)$. Then $\exists \delta > 0$ such that f is bounded above on $[b - \delta, b]$

Proof. By Definition 32″, $\exists \delta = \delta(b, 1)$ such that

$$0 < |x - b| < \delta \implies |f(x) - f(b)| < 1$$

Therefore, for such x,

$$\begin{aligned} f(x) &= f(x) - f(b) + f(b) \\ &\leq |f(x) - f(b)| + |f(b)| \\ &< 1 + |f(b)| \end{aligned}$$

$f(x) < f(b) + 1\ \forall x \in \left[b - \frac{\delta}{2}, b\right]$ $\square$

Lemma. (43″) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $a \in \mathbb{R}$. Suppose $\lim_{x \searrow a} f(x) = f(a)$. Then $\exists \delta > 0$ such that f is bounded above on $[a, a + \delta]$

Proof Theorem 37. As in the proof of Theorem 36, consider the set

$$A = \{x \in [a, b] : f \text{ is bounded above on } [a, x]\}$$

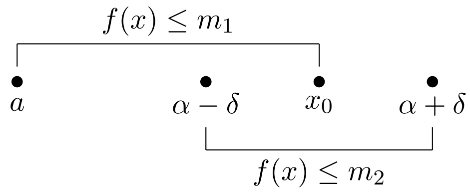
Since $a \in A$, we know $a \neq \emptyset$. Moreover, the point b is an upper bound for A, so $\sup A = \alpha$ exists.

Our objective is to show that $\alpha = b$.

Suppose for contradiction that $\alpha < b$. (Note that we must have $a < \alpha$. We can't have $a > \alpha$ since $a \in A$. and $\sup A \geq a$. If $\alpha = a$, then $A = \{a\}$, but we know from Lemma $43''$ that $\exists \delta > 0$ such that $[a, a + \delta] \subseteq A$)

By assumption $a < \alpha < b$ and so Lemma 43 $\implies$ $\exists \delta > 0$ such that f isbounded on $[\alpha - \delta, \alpha + \delta]$. Let's say $f(x) \leq m_2$ on this interval $[\alpha - \delta, \alpha + \delta]$.

By Lemma 28 (Alternate definition of supremum) $\exists x_0 \in A$ such that $\alpha - \delta < x_0 \leq \alpha$. f is bounded above on $[a, x_0]$ (by the definition of A). say $f(x) \leq m_1$ on $[a, x_0]$



Thus, $f(x) \leq \max\{m_1, m_2\}$ $\forall x \in [a, \alpha + \delta]$ We deduce that $\alpha + \delta \in A$ and $\alpha + \delta > \alpha = \sup A$. Hence,

$$\begin{aligned} \alpha = b &\iff \sup A = b \\ &\implies f \text{ is bounded above on } [a, b] \text{ for every } x < b \text{ } 1 \end{aligned}$$

Finally, using continuity at the point b by Lemma $43'$ $\exists \delta'$ such that f is bounded on $[b - \delta', b]$ 2.

Hence, choosing $x = b - \delta'$ in 1, $\exists M$ such that $f(x) \leq M$, $\forall x \in [a, b - \delta']$. and by 2, $\exists M_2$ such that $f(x) \leq M_2$, $\forall x \in [b - \delta', b]$. So, $f(x) \leq \max\{M, M_2\}$ $\forall x \in [a, b]$. $\square$

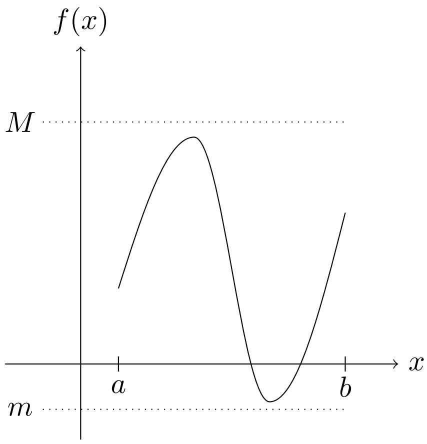
Summarize steps:

(i) define a good set A

(ii) show $b = \sup A$

(iii) show $b \in A$

The picture is



Whenever f is continuous on $[a, b]$, $\exists M > m$ such that $m \leq f(x) \leq M \ \forall x \in [a, b]$

Note: We must be careful aboue being continuous on $[a, b]$, and mot just (a, b). Indeed, $\underset{x \mapsto \frac{1}{x}}{f:(0,1)\to(0,\infty)}$, f is continuous on $[\tilde{x}, \infty)$ for every $\tilde{x} > 0$, but it is <u>not</u> continuous on $[0, \infty)$.

Question: does these exists $\xi_1, \xi_2 \in [a, b]$ such that

$$f(\xi_1) = \inf_{[a,b]} f \text{ and } f(\xi_2) = \sup_{[a,b]} f$$

Anwer: Yes

Later on, when we discuss differentiability, if sup/inf is achieved in (a, b), then $f' = 0$ at such points. This we will prove later.

Proof of Theorm 38. We already know from Theorem 37 that f is bounded on $[a, b]$, i.e., the set $B = f([a, b]) = \{f(x) : x \in [a, b]\}$ is bounded. This set is nonempty and so $\beta = \sup B$ is well-defined; Since $\beta \geq f(x) \ \forall x \in [a, b]$ it suffices to show that $\exists \xi \in [a, b]$ such that $f(\xi) = \beta$.

Suppose for contradiction that this is not the case, i.e., $\beta \neq f(y) \ \forall y \in [a, b]$ Then the function $g : [a, b] \to \mathbb{R}$, defined by $g(x) = \frac{1}{\beta - f(x)} \forall x \in [a, b]$, is well-defined and g is continuous on $[a, b]$ by virtue of Lemma 33

Since g is continuous, by Theorem 37 $\implies$ g is bounded above on $[a, b]$ However, by Lemma 28, given any $n \in \mathbb{N}, \exists x_n \in [a, b]$ such that

$$\beta - \frac{1}{n} < f(x_n) \leq \beta \implies g(x_n) \geq \frac{1}{\beta - \left(\beta - \frac{1}{n}\right)} = n$$

Hence given any $n \in \mathbb{N}, \exists x_n \in [a, b]$ such that $g(x_n) \geq n$ and therefore g is unbounded on $[a, b]$. $\square$

We've actually proved

Corollary 44. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$. Then $\exists \xi \in [a, b]$ such that $f(\xi) = \sup\{f(x) : x \in [a, b]\}$ (we often write with the shorthand $\sup_{[a,b]} f$)

Corollary 45. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$. Then $\exists \xi \in [a, b]$ such that $f(\xi) = \inf\{f(x) : x \in [a, b]\}$

Proof. Apploy Corollary 44 to the function $-f$ and use the result $\inf B = -\sup(-B)$. $\square$

3.5 Usage of 3 Hard Theorem

Example 46. Suppose f, g are continuous on $[a, b]$ and $f(a) < g(a)$ and $f(b) > g(b)$. Then $\exists x \in [a, b]$ such that $f(x) = g(x)$ (in actual fact, $x \in (a, b)$)

Proof. define $h(x) = f(x) - g(x)$. Then h is continuous on $[a, b]$, $h(a) < 0 < h(b)$ so from Theorem 36, $\exists \xi \in (a, b)$ such that $h(\xi) = 0 \implies f(\xi) = g(\xi)$ $\square$

Example 47. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[0, 1]$ and suppose $0 \leq f(x) \leq 1$ $\forall x \in [0, 1]$. Then $\exists x_0 \in [0, 1]$ such that $f(x_0) = x_0$ (we can imagine that f cross $y = x$)

Proof. Note that if $f(0) = 0$ on if $f(1) = 1$, then we are done. Suppose that $f(0) \neq 0$ and $f(1) \neq 1$ then $0 < f(0)$ and $f(1) < 1$ Let $g(x) = x - f(x)$. Then, $g(0) = 0 - f(0) < 0$ and $g(1) = 1 - f(1) > 0$. So, g is continuous and $g(0) < 0 < g(1)$, where Theorem 36 $\exists x_0 \in [0, 1]$ such that $g(x_0) = 0$ and hence $x_0 = f(x_0)$ $\square$

Example 48. There are 3 sub-examples here:

(a) Suppose $f : \mathbb{R} \to \mathbb{R}$ satsfies $|f(x)| \leq |x|$ for all $x \in \mathbb{R}$. Then f is continuous at 0

(b) There exists a function which satisfies the assumption of a.) but is not continuous at any other points other than $x = 0$

(c) Suppose g is continuous at 0 and $g(0) = 0$ and suppose $|f(x)| \leq |g(x)|$ $\forall x \in \mathbb{R}$. Then f is continuous at 0.

Proof. We will prove each separately:

(a) The inequallity implies $f(0) = 0$. Let $\varepsilon > 0$ be given, then the inequallity show that

$$|f(x) - f(0)| = |f(x)| \le |x - 0|$$

so letting $\delta = \varepsilon$, we see that

$$|x - 0| < \delta \implies |f(x) - f(0)| < \varepsilon$$

so f is continuous at 0

(b)

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then $|f(x)| \le |x| \ \forall x$ but f is not continuous at any points other than 0

(c) Since $g(0) = 0$, we immediately get $f(0) = 0$. Let $\varepsilon > 0$ be given. Since g is continuous at 0, $\exists \delta = \delta(\varepsilon, 0) > 0$ such that

$$|x - 0| < \delta \implies |g(x) - g(0)| \le \varepsilon$$

but then, in view of the bound $|f(x)| \le |g(x)| \ \forall x$, we see that

$$|x - 0| < \delta \implies |f(x) - f(0)| = |f(x)| \le |g(x)| = |g(x) - g(0)| < \varepsilon$$

□

Example 49. This exercis is here to help us gain more familiarity with limits– it's not concern with continuous functions per se.

(i) Let $f, g : \mathbb{R} \to \mathbb{R}$ and suppose $f(x) \le g(x) \ \forall x \in \mathbb{R}$ and suppose $\mu := \lim_{x \to a} f(x), \nu := \lim_{x \to a} g(x)$ Show that $\mu \le \nu$

(ii) Now suppose $f(x) < g(x) \ \forall x \in \mathbb{R}$. Does this guarantee $\mu < \nu$?

Proof. We will prove each separately:

(i) Let $\varepsilon > 0$ be given. Then $\exists \delta_1 = \delta_1(\varepsilon, a, \mu) > 0$ and $\exists \delta_2 = \delta_2(\varepsilon, a, \nu) > 0$ such that

$$|x - a| < \delta_1 \implies |f(x) - \mu| < \frac{\varepsilon}{2},$$
$$|x - a| < \delta_2 \implies |g(x) - \nu| < \frac{\varepsilon}{2}$$

Set $\delta := \min(\delta_1, \delta 2)$ Then, provided $|x - a| < \delta$, we have

$$\begin{aligned} \nu - \mu &= (\nu - g(x)) + (g(x) - f(x)) + (f(x) - \mu) \\ &\ge \underbrace{g(x) - f(x)}_{\ge 0} - \underbrace{|\nu - g(x)|}_{< \frac{\varepsilon}{2}} - \underbrace{|\mu - f(x)|}_{< \frac{\varepsilon}{2}} \\ &> -\varepsilon \end{aligned}$$

So, $\nu - \mu > -\varepsilon$ for all $\varepsilon > 0 \implies \nu - \mu \ge 0$

(ii) NO: Suppose $f(x) = 0$ and $g(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ \frac{1}{x} & \text{if } |x| \geq 1 \end{cases}$

Then $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$

□

Example 50. Let $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & x = 0 \end{cases}$

(a) Show that f is not continuous on $[-1, 1]$

(b) Show that f satisfies the conclusion of Theorem 36 (IVT)

Proof.

(a) for every $\delta > 0$, $n_\delta := \max\left(\lceil \frac{1}{2\pi}\delta^{-1} \rceil, 1\right) \in \mathbb{N}$ such that

$$\frac{1}{\frac{\pi}{2} + 2\pi n_\delta} < \delta \text{ and } x_\delta := \frac{1}{\frac{\pi}{2} + 2\pi n_\delta}$$

we get $0 < x_\delta < \delta$ and

$$|f(x_\delta) - f(0)| = |\sin(\frac{\pi}{2} + 2\pi n_\delta)| = 1$$

so, for all $\delta > 0, \exists x_\delta$ such that $0 < x_\delta < \delta$ and $|f(x_\delta) - f(0)| = 1$, so f is not continuous at 0.

(b) f is not continuous at 0, however f is continuous on $(-1, 0)$ and on $(0, 1]$ and so Theorem 36 holds on any interval of the form $[-1, y]$ and $[x, 1]$ for $y < 0$ and $x > 0$

It remains to check that

*Suppose $a > 0$ and $f(a) > 0$. Then, for every $c \in [0, f(a)]$, $\exists \xi_c \in [0, a]$ such that $f(\xi_c) = c$

Note that $f(a) \leq 1$, Indeed $\xi = \frac{1}{\arcsin(c)}$ is such that

$$f(\xi) = c$$
$$\sin\left(\frac{1}{\xi}\right) = \sin(\arcsin(c))$$

So the only remaining issue is that we do not necessarily have $\xi \in [0, a]$.

To this end, notice that, for every $N \in \mathbb{N}$, $\xi = \frac{1}{2\pi N + \arcsin(c)}$ also satisfies $f(\xi) = c$ and hence, choosing N sufficiently large such that $\frac{1}{2\pi N + \arcsin(c)} \leq a$, we have that $\xi = \frac{1}{2\pi N + \arcsin(c)}$ is a point that verifies *

□