

MATH 521 Lecture Notes

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1 Intro to naive set theory

Definition 1. We call B is a **subset** of a set A , $B \subseteq A$ if for every $x \in B$, $x \in A$.

We call B is a **proper subset** of a set A , $B \subsetneq A$ if $B \subseteq A$ and $B \neq A$.

Definition 2. empty set denoted as \emptyset

Definition 3. Given A , a set,

$$P(A) = \{B \mid B \subseteq A\}$$

called **power set** of A .

1.1 Cardinality

Definition 4. Given a set A, B , then if there exists bijection from A to B , then A and B have the same **cardinality**, denoted as $|A| = |B|$.

Definition 5. If there is an injection from A to B then we write $|A| \leq |B|$. If $|A| \leq |B|$ and $|A| \neq |B|$ then we write $|A| < |B|$.

Theorem 6. Let A and B be two sets, then either $|A| \leq |B|$, or $|B| \leq |A|$. Furthermore, if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

Theorem 7. Let A be a set, then $|A| < |P(A)|$.

1.1.1 Finite set and infinite set

Theorem 8. Let A be a set. The followings are equivalent:

1. There is a bijection between A and a proper subset of A .
2. There is an injection from \mathbb{N} to A .
3. There isn't a bijection from A to $\{x \in \mathbb{N} \mid x < n\}$.

Definition 9. If a set A satisfies any of the 3 conditions in the theorem above we call it an **infinite set**. Otherwise we call it a **finite set**.

1.1.2 Countable infinite set

Definition 10. If $|A| = |\mathbb{N}|$, we call A a **countable infinite set**. If $|A| > |\mathbb{N}|$ we call A an **uncountable set**.

2 Real Numbers

2.1 Ordered fields

Definition 11. A set F , $0, 1 \in F$, $+, \times : F \times F \rightarrow F$ and a linear order \leq called an **ordered field** if it satisfies

- $(F, +)$ is a commutative group with identity 0.
- (F, \times) is a commutative monoid with identity 1.
- for any $a, b, c \in F$, $a(b + c) = ab + ac$
- if $a \leq b$ then $a + c \leq b + c$ for any $c \in F$.
- if $0 \leq a, 0 \leq b$ then $0 \leq ab$.

Lemma 12. Let F be an ordered field, for any $a, b \in F$

- if $a \leq b$ then $-b \leq -a$
- if $a > b > 0$ then $0 < a^{-1} < b^{-1}$

Theorem 13. Let F be an ordered field, $a < b$, $a, b \in F$ Then we can define intervals

- $[a, b] = \{x \in F \mid a \leq x \leq b\}$
- $[a, b) = \{x \in F \mid a \leq x < b\}$
- $(a, b] = \{x \in F \mid a < x \leq b\}$
- $(a, b) = \{x \in F \mid a < x < b\}$

The **length** of these intervals are defined as $b - a$

2.2 Least upper bound property

Definition 14. least upper bound or supremum of A denote as $\sup(A)$, that satisfies

- for any $a \in A$, $a \leq \sup(A)$
- if for any $m \in F$ and for any $a \in A$, $a \leq m$ then $\sup(A) \leq m$

2.3 Archimedean property of real numbers

Theorem 15 (Archimedean property). for any $a, b \in \mathbb{R}$, $a, b > 0$ there exists $n \in \mathbb{N}$ such that

$$an > b$$

Theorem 16. for any $x \in \mathbb{R}$ there exists $n \in \mathbb{Z}, q \in [0, 1)$ such that

$$x = n + q$$

Theorem 17 (\mathbb{Q} dense in \mathbb{R}). for any $x \in \mathbb{R}$, for any $\varepsilon > 0$ there exists $q \in \mathbb{Q}$ such that

$$q \in (x - \varepsilon, x + \varepsilon)$$

2.4 Infinite decimal expansions

Definition 18. There is a bijection between $[0, 1)$ and the set

$$S = \{a \in \{0, 1, \dots, 9\}^{\mathbb{N}} \mid \forall n, \exists m > n, a(m) < 9\}$$

3 Metric Spaces

3.1 Definition

Definition 19. $(X, d), d : X \times X \rightarrow [0, \infty)$ is a metric space if: $\forall x, y, z \in X$

- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

Theorem 20. Given (X, d) metric space, $(X, f \circ d)$ is a metric space if

- $f(x) = 0 \iff x = 0$
- $f(x + y) \leq f(x) + f(y)$
- f is non-decreasing

Theorem 21. Let (X, d_x) and (Y, d_y) be metric spaces. Then $(X \times Y, d_{\text{sup}})$ is a metric space where

$$d_{\text{sup}}((x, y), (x', y')) = \max(d_x(x, x'), d_y(y, y'))$$

Theorem 22. Given (X, d) metric space, and S be non-empty set. Let

$$Y = \{f \in \text{Map}(S, X) \mid \text{diam}(f(S)) < \infty\}$$

and

$$\begin{aligned} d_{\text{sup}} : Y \times Y &\rightarrow [0, \infty) \\ f, g &\mapsto \sup\{d(f(s), g(s)) \mid s \in S\} \end{aligned}$$

Then (Y, d_{sup}) is a metric space.

3.2 Open sets and closed sets

3.2.1 Definition

Definition 23 (Open Ball). Let (X, d) be a metric space.

$$B_X(x, r) = \{y \in X \mid d(x, y) < r\}$$

is called an open ball centered at x with radius r .

Definition 24 (Open Set). Let (X, d) be a metric space. $A \subseteq X$ is called an open set if

$$\forall a \in A, \exists r > 0, B_X(a, r) \subseteq A$$

Definition 25 (Closed Set). Let (X, d) be a metric space. $A \subseteq X$ is called a closed set if

$$\forall a \in X \setminus A, \exists r > 0, B_X(a, r) \cap A = \emptyset$$

(or equivalently, $X \setminus A$ is an open set)

Theorem 26. Any open ball is open

Theorem 27. Any closed ball is closed. Furthermore any set consisting of a single point is closed.

3.2.2 Basic properties

Theorem 28. The union of a set of open sets is open.

Definition 29. Let (X, d) be a metric space, $A \subseteq X$. We define:

- The **closure** of A as

$$\overline{A} = \bigcap \{V \subseteq X \mid V \text{ closed}, A \subseteq V\}$$

- The **interior** of A as

$$A^\circ = \bigcup \{U \subseteq X \mid U \text{ open}, U \subseteq A\}$$

- The **boundary set** of A is

$$\partial A = \overline{A} \setminus A^\circ$$

Theorem 30. The intersection of finitely many open sets is open

Theorem 31. A non-empty subset of a metric space (X, d) is open \iff it is a union of open balls

3.2.3 Openness in subspaces

Theorem 32. If (X, d) is a metric space, $(Y, d|_{Y \times Y})$ a subspace.

- $A \subseteq Y$ is open in $Y \iff$ there is some open set A' in X where $A = A' \cap Y$
- $A \subseteq Y$ is closed in $Y \iff$ there is some closed set A' in X where $A = A' \cap Y$

3.2.4 Denseness

Definition 33. A subset $A \subseteq X$ is dense if any non-empty open subset of X has non-empty intersection with A .

3.2.5 Open sets and closed sets in \mathbb{R}

Lemma 34. Let $d_e(x, y) = |x - y|$

- Any finite open interval (a, b) , where $b > a$, is open under metric d_e .
- Any infinite open interval $(-\infty, a)$, (a, ∞) or $(-\infty, \infty)$ must be open under metric d_e .

Theorem 35. If $A \subset \mathbb{R}$ is non-empty and $\sup(A) < \infty$. Then

- If A is open, then $\sup(A) \notin A$.
- If A is closed, then $\sup(A) \in A$.

Theorem 36. $A \subseteq \mathbb{R}$, $A \neq \emptyset$, is open under $d(x, y) = |x - y| \iff$ it is the disjoint union of finite or countably infinitely many (finite or infinite) open intervals

3.3 Continuity

3.3.1 Definition

Definition 37. Given (X, d) and (Y, d') metric spaces, a function $f : X \rightarrow Y$

- is called continuous if for any open set $U \subseteq Y$, $f^{-1}(U)$ is open in X
- is called continuous at $x \in X$ if for any open set $U \subseteq Y$, $f(x) \in U$, there is an open set $V \subseteq X$, $x \in V$ such that $f(V) \subseteq U$

3.3.2 Localness and $\varepsilon - \delta$ characterization

Theorem 38. $f : X \rightarrow Y$ is continuous \iff for any $x \in X$, f is continuous at x .

Theorem 39. $f : X \rightarrow Y$ is continuous at $x \in X \iff$ for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(B_X(x, \delta)) \subseteq B_Y(f(x), \varepsilon)$$

Theorem 40. A map f from (X, d) to (Y, d') is continuous if for any $x, y \in X$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies d'(f(x), f(y)) < \varepsilon$$

Definition 41. If a bijection between two metric space and its inverse are both continuous, we call it a **homeomorphism**

3.3.3 Real valued continuous functions

Theorem 42 (Intermediate Value Theorem). if $f : [a, b] \rightarrow \mathbb{R}$ is continuous map, $m \in \mathbb{R}$, $(f(a) - m)(f(b) - m) < 0$ then there is some $c \in (a, b)$ such that $f(c) = m$

3.4 Compactness

Definition 43. A metric space X is called **compact**, if for every set of open subset C that $\bigcup C = X$, there is a finite subset C' such that $\bigcup C' = X$

Theorem 44. Any compact metric space has finite diameter

Theorem 45. If (X, d) is a metric space, $A \subset X$ is compact under the subspace metric, then A is closed in X .