MATH 521 Lecture Notes

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Contents

1 Intro to naive set theory

Definition 1. We call B is a **subset** of a set A, $B \subseteq A$ if for every $x \in B$, $x \in A$.

We call B is a **proper subset** of a set A, $B \subsetneq A$ if $B \subseteq A$ and $B \neq A$.

Definition 2. empty set denoted as $\emptyset$

Definition 3. Given A, a set,

$$P(A) = \{B \mid B \subseteq A\}$$

called **power set** of A.

1.1 Cardinality

Definition 4. Given a set A, B, then if there exists bijection from A to B, then A and B have the same **cardinality**, denoted as $|A| = |B|$.

Definition 5. If there is an injection from A to B then we write $|A| \leq |B|$. If $|A| \leq |B|$ and $|A| \neq |B|$ then we write $|A| < |B|$.

Theorem 6. Let A and B be two sets, then either $|A| \leq |B|$, or $|B| \leq |A|$. Furthermore, if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

Theorem 7. Let A be a set, then $|A| < |P(A)|$.

1.1.1 Finite set and infinite set

Theorem 8. Let A be a set. The followings are equivalent:

1. There is a bijection between A and a proper subset of A.
2. There is an injection from $\mathbb{N}$ to A.
3. There isn't a bijection from A to $\{x \in \mathbb{N} \mid x < n\}$.

Definition 9. If a set A satisfies any of the 3 conditions in the theorem above we call it an **infinite set**. Otherwise we call it a **finite set**.

1.1.2 Countable infinite set

Definition 10. If $|A| = |\mathbb{N}|$, we call A a **countable infinite set**. If $|A| > |\mathbb{N}|$ we call A an **uncountable set**.

2 Real Numbers

2.1 Ordered fields

Definition 11. A set F, $0, 1 \in F$, $+, \times : F \times F \to F$ and a linear order $\leq$ called an **ordered field** if it satisfies

- $(F, +)$ is a commutative group with identity 0.
- $(F, \times)$ is a commutative monoid with identity 1.
- for any $a, b, c \in F, a(b + c) = ab + ac$
- if $a \leq b$ then $a + c \leq b + c$ for any $c \in F$.
- if $0 \leq a, 0 \leq b$ then $0 \leq ab$.

Lemma 12. Let F be an ordered field, for any $a, b \in F$

- if $a \leq b$ then $-b \leq -a$
- if $a > b > 0$ then $0 < a^{-1} < b^{-1}$

Theorem 13. Let F be an ordered field, $a < b$, $a, b \in F$ Then we can define intervals

- $[a, b] = \{x \in F \mid a \leq x \leq b\}$
- $[a, b) = \{x \in F \mid a \leq x < b\}$
- $(a, b] = \{x \in F \mid a < x \leq b\}$
- $(a, b) = \{x \in F \mid a < x < b\}$

The **length** of these intervals are defined as $b - a$

2.2 Least upper bound property

Definition 14. least upper bound or supremum of A denote as $\sup(A)$, that satisfies

- for any $a \in A, a \leq \sup(A)$
- if for any $m \in F$ and for any $a \in A$, $a \leq m$ then $\sup(A) \leq m$

2.3 Archimedean property of real numbers

Theorem 15 (Archimedean property)**.** for any $a, b \in \mathbb{R}$, $a, b > 0$ there exists $n \in \mathbb{N}$ such that

$$an > b$$

Theorem 16. for any $x \in \mathbb{R}$ there exists $n \in \mathbb{Z}, q \in [0, 1)$ such that

$$x = n + q$$

Theorem 17 ($\mathbb{Q}$ dense in $\mathbb{R}$)**.** for any $x \in \mathbb{R}$, for any $\varepsilon > 0$ there exists $q \in \mathbb{Q}$ such that

$$q \in (x - \varepsilon, x + \varepsilon)$$

2.4 Infinite decimal expansions

Definition 18. There is a bijection between $[0, 1)$ and the set

$$S = \{a \in \{0, 1, \ldots, 9\}^{\mathbb{N}} \mid \forall n, \exists m > n, a(m) < 9\}$$

3 Metric Spaces

3.1 Definition

Definition 19. $(X, d), d : X \times X \to [0, \infty)$ is a metric space if: $\forall x, y, z \in X$

- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

Theorem 20. Given (X, d) metric space, $(X, f \circ d)$ is a metric space if

- $f(x) = 0 \iff x = 0$
- $f(x + y) \leq f(x) + f(y)$
- f is non-decreasing

Theorem 21. Let (X, d_x) and (Y, d_y) be metric spaces. Then $(X \times Y, d_{\text{sup}})$ is a metric space where

$$d_{\text{sup}}((x, y), (x', y')) = \max(d_x(x, x'), d_y(y, y'))$$

Theorem 22. Given (X, d) metric space, and S be non-empty set. Let

$$Y = \{f \in Map(S, X) \mid diam(f(S)) < \infty\}$$

and

$$\begin{aligned} d_{\text{sup}} : Y \times Y &\to [0, \infty) \\ f, g &\mapsto \sup\{d(f(S), g(S)) \mid s \in S\} \end{aligned}$$

Then (Y, d_{sup}) is a metric space.

3.2 Open sets and closed sets

3.2.1 Definition

Definition 23 (Open Ball)**.** Let (X, d) be a metric space.

$$B_X(x, r) = \{y \in X \mid d(x, y) < r\}$$

is called an open ball centered at x with radius r.

Definition 24 (Open Set)**.** Let (X, d) be a metric space. $A \subseteq X$ is called an open set if

$$\forall a \in A, \exists r > 0, B_X(a, r) \subseteq A$$

Definition 25 (Closed Set)**.** Let (X, d) be a metric space. $A \subseteq X$ is called an closed set if

$$\forall a \in X \setminus A, \exists r > 0, B_X(a, r) \cap A = \emptyset$$

(or equivalently, $X \setminus A$ is an open set)

Theorem 26. Any open ball is open

Theorem 27. Any closed ball is closed. Furthermore any set consisting of a single point is closed.

3.2.2 Basic properties

Theorem 28. The union of a set of open sets is open.

Definition 29. Let (X, d) be a metric space, $A \subseteq X$. We define:

- The **closure** of A as
$$\overline{A} = \bigcap \{V \subseteq X \mid V \text{ closed }, A \subseteq V\}$$
- The **interior** of A as
$$A^\circ = \bigcup \{U \subseteq X \mid U \text{ open }, U \subseteq A\}$$
- The **boundary set** of A is
$$\partial A = \overline{A} \setminus A^\circ$$

Theorem 30. The intersection of finitely many open sets is open

Theorem 31. A non-empty subset of a metric space (X, d) is open $\iff$ it is a union of open balls

3.2.3 Openness in subspaces

Theorem 32. If (X, d) is a metric space, $(Y, d|_{Y\times Y})$ a subspace.

- $A \subseteq Y$ is open in $Y \iff$ there is some open set A' in X where $A = A' \cap Y$
- $A \subseteq Y$ is closed in $Y \iff$ there is some closed set A' in X where $A = A' \cap Y$

3.2.4 Denseness

Definition 33. A subset $A \subseteq X$ is dense if any non-empty open subset of X has non-empty intersection with A.

3.2.5 Open sets and closed sets in $\mathbb{R}$

Lemma 34. Let $d_e(x, y) = |x - y|$

- Any finite open interval (a, b), where $b > a$, is open under metric d_e.
- Any infinite open interval $(-\infty, a), (a, \infty)$ or $(-\infty, \infty)$ must be open under metric d_e.

Theorem 35. If $A \subset \mathbb{R}$ is non-empty and $sup(A) < \infty$. Then

- If A is open, then $\sup(A) \notin A$.
- If A is closed, then $\sup(A) \in A$.

Theorem 36. $A \subseteq \mathbb{R}$, $A \neq \emptyset$, is open under $d(x, y) = |x - y| \iff$ it is the disjoint union of finite or countably infinitely many (finite or infinite) open intervals

3.3 Continuity

3.3.1 Definition

Definition 37. Given (X, d) and (Y, d') metric spaces, a function $f : X \to Y$

- is called continuous if for any open set $U \subseteq Y, f^{-1}(U)$ is open in X
- is called continuous at $x \in X$ if for any open set $U \subseteq Y, f(x) \in U$, there is an open set $V \subseteq X, x \in V$ such that $f(V) \subseteq U$

3.3.2 Localness and $\varepsilon - \delta$ characterization

Theorem 38. $f : X \to Y$ is continuous $\iff$ for any $x \in X$, f is continuous at x.

Theorem 39. $f : X \to Y$ is continuous at $x \in X \iff$ for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(B_X(x, \delta)) \subseteq B_Y(f(x), \varepsilon)$$

Theorem 40. A map f from (X, d) to (Y, d') is continuous if for any $x, y \in X$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies d'(f(x), f(y)) < \varepsilon$$

Definition 41. If a bijection between two metric space and its inverse are both continuous, we call it a **homeomorphism**

3.3.3 Real valued continuous functions

Theorem 42 (Intermediate Value Theorem)**.** if $f : [a, b] \to \mathbb{R}$ is continuous map, $m \in \mathbb{R}, (f(a) - m)(f(b) - m) < 0$ then there is some $c \in (a, b)$ such that $f(c) = m$

3.4 Compactness

Definition 43. A metric space X is called **compact**, if for every set of open subset C that $\bigcup C = X$, there is a finite subset C' such that $\bigcup C' = X$

Theorem 44. (X, d) is compact $\iff \forall C \subseteq \{B_X(x, r) \mid x \in X, r \in (0, \infty)\}, \bigcup C = X \implies \exists C' \subseteq C$ such that $\bigcup C' = X$ and C' is finite

Theorem 45. Any compact metric space has finite diameter ($diam(X) < \infty$)

Theorem 46. If (X, d) is a metric space, $Y \subseteq X$, a non-empty subset, with subspace metric. Then, $(Y, d|_{Y \times Y})$ is compact $\implies$ Y is closed.

Theorem 47. If (X, d) is a compact metric space $V \subseteq X$ is closed, then V under subspace metric is also compact

Theorem 48. $f : (X, d) \to (Y, d')$ is continuous and surjection. If X is compact, then Y is compact.

Theorem 49. If $(X, d), (Y, d')$ are both compact metric spaces, then $(X, Y, d_{\sup})$ is also compact

Theorem 50. A subset of $\mathbb{R}^n$ is compact $\iff$ it is bounded and closed

4 Limits

Definition 51. (X, d) a metric space, a sequence in X is a function $a : \mathbb{N} \to X$. We often dente $a(n)$ as a_n and write the sequence as $\{a_n\}$.

Definition 52. We say that

$$\lim_{n\to\infty} a_n = a$$

for $a \in X$ if for any $\varepsilon > 0$, there exists N such that

$$n > N \implies a_n \in B_X(a, \varepsilon)$$

Theorem 53. If $\{a_n\}$ converges, the limit is unique

Theorem 54. If $\{a_n\}$ and $\{b_n\}$ are 2 sequences on X. $\exists M \in \mathbb{N}$ such that $a_n = b_n$ if $n > M$ ($a_n = b_n$ for all but finitely many n) Then

1. $\{a_n\}$ converges $\iff$ $\{b_n\}$ converges
2. If $\{a_n\}, \{b_n\}$ converge, then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$

4.1 Limit, open sets and closed sets

Theorem 55. If (X, d) is a metric space, $\{a_n\}$ a sequence, If $U \subseteq X$ open and $\lim\limits_{n\to\infty} a_n \in U$, then $\exists N \in \mathbb{N}$ such that $n > N \implies a_n \in U$

Theorem 56. If (X, d) is a metric space, $\{a_n\}$ a sequence, $\lim\limits_{n\to\infty} a_n = b$, $V \subseteq X$ closed then $a_n \in V\ \forall n \implies b \in V$.

Definition 57 (isolated point)**.** Let (X, d) be a metric space, $p \in X$, if $\exists r > 0, B_X(p, r) = \{p\}$ we call p an **isolated point**

Theorem 58. if $\lim\limits_{n\to\infty} a_n = p$, p isolated, $\exists N, n > N \implies a_n = p$ ($a_n = p$ for all but finitely many n)

Theorem 59. (X, d) metric space, $U \subseteq X$, if for any sequence $\{a_n\}$ in X

$$\lim_{n\to\infty} a_n \in U \implies a_n \in U \text{ for all but finitely many } n$$

then U is open

Theorem 60. (X, d) metric space, $V \subseteq X$, If for any is converged sequence $\{a_n\}$,

$$a_n \in V \ \forall n \in \mathbb{N} \implies \lim_{n\to\infty} a_n \in V$$

then V is closed

Theorem 61. $f : (X, d) \to (X, d')$ a funciton, for $x \in X$, the following one equivalent.

1. f is continuous at x
2. if $\lim_{n\to\infty} a_n = x$ then $\lim_{n\to\infty} f(a_n) = f(x)$
3. if $\lim_{n\to\infty} a_n = x$ then $\lim_{n\to\infty} f(a_n)$ exists