# MATH 541 Lecture Notes

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## Contents

- Book: Dujmit Foote "Modern Algebra 3rd ed"
- Midterm 3/23 in class
- Final 5/8
- Homeworks: weekly
- Honors Credit: Extra sections + homeworks

# 1 Groups

Operations often modeled: $+, \cdot$

composition: space of thing that you are looking at $\leftarrow$ alomst always not commutative

**Groups**: One operation $\cdot$

**Rings**: 2 operations: $+, \cdot$ that play nice

## 1.1 Axioms of Groups

By "operation" on $S$, I mean a function $\cdot S \times S \to S$

Instead of $\cdot(a, b)$, we write $a \cdot b$

A group is a set $G$ with an operation $\cdot$ satisfying:

1. Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. There is an identity element: there is one special element $1 \in G$ so $1 \cdot a = a$ for any $a \in G$ and $a \cdot 1 = a$ for any $a \in G$
3. Inverses: For any $a \in G$, there is a $b \in G$ so $a \cdot b = b \cdot a = 1$

**Note**: $a \cdot b = b \cdot a$ is <u>not</u> an axiom.

If $G$ satisfies this, we call it an abelian group

**Example 1.** $(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +), (\mathbb{C}, +)$

1. 0 is the identity
2. inverses: $-a$ is the inverse of $a$

**Example 2.** $(\mathbb{Q} \setminus \{0\}, \cdot), (\mathbb{R} \setminus \{0\}, \cdot), (\mathbb{C} \setminus \{0\}, \cdot)$

1. 1 is the identity
2. Inverses: $\frac{1}{a}$ is the inverse of $a$

**Note:** $(\mathbb{Z} \setminus \{0\}, \cdot)$ is not a group

$(V, +)$ is a group

**Example 3.** For $n$, a natural number, $(\mathbb{Z}/n\mathbb{Z}, +)$ is a group

On $\mathbb{Z}$, we say $a, b$ are (mod $n$) equivalent (written $a \equiv b (\text{mod } n)$) if $n$ divides $a - b$

$\mathbb{Z}/n\mathbb{Z}$ is the set of equivalence classes mod $n$

**Example 4.** $n = 2$: (odds, evens) which is $\{0_{\text{mod } 2}, 1_{\text{mod } 2}\}$

$17_{\text{mod } 2} + 64_{\text{mod } 2} = 81_{\text{mod } 2} = 1_{\text{mod } 2}$

**Example 5.** $\mathbb{Z}/3\mathbb{Z} = \{0_{\text{mod } 3}, 1_{\text{mod } 3}, 2_{\text{mod } 3}\}$

**Example 6.** $(2\mathbb{Z}, +)$ is a group (even numbers)

**Example 7.** If $(G, \cdot_G)$ and $(H, \cdot_H)$ are groups, then $(G \times H, \cdot_G \times \cdot_H)$ is a group

- $(g_1, h_1) \cdot_{G \times H} (g_2, h_2) = (g_1 \cdot_G g_2, h_1 \cdot_H h_2)$
- Identity: $1_{G \times H} = (1_G, 1_H)$
- Inverse of $(g, h)$: $(g^{-1}, h^{-1})$

### 1.1.1 Properties

- $G$ has exactly 1 identity
- Each $g \in G$, there is exactly 1 inverse of $g$ we write this $g^{-1}$ (i.e. $^{-1} : G \to G$)
- $(g^{-1})^{-1} = g$
- $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$
- $(a_1 \cdot a_2 \cdot \ldots \cdot a_m)^{-1} = a_m^{-1} \cdot a_{m-1}^{-1} \cdot \ldots \cdot a_1^{-1}$

*Proof.*

- Suppose $a, b$ are both identities in $G$. Then $a = a \cdot b = b$
- Suppose $a, b$ are both inverses of $g$. i.e $a \cdot g = g \cdot a = 1$ and $b \cdot g = g \cdot b = 1$ Then $b = 1 \cdot b = (a \cdot g) \cdot b = a \cdot (g \cdot b) = a \cdot 1 = a$

- know $g \cdot g^{-1} = g^{-1} \cdot g = 1$ so $(g^{-1})^{-1} = g$
- $(a \cdot b)^{-1}$ satisfies: $x \cdot (a \cdot b) = (a \cdot b) \cdot x = 1$ we check $b^{-1}a^{-1}$ does this

  $(b^{-1}a^{-1}) \cdot (a \cdot b) = b^{-1}(a^{-1} \cdot a)b = b^{-1} \cdot 1 \cdot b = b^{-1}b = 1$

  $(ab)(b^{-1}a^{-1}) = a(b \cdot (b^{-1}) \cdot a^{-1} = a \cdot 1 \cdot a^{-1} = aa^{-1} = 1$

□

**Theorem 8.** In $G$, there is exactly 1 solution to the equation $ax = b$ for a fixed $a, b \in G$

**Corollary 9.** Cancellation laws:

$$ax = ay \implies x = y$$

$$xa = ya \implies x = y$$

*Proof.* If $a \cdot x = b$

$$\begin{aligned} a^{-1} \cdot a \cdot x &= a^{-1} \cdot b \\ (a^{-1} \cdot a) \cdot x &= a^{-1} \cdot b \\ 1x = x &= a^{-1} \cdot b \end{aligned}$$

□

**Definition 10.** For $x \in G$, the order of $x$, written $|x|$, is the least $n > 0$ so

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n} = 1_G$$

If there is no such $n$, $x$ has "infinite order"

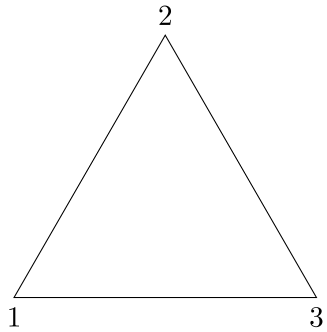
**Example 11.** In $(\mathbb{R} \setminus \{0\}, \cdot)$, $|5| = \infty$, $|-1| = 2$, $|1| = 1$

**Example 12.** $(\mathbb{Z}/6\mathbb{Z}, +)$, $|1_{\text{mod } 6}| = 6$, $|2_{\text{mod } 6}| = 3$, $|3_{\text{mod } 6}| = 2$, $|4_{\text{mod } 6}| = 3$, $|5_{\text{mod } 6}| = 2$

## 1.2 Dihedral Groups

### 1.2.1 Triangle

Look at the collection of symmetries of an equilateral Triangle




Rotation right

- $1 \to 2$
- $2 \to 3$
- $3 \to 1$

$r$

Rotation Left

- $1 \to 3$
- $2 \to 1$
- $3 \to 2$

$r^2$

Reflection around 2

- $1 \to 3$
- $2 \to 2$
- $3 \to 1$

$r^2 \circ s$

Reflection around 1

- $1 \to 1$
- $2 \to 3$
- $3 \to 2$

$s$

Reflection around 3

- $1 \to 2$
- $2 \to 1$
- $3 \to 3$

$r \circ s = s \circ r^2$

Identity

- $1 \to 1$
- $2 \to 2$
- $3 \to 3$

$r^3, s^2$

$$\begin{aligned} r^2 s &= r \cdot (r \cdot s) \\ &= (r \cdot s) \cdot r^{-1} \\ &= s \cdot (r^{-1} \cdot r^{-1}) \\ &= s \cdot (r^{-1})^2 \end{aligned}$$

(Symmetry of $\triangle, \circ$) $= D_6$

### 1.2.2 n-gon

Rotation right

- $k \to k+1$ (for $k < n$)
- $n \to 1$

$r, |r| = n$

Reflection around 1

- $k \to n+2-k$
- $1 \to 1$

$s, |r| = n$

My Symmetry

- $1 \to k$
- $2 \to k+1$

$r^k$

So, $\{r, s\}$ generates the group of sym of regular n-gon

(Symmetry of a regular n-gon, $\circ$) $= D_{2n}$

### 1.2.3 Definition

Rules of dihedral group multiplication in $D_{2n}$ $\{r, s\}$

a) $r^n = 1$

b) $s^2 = 1$

c) $r \cdot s = s \cdot r^{-1}$

When you have generators $S$ for $G$ and can list $R_1, R_2, R_3$ <u>all</u> the rules you need to know to do multiplication in $G$ Then $\langle S, R_1, R_2, R_3 \rangle$ is a "presentation of the group $G$"

$D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, rs = sr^{-1} \rangle = \{1, r, \ldots, r^{n-1}, s, rs, \ldots, rs^{n-1}\}$

Fact: There is a finite set of rule $R_1, \ldots, R_{2000}$ so $\langle a, b | R_1, \ldots, R_{2000} \rangle$ "undecidable word problem"

## 1.3 Symmetric Group

Given $\Omega$ any set, $S_\Omega =$ The permutations of $\Omega =$ The bijections $f : \Omega \to \Omega$

**Example 13.** $\Omega = \{1, 2, 3\}$

$S_n = S_{\{1,2,\ldots,n\}}$ has $n!$ elements

$|S_3| = 6, |D_6| = 6, D_6 \subseteq S_3$

$|D_{2n}| = 2n$

$|S_n| = n!$

### 1.3.1 Cycle Decomposition

$1 \to 4, 2 \to 1, 3 \to 2, 4 \to 3, 5 \to 5$ can be written as $(1432)(5)$

$(a_1 \ldots a_{m_1})(a_{m_1+1} \ldots a_{m_2})$ with $a_i$ is disjoint represents the function which satisfies

- $a_i$ to $a_{i+1}$ unless $i = m_j$ for some $j$
- $a_{m_j}$ to $a_{m_{j-1}} + 1$ $j \neq 1$
- $a_{m_1}$ to $a_1$

$(1)(2)(3)(4)(5)(6)(7) = 1$

$(1442) \circ (3421) = (124)$

$|(123)(45)| = 6$

Order of a product of disjoint cycles is the lcm(lengths of the cycles)

## 1.4 Homomorphisms and Isomorphisms

**Definition 14.** A homorphism from $(G, \cdot_G)$ to $(H, \cdot_H)$ is a function $f : G \to H$ such that

$$f(x \cdot_G y) = f(x) \cdot_H f(y)$$

for all $x, y \in G$

- $f(x^{-1}) = f(x)^{-1}$

$$\begin{aligned}
f(x) &= f(1_G \cdot_G x) \\
&= f(1_G) \cdot_H f(x) \\
f(x) \cdot_H (f(x))^{-1} &= f(1_G) \cdot_H f(x) \cdot_H (f(x))^{-1} \\
1_H &= f(1_G)
\end{aligned}$$

$$1_H = f(1_G) = f(x \cdot_G x^{-1}) = f(x) \cdot_H f(x^{-1})$$
$$1_H = f(1_G) = f(x^{-1} \cdot_G x) = f(x^{-1}) \cdot_H f(x)$$

**Definition 15.** If $f$ is a bijection and a homorphism, then $f$ is an isomorphism

**Example 16.** $\cdot id : G \to G$

$\cdot^{-1} : G \to G, x \mapsto x^{-1}$

$(x \cdot y)^{-1} = (x^{-1}) \cdot (y^{-1})$

is an isomorphism if and only if $G$ is abelian

$xyx^{-1}y^{-1} = 1$

**Example 17.** $e^x : (\mathbb{R}, +) \to (\mathbb{R}, \cdot), f(x+y) = f(x) \cdot f(y)$ is an isomorphism

**Example 18.** $f : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$

- $0 \to 0$
- $1 \to 1$
- $2 \to 2$
- $3 \to 0$
- $4 \to 1$
- $5 \to 2$

is a homorphism NOT an isomorphism

**Definition 19.** $G$ and $H$ is isomorphic if there is a $f : G \to H$ which is an isomorphism (written $G \cong H$)

If $G \cong H$ then

- $G$ is a belian iff $H$ is abelian

**Abelian:** For every $x, y$ $x \cdot_G y = y \cdot_G x$

$$f(y) \cdot_H f(x) = f(y \cdot_G x) = f(x \cdot_G y) = f(x) \cdot_H f(y)$$

So, any 2 elements

If $f$ is a $\cong$, $f : G \to H$ and $x \in G$ has order 2 Then $f(x) \in H$ has order 2

$$x^2 = 1_G$$
$$(f(x))^2 = f(x) \cdot f(x) = f(x \cdot x) = f(1_G) = 1_H$$

**Recall** $D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, rs = sr^{-1} \rangle$

If $G = \langle g_1, \ldots, g_n \mid R_1, R_2, \ldots \rangle$ and $h_1, \ldots, h_n \in H$ so $R_1(h_1 \ldots h_n) \ldots$ Then $f : g_i \mapsto h_i$ is a homomorphism

## 1.5 Group Actions

**Definition 20.** A group action is a function

$$\alpha : G \times A \to A$$

so

$$\alpha(g, \alpha(h, a)) = \alpha(g \cdot h, a)$$

We write $g \cdot a$ for $\alpha(g, a)$

$$g \cdot (h \cdot a) = (g \cdot h) \cdot a$$

- $1_G \cdot a = a$ for any $a \in A$

For any $g \in G$ the function $g\cdot : A \to A$, $a \mapsto g \cdot a$ is a bijection of $a$.

$$\begin{aligned} (g \cdot (g^{-1} \cdot_G)) &: A \to A \\ &= (g \cdot g^{-1}) \cdot a \\ &= 1_G \cdot a = a \\ g^{-1}(g \cdot a) &= a \end{aligned}$$

Since this function has an inverse (as a funciton) it is bijective

**Recall:** $S_A$ is the group of all permutations of $A$

Get a function $\sigma : G \to S_A$ and $\sigma(g) =$ the function $a \mapsto g \cdot a$

**Observation:** $\sigma$ is a homomorphism

$$\sigma(g \cdot h) = \sigma(g) \cdot \sigma(h)$$

**Example 21.** $(\mathbb{R}, +)$ acts on $A = \{1, 2, 3\}$

$g \cdot a = a$

$\sigma : \mathbb{R} \to S_3$, $g \mapsto 1_{S_3}$

# 2 Subgroups

## 2.1 Definition and Examples

**Definition 22.** Let $G$ be a group. The subset $H$ of $G$ is a subgroup of $G$ if

- $1_G \in H$
- $\forall x, y \in H$, $x \cdot y \in H$
- $\forall x \in H$, $x^{-1} \in H$

We write $H \leq G$ to indicate that $H$ is a subgroup of $G$.

**Proposition 23.** A subset $H$ of a group $G$ is a subgroup of $G$ if and only if

- $H \neq \emptyset$
- $\forall x, y \in H, xy^{-1} \in H$

## 2.2 Centralizers and Normalizers, Stabilizers and Kernels

**Definition 24** (Centralizer)**.** Let $G$ be a group and $A$ be a subset of $G$. The centralizer of $A$ in $G$ is

$$C_G(A) = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$$

Moreover, $C_G(A)$ is a subgroup of $G$.

**Definition 25** (Center)**.** Let $G$ be a group. The center of $G$ is

$$Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$$

**Definition 26** (Normalizer)**.** Let $G$ be a group and $A$ be a subset of $G$. Let

$$gAg^{-1} = \{gag^{-1} \mid a \in A\}$$

The Normalizer of $A$ in $G$ is

$$N_G(A) = \{g \in G \mid gAg^{-1} = A\}$$

**Definition 27** (Stabilizer)**.** If $G$ is a group acting on a set $S$ and $s$ is some fixed element of $S$ the stabilizer of $s$ is

$$G_s = \{g \in G \mid g \cdot s = s\}$$

## 2.3 Cyclic groups

**Definition 28.** A group $H$ is a cyclic if $H$ can be generated by a single element. i.e., $H = \langle x \rangle = \{x^n \mid n \in \mathbb{Z}\}$ for some $x \in H$.

**Proposition 29.** If $H = \langle x \rangle$, then $|x| = n$.

**Proposition 30.** Let $G$ be an arbitrary group, $x \in G$ and let $m, n \in \mathbb{Z}$. If $x^n = 1$ and $x^m = 1$, then $x^d = 1$, where $d = (m, n)$.

*Proof.* By the Euclidean Algorithm, there exists $q, r \in \mathbb{Z}$ such that $d = mr + ns$ where $d = (m, n)$. Thus

$$x^d = x^{mr+ns} = (x^m)^r (x^n)^s = 1^r 1^s = 1$$

□

**Theorem 31.** If $H_1, H_2$ is cyclic groups and $|H_1| = |H_2|$ then $H_1 \cong H_2$.

**Proposition 32.** Let $G$ be a group, let $x \in G$ and let $a \in \mathbb{Z} - \{0\}$.

1. If $|x| = \infty$, then $|x^a| = \infty$
2. If $|x| = n < \infty$, then $|x^a| = \frac{n}{(n,a)}$
3.

**Theorem 33.** If $H = \langle x \rangle$ and $|x| = n$ then $x^a = 1$ if and only if $n \mid a$.

**Theorem 34.** If $H = \langle x \rangle$ and $K \leq H$. Then $K$ is cyclic

*Proof.* Let $a$ be the least positive integer such that $x^a \in K$, let $y = x^a$

Then we want to show $\langle y \rangle = K$.

- $\langle y \rangle \subseteq K$ Obvious

- $\langle y \rangle \supseteq K$ Given $x^b \in K$ we can write $b = am + r$ with $a \leq r < a$

$$\begin{aligned} x^b &= x^{am+r} = (x^a)^m x^r \\ &= y^m \underbrace{x^r}_{\in K} \\ x^r &= \underbrace{y^{-m}}_{\in K} \underbrace{x^b}_{\in K} \end{aligned}$$

  So, $x^r \in K$ so $r = 0$, $x^b = y^m$

Therefore $\langle y \rangle = K$ $\square$

# 3 Quotient Groups and Homomorphisms

## 3.1 Definition and Examples

**Definition 35.** If $\varphi : G \to H$ is a homomorphism then $\ker(\varphi) = \{x \in G \mid \varphi(x) = 1_H\}$

**Lemma 36.** $\ker(\varphi) \leq G$

*Proof.* Proof eash properties of subgroup

- Closed identity, Since $\varphi(1_G) = 1_H$

$$\varphi(1_G) = \varphi(1_G 1_G) = \varphi(1_G) \cdot \varphi(1_G) = 1$$

  So, $1_G \in \ker(\varphi)$

- Closed under inverses, if $x \in ker(\varphi)$

$$\begin{aligned} \varphi(x^{-1}) = (\varphi(x))^{-1} = (1_H)^{-1} = 1_H \\ 1_H = \varphi(1_G) = \varphi(x^{-1}x) = \varphi(x) \cdot \varphi(x^{-1}) \end{aligned}$$

  So, $x^{-1} \in \ker(\varphi)$

- Closed under multiplication, if $x, y \in \ker(\varphi)$

$$\begin{aligned} \varphi(xy) &= \varphi(x) \cdot \varphi(y) \\ &= 1_H \cdot 1_H = 1_H \end{aligned}$$

  So, $xy \in \ker(\varphi)$

□

**Definition 37.** Given $\varphi : G \to H$ a homomorphism and $K = \ker(\varphi)$ For any $a \in H$, let

$$X_a = \{x \in G \mid \varphi(x) = a\}$$

then

$$G/K = (\{X_a \mid a \in H\}, \circ)$$

where

$$X_a \circ X_b = X_{ab}$$

**Lemma 38.** If $\varphi : G \to H$ is a homomorphism, $K = \ker(\varphi)$, and $\varphi(b) = a$ then $X_a = bK$ where $bK = \{b \cdot z \mid z \in K\}$

*Proof.* The goal is to show $X_a = bK$

- $X_a \supseteq bK$, Given $y \in bK, y = b \cdot z$ for some $z \in K$

$$\varphi(y) = \varphi(b \cdot z) = \varphi(b) \cdot \varphi(z) = a \cdot 1_H = a$$

- $X_a \subseteq bK$, Given $\varphi(y) = a$

$$\varphi(b^{-1}y) = \varphi(b^{-1})\varphi(y) = (\varphi(b))^{-1} \cdot \varphi(y) = a^{-1} \cdot a = 1$$

Therefore $X_a = bK$ □

**Definition 39.** For any $N \leq G$ and for any $g \in G$ let

$$gN = \{gn \mid n \in N\}$$

and

$$Ng = \{ng \mid n \in N\}$$

**Theorem 40.** Let $G$ be a group and $K$ be the kernel of some homomorphism. Then the set whose elements are the left cosets of $K$ in $G$ with operation defined by

$$uK \circ vK = (uv)K$$

forms a group $G/K$.

*Proof.* Let $X, Y \in G/K$ and let $Z = XY$ in $G/K$. Since $K$ is the kernel of some homomorphism, $\varphi : G \to H$, so $X = \varphi^{-1}(a)$ and $Y = \varphi^{-1}(b)$ for some $a, b \in H$. By definition of the operation in $G/K$, $Z = \varphi^{-1}(ab)$.

Let $u, v$ be arbitrary representatives of $X, Y$ ($\varphi(u) = a, \varphi(v) = b$ and $X = uK, Y = vK$)

GOAL: show $uv \in Z$

$$\begin{aligned} uv \in Z &\iff uv \in \varphi^{-1}(a, b) \\ &\iff \varphi(uv) = ab \\ &\iff \varphi(u)\varphi(v) = ab \end{aligned}$$

Therefore $Z$ is the (left) coset $(uv)K$. □

**Proposition 41.** If $N \leq G$ then for all $u, v \in G, uK = vK$ if and only if $v^{-1}u \in K$

*Proof.*

- $G = \bigcup_{b \in G} bK$, $1 \in K \to b \in bK$
- If $a \in uK \cap vK$ then for $k_1, k_2 \in K$,

$$\begin{aligned} u \cdot k_1 = v \cdot k_2 = a \\ v^{-1}u = k_2^{-1}k_1 \in K \end{aligned}$$

  Given any $l \in K$,

$$ul = v \cdot (v^{-1} \cdot u \cdot l) \in vK$$

□

**Theorem 42.** For $K \leq G$, the following are equivalent

- $K \trianglelefteq G$
- $N_G(N) = G$
- $gN = Ng$ for all $g \in G$
- $gNg^{-1} \subseteq N$ for all $g \in G$