MATH 629 Lecture Notes

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Contents

1 From Riemann to Lebesgue

§1.1 Riemann Integral

Definition 1.1.1. $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a, b]$.

Definition 1.1.2. If P, P' are partitions of $[a, b]$ and $P \subseteq P'$, then P' is a refinement of P.

Definition 1.1.3. Given a bounded function $f : [a, b] \to \mathbb{R}$ and a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ we define

$$m_i(f) = \inf_{t \in [x_{i-1}, x_i]} f(t)$$

$$M_i(f) = \sup_{t \in [x_{i-1}, x_i]} f(t)$$

define the lower sum as

$$L(f, P) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1})$$

and the upper sum as

$$U(f, P) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$$

Lemma 1.1.4

Given a bounded function $f : [a, b] \to \mathbb{R}$ and partitons P of $[a, b]$. Suppose that P' is a refinement of P then

$$(b - a) \inf_{t \in [a,b]} f(t) \leq L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P) \leq (b - a) \sup_{t \in [a,b]} f(t)$$

Corollary 1.1.5

Suppose that P_1, P_2 are partitions of $[a, b]$ then $L(f, P_1) \leq U(f, P_2)$

Proof. Let $P' = P_1 \cup P_2$ then P' is a refinement of P_1 and P_2 and use Lemma 1.1.4 $\square$

Lemma 1.1.6

Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Then

$$(b-a) \inf_{t \in [a,b]} f(t) \leq \sup_P L(f, P) \leq \inf_P U(f, P) \leq (b-a) \sup_{t \in [a,b]} f(t)$$

Definition 1.1.7. A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if

$$\sup_P L(f, P) = \inf_P U(f, P)$$

and the common value is called the Riemann integral of f and is denoted by $\int_a^b f$

Lemma 1.1.8

Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Then f is Riemann integrable if and only if for any $\varepsilon > 0$ there exists a partition P such that

$$U(f, P) - L(f, P) < \varepsilon$$

Proof. ($\Rightarrow$) For any $\varepsilon > 0$. Suppose that f is Riemann integrable. Then there exists P_1, P_2 such that

$$L(f, P_1) \geq \int_a^b f - \frac{\varepsilon}{2}$$

$$U(f, P_2) \leq \int_a^b f + \frac{\varepsilon}{2}$$

let $P = P_1 \cup P_2$ then

$$U(f, P) - L(f, P) \leq \varepsilon$$

($\Leftarrow$) For any $\varepsilon > 0$, there exists P_ε such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

since ε is arbitrary, we have

$$\sup_P L(f, P) = \inf_P U(f, P)$$

□

Theorem 1.1.9

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ then f is Riemann integrable.

Proof. f is continuous on a compact set, so, f is uniformly continous. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in [a, b]$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{(b-a)}$. Let N be such that $\frac{(b-a)}{N} < \delta$ and let $P = \{x_i := a + \frac{(b-a)i}{N}\}$ then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{N} (M_i(f) - m_i(f)) \frac{(b-a)}{N} \\ &\leq \sum_{i=1}^{N} \frac{\varepsilon}{(b-a)} \frac{(b-a)}{N} \\ &= \varepsilon \end{aligned}$$

□

Remark 1.1.10. Let $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ defined on the $[0, 1]$. Then $U(f, P) = 1$ and $L(f, P) = 0$ for any partition P. So, f is not Riemann integrable.

§1.2 Lebesgue null sets

Definition 1.2.1. For the closed interval $I = [a, b]$, the length of I, denoted as $\ell(I)$ is defined as $\ell(I) = b - a$

Definition 1.2.2. A set E is said to be a Lebesgue null set if for any $\varepsilon > 0$ there exists a sequence of intervals $\{I_n\}_{n \in \mathbb{N}}$ such that

$$E \subseteq \bigcup_{n=1}^{\infty} I_n$$

and

$$\sum_{n=1}^{\infty} \ell(I_n) < \varepsilon$$

Lemma 1.2.3

Countable unions of Lebesgue null sets are Lebesgue null sets.

Proof. For any $\varepsilon > 0$ and for each Lebesgue null sets E_n there exists $I_{E_n,i}$ such that

$$E_n \subseteq \bigcup_{i=1}^{\infty} I_{E_n,i}$$

and

$$\sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \frac{\varepsilon}{2^n}$$

then

$$\sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \varepsilon$$

□

Definition 1.2.4. A set $E \subseteq [a,b]$ has content zero if for any $\varepsilon > 0$ there exists $I_1, I_2, \ldots, I_n$ such that

$$E \subseteq \bigcup_{i=1}^{n} I_i$$

and

$$\sum_{i=1}^{n} \ell(I_i) < \varepsilon$$

Lemma 1.2.5

Suppose that $E \subseteq [a,b]$ is a compact Lebesgue null set then E has content zero.

Proof. For any $\varepsilon > 0$ there exists a sequnce of interval $\{I_n\}_{n\in\mathbb{N}}$ such that $E \subseteq \bigcup I_n$ and $\sum \ell(I_n) < \frac{\varepsilon}{2}$. Suppose that $I_n = [a_n, b_n]$, then let

$$J_n = \left(a_n - \frac{\varepsilon}{2^{n+3}}, b_n + \frac{\varepsilon}{2^{n+3}}\right) \supseteq E_n$$

then from the compactness of E, there exists a finite subcover $J_{n_1}, J_{n_2}, \ldots, J_{n_k}$ such that $E \subseteq \bigcup J_{n_i}$ then we construct a finite closed interval K_i by

$$K_i = \left[a_{n_i} - \frac{\varepsilon}{2^{n_i+2}}, b_{n_i} + \frac{\varepsilon}{2^{n_i+2}}\right]$$

then $E \subseteq \bigcup K_i$ and $\sum \ell(K_i) < \varepsilon$ □

Corollary 1.2.6

if $a < b$ then $[a,b]$ is not a Lebesgue null set.

Proof. By contradiction, since $[a,b]$ is compact, then $[a,b]$ has content zero, but $[a,b]$ don't have content zero. □

§1.3 Oscillation and Discontinuity

Definition 1.3.1. Suppose that $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ for any $x \in X$ and $\delta > 0$, define

$$M_{f,\delta}(x) := \sup\{f(y) : d(x,y) < \delta\}$$

$$m_{f,\delta}(x) := \inf\{f(y) : d(x,y) < \delta\}$$

then we define

$$\mathrm{osc}_f(x) := \lim_{\delta \to 0+} M_{f,\delta}(x) - m_{f,\delta}(x)$$

Lemma 1.3.2

f is continuous at x if and only if $\mathrm{osc}_f(x) = 0$.

Proof. ($\Rightarrow$) Suppose that f is continuous at x, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x,y) < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Then

$$M_{f,\delta}(x) - m_{f,\delta}(x) \leq \sup\{f(y) : d(x,y) < \delta\} - \inf\{f(y) : d(x,y) < \delta\} < \varepsilon$$

($\Leftarrow$) Suppose that $\mathrm{osc}_f(x) = 0$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $M_{f,\delta}(x) - m_{f,\delta}(x) < \varepsilon$. Then for any $y \in X$ such that $d(x,y) < \delta$, we have $|f(x) - f(y)| < \varepsilon$ then f is continuous at x. $\square$

Before we prove this theorem, we need to prove the following lemma.

Lemma 1.3.3

$\{x \in [a,b] : \mathrm{osc}_f(x) \geq \gamma\}$ is closed.

Proof. We need to show that $\{x : \mathrm{osc}_f(x) < \gamma\}$ is open. Fix x in that set. Let $\varepsilon = \gamma - \mathrm{osc}_f(x)$ then

$$\sup_{|w-x|<\delta} f(w) - \inf_{|w-x|<\delta} f(w) < \mathrm{osc}_f(x) < \gamma$$

then for any $w \in (x-\delta, x+\delta)$ if $|w - x| < \frac{\delta}{2}$ then

$$\mathrm{osc}(w) \leq \sup_{|y-w|<\frac{\delta}{2}} f(y) - \inf_{|y-w|<\frac{\delta}{2}} f(y) < \gamma$$

So, $B\left(x, \frac{\delta}{2}\right) \subseteq \{x : \mathrm{osc}_f(x) < \gamma\}$ $\square$

we observe that

(i) If the set of discontinuities is a Lebesque null set, then $\{x : \operatorname{osc}_f(x) \geq \gamma\}$ is a set of content zero.

(ii) If $\{x : \operatorname{osc}_f(x) \geq \gamma\}$ is a Lebesgue null set, then the set of discontinuities is also a Lebesque null set.

Lemma 1.3.4

Suppose that f is defined on $[c, d]$, assume that $\operatorname{osc}_f(x) < \gamma$ then we can find a partition

$$U(f, P) - L(f, P) < \gamma(b - a)$$

Proof. For every $x \in [c, d]$, there exists $\delta_x > 0$ such that

$$\sup_{|w-x|<\delta_x} f(w) - \inf_{|w-x|<\delta_x} f(x) < \gamma$$

construct a cover by

$$B(x, \delta_x) = \{w \in [c, d] : |w - x| < \delta_x\}$$

since $[c, d]$ is compact, there exists a finite subcover $B(p_1, \delta_{p_1}), \ldots, B(p_n, \delta_{p_n})$ then let $\delta_0 = \frac{\min\{\delta_{p_i}\}}{100}$ then we can construct a partition $P = \{c = x_0 < x_1 < \cdots < x_n = d\}$ such that $|x_i - x_{i-1}| < \delta_0$ then $M_i - m_i < \gamma$ and

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &< \gamma \sum_{i=1}^{n} (x_i - x_{i-1}) \\ &= \gamma(d - c) \end{aligned}$$

□

Theorem 1.3.5

Suppose that $f : [a, b] \to \mathbb{R}$ then $f \in \mathcal{R}([a, b])$ if and only if f is bounded and the set of discontinuity of f is a Lebesgue null set.

Proof. ($\Rightarrow$) We want to show that for every $n \in \mathbb{N}$,

$$\mathcal{D}_n = \left\{x : \operatorname{osc}_f(x) \geq \frac{1}{n}\right\}$$

is a Lebesque null set. For any $\varepsilon > 0$, since f is Riemann integrable, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1})(M_i - m_i) \leq \frac{\varepsilon}{n}$$

where $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ and $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$. in particular

$$\sum_{[x_{i-1}, x_i] \cap \mathcal{D}_n \neq \emptyset} (x_i - x_{i-1})(M_i - m_i) \leq \frac{\varepsilon}{n}$$

$$\frac{1}{n} \sum_{[x_{i-1}, x_i] \cap \mathcal{D}_n \neq \emptyset} \ell([x_{i-1}, x_i]) \leq \frac{\varepsilon}{n}$$

So, this interval cover the set $\mathcal{D}_n$

($\Leftarrow$) pick $\varepsilon_1 \ll \varepsilon$, consider the set $D(\varepsilon_1) = \{x \in [a, b] : \operatorname{osc}_f(x) \geq \varepsilon_1\}$ closed set. Since $D(\varepsilon_1)$ is a Lebesgue null set from the Lemma 1.2.5 it has content zero so we can find $I_1, \ldots, I_n$ such that

$$\sum_{j=1}^{n} \ell(I_j) < \varepsilon_1 \text{ and } D(\varepsilon_1) \subseteq \bigcup_{j=1}^{n} I_j$$

We form a partition of $[a, b]$, $a = x_0 < x_1 < \cdots < x_N = b$ from I_j. There are two cases that we need to consider

1) if $[x_{i-1}, x_i] \subseteq I_j$ for some j then set $P_i = [x_{i-1}, x_i]$
2) if $[x_{i-1}, x_i] \cap I_j = \emptyset$ for all j then $\operatorname{osc}(x) < \varepsilon_1$ for all $x \in [x_{i-1}, x_i]$. We want to partition further the interval $[x_{i-1}, x_i]$ by partition P_i. Using Lemma 1.3.4 we can find a partition P_i of $[x_{i-1}, x_i]$ such that

$$U(f, P_i) - L(f, P_i) < \varepsilon_1(x_i - x_{i-1})$$

We form a partition $P = P_1 \cup \cdots \cup P_N$ then

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^{N} (U(f, P_i) - L(f, P_i)) \\
&= \sum_{i:\text{case } 1} (U(f, P_i) - L(f, P_i)) + \sum_{i:\text{case } 2} (U(f, P_i) - L(f, P_i)) \\
&\leq 2M \sum_{i:\text{case } 1} (x_i - x_{i-1}) + \varepsilon_1 \sum_{i:\text{case } 2} (x_i - x_{i-1}) \\
&\leq 2M\varepsilon_1 + \varepsilon_1(b - a) \\
&= \varepsilon_1(2M + b - a)
\end{aligned}$$

$\square$

2 Measures

§2.1 Introduction

We define the $\ell([c, d]) = d - c$ and If $E = [c_1, d_1] \cup [c_2, d_2]$ where $d_1 < c_2$ then $\ell(E) = d_1 - c_1 + d_2 - c_2$. This is consistent with the definition

$$\ell(E) = \int \mathbb{1}_E(x) \, \mathrm{d}x$$

where the integral denotes the Riemann integral.

if $E \subseteq [a, b]$ reference interval is

$$\int_a^b \mathbb{1}_E \, \mathrm{d}x$$

Remark 2.1.1. The consistency of the definition also works with the set (c, d), $[c, d)$, and $(c, d]$, where the length of all of them is $d - c$.

Remark 2.1.2. we defnote $\mathbb{1}_E$ to be

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Example 2.1.3

Let $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ defined on the $[0, 1]$. Then $U(f, P) = 1$ and $L(f, P) = 0$ for any partition P.

Fix the reference interval $[a, b]$ and consider subset of $[a, b]$

Let $\mathcal{A}$ = collection of sets for which $\int_{[a,b]} \mathbb{1}_E \, \mathrm{d}x$ exists.

If $A_1, \ldots, A_n \in \mathcal{A}$, we can make the set to be mutually disjoint by taking $E_1 = A_1$, $E_2 = A_2 \setminus A_1$, $E_3 = A_3 \setminus (A_1 \cup A_2)$, and so on.

Example 2.1.4

For $E_1, E_2 \in \mathcal{A}$, we have

$$\mathbb{1}_{E_1 \cap E_2}(x) = \mathbb{1}_{E_1}(x)\mathbb{1}_{E_2}(x)$$

Example 2.1.5

For the Riemann integral, we have

$$\int_a^b f(y) = \int_{a-v}^{b-v} f(v+y)$$

and we want

$$\int \mathbb{1}_E(x)\ \mathrm{d}x = \int \mathbb{1}_{v+E}$$

where $v + E = \{v + x : x \in E\}$

Let $E = \mathbb{Q} \cap [0, 1]$ countable set, we can enumerate $r_1, r_2, r_3, \ldots$ such that

$$E = \bigcup_{n=1}^{\infty} \{r_n\}$$

and

$$\int \mathbb{1}_{\{r_k\}} = 0$$

E should have length zero but according $\mathbb{1}_E$ is not Riemann integrable.

§2.2 Construction of Measure

Suppose that $\mathcal{C}$ be a collection of sets.

Can we define on suitable large collection of subset of $\mathbb{R}$?

a set function $\mu : \mathcal{C} \to [0, \infty]$ such that if $\{E_j\}_{j=1}^{\infty}$ is a sequence of disjoint set in $\mathcal{C}$ then

$$\mu\left(\bigcup_{i=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

$\mu([a, b]) = b - a$, $\mu([0, 1)) = 1$

Can we do this for the collection of all subset of $\mathbb{R}$?

Answer: No, Vitali set.

Theorem 2.2.1

We cannot define a measure on the collection of all subset of $\mathbb{R}$. i.e., there does not exist a set function $\mu : \mathfrak{P}(\mathbb{R}) \to [0, \infty]$ such that

(i) $\mu(v + E) = \mu(E)$ for all $E \subseteq \mathbb{R}$ and $v \in \mathbb{R}$

(ii) $\mu([0, 1]) = 1$

(iii) $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$ for all disjoint $A_j \subseteq \mathbb{R}$

Before we prove that theorem, we need to define something and prove the following lemma.

Definition 2.2.2. We define a Vitali set V from picking an element $x \in [0, 1)$ from each equivalence class of the relation $x \sim y$ if $x - y \in \mathbb{Q}$. (e.g, pick $x \in O_x$ for $O_x \in \mathbb{R}/\mathbb{Q}$)

Lemma 2.2.3

Suppose that V is a Vitali set then

$$V \cap V + q = \emptyset$$

For all $q \in \mathbb{Q} \setminus \{0\}$

Proof. Suppose not, there exists $a \in V$ such that $a \in V + q \implies a - q \in V$ but we only pick 1 element in each equivalence class. contradiction. □

Lemma 2.2.4

Let V be a Vitali set and let $W = \{q \in [-1, 1] : q \in \mathbb{Q}\}$ and

$$E = \bigcup_{w \in W} V + w$$

then

$$[0, 1] \subseteq E \subseteq [-1, 2]$$

Proof. Consider $E \subseteq [-1, 2]$. Since $V \subseteq [0, 1)$, then for any $v \in V$, $v \in [0, 1) \implies v + w \in [-1, 2]$.

For the $[0, 1] \subseteq E$, for any $x \in [0, 1]$ there exists $O_x \in \mathbb{R}/\mathbb{Q}$ such that $x \in O_x$. then there exists $v \in C_x$ such that $v \in [0, 1)$ and $v \in V$, since both are from the same equivalence class, then $x - v \in \mathbb{Q}$ and $|x - v| < 1 \implies x - v \in (-1, 1)$. Hence, there exists $w \in W$ such that $w = x - v$ so $v + w = x$. □

Proof of the theorem. Suppose that μ exists then using the result from Lemma 2.2.4 we get that

$$\mu([0,1]) \leq \mu(E) \leq \mu([-1,2])$$

from Lemma 2.2.3 we know that each $V + w$ is disjoint, so

$$\mu([0,1]) \leq \sum_{w \in W} \mu(V) \leq \mu([-1,2])$$

$$1 \leq \sum_{w \in W} \mu(V) \leq 3$$

if $\mu(V) = 0$ then $\mu(E) = 0$ and if $\mu(V) > 0$ then $\mu(E) = \infty$. Both are contradiction. □

§2.3 σ-algebra

Definition 2.3.1. Given a reference X. An **algebra** is a collection of subsets of X, $\mathcal{A}$, such that

(i) $X \in \mathcal{A}$

(ii) If $A \in \mathcal{A}$ then the complement $A^{\complement} = X \setminus A \in \mathcal{A}$

(iii) If $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$

Remark 2.3.2.

- $\emptyset \in \mathcal{A}$ because $\emptyset = X^{\complement}$
- $A_1, A_2 \in \mathcal{A}$, $A_1 \setminus A_2 = A_1 \cap A_2^{\complement} \in \mathcal{A}$
- Observe that if $A_1, A_2 \in \mathcal{A}$ then $A_1 \cap A_2 \in \mathcal{A}$ because $(A_1 \cap A_2)^{\complement} = A_1^{\complement} \cup A_2^{\complement}$

Example 2.3.3

$X = [a, b]$ and $\mathcal{A}$ is the collection of all sets $E \subseteq [a, b]$ such that the Riemann integral $\int \mathbb{1}_E(t)\ \mathrm{d}t$ exists

Definition 2.3.4. A σ-algebra $\mathcal{M}$ on X is

(i) an algebra of subsets of X

(ii) If $A_1, A_2, A_3, \ldots$ is a sequence of set in $\mathcal{M}$ then

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$$

$(X, \mathcal{M})$ is called a "**measurable space**".

> **Remark 2.3.5.** $\mathcal{M}$ is a σ-algebra on X then it satisfies
>
> (i) $X \in \mathcal{M}$
>
> (ii) If $A \in \mathcal{M}$ then $A^{\complement} \in \mathcal{M}$
>
> (iii) countable union of sets in $\mathcal{M}$ is in $\mathcal{M}$

Definition 2.3.6. Let $(X, \mathcal{M})$ be a measurable set. Then a measure μ is a set function $\mu : \mathcal{M} \to [0, \infty], E \mapsto \mu(E)$ such that

(i) $\mu(\emptyset) = 0$

(ii) If $E_1, E_2, E_3, \dots$ is a sequence of disjoint set in $\mathcal{M}$ then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

called σ-additivity.

$(X, \mathcal{M}, \mu)$ is called a "**measure space**".

> **Remark 2.3.7.**
>
> $$\left(\bigcap_{j=1}^{\infty} A_j\right) = \left(\bigcup_{j=1}^{\infty} A_j^{\complement}\right)^{\complement} \in \mathcal{M}$$

Example 2.3.8

examples of σ-algebra

(i) $\mathcal{M} = \{\emptyset, X\}$

(ii) $\mathcal{M} = \mathfrak{P}(X) =$ collection of all subsets of X

$\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mu(E) = |E|$ (the cardinality of E) if E is finite and $\mu(E) = \infty$ if E is infinite.

(iii) X write X as a disjoint (countable) union of sets A_j. Then $\mathcal{M} =$ all countable unions of A_j.

(iv) Let X be a set. Let $\mathcal{M}$ be the collection of all sets A, $A \subseteq X$ such that A is countable or $A^{\complement}$ is countable.

(v) $X = \mathbb{R}$ (or $\mathbb{R}^n$), $\mathcal{B}_{\mathbb{R}}$ is the smallest σ-algebra containing all open sets.

More generally if $\mathcal{E}$ is a collection of subsets of X then $\mathfrak{M}(\mathcal{E})$ is the smallest σ-algebra that contains all sets in $\mathcal{E}$.

If $\mathcal{M}_1, \mathcal{M}_2$ are two σ-algebras, then $\mathcal{M}_1 \cap \mathcal{M}_2$ is also a σ-algebra.

If $\{\mathcal{M}_\alpha\}_{\alpha \in \mathcal{I}}$ is a collection of σ-algebras, their intersection is also a σ-algebra.

Generating σ-algebra

Definition 2.3.9. $\mathfrak{M}(\mathcal{E})$ = intersection of all σ-algebra that contain the collection $\mathcal{E}$ We call it the σ-algebra generated by $\mathcal{E}$.

Remark 2.3.10. If $\mathcal{E} \subset \mathcal{F} \implies \mathfrak{M}(\mathcal{E}) \subset \mathfrak{M}(\mathcal{F})$

Lemma 2.3.11

If $\mathcal{E} \subseteq \mathfrak{M}(\mathcal{F})$ then $\mathfrak{M}(\mathcal{E}) \subseteq \mathfrak{M}(\mathcal{F})$

Proof. $\mathfrak{M}(\mathcal{F})$ is a σ-algebra that contains $\mathcal{E}$ It contains the intersection of all σ-algebras which contain $\mathcal{E}$ $\square$

Example 2.3.12

$\mathcal{B}_{\mathbb{R}}$ = σ-algebra on $\mathbb{R}$ containing all open sets $\mathcal{E}$ a collection of all open intervals, $\mathcal{E} \subseteq \mathcal{O}$ = collection of all open sets in $\mathbb{R}$, $\mathcal{B}_{\mathbb{R}} = \mathfrak{M}(\mathcal{O})$. $\mathfrak{M}(\mathcal{E}) \subseteq \mathcal{B}_{\mathbb{R}}$. Each open set is a countable union of open intervals. Each open set is contained in $\mathfrak{M}(\mathcal{E})$.

Since $\mathcal{O} \subseteq \mathfrak{M}(\mathcal{E}) \implies \mathfrak{M}(\mathcal{O}) \subseteq \mathfrak{M}(\mathcal{E})$. get $\mathfrak{M}(\mathcal{O}) = \mathfrak{M}(\mathcal{E})$.

Definition 2.3.13. Given $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2), \ldots, (X_n, \mathcal{M}_n)$ measurable spaces. Define a "product σ-algebra" on $X_1 \times X_2 \times \cdots \times X_n$ denoted by

$$\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_n = \bigoplus_{j=1}^{n} \mathcal{M}_j$$

defined as the σ-algebra generated by the sets $E_1 \times E_2 \times \cdots \times E_n$ where $E_j \in \mathcal{M}_j$.

i.e., define $\mathcal{E} := \{(E_1 \times E_2 \times \cdots \times E_n) : E_j \in \mathcal{M}_j\}$ then

$$\bigoplus_{j=1}^{n} \mathcal{M}_j := \mathfrak{M}(\mathcal{E})$$

Remark 2.3.14. Folland defines it the σ-algebra generated by

$$(X_1 \times X_2 \times \cdots \times X_{n-1} \times E_n)$$

where $E_n \in \mathcal{M}_n$,

$$(X_1 \times X_2 \times \cdots E_{n-1} \times X_n)$$

where $E_{n-1} \in \mathcal{M}_{n-1}$. and so on. To be clear, let

$$\mathcal{E}' := \bigcup_{j=1}^{n} \{(X_1 \times \cdots \times X_{j-1} \times E_j \times X_{j+1} \times \cdots \times X_n) : E_j \in \mathcal{M}_j\}$$

then

$$\bigoplus_{j=1}^{n} \mathcal{M}_j := \mathfrak{M}(\mathcal{E}')$$

Claim 2.3.15 — Both defintions on product of σ-algebra are equivalent.

Proof. The goal is to show that $\mathfrak{M}(\mathcal{E}) = \mathfrak{M}(\mathcal{E}')$.

($\supseteq$) Obviously, $\mathcal{E}' \subseteq \mathcal{E}$ so $\mathfrak{M}(\mathcal{E}') \subseteq \mathfrak{M}(\mathcal{E})$.

($\subseteq$) We want to show that $\mathcal{E} \subseteq \mathfrak{M}(\mathcal{E}')$.

□

Theorem 2.3.16

Given $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2)$ measurable spaces. Assume that $\mathcal{M}_1$ is generated by a collection $\mathcal{E}_1$ and $\mathcal{M}_2$ is generated by a collection $\mathcal{E}_2$. Then $\mathcal{M}_1 \oplus \mathcal{M}_2$ is generated by the sets $E_1 \times X_2$, $X_1 \times E_2$, where $E_1 \in \mathcal{E}_1$ and $E_2 \in \mathcal{E}_2$.

Proof. Let $\mathcal{P} = \{E_1 \times E_2 : E_i \in \mathcal{E}_i\}$, $\mathfrak{M}(\mathcal{P}) \subseteq \mathcal{M}_1 \oplus \mathcal{M}_2$. We need to show that $\mathcal{M}_1 \oplus \mathcal{M}_2 \subseteq \mathfrak{M}(\mathcal{P})$. Define

$$\mathcal{G}_1 = \{E_1 \subseteq X_1 : E_1 \times X_2 \in \mathfrak{M}(\mathcal{P})\}$$

$$\mathcal{G}_2 = \{E_2 \subseteq X_2 : X_1 \times E_2 \in \mathfrak{M}(\mathcal{P})\}$$

then $\mathcal{G}_1$ is a σ-algebra consistion of subset of X_1 which contains $\mathcal{E}_1$, $\mathcal{E}_1 \subseteq \mathcal{G}_1$. $\mathcal{E}_1$ generates $\mathcal{M}_1$ so $\mathfrak{M}(\mathcal{E}_1) = \mathcal{M}_1 \subseteq \mathcal{G}_1$. So, we have $E_1 \times X_2 \in \mathfrak{M}(\mathcal{P})$ for all $E_1 \in \mathcal{M}_1$ and $X_1 \times E_2 \in \mathfrak{M}(\mathcal{P})$ for all $E_2 \in \mathcal{M}_2$. The σ-algebra generated by the sets $E_1 \times X_2$, $X_1 \times E_2$ is contained $\mathcal{M}_1 \oplus \mathcal{M}_2 \in \mathfrak{M}(\mathcal{P})$. □

Claim 2.3.17 — $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$

Consider the collection of all open rectangle of the form $(a_1, b_1) \times (a_2, b_2)$ such $a_i, b_i \in \mathbb{Q}$. which are contained in $O \subseteq \mathbb{R}^2$

Definition 2.3.18 (The Borel σ algebra on the extended real line)**.** We use the notion $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$. One possibility to define "$\mathcal{B}_{\overline{\mathbb{R}}}$" is the σ-algebra generated by open sets in $\mathbb{R}, \{\infty\}, \{-\infty\}$ open intervals should be $(a,b), (a,\infty], [-\infty, b)$ for $-\infty \leq a < b \leq \infty$. Then define $d(x,y) = |\arctan(x) - \arctan(y)|$ and $\arctan(\infty) = \pi/2$, $\arctan(-\infty) = -\pi/2$.

§2.4 Measures

Definition 2.4.1. Measures are σ-additive set functions, $\mu(\emptyset) = 0$ and

$$\mu\left(\biguplus_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

where $E_1, E_2, \ldots$ is a sequence of disjoint sets.

Remark 2.4.2. There is some property

$E \subseteq F \implies \mu(E) \leq \mu(F)$

$F = E \uplus (F \setminus E) \implies \mu(F) = \mu(E) + \mu(F \setminus E)$

$\mu(\bigcup A_j) \leq \sum \mu(A_j)$ we can write $\bigcup A_j$ as a disjoint union, i.e., $E_1 = A_1$, $E_2 = A_2 \setminus A_1$, $E_3 = A_3 \setminus (A_1 \cup A_2)$, and so on then $\mu(\bigcup A_j) = \mu(\bigcup E_j) = \sum \mu(E_j) \leq \mu(A_j)$

The monotone convergence theorem for sets (continuity from below)

Theorem 2.4.3
If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j \to \infty} \mu(E_j)$$

Proof. $\bigcup E_j = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus (E_1 \cup E_2)) \cup \cdots$ ∎