MATH 629 (Measure Theory) Lecture Notes

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Contents

1 From Riemann to Lebesgue

§1.1 Riemann Integral

Definition 1.1.1. $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a, b]$.

Definition 1.1.2. If P, P' are partitions of $[a, b]$ and $P \subseteq P'$, then P' is a refinement of P.

Definition 1.1.3. Given a bounded function $f : [a, b] \to \mathbb{R}$ and a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ we define

$$m_i(f) = \inf_{t \in [x_{i-1}, x_i]} f(t)$$

$$M_i(f) = \sup_{t \in [x_{i-1}, x_i]} f(t)$$

define the lower sum as

$$L(f, P) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1})$$

and the upper sum as

$$U(f, P) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$$

Lemma 1.1.4

Given a bounded function $f : [a, b] \to \mathbb{R}$ and partitons P of $[a, b]$. Suppose that P' is a refinement of P then

$$(b - a) \inf_{t \in [a,b]} f(t) \leq L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P) \leq (b - a) \sup_{t \in [a,b]} f(t)$$

Corollary 1.1.5

Suppose that P_1, P_2 are partitions of $[a, b]$ then $L(f, P_1) \leq U(f, P_2)$

Proof. Let $P' = P_1 \cup P_2$ then P' is a refinement of P_1 and P_2 and use Lemma 1.1.4 $\square$

Lemma 1.1.6

Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Then

$$(b-a) \inf_{t \in [a,b]} f(t) \leq \sup_P L(f, P) \leq \inf_P U(f, P) \leq (b-a) \sup_{t \in [a,b]} f(t)$$

Definition 1.1.7. A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if

$$\sup_P L(f, P) = \inf_P U(f, P)$$

and the common value is called the Riemann integral of f and is denoted by $\int_a^b f$

Lemma 1.1.8

Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Then f is Riemann integrable if and only if for any $\varepsilon > 0$ there exists a partition P such that

$$U(f, P) - L(f, P) < \varepsilon$$

Proof. ($\Rightarrow$) For any $\varepsilon > 0$. Suppose that f is Riemann integrable. Then there exists P_1, P_2 such that

$$L(f, P_1) \geq \int_a^b f - \frac{\varepsilon}{2}$$

$$U(f, P_2) \leq \int_a^b f + \frac{\varepsilon}{2}$$

let $P = P_1 \cup P_2$ then

$$U(f, P) - L(f, P) \leq \varepsilon$$

($\Leftarrow$) For any $\varepsilon > 0$, there exists P_ε such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

since ε is arbitrary, we have

$$\sup_P L(f, P) = \inf_P U(f, P)$$

□

Theorem 1.1.9

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ then f is Riemann integrable.

Proof. f is continuous on a compact set, so, f is uniformly continous. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in [a, b]$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{(b-a)}$. Let N be such that $\frac{(b-a)}{N} < \delta$ and let $P = \{x_i := a + \frac{(b-a)i}{N}\}$ then

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{N} (M_i(f) - m_i(f)) \frac{(b-a)}{N} \\ &\leq \sum_{i=1}^{N} \frac{\varepsilon}{(b-a)} \frac{(b-a)}{N} \\ &= \varepsilon \end{aligned}$$

$\square$

> **Remark 1.1.10.** Let $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ defined on the $[0, 1]$. Then $U(f, P) = 1$ and $L(f, P) = 0$ for any partition P. So, f is not Riemann integrable.

§1.2 Lebesgue null sets

Definition 1.2.1. For the closed interval $I = [a, b]$, the length of I, denoted as $\ell(I)$ is defined as $\ell(I) = b - a$

Definition 1.2.2. A set E is said to be a Lebesgue null set if for any $\varepsilon > 0$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E \subseteq \bigcup_{n=1}^{\infty} I_n$$

and

$$\sum_{n=1}^{\infty} \ell(I_n) < \varepsilon$$

Lemma 1.2.3
Countable unions of Lebesgue null sets are Lebesgue null sets.

Proof. For any $\varepsilon > 0$ and for each Lebesgue null sets E_n there exists $I_{E_n,i}$ such that

$$E_n \subseteq \bigcup_{i=1}^{\infty} I_{E_n,i}$$

and
$$\sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \frac{\varepsilon}{2^n}$$
then
$$\sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \varepsilon$$

□

Definition 1.2.4. A set $E \subseteq [a,b]$ has content zero if for any $\varepsilon > 0$ there exists $I_1, I_2, \ldots, I_n$ such that
$$E \subseteq \bigcup_{i=1}^{n} I_i$$
and
$$\sum_{i=1}^{n} \ell(I_i) < \varepsilon$$

Lemma 1.2.5

Suppose that $E \subseteq [a,b]$ is a compact Lebesgue null set then E has content zero.

Proof. For any $\varepsilon > 0$ there exists a sequnce of interval $\{I_n\}_{n\in\mathbb{N}}$ such that $E \subseteq \bigcup I_n$ and $\sum \ell(I_n) < \frac{\varepsilon}{2}$. Suppose that $I_n = [a_n, b_n]$, then let
$$J_n = \left(a_n - \frac{\varepsilon}{2^{n+3}}, b_n + \frac{\varepsilon}{2^{n+3}}\right) \supseteq E_n$$
then from the compactness of E, there exists a finite subcover $J_{n_1}, J_{n_2}, \ldots, J_{n_k}$ such that $E \subseteq \bigcup J_{n_i}$ then we construct a finite closed interval K_i by
$$K_i = \left[a_{n_i} - \frac{\varepsilon}{2^{n_i+2}}, b_{n_i} + \frac{\varepsilon}{2^{n_i+2}}\right]$$
then $E \subseteq \bigcup K_i$ and $\sum \ell(K_i) < \varepsilon$ □

Corollary 1.2.6

if $a < b$ then $[a,b]$ is not a Lebesgue null set.

Proof. By contradiction, since $[a,b]$ is compact, then $[a,b]$ has content zero, but $[a,b]$ don't have content zero. □

§1.3 Oscillation and Discontinuity

Definition 1.3.1. Suppose that $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ for any $x \in X$ and $\delta > 0$, define

$$M_{f,\delta}(x) := \sup\{f(y) : d(x,y) < \delta\}$$

$$m_{f,\delta}(x) := \inf\{f(y) : d(x,y) < \delta\}$$

then we define

$$\operatorname{osc}_f(x) := \lim_{\delta \to 0+} M_{f,\delta}(x) - m_{f,\delta}(x)$$

Lemma 1.3.2

f is continuous at x if and only if $\operatorname{osc}_f(x) = 0$.

Proof. ($\Rightarrow$) Suppose that f is continuous at x, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x,y) < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Then

$$M_{f,\delta}(x) - m_{f,\delta}(x) \leq \sup\{f(y) : d(x,y) < \delta\} - \inf\{f(y) : d(x,y) < \delta\} < \varepsilon$$

($\Leftarrow$) Suppose that $\operatorname{osc}_f(x) = 0$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $M_{f,\delta}(x) - m_{f,\delta}(x) < \varepsilon$. Then for any $y \in X$ such that $d(x,y) < \delta$, we have $|f(x) - f(y)| < \varepsilon$ then f is continuous at x. $\square$

Before we prove this theorem, we need to prove the following lemma.

Lemma 1.3.3

$\{x \in [a,b] : \operatorname{osc}_f(x) \geq \gamma\}$ is closed.

Proof. We need to show that $\{x : \operatorname{osc}_f(x) < \gamma\}$ is open. Fix x in that set. Let $\varepsilon = \gamma - \operatorname{osc}_f(x)$ then

$$\sup_{|w-x|<\delta} f(w) - \inf_{|w-x|<\delta} f(w) < \operatorname{osc}_f(x) < \gamma$$

then for any $w \in (x - \delta, x + \delta)$ if $|w - x| < \frac{\delta}{2}$ then

$$\operatorname{osc}(w) \leq \sup_{|y-w|<\frac{\delta}{2}} f(y) - \inf_{|y-w|<\frac{\delta}{2}} f(y) < \gamma$$

So, $B\left(x, \frac{\delta}{2}\right) \subseteq \{x : \operatorname{osc}_f(x) < \gamma\}$ $\square$

we observe that

(i) If the set of discontinuities is a Lebesque null set, then $\{x : \mathrm{osc}_f(x) \geq \gamma\}$ is a set of content zero.

(ii) If $\{x : \mathrm{osc}_f(x) \geq \gamma\}$ is a Lebesgue null set, then the set of discontinuities is also a Lebesque null set.

Lemma 1.3.4

Suppose that f is defined on $[c, d]$, assume that $\mathrm{osc}_f(x) < \gamma$ then we can find a partition

$$U(f, P) - L(f, P) < \gamma(b - a)$$

Proof. For every $x \in [c, d]$, there exists $\delta_x > 0$ such that

$$\sup_{|w-x|<\delta_x} f(w) - \inf_{|w-x|<\delta_x} f(x) < \gamma$$

construct a cover by

$$B(x, \delta_x) = \{w \in [c, d] : |w - x| < \delta_x\}$$

since $[c, d]$ is compact, there exists a finite subcover $B(p_1, \delta_{p_1}), \ldots, B(p_n, \delta_{p_n})$ then let $\delta_0 = \frac{\min\{\delta_{p_i}\}}{100}$ then we can construct a partition $P = \{c = x_0 < x_1 < \cdots < x_n = d\}$ such that $|x_i - x_{i-1}| < \delta_0$ then $M_i - m_i < \gamma$ and

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &< \gamma \sum_{i=1}^{n} (x_i - x_{i-1}) \\ &= \gamma(d - c) \end{aligned}$$

□

Theorem 1.3.5

Suppose that $f : [a, b] \to \mathbb{R}$ then $f \in \mathcal{R}([a, b])$ if and only if f is bounded and the set of discontinuity of f is a Lebesgue null set.

Proof. ($\Rightarrow$) We want to show that for every $n \in \mathbb{N}$,

$$\mathcal{D}_n = \left\{x : \mathrm{osc}_f(x) \geq \frac{1}{n}\right\}$$

is a Lebesgue null set. For any $\varepsilon > 0$, since f is Riemann integrable, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1})(M_i - m_i) \leq \frac{\varepsilon}{n}$$

where $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ and $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$. in particular

$$\sum_{[x_{i-1},x_i] \cap \mathcal{D}_n \neq \emptyset} (x_i - x_{i-1})(M_i - m_i) \leq \frac{\varepsilon}{n}$$

$$\frac{1}{n} \sum_{[x_{i-1},x_i] \cap \mathcal{D}_n \neq \emptyset} \ell([x_{i-1}, x_i]) \leq \frac{\varepsilon}{n}$$

So, this interval cover the set $\mathcal{D}_n$

($\Leftarrow$) pick $\varepsilon_1 \ll \varepsilon$, consider the set $D(\varepsilon_1) = \{x \in [a,b] : \operatorname{osc}_f(x) \geq \varepsilon_1\}$ closed set. Since $D(\varepsilon_1)$ is a Lebesgue null set from the Lemma 1.2.5 it has content zero so we can find $I_1, \ldots, I_n$ such that

$$\sum_{j=1}^{n} \ell(I_j) < \varepsilon_1 \text{ and } D(\varepsilon_1) \subseteq \bigcup_{j=1}^{n} I_j$$

We form a partition of $[a,b]$, $a = x_0 < x_1 < \cdots < x_N = b$ from I_j. There are two cases that we need to consider

1) if $[x_{i-1}, x_i] \subseteq I_j$ for some j then set $P_i = [x_{i-1}, x_i]$

2) if $[x_{i-1}, x_i] \cap I_j = \emptyset$ for all j then $\operatorname{osc}(x) < \varepsilon_1$ for all $x \in [x_{i-1}, x_i]$. We want to partition further the interval $[x_{i-1}, x_i]$ by partition P_i. Using Lemma 1.3.4 we can find a partition P_i of $[x_{i-1}, x_i]$ such that

$$U(f, P_i) - L(f, P_i) < \varepsilon_1(x_i - x_{i-1})$$

We form a partition $P = P_1 \cup \cdots \cup P_N$ then

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i=1}^{N} (U(f,P_i) - L(f,P_i)) \\
&= \sum_{i:\text{case 1}} (U(f,P_i) - L(f,P_i)) + \sum_{i:\text{case 2}} (U(f,P_i) - L(f,P_i)) \\
&\leq 2M \sum_{i:\text{case 1}} (x_i - x_{i-1}) + \varepsilon_1 \sum_{i:\text{case 2}} (x_i - x_{i-1}) \\
&\leq 2M\varepsilon_1 + \varepsilon_1(b-a) \\
&= \varepsilon_1(2M + b - a)
\end{aligned}$$

$\square$

2 Measures

§2.1 Introduction

We define the $\ell([c,d]) = d - c$ and If $E = [c_1, d_1] \cup [c_2, d_2]$ where $d_1 < c_2$ then $\ell(E) = d_1 - c_1 + d_2 - c_2$. This is consistent with the definition

$$\ell(E) = \int \mathbb{1}_E(x) \, \mathrm{d}x$$

where the integral denotes the Riemann integral.

if $E \subseteq [a,b]$ reference interval is

$$\int_a^b \mathbb{1}_E \, \mathrm{d}x$$

Remark 2.1.1. The consistency of the definition also works with the set (c,d), $[c,d)$, and $(c,d]$, where the length of all of them is $d - c$.

Remark 2.1.2. we defnote $\mathbb{1}_E$ to be

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Example 2.1.3

Let $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ defined on the $[0,1]$. Then $U(f,P) = 1$ and $L(f,P) = 0$ for any partition P.

Fix the reference interval $[a,b]$ and consider subset of $[a,b]$

Let $\mathcal{A}$ = collection of sets for which $\int_{[a,b]} \mathbb{1}_E \, \mathrm{d}x$ exists.

If $A_1, \ldots, A_n \in \mathcal{A}$, we can make the set to be mutually disjoint by taking $E_1 = A_1$, $E_2 = A_2 \setminus A_1$, $E_3 = A_3 \setminus (A_1 \cup A_2)$, and so on.

Example 2.1.4

For $E_1, E_2 \in \mathcal{A}$, we have

$$\mathbb{1}_{E_1 \cap E_2}(x) = \mathbb{1}_{E_1}(x)\mathbb{1}_{E_2}(x)$$

Example 2.1.5

For the Riemann integral, we have

$$\int_a^b f(y) = \int_{a-v}^{b-v} f(v+y)$$

and we want

$$\int \mathbb{1}_E(x)\ \mathrm{d}x = \int \mathbb{1}_{v+E}$$

where $v + E = \{v + x : x \in E\}$

Let $E = \mathbb{Q} \cap [0,1]$ countable set, we can enumerate $r_1, r_2, r_3, \ldots$ such that

$$E = \bigcup_{n=1}^{\infty} \{r_n\}$$

and

$$\int \mathbb{1}_{\{r_k\}} = 0$$

E should have length zero but according $\mathbb{1}_E$ is not Riemann integrable.

§2.2 Construction of Measure

Suppose that $\mathcal{C}$ be a collection of sets.

Can we define on suitable large collection of subset of $\mathbb{R}$?

a set function $\mu : \mathcal{C} \to [0, \infty]$ such that if $\{E_j\}_{j=1}^{\infty}$ is a sequence of disjoint set in $\mathcal{C}$ then

$$\mu\left(\bigcup_{i=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

$\mu([a,b]) = b - a$, $\mu([0,1)) = 1$

Can we do this for the collection of all subset of $\mathbb{R}$?

Answer: No, Vitali set.

Theorem 2.2.1

We cannot define a measure on the collection of all subset of $\mathbb{R}$. i.e., there does not exist a set function $\mu : \mathfrak{P}(\mathbb{R}) \to [0, \infty]$ such that

(i) $\mu(v + E) = \mu(E)$ for all $E \subseteq \mathbb{R}$ and $v \in \mathbb{R}$

(ii) $\mu([0, 1]) = 1$

(iii) $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$ for all disjoint $A_j \subseteq \mathbb{R}$

Before we prove that theorem, we need to define something and prove the following lemma.

Definition 2.2.2. We define a Vitali set V from picking an element $x \in [0, 1)$ from each equivalence class of the relation $x \sim y$ if $x - y \in \mathbb{Q}$. (e.g, pick $x \in O_x$ for $O_x \in \mathbb{R}/\mathbb{Q}$)

Lemma 2.2.3

Suppose that V is a Vitali set then

$$V \cap V + q = \emptyset$$

For all $q \in \mathbb{Q} \setminus \{0\}$

Proof. Suppose not, there exists $a \in V$ such that $a \in V + q \implies a - q \in V$ but we only pick 1 element in each equivalence class. contradiction. □

Lemma 2.2.4

Let V be a Vitali set and let $W = \{q \in [-1, 1] : q \in \mathbb{Q}\}$ and

$$E = \bigcup_{w \in W} V + w$$

then

$$[0, 1] \subseteq E \subseteq [-1, 2]$$

Proof. Consider $E \subseteq [-1, 2]$. Since $V \subseteq [0, 1)$, then for any $v \in V$, $v \in [0, 1) \implies v + w \in [-1, 2]$.

For the $[0, 1] \subseteq E$, for any $x \in [0, 1]$ there exists $O_x \in \mathbb{R}/\mathbb{Q}$ such that $x \in O_x$. then there exists $v \in C_x$ such that $v \in [0, 1)$ and $v \in V$, since both are from the same equivalence class, then $x - v \in \mathbb{Q}$ and $|x - v| < 1 \implies x - v \in (-1, 1)$. Hence, there exists $w \in W$ such that $w = x - v$ so $v + w = x$. □

Proof of the theorem. Suppose that μ exists then using the result from Lemma 2.2.4 we get that

$$\mu([0,1]) \le \mu(E) \le \mu([-1,2])$$

from Lemma 2.2.3 we know that each $V + w$ is disjoint, so

$$\mu([0,1]) \le \sum_{w \in W} \mu(V) \le \mu([-1,2])$$
$$1 \le \sum_{w \in W} \mu(V) \le 3$$

if $\mu(V) = 0$ then $\mu(E) = 0$ and if $\mu(V) > 0$ then $\mu(E) = \infty$. Both are contradiction. □

§2.3 σ-algebra

Definition 2.3.1. Given a reference X. An **algebra** is a collection of subsets of X, $\mathcal{A}$, such that

(i) $X \in \mathcal{A}$

(ii) If $A \in \mathcal{A}$ then the complement $A^{\complement} = X \setminus A \in \mathcal{A}$

(iii) If $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$

Remark 2.3.2.
- $\emptyset \in \mathcal{A}$ because $\emptyset = X^{\complement}$
- $A_1, A_2 \in \mathcal{A}$, $A_1 \setminus A_2 = A_1 \cap A_2^{\complement} \in \mathcal{A}$
- Observe that if $A_1, A_2 \in \mathcal{A}$ then $A_1 \cap A_2 \in \mathcal{A}$ because $(A_1 \cap A_2)^{\complement} = A_1^{\complement} \cup A_2^{\complement}$

Example 2.3.3

$X = [a, b]$ and $\mathcal{A}$ is the collection of all sets $E \subseteq [a, b]$ such that the Riemann integral $\int \mathbb{1}_E(t)\ dt$ exists

Definition 2.3.4. A σ-algebra $\mathcal{M}$ on X is

(i) an algebra of subsets of X

(ii) If $A_1, A_2, A_3, \ldots$ is a sequence of set in $\mathcal{M}$ then

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$$

$(X, \mathcal{M})$ is called a "**measurable space**".

Remark 2.3.5. $\mathcal{M}$ is a σ-algebra on X then it satisfies

(i) $X \in \mathcal{M}$

(ii) If $A \in \mathcal{M}$ then $A^{\complement} \in \mathcal{M}$

(iii) countable union of sets in $\mathcal{M}$ is in $\mathcal{M}$

Definition 2.3.6. Let $(X, \mathcal{M})$ be a measurable set. Then a measure μ is a set function $\mu : \mathcal{M} \to [0, \infty], E \mapsto \mu(E)$ such that

(i) $\mu(\emptyset) = 0$

(ii) If $E_1, E_2, E_3, \ldots$ is a sequence of disjoint set in $\mathcal{M}$ then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

called σ-additivity.

$(X, \mathcal{M}, \mu)$ is called a "**measure space**".

Remark 2.3.7.

$$\left(\bigcap_{j=1}^{\infty} A_j\right) = \left(\bigcup_{j=1}^{\infty} A_j^{\complement}\right)^{\complement} \in \mathcal{M}$$

Example 2.3.8

examples of σ-algebra

(i) $\mathcal{M} = \{\emptyset, X\}$

(ii) $\mathcal{M} = \mathfrak{P}(X) =$ collection of all subsets of X

$\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mu(E) = |E|$ (the cardinality of E) if E is finite and $\mu(E) = \infty$ if E is infinite.

(iii) X write X as a disjoint (countable) union of sets A_j. Then $\mathcal{M} =$ all countable unions of A_j.

(iv) Let X be a set. Let $\mathcal{M}$ be the collection of all sets A, $A \subseteq X$ such that A is countable or $A^{\complement}$ is countable.

(v) $X = \mathbb{R}$ (or $\mathbb{R}^n$), $\mathcal{B}_{\mathbb{R}}$ is the smallest σ-algebra containing all open sets.

More generally if $\mathcal{E}$ is a collection of subsets of X then $\mathfrak{M}(\mathcal{E})$ is the smallest σ-algebra that contains all sets in $\mathcal{E}$.

If $\mathcal{M}_1, \mathcal{M}_2$ are two σ-algebras, then $\mathcal{M}_1 \cap \mathcal{M}_2$ is also a σ-algebra.

If $\{\mathcal{M}_\alpha\}_{\alpha \in \mathcal{I}}$ is a collection of σ-algebras, their intersection is also a σ-algebra.

Generating σ-algebra

Definition 2.3.9. $\mathfrak{M}(\mathcal{E})$:= intersection of all σ-algebra that contain the collection $\mathcal{E}$ We call it the σ-algebra generated by $\mathcal{E}$. i.e.

$$\mathfrak{M}(\mathcal{E}) = \bigcap_{\substack{\mathcal{F} \in \mathcal{M} \\ \mathcal{E} \subseteq \mathcal{F}}} \mathcal{F}$$

> **Remark 2.3.10.** If $\mathcal{E} \subseteq \mathcal{F} \implies \mathfrak{M}(\mathcal{E}) \subseteq \mathfrak{M}(\mathcal{F})$

Lemma 2.3.11

If $\mathcal{E} \subseteq \mathfrak{M}(\mathcal{F})$ then $\mathfrak{M}(\mathcal{E}) \subseteq \mathfrak{M}(\mathcal{F})$

Proof. $\mathfrak{M}(\mathcal{F})$ is a σ-algebra that contains $\mathcal{E}$ It contains the intersection of all σ-algebras which contain $\mathcal{E}$ □

Example 2.3.12

$\mathcal{B}_\mathbb{R}$ = σ-algebra on $\mathbb{R}$ containing all open sets $\mathcal{E}$ a collection of all open intervals, $\mathcal{E} \subseteq \mathcal{O}$ = collection of all open sets in $\mathbb{R}$, $\mathcal{B}_\mathbb{R} = \mathfrak{M}(\mathcal{O})$. $\mathfrak{M}(\mathcal{E}) \subseteq \mathcal{B}_\mathbb{R}$. Each open set is a countable union of open intervals. Each open set is contained in $\mathfrak{M}(\mathcal{E})$.

Since $\mathcal{O} \subseteq \mathfrak{M}(\mathcal{E}) \implies \mathfrak{M}(\mathcal{O}) \subseteq \mathfrak{M}(\mathcal{E})$. get $\mathfrak{M}(\mathcal{O}) = \mathfrak{M}(\mathcal{E})$.

Definition 2.3.13. Given $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2), \ldots, (X_n, \mathcal{M}_n)$ measurable spaces. Define a "product σ-algebra" on $X_1 \times X_2 \times \cdots \times X_n$ denoted by

$$\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_n = \bigoplus_{j=1}^{n} \mathcal{M}_j$$

defined as the σ-algebra generated by the sets $E_1 \times E_2 \times \cdots \times E_n$ where $E_j \in \mathcal{M}_j$.

i.e., define $\mathcal{E} := \{(E_1 \times E_2 \times \cdots \times E_n) : E_j \in \mathcal{M}_j\}$ then

$$\bigoplus_{j=1}^{n} \mathcal{M}_j := \mathfrak{M}(\mathcal{E})$$

Remark 2.3.14. Folland defines it the σ-algebra generated by

$$(X_1 \times X_2 \times \cdots \times X_{n-1} \times E_n)$$

where $E_n \in \mathcal{M}_n$,

$$(X_1 \times X_2 \times \cdots E_{n-1} \times X_n)$$

where $E_{n-1} \in \mathcal{M}_{n-1}$. and so on. To be clear, let

$$\mathcal{E}' := \bigcup_{j=1}^{n} \{(X_1 \times \cdots \times X_{j-1} \times E_j \times X_{j+1} \times \cdots \times X_n) : E_j \in \mathcal{M}_j\}$$

then

$$\bigoplus_{j=1}^{n} \mathcal{M}_j := \mathfrak{M}(\mathcal{E}')$$

Claim 2.3.15 — Both defintions on product of σ-algebra are equivalent.

Proof. The goal is to show that $\mathfrak{M}(\mathcal{E}) = \mathfrak{M}(\mathcal{E}')$.

($\supseteq$) Obviously, $\mathcal{E}' \subseteq \mathcal{E}$ so $\mathfrak{M}(\mathcal{E}') \subseteq \mathfrak{M}(\mathcal{E})$.

($\subseteq$) We want to show that $\mathcal{E} \subseteq \mathfrak{M}(\mathcal{E}')$. Fix $(E_1 \times E_2 \times \cdots \times E_n) \in \mathcal{E}$ then from the definition of σ-algebra generated by a collection, which is closed under intersection, so we can pick an element from the construction of $\mathcal{E}'$ and do the intersection, so $(E_1 \times E_2 \cdots \times E_n) \in \mathfrak{M}(\mathcal{E})$.

$\square$

Theorem 2.3.16

Given $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2)$ measurable spaces. Assume that $\mathcal{M}_1$ is generated by a collection $\mathcal{E}_1$ and $\mathcal{M}_2$ is generated by a collection $\mathcal{E}_2$. Then $\mathcal{M}_1 \oplus \mathcal{M}_2$ is generated by the sets $E_1 \times X_2$, $X_1 \times E_2$, where $E_1 \in \mathcal{E}_1$ and $E_2 \in \mathcal{E}_2$.

Proof. Let $\mathcal{P} := \{E_1 \times E_2 : E_i \in \mathcal{E}_i\}$, obviously $\mathfrak{M}(\mathcal{P}) = \mathfrak{M}(\{E_1 \times X_2 : E_1 \in \mathcal{E}_1\} \cup \{X_1 \times E_2 : E_2 \in \mathcal{E}_2\})$ and $\mathfrak{M}(\mathcal{P}) \subseteq \mathcal{M}_1 \oplus \mathcal{M}_2$. We need to show that $\mathcal{M}_1 \oplus \mathcal{M}_2 \subseteq \mathfrak{M}(\mathcal{P})$. Define

$$\mathcal{G}_1 = \{E_1 \subseteq X_1 : E_1 \times X_2 \in \mathfrak{M}(\mathcal{P})\}$$
$$\mathcal{G}_2 = \{E_2 \subseteq X_2 : X_1 \times E_2 \in \mathfrak{M}(\mathcal{P})\}$$

then $\mathcal{G}_1$ is a σ-algebra consisting of subset of X_1 which contains $\mathcal{E}_1$, $\mathcal{E}_1 \subseteq \mathcal{G}_1$. $\mathcal{E}_1$ generates $\mathcal{M}_1$ so $\mathfrak{M}(\mathcal{E}_1) = \mathcal{M}_1 \subseteq \mathcal{G}_1$. So, we have $E_1 \times X_2 \in \mathfrak{M}(\mathcal{P})$ for all $E_1 \in \mathcal{M}_1$ and $X_1 \times E_2 \in \mathfrak{M}(\mathcal{P})$ for all $E_2 \in \mathcal{M}_2$. The σ-algebra generated by the sets $E_1 \times X_2$, $X_1 \times E_2$ is contained $\mathcal{M}_1 \oplus \mathcal{M}_2 \in \mathfrak{M}(\mathcal{P})$. $\square$

Claim 2.3.17 — $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$.

where $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}}$ is generated by $E_1 \times E_2$, where $E_1, E_2 \in \mathcal{B}_{\mathbb{R}}$. and $\mathcal{B}_{\mathbb{R}^2}$ is generated by the open sets in $\mathbb{R}^2$.

Proof. $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{B}_{\mathbb{R}^2}$. Want $\mathcal{B}_{\mathbb{R}^2} \subseteq \mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}}$. Consider the collection of all open rectangle of the form $(a_1, b_1) \times (a_2, b_2)$ such $a_i, b_i \in \mathbb{Q}$. which are contained in $O \subseteq \mathbb{R}^2$ $\square$

Definition 2.3.18 (The Borel σ algebra on the extended real line)**.** We use the notion $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$. One possibility to define "$\mathcal{B}_{\overline{\mathbb{R}}}$" is the σ-algebra generated by open sets in $\mathbb{R}, \{\infty\}, \{-\infty\}$ open intervals should be $(a, b), (a, \infty], [-\infty, b)$ for $-\infty \leq a < b \leq \infty$. Then define $d(x, y) = |\arctan(x) - \arctan(y)|$ and $\arctan(\infty) = \pi/2$, $\arctan(-\infty) = -\pi/2$.

§2.4 Measures

Definition 2.4.1. Measures are σ-additive set functions, $\mu(\emptyset) = 0$ and

$$\mu\left(\biguplus_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

where $E_1, E_2, \ldots$ is a sequence of disjoint sets.

Remark 2.4.2. There is some property

$E \subseteq F \implies \mu(E) \leq \mu(F)$

$F = E \uplus (F \setminus E) \implies \mu(F) = \mu(E) + \mu(F \setminus E)$

$\mu(\bigcup A_j) \leq \sum \mu(A_j)$ we can write $\bigcup A_j$ as a disjoint union, i.e., $E_1 = A_1$, $E_2 = A_2 \setminus A_1$, $E_3 = A_3 \setminus (A_1 \cup A_2)$, and so on then $\mu(\bigcup A_j) = \mu(\bigcup E_j) = \sum \mu(E_j) \leq \mu(A_j)$

The monotone convergence theorem for sets (continuity from below)

Theorem 2.4.3
If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j \to \infty} \mu(E_j)$$

Proof.

$$\bigcup_{j=1}^{\infty} E_j = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus (E_1 \cup E_2)) \cup \cdots$$

So, we define $B_1 = E_1, B_n = E_n \setminus E_{n-1}$ for $n \geq 2$ then all B_j are disjoint.

$$\begin{aligned}
\bigcup_{j=1}^{\infty} E_j &= \bigcup_{j=1}^{\infty} B_j \\
\mu\left(\bigcup_{j=1}^{\infty} E_j\right) &= \mu\left(\bigcup_{j=1}^{\infty} B_j\right) \\
&= \sum_{j=1}^{\infty} \mu(B_j) \\
&= \mu(E_1) + \sum_{j=2}^{\infty} \mu(E_j \setminus E_{j-1}) \\
&= \mu(E_1) + \sum_{j=2}^{\infty} \mu(E_j) - \mu(E_{j-1}) \\
&= \lim_{n\to\infty} \mu(E_n)
\end{aligned}$$

□

Remark 2.4.4. If we prove something for the set then we can prove it for the complement.

$$\mu(A) + \mu(A^{\complement}) = \mu(X)$$

Theorem 2.4.5

If $\mu(X) < \infty$ then if $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$, $E_n \supseteq E_{n+1}$ for all n then

$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$$

Proof. Assume E_j are decreasing, i.e.,

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$$

then $E_1^\complement \subseteq E_2^\complement \subseteq \cdots$ then

$$\begin{aligned}
\mu\left(\bigcup_{j=1}^{\infty} E_j^\complement\right) &= \lim_{j\to\infty} \mu(E_j^\complement) \\
\mu(X) - \mu\left(\left(\bigcup_{j=1}^{\infty} E_j^\complement\right)^\complement\right) &= \lim_{j\to\infty} (\mu(X) - \mu(E_j)) \\
\mu(X) - \mu\left(\bigcap_{j=1}^{\infty} E_j\right) &= \lim_{j\to\infty} (\mu(X) - \mu(E_j))
\end{aligned}$$

□

Example 2.4.6

$\mathbb{N}$ with counting measure, $E_j = \{j, j+1, j+2, \dots\}$, $\mu(E_j) = \infty$, $\bigcap E_j = \emptyset$ has measure 0.

Definition 2.4.7. If $A_1, A_2, A_3, \dots$ is an arbitrary sequence of measurable sets. We can define

$$\limsup A_j := \bigcap_{n=1}^{\infty} \bigcup_{j\geq n} A_j = \{x : x \in A_n \text{ for infinitely many } n\}$$

$$\liminf A_j := \bigcup_{n=1}^{\infty} \bigcap_{j\geq n} A_j = \{x : x \text{ belong to all but finitely many}\}$$

Lemma 2.4.8 (Borel-Cantelli Lemma)

If $\{A_j\}$ is a sequence of measurable sets such that

$$\sum_{j=1}^{\infty} \mu(A_j) < \infty$$

then almost every x (meaning all x except in a null set) belong to on A_n for only finitely many n. Or equivalently,

$$\mu\left(\limsup A_n\right) = 0$$

Proof. $\bigcup_{j\geq n} A_j$ are decreasing. In Borel Cantelli, we have $\sum \mu(A_j) < \infty$, so $\mu\left(\bigcup A_n\right) = 0$.

use "continuity from above"

$$\mu(\limsup A_n) = \lim_{n\to\infty} \mu\left(\bigcup_{j\geq n} A_j\right)$$

$$\mu\left(\bigcup_{j\geq n} A_j\right) \leq \sum_{j\geq n} \mu(A_j) \to 0$$

as $n \to \infty$. $\square$

Completion of a σ-algebra (when a measure μ is given), $(X, \mathcal{M}, \mu)$ $\overline{\mathcal{M}}$ consists of all unions $E \cup F$, where $E \in \mathcal{M}$ and $F \subseteq N \in \mathcal{M}$ for some null set N, $\mu(N) = 0$.

Define $\overline{\mu}$ by $\overline{\mu}(E \cup F) = \mu(E)$.

§2.5 Measurable Functions

Definition 2.5.1. $f : X \to Y$ where $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces. f is $(\mathcal{M}, \mathcal{N})$-measurable if for every $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$. where $f^{-1}(E) = \{x \in X : f(x) \in E\}$.

Lemma 2.5.2

Let $\mathcal{E}$ generate $\mathcal{N}$ (i.e., $\mathcal{N} = \mathfrak{M}(\mathcal{E})$). Then f is $(\mathcal{M}, \mathcal{N})$-measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

Proof. Define $\mathcal{C} = \{E \in \mathcal{E} : f^{-1}(E) \in \mathcal{M}\}$, observe that $\mathcal{C}$ is a σ-algebra. then

$$f(x) = \bigcup E_j \iff x \in f^{-1}\left(\bigcup E_j\right) \iff x \in \bigcup f^{-1}(E_j) \iff \bigcup \{x : f(x) \in E_j\}$$

$\square$

Claim 2.5.3 — $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable, $g : Y \to Z$ is $(\mathcal{N}, \mathcal{R})$-measurable then $g \circ f : X \to Z$ is $(\mathcal{M}, \mathcal{R})$-measurable.

Proof. $(g \circ f)^{-1}(E) = \{x \in X : g(f(x)) \in E\} = f^{-1}(g^{-1}(E)) = \{x \in X : f(x) \in g^{-1}(E)\}$ $\square$

Claim 2.5.4 — $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable then f^2 is $\mathcal{M}$-measurable.

Proof. $(f^2)^{-1}(-\infty, a) = \{x : f^2(x) < a\} = \{x : f(x) < \sqrt{a}\} \cup \{x : f(x) > -\sqrt{a}\}$ □

Claim 2.5.5 — $f : X \to \mathbb{R}, g : X \to \mathbb{R}$ are $\mathcal{M}$-measurable then $f + g$ and $f \cdot g$ are $\mathcal{M}$-measurable.

Proof.

$$(f + g)^{-1}(-\infty, a) = \bigcup_{r \in \mathbb{Q}} \left(f^{-1}(-\infty, a + r) \cap g^{-1}(-\infty, r)\right)$$

$$\begin{aligned} (f + g)^2 &= f^2 + 2fg + g^2 \\ fg &= \frac{1}{2}\left((f + g)^2 - f^2 - g^2\right) \end{aligned}$$

□

Claim 2.5.6 — vector-valued-function $f : X \to (Y_1 \times Y_2 \times \cdots \times Y_n)$ and defined by $x \mapsto (f_1(x), f_2(x), \ldots, f_n(x))$ where $f_j : X \to Y_j$ is $(\mathcal{M}, \mathcal{N}_j)$-measurable.

Then f is $(\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_n)$ if and only if $f_i(\mathcal{M}_i, \mathcal{N}_i)$-measurable.

Proof.

$$\begin{aligned} f^{-1}(E_1 \times E_2 \times \cdots \times E_n) &= f_1^{-1}(E_1) \cap f_2^{-1}(E_2) \cap \cdots \cap f_n^{-1}(E_n) \\ &= \bigcap_{j=1}^{n} f_j^{-1}(E_j) \end{aligned}$$

□

Claim 2.5.7 — $M(x) = \max\{f(x), g(x)\}$, $f, g : X \to \mathbb{R}$, $\mathcal{M}$-measurable.

Proof. $M^{-1}(-\infty, a) = \{x : M(x) < a\} = \{x : f(x) < a, g(x) < a\} = f^{-1}(-\infty, a) \cap g^{-1}(-\infty, a)$ □

Claim 2.5.8 — $f_n : X \to \mathbb{R}$, $\mathcal{M}$-measurable, then $S(x) = \sup_{n \in \mathbb{N}} f_n$ is $\mathcal{M}$-measurable.

Proof. $S^{-1}(-\infty, a) = \{x : S(x) < a\} = \{x : \sup f_n(x) < a\} = \bigcap_n \{x : f_n(x) < a\}$ □

Remark 2.5.9. We use the similar proof for min and inf.

Definition 2.5.10. If $f_n : X \to \mathbb{R}$, $\mathcal{M}$-measurable then

$$\limsup f_n = \inf_k \sup_{n \geq k} f_n$$

$$\liminf f_n = \sup_k \inf_{n \geq k} f_n$$

Claim 2.5.11 — $\limsup f_n$ and $\liminf f_n$ are $\mathcal{M}$-measurable.

Proof. For $\limsup f_n$, fix k then $\sup_{n \geq k} f_n$ is $\mathcal{M}$-measurable, $\inf_k \sup_{n \geq k} f_n$ is $\mathcal{M}$-measurable. Similarly for $\liminf f_n$. □

Theorem 2.5.12

Let $(X, \mathcal{M})$ be a measurable space, $f_n : X \to \mathbb{C}$ be $\mathcal{M}$-measurable functions. Define

$$E_{lim} = \{x \in X : \lim_{n \to \infty} f_n(x) \text{ exists}\}$$

then $E_{lim} \in \mathcal{M}$.

Proof. We can rewrite E_{lim} as

$$E_{lim} = \{x \in X : \{f_n(x)\}_{n \in \mathbb{N}} \text{ is a Cauchy sequence}\}$$

Define

$$A_{n,m}(k) = \left\{x \in X : |f_n(x) - f_m(x)| < \frac{1}{k}\right\}$$

then $A_{n,m}(k) \in \mathcal{M}$ for all n, m, k. then

$$E_{lim} = \bigcup_{k \geq 1} \bigcap_{N \geq 1} \bigcup_{m \geq N, n \geq N} A_{m,n}(k)$$

□

3 Integration

§3.1 Simple Functions

Definition 3.1.1. nonnegative simple function are measurable function with finitely many values in $\mathbb{R}$ (NOT on $\overline{\mathbb{R}}$). $s : X \to \mathbb{R}$, $s(x) = \sum z_j \mathbb{1}_{x, s(z)=z_j}(x) = \sum z_j \mathbb{1}_{f^{-1}(z_j)}$ If values of s are $\{z_1, \ldots, z_n\}$

Theorem 3.1.2

Consider nonnegative measurable function f. There exist a sequence of simple function s_n such that

- $0 \leq s_n \leq s_{n+1} \leq f$ (i.e, $s_n(x) \leq s_{n+1}(x)$)
- $\lim\limits_{n\to\infty} s_n(x) = f(x)$ for all x
- The convergence is uniform on all sets where f is bounded. If E is such that $|f(x)| \leq M$ for all $x \in E$ then

$$\sup_{x\in E} f(x) - s_n(x) \to 0$$

Proof. s_n is defined so that if takes value in $[0, 2^n)$. Consider the segment $\frac{k}{2^n}$ on y-axis, then

$$s_n(x) = \begin{cases} k \cdot 2^{-n} & \text{if } k2^{-n} \leq f(x) < (k+1)2^{-n}, 0 \leq k \leq 4^n - 1 \\ 2^n & \text{if } f(x) \geq 2^n \end{cases}$$

If $f(x) < 2^n$ then $0 \leq f(x) - s_n(x) \leq 2^{-n}$. We can see that $s_n(x) \leq s_{n+1}(x)$ because each step of s_{n+1} is a refinement of s_n. □

We first define the integral for simple function (in analogy to the definition of Riemann-integral for stop functions)

Definition 3.1.3. Define $s(x) = \sum_j c_j \mathbb{1}_{E_j}$ where the E_j are pairwise disjoint, $\biguplus E_j = X$, then

$$\int s \, \mathrm{d}\mu = \sum_j c_j \mu(E_j)$$

Claim 3.1.4 —

$$s(x) = \sum_{j=1}^{n} c_j \mathbb{1}_{E_j}(x) = \sum_{k=1}^{m} d_k \mathbb{1}_{E_k}(x)$$

where $X = \biguplus E_j = \biguplus E_k$. If $x \in E_j \cap E_k$ then $c_j = d_k$.

Proof. We know that $\biguplus_{j,k} E_j \cap E_k = X$ and $E_j = \biguplus_k E_j \cap E_k$

GOAL: $\sum_{j=1}^{n} c_j \mu(E_j) = \sum_{k=1}^{m} d_k \mu(F_k)$

$$\begin{aligned} \text{LHS} = \sum_{j=1}^{n} c_j \sum_{k=1}^{\infty} \mu(E_j \cap F_k) &= \sum_{k=1}^{m} \sum_{j=1}^{n} d_k \mu(E_j \cap E_k) \\ &= \sum_{k=1}^{m} d_k \mu(F_k) \end{aligned}$$

□

Lemma 3.1.5

Suppose s, t are simple functions then

$$\int (s+t)\ \mathrm{d}\mu = \int s\ \mathrm{d}\mu + \int t\ \mathrm{d}\mu$$

Remark 3.1.6. Can shortly write

$$\int s + t = \int s + \int t$$

Proof.

$$\begin{aligned} s = \sum_{j=1}^{n} c_j \mathbb{1}_{E_j} &= \sum_j \sum_k c_j \mathbb{1}_{E_j \cap F_k} \\ t = \sum_{k=1}^{m} d_k \mathbb{1}_{F_k} &= \sum_j \sum_k d_k \mathbb{1}_{E_j \cap F_k} \\ s + t &= \sum_{j,k} (c_j + d_k) \mathbb{1}_{E_j \cap F_k} \end{aligned}$$

$$\int s \ \mathrm{d}\mu = \sum_{j,k} c_j \mu(E_j \cap F_k)$$

$$\int t \ \mathrm{d}\mu = \sum_{j,k} d_k \mu(E_j \cap F_k)$$

$$\int (s+t) \ \mathrm{d}\mu = \sum_{j,k} (c_j + d_k) \mu(E_j \cap F_k)$$

□

Lemma 3.1.7

$\nu(E) = \int_E s \ \mathrm{d}\mu = \int s \mathbb{1}_E \ \mathrm{d}\mu = \sum c_j \mu(E_j \cap E)$ this defines a measure on $\mathcal{M}$ (given σ-algebra)

Proof. If E^l is a sequence of pairwise disjoint measureable set, check

$$\nu\left(\biguplus E^l\right) = \sum \nu(E^l)$$

$$\begin{aligned}
\nu\left(\biguplus E^l\right) &= \sum c_j \mu(E_j \cap \biguplus E^l) \\
&= \sum_{j=1}^{n} c_j \sum_l \mu(E_j \cap E^l) \\
&= \sum_l \sum_j c_j \mu(E_j \cap E^l) \\
&= \sum_l \nu(E^l)
\end{aligned}$$

□

§3.2 Non-negative Measurable Functions

Definition 3.2.1. For any non-negative f, a measurable function, define

$$\int f \ \mathrm{d}\mu = \sup_{\substack{s \le f \\ s \text{ simple}}} \int s \ \mathrm{d}\mu$$

Remark 3.2.2. If $0 \leq f \leq g$ then $\int f \, \mathrm{d}\mu \leq \int g \, \mathrm{d}\mu$

Theorem 3.2.3 (Monotone Convergence Theorem)

If $\{f_n\}$ is a sequence of measurable function, and $0 \leq f_n \leq f_{n+1}$ for all n. (that means $f(x) = \lim_{n\to\infty} f_n(x)$)Then

$$\int f \, \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu$$

Proof. Since $f_n \leq f_{n+1} \leq f$ then

$$\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu \leq \int f \, \mathrm{d}\mu$$

We need to show that

$$\int f \, \mathrm{d}\mu \leq \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu$$

So, it suffices to show that for any $0 \leq \underset{\text{simple}}{s} \leq f$, that

$$\int s \, \mathrm{d}\mu \leq \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu$$

It suffices to show that for any $\varepsilon > 0$,

$$(1-\varepsilon) \int s \, \mathrm{d}\mu \leq \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu$$

Define $E_n = \{x : (1-\varepsilon)s(x) \leq f_n(x)\}$, any x will be in one of the E_n. Then for any $x \in E_n$,

$$s(x) \leq \frac{f_n(x)}{1-\varepsilon}$$

Consider the measure defined by

$$\nu(E) = \int_E s \, \mathrm{d}\mu$$

(we already show this is a measure in 3.1.7). We have $E_n \subseteq E_{n+1}$ and $E_n \to X$. By continuity from below 2.4.3,

$$\lim_{n\to\infty} \nu(E_n) = \nu(X) = \int s \, \mathrm{d}\mu$$

We get that

$$\nu(E_n) = \int_{E_n} s \, \mathrm{d}\mu \leq \int_{E_n} \frac{f_n(x)}{1-\varepsilon} \, \mathrm{d}\mu \leq \int \frac{f_n(x)}{1-\varepsilon} \, \mathrm{d}\mu = \frac{1}{1-\varepsilon} \int f_n(x) \, \mathrm{d}\mu$$

Finally, we take limit on both sides and we have

$$\lim_{n\to\infty} \nu(E_n) = \nu(\mathbb{R}) = \int s \, \mathrm{d}\mu \leq \lim_{n\to\infty} \frac{1}{1-\varepsilon} \int f_n \, \mathrm{d}\mu$$

□

Lemma 3.2.4

If f, g are non negative measurable function then

$$\int (f+g) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$$

Proof. Now we have a tool

- Monotone Convergence Theorem
- Existence of $s_n \gg f, t_n \gg g$

$$\int (s_n + t_n) \, \mathrm{d}\mu = \int s_n \, \mathrm{d}\mu + \int t_n \, \mathrm{d}\mu$$
$$\int (f+g) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$$

□

Lemma 3.2.5

$f_k \geq 0$, f_k is measurable

$$\int \sum_{k=1}^{\infty} f_k(x) \, \mathrm{d}\mu = \sum_{k=1}^{\infty} \int f_k \, \mathrm{d}\mu$$

Proof. Just apply the Monotone Convergence Theorem.

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \to \sum_{k=1}^{\infty} f_k(x)$$

□

Remark 3.2.6. We cannot always interchange integrals and limits (monotonicity is key) $f_n(x) = \frac{1}{n}\mathbb{1}_{[0,n]}, \int f_n \, \mathrm{d}\mu = 1$ but $\lim\limits_{n\to\infty} f_n(x) = 0$.

$$0 = \int \lim_{n\to\infty} f_n(x) \, \mathrm{d}\mu < \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu$$

Or on $[0,1]$, $f_n(x) = n\mathbb{1}_{[0,1/n]}$, $\int f_n \, \mathrm{d}\mu = 1$ but $\lim\limits_{n\to\infty} f_n(x) = 0$.

$$\lim_{n\to\infty} f_n(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Lemma 3.2.7 (Fatou's Lemma)
If $\{f_j\}$ is a sequnce of measurable functions

$$\int \liminf_{j\to\infty} f_j(x) \, \mathrm{d}\mu \le \liminf_{j\to\infty} \int f_j \, \mathrm{d}\mu$$

meaning

$$\int \lim_{k\to\infty} \underbrace{\inf_{j\ge k} f_j(x)}_{\text{increasing on } k} \, \mathrm{d}\mu \le \lim_{k\to\infty} \inf_{j\ge k} \int f_j \, \mathrm{d}\mu$$

Proof.

$$\int \lim_{k\to\infty} \inf_{j\ge k} f_j(x) \, \mathrm{d}\mu \underset{\text{MCT}}{=} \lim_{k\to\infty} \int \inf_{j\ge k} f_j(x) \, \mathrm{d}\mu$$

Take any $l \ge k$, then $\inf_{j\ge k} f_j(x) \le f_l(x)$, then for $l \ge k$

$$\int \inf_{j\ge k} f_j(x) \, \mathrm{d}\mu \le \int f_l(x) \, \mathrm{d}\mu$$
$$\int \inf_{j\ge k} f_j(x) \, \mathrm{d}\mu \le \inf_{j\ge k} \int f_j(x) \, \mathrm{d}\mu$$

□

§3.3 General Measurable Functions

Integral for "general" measurable functions.

Definition 3.3.1. Given a measurable function f, we define the **positive part** of f as

$$f^+(x) = \max\{f(x), 0\}$$

and the **negative part** of f as

$$f^-(x) = \max\{-f(x), 0\}$$

Then we get that

$$f = f^+ - f^-$$

Definition 3.3.2. $f : X \to \mathbb{R}$ (or $\overline{\mathbb{R}}$) Suppose that f is a measurable function, then we define

$$\int f \, \mathrm{d}\mu = \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu$$

provided that at least one of $\int f^\pm \, \mathrm{d}\mu$ is finite

Definition 3.3.3. $f : X \to \mathbb{R}$ (or $\overline{\mathbb{R}}$) f is **integrable** if $\int f^+ \, \mathrm{d}\mu$, $\int f^- \, \mathrm{d}\mu$ is finite $\iff$ $\int |f| \, \mathrm{d}\mu$ is finite

$\mathcal{L}^1$ is the class of integrable function

Definition 3.3.4. $f : X \to \mathbb{C}$ is measureable ($\iff$ $\Re(f)$ and $\Im(f)$ are measurable) Assumeing that $\Re f \in \mathcal{L}^1$ and $\Im f \in \mathcal{L}^1$ then

$$\int f \, \mathrm{d}\mu = \int \Re f \, \mathrm{d}\mu + i \int \Im f \, \mathrm{d}\mu$$

Claim 3.3.5 — Suppose that f, g are measurable then

$$\int f + g \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$$

$$\int \alpha f \, \mathrm{d}\mu + \alpha \int f \, \mathrm{d}\mu$$

Lemma 3.3.6

$f : X \to \overline{\mathbb{R}}$ is measurable, and $\int |f| \, \mathrm{d}\mu = 0$ then $f = 0$ almost everywhere.

Proof. Define $E_n = \left\{x : |f(x)| > \frac{1}{n}\right\}$ then from continuity from below, we get that

$$\lim_{n \to \infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$$

define $E = \bigcup E_n$ and we can write $E = \{x : |f(x)| > 0\}$then we know that

$$|f| \geq |f| \mathbb{1}_{E_n} \geq \frac{1}{n} \mathbb{1}_{E_n}$$

then

$$\begin{aligned} \int |f| \, \mathrm{d}\mu &\geq \int \frac{1}{n} \mathbb{1}_{E_n} \, \mathrm{d}\mu \\ &= \frac{1}{n} \mu(E_n) \end{aligned}$$

we get that $\mu(E_n) = 0$ for all n then $\mu(E) = 0$. Therefore $f = 0$ almost everywhere. $\square$

Remark 3.3.7. $\|f\| = \int |f| \, \mathrm{d}\mu$ satisfies

- $\|f + g\| \leq \|f\| + \|g\|$
- $\|cf\| = |c|\|f\|$
- $\|f\| = 0 \iff f = 0$ almost everywhere

Remark 3.3.8. Almost everywhere equal is an equivalence relation.

$$f \sim g \underset{\text{def}}{\iff} f(x) = g(x) \ \mu\text{-almost everywhere}$$

$N = \{f \in \mathcal{L}^1 : f(x) = 0 \text{ almost everywhere}\}$ is a linear subspace of $\mathcal{L}^1$ vector. $\mathcal{L}^1/N$ is the set of equivalence classes of $\mathcal{L}^1$.

$f_n \to f$ almost everywhere, $f_n \geq 0$, f_n measurable, Can we define $\int f \, \mathrm{d}\mu$? f may not be measurable. This problem is fixed if f we work in a complete measurable space $(X, \mathcal{M}, \mu) \to (X, \overline{\mathcal{M}}, \overline{\mu})$ where

$$\overline{\mathcal{M}} = \{A \cup B : A \in \mathcal{M}, B \text{ a subset of a set of measure } 0\}$$

Lemma 3.3.9

$f \in \mathcal{L}^1$, $\int |f| \, \mathrm{d}\mu < \infty$. If f is real valued $f = f^+ - f^-$,

$$\left| \int f \, \mathrm{d}\mu \right| \leq \int |f| \, \mathrm{d}\mu$$

Proof.

$$\begin{aligned}
\left| \int f \, \mathrm{d}\mu \right| &= \left| \int f^+ - f^- \, \mathrm{d}\mu \right| \\
&\leq \left| \int f^+ \, \mathrm{d}\mu \right| + \left| \int f^- \, \mathrm{d}\mu \right| \\
&= \int f^+ \, \mathrm{d}\mu + \int f^- \, \mathrm{d}\mu \\
&= \int |f| \, \mathrm{d}\mu
\end{aligned}$$

□

Remark 3.3.10. If f is complex valued, then $|f| = \sqrt{(\Re f)^2 + (\Im f)^2}$. Then

$$\left| \int \Re f \right| \leq \int |\Re f| \leq \int |f|$$

$$\left|\int \Im f\right| \leq \int |\Im f| \leq \int |f|$$

So,

$$\left|\int f \, \mathrm{d}\mu\right| \leq 2\int |f|$$

Remark 3.3.11. Estimate $\int f \, \mathrm{d}\mu = \alpha + i\beta = re^{i\phi}$, then $e^{-i\phi}\int f \, \mathrm{d}\mu$ is real and nonnegative.

$$\begin{aligned}\left|\int f \, \mathrm{d}\mu\right| &= \left|e^{-i\phi}\int f \, \mathrm{d}\mu\right| \\ &= \Re\int e^{-i\phi} f \, \mathrm{d}\mu \\ &\leq \int |e^{-i\phi} f| \, \mathrm{d}\mu \\ &= \int |f| \, \mathrm{d}\mu\end{aligned}$$

Lemma 3.3.12

$f \in \mathcal{L}^+$ means non-negative, then $\nu(E) = \int_E f \, \mathrm{d}\mu$ define measure

Proof. Check the σ-additivity $E = \biguplus_{n=1}^{\infty} E_n$,

$$\begin{aligned}\nu\left(\biguplus_{n=1}^{\infty} E_n\right) &= \int_{\biguplus E_n} f \, \mathrm{d}\mu \\ &= \int f \mathbb{1}_{\biguplus E_n} \, \mathrm{d}\mu \\ &= \int f\left(\sum_{n=1}^{\infty} \mathbb{1}_{E_n}\right) \mathrm{d}\mu \\ &= \sum_{n=1}^{\infty}\int f\mathbb{1}_{E_n} \, \mathrm{d}\mu \\ &= \sum_{n=1}^{\infty} \nu(E_n)\end{aligned}$$

□

Claim 3.3.13 — If $f \in \mathcal{L}^1 \cap \mathcal{L}^+$ then ν is a finite measure.

If $\nu(E) = intf \, \mathrm{d}\mu$ How does $\int g \, \mathrm{d}\nu$ look like? $\nu(E) = \int f \, \mathrm{d}\mu = \int E \, \mathrm{d}\nu$ We want "$f \, \mathrm{d}\mu = \mathrm{d}\nu$"

Lemma 3.3.14

If $f \in \mathcal{L}^+$ and $\nu(E) = \int_E f \, \mathrm{d}\mu$ then for any $g \in \mathcal{L}^+$ or $g \in \mathcal{L}^1$ then,

$$\int g \, \mathrm{d}\nu = \int gf \, \mathrm{d}\mu$$

Proof.

- True for characteristic functions of measure set by the definition of ν. Fix $g = \mathbb{1}_E$ for some $E \in \mathcal{M}$

$$\int g \, \mathrm{d}\nu = \int \mathbb{1}_E \, \mathrm{d}\mu = \nu(E) = \int_E f \, \mathrm{d}\mu = \int \mathbb{1}_E f \, \mathrm{d}\mu = \int gf \, \mathrm{d}\mu$$

- By linearity of the integral, it is true for simple function. Fix $g = \sum_{j=1}^n c_j \mathbb{1}_{E_j}$, then

$$\int g \, \mathrm{d}\nu = \sum_{j=1}^n c_j \nu(E_j) = \sum_{j=1}^n c_j \int_{E_j} f \, \mathrm{d}\mu = \int gf \, \mathrm{d}\mu$$

- $s_n \nearrow g$ if $g \in \mathcal{L}^+$, by Monotone convergence theorem,

$$\int s_n \underset{\text{MCT}}{\nearrow} \int g$$

$$\int s_n \, \mathrm{d}\nu = \int s_n f \, \mathrm{d}\mu$$
$$\int g \, \mathrm{d}\nu = \int gf \, \mathrm{d}\mu$$

Then extend to general g by linearity

□

Theorem 3.3.15

If X is a finite measure space, if f_n measurable, $f_n \in \mathcal{L}^1$ (integrable) and $f_n \to f$ uniformly on X. then

$$\int |f_n - f| \, \mathrm{d}\mu \to 0$$

and

$$\int f_n \, \mathrm{d}\mu \to \int f \, \mathrm{d}\mu$$

Remark 3.3.16. Uniform convergence means

$$\lim_{n\to\infty} \sup_{x\in X} |f_n(x) - f(x)| = 0$$

Proof. We can rewrite that term as

$$\begin{aligned}\int |f_n - f| \, \mathrm{d}\mu &\leq \int \sup_{x\in X} |f_n - f| \, \mathrm{d}\mu \\ &= \mu(X) \sup_{x\in X} |f_n - f| \to 0\end{aligned}$$

We can rewrite f as $f = (f - f_n) + (f_n)$ since $f - f_n$ converge and f_n integrable so f must be integrable.

$$\begin{aligned}\left|\int f_n - \int f\right| &= \left|\int (f_n - f) \, \mathrm{d}\mu\right| \\ &\leq \int |f_n - f| \, \mathrm{d}\mu\end{aligned}$$

□

Definition 3.3.17. Suppose that f_n, f are measurable $f_n \to f$ almost uniformly if for every $\varepsilon > 0$ there is a measurable set E such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on $E^\complement$ ($\sup_{x\in E^\complement} |f_n(x) - f(x)| \to 0$)

Theorem 3.3.18 (Egorov's Theorem)

If $\mu(X) < \infty$ and if $f_n \to f$ almost everywhere then $f_n \to f$ almost uniformly

Remark 3.3.19. $f_n(x) \to f(x)$ if for every k there exists $n = n(k)$ such that $|f_m(x) - f(x)| < \frac{1}{k}$ for all $m \geq n(k)$

Proof. Fix $\varepsilon > 0$, define

$$\begin{aligned}E_n(k) &:= \left\{x : |f_m(x) - f(x)| \geq \frac{1}{k} \text{ for some } m \geq n\right\} \\ &= \bigcup_{m\geq n} \left\{x : |f_m(x) - f(x)| \geq \frac{1}{k}\right\}\end{aligned}$$

(Given x For sufficiently large n, $x \notin E_n(k)$), $E_n(k) \supseteq E_{n+1}(k)$ $\bigcap_n E_n(k) = \emptyset$ because of $f_n \to f$ everywhere. Form the continuity from above 2.4.5, we get that

$\lim_{n\to\infty} \mu(E_n(k)) = 0$. Find $n(k)$ such that $\mu(E_{n(k)}(k)) < \frac{\varepsilon}{2^k}$, then $E = \bigcup_k E_{n(k)}(k)$ has measure $< \varepsilon$.

For $x \in \left(\bigcup_k E_{n(k)}(k)\right)^\complement = \bigcap_k E_{n(k)}(k)^\complement$ I have for all k $|f_m(x) - f(x)| < \frac{1}{k}$ for all $m \geq n(k)$. So, we get $f_n \to f$ uniformly on $E^\complement$. $\square$

Theorem 3.3.20 (Baby Dominated Convergence Theorem)
Given $(X, \mathcal{M}, \mu)$ where μ is a finite measure ($\mu(X) < \infty$). Let $\{f_n\}$ be measurable functions, $f_n \to f$ everywhere.

$$|f_n| \leq C \implies \int |f_n - f| \, \mathrm{d}\mu \to 0$$

i.e. f_n converges with respect to L^1-(semi-)norm.

Corollary 3.3.21

$$\int_X f_n \, \mathrm{d}\mu \to \int_X f \, \mathrm{d}\mu$$

Proof. Tools:

(i) If $f_n \to f$ uniformly then $\int |f_n - f| \, \mathrm{d}\mu \to 0$

(ii) Egorov's Theorem

$|f(x)| \leq C$, f is measurable. Given any $\varepsilon > 0$, By Egorov's Theorem, find a set of measure E that $\mu(E) < \frac{\varepsilon}{4C}$ such that $f_n \to f$ uniformly on $E^\complement$. Then

$$\int |f_n - f| \, \mathrm{d}\mu \leq \int_E |f_n - f| \, \mathrm{d}\mu + \int_{E^\complement} |f_n - f| \, \mathrm{d}\mu$$

we know that $|f_n - f| \leq |f_n| + |f| \leq 2C$ then

$$\begin{aligned}\int |f_n - f| \, \mathrm{d}\mu &\leq 2C\mu(E) + \int_{E^\complement} |f_n - f| \, \mathrm{d}\mu \\ &\leq \frac{\varepsilon}{2} + \int_{E^\complement} |f_n - f| \, \mathrm{d}\mu\end{aligned}$$

so for large n, the second term will be $< \frac{\varepsilon}{2}$. $\square$

Theorem 3.3.22 (Dominated Convergence Theorem)

Given $(X, \mathcal{M}, \mu)$ where μ is a finite measure ($\mu(X) < \infty$). Let $\{f_n\}$ be measurable functions, $f_n \to f$ almost everywhere.

$$\sup_n |f_n| \in \mathcal{L}^1 \implies \int |f_n - f| \, \mathrm{d}\mu \to 0$$

Proof. Define $g(x) = \sup_n |f_n(x)|$ The trick is

$$|f_n - f| = \begin{cases} \frac{|f_n - f|}{g} g & \text{if } g > 0 \\ 0 & \text{if } g = 0 \end{cases}$$

define a new measure $\nu(E) = \int_E g \, \mathrm{d}\mu$. Then ν is a finite measure, and

$$g \, \mathrm{d}\mu = \mathrm{d}\nu$$
$$\int h \, \mathrm{d}\nu = \int hg \, \mathrm{d}\mu$$

then define

$$h_n = \begin{cases} \frac{|f_n - f|}{g} & \text{if } g > 0 \implies |h_n(x)| \le 2 \\ 0 & \text{if } g = 0 \implies h_n(x) \to 0 \end{cases}$$

Then

$$\begin{aligned} \int |f_n - f| \, \mathrm{d}\mu &= \int h_n g \, \mathrm{d}\mu \\ &= \int h_n \, \mathrm{d}\nu \to 0 \end{aligned}$$

By Baby Dominated Convergence Theorem □

§3.4 Integration from Riemann to Lebesgue

Theorem 3.4.1

If f is Riemann integrable on $[a, b]$ then f is Lebesgue integrable.

$$\int_a^b f(x) \, \mathrm{d}x = \int_{[a,b]} f \, \mathrm{d}\mu$$

where μ is Lebesgue measure.

Proof. Define

$$U_P f(x) = \begin{cases} M_j & \text{if } x \in [x_{j-1}, x_j) \\ M_n & \text{if } x \in [x_{n-1}, x_n] \end{cases}$$

Similarly for the lower sum $L_P f(x)$. If P' is a refinement of P then $U_{P'} f(x) \leq U_P f(x)$ and $L_{P'} f(x) \geq L_P f(x)$. Since f is Riemann integrable, then

$$\inf_P U(f, P) =: \overline{\mathcal{I}}_a^b(f) = \underline{\mathcal{I}}_a^b(f) := \sup_P L(f, P)$$

Choose a sequence of partitions P_n such that

$$\int U_{P_n} f \to \overline{\mathcal{I}}_a^b(f)$$
$$\int L_{P_n} f \to \underline{\mathcal{I}}_a^b(f)$$

Since that P_{n+1} is a refinement of the P_n then $U_{P_n} f \searrow U(x)$ and $L_{P_n} f \nearrow L(x)$ and $L(x) = U(x)$. Notice that from Riemann integrable, $|f| < C$, then

$$\int_{[a,b]} U_{P_n} f \to \overline{\mathcal{I}}_a^b(f) = \int_{[a,b]} U(x) \ \mathrm{d}m$$
$$\int_{[a,b]} L_{P_n} f \to \underline{\mathcal{I}}_a^b(f) = \int_{[a,b]} L(x) \ \mathrm{d}m$$

If f is Riemann integrable,

$$\int U \ \mathrm{d}m = \int L \ \mathrm{d}m = \int_a^b f(x) \ \mathrm{d}x$$

and $U \geq L$ then

$$\int (U - L) \ \mathrm{d}m = 0 \implies U(x) = L(x)$$

almose everywhere, $L(x) \leq f(x) \leq U(x) \implies f = L$ almost everywhere and $f = U$ almost everywhere. Then f is Lebesgue integrable and

$$\int f \ \mathrm{d}m = \int L \ \mathrm{d}m = \int U \ \mathrm{d}m$$

□

Definition 3.4.2 (Improper Riemann integrals)**.**

$$\int_0^\infty f(x) \ \mathrm{d}x, \int_1^\infty f(x) \ \mathrm{d}x, \int_0^1 f(x) \ \mathrm{d}x$$

if f is not Riemann-integrable on the domain but on every compact subinterval. We can define as

$$\int_1^\infty f(x) \ \mathrm{d}x = \lim_{R \to \infty} \int_1^R f(x) \ \mathrm{d}x$$

Example 3.4.3

$$\int_1^\infty \frac{\sin x}{x}\,\mathrm{d}x = \lim_{R\to\infty}\int_1^R \frac{\sin x}{x}\,\mathrm{d}x$$

$I_k = [2k\pi + \frac{\pi}{4}, 2k\pi + \frac{3\pi}{4}]$, $\sin x \geq \frac{1}{\sqrt{2}}$, so, $\frac{\sin x}{x} \geq \frac{1}{x\sqrt{2}} \cdot \frac{1}{2k\pi + \frac{3\pi}{4}}$. We can do integration by parts

$$\int_1^R \frac{\sin x}{x}\,\mathrm{d}x = -\frac{\cos x}{x}\Big|_1^R - \int_1^R -\frac{\cos x}{x^2}\,\mathrm{d}x$$

Example 3.4.4

$$\int_0^\infty \sin(x^2)\,\mathrm{d}x$$

consider $\sin(x^2)$

$$\sqrt{2k\pi + \frac{\pi}{2}} \leq \sqrt{x^2} \leq \sqrt{2k\pi + \frac{3\pi}{4}}$$

$$\sqrt{2k\pi + \frac{3\pi}{4}} - \sqrt{2k\pi + \frac{\pi}{4}} \approx \frac{1}{\sqrt{k}}$$

Lemma 3.4.5

Suppose that if $\int_1^\infty |f(x)|\,\mathrm{d}x < \infty$ then $f \in \mathcal{L}^1$.

Proof.

$$\begin{aligned}\int_1^\infty |f(x)|\,\mathrm{d}x &= \int_1^\infty \lim_{n\to\infty} |f(x)|\mathbb{1}_{[1,n]}(x)\,\mathrm{d}x \\ &= \lim_{n\to\infty}\int_1^n |f(x)|\,\mathrm{d}x\end{aligned}$$

□

Theorem 3.4.6

If f is integrable on $\mathbb{R}$,

$$\int_{\mathbb{R}} |f(x)| \, \mathrm{d}x < \infty$$

$f \in \mathcal{L}^1$ then for every $\varepsilon > 0$, there is a continuous function (C^∞) g, vanishes off a compact set,

$$\int |f - g| \, \mathrm{d}m < \varepsilon$$

§3.5 Introduction to Outer Measures

Definition 3.5.1. In our axiometic theorem on the Lebesgue measure, $m(I) = \ell(I)$, $m((a, b]) = b - a$ and for a general Borel set on $\mathbb{R}$, m is given by the **outer measure** induced by the collection of intervals

$$\varrho(E) = \inf \sum_{k=1}^{\infty} \ell(I_k)$$

where the inf is taken over collections $\{I_k\}$, such that $E \subseteq \bigcup_{k=1}^{\infty} I_k$

Remark 3.5.2. $\widetilde{\varrho}$ defined similarly but we only admit open intervals in the infimum. Obviously, $\widetilde{\varrho}(E) \geq \varrho(E)$ Need to show that $\widetilde{\varrho}(E) \leq \varrho(E)$ we may assume that $\varrho(E) < \infty$, show $\widetilde{\varrho}(E) \leq \varrho(E) + \varepsilon$. There is a collection of intervals I_k such that

$$\sum_k \ell(I_n) < \varrho(E) + \frac{\varepsilon}{2}$$

If $I_k = [a_k, b_k]$, then define $J_k = (a_k - \frac{\varepsilon}{2^{k+2}}, b_k + \frac{\varepsilon}{2^{k+2}})$ Then $\ell(J_k) = \ell(I_k) + \frac{\varepsilon}{2^{k+1}}$ then

$$\begin{aligned}
\widetilde{\varrho}(E) \leq \sum_{k=1}^{\infty} \ell(J_k) &\leq \sum_{k=1}^{\infty} \ell(J_k) + \varepsilon 2^{-k-1} \\
&\leq \varrho(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
\end{aligned}$$

Lemma 3.5.3

$m(E) = \sup\{m(K) : K \subseteq E, K \text{ compact}\}$

Proof. Case where $E = \overline{E}$ and E is bounded, there is nothing to show

Assume that E is bounded, GOAL: find $K \subseteq E$ such that $m(E \setminus K) < \varepsilon$. Consider $\overline{E} \setminus E$, find $O \supseteq \overline{E} \setminus E$, $m(O \setminus (\overline{E} \setminus E)) < \varepsilon$. then $O^{\complement} \cap \overline{E} \subseteq E$ because if $x \in O^{\complement}$, either $x \in \overline{E}$ or

$x \in E$. $E \setminus K = E \cap K^{\complement} \subseteq O \cup \overline{E}^{\complement}$. Since $E \subseteq \overline{E}$ then $E \setminus K \subseteq O$ and $E \setminus K \subseteq O \setminus (\overline{E} \setminus E)$ has measure $< \varepsilon$. □

Theorem 3.5.4

For every Borel set E, $m(E) < \infty$, there is an open set $O \supseteq E$ such that $m(O \setminus E) < \varepsilon$. where $m(E) = \inf \sum \ell(I_n)$ where inf take over I_k, I_k are open, $E \subseteq \bigcup I_k$

Proof. Define $E_n = E \cap \overline{B}(0, n)$. Find compact set $K_n \subseteq E_n \setminus E_{n-1}$ then $m((E_n \setminus E_{n-1}) \setminus K_n) < \varepsilon 2^{-n-1}$. The set $H_l = K_1 \cup \cdots \cup K_l$ is compact and increasing, $H_l \subseteq E_l$ and

$$m(E_l) - \varepsilon \leq m(H_l) \leq m(E_l) \to m(E)$$

□

Theorem 3.5.5

Given an open set O, we can decompose O as a disjoint union of "dyadic cubes"

Theorem 3.5.6

We can choose the cubes a dyadic cubes such that if $O \neq \mathbb{R}^n$ such that

$$\operatorname{diam}(Q) < \operatorname{dist}(Q, O^C) \leq 4\operatorname{diam}(Q)$$

Remark 3.5.7. If side length of Q is 2^{-k} then the diameter is $\sqrt{n}2^{-k}$.

Theorem 3.5.8 (Whitney decomposition theorem)

Given Ω open set in $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, there is a family $\mathcal{F}$ of dyadic cubes such that

- they are disjoint
- $\biguplus_{Q \in \mathcal{F}} Q = \Omega$
- For every $Q \in \mathcal{F}$, $C\operatorname{diam}(Q) < \operatorname{dist}(Q, \Omega^C) \leq (2C + 2)\operatorname{diam}(Q)$

Proof. Define for each k a family of dyadic cubes $\mathcal{F}_k$ of side length 2^{-k} (i.e., the diameter is $\sqrt{n}2^{-k}$) intersecting the region

$$\Omega_k = \{x : A2^{-k}\sqrt{n} \leq \operatorname{dist}(x, \Omega^{\complement}) \leq 2A \cdot 2^{-k}\sqrt{n}\}$$

Pick a cube in $\mathcal{F}_k$, Q_1 it contains an $x_Q \in \Omega_k$

$$\operatorname{dist}(Q, \Omega^\complement) \le \operatorname{dist}(x_Q, \Omega^\complement) \le 2A \cdot 2^{-k}\sqrt{n} - 2A\operatorname{diam}(Q)$$

$$\begin{aligned}\operatorname{dist}(Q, \Omega^\complement) &\ge \operatorname{dist}(x_Q, \Omega^\complement) - \operatorname{diam}(Q)\\ &\ge A2^{-k}\sqrt{n} - 2A\operatorname{diam}(Q)\\ &= (A-1)\cdot \operatorname{diam}(Q)\end{aligned}$$

Then $\mathcal{F}_T = \bigcup \mathcal{F}_k$ and finally $\mathcal{F}$ = collection of all maximal (with respect to inclusion) cubes in $\mathcal{F}_T$. Fix Q, Q' and assume that $Q \subseteq Q'$ then

$$(A-1)\operatorname{diam}Q' \le \operatorname{dist}(Q', \Omega^\complement) \le \operatorname{dist}(Q, \Omega^\complement) \le 2A \cdot \operatorname{diam}(Q)$$

then $\operatorname{diam}(Q') \le \frac{2A}{A-1}\operatorname{diam}(Q)$ $\square$

we know that for every $\varepsilon_1 > 0$ we can find a simple function s such that $\int |f - s| \; \mathrm{d}m < \varepsilon_1$. (For non-negative f use MCT $s_n \nearrow f$ and $s_n \le f$ so $\int s_n \nearrow \int f \implies \int f - s_n \to 0$ and then we use $f = f_+ - f_-$)

$s = \sum c_j \mathbb{1}_{E_j}$, $E_j \subseteq O_j$, $m(O_j \setminus E_j) < \varepsilon$ $\widetilde{s} = \sum c_j \mathbb{1}_{O_j}$

$$\begin{aligned}\int \widetilde{s} - s \; \mathrm{d}m &= \left|\int \sum c_j \mathbb{1}_{O_j} - \mathbb{1}_{E_j} \; \mathrm{d}m\right|\\ &\le \sum_{j=1}^{N} |c_j||m(O_j \setminus E_j)|\\ &\le \varepsilon_2\end{aligned}$$

Then $\mathbb{1}_{O_j} = \sum_\nu \mathbb{1}_{Q_\nu}$ where $\{Q_\nu\}$ are the Whitney cubes in Whitney Theorem. $|O_j| = \sum_{\nu \in I} m(Q_\nu)$ There is a finite $\widetilde{I}_j$ such that

$$\int \left|\mathbb{1}_{O_j} - \sum_{\nu \in \widetilde{I}_j} \mathbb{1}_{Q_\nu}\right| < \varepsilon_3$$

replace $\sum_{j=1}^{N} c_j \mathbb{1}_{O_j}$ by $\sum_{j=1}^{N} c_j \mathbb{1}_{\bigcup_{\nu \in \widetilde{I}_j} Q_\nu}$

then

$$\mathbb{1}_{Q_\nu}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \mathbb{I}_{\nu, \mathbb{i}}(x_i)$$

Lemma 3.5.9

For any $f \in L^1$ there exists s a step function such that $\int |f - s| \, \mathrm{d}m < \varepsilon$

Proof. Suppose that $f \in L^1$ I want to show that there exists s a step function such that $\int |f - s| \, \mathrm{d}m < \varepsilon$ for any $\varepsilon > 0$. Since $f = f^+ - f^-$, WLOG, $f \geq 0$ (otherwise we can do each positive and negative part and do the sum of both step functions with $\frac{\varepsilon}{2}$ bound). Given any $\varepsilon > 0$, there exists s' a simple function such that $\int |f - s'| \, \mathrm{d}m < \frac{\varepsilon}{2}$. Then we can write $s' = \sum_{j=1}^{N} c_j \mathbb{1}_{E_j}$, then there exists O_j open set such that $E_j \subseteq O_j$ and $m(O_j \setminus E_j) < \frac{\varepsilon}{4|c_j|N}$. Since O_j is an open set, then $O_j = \bigcup (a_i, b_i)$ then define $K_n = \bigcup_{i=1}^{n} (a_i, b_i)$ from continuity from below, there exists n' such that $m(K_{n'}) > m(O_j) - \frac{\varepsilon}{4|c_j|N}$ and $K_{n'}$ contain finite interval, then we define $O'_j := K_{n'}$. Define $s = \sum_{j=1}^{N} c_j \mathbb{1}_{O'_j}$ then

$$\begin{aligned}
\int |f - s| \, \mathrm{d}m &\leq \int |f - s'| \, \mathrm{d}m + \int |s' - s| \, \mathrm{d}m \\
&\leq \frac{\varepsilon}{2} + \int \left| \sum_{j=1}^{N} c_j (\mathbb{1}_{E_j} - \mathbb{1}_{O'_j}) \right| \, \mathrm{d}m \\
&\leq \frac{\varepsilon}{2} + \sum_{j=1}^{N} |c_j| \int \left| \mathbb{1}_{E_j} - \mathbb{1}_{O'_j} \right| \, \mathrm{d}m \\
&= \frac{\varepsilon}{2} + \sum_{j=1}^{N} |c_j| (m(E_j \setminus O'_j) + m(O'_j \setminus E_j)) \\
&\leq \frac{\varepsilon}{2} + \sum_{j=1}^{N} |c_j| (m(O_j \setminus O'_j) + m(O_j \setminus E_j)) \\
&\leq \frac{\varepsilon}{2} + \sum_{j=1}^{N} |c_j| \left(\frac{\varepsilon}{4|c_j|N} + \frac{\varepsilon}{4|c_j|N} \right) \\
&< \varepsilon
\end{aligned}$$

$\square$

4 L^p Spaces

§4.1 normed spaces

Remark 4.1.1. If $f_n \to f$ almost everywhere, do we have $\int |f_n - f| \, d\mu \to 0$?

- No, if $f_n = \mathbb{1}_{[n,n+1]}$ then $f_n \to 0$ almost everywhere but $\int |f_n - 0| \, d\mu = 1$ and $\int |f_n - f_m| \, d\mu = 2$.
- No, if $f_n = \mathbb{1}_{[0,\frac{1}{n}]}$

Remark 4.1.2. Convergence in L^1 implies convergence almost everywhere? No, If $2^k \le n \le 2^{k+1}$ where $n = 2^k + j$, $j = 0, \dots, 2^k - 1$ $f_{2^k+1} = \mathbb{1}_{[i2^{-k},(i+1)2^{-k}]}$ for $i = 0, \dots, 2^k - 1$. For $2^k \le n \le 2^{k+1}$, $\|f_n\|_{L^1} = 2^{-k}$

Claim 4.1.3 — If $f_n \to f$ in L^1 ($\int |f_n - f| \, d\mu \to 0$) then there is a subsequence $f_{n_k} \to f$ almost everywhere.

Proof. Consider the normed space L^1 space of semi-nromed space $\mathcal{L}^1$. (define as a equivalence class of almost everywhere where $f \underset{\text{a.e.}}{\sim} g$ if $f = g$ almost everywhere) Construct a convergence subsequence (a.e. and also in Norm) Choose $\varepsilon = \frac{1}{2^k}$ there exists number $N(k)$ such that $\|f_l - f_m\| < \frac{1}{2^k}$ for $l, m \ge N(k)$ for $l, m \ge N(k)$ then $\|f_{N(k)} - f_{N(k+1)}\| \le \frac{1}{2^k}\|$ Define

$$G(x) = |f_{N(1)}(x)| + \sum_{k=1}^{\infty} |f_{N(k+1)}(x) - f_{N(k)}(x)|$$

then

$$\int G(x) \, d\mu = \int |f_{N(1)}(x)| \, d\mu + \sum_{k=1}^{\infty} \int |f_{N(k+1)}(x) - f_{N(k)}(x)| \, d\mu \quad \le \|f_{N(1)}\|_1 + \sum_{k=1}^{\infty} \frac{1}{2^k}$$

So, G is integrable, $\int |G(x)| \, d\mu < \infty$ then $G(x) < \infty$ almost everywhere. We se that for almost everywhere,

$$f_{N_1}(x) + \sum_{k=1}^{\infty} f_{N(k+1)} - f_{N(k)}(x)$$

converges for almost everywhere x, define

$$s_M(x) = f_{N(1)}(x) + f_{N(2)}(x) - f_{N(1)}(x) + \ldots + f_{N(M+1)}(x) - f_{N(M)}(x) = f_{N(M+1)}(x)$$

then $s_{M-1}(x) = f_{N(M)}(x)$ and as $M \to \infty$, this is converges for almost everywhere x.

$f(x) = \lim_{M\to\infty} f_{N(M)}$

$$f(x) = f_{N_1}(x) + \sum_{M=1}^{\infty} f_{N(M+1)}(x) - f_{N(M)}(x)$$

$$\begin{aligned}
\int |f(x) - f_{N_1}(x)| \, \mathrm{d}\mu &= \int \sum_{M=1}^{\infty} f_{N(M+1)}(x) - f_{N(M)}(x) \, \mathrm{d}\mu \\
&\leq \int |f_{N(M+1)}(x) - f_{N(M)}(x)| + \int |f_{N(M+2)} - f_{N(M+1)}| + \dots \\
&= \sum_{k=M}^{\infty} \int |f_{N(k+1)}(x) - f_{N(k)}(x)| \, \mathrm{d}\mu \\
&\leq 2^{1-M}
\end{aligned}$$

This shows convergence of $f_{N(M)} \to f$ in L^1. What happens with $l \geq N(k)$,

$$\|f_l - f\| \leq \|f_l - f_{N(k)}\| + \|f_{N(k)} - f\| \leq \frac{1}{2^k}, \to 0$$

□

L^1 or $(\mathcal{L}^1)$ are complete, in the sense that every Cauchy sequence converges. $\{f_n\}$ cxauchy, For every $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for $l, m \geq N(\varepsilon)$ then $\|f_l - f_m\| < \varepsilon$

§4.2 Functional space

Definition 4.2.1.

$$\|f\|_p = \left(\int |f|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}}$$

where L^P is space of equivalence class and $\mathcal{L}^p$ is space of functions, $f \in \mathcal{L}^p$ if $\|f\|_p < \infty$

Theorem 4.2.2

$\|f\|_p$ is a norm on L^p, if $p \geq 1$ (not a nrom if $p < 1$ because triangle inquality fails)

Proof. for any $f, g \in L^p$,

$$\begin{aligned}
\int |f+g|^p \, \mathrm{d}\mu &\leq \int (2 \max |f|, |g|)^p \, \mathrm{d}\mu \\
&= 2 \left(\int \max |f|^p, |g|^p\right) \, \mathrm{d}\mu \\
&\leq 2 \int |f|^p + |g|^p \, \mathrm{d}\mu
\end{aligned}$$

□

Remark 4.2.3. $\|f+g\|_p \leq 2^{\frac{1}{p}}(\|f\|_p + \|g\|_p)$

Theorem 4.2.4

For $p \leq 1$ we have inequality

$$\|f+g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$$

Proof. we claim that

$$\int |f+g|^p \, \mathrm{d}\mu \leq \int |f|^p \, \mathrm{d}\mu + \int |g|^p \, \mathrm{d}\mu$$

for $a, b \in [0, \infty)$, $(a+b)^p \leq a^p + b^p$ WLOG $b \leq a$, $f(x) = 1 + x^p - (1-x)^p$, $f'(x) \geq 0 \implies (1+x)^p \leq 1 + x^p$ for $0 < p \leq 1$. set $x = b/a$ Then

$$(a+b)^p \leq a^p + b^p$$

then setting $a = |f(x)|, b = |g(x)|$ and we are done. □

Remark 4.2.5. For $p < 1$ we do not get$\|f+g\|_p \leq \|f\|_p + \|g\|_p$ for $x, y \in \mathbb{R}^2$ want to disprove $\|x+y\|_p \leq \|x\|_p + \|y\|_p, p < 1$

$$2^{\frac{1}{p}} = (1^p + 1^p)^{\frac{1}{p}}$$

(it is because failure of convexity of the norm $p < 1$)

Claim 4.2.6 — For $0 < \theta < 1$, $a, b \geq 0$, then $a^{1-\theta} b^\theta \leq (1-\theta)a + \theta b$

Proof. Generalized AM-GM inequality ($\sqrt{ab} \leq \frac{a+b}{2}$) then put for $0 < \theta < 1$ then $a^{1-\theta} b^\theta \leq (1-\theta)a + \theta b$ WLOG $b \leq a$ then

$$\left(\frac{b}{a}\right)^\theta \leq 1 - \theta + \theta \frac{b}{a}$$

let $x = \frac{b}{a}$ for $0 \leq x \leq 1$ we need to show that $g(x) = 1 - \theta + \theta x - x^\theta \geq 0$ then $g'(x) = -1 + \theta - \theta x^{\theta - 1} \leq 0$ (because $0 \leq \theta \leq 1$) □

Claim 4.2.7 (Holder's inequality) — Given $p > 1$, q to be such that

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \left(q = \frac{p}{p-1}\right)$$

for $f \in L^p, g \in L^q$, then $fg \in L^1$ and

$$\int |fg| \, \mathrm{d}\mu \leq \|f\|_p \|g\|_q$$

Proof. Rewrite AM-GM (generalized) as "Young's inquality" substitute $a = u^p, 1 - \theta = \frac{1}{p}, b = v^q, \theta = \frac{1}{q}$ then we get

$$uv \leq \frac{1}{p} u^p + \frac{1}{q} v^q$$

apply $f(x)g(x)$

$$\int |f(x)||g(x)| \, \mathrm{d}\mu \leq \int \frac{|f(x)|^p}{p} \, \mathrm{d}\mu + \int \frac{|g(x)|^q}{q} \, \mathrm{d}\mu = \frac{\|f\|_p^p}{p} + \frac{\|g\|_q^q}{q}$$

(This is Holder when two norms are normalized $\|f\|_p = 1 = \|g\|_q$)

Then $\frac{f(x)}{\|f\|_p}$ has "p-norm" equal to 1 because

$$\left(\int \left|\frac{f(x)}{\|f\|_p}\right|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} = \frac{1}{\|f\|_p} \left(\int |f(x)|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}}$$

Substitute $f = \frac{f}{\|f\|_p}$ and $g = \frac{g}{\|g\|_q}$ then we get

$$\int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \, \mathrm{d}\mu \leq \frac{1}{p} + \frac{1}{q} = 1$$
$$\int |fg| \, \mathrm{d}\mu \leq \|f\|_p \|g\|_q$$

□

Theorem 4.2.8 (Minkowski's inquality)
$p \geq 1$ We do have a triangle inquality $\|f + g\|_p \leq \|f\|_p + \|g\|_p$

$$\left(\int |f+g|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} \leq \left(\int |f|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int |g|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}}$$

Proof. It is enough to show that

$$\|f+g\|_p^p \leq (\|f\|_p + \|g\|_p)\|f+g\|_p^{p-1}$$

$$\begin{aligned}\int |f+g|^{p-1+1}\,\mathrm{d}\mu &= \int |f+g|^{p-1}|f|\,\mathrm{d}\mu + \int |f+g|^{p-1}|g|\,\mathrm{d}\mu \\ &\leq \left(\int |f|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int |f+g|^p\,\mathrm{d}\mu\right)^{\frac{p-1}{p}} + \left(\int |g|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int |f+g|^p\,\mathrm{d}\mu\right)^{\frac{p-1}{p}} \\ &= (\|f\|_p + \|g\|_p)\|f+g\|_p^{p-1}\end{aligned}$$

□

Remark 4.2.9. Holder's inequality holds

$$\int fg\,\mathrm{d}\mu \leq \|f\|_p\|g\|_q$$

where $\frac{1}{p} + \frac{1}{q} = 1$ can be generalized to several factors

$$\int f_1 f_2 \cdots f_n\,\mathrm{d}\mu \leq \prod_{j=1}^n \|f_j\|_{p_j}$$

where $\sum_{j=1}^n \frac{1}{p_j} = 1$

§4.3 Application

Lemma 4.3.1 (Shebyshev's inequality)
This is an inequality for the distribution function (given a measure space $(X, \mathcal{M}, \mu)$) $\mu_f(\alpha) = \mu(\{x : |f(x)| \geq \alpha\}), E_\alpha = \{x : |f(x)| \geq \alpha\}$ The Shebyshev's inequality is

$$\mu_f(\alpha) \leq \frac{\|f\|_p^p}{\alpha^p}$$

Proof. for $x \in E_\alpha$,

$$1 \leq \frac{|f(x)|^p}{\alpha^p}$$

$$\begin{aligned}\mu_f(\alpha) &= \int_{E_\alpha} \mathbb{1} \,\mathrm{d}\mu \\ &\leq \int_{E_\alpha} \frac{|f(x)|^p}{\alpha^p} \,\mathrm{d}\mu \\ &\leq \frac{1}{\alpha^p} \int |f(x)|^p \,\mathrm{d}\mu\end{aligned}$$

(Probabilitiy may call this Markov's inequality) □

$\mu_{cf}(\alpha) = \mu(\{x : |cf(x)| \geq \alpha\}) = \mu(\{x : |f(x)| \geq \frac{\alpha}{|c|}\})$

$\alpha\mu_{cf}(\alpha)^{\frac{1}{p}} = |c|\frac{\alpha}{|c|}\mu_f\left(\frac{\alpha}{|c|}\right)^{\frac{1}{p}}$

Claim 4.3.2 — If $\delta_0 + \delta_1 = 1$, $\delta_0, \delta_1 \geq 0$, then

$$E_\alpha(f+g) \subseteq E_{\alpha\delta_0}(f) \cup E_{\alpha\delta_1}(f)$$

Theorem 4.3.3

If $\mu(X) < \infty$ then $L^q \subseteq L^p$ for $p \leq q$

Proof. We need an inequality $\|f\|_p \leq C\|f\|_q$

$$\left(\int 1|f(x)|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \leq C\left(\int |f(x)|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}}$$

Apply Holder with exponent $\frac{q}{p} > 1$, $\left(\frac{q}{p}\right)'$ where

$$\frac{1}{\left(\frac{q}{p}\right)} + \frac{1}{\left(\frac{q}{p}\right)'} = 1$$

$$\begin{aligned}\int |f|^p \cdot 1 \,\mathrm{d}\mu &\leq \left(\int (|f|^p)^{\frac{q}{p}} \,\mathrm{d}\mu\right)^{\frac{p}{q}} \left(\int_X 1^{\left(\frac{q}{p}\right)'} \,\mathrm{d}\mu\right)^{\frac{1}{\left(\frac{q}{p}\right)'}} \\ &= \|f\|_q^p \mu(X)^{\frac{1}{\left(\frac{q}{p}\right)'}} \\ &= \left[\|f\|_q \mu(X)^{\frac{1}{\left(\frac{q}{p}\right)'}}\right]^p\end{aligned}$$

□

Another extreme case would be $\mathbb{N}$ with counting measure In this $L^p(\mathbb{N}, \mu)$ is denoted by $\ell^p(\mathbb{N})$

Theorem 4.3.4

For $p \geq 1$, $\ell^p \subseteq \ell^q$ for $p \leq q$

Proof. We wan to prove $\|f\|_{\ell^q} \leq C\|f\|_{\ell^p}$ If $\|f\|_{\ell^p} < 1$, this means

$$\sum_{n=1}^{\infty} |f(n)|^p \leq 1$$

$\implies$ $|f(n)|^p \leq 1$ for all n

$$\sum_{n=1}^{\infty} |f(n)|^q \leq \sum_{n=1}^{\infty} |f(n)|^p$$

provided that $|f(n)|^q \leq |f(n)|^p$

For $f \in \ell^p$, $\frac{f}{\|f\|_p}$ has ℓ^p norm equal to 1 therefore $\left\|\frac{f}{\|f\|_p}\right\|_q \leq 1$ then $\|f\|_q \leq \|f\|_p$ $\square$

Theorem 4.3.5 (Littlewood Theorem)

Every measurable function is nearly continuous. i.e., $f \in L^1(A)$ there exists g continuous such that

$$\int |f(x) - g(x)| \, \mathrm{d}m < \varepsilon$$

Theorem 4.3.6 (Lusin's Theorem)

$f : [a, b] \to \mathbb{C}$ (or $\mathbb{R}$) almost everywhere, then there is a compact set $K \subseteq [a, b]$ such that $f|_K$ is continuous and $\mu([a, b] \setminus K) = 0$

Example 4.3.7

$f(x) = \mathbb{1}_{\mathbb{Q}'}(x)$ let $\{r_k\}$ be enumeration of rational number in $[a, b]$. Define

$$O = \bigcup_{k=1}^{\infty} \left(r_k - \frac{\varepsilon}{2^{k+2}}, r_k + \frac{\varepsilon}{2^{k+2}}\right)$$

then $m(O) < \varepsilon$, $K = [a, b] \setminus O$, $f|_K = 1$ is continuous.

Example 4.3.8

$$\sum_{k=1}^{\infty} \frac{1}{|x - r_k|^{10}} 2^{-k} \mathbb{1}_{[a,b]}$$

(this function is in L^p, if $p < \frac{1}{10}$). The Challenging part is to find a K as in Lusin's theorem such that $f = f|_K$ is continuous.

Proof. $f : A \to \mathbb{R}$ (or $\mathbb{C}$), $m(A) < \infty$ The for every $\varepsilon > 0$, there is a compact set $K \subseteq A$, such that $m(A \setminus K) < \varepsilon$ and $f|_K$ is continuous.

1. We can find a set E_1 such that $m(A \setminus E_1) < \frac{\varepsilon}{3}$ and $f|_{E_1}$ is bounded.

   $$S_\alpha = \{x \in A : |f(x)| > \alpha\}$$

   Then $\bigcup S_\alpha = \bigcup S_{2^M}$ has measure zero by the assumption. From continuity from above, we get

   $$m(S_\alpha) = m\left(\bigcup S_{2^M}\right) = \lim_{M\to\infty} m(S_{2^M})$$

   because $m(A) < \infty$. For large M, $m(S_M) < \frac{\varepsilon}{3}$, let $A \setminus E_1 = S_{2^M}$ (M large)

2. We know that f is bounded on E_1 Can find a sequence g_n of continuous functions $g_n \to f$ almost everywhere on E_1, $m(E_1) < \infty$.

3. Using Egorov's theorem: $g_n \to f$ almost uniformly on E_1. Find $E_2 \subseteq E_1$ such that $m(E_1 \setminus E_2) < \frac{\varepsilon}{3}$ and $g_n \to f$ uniformly on E_2. $f|_{E_2}$ is continuous on E_2, we can find $E_3 = K$ compact, $E_3 \subseteq E_2$, $m(E_2 \setminus E_3) < \frac{\varepsilon}{3}$ and $f|_K$ is continuous.

   $$m(A \setminus K) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$\square$

5 Construction of Measures

§5.1 Abstract Outer Measure

Definition 5.1.1. Given X, $\mathcal{E} \subseteq \mathfrak{P}(X)$ where $\emptyset \in \mathcal{E}$ and $X \in \mathcal{E}$, $u : \mathcal{E} \to [0, \infty]$. For any set

$$u^*(E) = \inf_{\bigcup E_j \supseteq E} \sum_{j=1}^{\infty} u(E_j)$$

where $E_j \in \mathcal{E}$, called the (concrete) outer measure induced by $\mathcal{E}$. ($u^*(E) = \infty$ if we cannot cover E with a countable collection of sets in $\mathcal{E}$).

> **Remark 5.1.2.** It is not necessary to assume that $X \in \mathcal{E}$. In this case, $u^*(E) = \infty$ if we cannot cover E with a countable collection of sets in $\mathcal{E}$.

Lemma 5.1.3

An outer measure u^* incuced by the collection $\mathcal{E}$ satisfies

(i) $u^*(\emptyset) = 0$

(ii) If $A \subseteq B$ then $u^*(A) \leq u^*(B)$

(iii) $u^*(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} u^*(E_j)$

E.g., A (concrete) outer measure is an abstract outer measure

Proof. $u^*(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} u^*(E_j)$. WLOG, LHS $< \infty$, for each j find a collection $\{E_k^j\}_{k \in \mathbb{N}}$ such that

$$\sum u(E_k^j) < u^*(A_j) + \varepsilon 2^{-j-1}$$

$\{E_k^j\}_{j,k=1}^{\infty}$ covers $\bigcup A_j$

$$u^*\left(\bigcup A_j\right) \leq \sum_j \sum_k u(E_k^j) \leq \sum_j u^*(A_j) + \varepsilon \sum_j 2^{-j-1}$$

□

Definition 5.1.4 (Abstract Outer Measure)**.** A set function $\varrho : \mathfrak{P}(X) \to [0, \infty]$ is an abstract outer measure if

(i) $\varrho(\emptyset) = 0$

(ii) Monotonicity: $A \subseteq B \implies \varrho(A) \leq \varrho(B)$

(iii) Subadditivity: $\varrho\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \varrho(E_j)$

§5.2 Caratheodory's Construction

Definition 5.2.1. Given ϱ abstract outer measure, a set A is Caratheodory measure or ϱ-measurable (in the sence of) if **for all** $E \in \mathfrak{P}(X)$,

$$\varrho(E) = \varrho(E \cap A) + \varrho(E \cap A^{\complement})$$

Remark 5.2.2. It is trivial that

$$\varrho(E) \leq \varrho(E \cap A) + \varrho(E \cap A^{\complement})$$

By subaddictivity, so we only need

$$\varrho(E) \geq \varrho(E \cap A) + \varrho(E \cap A^{\complement})$$

to show the equality of the outer measure.

Theorem 5.2.3 (Caratheodory's Theorem)
The collection of ϱ-measurable sets is a σ-algebra $\mathcal{M}$,

$$\mathcal{M} = \{A \subseteq X : \forall E \subseteq X, \varrho(E) = \varrho(E \cap A) + \varrho(E \cap A^{\complement})\}$$

$\varrho|_{\mathcal{M}}$ is a measure (in fact, a complete measure)

Proof. First, I will show that $\mathcal{M}$ is an algebra.

(i) $\varrho(E) = \varrho(E \cap X) + \varrho(E \cap X^{\complement})$, so $X \in \mathcal{M}$.

(ii) If $A \in \mathcal{M}$, then for any $E \in \mathfrak{P}(X)$,

$$\begin{aligned}\varrho(E) &= \varrho(E \cap A) + \varrho(E \cap A^{\complement}) \\ &= \varrho(E \cap A^{\complement}) + \varrho(E \cap (A^{\complement})^{\complement})\end{aligned}$$

so $A^{\complement} \in \mathcal{M}$.

(iii) For $A, B \in \mathcal{M}$, it is enough to show that $A \cup B \in \mathcal{M}$. For any $E \in \mathfrak{P}(X)$,

$$\begin{aligned}\varrho(E) &= \varrho(E \cap A) + \varrho(E \cap A^{\complement}) \\ &= \varrho(E \cap A \cap B) + \varrho(E \cap A \cap B^{\complement}) + \varrho(E \cap A^{\complement} \cap B) + \varrho(E \cap A^{\complement} \cap B^{\complement})\end{aligned}$$

We know that $A \cup B = (A \cap B) \cup (A \cap B^{\complement}) \cup (A^{\complement} \cap B)$, so from the subaddictivity of ϱ, we get that

$$\varrho(E \cap (A \cup B)) \leq \varrho(E \cap (A \cap B)) + \varrho(E \cap (A \cap B^{\complement})) + \varrho(E \cap (A^{\complement} \cap B))$$

hence,

$$\begin{aligned}\varrho(E) &\geq \varrho(E \cap (A \cup B)) + \varrho(E \cap A^{\complement} \cap B^{\complement}) \\ &= \varrho(E \cap (A \cup B)) + \varrho(E \cap (A \cup B)^{\complement})\end{aligned}$$

Now, we can conclude that $\mathcal{M}$ is an algebra and we know that for any $A, B \in \mathcal{M}$, if $A \cap B = \emptyset$ then

$$\varrho(A \uplus B) = \varrho((A \uplus B) \cap A) + \varrho((A \uplus B) \cap A^{\complement}) = \varrho(A) + \varrho(B)$$

so ϱ is finite additive on $\mathcal{M}$.

To show that $\mathcal{M}$ is a σ-algebra, it is enough to show that $\mathcal{M}$ is closed under countable disjoint unions. Considering $\{A_j\}_{j=1}^{\infty}$ pairwise disjoint, $A_j \in \mathcal{M}$, define $B_n = \biguplus_{j=1}^{n} A_j$ and $B = \biguplus_{j=1}^{\infty} A_j$. For any $E \in \mathfrak{P}(X)$,

$$\varrho(E \cap B_n) = \varrho(E \cap B_n \cap A_n) + \varrho(E \cap B_n \cap A_n^{\complement}) = \varrho(E \cap A_n) + \varrho(E \cap B_{n-1})$$

then from iteration, we get that

$$\varrho(E \cap B_n) = \sum_{j=1}^{n} \varrho(E \cap A_j)$$

and

$$\varrho(E) = \varrho(E \cap B_n) + \varrho(E \cap B_n^{\complement}) \geq \sum_{j=1}^{n} \varrho(E \cap A_j) + \varrho(E \cap B^{\complement})$$

then $n \to \infty$ gives that

$$\begin{aligned}\varrho(E) &\geq \sum_{j=1}^{\infty} \varrho(E \cap A_j) + \varrho(E \cap B^{\complement}) \geq \varrho\left(\biguplus_{j=1}^{\infty} E \cap A_j\right) + \varrho(E \cap B^{\complement}) \\ &= \varrho(E \cap B) + \varrho(E \cap B^{\complement}) \geq \varrho(E)\end{aligned}$$

Therefore, $B \in \mathcal{M}$ and finally we get that

$$\varrho\left(\biguplus_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \varrho(A_j)$$

Now, we can conclude that $\mu := \varrho|_{\mathcal{M}}$ is a measure on $\mathcal{M}$. then we need to show that μ is complete. Let $A \in \mathcal{M}$ such that $\mu(A) = 0$. For any $E \in \mathfrak{P}(X)$, we have that

$$\varrho(E) \leq \varrho(E \cap A) + \varrho(E \cap A^{\complement}) = \varrho(E \cap A^{\complement}) \leq \varrho(E)$$

Therefore, μ is complete. □

Definition 5.2.4. ϱ is complete if for $A \in \mathcal{M}$, if $\varrho(A) = 0$ then $\varrho(N) = 0$ for $N \subseteq A$.

Example 5.2.5

For any X there exists a trivial σ-algebra $\mathcal{M} = \{\emptyset, X\}$ satisfies ϱ-measurable sets. Fix $c > 0$, define

$$\varrho(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ c & \text{if } E \neq \emptyset \end{cases}$$

§5.3 Rings and semirings

"Intervals" or "I-cells" half open intervals of the form $(a, b]$

$$(a, b] \uplus (b, c] = (a, c]$$

n-dimensional analogy n-cells

$$(a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n]$$

Definition 5.3.1 (Semiring)**.** Given X, a collection of subsets $\mathcal{S}$ of X is a semiring if

(i) $\emptyset \in \mathcal{S}$

(ii) $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$

(iii) $A, B \in \mathcal{S} \implies A \setminus B = \biguplus_{j=1}^{n} C_j$ where $C_j \in \mathcal{S}$

Example 5.3.2

$\mathcal{S} = \{(a, b] \mid a, b \in \mathbb{R}\}$ is a semiring.

Definition 5.3.3 (Ring)**.** A collection of subsets $\mathcal{R}$ is a ring if

(i) $\emptyset \in \mathcal{R}$

(ii) $A, B \in \mathcal{R} \implies A \setminus B \in \mathcal{R}$

(iii) $A, B \in \mathcal{R} \implies A \cup B \in \mathcal{R}$

Remark 5.3.4. $A \cap B = A \setminus (A \setminus B)$, $A \triangle B = (A \setminus B) \cup (B \setminus A)$

The ring is equivalent to for $A, B \in \mathcal{R}$

(i) $\emptyset \in \mathcal{R}$

(ii) $A \cap B \in \mathcal{R}$

(iii) $A \triangle B \in \mathcal{R}$

where $\cap$ is multiplication and $\triangle$ is addition.

Finite $\implies$ unions of disjoint sets in $\mathcal{R}$ are in $\mathcal{R}$.

$$A \cup B = (A \cap B) \uplus (A \triangle B)$$

$$(A \triangle B) \cap A = (A \setminus B \cup B \setminus A) \cup A = A \setminus B$$

Definition 5.3.5. $\mathcal{E}$ a collection. $\mathfrak{R}(\mathcal{E})$ is the smallset ring that contains the collection.

$$\mathfrak{R}(\mathcal{E}) = \bigcap_{\substack{\mathcal{E} \subseteq \mathcal{R} \\ \mathcal{R} \text{ ring}}} \mathcal{R}$$

Lemma 5.3.6

Suppose that $\mathcal{E}$ any collection and $\mathcal{R}$ is a ring. If $\mathcal{E} \subseteq \mathcal{R}$ then $\mathfrak{R}(\mathcal{E}) \subseteq \mathcal{R}$

Proof. Obvious □

Theorem 5.3.7

Let $\mathcal{S}$ be a semiring (of subsets of X). Then $\mathfrak{R}(\mathcal{S})$ is a ring generated by $\mathcal{S}$, is the collection of finite disjoint unions of sets in $\mathcal{S}$.

Proof. Define $\mathcal{R}$ to be the collection of finite disjoint unions of sets in $\mathcal{S}$. Obviously, $\mathcal{R} \subseteq \mathfrak{R}(\mathcal{S})$. want to show that $\mathfrak{R}(\mathcal{S}) \subseteq \mathcal{R}$. If is obvious that $\mathcal{S} \subseteq \mathcal{R}$. So, we only need to show that $\mathcal{R}$ is a ring.

Obviously, $\emptyset \in \mathcal{R}$.

GOAL: $A \setminus B \in \mathcal{R}$.

$$\biguplus A_j \setminus \biguplus B_k = \biguplus_j (A_j \setminus \biguplus B_k)$$

Neet to check that for each j, $(A_j \setminus \biguplus B_k)$ is a disjoint union of $\mathcal{S}$. Take $A \in \mathcal{S}$, $B_1, \ldots, B_n \in \mathcal{S}$, B_i are disjoint

Claim$_n$: $A \setminus \biguplus B_k \in$ disjoint union of sets in $\mathcal{S}$. By induction $n = 1$, by definition (iii) then Claim$_{n-1} \implies$ Claim$_n$. Assume

$$\begin{aligned} A \setminus \biguplus_{k=1}^{n-1} B_k &= \biguplus_{l=1}^{M} C_l \\ A \setminus \left(\biguplus_{k=1}^{n-1} B_k \right) \setminus B_n &= \bigcup_{l=1}^{M} (C_l \setminus B_n) \\ A \setminus \left(\biguplus_{k=1}^{n} B_k \right) &= \biguplus_{l=1}^{M} \biguplus_{j=1}^{M(l,n)} C_{l,n,j} \end{aligned}$$

The hard part is to show that $\mathcal{R}(\mathcal{S})$ is a ring. $A = \biguplus A_j, B = \biguplus B_k$ then

$$A \cup B = \biguplus_{j,k} \underbrace{(A_j \cap B_k)}_{\in \mathcal{S}}$$

□

Lemma 5.3.8

If $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n$ are semirings, then the collection $(A_1 \times A_2 \times \cdots \times A_n), A_i \in \mathcal{S}_i$ form a semiring

Proof. By induction, it suffices to check this for $n = 2$.

- $\emptyset \times \emptyset \in \mathcal{S}_1 \times \mathcal{S}_2$
- $A_1 \times A_2 \cap B_1 \times B_2 = (A_1 \cap B_1) \times (A_2 \cap B_2) \in \mathcal{S}_1 \times \mathcal{S}_2$
- $$\begin{aligned} (A_1 \times A_2) \setminus (B_1 \times B_2) &= (A_1 \setminus B_1) \times A_2 \uplus (A_1 \cap B_1) \times (A_2 \setminus B_2) \\ &= \biguplus_{k=1}^{M_1} (C_{1,k} \times A_2) \uplus \biguplus_{l=1}^{M_2} (A_1 \cap B_1) \times (C_{2,l}) \end{aligned}$$

□

§5.4 Contents, premeasures, and their extensions

Definition 5.4.1. A **content** on a semiring $\mathcal{S}$ (ring) is $\varrho : \mathcal{S} \to [0, \infty]$

(i) $\varrho(\emptyset) = 0$

(ii) If $\{A_k\}_{k=1}^N$ disjoint, $A_k \in \mathcal{S}$ and if $\biguplus_{k=1}^N A_k \in \mathcal{S}$ then $\varrho\left(\biguplus_{k=1}^N A_k\right) = \sum_{k=1}^N \varrho(A_k)$

Remark 5.4.2. In a ring $\mathcal{R}$ same definition but we have $\biguplus_{k=1}^N A_j \in \mathcal{R}$ if $A_j \in \mathcal{R}$

Definition 5.4.3. A **premeasure** on a semiring (ring) is $\nu : \mathcal{S} \to [0, \infty]$ such that

(i) $\nu(\emptyset) = 0$

(ii) If $\{A_k\}_{k=1}^\infty$ disjoint, $A_k \in \mathcal{S}$ and if $\biguplus_{k=1}^N A_k \in \mathcal{S}$ then $\nu\left(\biguplus_{k=1}^N A_k\right) = \sum_{k=1}^N \nu(A_k)$

Example 5.4.4

$S_1 =$ intervals of the form $(a, b]$ for $a, b \in \mathbb{R}$ $\varrho((a, b]) = b - a$ is a premeasure on S. First check ϱ is a content on S_1. $(a, b] = \bigcup_{j=1}^M (a_j, a_{j+1}]$ ordered $a_1 < a_2 < \cdots < a_{M+1} = b$ $\sum a_{j+1} - a_j = b - a$

Theorem 5.4.5

Suppose that $\mathcal{S}$ is a semiring and $\varrho : \mathcal{S} \to [0, \infty]$ be content. There exists unique extension $\nu : \mathfrak{R}(\mathcal{S}) \to [0, \infty]$ such that for any $A \in \mathfrak{R}(\mathcal{S})$, $A = \biguplus_{j=1}^N A_j$, $A_j \in \mathcal{S}$

$$\nu(A) = \sum_{j=1}^m \varrho(A_j)$$

Proof. Have to check well-defined. Let $A \in \mathfrak{R}(\mathcal{S})$

$$A = \biguplus_{j=1}^{M_1} A_j = \biguplus_{k=1}^{M_2} B_k$$

Want to show that

$$\sum_{j=1}^{M_1} \varrho(A_j) = \sum_{k=1}^{M_2} \varrho(B_k)$$

$$A_j = \biguplus_{k=1}^{M_2} (A_j \cap B_k)$$
$$B_k = \biguplus_{j=1}^{M_1} (A_j \cap B_k)$$
$$\sum_{j=1}^{M_1} \varrho(A_j) = \sum_{j=1}^{M_2} \sum_{k=1}^{M_2} \varrho(A_j \cap B_k)$$
$$\sum_{k=1}^{M_2} \varrho(B_k) = \sum_{k=1}^{M_2} \sum_{j=1}^{M_1} \varrho(A_j \cap B_k)$$

□

Claim 5.4.6 — Half open interval length is a premeasure.

Proof. Preliminary consideration: If $(a_j, b_j], (a, b] \subseteq \bigcup (a_j, b_j]$ then $b - a \leq \sum (b_j - a_j)$

Now $(a, b] = \biguplus_{j=1}^{\infty} (a_j, b_j]$, $a_j < b_j$ then

Claim:
$$\sum_{j=1}^{N} (b_j - a_j) \leq b - a$$
for all N. From the monotonicity property of a content.

Now show
$$\sum (b_j - a_j) \geq b - a - C\varepsilon$$
$[a + \varepsilon, b]$ this is covered by $(a_j, b_j]$ and in fact by the open set $(a_j, b_j + \varepsilon 2^{-j-1})$ There is a finite subcover of $[a + \varepsilon, b]$ by $(a_{j_i}, b_{j_i} + \varepsilon 2^{-j_i-1})$ for $i = 1, \ldots, M$

$$\sum_{i=1}^{M} (b_{j_i} + \varepsilon 2^{-j_i} - a_{j_i}) \leq \sum_{j} (b_j - a_j) + \sum_{j=1}^{\infty} \varepsilon 2^{-j-1}$$

□

§5.5 Extend premeasures to measure on a σ-algebra

Definition 5.5.1. Suppose that ν is a content on semiring $\mathcal{S}$. Define outer measure $\nu^* : \mathfrak{P}(X) \to [0, \infty]$ by

$$\nu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \nu(A_j) : A_j \in \mathcal{S}, E \subseteq \bigcup_{j=1}^{\infty} A_j \right\}$$

Definition 5.5.2. Suppose that ν is e the extension of a content on a semirig $\mathcal{S}$. Define outer measure $\nu^{**} : \mathfrak{P}(X) \to [0, \infty]$ by

$$\nu^{**}(E) = \inf \left\{ \sum_{j=1}^{\infty} \nu(A_j) : A_j \in \mathfrak{R}(\mathcal{S}), E \subseteq \bigcup_{j=1}^{\infty} A_j \right\}$$

Lemma 5.5.3

$\nu^{**}(E) = \nu^*(E)$ if ν is a <u>content</u>.

Proof. Obviously, $\nu^{**}(E) \leq \nu^*(E)$.

We only need to show that $\nu^*(E) \leq \nu^{**}(E)$, WLOG $\nu^{**}(E) < \infty$. Given $\varepsilon > 0$ find $A_j \in \mathfrak{R}(\mathcal{S})$ such that $\sum_{j=1}^{\infty} \nu(A_j) < \nu^{**}(E) + \varepsilon$ For each A_j we can decompose it as

$$A_j = \biguplus_{k=1}^{M_j} C_{j,k}$$

then

$$\nu(A_j) = \sum_{n=1}^{M_j} \nu(C_{j,k})$$

so,

$$\nu^*(E) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{M_j} \nu(C_{j,k}) = \sum_{j=1}^{\infty} \nu(A_j) < \nu^{**}(E) + \varepsilon$$

□

Theorem 5.5.4 (Hahn-Kolomogorov-Caratheodory-Frechet) (i) If ν is a content on $\mathcal{S}$ (semiring) over X, (and therefore on $\mathfrak{R}(\mathcal{S})$). Then the Caratheodory σ-algebra $\mathcal{M}^*$ of ν^*-measurable set contains $\mathcal{S}$ and $\mathfrak{R}(\mathcal{S})$ and $\mathfrak{M}(\mathcal{S})$

(ii) If ν is a premeasure then $\nu^*|_{\mathcal{S}} = \nu$ and $\mu = \nu^*$ is a measure (on $\mathcal{M} = \mathfrak{M}(\mathcal{S})$)

Proof. Show that $\mathcal{S} \subseteq \mathcal{M}^*$. It is enough to show that for any $A \in \mathcal{S}$, for any $E \subseteq X$,

$$\nu^*(E) \geq \nu^*(E \cap A) + \nu^*(E \cap A^\complement)$$

Let $A \in \mathcal{S}$. Fix $E \subseteq X$, work with an ε-efficient cover, i.e., find a collection $\{A_j\}_{j=1}^\infty$ such that $A_j \in \mathcal{S}, E \subseteq \bigcup_{j=1}^\infty A_j : \sum_{j=1}^\infty \nu(A_j) \leq \nu^*(E)+\varepsilon$ We get that $A_j = (A_j \cap A) \uplus (A_j \cap A^\complement)$, then $\nu(A_j) = \nu(A_j \cap A) + \nu(A_j \cap A^\complement)$

$$\nu^*(E) + \varepsilon \geq \sum_{j=1}^\infty \nu(A_j) = \sum_{j=1}^\infty \nu(A_j \cap A) + \sum_{j=1}^\infty \nu(A_j \cap A^\complement)$$

We know that $E \cap A \subseteq \bigcup_{j=1}^\infty (A_j \cap A)$ and $E \cap A^\complement \subseteq \bigcup_{j=1}^\infty (A_j \cap A^\complement)$ then

$$\nu^*(E) + \varepsilon \geq \nu^*(E \cap A) + \nu^*(E \cap A^\complement)$$

For $A \in \mathcal{S}$ show $\nu^*(A) = \nu(A)$.

It is easy to show that $\nu^*(A) \leq \nu(A)$ since $\{A_j\}_{j=1}^\infty$ is a cover of A.

We need $\nu(A) \leq \nu^*(A) + \varepsilon$ (for all $\varepsilon > 0$)

Pick $\{A_j\}_{j=1}^\infty$, $\bigcup_{j=1}^\infty A_j \supseteq A$, $\sum_{j=1}^\infty \nu(A_j) < \nu^*(A) + \varepsilon = \nu^{**}(A) + \varepsilon$. Define $B_1 = A_1, B_2 = A_2 \setminus A_1, B_3 = A_3 \setminus (A_1 \cup A_2), \ldots$ B_j is disjoint and $B_n \in \mathfrak{R}(\mathcal{S})$ (but might not in the semiring) We claim that premeasure assumption We know that $\bigcup_{n=1}^\infty (A \cap B_n) = A$

$$\nu(A) = \sum_n \nu(A \cap B_n) \leq \sum_n \nu(A_n) \leq \nu^{**}(E) + \varepsilon$$

$\square$

Example 5.5.5

$E \subseteq \mathbb{N}$

$$\varrho(E) = \limsup_{n \to \infty} \frac{|E \cap [1, n]|}{n}$$

If E is finite $\varrho(E) = 0$, if $E = \mathbb{N}$ then $\varrho(E) = 1$ But

$$\varrho(\mathbb{N}) \not\leq \sum_{n=1}^\infty \varrho(\{n\})$$

So, Subaddictivity fails, not a premeasure

Extenshion of a premeasure ν on a (semi) ring $\mathcal{S}$ to a measure μ on the σ-algebra generated by $\mathcal{S}$.

1. If $\widetilde{\mu}$ is an extension of ν and $\mu = \nu^*|_{\mathcal{M}}$ then $\widetilde{\mu}(E) \leq \mu(E)$.

If $E \in \mathcal{M}$ is such $\mu(E) < \infty \implies \mu(E) = \widetilde{\mu}(E)$.

If ν is a σ-finite premeasure on $\mathcal{S}$ then $\mu = \widetilde{\mu}$ $X = \bigcup_{j=1}^{\infty} X_j, X_j \in \mathcal{S}, \nu(X_j) < \infty$.

Proof. 1. WLOG $\mu(E) < \infty$, $\mu(E) = \nu^*(E) = \inf_{A_K \in \mathcal{S}, \bigcup A_k \supseteq E} \sum_{k=1}^{\infty} \nu(A_k)$ or ∞ Given $\varepsilon > 0$ we find a $\{A_k\}$ of set in $\mathcal{S}$

$$\nu^*(E) = \sum_{k=1}^{\infty} \nu(A_k) < \nu^*(E) + \varepsilon$$

replace A_k by $A_k \setminus (A_1 \cup \cdots \cup A_{k-1})$ to get a disjoint collection (May assume that the A_k are disjoint)

$$\widetilde{\mu}(E) = \widetilde{\mu} \left(\biguplus A_k = \sum \widetilde{\mu}(A_k) = \sum \nu(A_k) \leq \nu^*(E) + \varepsilon \right)$$

$\square$

On <u>complete</u> union of set $\mathcal{R}(\mathcal{S})$, μ and $\widetilde{\mu}$ coincide.

Take $E \in \mathcal{M}$, $\mu(E) = \nu^*(E) < \infty$, show that $\widetilde{\mu}(E) = \mu(E)$

Find a countable cover of E with set $B_k \in \mathcal{R}(\mathcal{S})$ such that

$$\sum \nu(B_k) < \nu^*(E) + \varepsilon, B = \bigcup_{k=1}^{\infty} B_k, \mu(B) = \widetilde{\mu}(B)$$

$\mu(E) \leq \mu(B) = \widetilde{\mu}(B) = \widetilde{\mu}(E) + \widetilde{\mu}(B \setminus E) \leq \widetilde{\mu}(E) + \mu(B \setminus E)$

6 Product measure

§6.1 Introduction

Given two (or finitely many) measure space $(X_1, \mathcal{M}_1, \mu_1), (X_2, \mathcal{M}_2, \mu_2)$ $X = X_1 \times X_2$, On X there is a natural σ-algebra $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2 =:$ σ-algebra generated by set of the form. $E_1 \times E_2, E_1 \in \mathcal{M}_1, E_2 \in \mathcal{M}_2$ and $\mu(E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2)$

Extend the set function μ, defined on the general rectangles to all of $\mathcal{M}_1 \otimes \mathcal{M}_2$.

Claim 6.1.1 — The rectangles form a semiring (use cartesian product $\mathcal{M}_1 \times \mathcal{M}_2$ of semirings are semiring)

Theorem 6.1.2

Given $(X_1, \mathcal{M}_1, \mu_1), (X_2, \mathcal{M}_2, \mu_2)$, σ-finite measure spaces, there exists a unique measure μ on $\mathcal{M}_1 \otimes \mathcal{M}_2$ such that $\mu(E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2)$ for all $E_1 \in \mathcal{M}_1, E_2 \in \mathcal{M}_2$

Proof. $\mathcal{S}_{\text{rect}} = \mathcal{M}_1 \times \mathcal{M}_2$ semiring of rectangles. Need to show that $\nu : \mathcal{S}_{\text{rect}} \to [0, \infty)$ is a premeasure verify σ-additivity on $\mathcal{S}_{\text{rect}}$ If

$$E_1 \times E_2 = \biguplus_{j=1}^{\infty} E_{j,1} \times E_{j,2}$$

then

$$\nu(E_1 \times E_2) = \sum_{j=1}^{\infty} \nu(E_{j,1} \times E_{j,2})$$

$$\mathbb{1}_{E_1 \times E_2}(x_1, x_2) = \sum_{j=1}^{\infty} \mathbb{1}_{E_{j,1} \times E_{j,2}}(x_1, x_2) \iff \mathbb{1}_{E_1}(x_1)\mathbb{1}_{E_2}(x_2) = \sum_{j=1}^{\infty} \mathbb{1}_{E_{j,1}}(x_1)\mathbb{1}_{E_{j,2}}(x_2)$$

$$\begin{aligned}
\mathbb{1}_{E_1}(x_1)\mathbb{1}_{E_2}(x_2) &= \sum_{j=1}^{\infty} \mathbb{1}_{E_{j,1}}(x_1)\mathbb{1}_{E_{j,2}}(x_2) \\
\int \mathbb{1}_{E_1}(x_1)\mathbb{1}_{E_2}(x_2)\ \mathrm{d}\mu_2 &= \int \sum_{j=1}^{\infty} \mathbb{1}_{E_{j,1}}(x_1)\mathbb{1}_{E_{j,2}}(x_2)\ \mathrm{d}\mu_2 \\
\mathbb{1}_{E_1}(x_1)\mu_2(E_2) &= \sum_{j=1}^{\infty} \int \mathbb{1}_{E_{j,1}}(x_1)\mathbb{1}_{E_{j,2}}(x_2)\ \mathrm{d}\mu_2 \\
&= \sum_{j=1}^{\infty} \mathbb{1}_{E_{j,1}}(x_1)\mu_2(E_{j,2}) \\
\int \mathbb{1}_{E_1}(x_1)\mu_2(E_2)\ \mathrm{d}\mu_1 &= \int \sum_{j=1}^{\infty} \mathbb{1}_{E_{j,1}}(x_1)\mu_2(E_{j,2})\ \mathrm{d}\mu_1 \\
\mu_1(E_1)\mu_2(E_2) &= \sum_{j=1}^{\infty} \mu_1(E_{j,1})\mu_2(E_{j,2}) \\
\nu(E_1 \times E_2) &= \sum_{j=1}^{\infty} \nu(E_{j,1} \times E_{j,2})
\end{aligned}$$

□

Lemma 6.1.3

If μ_1 is σ-finite and μ_2 is σ-finite, then ν is σ-finite.

Proof. $X_1 = \biguplus_{j=1}^{\infty} X_{1,j}$, $X_2 = \biguplus_{k=1}^{\infty} X_{2,k}$, $\mu_1(X_{1,j}) < \infty$, $\mu_2(X_{2,k}) < \infty$ $X_1 \times X_2 = \biguplus_{j,k} X_{1,j} \times X_{2,k}$, $\nu(X_{1,j} \times X_{2,k}) = \mu_1(X_{1,j})\mu_2(X_{2,k}) < \infty$ □

Remark 6.1.4. In Summary, we have proved that the product pre-measure extends uniquely to $\mathcal{M}_1 \otimes \mathcal{M}_2$. (under the assumption of σ-finiteness of μ_1, μ_2)

Remark 6.1.5. Constructed a product measure on $\mathcal{M}_1 \otimes \mathcal{M}_2$ by $\mu(E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2)$ (this is unique if $\mathcal{M}_1, \mathcal{M}_2$ are σ-finite)

Lemma 6.1.6

Let $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$, Define

$$sl_{x_1}(E) = \{x_2 : (x_1, x_2) \in E\}$$

$$sl^{x_2}(E) = \{x_1 : (x_1, x_2) \in E\}$$

Then $sl_{x_1}(E) \in \mathcal{M}_2$, $sl^{x_2}(E) \in \mathcal{M}_1$

Proof. Considering the rectangle case, this is "easy"

$$sl_{x_1}(E_1 \times E_2) = \{x_2 : (x_1, x_2) \in E_1 \times E_2\} = \begin{cases} \emptyset, & x_1 \notin E_1 \\ E_2, & x_1 \in E_1 \end{cases}$$

Define

$$\mathfrak{C}_{x_1} = \{E \in \mathfrak{P}(X_1 \times X_2) : sl_{x_1}(E) \in \mathcal{M}_2\}$$

we know that $\mathcal{S}_{\text{rect}} \subseteq \mathfrak{C}_{x_1}$

Check $\mathfrak{C}_{x_1}$ is a σ-algebra (If so, $\mathfrak{C}_{x_1}$ contains $\mathfrak{M}(\mathcal{S}_{\text{rect}}) = \mathcal{M}_1 \otimes \mathcal{M}_2$)

(i) $sl_{x_1}(X_1 \times X_2) = X_2 \in \mathcal{M}_2$, so, $X_1 \times X_2 \in \mathfrak{C}_{x_1}$

(ii) Let $E \in \mathfrak{C}_{x_1}$, we know that $sl_{x_1}(E) \in \mathcal{M}_2$, consider $E^{\complement} = X \setminus E$

$$sl_{x_1}(E^{\complement}) = \{x_2 : (x_1, x_2) \notin E\} = \{x_2 : x_2 \notin sl_{x_1}(E)\} = (sl_{x_1}(E))^{\complement} \in \mathcal{M}_2$$

(iii) For $E_j \in \mathfrak{C}_{x_1}$, we know that $sl_{x_1}(E_j) \in \mathcal{M}_2$, then

$$sl_{x_1}\left(\bigcup_{j=1}^{\infty} E_j\right) = \bigcup_{j=1}^{\infty} sl_{x_1}(E_j) \in \mathcal{M}_2$$

So, $\mathfrak{C}_{x_1}$ is a σ-algebra, and $\mathfrak{C}_{x_1}$ contains $\mathcal{S}_{\text{rect}}$, so $\mathfrak{C}_{x_1}$ contains $\mathcal{M}_1 \otimes \mathcal{M}_2$. Hence, for any $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$, $sl_{x_1}(E) \in \mathcal{M}_2$. $\square$

Lemma 6.1.7

Let f be a $(\mathcal{M}_1 \otimes \mathcal{M}_2)$-measurable. Then $f_{x_1}(x_2) = f(x_1, x_2)$ is an $\mathcal{M}_2$-measurable, and $f^{x_2}(x_1) = f(x_1, x_2)$ is an $\mathcal{M}_1$-measurable

Proof. Check: $f_{x_1}^{-1}(I) \in \mathcal{M}_2$, for any Borel set I

$$\begin{aligned} f_{x_1}^{-1}(I) &= \{x_2 : f_{x_1}(x_2) = f(x_1, x_2) \in I\} \\ &= sl_{x_1}(\{(x_1, x_2) : f(x_1, x_2) \in I\}) \\ &= sl_{x_1}(\underbrace{f^{-1}(I)}_{\in \mathcal{M}_1 \otimes \mathcal{M}_2}) \end{aligned}$$

□

§6.2 Extension of Measure on Integral

Theorem 6.2.1 (Cavalieri's Principle)
Let $(X_1, \mathcal{M}_1, \mu_1), (X_2, \mathcal{M}_2, \mu_2)$ be σ-finite and let μ be a product measure of two measure space. For any $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$, then

(i) $g_E : x_1 \mapsto \mu_2(sl_{x_1}(E))$ is $\mathcal{M}_1$-measurable

(ii) $h_E : x_2 \mapsto \mu_1(sl^{x_2}(E))$ is $\mathcal{M}_2$-measurable

and

(i) $\mu(E) = \int g_E \, \mathrm{d}\mu_1 = \int \mu_2(sl_{x_1}(E)) \, \mathrm{d}\mu_1$

(ii) $\mu(E) = \int h_E \, \mathrm{d}\mu_2 = \int \mu_1(sl^{x_2}(E)) \, \mathrm{d}\mu_2$

Proof. First, consider the case that $\mu_1(X_1) < \infty$ and $\mu_2(X_2) < \infty$. Similarly, considering the rectangle case, for any $E \in \mathcal{S}_{\text{rect}}$,

$$g_E(x_1) = \begin{cases} \mu_2(E_2) & \text{if } x_1 \in E_1 \\ 0 & \text{if } x_1 \notin E_1 \end{cases}$$

So, g_E is $\mathcal{M}_1$-measurable. Moreover,

$$\int g_E \, \mathrm{d}\mu_1 = \int_{E_1} \mu_2(E_2) \, \mathrm{d}\mu_1 = \mu_1(E_1)\mu_2(E_2) = \mu(E)$$

proved the theorem for the rectangle case.

Next, define

$$\mathfrak{C} = \left\{ E \in \mathfrak{P}(X) : g_E \text{ is } \mathcal{M}_1\text{-measurable}, \ \mu(E) = \int g_E \, \mathrm{d}\mu_1 \right\}$$

I will show that $\mathfrak{C}$ is a Dynkin system. For the case $\mathcal{M}_1$-measurable,

(i) $g_X(x_1) = \mu_2(sl_{x_1}(X)) = \mu_2(X_2)$ is a constant function, so it is $\mathcal{M}_1$-measurable.

(ii) For $E \in \mathfrak{C}$, $g_{E^\complement}(x_1) = \mu_2(sl_{x_1}(E^\complement)) = \mu_2(X_2 \setminus sl_{x_1}(E)) = \mu_2(X_2) - \mu_2(sl_{x_1}(E))$ is $\mathcal{M}_1$-measurable.

(iii) For $E_j \in \mathfrak{C}$, where E_j are disjoint,

$$g_{\biguplus_{j=1}^\infty E_j}(x_1) = \mu_2\left(sl_{x_1}\left(\biguplus_{j=1}^\infty E_j\right)\right) = \sum_{j=1}^\infty \mu_2(sl_{x_1}(E_j))$$

is $\mathcal{M}_1$-measurable.

For the case of $\mu(E) = \int g_E \, \mathrm{d}\mu_1$,

(i)
$$\int g_X \, \mathrm{d}\mu_1 = \int \mu_2(sl_{x_1}(X)) \, \mathrm{d}\mu_1 = \mu_1(X_1)\mu_2(X_2) = \mu(X)$$

(ii) For $E \in \mathfrak{C}$,

$$\begin{aligned}\int g_{E^\complement} \, \mathrm{d}\mu &= \int \mu_2(sl_{x_1}(E^\complement)) \, \mathrm{d}\mu = \int \mu_2(X_2) - \mu_2(sl_{x_1}(E)) \, \mathrm{d}\mu \\ &= \mu(X) - \mu(E) = \mu(E^\complement)\end{aligned}$$

(iii) For $E_j \in \mathfrak{C}$, where E_j are disjoint,

$$\begin{aligned}\int g_{\biguplus_{j=1}^\infty E_j} \, \mathrm{d}\mu &= \int \sum_{j=1}^\infty \mu_2(sl_{x_1}(E_j)) \, \mathrm{d}\mu = \sum_{j=1}^\infty \int \mu_2(sl_{x_1}(E_j)) \, \mathrm{d}\mu \\ &= \sum_{j=1}^\infty \mu(E_j) = \mu\left(\biguplus_{j=1}^\infty E_j\right)\end{aligned}$$

We have a theorem stating that if $\mathcal{E}$ is $\cap$-stable then $\mathcal{D}(\mathcal{E}) = \mathfrak{M}(\mathcal{E})$. I will show that $\mathfrak{C}$ is a Dynkin system containing $\mathcal{S}_{\text{rect}}$ and $\mathfrak{M}(\mathcal{S}_{\text{rect}}) \subseteq \mathfrak{C}$. Since the ractangle cases are $\mathcal{M}_1$-measurable, $\mathfrak{C}$ contains $\mathcal{S}_{\text{rect}}$. Moreover, since $\mathcal{S}_{\text{rect}}$ is semiring, so it is $\cap$-stable, and from the definition of Dynkin system, $\mathfrak{M}(\mathcal{S}_{\text{rect}}) = \mathcal{D}(\mathcal{S}_{\text{rect}}) \subseteq \mathfrak{C}$. Hence, for any $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$, g_E is $\mathcal{M}_1$-measurable and $\mu(E) = \int g_E \, \mathrm{d}\mu_1$.

$\square$

Theorem 6.2.2 (Tonelli's Theorem)

Let $(X_i, \mathcal{M}_i, \mu_i), i = 1, 2$ σ-finite measure spaces, $\mu = \mu_1 \times \mu_2$ be a product measure on $\mathcal{M}_1 \otimes \mathcal{M}_2$. Let $f \in \mathcal{L}^+(X, \mu)$ then

- $x_1 \mapsto \int f_{x_1} \, \mathrm{d}\mu_2$ belongs to $\mathcal{L}^+(X_1, \mu_1)$
- $x_2 \mapsto \int f^{x_2} \, \mathrm{d}\mu_1$ belongs to $\mathcal{L}^+(X_2, \mu_2)$

and

$$\int f \, \mathrm{d}\mu = \int \int f_{x_1} \, \mathrm{d}\mu_2 \mathrm{d}\mu_1$$

Proof. We have proved this theorem for simple functions / indicator functions of measureable sets. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of simple functions such that $s_n(x) \nearrow f(x)$ for all x. define

$$g_n(x_1) = \int s_n(x_1, x_2) \, \mathrm{d}\mu_2, \; h_n(x_2) = \int s_n(x_1, x_2) \, \mathrm{d}\mu_1$$

$$g(x_1) = \int f(x_1, x_2) \, \mathrm{d}\mu_2, \; h(x_2) = \int f(x_1, x_2) \, \mathrm{d}\mu_1$$

By Monotone Convergence Theorem, we have $g_n \nearrow g$, $h_n \nearrow h$ and limit of measureable functions is measureable. By the fact that $g_n \nearrow g$,

$$\lim_{n\to\infty} \int g_n \, \mathrm{d}\mu_1 = \int g \, \mathrm{d}\mu_1 = \int \int f(x_1, x_2) \, \mathrm{d}\mu_2 \mathrm{d}\mu_1$$

$$\lim_{n\to\infty} \int g_n \, \mathrm{d}\mu_1 = \lim_{n\to\infty} \int \int s_n(x_1, x_2) \, \mathrm{d}\mu_2 \mathrm{d}\mu_1 \overset{Cavalieri}{=} \lim_{n\to\infty} \int s_n \, \mathrm{d}\mu \overset{\text{def}}{=} \int f \, \mathrm{d}\mu$$

□

Theorem 6.2.3 (Fubini's Theorem)

$f \in \mathcal{L}^1(X, \mu)$ then $f_{x_1} \in \mathcal{L}^1(X_2, \mu_2)$ for μ_1 almost every x_1 and $f^{x_2} \in \mathcal{L}^1(X_1, \mu_1)$ for μ_2 almost every x_2 and

$$\int f \, \mathrm{d}\mu = \int \int f_{x_1} \, \mathrm{d}\mu_2 \, \mathrm{d}\mu_1 = \int \int f^{x_2} \, \mathrm{d}\mu_1 \, \mathrm{d}\mu_2$$

§6.3 Distribution Function

Definition 6.3.1. Given $(X, \mathcal{M}, \mu)$, given measurable function f, define

$$\alpha \mapsto \mu_f(\alpha) = \mu(\{x : |f| > \alpha\})$$

$\alpha > 0$

Theorem 6.3.2

Given $(X, \mathcal{M}, \mu)$,

$$\int |f|^p \, \mathrm{d}\mu = \int_0^\infty p\alpha^{p-1}\mu_f(\alpha) \, \mathrm{d}\alpha$$

Example 6.3.3

example where μ_f shows

$$\mu_f(\alpha) = \frac{\|f\|_p^p}{\alpha^p}$$

Chebyshev's inequality

$L^{p,\infty}$ = "weake type p-spaces" $\iff$ f measurable $\iff$ $\sup_\alpha \alpha^p \mu_f(\alpha) < \infty$

Proof. We first assume μ is σ-finite. Considering Right-hand-side

$$\begin{aligned}\int_0^\infty p\alpha^{p-1}\mu_f(\alpha) \, \mathrm{d}\alpha &= \int_0^\infty p\alpha^{p-1} \int_{|f(x)|>\alpha} \, \mathrm{d}\mu \mathrm{d}\alpha \\ &= \int_0^\infty \int_X p\alpha^{p-1} \mathbb{1}_{\{x:|f(x)|-\alpha>0\}}(x,\alpha) \, \mathrm{d}\mu \mathrm{d}\alpha\end{aligned}$$

$x \mapsto f(x)$ is measurable as a function on X, and as a function on $X \times [0, \infty)$. define $g : \mathbb{R}^2 \to \mathbb{R}, (t, \alpha) \mapsto (|t|-\alpha)\mathbb{1}_{\alpha>0}$ then $g \circ f$ is $\mathcal{M}$-measurable because $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ then

$$\int_0^\infty p\alpha^{p-1}\mu_f(\alpha) \, \mathrm{d}\alpha = \int_X \underbrace{\int_{\alpha=0}^{|f(x)|} p\alpha^{p-1} \, \mathrm{d}\alpha}_{|f(x)|^p} \, \mathrm{d}\mu$$

This proves the formula for σ-finite X.

For the general cases, let assume that there exists α such that $\mu_f(\alpha) = \infty$, $\mu(\{x : |f(x) > \alpha\}) = \infty \implies \mu_f(\beta) = \infty$ for $0 < \beta < \alpha$. In this case, we have $\infty = \infty$

Assume $\mu_f(\alpha) < \infty$ for all $\alpha > 0$

claim: $\{x : |f(x)| > 0\} = \bigcup_n \{x : |f(x) > \frac{1}{n}\}$, σ-finite because $\mu(\{x : |f(x) > \frac{1}{n}\}) = \mu_f(\frac{1}{n}) < \infty$ $\square$

Example 6.3.4

$$\int_{-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x = \sqrt{\pi}$$

equivalent to

$$\int_{0}^{\infty} e^{-x^2} \, \mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

Proof. Let

$$F(x,y) = \begin{cases} ye^{-y^2(1+x^2)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int F \, \mathrm{d}m(x,y) = \int_{y=0}^{\infty} e^{-y^2} \int_{x=0}^{\infty} e^{-y^2x^2} y \, \mathrm{d}x\mathrm{d}y$$

By changing variable, let $w = yx, \mathrm{d}w = y \, \mathrm{d}x$ then

$$\int_{y=0}^{\infty} e^{-y^2} \int_{x=0}^{\infty} e^{-y^2x^2} y \, \mathrm{d}x\mathrm{d}y = \int_{y=0}^{\infty} e^{-y^2} \, \mathrm{d}y \int_{w=0}^{\infty} e^{-w^2} \, \mathrm{d}w$$

Considering LHS

$$\int F \, \mathrm{d}m = \int_{x=0}^{\infty} \int_{y} y \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} e^{-y^2(1+x^2)} \, \mathrm{d}y\mathrm{d}x$$

let $w = y\sqrt{1+x^2}$, $\mathrm{d}w = \sqrt{1+x^2} \, \mathrm{d}y$ then

$$\begin{aligned} \int F \, \mathrm{d}m &= \int_0^{\infty} \frac{1}{1+x^2} \int_0^{\infty} \frac{2w}{2} e^{-w^2} \, \mathrm{d}w\mathrm{d}x \\ &= \frac{1}{2} \int_0^{\infty} \frac{1}{1+x^2} \, \mathrm{d}x = \frac{\pi}{4} \end{aligned}$$

Considering the first integration and the second integration, we get that

$$\begin{aligned} \left(\int_0^{\infty} e^{-s^2} \, \mathrm{d}s \right)^2 &= \frac{\pi}{4} \\ \int_0^{\infty} e^{-s^2} \, \mathrm{d}s &= \frac{\sqrt{\pi}}{2} \end{aligned}$$

□

Example 6.3.5

$$\int_0^\infty e^{-ax}(\sin x)^2 \frac{\mathrm{d}x}{x}$$

Hint: To apply Fubini-Tonelli,

$$\iint_{[0,\infty]\times[0,1]} e^{-ax}\sin(2xy)\ \mathrm{d}m(x,y)$$

(value $1/4\ln(1+4/a^2)$)

§6.4 Linear Change of Variable

Theorem 6.4.1

Let $f \in L^1(\mathbb{R}^n)$ or $(L^+(\mathbb{R}^n))$ and let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible $n \times n$ matrix. Then

$$|\det A| \int f(Ax)\ \mathrm{d}m(x) = \int f(x)\ \mathrm{d}m(x)$$

Moreover, $m(AE) = |\det A| m(E)$ for every Lebesgue measureable set E.

Proof. We first assume that f is Borel measurable, then $x \mapsto f(Ax)$ is Borel measurable. (since $x \mapsto Ax$ is continuous) We consider a number of special cases. In what follows write $\mathbb{R}^n \ni (x_1, x')$ where $x_1 \in \mathbb{R}, x' \in \mathbb{R}^{n-1}$.

Case 1:

$$A = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$\lambda \neq 0$ then $\det A = \lambda$. then

$$|\lambda| \int f(Ax)\ \mathrm{d}m(x) = |\lambda| \int_{\mathbb{R}^{d-1}} \left[\int_{\mathbb{R}} f(\lambda x, x')\ \mathrm{d}m_1(x_1) \right] \mathrm{d}m_{n-1}(x')$$

$$\int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} f(cx, x')\ \mathrm{d}m_1(x_1) \right] \mathrm{d}m_{n-1}(x') = \int f(x)\ \mathrm{d}m(x)$$

Case 2:

$$A = \begin{bmatrix} 1 & \lambda & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Then $\det A = 1$

$$1\int f(Ax)\ \mathrm{d}m(x) = \int_{\mathbb{R}^{d-1}} \left[\int_{\mathbb{R}} f(x_1+\lambda x_2, x')\ \mathrm{d}m_1(x_1)\right] \mathrm{d}m_{n-1}(x')$$

$$\int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} f(x_1+\lambda x_2, x')\ \mathrm{d}m_1(x_1)\right] \mathrm{d}m_{n-1}(x') = \int f(x)\ \mathrm{d}m(x)$$

Case 3: A permutes two coordinates: there exist $j \neq k$ where $A(e_i) = e_i$ if $i \notin \{j,k\}$, $A(e_j) = e_k$, $A(e_k) = e_j$. Then $\det A = -1$.

$$1\int f(Ax)\ \mathrm{d}m(x) = \int f(x)\ \mathrm{d}m(x)$$

General case: In general we can write $A = E_1 \dots E_n$ where each E_j is in Case 1, 2, or 3.

$$\begin{aligned}|\det A| \int f(Ax)\ \mathrm{d}x &= |\det(E_1 \cdots E_{n-1})||\det E_n| \int f(f \circ E_1 \dots E_{n-1})(E_n x)\ \mathrm{d}x \\ &= |\det(E_1 \cdots E_{n-1})| \int f(E_1 \cdots E_{n-1}x)\ \mathrm{d}x \\ &= \cdots \int f(x)\ \mathrm{d}x\end{aligned}$$

This proves the theorem when f is Borel measurable. In particular, if E is Borel measurable then

$$\begin{aligned}\divideontimes m(AE) &= \int \mathbb{1}_{AE}(x)\ \mathrm{d}m(x) \\ &= |\det A| \int \mathbb{1}_{AE}(Ax)\ \mathrm{d}m(x) \\ &= |\det A| \int \mathbb{1}_{E}(x)\ \mathrm{d}m(x) \\ &= |\det A| m(E)\end{aligned}$$

Finally consider the case when f is Lebesgue measurable. Then there is a Borel measurable function g and a Borel measurable set E with $m(E) = 0$ and $f = g$ on $\mathbb{R}^d \setminus E$ (so $f = g$ almost everywhere). Then $f \circ A = g \circ A$ on $\mathbb{R}^d \setminus A^{-1}(E)$. By $\divideontimes$ $m(A^{-1}E) = 0$ So $f \circ A$ is Lebesbue measurable and $f \circ A = g \circ A$ almost everywhere. So

$$|\det A| \int f(Ax)\ \mathrm{d}m(x) = |\det A| \int g(Ax)\ \mathrm{d}m(x)$$

$$\int g(x)\ \mathrm{d}m(x) = \int f(x)\ \mathrm{d}m(x)$$

Also, arguing exactly as in $\divideontimes$ we have $m(AE) = |\det A| m(E)$ for every Lebesgue measurable set E. $\square$

Theorem 6.4.2

Assume $f, g : [a,b] \to \mathbb{R}$ are Lebesgue integrable. Define

$$F(x) = \int_{[a,x]} f \, \mathrm{d}m$$

$$G(x) = \int_{[a,x]} g \, \mathrm{d}m$$

then

$$\int_{[a,b]} Fg \, \mathrm{d}m = F(b)G(b) - \int_{[a,b]} fG \, \mathrm{d}m$$

Remark 6.4.3. if $u, v : [a,b] \to \mathbb{R}$ are continuously differentiable with $u(a) = v(a) = 0$, then

$$u(x) = \int_{[a,x]} u' \, \mathrm{d}m$$

$$v(x) = \int_{[a,x]} v' \, \mathrm{d}m$$

So, the above theorem implies

$$\int_{[a,b]} uv' \, \mathrm{d}m = u(b)v(b) - \int_{[a,b]} u'v \, \mathrm{d}m$$

Proof. Let $T = \{(x,t) : a \le x \le b, a \le t \le x\} = \{(x,t) : a \le t \le b, t \le x \le b\}$ Then T is a Lebesgue measurable set in $\mathbb{R}^2$ and $(x,t) \mapsto f(t)g(x)\mathbb{1}_T(x,t)$ is Lebesgue measurable. Also by Tonelli's Theorem,

$$\begin{aligned}\int |f(t)g(x)\mathbb{1}_T(x,t)| \, \mathrm{d}m_2(x,t) &= \int_{[a,b]} |g(x)| \left[\int_{[a,x]} |f(t)| \, \mathrm{d}t\right] \mathrm{d}x \\ &\le \int_{[a,b]} |g| \, \mathrm{d}m \int_{[a,b]} |f| \, \mathrm{d}m < \infty\end{aligned}$$

So, by Fubini's Theorem,

$$\begin{aligned}\int f(t)g(x)\mathbb{1}_T(x,t) \, \mathrm{d}m_2(x,t) &= \int_{[a,b]} g(x) \left[\int_{[a,x]} f(t) \, \mathrm{d}t\right] \mathrm{d}x \\ &= \int_{[a,b]} gF \, \mathrm{d}m\end{aligned}$$

and

$$\begin{aligned}\int f(t)g(x)\mathbb{1}_T(x,t)\ \mathrm{d}m_2(x,t) &= \int_{[a,b]} f(t)\left[\int_{[t,b]} g(x)\ \mathrm{d}x\right]\mathrm{d}t \\ &= \int_{[a,b]} f(t)\left[\int_{[a,b]} g(x)\ \mathrm{d}x - \int_{[a,t]} g(x)\ \mathrm{d}x\right]\mathrm{d}t \\ &= \int_{[a,b]} f(t)\left[\int_{[a,b]} g(x)\ \mathrm{d}x\right]\mathrm{d}t \\ &- \int_{[a,b]} f(t)\left[\int_{[a,t]} g(x)\ \mathrm{d}x\right]\mathrm{d}t \\ &= F(b)G(a) - \int_{[a,b]} fG\ \mathrm{d}m\end{aligned}$$

So,

$$\int_{[a,b]} Fg\ \mathrm{d}m = F(b)G(b) - \int_{[a,b]} fG\ \mathrm{d}m$$

□

§6.5 Change of Variable

We konw $A \in GL(n,\mathbb{R})$ A is $n\times n$ matrix, A is invertible $\iff$ $\det A = 0$. If $f \in \mathcal{L}^+$ or $f \in \mathcal{L}^1$ then

$$\int f(x)\ \mathrm{d}m(x) = \int f(Ax)|\det A|\ \mathrm{d}m(x)$$

$$A\Omega = \{Ax : x \in \Omega\}$$

The step is to replace linear transformation to non-linear transformation.

$\phi(x) = \phi(x_1,\dots,x_n)$, ϕ, $\mathbb{R}^n$-valued, $\phi = (\phi_1,\dots,\phi_n)$. we know that ϕ is differentiable on Ω. if $\frac{\partial\phi_j}{\partial x_k}$ exist and are continuous.

$$\phi'(x) = \begin{pmatrix} \frac{\partial\phi_1}{\partial x_1} & \cdots & \frac{\partial\phi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial\phi_n}{\partial x_1} & \cdots & \frac{\partial\phi_n}{\partial x_n}\end{pmatrix}$$

$\phi(x) = Ax, \phi'(x) = A$

Definition 6.5.1. Let $\Omega_1 \subseteq \mathbb{R}^n$, $\Omega_2 \subseteq \mathbb{R}^n$. We say that $\phi : \Omega_1 \to \Omega_2$ is a C^1-diffeomorphism. if

(i) ϕ is bijective

(ii) $\phi \in C^1$ (or $\frac{\partial \phi_j}{\partial x_k}$ exist and are continuous)

(iii) $\det \phi'(x) \neq 0$ (i.e., $\phi'(x)$ is invertible)

Remark 6.5.2. We say that ϕ_1 and ϕ_2 are diffeomorphially equivalent.

1. $\Omega_1 \sim \Omega_1$, $\phi(x) = x$, $\phi'(x) = I$
2. $\phi : \Omega_1 \to \Omega_2$, C^1-diffeomorphism, $\phi^{-1}(x)$ is also a C^1-diffeomorphism. (use the inverse function theorem which proves the differentiability of ϕ^{-1}) $(\phi^{-1})'(y) = (\phi'(\phi^{-1}(y)))^{-1} = (\phi'(x))^{-1}$ where $x = \phi^{-1}(y)$
3.

Example 6.5.3

$\phi : \underbrace{(0,2) \times (0,\pi/4)}_{\Omega_1} \to \Omega_2$ defined by $\phi(x_1, x_2) = (x_1 \sin x_2, x_1 \cos x_2)$ $\phi(r,\theta) = (r\sin\theta, r\cos\theta)$

$$\phi'(x) = \begin{pmatrix} \sin x_2 & x_1 \cos x_2 \\ \cos x_2 & -x_1 \sin x_2 \end{pmatrix}$$

$\det \phi'(x) = -x_1 = -r$

Theorem 6.5.4

Let Ω_1, Ω_2 are open sets in $\mathbb{R}^n$, $\phi : \Omega_1 \to \Omega_2$ is a C_1-diffeomorphism. $f \in \mathcal{L}^1(\Omega_2)$ or $f \in \mathcal{L}^+(\Omega_2)$ then

$$\int_{\Omega_2} f(y)\ \mathrm{d}m(y) = \int_{\Omega_1} f(\phi(x)) |\det \phi'(x)|\ \mathrm{d}m(x)$$

Remark 6.5.5. In calculus, we have substitution rule

$$\int_{\phi(a)}^{\phi(b)} f(y)\ \mathrm{d}y = \int_a^b f(\phi(x))\phi'(x)\ \mathrm{d}x$$

Proof. It is enough to show that

$$\int_{\Omega_2} f(y)\ \mathrm{d}m(y) \leq \int_{\Omega_1} f(\phi(x)) |\det \phi'(x)|\ \mathrm{d}m(x)$$

Then we may apply this inequality for the diffeomorphism $\phi^{-1} : \Omega_2 \to \Omega_1$ to get the reverse inequality.

$$\int_{\Omega_1} g(x)\ \mathrm{d}m(x) \leq \int_{\Omega_2} g(\phi^{-1}(y)) |\det(\phi^{-1})'(y)|\ \mathrm{d}m(y)$$

Apply

$$g(x) = f(\phi(x))|\det \phi'(x)|$$

$$\int_{\Omega_1} g(x)\ \mathrm{d}m(x) \le \int_{\Omega_2} f(\phi(\phi^{-1}(y)))|\det \phi'(\phi^{-1}(y))||\det(\phi^{-1})'(y)|\ \mathrm{d}m(y)$$

we know that $\phi(\phi^{-1}(y)) = y$, $\phi^{-1}(\phi'(x)) = x$ then take the differentiable, we get

$$(\phi^{-1})'(\phi(x))\phi'(x) = I$$

$$\phi'(\phi^{-1}(y))(\phi^{-1})'(y) = I$$

If we can prove $f \in L^+(\Omega_2)$ we can also prove it for general $f \in L^1$

Noted that it suffices to prove the $\le$

First step: $f = \mathbb{1}_{\phi(E)}$ we need to prove

$$m(\phi(E))^{1} \le \int_E |\det \phi'(x)|\ \mathrm{d}x$$

Assume that E is a cube Q away from the boundary $\operatorname{dist}(Q, \Omega_1^{\complement}) > 0$ For any $\varepsilon > 0$, let $\delta > 0$,

$$\|\phi'(\widetilde{x})^{-1}\phi'(x) - I\|_{\infty-\infty} < \varepsilon, \quad \text{where } |x - \widetilde{x}|_\infty < \delta$$

where $|x - \widetilde{x}|_\infty = \max_{1\le j\le n} |x_j - \widetilde{x}_j|$ and $\|A\|_{\infty-\infty} = \max_i \sum_{k=1}^n |a_{ik}|$ we decompose $Q = \biguplus Q^\nu$ where $\operatorname{diam} Q^\nu < \delta$ then $|\det \phi'(x) - \det \phi'(\widetilde{x})| < \varepsilon$ if $|x - \widetilde{x}| < \delta$ Now, we get that $\phi(Q_\nu)$ are all disjoint then

$$m(\phi(Q)) = \sum_\nu m(\phi(Q_\nu))$$

$$\phi(Q_\nu) = \phi'(x_\nu)\phi'(x_\nu)^{-1}\phi$$

then

$$\Psi^\nu = x \mapsto \phi'(x_\nu)^{-1}\phi(x)$$

for $x \in Q^\nu$, $\|(\Psi^\nu)' - I\|_{\infty-\infty} < \varepsilon$

Claim: $\Psi^\nu(Q^\nu)$ contain in a cube centered at $\Psi^\nu(x^\nu)$ with side length $= (1+\varepsilon)$ side length(Q_ν). Thus $m(\Psi^n u(Q^\nu)) \le (1+\varepsilon)^n m(Q_\nu)$. (Need to estimate $\Psi^\nu(x) - \Psi^\nu(x^\nu)$ for $x \in Q_\nu$)

Assuming the claim.

$$m(\phi(Q_\nu)) = |\det \phi'(x_\nu)| m(\Psi^\nu(Q))$$

[1]Note that $\mathbb{1}_{\phi(E)}(\phi(x)) = \mathbb{1}_E(x)$

by the Corollary for linear differentiation then

$$\begin{aligned}
m(\phi(Q_\nu)) &\leq |\det \phi'(x_\nu)|(1+\varepsilon)^n \int_{Q_\nu} \mathrm{d}m(x) \\
&= (1+\varepsilon)^n \sum_\nu \int_{Q_\nu} \underbrace{|\det \phi'(x) - (\det \phi'(x) - \det \phi'(x^\nu))|}_{\leq |\det \phi'(x)| + \varepsilon} \mathrm{d}m(x) \\
&\leq (1+\varepsilon)^n \sum_\nu \int_{Q^\nu} |\det \phi'(x)| + (1+\varepsilon)^n \sum_\nu \int_{Q_\nu} \varepsilon \\
&\leq (1+\varepsilon)^n \int_Q |\det \phi'(x)| \, \mathrm{d}m(x) + (1+\varepsilon)^n \varepsilon m(Q)
\end{aligned}$$

For the **Claim**: we need to check

$$|\Psi_j^\nu(x) - \Psi_j^\nu(x^\nu)| \leq (1+\varepsilon) \max_{1 \leq k \leq n} |x_k - x_k^\nu|$$

$$\begin{aligned}
|\Psi_j^\nu(x) - \Psi_j^\nu(x^\nu)| &= \left| \int_{s=0}^1 \sum_{k=1}^n (x_k - x_k^\nu) \frac{\partial \Psi_j^\nu}{\partial x_k}(x^\nu + s(x - x^\nu)) \, \mathrm{d}s \right| \\
&= \left| (x_j - x_j^\nu) \int_0^1 \sum_{k=1}^n (x_k - x_k^\nu) \left(\frac{\partial \Psi_j^\nu}{\partial x_k} - I_{jk} \right) (x^\nu + s(x - x^\nu)) \, \mathrm{d}s \right| \\
&\leq |x_j - x_j^\nu| + \int_0^1 \|x - x^\nu\|_\infty \underbrace{\sum_k \left| \frac{\partial \Psi_j^\nu}{\partial x_k} - I_{jk} \right| (x^\nu + s(x - x^\nu))}_{\leq \varepsilon} \, \mathrm{d}s \\
&\leq |x_j - x_j^\nu| + \varepsilon \|x - x^\nu\|_\infty
\end{aligned}$$

So, $\|\Psi^\nu(x) - \Psi^\nu(x^\nu)\|_\infty\|_\infty \leq (1+\varepsilon)\|x - x^\nu\|_\infty$

If E is an open set $\mathcal{O} \subseteq \Omega_1$ then by Whitney theorem, we can decompose

$$\mathcal{O} = \biguplus_{l=1}^\infty Q_l$$

where the cubes Q_l are disjoint and aways stay from the boundary of Ω_1. then

$$m(\phi(\mathcal{O})) = \sum m(\phi(\mathcal{O}_l)) \leq \sum_l \int_{Q_l} |\det \phi'(x)| \, \mathrm{d}m = \int_{\mathcal{O}} |\det \phi'(x)| \, \mathrm{d}x$$

Let $R \in \mathbb{N}$ then

$$\Omega_{1,R}\{x : |x| \leq R, \operatorname{dist}(x_1, \Omega_1^\complement) \geq \frac{1}{R}\}$$

and we get that $\overline{\Omega_{1,R}}$ is compact and

$$\int_{\Omega_{1,R}} |\det \phi'(x)| < \infty$$

Let $E \subseteq \Omega_{1,R}$ then find an $\mathcal{O}_k, m(\mathcal{O}_k) \le m(E) + 2^{-k}$. We can construct such that $\mathcal{O}_1 \supseteq \mathcal{O}_2 \supseteq \cdots$. we canuse continuity from abouve or dominated convergence theorem to get

$$m(E) \le \int_{\mathcal{O}_k} |\det \phi'(x)| \, \mathrm{d}m(x) \to \int_E |\det \phi'(x)| \, \mathrm{d}m(x)$$

Main Step: (special case of $f = \mathbb{1}_E$)

$$m(\phi(E)) \le \int_E |\det \phi'| \, \mathrm{d}m$$

for $E \subseteq \Omega_1$ open set. replace $\phi(E)$ by A and E by $\phi^{-1}(A)$ $A_1, \dots, A_j$ are given, $c_1, \dots, c_j$, $c_j > 0$, let

$$s(x) = \sum_{i=1}^{j} c_i \mathbb{1}_{A_i}$$

then

$$\sum_{i=1}^{j} c_k \int \mathbb{1}_{A_i} \, \mathrm{d}m \le \sum_{i=1}^{j} c_i \int_{\phi^{-1}(A_i)} |\det \phi'(x)| \, \mathrm{d}m$$

$$x \in \phi^{-1}(A_i) \iff \phi(x) \in A_i$$

then $\mathbb{1}_{\phi^{-1}(A_i)} = \mathbb{1}_{A_i}(\phi(x))$ so

$$\int s(y) \, \mathrm{d}m(y) \le \int \sum c_i \mathbb{1}_{\phi^{-1}(A_i)}(x) |\det \phi'(x)| \, \mathrm{d}m(x) \quad \le \int s(\phi(x)) |\det \phi'(x)| \, \mathrm{d}m(x)$$

then $s_n \nearrow f$ apply MCT given the formula for $f \in \mathcal{L}^+$

$$f \in L^1 \implies f = (\Re f)_+ - (\Re f)_- + i((\Im f)_+ - (\Im f)_-)$$

$\square$

7 Additional Topics

§7.1 Lebesgue Differentiation Theorem

In Calculus, $f \in C[a,b]$ we have theorem that

$$F(x) = \int_a^x f(t)\ \mathrm{d}t$$

then F is differentiable at all $x \in [a,b]$ and $F' = f$ we use the prove that

$$\lim_{h \to 0^+} \frac{F(x+h) - F(x)}{h} = f(x), x < b$$

$$\lim_{h \to 0^-} \frac{F(x+h) - F(x)}{h} = f(x), x > a$$

rewrite the limit as

$$\frac{F(x+h) - f(x)}{h} = \frac{1}{h} \int_x^{x+h} (f(t) - f(x) + f(x))\ \mathrm{d}t = f(x) + \frac{1}{h} \int_x^{x+h} f(t) - f(x)\ \mathrm{d}t$$

For t close to x, use $|f(t) - f(x)| < \varepsilon$, $|t| < \delta$ for given similarly for negative

Theorem 7.1.1 (Lebesgue Differentiation Theorem)
If $f \in L^1([a,b])$,

$$F(x) = \int_{[a,x]} f\ \mathrm{d}m$$

then F is differentiable almost everywhere and $F' = f$ almost everywhere.

Definition 7.1.2. In $\mathbb{R}^n$ averages over ball $B(x,r) = \{y : |y - x| < r\}$

$$\mathcal{A}_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} |f|\ \mathrm{d}m$$

Definition 7.1.3.

$$M_{\mathrm{HL}} f(x) = \sup_{r>0} \mathcal{A}_r |f|$$

Remark 7.1.4. Note the function $x \mapsto \mathcal{A}_r f(x)$ is continuous

$$\left| \int_{B(x,r)} |f| - \int_{B(\widetilde{x},r)} |f| \right| \leq \int_{B(x,r) \triangle B(\widetilde{x},r)} |f| \to 0$$

as $x \to \widetilde{x}$ Observe that

$$\Omega_\alpha = \left\{ x : \sup_{r>0} \mathcal{A}_r f(x) > \alpha \right\}$$

is an open set because union of open sets are open set

Theorem 7.1.5 (Hardy and Littlewood)

M satisfies the inequality

(i) $m\left(\{x : M_{\mathrm{HL}} f(x) > \alpha\}\right) \leq C_n \frac{\|f\|_1}{\alpha}$

(ii) If $p > 1$, $\|M_{\mathrm{HL}} f\|_p \leq C_{p,n} \|f\|_p$

(iii) is clearly true for $p = \infty$ (for simplicity for continuous f)

In general part (ii) is an instance of an interpolation.

We say that a constant A essentially bounds a function f if $|f(x)| \leq A$ for almost every x.

$$\operatorname{esssup} |f(x)| = \inf\{A : A \text{ essentially bounds } |f|\}$$

$$|f(x)| \leq B + \frac{1}{n} \text{ for } x \in X \setminus N_n \implies |f(x)| < B \text{ for } x \in X \setminus \bigcup N_n$$

Proof. $\Omega_\alpha = \{x : M_{\mathrm{HL}} f(x) > \alpha\}$, $m(\Omega_\alpha) = \sup_{K \subseteq \Omega_\alpha} m(K)$. It suffices to show that for every compact set K contained in Ω_α, $m(K) \leq 3^n \frac{\|f\|_1}{\alpha}$ fix $x \in K$ then there exists a ball $B(x, r(x))$ such that

$$\frac{1}{m(B(x,r(x)))} \int_{B(x,r(x))} |f| \, \mathrm{d}m > \alpha \implies m(B(x, r(x))) < \frac{1}{\alpha} \int_{B(x,r(x))} |f| \, \mathrm{d}m$$

there is a finite family of ball $B(x_i, r(x_i))$, for $i = 1, \ldots, L$ such that

$$K \subseteq \bigcup_{B \in \mathfrak{B}} B$$

K is compact

Covering Lemma Given ball $\mathfrak{B} = \{B_1, \ldots, B_L\}$ there exists subcollection $\widetilde{\mathfrak{B}} \subseteq \mathfrak{B}$ of ball $\{B : B \in \widetilde{B}\}$ which is pairwise disjoint such that the three fold balls $\{3B : B \in \widetilde{B}\}$ contain

$$\bigcup_{B \in \mathfrak{B}} B$$

(union of the original ball) then Notice that $B^* \equiv 3B \equiv$ ball of radius 3 radius of B centered at the center of B. $(x_B + 3(B - x_B))$

Proof by induction on the cardinality of $\mathfrak{B}$. $|\mathfrak{B}| = N > 1$, find ball $B_1 \in \mathfrak{B}$ with the maximal radius. Observe that for any $B \in \mathfrak{B}$, if $B \cap B_1 \neq \emptyset$ then $B \subseteq B_1^*$. Consider the collection $\mathfrak{B}_2 = \{$all ball in $\mathfrak{B}$ which are disjoint from $B_1\}$ By the induction hypothesis (and $|\mathfrak{B}_2| < |\mathfrak{B}| = N$) there are finitely many balls $B_2, \ldots, B_M$ disjoint and

$$\bigcup_{j=2}^{M} B_j^* \supseteq \bigcup_{B \in \mathfrak{B}_2} B$$

All balls in $\mathfrak{B}$, that intersect B_1 are contained in B_1^*.

$$m(B(x, r(x_j))) \leq \frac{1}{\alpha} \int_{B(x_j, r(x_j))} |f|$$

Apply the covering lemma, we get balls

$$B_1 = B(x_1, r(x_1)), \ldots, B_M = B(x_M, r(x_M))$$

disjoint, and

$$\bigcup B_i^* \supseteq K$$

then

$$\begin{aligned} m(K) &\leq m\left(\bigcup_{i=1}^{M} B_i^*\right) \leq \sum_{i=1}^{M} m(B_i^*) = 3^n \sum_{i=1}^{M} m(B_i) \\ &\leq 3^n \sum_{i=1}^{n} \frac{1}{\alpha} \int_{B_i} |f| \ \mathrm{d}m \leq \frac{3^n}{\alpha} \int |f| \ \mathrm{d}m = 3^n \alpha^{-1} \|f\|_1 \end{aligned}$$

□

Claim in the Lebesgue differentiation theorem Let $f \in L^1$ then for almost every where $x \in \mathbb{R}^n$

$$\frac{1}{m(B(x, r))} \int_{B(x,r)} f(y) \ \mathrm{d}m(y) = f(x)$$

slightly strong statement

$$\frac{1}{m(B(x, r))} \int_{B(x,r)} |f(y) - f(x)| \ \mathrm{d}m(y) \to 0$$

for almost everywhere x (If this is true for a specific $x \in \mathbb{R}^n$ then we call x a Lebesgue point)

Theorem 7.1.6

for $f \in L^1$ then for almost every $x \in \mathbb{R}^n$

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}m(y) = 0$$

Proof. Define

$$E_\alpha = \left\{ x : \limsup_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}m(y) > \alpha \right\}$$

The goal is to show that for any $\alpha > 0$,

$$m\left(E_\alpha\right) = 0$$

We know that for every $\delta > 0$ there exists g continuous with compact support that

$$\|f - g\|_1 < \delta$$

So, we can get

$$\frac{1}{m(B(x,r))} \int_{m(B(x,r))} |f(y) - g(y)| \, \mathrm{d}m(y) \to 0, \; r \to 0$$

since $f(y) - f(x) = (f(y) - g(y)) + (g(y) - g(x)) + (g(x) - f(x))$ then

$$\begin{aligned}
&\limsup_{r \to 0} \frac{1}{m(B(x,r))} \int_{m(B(x,r))} |f(y) - f(x)| \, \mathrm{d}m(y) \\
&= \limsup_{r \to 0} \frac{1}{m(B(x,r))} \int_{m(B(x,r))} |(f-g)(y) - (f-g)(x)| \, \mathrm{d}m(y) \\
&\leq \limsup_{r \to 0} \frac{1}{m(B(x,r))} \int_{m(B(x,r))} |(f-g)(y)| \, \mathrm{d}m(y) + |f(x) - g(x)| \\
&\leq M_{\mathrm{HL}}(f-g)(x) + |f(x) - g(x)|
\end{aligned}$$

then

$$\begin{aligned}
m(E_\alpha) &\leq m\left(\{x : M_{\mathrm{HL}}(f-g)(x) > \alpha/2\}\right) + m\left(\{x : |f(x) - g(x)| > \alpha/2\}\right) \\
&\leq \frac{C}{\alpha/2}\|f - g\|_1 + \underbrace{\frac{1}{\alpha/2}\|f - g\|_1}_{\text{from Chebyshev}} \\
&\leq \frac{2C + 2}{\alpha}\|f - g\|_1 \\
&\leq \frac{2C + 2}{\alpha}\delta \\
&< \varepsilon
\end{aligned}$$

□

$$\int_K |f| \, \mathrm{d}m < C_K$$

for all compact set. Fix any ball K of radius 1. show that

$$\int_{B(x,r)} ||f(y) - f(x)| \, \mathrm{d}m \to 0 \text{ as } r \to 0$$

$f = f\mathbb{1}_{K^*} + f\mathbb{1}_{\mathbb{R}^n \setminus K^*}$ If $x \in K$ and r is sufficiently small then

$$\int_{B(x,r)} |f(x) - f(y)| \mathrm{d}m(y) = \int |f\mathbb{1}_{K^*}(x) - f\mathbb{1}_{K^*}(y)| \, \mathrm{d}m(y)$$

Extension, if $E_r(x)$ is a sequence of sets such that $m(E_r(x)) \geq \gamma m(B(x,r))$ then we shall have

$$\int_{E_r(x)} |f(x) - f(y)| \, \mathrm{d}m(y) \to 0$$

when $r \to 0$ a.e.

We can replace $B(x,r)$ b

"Trivial inequality"

$$\|Mf\|_\infty \leq \|f\|_\infty$$

where $\| \cdot \| = \text{esssup}$

Theorem 7.1.7

For $p > 1$

$$\|Mf\|_p \leq C_p \|f\|_p$$

More generally, let T defined on L^1, L^∞ and therefore $L^1 + L^\infty$. and let T be sublinear (subadditive) $T(f+g)(x) \leq T(f)(x) + T(g)(x)$ Assume

(i) weak type (1, 1) inequality

$$m(\{x : |Tf(x)| > \alpha\}) \leq A \frac{\|f\|_1}{\alpha}$$

(ii) $\|Tf\|_\infty \leq B\|f\|_\infty$

for all $f \in L^1 \cup L^\infty$ Then there is C_p such that

$$\|Tf\|_p \leq C_p A^{1/p} B^{1-1/p} \|f\|_p$$

UseLayer-cake formula for L^p-norm of g.

$$\int |g|^p \, \mathrm{d}\mu = \int_0^\infty p\alpha^{p-1} \mu(\{x : |g| > \alpha\}) \, \mathrm{d}\alpha$$

We apply this for $Tf, \beta = \beta(\alpha)$

$$f_\beta(x) = \begin{cases} f(x) & |f(x)| \le \beta \\ 0 & |Tf(x)| > \beta \end{cases}$$

$$f^\beta(x) = \begin{cases} 0 & |f(x)| \le \beta \\ f(x) & |f(x)| > \beta \end{cases}$$

then $f = f_\beta + f^\beta$

$$|Tf| \le |Tf_{\beta(\alpha)} + |Tf^{\beta(\alpha)}|$$

we know that

$$\{x : |Tf| > \alpha\} \subseteq \{x : |Tf_\beta| > \frac{\alpha}{2}\} \cup \{x : |Tf^\beta| > \frac{\alpha}{2}\}$$

then

$$m(\{x : |Tf| > \alpha\}) \le m(\{x : |Tf_\beta| > \frac{\alpha}{2}\}) + m(\{x : |Tf^\beta| > \frac{\alpha}{2}\})$$

$|f_\beta| \le \beta$ a.e $\implies |Tf_\beta| \le B\beta$ a.e.

Define $\beta = \beta(\alpha) = \frac{\alpha}{2B}$ then the first term $m(\{x : |Tf_\beta| > \frac{\alpha}{2}\}) = 0$.

$$\begin{aligned}
\|Tf\|_p^p &\le p\int_0^\infty \alpha^{p-1} m(\{x : |Tf^{\beta(\alpha)}(x)| \ge \frac{\alpha}{2}\})\ \mathrm{d}\alpha \\
&\le p\int_0^\infty \alpha^{p-1}\frac{A}{\alpha}\int_{\{|f(x)|>\beta(\alpha)\}} |f^{\beta(\alpha)}(x)|\ \mathrm{d}m\ \mathrm{d}\alpha \\
&= \int_X \int_{\alpha=0}^{2B|f(x)|} p\alpha^{p-2}\ \mathrm{d}\alpha |f(x)|\ \mathrm{d}m(x) \\
&= \int_X |f(x)| A\frac{p}{p-1}(2B|f(x)|)^{p-1}\ \mathrm{d}m(x) \\
&= \|f\|_p^p AB^{p-1}\underbrace{\frac{2^{p-1}p}{p-1}}_{C_p^p}
\end{aligned}$$

Weak L^1 also $L^{1,\infty}$

$$\|f\|_{L^{1,\infty}} = \sup_{\alpha > 0} \alpha\mu(x : |f(x)| > \alpha)$$

then

- $\mu_f(\alpha) \le \frac{1}{\alpha}\|f\|_{L^{1,\infty}}$
- $\|cf\|_{L^{1,\infty}} = |c|\|f\|_{L^{1,\infty}}$
- $\|f + g\|_{L^{1,\infty}} \le 2(\|f\|_{L^{1,\infty}} + \|g\|_{L^{1,\infty}})$

Q: Is there a norm $\|\cdot\|$ such that $\|f\| \approx \|f\|_{L^{1,\infty}}$?

A: No

Argue by contradiction, let $\|f\|_*$ be a norm on $L^{1,\infty}(\mathbb{R}, m)$ Goal is to show that there is a sequence $f_N \in L^{1,\infty}$ such that

$$\frac{\|f_N\|_{L^{1,\infty}}}{\|f_N\|_*} \to \infty$$

Simple building block is $\frac{1}{|x|}$, $m\left(\{x : \frac{1}{|x|} > \alpha\}\right) = \frac{2}{\alpha}$ $\frac{1}{|x-k|} \in L^{1,\infty}$ and $g_k(x) = \frac{1}{|x-k|}$ then $\|g_k\|_{L^{1,\infty}} = c$ then

$$f_N = \sum_{k=1}^{N} g_k(x) = \sum_{k=1}^{N} \frac{1}{|x-k|}$$

$$\|f_N\|_* \leq \sum_{k=1}^{N} \|g_k\|_* \leq cN$$

Consider this f_N in $[0, N]$, for $x \in [\frac{N}{4}, \frac{N}{2}]$

$$\sum_{k=1}^{N} \frac{1}{|x-k|} > c \sum_{1 \leq l \leq N/2} \frac{1}{l} \geq c^1 \log N$$

Conclusion on a set of measure $N/4$, $|f_N(x)| > c^1 \log N$. Let $\alpha = c^1 \log N$ then

$$\begin{aligned} \|f_N\|_{L^{1,\infty}} &\geq c^1 \log N \mu(\{x : |f_N(x)| > c^1 \log N\}) \\ &> \frac{N}{4} c^1 \log N \end{aligned}$$

§7.2 Convolution

or approximation of the identity. Let take $\phi \in L^1$ (or much nicer)

$$\int \phi = 1, \ \int \frac{1}{t^n} \phi\left(\frac{y}{t}\right) \ \mathrm{d}m(y) = \int \phi = 1$$

$$f * \phi_t = \int f(x-y) \frac{1}{t^n} \phi\left(\frac{y}{t}\right) \mathrm{d}y, \ \phi_t = \frac{1}{t^n} \phi\left(\frac{n}{t}\right)$$

If f is continuous, $\lim_{|x| \to \infty} f(x) = 0$

$$\int f(x-y) \frac{1}{t^n} \phi\left(\frac{y}{t}\right) \ \mathrm{d}t - f(x) = \int f(x-y) - f(x) \frac{1}{t^n} \phi\left(\frac{y}{t}\right) \ \mathrm{d}m(y)$$

Claim: f is uniformly continuous on $\mathbb{R}$

Given ε, there is δ such that $|f(x-y)-f(x)|<\varepsilon$ if $|y|<\delta$

$$\left|\int_{\mathbb{R}} f(x-y)-f(x)\frac{1}{t^n}\phi\left(\frac{y}{t}\right)\ \mathrm{d}t\right| =$$

$$\int_{|y|<\delta}\underbrace{|f(x-y)-f(x)|}_{<\varepsilon}\left|\frac{1}{t^n}\phi\left(\frac{y}{t}\right)\right|\ \mathrm{d}t+\int_{|y|>\delta}\underbrace{|f(x-y)-f(x)|}_{<\varepsilon}\left|\frac{1}{t^n}\phi\left(\frac{y}{t}\right)\right|\ \mathrm{d}t$$

$$\leq \varepsilon\int\left|\frac{1}{t^n}\phi(\frac{y}{t})\right|\mathrm{d}t+\int_{|y|\geq\delta}2\|f\|_\infty\frac{1}{t^n}|\phi(\frac{y}{t})|\ \mathrm{d}t$$

then $\int_{|y|\geq\delta}\frac{1}{t^n}|\phi(\frac{y}{t})|\ \mathrm{d}t$ change variable $\int_{|w|>\delta/t}|\phi(w)|\mathrm{d}w\to 0$ on $t\to\infty$

$f\in L^1$, $\|f(\cdot-y)-f(\cdot)\|_{L^1}\to 0$ as $|y|\to 0$

There exists g with compact support such that $\|f-g\|_{L^1}<\varepsilon$ then

$$\|f(\cdot-g)-f(\cdot)-(g(\cdot y)-g(\cdot))+g(\cdot-y)-g(\cdot)\|_{L^1}\|_1+\|g(\cdot-y)-g(\cdot)\|_1\to 0$$

as $y\to 0$

$f\in L^1$, $\phi\in L^1$, $\frac{1}{t^n}\phi(\frac{\cdot}{t})=\phi_t$, $\int\phi=1$ then

$$\begin{aligned}
&=\int_x\int_y|f(x-y)-f(x)||\phi_t(y)|\ \mathrm{d}y\ \mathrm{d}x\\
&=\int_{|y|\leq\delta}|\phi_t(y)|\int_x|f(x-y)-f(x)|\ \mathrm{d}x\ \mathrm{d}y-\int_{|y|>\delta}|\phi_t(y)|\int|f(x-y)-f(x)|\ \mathrm{d}x\ \mathrm{d}y\\
&\leq\varepsilon\|\phi\|_{L^1}+\|f\|_1 2\int_{|y|>\delta}|\phi_t(y)|\ \mathrm{d}y
\end{aligned}$$

For pointwise convergence, we would have to prove an estimate

$$\sup_t|\phi_t*f(x)|\leq M_{\mathrm{HL}}f(x)$$

Given $E, m(E)=1$ star like with respect to the origin. If $f\in L^p, p>1$

$$\frac{1}{m(rE)}\int_{rE}f(x+y)\ \mathrm{d}m(y)\to f(x)$$

a.e.

$$M_Ef=\sup_{r>0}\frac{1}{m(rE)}\int_{rE}|f(x+y)|\ \mathrm{d}m$$

Show: $M_E:L^p\to L^p, p>1$

For $1 < p < \infty$

$$\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}} \le 1} \left| \int f(x) g(x) \, \mathrm{d}x \right|$$

where $g \in L^{p'}$, $1/p + 1/p' = 1$

$$\begin{aligned}
\left\| \int_I f(x, \varphi) \, \mathrm{d}\varphi \right\|_{L^p} &= \sup_{\|g\|_{L^{p'}} \le 1} \left| \int \int f(x,\varphi) \, \mathrm{d}\varphi g(x) \, \mathrm{d}x \right| \\
&\le \sup_{\|g\|_{L^{p'}} \le 1} \left| \int_I \left| \int_x f(x,\varphi) g(x) \, \mathrm{d}x \right| \, \mathrm{d}\varphi \right| \\
&\le \int_I \sup_{\|g\|_{L^{p'}} \le 1} \int |f(x,\varphi) g(x)| \, \mathrm{d}x \, \mathrm{d}\varphi \\
&\le \int_I \sup_{\|g\|_{L^{p'}} \le 1} \left(\int_x |f(x,p)|^p \, \mathrm{d}x \right)^{1/p} \left(\int |g(x)|^{p'} \right)^{1/p'} \\
&\le \int_I \left(\int |f(x,\varphi)|^p \, \mathrm{d}x \right)^{1/p} \, \mathrm{d}\varphi \\
&= \int_I \|f(\cdot, \varphi)\|_{L^p} \, \mathrm{d}\varphi
\end{aligned}$$

§7.3 Inner Product Spaces

Definition 7.3.1. Given $(X, \mathcal{M}, \mu)$ measure space

$$L^2(X, \mu) \equiv L^2 = \left\{ f : \left(\int |f|^2 \mathrm{d}\mu \right)^{1/2} < \infty \right\}$$

called the Hilbert space of square integrable functions.

Definition 7.3.2. we defnote the inner product space

$$\langle f, g \rangle = \int f(x) \overline{g(x)} \, \mathrm{d}\mu$$

> **Remark 7.3.3.** We can write
>
> $$\|f\|_2 = \langle f, f \rangle$$

Definition 7.3.4. The following properties hold

(i) Fix g then under the map $f \mapsto \langle f, g \rangle$ is a linear functional on L^2

(ii) $\langle f, g \rangle = \overline{\langle g, f \rangle}$

(iii) $\langle f, f \rangle \ge 0$ and $\langle f, f \rangle = 0$ if $f = 0$ in L^2 if and only if $f = 0$ a.e.

Example 7.3.5

The following are examples of inner products

- Choose μ to be Lebesgue measure or $\mathrm{d}\mu = w\ \mathrm{d}x$, w non-negative
- $X = \{1, 2, \ldots, n\}$, μ counting measure

$$\langle f, g\rangle = \sum_{j=1}^{n} f(j)\overline{g(j)}$$

In general a vector space (over $\mathbb{R}$ or $\mathbb{C}$) with an inner product is called inner product space or Euclidean space.

Lemma 7.3.6

Properties of inner products

(i) Cauchy-Schwarz inequality

$$\langle f, g\rangle \le \sqrt{\langle f, f\rangle}\sqrt{\langle g, g\rangle}$$

(ii) $\|f\| = \sqrt{\langle f, f\rangle}$ is a norm on L^2

Proof. (i) We will consider the case that $\langle f, g\rangle \in \mathbb{R}$.

$$\begin{aligned}\langle f + tg, f + tg\rangle &= \langle f, f\rangle + t\langle g, f\rangle + t\langle f, g\rangle + t^2\langle g, g\rangle \\ &= \langle f, f\rangle + 2t\langle f, g\rangle + t^2\langle g, g\rangle\end{aligned}$$

If $\langle g, g\rangle = 0$ then $g = 0$ so $\langle f, g\rangle = 0$.

Define $p(t) := \langle f, f\rangle + 2t\langle f, g\rangle + t^2\langle g, g\rangle$ then the minimum of $p(t)$ is at $t_0 = -\frac{\langle f,g\rangle}{\langle g,g\rangle}$ and $p(t) \ge 0$ for all t then

$$\begin{aligned}0 &\le \langle f, f\rangle + 2t_0\langle f, g\rangle + t_0^2\langle g, g\rangle \\ &= \langle f, f\rangle - \frac{|\langle f, g\rangle|^2}{\langle g, g\rangle} \\ |\langle f, g\rangle|^2 &\le \langle f, f\rangle\langle g, g\rangle \\ |\langle f, g\rangle| &\le \sqrt{\langle f, f\rangle}\sqrt{\langle g, g\rangle}\end{aligned}$$

For general cases, then there exists $c \in \mathbb{C}$ such that $|c| = 1$ and $c\langle f, g\rangle = \langle cf, g\rangle \in \mathbb{R}$ then

$$|\langle f, g\rangle| = c\langle f, g\rangle = \langle cf, g\rangle \le \sqrt{\langle cf, cf\rangle}\sqrt{\langle g, g\rangle} = \sqrt{\langle f, f\rangle}\sqrt{\langle g, g\rangle}$$

(ii)

$$\begin{aligned}\|f+g\|^2 &= \langle f+g, f+g\rangle \\ &= \langle f, f+g\rangle + \langle g, f+g\rangle \\ &\leq \sqrt{\langle f,f\rangle}\sqrt{\langle f+g, f+g\rangle} + \sqrt{\langle g,g\rangle}\sqrt{\langle f+g, f+g\rangle} \\ &= \|f\|\|f+g\| + \|g\|\|f+g\| \\ &\leq (\|f\| + \|g\|)\|f+g\| \\ \|f+g\| &\leq \|f\| + \|g\|\end{aligned}$$

□

Definition 7.3.7. An inner product space H is a Hilbert space if H is complete as a normed space

$$\|f\| = \langle f, f\rangle^{1/2}$$

Lemma 7.3.8

Let $\lambda_g(f) = \langle f, g\rangle$ then λ_g is a bounded linear functional on H. and

$$\|\lambda_g\|_{H'} = \|g\|_H$$

Remark 7.3.9. H' is the dual space of bounded linear function of H.

Proof. From the Cauchy-Schwarz inequality, for any $f \in H$,

$$|\lambda_g(f)| = |\langle f, g\rangle| \leq \|g\|_H\|f\|_H$$

Thus,

$$\|\lambda_g\|_{H'} \leq \|g\|_H$$

Next,

$$|\lambda_g(g)| = |\langle g, g\rangle| = \|g\|_H^2$$

So, we get $\|\lambda_g\|_{H'} = \|g\|_H$ □

Theorem 7.3.10 (Characterizations of all bounded linear functionals)

If H is a Hilbert space then all bounded linear functions on H are of the form λ_g for some $g \in H$.

Example 7.3.11

$\mathbb{V} = \mathcal{C}[-1, 1]$,

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\overline{g(x)}\, \mathrm{d}x$$

$$g = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0 \end{cases}$$

$$\lambda_g(f) = \int_0^1 f(x)\, \mathrm{d}x - \int_{-1}^{0} f(x)\, \mathrm{d}x$$

Proof. Let $\lambda \in H'$. WLOG, $\|\lambda\| = 1$. (Otherwise, using λ devided by $\|\lambda\|$) where

$$\|\lambda\| = \sup_{f \neq 0} \frac{|\lambda(f)|}{\|f\|} = \sup_{\|f\|=1} |\lambda(f)|$$

The idea is to find a g such that $\lambda(g) = \|\lambda\|_H$

Parallogram identity

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2)$$

Find a sequence $u_n \in H$, $\|u_n\| = 1$ so that

$$|\lambda(u_n)| \geq \|\lambda\| - \frac{1}{n} = 1 - \frac{1}{n}$$

$$\lambda(u_n) = e^{i\theta_n}|\lambda(u_n)|$$
$$|\lambda(u_n)| = \lambda(e^{-i\theta_n}u_n) \in [0, 1] \geq 1 - \frac{1}{n}$$

$g_n = e^{i\theta_n}u_n$, $\|g_n\| = 1$, $\lambda(g_n) \geq 1 - \frac{1}{n}$

WTS: g_n is a Cauchy sequence in H. Estimate: $\|g_n - g_m\|^2$

$$\|g_n - g_m\|^2 = \underbrace{2\|g_n\|^2 + 2\|g_m\|^2}_{4} - \|g_n - g_m\|^2$$

$$\underbrace{\|\lambda\|}_{1} \cdot \|g_n + g_m\|_H \geq \lambda(g_n + g_m) = \lambda(g_n) + \lambda(g_n) \geq 1 - \frac{1}{n} + 1 - \frac{1}{m}$$

So,

$$\|g_n + g_m\|_H^2 \geq \left(2 - \frac{1}{n} - \frac{1}{m}\right)^2$$

$$\|g_n + g_m\|_H^2 \leq 4 - \left(2 - \frac{1}{n} - \frac{1}{m}\right)^2 = 4\left(\frac{1}{n} + \frac{1}{m}\right) - \left(\frac{1}{n} + \frac{1}{m}\right)^2$$

Now, g_n Cauchy $\implies g_n \to g \in H$, $\|g\| = 1$, $\lambda(g) = 1$

Show: $\lambda = \lambda_g$, Do this by showing for all $f \in H$, $\Re\lambda(f) = \Re\langle f, g\rangle$, $\Im\lambda(f) = \Im\langle f, g\rangle$

$$\begin{aligned}
\Re\lambda(f) &= \frac{\Re\lambda(g+tf) - \Re\lambda(g)}{t} \leq \frac{\|g+tf\| - \|g\|}{t} \\
\Re\lambda(f) &= \frac{\Re\lambda(g) - \Re\lambda(g-tf)}{t} = \frac{\Re\lambda(g-tf) - \Re\lambda(g)}{-t} \\
&\geq \frac{\|g\| - \|g-tf\|}{t} = \frac{\|g-tf\| - \|g\|}{-t}
\end{aligned}$$

So, we can bound the function $\Re\lambda(f)$ by

$$\begin{aligned}
\frac{\|g-tf\|^2 - \|g\|^2}{-t} &\leq \Re\lambda(f) \leq \frac{\|g+tf\|^2 - \|g\|^2}{t} \\
\frac{\|g-tf\| - \|g\|}{-t(\|g-tf\| + \|g\|)} &\leq \Re\lambda(f) \leq \frac{\|g+tf\| - \|g\|}{t(\|g+tf\| + \|g\|)}
\end{aligned}$$

We know that $\|g \pm tf\| = \|g\|^2 \pm 2t\Re\langle g, f\rangle + t^2\|f\|^2$ and $\|g\| = 1$ we obtain

$$\frac{2t\Re\langle g, f\rangle - t\|f\|^2}{\|g-tf\| + \|g\|} \leq \Re\lambda(f) \leq \frac{2t\Re\langle g, f\rangle + t\|f\|^2}{\|g+tf\| + \|g\|}$$

then for $t \to 0^+$, we get $\Re\lambda(f) = \Re\langle f, g\rangle$

For complex case, we derive

$$\begin{aligned}
\Im\lambda(f) &= -\Re(i\lambda(f)) \\
&= -\Re\lambda(if) = -\Re\langle if, g\rangle = -\Re i\langle f, g\rangle \\
&= \Im\langle f, g\rangle
\end{aligned}$$

Since $\|\lambda\| = 1$, drop the assumption $\|\lambda\| = 1$, $\frac{\lambda}{\|\lambda\|}$ then there is a g such that

$$\frac{\lambda(f)}{\|\lambda\|} = \langle f, g\rangle \implies \lambda(f) = \langle f, \|\lambda\| g\rangle$$

for all $f \in H$. $\square$

A Practice Exam

§A.1 Practice Exam 1

Problem A.1.1. Let E_n be Lebesgue measurable subsets of $[0,1]$ such that $E_{n+1} \subseteq E_n$. What can you say about the Lebesgue measure of $\bigcap_n E_n$? Does your answer necessarily hold when $[0,1]$ is replaced by $[0,\infty)$?

solution. We can use continuity from above because $\mu([0,1]) < \infty$. We can say that

$$\mu\left(\bigcap_n E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

In case of $[0,\infty)$, we can't use continuity from below becuase if $E_n = [n, n+1)$ then $\mu(E_n) = 1$ but $\bigcap_n E_n = \emptyset$, so, $\lim_{n\to\infty} \mu(E_n) = 1$ but $\mu\left(\bigcap_n E_n\right) = 0$. $\square$

Problem A.1.2. Let

$$\begin{aligned} E = \{x = (x_1, x_2) \in \mathbb{R}^2 : \\ &\frac{1}{1+(x_1-x_2)^3} \le e^{\sin x_1} \text{ if } x_1^{23} < 3|\cos(x_1+x_2)|, \\ &\sqrt{1+e^{|x_2|+|x_1|}} > e^{x_1^2} \text{ if } x_1^{23} > 3|\cos(x_1+x_2)| \text{ and } x_2 \in \mathbb{R}\setminus\mathbb{Q}, \\ &\sqrt{\cos(|x_1x_2|)}\sin(x_1x_2) > 0 \text{ if } x_1^{23} > 3|\cos(x_1+x_2)| \text{ and } x_2 \in \mathbb{Q}\} \end{aligned}$$

(i) Is the characteristic function of E Borel measurable?

(ii) If $\mathcal{M}$ denote the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$ does E belong to $\mathcal{M} \oplus \mathcal{M}$?

solution. (i) Let $f_1(x_1,x_2) = \frac{1}{1+(x_1-x_2)^3} - e^{\sin x_1}$, $f_2(x_1,x_2) = \sqrt{1+e^{|x_2|+|x_1|}} - e^{x_1^2}$, $f_3(x_1,x_2) = \sqrt{\cos(|x_1x_2|)}\sin(x_1x_2)$ and $g(x_1,x_2) = x_1^{23} - 3|\cos(x_1+x_2)|$ Then $E = \{x : f_1(x) \le 0 \text{ if } g(x) < 0, f_2(x) > 0 \text{ if } g(x) > 0 \text{ and } x_2 \in \mathbb{R}\setminus\mathbb{Q} \ldots\}$ $E = (f_1^{-1}(-\infty,0] \cap g^{-1}(-\infty,0)) \cup (f_2^{-1}(0,\infty) \cap g^{-1}((0,\infty)) \cap \mathbb{R}\times(\mathbb{R}\setminus\mathbb{Q})) \cup \ldots$ Then f_1, f_2, f_3, g are Borel measurable (because it is continuous) functions then E is Borel measurable.

(ii)

$\square$

Problem A.1.3. For each of the statements give a proof of find a counterexample.

(i) For each $r \in \mathbb{R}$ let $f_r : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define $G(x) = \sup_{r\in\mathbb{R}} f_r(x)$. Is G a Borel measurable function?

(ii) For each $r \in \mathbb{R}$ let $f_r : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Define $G(x) = \sup_{r\in\mathbb{R}} f_r(x)$. Is G a Borel measurable function?

(iii) For each $r \in \mathbb{R}$ let $f_r : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Define $G(x) = \sup_{r\in\mathbb{Q}} f_r(x)$. Is G a Borel measurable function?

(iv) For each $r \in \mathbb{Q}$ let $f_r : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Define $S(x) = \sum_{r\in\mathbb{Q}} f_r(x)$. Is S a Borel measurable function?

(v) What happens if you replace Borel measurable by Lebesgue measurable in the above statements?

solution. (i) $G^{-1}((-\infty, x]) = \bigcap_{r\in\mathbb{R}} f_r^{-1}((-\infty, x])$ and f_r is continuous so, G is Borel measurable.

(ii) No, $f_r = \mathbb{1}_{\{c_r\}}$ where $c_r \in \mathbb{Q} + r$ and $c_r \in [0, 1)$ then $G^{-1}(\{1\})$ is Vitali set.

(iii) For $E \in \mathcal{M}$, $G^{-1}(E) = \bigcap_{r\in\mathbb{Q}} f_r^{-1}(E)$ and f_r is Borel measurable so, G is Borel measurable.

(iv) Suppose that $\{q_r\}$ is an enumeration of $\mathbb{Q}$ then define $g_n = \sum_{r=1}^{n} f_{q_r}$ then suppose that limit exists then $\limsup g_n$ and $\liminf g_n$ are Borel measurable then S is Borel measurable.

□

Problem A.1.4. Assume that f_n is a sequence of $L^1(\mu)$ functions.

(i) If in addition $\mu(X) < \infty$ and

$$\lim_{n\to\infty} \sup_{x\in X} |f_n(x) - f(x)| = 0$$

then you have learned that $f \in L^1(\mu)$ and that $\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$. Review the proof.

(ii) Show that if $\mu(X) = \infty$ both conclusions may fail in (i).

(iii) If $\mu(X) < \infty$ if $f_n \to f$ converges just pointwise show that for every $\varepsilon > 0$ there exists a set A of measure $< \varepsilon$ such that

$$\int_{A^\complement} |f_n - f| \, \mathrm{d}\mu < \varepsilon$$

solution. (i) Suppose that $\lim_{n\to\infty} \|f_n - f\|_{\sup} \to 0$ then

$$\lim_{n\to\infty} \int |f_n - f| \, \mathrm{d}\mu \le \lim_{n\to\infty} \mu(X)\|f_n - f\|_{\sup} = 0$$

and we know that $f = (f - f_n) + (f_n)$ then

$$\int |f| \, \mathrm{d}\mu = \int |f - f_n + f_n| \, \mathrm{d}\mu \le \int |f - f_n| \, \mathrm{d}\mu + \int |f_n| \, \mathrm{d}\mu$$

so, $f \in L^1$ and

$$\left| \int f_n - f \, \mathrm{d}\mu \right| \le \int |f_n - f| \, \mathrm{d}\mu$$

So,

$$\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$$

(ii) Let $f_n(x) = \frac{1}{x}\mathbb{1}_{[1,n]}$ then $f_n \in L^1$ but $f_n \to f = \frac{1}{x}\mathbb{1}_{[1,\infty)}$ and $f \notin L^1$.

(iii) By Egorov's theorem, for any $\varepsilon > 0$ there exists $A \in \mathcal{M}$ such that $\mu(A) < \varepsilon$ and $f_n \to f$ uniformly on $A^\complement$. Then select n such that $\|f_n - f\| < \frac{\varepsilon}{\mu(X)}$ then

$$\int_{A^\complement} |f_n - f| \mathrm{d}\mu < \varepsilon$$

□

Problem A.1.5. Let $\mathcal{M}$ be a σ-algebra on X.

(i) Consider a function $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : X \to \mathbb{R}^2$ such that for all rational number r_1, r_2 the sets

$$\{x \in X : r_1 < f_1(x), f_2(x) < r_2\}$$

belong to $\mathcal{M}$. Is f $(\mathcal{M}, \mathcal{B}(\mathbb{R}^2))$-measurable?

(ii) Let $f_n : X \to \overline{\mathbb{R}}$ a sequence of measurable functions. Show that the set

$$E = \{x \in X : \{f_n(x)\}_{n=1}^\infty \text{ is a nondecreasing sequence}\}$$

is measurable.

solution. (i) $\{x : r_1 < f_1(x), f_2(x) < r_2\} = f_1^{-1}((r_1,\infty)) \cap f_2^{-1}((-\infty, r_2))$ It is enough to show that for any $p \in \mathbb{R}$, $f_1^{-1}(p,\infty) \in \mathcal{M}$ and $f_2^{-1}(-\infty, p) \in \mathcal{M}$.

- Fix p, there exists a sequence $\{x_n\}, \{y_n\}$ of $\mathbb{Q}$ such that $x_n \to p^+$ and $y_n \to \infty$ Then

$$f_1^{-1}(p,\infty) = \bigcup_{n=1}^\infty f_1^{-1}((x_n,\infty)) \cap f_2^{-1}((-\infty, y_n)) \in \mathcal{M}$$

- Similarly, there exists a sequence $\{x_n\}, \{y_n\}$ of $\mathbb{Q}$ such that $x_n \to -\infty$ and $y_n \to p^-$ Then

$$f_2^{-1}(-\infty, p) = \bigcup_{n=1}^{\infty} f_1^{-1}((x_n, \infty)) \cap f_2^{-1}((-\infty, y_n)) \in \mathcal{M}$$

(ii) Let $g_k = f_k - f_{k+1}$ then g_k is measurable. So,

$$E = \bigcap_{k=1}^{\infty} \{x \in X : g_k(x) \geq 0\}$$

□

Problem A.1.6. Let f be a Lebesgue measurable function on $\mathbb{R}^n$. Let m denote Lebesgue measure on $\mathbb{R}^n$.

Show that the following three statements are equivalent:

(a) f is integrable (i.e. belong to $L^1(\mathbb{R}^n)$).

(b) $\sum_{k\in\mathbb{Z}} 2^k m(\{x : |f(x)| > 2^k\}) < \infty$.

(c) $\sum_{k\in\mathbb{Z}} 2^k m(\{x : 2^k \leq |f(x)| < 2^{k+1}\}) < \infty$.

solution.

- (a) ⇒ (c):

$$\sum_{k\in\mathbb{Z}} 2^k m(\{x : 2^k \leq |f(x)| < 2^{k+1}\}) \leq \int |f| \, \mathrm{d}m < \infty$$

- (b) ⇒ (c):

$$\sum_{k\in\mathbb{Z}} 2^k m(\{x : 2^k \leq |f(x)| < 2^{k+1}\}) < \sum_{k\in\mathbb{Z}} 2^k m(\{x : |f(x)| > 2^{k-1}\}) < \infty$$

- (c) ⇒ (a):

$$\int |f| \, \mathrm{d}m \leq \sum_{k\in\mathbb{Z}} 2^{k+1} m(\{x : 2^k \leq |f(x)| < 2^{k+1}\}) < \infty$$

- (c) $\Rightarrow$ (b): Let $E_k = \{x : 2^k \leq |f(x)| < 2^{k+1}\}$ then

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} 2^k m(\{x : |f(x)| > 2^k\}) &\leq \sum_{k\in\mathbb{Z}} 2^k \sum_{j=k-1}^{\infty} E_j \\
&= \left(\sum_{k\in\mathbb{Z}} 2^k E_{k-1} + E_k\right) + \left(\sum_{k\in\mathbb{Z}} 2^k \sum_{j=k+1}^{\infty} E_j\right) \\
&= 3\sum_{k\in\mathbb{Z}} 2^k E_k + \left(\sum_{j\in\mathbb{Z}} \sum_{k=-\infty}^{j-1} 2^k E_j\right) \\
&\leq 3\sum_{k\in\mathbb{Z}} 2^k E_k + \left(\sum_{j\in\mathbb{Z}} 2^j E_j\right) \\
&= 4\sum_{k\in\mathbb{Z}} 2^k E_k
\end{aligned}$$

□

Problem A.1.7. Let m be Lebesgue measure on $\mathbb{R}$ and f be a Lebesgue measurable function with $\int |f| \; \mathrm{d}m < \infty$. Define $G(x) = \int_{-\infty}^{x} f \; \mathrm{d}m$. Prove that G is uniformly continuous on $\mathbb{R}$.

solution. I want to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $E \in \mathcal{M}$ with $m(E) < \delta$, implies that $\int_E |f| \; \mathrm{d}m < \varepsilon$. Define $E_n = \{x : |f(x)| > n\}$. Define $f_n = 1_{E_n}|f|$ then $|f_n| \leq |f|$. Then by Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int f_n \; \mathrm{d}m = 0$$

There exists k such that $\int f_k \; \mathrm{d}m < \varepsilon/2$, select $\delta = \frac{\varepsilon}{2k}$. Given any $E \in \mathcal{M}$ then

$$\begin{aligned}
\int_E |f| \; \mathrm{d}m &= \int_{E\cap E_n} |f| \; \mathrm{d}m + \int_{E\cap E_n^{\complement}} |f| \; \mathrm{d}m \\
&\leq \frac{\varepsilon}{2} + n\mu(E) \\
&\leq \varepsilon
\end{aligned}$$

For any $\varepsilon > 0$, pick δ to be in this theorem, then if $|x-y| < \delta$ then $|G(x) - G(y)| < \varepsilon$. □

Problem A.1.8. Determine the limits

(i) $\lim_{n\to\infty} \int_0^{n^{99/100}} (x/n)^n \; \mathrm{d}x$

(ii) $\lim_{n\to\infty} \int_0^{n} (1 - \frac{x}{n})^n e^{-2x} \; \mathrm{d}x$

and, in both cases carefully justify your computation.

solution. (i) define $f_n = \mathbb{1}_{[0,n^{99/100}]}(x/n)^n$ and $f =$ then $|f_n| \leq |f| \in L^1([0,\infty))$ then by Dominated Convergence Theorem,

□

Problem A.1.9. Let $f \in \mathcal{L}^1(\mathbb{R}^n)$. Let m be Lebesgue measure in $\mathbb{R}^n$. Prove that for $t > 0$

$$t^n \int f(tx) \, \mathrm{d}m = \int f(x) \, \mathrm{d}x$$

Hint: First prove this for indicator functions of cubes, then for indicator functions of sets of finite measure.

solution. Suppose that $f = \mathbb{1}_E$, then

$$\mathbb{1}_E(tx) = \begin{cases} 1 & \text{if } tx \in E \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } x \in \frac{1}{t}E \\ 0 & \text{otherwise} \end{cases} = \mathbb{1}_{\frac{1}{t}E}(x)$$

and $m(\frac{1}{t}E) = t^{-n}m(E)1$ □

Problem A.1.10. Let $f \in \mathcal{L}^1(\mathbb{R}^n)$. Then

(i) $\lim_{|h|\to 0} \int |f(x+h) - f(x)| \, \mathrm{d}m = 0$.

(ii) $\lim_{t\to 1} \int |f(tx) - f(x)| \, \mathrm{d}m = 0$.

(iii) Can the Lebesgue dominated convergence be used for the proof of (i) or (ii)?

solution. (i) there exists g continuous function with compact support such that $\int |f - g| < \frac{\varepsilon}{3}$ then

$$|f(x+h) - f(x)| \leq |f(x+h) - g(x+h)| + |g(x+h) - g(x)| + |g(x) - f(x)|$$

Obviously, $\int |f(x+h) - g(x+h)| \, \mathrm{d}m < \frac{\varepsilon}{3}$ and $\int |g(x) - f(x)| \, \mathrm{d}m < \frac{\varepsilon}{3}$ then

$$\int |g(x+h) - g(x)| \, \mathrm{d}m \leq \varepsilon \cdot 2\mu(K)$$

(ii)

□

Problem A.1.11. Let $I = [a,b], f \in \mathcal{L}^1(I)$. Show that

$$\lim_{n\to\infty} \int_I f(x) \sin(nx) \, \mathrm{d}m(x) = 0$$

solution. There exists g step such that $\int |f-g| \, \mathrm{d}m < \varepsilon$ then

$$\left|\int f(x)\sin(nx) - g(x)\sin(nx) \, \mathrm{d}m\right| \leq \int |f(x)-g(x)||\sin(nx)| \, \mathrm{d}m$$

$$\left|\int f(x)\sin(nx) \, \mathrm{d}m\right| - \left|\int g(x)\sin(nx) \, \mathrm{d}m\right| \leq \varepsilon$$

then fix some interval c then

$$\begin{aligned}\left|\int c\sin(nx) \, \mathrm{d}m\right| &= \frac{1}{n}\left|\int c\sin(x) \, \mathrm{d}m\right| \\ &\to 0\end{aligned}$$

as $n \to \infty$. □

Problem A.1.12. Recall the monotone convergence theorem and Fatou's lemma.

(i) Show that Fatou's lemma implies the monotone convergence theorem.

(ii) Show that the monotone convergence theorem implies Fatou's lemma.

solution. (i) $f_n \leq f \implies \int f_n \leq \int f$ So, $\lim_{n\to\infty} \int f_n \leq \int f$. Then from Fatou's lemma, we have

$$\int \liminf_{k\to\infty} f_k \, \mathrm{d}\mu \leq \liminf_{k\to\infty} \int f_k \, \mathrm{d}\mu$$

Since f_n is non-decreasing, so, $\int f_n$ is also non-decreasing. So, we have $\liminf\limits_{k\to\infty} f_k = \lim\limits_{k\to\infty} f_k$ then

$$\int \lim_{k\to\infty} f_k \, \mathrm{d}\mu \leq \lim_{k\to\infty} \int f_k \, \mathrm{d}\mu$$

(ii) we want to show that

$$\int \liminf_{k\to\infty} f_k \, \mathrm{d}\mu \leq \liminf_{k\to\infty} \int f_k \, \mathrm{d}\mu$$

It is enough to show that

$$\int \lim_{n\to\infty} \inf_{m\geq n} f_m \, \mathrm{d}\mu \leq \lim_{n\to\infty} \inf_{m\geq n} \int f_m \, \mathrm{d}\mu$$

From the monotone convergence theorem, we have

$$\int \lim_{n\to\infty} \inf_{m\geq n} f_m \, \mathrm{d}\mu \underset{\text{MCT}}{=} \lim_{n\to\infty} \int \inf_{m\geq n} f_m \, \mathrm{d}\mu$$

Then for any $k \geq n$, we have $\inf_{m\geq n} f_m \leq f_k$ then

$$\int \inf_{m\geq n} f_m \, \mathrm{d}\mu \leq \int f_k \, \mathrm{d}\mu$$

So, we have

$$\int \lim_{n\to\infty} \inf_{m\geq n} f_m \, \mathrm{d}\mu \leq \lim_{n\to\infty} \inf_{m\geq n} \int f_m \, \mathrm{d}\mu$$

$\square$

Problem A.1.13. Determine

$$\lim_{n\to\infty} \int_0^\infty \frac{2n\sin(x/n)}{x(1+x^2)} \, \mathrm{d}x$$

Provide justifications.

solution. We know that $|n/x\sin(x/n)| \leq 1$ then define

$$f_n = \frac{2n\sin(x/n)}{x(1+x^2)}$$

Then $|f_n| \leq \frac{2n}{x(1+x^2)}$ and $\int \frac{2n}{x(1+x^2)} \, \mathrm{d}x < \infty$ then by Dominated Convergence Theorem,

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty \frac{2n\sin(x/n)}{x(1+x^2)} \, \mathrm{d}x &= \int_0^\infty \lim_{n\to\infty} \frac{2n\sin(x/n)}{x(1+x^2)} \, \mathrm{d}x \\
&= \int_0^\infty \frac{2}{1+x^2} \, \mathrm{d}x \\
&= \pi
\end{aligned}$$

$\square$

Problem A.1.14. Let $f(x) = \sin(x^2)$ on the measure space $X = [1, \infty)$ (with Lebesgue measure m). Prove:

(i) $\int_{[1,\infty)} |f| \, \mathrm{d}m = \infty$

(ii) $\lim_{R\to\infty} \int_{[0,R]} f \, \mathrm{d}m$ exists (and is finite).

Hint: For part (ii) use that $2x\sin(x^2)$ is the derivative of $-\cos(x^2)$.

solution. (i) Constructing triangles under the curve, we have

$$\begin{aligned}
\int_{[0,\infty)} |f| \, \mathrm{d}m &\geq \sum_{k=1}^{\infty} \frac{1}{2}\sqrt{(k+1)\pi} - \sqrt{k\pi} \\
&= \frac{1}{2}\sqrt{\pi} \sum_{k=1}^{\infty} \sqrt{k+1} - \sqrt{k} \\
&= \frac{1}{2}\sqrt{\pi} \left(\lim_{k\to\infty} \sqrt{k} - \sqrt{1} \right) \\
&= \infty
\end{aligned}$$

(ii) Define $u = 1/2x$ then $dv = 2x\sin(x^2)\ \mathrm{d}x$ then $v = -\cos(x^2)$ and $du = -1/2x^2\ \mathrm{d}x$ then

$$\begin{aligned}\int_0^R f\ \mathrm{d}x &= uv\Big|_0^R - \int_0^R v\ \mathrm{d}u \\ &= -\frac{1}{2}\cos(R^2) + \frac{1}{2}\cos(0) - \int_0^R -\cos(x^2)\ \mathrm{d}u\end{aligned}$$

□

Problem A.1.15. Let $f : X \to \overline{\mathbb{R}}$ be a nonnegative measurable function on the measure space $(X, \mathcal{M}, \mu)$ and assume $\mu(X) < \infty$.

(i) Let $E_R = \{x \in X : |f(x)| > R\}$. Prove: If $|f(x)| < \infty$ for almost every $x \in X$ then $\lim_{R\to\infty} \mu(E_R) = 0$.

(ii) Is the conclusion in (i) still valid if we drop the assumption of finite measure space? Give a proof or counterexample.

solution. (i) Using continuity from above, we have

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right)$$

and $|f(x)| < \infty$ for almost every $x \in X$ then $\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = 0$, so $\lim_{R\to\infty} \mu(E_R) = 0$.

(ii) $f(x) = x$

□

Problem A.1.16. Let $p > 0$. For $x \in \mathbb{R}^n$ let $|x|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. Let $\Omega = \{x \in \mathbb{R}^n : |x|_p > 3\}$. Show that

$$\int_\Omega |x|_p^{-\alpha}\ \mathrm{d}m < \infty$$

if and only if $\alpha > n$. What is the result if you replace Ω by $\Omega^{\complement}$?

solution. Define $E_k = \{x \in \mathbb{R}^n : 3^k \leq |x|_p < 3^{k+1}\}$ then

$$\begin{aligned}\int_\Omega |x|_p^{-\alpha}\,\mathrm{d}m &= \sum_{k=1}^{\infty} \int_{E_k} |x|_p^{-\alpha}\,\mathrm{d}m \\ &\leq \sum_{k=1}^{\infty} c_1 3^{-\alpha k} \mu(E_k) \\ &\leq \sum_{k=1}^{\infty} c_1 3^{-\alpha k} c_2 3^{kn} \\ &= c_1 c_2 \sum_{k=1}^{\infty} 3^{k(n-\alpha)}\end{aligned}$$

So, $n - \alpha < 0$ then $\alpha > n$. For the converse use the same bound (but lower bound)

For $\Omega^\complement$, define $E_k = \{x \in \mathbb{R}^n : 3^{-k} \leq |x|_p < 3^{-k+1}\}$ then

$$\begin{aligned}\int_{\Omega^\complement} |x|_p^{-\alpha}\,\mathrm{d}m &= \sum_{k=1}^{\infty} \int_{E_k} |x|_p^{-\alpha}\,\mathrm{d}m \\ &\leq \sum_{k=1}^{\infty} c_1 3^{\alpha k} \mu(E_k) \\ &\leq \sum_{k=1}^{\infty} c_1 3^{\alpha k} c_2 3^{-kn} \\ &= c_1 c_2 \sum_{k=1}^{\infty} 3^{k(\alpha-n)}\end{aligned}$$

So, $\alpha - n < 0$ then $\alpha < n$. □

§A.2 Practice Exam 2

Problem A.2.1. Let $a \in \mathbb{R}$. On $\mathbb{N}$ consider the measure μ_a defined by $\mu_a(E) = \sum_{n\in E} n^{-a}$ where $E \subseteq \mathbb{N}$. For every pair (p_1, p_2) determine whether the embedding $L^{p_1}(\mathbb{N}, \mu_a) \subseteq L^{p_2}(\mathbb{N}, \mu_a)$ is true.

solution. For any $a_n \in L^{p_1}(\mathbb{N}, \mu_a)$, we have

$$\|a_n\|_{p_1} < \infty \implies \sum_{n\in\mathbb{N}} n^{-a} |a_n|^{p_1} < \infty$$

From ratio test, considering

$$\lim_{n\to\infty} \frac{|a_{n+1}|^{p_1}}{(n+1)^a} \frac{n^a}{|a_n|^{p_1}} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|^{p_1} \left(\frac{n}{n+1}\right)^a$$

For $a > 1$, we know that $(n/(n+1))^a \to 1$ then we have

- If $\lim_{n\to\infty} |a_{n+1}/a_n|^{p_1} < 1$ then
- If $\lim_{n\to\infty} |a_{n+1}/a_n|^{p_1} = 1$ then $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$ and p series converge

□

Problem A.2.2. Let $b \in \mathbb{R}$. On $[0,1]$ (with the Lebesgue σ-algebra) consider the measure ν_b defined by $\nu_b(E) = \int_E (\sin x)^{-b} \,\mathrm{d}m(x)$ where E is a Lebesgue measurable subset of $[0,1]$. For every pair (p_1, p_2) determine whether the embedding $L^{p_1}([0,1],\nu_b) \subseteq L^{p_2}([0,1],\nu_b)$ is true.

Problem A.2.3. Let f be continuous on $[0,1]$

(i) Show that $\lim_{n\to\infty} \int_0^1 f(x)\sin(2\pi nx) \,\mathrm{d}x = 0$.

(ii) *Does the limit

$$\lim_{n\to\infty} \int_0^1 f(x)|\sin(2\pi nx)| \,\mathrm{d}x$$

exist? If yes, determine its value.

(iii) What can be said about the above if we just assume $f \in L^1([0,1])$?

Problem A.2.4. Let $f \in L^1([0,2\pi])$.

(i) Prove that

$$\lim_{n\to\infty} \int_0^{2\pi} f(x)e^{-inx} \,\mathrm{d}x = 0$$

(ii) Prove that for a set $E \subseteq [0,2\pi]$ of positive Lebesgue measure $m(E)$,

$$\lim_{n\to\infty} \int_E \cos^2(nx+\gamma_n) \,\mathrm{d}x = \frac{m(E)}{2}$$

for any sequence γ_n.

(iii) * Let $F_n(x) = a_n \cos nx + b_n \sin nx$. Assume that the series $\sum_{n=0}^\infty F_n(x)$ converges on a set E of positive measure. Show that $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 0$.

Problem A.2.5. (i) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and compactly supported. Let $P_t f(x) = \int f(x-y)t^{-n/2}e^{-|x|^2/t} \,\mathrm{d}x$.

Show that $\lim_{t\to 0} P_t f(x)$ exist and determine the limit.

(ii) If $f \in L^1(\mathbb{R}^n)$ show that $P_t f$ converges in L^1 as $t \to 0$.

(iii) Show that for all $t > 0$ and $x \in \mathbb{R}^n$

$$|P_t f(x)| \le CMf(x)$$

where Mf is the Hardly-Littlewood maximal function and C is some constant independent of f. Show that $P_t f(x)$ converges almost everywhere as $t \to 0$.

Problem A.2.6. In $\mathbb{R}^n$ let, for $j = 1, \ldots, n$, $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projections

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

Let $g_j, j = 1, \ldots, j_n$ be functions in $L^{n-1}(\mathbb{R}^{n-1})$.

Prove the inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} g_j(\pi_j(x)) \, \mathrm{d}x \le \prod_{j=1}^{n} \|g_j\|_{L^{n-1}(\mathbb{R}^{n-1})}$$

Hint: It may help to consider the cases $n = 2, 3$ first.

Problem A.2.7. Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set of positive Lebesgue measure, $m(E) > 0$. Show that for any $\beta < 1$, there is an open interval I such that $m(E \cap I) \ge \beta m(I)$.

Hint: Argue by contradiction and think about the definition of the outer measure that determines Lebesgue measure.

Problem A.2.8. Review the proof of *Steinhaus' theorem*: Let $E \subseteq [0, 1]$ be a Lebesgue measurable set with $m(E) > 0$. Then the difference set

$$E - E = \{x_1 - x_2 : x_1 \in E, x_2 \in E\}$$

contains an interval centered at the origin.

Proof. We may assume that E is compact (why?). Moreover given (small) $a > 0$ there is an open set $U \supseteq E$ such that

$$m(U) < (1 + a)m(E)$$

Let $\varepsilon = \frac{1}{2}\mathrm{dist}(E, U^\complement) > 0$. We show that $(-\varepsilon, \varepsilon) \subseteq E - E$; in other words we show that for any t with $|t| < \varepsilon$ there exists $x \in E$ such that $x + t$ is also in E. Hence we need to show that $E \cap (E - t)$ is nonempty, and we do this by showing that $m(E \cap (E - t)) > 0$.

By our definition of ε, we have $(E - t) \cap U^\complement = \emptyset$ if $|t| < \varepsilon$. Hence

$$m(U \setminus (E - t)) = m(U) - m(E - t) = m(U) - m(E) < (1 + a)m(E) - m(E) \le am(E)$$

Therefore

$$\begin{aligned}m(E\cap(E-t)) &\geq m(U)-m(U\setminus E)-m(U\setminus(E-t))\\ &\geq m(u)-2am(E)\geq(1-2a)m(E)\end{aligned}$$

and we get $m(E\cap m(E-t))>0$ if we specify above that $a<1/2$. $\square$

Problem A.2.9. Let $E\subseteq\mathbb{R}$ be a set of positive Lebesgue measure. The following statements can be proved by an adaptation of the method used in Steinhaus' theorem. Work this out.

(i) Let $N\in\mathbb{N}$. Show that E contains an arthmetic progression of length N, i.e there is an $a>0$ and a real number x so that $x, x+a, x+2a,\dots,x+(N-1)a$ belong to E.

(ii) Show there exist $t>0$ and x such that $x, x+t$ and $x+t^2$ belong to E.

Problem A.2.10. Below always assume that $E\subseteq\mathbb{R}^n$ is Lebesgue measurable, with $m(E)=1$.

(i) Let f be bounded and continuous. Show that for all $x\in\mathbb{R}^n$

$$\lim_{t\to 0}\int_E f(x-ty)\ \mathrm{d}m(y)=f(x)$$

(ii) Let $f\in L^1(\mathbb{R}^n)$ and assume in addition that E is bounded. Show that $\lim_{t\to 0}\int_E f(x-ty)\ \mathrm{d}m(y)=f(x)$ for almost every x.

(iii) *Assume $n=2$ and $E\subseteq\mathbb{R}^2$ is a starlike set in $\mathbb{R}^2$, i.e. for every $x\in E$ the line connecting x with the origin belongs to E. Let $f\in L^p(\mathbb{R}^2)$ and $p>1$. Show $\lim_{t\to 0}\int_E f(x-ty)\ \mathrm{d}m(y)=f(x)$ almost everywhere.

(iii) also holds in higher dimensions. It is an open problem whether it holds for $p=1$.