

MATH 629 (Measure Theory) Lecture Notes

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1 From Riemann to Lebesgue

§1.1 Riemann Integral

Definition 1.1.1. $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a, b]$.

Definition 1.1.2. If P, P' are partitions of $[a, b]$ and $P \subseteq P'$, then P' is a refinement of P .

Definition 1.1.3. Given a bounded function $f : [a, b] \rightarrow \mathbb{R}$ and a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ we define

$$m_i(f) = \inf_{t \in [x_{i-1}, x_i]} f(t)$$

$$M_i(f) = \sup_{t \in [x_{i-1}, x_i]} f(t)$$

define the lower sum as

$$L(f, P) = \sum_{i=1}^n m_i(f)(x_i - x_{i-1})$$

and the upper sum as

$$U(f, P) = \sum_{i=1}^n M_i(f)(x_i - x_{i-1})$$

Lemma 1.1.4

Given a bounded function $f : [a, b] \rightarrow \mathbb{R}$ and partitions P of $[a, b]$. Suppose that P' is a refinement of P then

$$(b - a) \inf_{t \in [a, b]} f(t) \leq L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P) \leq (b - a) \sup_{t \in [a, b]} f(t)$$

Corollary 1.1.5

Suppose that P_1, P_2 are partitions of $[a, b]$ then $L(f, P_1) \leq U(f, P_2)$

Proof. Let $P' = P_1 \cup P_2$ then P' is a refinement of P_1 and P_2 and use Lemma 1.1.4 \square

Lemma 1.1.6

Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is bounded. Then

$$(b-a) \inf_{t \in [a, b]} f(t) \leq \sup_P L(f, P) \leq \inf_P U(f, P) \leq (b-a) \sup_{t \in [a, b]} f(t)$$

Definition 1.1.7. A function $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable if

$$\sup_P L(f, P) = \inf_P U(f, P)$$

and the common value is called the Riemann integral of f and is denoted by $\int_a^b f$

Lemma 1.1.8

Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is bounded. Then f is Riemann integrable if and only if for any $\varepsilon > 0$ there exists a partition P such that

$$U(f, P) - L(f, P) < \varepsilon$$

Proof. (\Rightarrow) For any $\varepsilon > 0$. Suppose that f is Riemann integrable. Then there exists P_1, P_2 such that

$$L(f, P_1) \geq \int_a^b f - \frac{\varepsilon}{2}$$

$$U(f, P_2) \leq \int_a^b f + \frac{\varepsilon}{2}$$

let $P = P_1 \cup P_2$ then

$$U(f, P) - L(f, P) \leq \varepsilon$$

(\Leftarrow) For any $\varepsilon > 0$, there exists P_ε such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

since ε is arbitrary, we have

$$\sup_P L(f, P) = \inf_P U(f, P)$$

□

Theorem 1.1.9

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ then f is Riemann integrable.

Proof. f is continuous on a compact set, so, f is uniformly continuous. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in [a, b]$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{(b-a)}$. Let N be such that $\frac{(b-a)}{N} < \delta$ and let $P = \{x_i := a + \frac{(b-a)i}{N}\}$ then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^N (M_i(f) - m_i(f)) \frac{(b-a)}{N} \\ &\leq \sum_{i=1}^N \frac{\varepsilon}{(b-a)} \frac{(b-a)}{N} \\ &= \varepsilon \end{aligned}$$

□

Remark 1.1.10. Let $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ defined on the $[0, 1]$. Then $U(f, P) = 1$ and $L(f, P) = 0$ for any partition P . So, f is not Riemann integrable.

§1.2 Lebesgue null sets

Definition 1.2.1. For the closed interval $I = [a, b]$, the length of I , denoted as $\ell(I)$ is defined as $\ell(I) = b - a$

Definition 1.2.2. A set E is said to be a Lebesgue null set if for any $\varepsilon > 0$ there exists a sequence of intervals $\{I_n\}_{n \in \mathbb{N}}$ such that

$$E \subseteq \bigcup_{n=1}^{\infty} I_n$$

and

$$\sum_{n=1}^{\infty} \ell(I_n) < \varepsilon$$

Lemma 1.2.3

Countable unions of Lebesgue null sets are Lebesgue null sets.

Proof. For any $\varepsilon > 0$ and for each Lebesgue null sets E_n there exists $I_{E_n, i}$ such that

$$E_n \subseteq \bigcup_{i=1}^{\infty} I_{E_n, i}$$

and

$$\sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \frac{\varepsilon}{2^n}$$

then

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \varepsilon$$

□

Definition 1.2.4. A set $E \subseteq [a, b]$ has content zero if for any $\varepsilon > 0$ there exists I_1, I_2, \dots, I_n such that

$$E \subseteq \bigcup_{i=1}^n I_i$$

and

$$\sum_{i=1}^n \ell(I_i) < \varepsilon$$

Lemma 1.2.5

Suppose that $E \subseteq [a, b]$ is a compact Lebesgue null set then E has content zero.

Proof. For any $\varepsilon > 0$ there exists a sequence of interval $\{I_n\}_{n \in \mathbb{N}}$ such that $E \subseteq \bigcup I_n$ and $\sum \ell(I_n) < \frac{\varepsilon}{2}$. Suppose that $I_n = [a_n, b_n]$, then let

$$J_n = \left(a_n - \frac{\varepsilon}{2^{n+3}}, b_n + \frac{\varepsilon}{2^{n+3}} \right) \supseteq I_n$$

then from the compactness of E , there exists a finite subcover $J_{n_1}, J_{n_2}, \dots, J_{n_k}$ such that $E \subseteq \bigcup J_{n_i}$ then we construct a finite closed interval K_i by

$$K_i = \left[a_{n_i} - \frac{\varepsilon}{2^{n_i+2}}, b_{n_i} + \frac{\varepsilon}{2^{n_i+2}} \right]$$

then $E \subseteq \bigcup K_i$ and $\sum \ell(K_i) < \varepsilon$

□

Corollary 1.2.6

if $a < b$ then $[a, b]$ is not a Lebesgue null set.

Proof. By contradiction, since $[a, b]$ is compact, then $[a, b]$ has content zero, but $[a, b]$ don't have content zero. □

§1.3 Oscillation and Discontinuity

Definition 1.3.1. Suppose that $X \subseteq \mathbb{R}$, $f : X \rightarrow \mathbb{R}$ for any $x \in X$ and $\delta > 0$, define

$$M_{f,\delta}(x) := \sup\{f(y) : d(x, y) < \delta\}$$

$$m_{f,\delta}(x) := \inf\{f(y) : d(x, y) < \delta\}$$

then we define

$$\text{osc}_f(x) := \lim_{\delta \rightarrow 0^+} M_{f,\delta}(x) - m_{f,\delta}(x)$$

Lemma 1.3.2

f is continuous at x if and only if $\text{osc}_f(x) = 0$.

Proof. (\Rightarrow) Suppose that f is continuous at x , then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Then

$$M_{f,\delta}(x) - m_{f,\delta}(x) \leq \sup\{f(y) : d(x, y) < \delta\} - \inf\{f(y) : d(x, y) < \delta\} < \varepsilon$$

(\Leftarrow) Suppose that $\text{osc}_f(x) = 0$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $M_{f,\delta}(x) - m_{f,\delta}(x) < \varepsilon$. Then for any $y \in X$ such that $d(x, y) < \delta$, we have $|f(x) - f(y)| < \varepsilon$ then f is continuous at x . \square

Before we prove this theorem, we need to prove the following lemma.

Lemma 1.3.3

$\{x \in [a, b] : \text{osc}_f(x) \geq \gamma\}$ is closed.

Proof. We need to show that $\{x : \text{osc}_f(x) < \gamma\}$ is open. Fix x in that set. Let $\varepsilon = \gamma - \text{osc}_f(x)$ then

$$\sup_{|w-x|<\delta} f(w) - \inf_{|w-x|<\delta} f(w) < \text{osc}_f(x) < \gamma$$

then for any $w \in (x - \delta, x + \delta)$ if $|w - x| < \frac{\delta}{2}$ then

$$\text{osc}(w) \leq \sup_{|y-w|<\frac{\delta}{2}} f(y) - \inf_{|y-w|<\frac{\delta}{2}} f(y) < \gamma$$

So, $B(x, \frac{\delta}{2}) \subseteq \{x : \text{osc}_f(x) < \gamma\}$ \square

we observe that

- (i) If the set of discontinuities is a Lebesgue null set, then $\{x : \text{osc}_f(x) \geq \gamma\}$ is a set of content zero.
- (ii) If $\{x : \text{osc}_f(x) \geq \gamma\}$ is a Lebesgue null set, then the set of discontinuities is also a Lebesgue null set.

Lemma 1.3.4

Suppose that f is defined on $[c, d]$, assume that $\text{osc}_f(x) < \gamma$ then we can find a partition

$$U(f, P) - L(f, P) < \gamma(b - a)$$

Proof. For every $x \in [c, d]$, there exists $\delta_x > 0$ such that

$$\sup_{|w-x|<\delta_x} f(w) - \inf_{|w-x|<\delta_x} f(x) < \gamma$$

construct a cover by

$$B(x, \delta_x) = \{w \in [c, d] : |w - x| < \delta_x\}$$

since $[c, d]$ is compact, there exists a finite subcover $B(p_1, \delta_{p_1}), \dots, B(p_n, \delta_{p_n})$ then let $\delta_0 = \frac{\min\{\delta_{p_i}\}}{100}$ then we can construct a partition $P = \{c = x_0 < x_1 < \dots < x_n = d\}$ such that $|x_i - x_{i-1}| < \delta_0$ then $M_i - m_i < \gamma$ and

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \\ &< \gamma \sum_{i=1}^n (x_i - x_{i-1}) \\ &= \gamma(d - c) \end{aligned}$$

□

Theorem 1.3.5

Suppose that $f : [a, b] \rightarrow \mathbb{R}$ then $f \in \mathcal{R}([a, b])$ if and only if f is bounded and the set of discontinuity of f is a Lebesgue null set.

Proof. (\Rightarrow) We want to show that for every $n \in \mathbb{N}$,

$$\mathcal{D}_n = \left\{x : \text{osc}_f(x) \geq \frac{1}{n}\right\}$$

is a Lebesgue null set. For any $\varepsilon > 0$, since f is Riemann integrable, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) = \sum_{i=1}^n (x_i - x_{i-1})(M_i - m_i) \leq \frac{\varepsilon}{n}$$

where $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ and $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$. in particular

$$\begin{aligned} \sum_{[x_{i-1}, x_i] \cap \mathcal{D}_n \neq \emptyset} (x_i - x_{i-1})(M_i - m_i) &\leq \frac{\varepsilon}{n} \\ \frac{1}{n} \sum_{[x_{i-1}, x_i] \cap \mathcal{D}_n \neq \emptyset} \ell([x_{i-1}, x_i]) &\leq \frac{\varepsilon}{n} \end{aligned}$$

So, this interval cover the set \mathcal{D}_n

(\Leftarrow) pick $\varepsilon_1 \ll \varepsilon$, consider the set $D(\varepsilon_1) = \{x \in [a, b] : \text{osc}_f(x) \geq \varepsilon_1\}$ closed set. Since $D(\varepsilon_1)$ is a Lebesgue null set from the Lemma 1.2.5 it has content zero so we can find I_1, \dots, I_n such that

$$\sum_{j=1}^n \ell(I_j) < \varepsilon_1 \text{ and } D(\varepsilon_1) \subseteq \bigcup_{j=1}^n I_j$$

We form a partition of $[a, b]$, $a = x_0 < x_1 < \dots < x_N = b$ from I_j . There are two cases that we need to consider

- 1) if $[x_{i-1}, x_i] \subseteq I_j$ for some j then set $P_i = [x_{i-1}, x_i]$
- 2) if $[x_{i-1}, x_i] \cap I_j = \emptyset$ for all j then $\text{osc}(x) < \varepsilon_1$ for all $x \in [x_{i-1}, x_i]$. We want to partition further the interval $[x_{i-1}, x_i]$ by partition P_i . Using Lemma 1.3.4 we can find a partition P_i of $[x_{i-1}, x_i]$ such that

$$U(f, P_i) - L(f, P_i) < \varepsilon_1(x_i - x_{i-1})$$

We form a partition $P = P_1 \cup \dots \cup P_N$ then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^N (U(f, P_i) - L(f, P_i)) \\ &= \sum_{i:\text{case 1}} (U(f, P_i) - L(f, P_i)) + \sum_{i:\text{case 2}} (U(f, P_i) - L(f, P_i)) \\ &\leq 2M \sum_{i:\text{case 1}} (x_i - x_{i-1}) + \varepsilon_1 \sum_{i:\text{case 2}} (x_i - x_{i-1}) \\ &\leq 2M\varepsilon_1 + \varepsilon_1(b - a) \\ &= \varepsilon_1(2M + b - a) \end{aligned}$$

□

2 Measures

§2.1 Introduction

We define the $\ell([c, d]) = d - c$ and If $E = [c_1, d_1] \cup [c_2, d_2]$ where $d_1 < c_2$ then $\ell(E) = d_1 - c_1 + d_2 - c_2$. This is consistent with the definition

$$\ell(E) = \int \mathbb{1}_E(x) \, dx$$

where the integral denotes the Riemann integral.

if $E \subseteq [a, b]$ reference interval is

$$\int_a^b \mathbb{1}_E \, dx$$

Remark 2.1.1. The consistency of the definition also works with the set (c, d) , $[c, d)$, and $(c, d]$, where the length of all of them is $d - c$.

Remark 2.1.2. we denote $\mathbb{1}_E$ to be

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Example 2.1.3

Let $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ defined on the $[0, 1]$. Then $U(f, P) = 1$ and $L(f, P) = 0$ for any partition P .

Fix the reference interval $[a, b]$ and consider subset of $[a, b]$

Let $\mathcal{A} =$ collection of sets for which $\int_{[a, b]} \mathbb{1}_E \, dx$ exists.

If $A_1, \dots, A_n \in \mathcal{A}$, we can make the set to be mutually disjoint by taking $E_1 = A_1$, $E_2 = A_2 \setminus A_1$, $E_3 = A_3 \setminus (A_1 \cup A_2)$, and so on.

Example 2.1.4

For $E_1, E_2 \in \mathcal{A}$, we have

$$\mathbb{1}_{E_1 \cap E_2}(x) = \mathbb{1}_{E_1}(x) \mathbb{1}_{E_2}(x)$$

Example 2.1.5

For the Riemann integral, we have

$$\int_a^b f(y) = \int_{a-v}^{b-v} f(v+y)$$

and we want

$$\int \mathbb{1}_E(x) \, dx = \int \mathbb{1}_{v+E}$$

where $v + E = \{v + x : x \in E\}$

Let $E = \mathbb{Q} \cap [0, 1]$ countable set, we can enumerate r_1, r_2, r_3, \dots such that

$$E = \bigcup_{n=1}^{\infty} \{r_n\}$$

and

$$\int \mathbb{1}_{\{r_k\}} = 0$$

E should have length zero but according $\mathbb{1}_E$ is not Riemann integrable.

§2.2 Construction of Measure

Suppose that \mathcal{C} be a collection of sets.

Can we define on suitable large collection of subset of \mathbb{R} ?

a set function $\mu : \mathcal{C} \rightarrow [0, \infty]$ such that if $\{E_j\}_{j=1}^{\infty}$ is a sequence of disjoint set in \mathcal{C} then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

$$\mu([a, b]) = b - a, \mu([0, 1)) = 1$$

Can we do this for the collection of all subset of \mathbb{R} ?

Answer: No, Vitali set.

Theorem 2.2.1

We cannot define a measure on the collection of all subset of \mathbb{R} . i.e., there does not exist a set function $\mu : \mathfrak{P}(\mathbb{R}) \rightarrow [0, \infty]$ such that

- (i) $\mu(v + E) = \mu(E)$ for all $E \subseteq \mathbb{R}$ and $v \in \mathbb{R}$
- (ii) $\mu([0, 1]) = 1$
- (iii) $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$ for all disjoint $A_j \subseteq \mathbb{R}$

Before we prove that theorem, we need to define something and prove the following lemma.

Definition 2.2.2. We define a Vitali set V from picking an element $x \in [0, 1)$ from each equivalence class of the relation $x \sim y$ if $x - y \in \mathbb{Q}$. (e.g, pick $x \in O_x$ for $O_x \in \mathbb{R}/\mathbb{Q}$)

Lemma 2.2.3

Suppose that V is a Vitali set then

$$V \cap V + q = \emptyset$$

For all $q \in \mathbb{Q} \setminus \{0\}$

Proof. Suppose not, there exists $a \in V$ such that $a \in V + q \implies a - q \in V$ but we only pick 1 element in each equivalence class. contradiction. \square

Lemma 2.2.4

Let V be a Vitali set and let $W = \{q \in [-1, 1] : q \in \mathbb{Q}\}$ and

$$E = \bigcup_{w \in W} V + w$$

then

$$[0, 1] \subseteq E \subseteq [-1, 2]$$

Proof. Consider $E \subseteq [-1, 2]$. Since $V \subseteq [0, 1)$, then for any $v \in V$, $v \in [0, 1) \implies v + w \in [-1, 2]$.

For the $[0, 1] \subseteq E$, for any $x \in [0, 1]$ there exists $O_x \in \mathbb{R}/\mathbb{Q}$ such that $x \in O_x$. then there exists $v \in C_x$ such that $v \in [0, 1)$ and $v \in V$, since both are from the same equivalence

class, then $x - v \in \mathbb{Q}$ and $|x - v| < 1 \implies x - v \in (-1, 1)$. Hence, there exists $w \in W$ such that $w = x - v$ so $v + w = x$. \square

Proof of the theorem. Suppose that μ exists then using the result from Lemma 2.2.4 we get that

$$\mu([0, 1]) \leq \mu(E) \leq \mu([-1, 2])$$

from Lemma 2.2.3 we know that each $V + w$ is disjoint, so

$$\begin{aligned} \mu([0, 1]) &\leq \sum_{w \in W} \mu(V) \leq \mu([-1, 2]) \\ 1 &\leq \sum_{w \in W} \mu(V) \leq 3 \end{aligned}$$

if $\mu(V) = 0$ then $\mu(E) = 0$ and if $\mu(V) > 0$ then $\mu(E) = \infty$. Both are contradiction. \square

§2.3 σ -algebra

Definition 2.3.1. Given a reference X . An **algebra** is a collection of subsets of X , \mathcal{A} , such that

- (i) $X \in \mathcal{A}$
- (ii) If $A \in \mathcal{A}$ then the complement $A^c = X \setminus A \in \mathcal{A}$
- (iii) If $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$

Remark 2.3.2. • $\emptyset \in \mathcal{A}$ because $\emptyset = X^c$

- $A_1, A_2 \in \mathcal{A}$, $A_1 \setminus A_2 = A_1 \cap A_2^c \in \mathcal{A}$
- Observe that if $A_1, A_2 \in \mathcal{A}$ then $A_1 \cap A_2 \in \mathcal{A}$ because $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$

Example 2.3.3

$X = [a, b]$ and \mathcal{A} is the collection of all sets $E \subseteq [a, b]$ such that the Riemann integral $\int \mathbb{1}_E(t) dt$ exists

Definition 2.3.4. A σ -algebra \mathcal{M} on X is

- (i) an algebra of subsets of X
- (ii) If A_1, A_2, A_3, \dots is a sequence of set in \mathcal{M} then

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$$

(X, \mathcal{M}) is called a “**measurable space**”.

Remark 2.3.5. \mathcal{M} is a σ -algebra on X then it satisfies

- (i) $X \in \mathcal{M}$
- (ii) If $A \in \mathcal{M}$ then $A^c \in \mathcal{M}$
- (iii) countable union of sets in \mathcal{M} is in \mathcal{M}

Definition 2.3.6. Let (X, \mathcal{M}) be a measurable set. Then a measure μ is a set function $\mu : \mathcal{M} \rightarrow [0, \infty], E \mapsto \mu(E)$ such that

- (i) $\mu(\emptyset) = 0$
- (ii) If E_1, E_2, E_3, \dots is a sequence of disjoint set in \mathcal{M} then

$$\mu \left(\bigcup_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \mu(E_j)$$

called σ -additivity.

(X, \mathcal{M}, μ) is called a “**measure space**”.

Remark 2.3.7.

$$\left(\bigcap_{j=1}^{\infty} A_j \right) = \left(\bigcup_{j=1}^{\infty} A_j^c \right)^c \in \mathcal{M}$$

Example 2.3.8

examples of σ -algebra

- (i) $\mathcal{M} = \{\emptyset, X\}$
- (ii) $\mathcal{M} = \mathfrak{P}(X)$ = collection of all subsets of X
 $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mu(E) = |E|$ (the cardinality of E) if E is finite and $\mu(E) = \infty$ if E is infinite.
- (iii) X write X as a disjoint (countable) union of sets A_j . Then \mathcal{M} = all countable unions of A_j .
- (iv) Let X be a set. Let \mathcal{M} be the collection of all sets A , $A \subseteq X$ such that A is countable or A^c is countable.
- (v) $X = \mathbb{R}$ (or \mathbb{R}^n), $\mathcal{B}_{\mathbb{R}}$ is the smallest σ -algebra containing all open sets.

More generally if \mathcal{E} is a collection of subsets of X then $\mathfrak{M}(\mathcal{E})$ is the smallest σ -algebra that contains all sets in \mathcal{E} .

If $\mathcal{M}_1, \mathcal{M}_2$ are two σ -algebras, then $\mathcal{M}_1 \cap \mathcal{M}_2$ is also a σ -algebra.

If $\{\mathcal{M}_\alpha\}_{\alpha \in \mathcal{I}}$ is a collection of σ -algebras, their intersection is also a σ -algebra.

Generating σ -algebra

Definition 2.3.9. $\mathfrak{M}(\mathcal{E}) :=$ intersection of all σ -algebra that contain the collection \mathcal{E} . We call it the σ -algebra generated by \mathcal{E} .

Remark 2.3.10. If $\mathcal{E} \subset \mathcal{F} \implies \mathfrak{M}(\mathcal{E}) \subset \mathfrak{M}(\mathcal{F})$

Lemma 2.3.11

If $\mathcal{E} \subseteq \mathfrak{M}(\mathcal{F})$ then $\mathfrak{M}(\mathcal{E}) \subseteq \mathfrak{M}(\mathcal{F})$

Proof. $\mathfrak{M}(\mathcal{F})$ is a σ -algebra that contains \mathcal{E} . It contains the intersection of all σ -algebras which contain \mathcal{E} . \square

Example 2.3.12

$\mathcal{B}_{\mathbb{R}} = \sigma$ -algebra on \mathbb{R} containing all open sets \mathcal{E} a collection of all open intervals, $\mathcal{E} \subseteq \mathcal{O} =$ collection of all open sets in \mathbb{R} , $\mathcal{B}_{\mathbb{R}} = \mathfrak{M}(\mathcal{O})$. $\mathfrak{M}(\mathcal{E}) \subseteq \mathcal{B}_{\mathbb{R}}$. Each open set is a countable union of open intervals. Each open set is contained in $\mathfrak{M}(\mathcal{E})$.

Since $\mathcal{O} \subseteq \mathfrak{M}(\mathcal{E}) \implies \mathfrak{M}(\mathcal{O}) \subseteq \mathfrak{M}(\mathcal{E})$. get $\mathfrak{M}(\mathcal{O}) = \mathfrak{M}(\mathcal{E})$.

Definition 2.3.13. Given $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2), \dots, (X_n, \mathcal{M}_n)$ measurable spaces. Define a “product σ -algebra” on $X_1 \times X_2 \times \dots \times X_n$ denoted by

$$\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \dots \oplus \mathcal{M}_n = \bigoplus_{j=1}^n \mathcal{M}_j$$

defined as the σ -algebra generated by the sets $E_1 \times E_2 \times \dots \times E_n$ where $E_j \in \mathcal{M}_j$.

i.e., define $\mathcal{E} := \{E_1 \times E_2 \times \dots \times E_n : E_j \in \mathcal{M}_j\}$ then

$$\bigoplus_{j=1}^n \mathcal{M}_j := \mathfrak{M}(\mathcal{E})$$

Remark 2.3.14. Folland defines it the σ -algebra generated by

$$(X_1 \times X_2 \times \cdots \times X_{n-1} \times E_n)$$

where $E_n \in \mathcal{M}_n$,

$$(X_1 \times X_2 \times \cdots \times E_{n-1} \times X_n)$$

where $E_{n-1} \in \mathcal{M}_{n-1}$. and so on. To be clear, let

$$\mathcal{E}' := \bigcup_{j=1}^n \{(X_1 \times \cdots \times X_{j-1} \times E_j \times X_{j+1} \times \cdots \times X_n) : E_j \in \mathcal{M}_j\}$$

then

$$\bigoplus_{j=1}^n \mathcal{M}_j := \mathfrak{M}(\mathcal{E}')$$

Claim 2.3.15 — Both definitions on product of σ -algebra are equivalent.

Proof. The goal is to show that $\mathfrak{M}(\mathcal{E}) = \mathfrak{M}(\mathcal{E}')$.

(\supseteq) Obviously, $\mathcal{E}' \subseteq \mathcal{E}$ so $\mathfrak{M}(\mathcal{E}') \subseteq \mathfrak{M}(\mathcal{E})$.

(\subseteq) We want to show that $\mathcal{E} \subseteq \mathfrak{M}(\mathcal{E}')$. Fix $(E_1 \times E_2 \times \cdots \times E_n) \in \mathcal{E}$ then from the definition of σ -algebra generated by a collection, which is closed under intersection, so we can pick an element from the construction of \mathcal{E}' and do the intersection, so $(E_1 \times E_2 \times \cdots \times E_n) \in \mathfrak{M}(\mathcal{E}')$.

□

Theorem 2.3.16

Given $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2)$ measurable spaces. Assume that \mathcal{M}_1 is generated by a collection \mathcal{E}_1 and \mathcal{M}_2 is generated by a collection \mathcal{E}_2 . Then $\mathcal{M}_1 \oplus \mathcal{M}_2$ is generated by the sets $E_1 \times X_2, X_1 \times E_2$, where $E_1 \in \mathcal{E}_1$ and $E_2 \in \mathcal{E}_2$.

Proof. Let $\mathcal{P} := \{E_1 \times E_2 : E_i \in \mathcal{E}_i\}$, obviously $\mathfrak{M}(\mathcal{P}) = \mathfrak{M}(\{E_1 \times X_2 : E_1 \in \mathcal{E}_1\} \cup \{X_1 \times E_2 : E_2 \in \mathcal{E}_2\})$ and $\mathfrak{M}(\mathcal{P}) \subseteq \mathcal{M}_1 \oplus \mathcal{M}_2$. We need to show that $\mathcal{M}_1 \oplus \mathcal{M}_2 \subseteq \mathfrak{M}(\mathcal{P})$. Define

$$\mathcal{G}_1 = \{E_1 \subseteq X_1 : E_1 \times X_2 \in \mathfrak{M}(\mathcal{P})\}$$

$$\mathcal{G}_2 = \{E_2 \subseteq X_2 : X_1 \times E_2 \in \mathfrak{M}(\mathcal{P})\}$$

then \mathcal{G}_1 is a σ -algebra consisting of subset of X_1 which contains \mathcal{E}_1 , $\mathcal{E}_1 \subseteq \mathcal{G}_1$. \mathcal{E}_1 generates \mathcal{M}_1 so $\mathfrak{M}(\mathcal{E}_1) = \mathcal{M}_1 \subseteq \mathcal{G}_1$. So, we have $E_1 \times X_2 \in \mathfrak{M}(\mathcal{P})$ for all $E_1 \in \mathcal{M}_1$ and $X_1 \times E_2 \in \mathfrak{M}(\mathcal{P})$ for all $E_2 \in \mathcal{M}_2$. The σ -algebra generated by the sets $E_1 \times X_2, X_1 \times E_2$ is contained $\mathcal{M}_1 \oplus \mathcal{M}_2 \in \mathfrak{M}(\mathcal{P})$. □

Claim 2.3.17 — $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$.

where $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}}$ is generated by $E_1 \times E_2$, where $E_1, E_2 \in \mathcal{B}_{\mathbb{R}}$. and $\mathcal{B}_{\mathbb{R}^2}$ is generated by the open sets in \mathbb{R}^2 .

Proof. $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{B}_{\mathbb{R}^2}$. Want $\mathcal{B}_{\mathbb{R}^2} \subseteq \mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}}$. Consider the collection of all open rectangle of the form $(a_1, b_1) \times (a_2, b_2)$ such $a_i, b_i \in \mathbb{Q}$. which are contained in $O \subseteq \mathbb{R}^2$ \square

Definition 2.3.18 (The Borel σ algebra on the extended real line). We use the notion $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$. One possibility to define “ $\mathcal{B}_{\overline{\mathbb{R}}}$ ” is the σ -algebra generated by open sets in \mathbb{R} , $\{\infty\}$, $\{-\infty\}$ open intervals should be (a, b) , $(a, \infty]$, $[-\infty, b)$ for $-\infty \leq a < b \leq \infty$. Then define $d(x, y) = |\arctan(x) - \arctan(y)|$ and $\arctan(\infty) = \pi/2$, $\arctan(-\infty) = -\pi/2$.

§2.4 Measures

Definition 2.4.1. Measures are σ -additive set functions, $\mu(\emptyset) = 0$ and

$$\mu \left(\biguplus_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \mu(E_j)$$

where E_1, E_2, \dots is a sequence of disjoint sets.

Remark 2.4.2. There is some property

$$E \subseteq F \implies \mu(E) \leq \mu(F)$$

$$F = E \uplus (F \setminus E) \implies \mu(F) = \mu(E) + \mu(F \setminus E)$$

$\mu(\bigcup A_j) \leq \sum \mu(A_j)$ we can write $\bigcup A_j$ as a disjoint union, i.e., $E_1 = A_1$, $E_2 = A_2 \setminus A_1$, $E_3 = A_3 \setminus (A_1 \cup A_2)$, and so on then $\mu(\bigcup A_j) = \mu(\bigcup E_j) = \sum \mu(E_j) \leq \sum \mu(A_j)$

The monotone convergence theorem for sets (continuity from below)

Theorem 2.4.3

If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$ then

$$\mu \left(\bigcup_{j=1}^{\infty} E_j \right) = \lim_{j \rightarrow \infty} \mu(E_j)$$

Proof.

$$\bigcup_{j=1}^{\infty} E_j = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus (E_1 \cup E_2)) \cup \dots$$

So, we define $B_1 = E_1, B_n = E_n \setminus E_{n-1}$ for $n \geq 2$ then all B_j are disjoint.

$$\begin{aligned} \bigcup_{j=1}^{\infty} E_j &= \bigcup_{j=1}^{\infty} B_j \\ \mu \left(\bigcup_{j=1}^{\infty} E_j \right) &= \mu \left(\bigcup_{j=1}^{\infty} B_j \right) \\ &= \sum_{j=1}^{\infty} \mu(B_j) \\ &= \mu(E_1) + \sum_{j=2}^{\infty} \mu(E_j \setminus E_{j-1}) \\ &= \mu(E_1) + \sum_{j=2}^{\infty} \mu(E_j) - \mu(E_{j-1}) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \end{aligned}$$

□

Remark 2.4.4. If we prove something for the set then we can prove it for the complement.

$$\mu(A) + \mu(A^c) = \mu(X)$$

Theorem 2.4.5

If $\mu(X) < \infty$ then if $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots, E_n \supseteq E_{n+1}$ for all n then

$$\mu \left(\bigcap_{j=1}^{\infty} E_j \right) = \lim_{j \rightarrow \infty} \mu(E_j)$$

Proof. Assume E_j are decreasing, i.e.,

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$$

then $E_1^c \subseteq E_2^c \subseteq \dots$ then

$$\begin{aligned}\mu\left(\bigcup_{j=1}^{\infty} E_j^c\right) &= \lim_{j \rightarrow \infty} \mu(E_j^c) \\ \mu(X) - \mu\left(\left(\bigcup_{j=1}^{\infty} E_j^c\right)^c\right) &= \lim_{j \rightarrow \infty} (\mu(X) - \mu(E_j)) \\ \mu(X) - \mu\left(\bigcap_{j=1}^{\infty} E_j\right) &= \lim_{j \rightarrow \infty} (\mu(X) - \mu(E_j))\end{aligned}$$

□

Example 2.4.6

\mathbb{N} with counting measure, $E_j = \{j, j+1, j+2, \dots\}$, $\mu(E_j) = \infty$, $\bigcap E_j = \emptyset$ has measure 0.

Definition 2.4.7. If A_1, A_2, A_3, \dots is an arbitrary sequence of measurable sets. We can define

$$\begin{aligned}\limsup A_j &:= \bigcap_{n=1}^{\infty} \bigcup_{j \geq n} A_j = \{x : x \in A_n \text{ for infinitely many } n\} \\ \liminf A_j &:= \bigcup_{n=1}^{\infty} \bigcap_{j \geq n} A_j = \{x : x \text{ belong to all but finitely many}\}\end{aligned}$$

Lemma 2.4.8 (Borel-Cantelli Lemma)

If $\{A_j\}$ is a sequence of measurable sets such that

$$\sum_{j=1}^{\infty} \mu(A_j) < \infty$$

then almost every x (meaning all x except in a null set) belong to A_n for only finitely many n . Or equivalently,

$$\mu(\limsup A_n) = 0$$

Proof. $\bigcup_{j \geq n} A_j$ are decreasing. In Borel Cantelli, we have $\sum \mu(A_j) < \infty$, so $\mu(\bigcup A_n) = 0$.

use “continuity from above”

$$\mu(\limsup A_n) = \lim_{n \rightarrow \infty} \mu \left(\bigcup_{j \geq n} A_j \right)$$

$$\mu \left(\bigcup_{j \geq n} A_j \right) \leq \sum_{j \geq n} \mu(A_j) \rightarrow 0$$

as $n \rightarrow \infty$. □

Completion of a σ -algebra (when a measure μ is given), (X, \mathcal{M}, μ) $\bar{\mathcal{M}}$ consists of all unions $E \cup F$, where $E \in \mathcal{M}$ and $F \subseteq N \in \mathcal{M}$ for some null set N , $\mu(N) = 0$.

Define $\bar{\mu}$ by $\bar{\mu}(E \cup F) = \mu(E)$.

§2.5 Measurable Functions

Definition 2.5.1. $f : X \rightarrow Y$ where (X, \mathcal{M}) and (Y, \mathcal{N}) are measurable spaces. f is $(\mathcal{M}, \mathcal{N})$ -measurable if for every $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$. where $f^{-1}(E) = \{x \in X : f(x) \in E\}$.

Lemma 2.5.2

Let \mathcal{E} generate \mathcal{N} (i.e., $\mathcal{N} = \mathfrak{M}(\mathcal{E})$). Then f is $(\mathcal{M}, \mathcal{N})$ -measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

Proof. Define $\mathcal{C} = \{E : f^{-1}(E) \in \mathcal{M}\}$, observe that \mathcal{C} is a σ -algebra. then

$$f(x) \in \bigcup E_j \iff x \in f^{-1}\left(\bigcup E_j\right) \iff x \in \bigcup f^{-1}(E_j) \iff \bigcup \{x : f(x) \in E_j\}$$

□

Claim 2.5.3 — $f : X \rightarrow Y$ is $(\mathcal{M}, \mathcal{N})$ -measurable, $g : Y \rightarrow Z$ is $(\mathcal{N}, \mathcal{R})$ -measurable then $g \circ f : X \rightarrow Z$ is $(\mathcal{M}, \mathcal{R})$ -measurable.

Proof. $(g \circ f)^{-1}(E) = \{x \in X : g(f(x)) \in E\} = f^{-1}(g^{-1}(E)) = \{x \in X : f(x) \in g^{-1}(E)\}$ □

vector-valued-function $f : X \rightarrow (Y_1 \times Y_2 \times \cdots \times Y_n)$ and defined by $x \mapsto (f_1(x), f_2(x), \dots, f_n(x))$ where $f_j : X \rightarrow Y_j$ is $(\mathcal{M}, \mathcal{N}_j)$ -measurable.

Then f is $(\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_n)$ if and only if $f_i(\mathcal{M}_i, \mathcal{N}_i)$ -measurable.

$$\begin{aligned} f^{-1}(E_1 \times E_2 \times \cdots \times E_n) &= f_1^{-1}(E_1) \cap f_2^{-1}(E_2) \cap \cdots \cap f_n^{-1}(E_n) \\ &= \bigcap_{j=1}^n f_j^{-1}(E_j) \end{aligned}$$

Let $f : X \rightarrow \mathbb{R}$, $g : X \rightarrow \mathbb{R}$ \mathcal{M} σ -algebra on X , f, g are \mathcal{M} -measurable What about $f(x) + g(x)$

$M(x) = \max\{f(x), g(x)\}$, $f, g : X \rightarrow \mathbb{R}$, \mathcal{M} -measurable. The collection (a, ∞) generates the Borel σ -algebra on \mathbb{R} . $(a, \infty) \setminus (b, \infty) = (a, b]$ for $a < b$ and $(a, c) = \bigcup_n (a, c - \frac{1}{n}]$
 $M^{-1}(a, \infty) = \{x : f(x) > a, g(x) > a\} = f^{-1}(a, \infty) \cap g^{-1}(a, \infty)$

$f_n : X \rightarrow \mathbb{R}$, \mathcal{M} -measurable, then $S = \sup_n f_n$ is \mathcal{M} -measurable. $S^{-1}(a, \infty) = \{x : \sup f_n(x) > a\} = \bigcup_n \{x : f_n(x) > a\} = \bigcup f_n^{-1}(a, \infty)$

Similar for min and inf.

Definition 2.5.4. If $f_n : X \rightarrow \mathbb{R}$, \mathcal{M} -measurable then

$$\limsup f_n = \inf_k \sup_{n \geq k} f_n$$

$$\liminf f_n = \sup_k \inf_{n \geq k} f_n$$

Claim 2.5.5 — $\limsup f_n$ and $\liminf f_n$ are \mathcal{M} -measurable.

Proof. f_n has a pointwise limit if $\lim_{n \rightarrow \infty} f_n(x)$ exists for all $x \in X$. $\{x \subseteq X : \lim f_n(x) \text{ exists}\} = \{x \subseteq X : \underbrace{\limsup f_n(x) - \liminf f_n(x)}_{D(x)} = 0\} = D^{-1}(\{0\})$ \square

§2.6 Integration

nonnegative simple function are measurable function with finitely many values in \mathbb{R} (NOT on $\bar{\mathbb{R}}$). $s : X \rightarrow \mathbb{R}$, $s(x) = \sum z_j \mathbb{1}_{x, s(x)=z_j}(x) = \sum z_j \mathbb{1}_{f^{-1}(z_j)}$ If values of s are $\{z_1, \dots, z_n\}$

Theorem 2.6.1

Consider nonnegative measurable function f . There exist a sequence of simple function s_n such that

- $0 \leq s_n \leq s_{n+1} \leq f$ (i.e, $s_n(x) \leq s_{n+1}(x)$)
- $\lim_{n \rightarrow \infty} s_n(x) = f(x)$ for all x
- The convergence is uniform on all sets where f is bounded. If E is such that $|f(x)| \leq M$ for all $x \in E$ then

$$\sup_{x \in E} f(x) - s_n(x) \rightarrow 0$$

Proof. s_n is defined so that it takes value in $[0, 2^n]$. Consider the segment $\frac{k}{2^n}$ on y-axis, then

$$s_n(x) = \begin{cases} k \cdot 2^{-n} & \text{if } k2^{-n} \leq f(x) < (k+1)2^{-n}, 0 \leq k \leq 4^n - 1 \\ 2^n & \text{if } f(x) \geq 2^n \end{cases}$$

If $f(x) < 2^n$ then $0 \leq f(x) - s_n(x) \leq 2^{-n}$. We can see that $s_n(x) \leq s_{n+1}(x)$ because each step of s_{n+1} is a refinement of s_n . \square

We first define the integral for simple function (in analogy to the definition of Riemann-integral for step functions)

Definition 2.6.2. Define $s(x) = \sum_j c_j \mathbb{1}_{E_j}$ where the E_j are pairwise disjoint, $\biguplus E_j = X$, then

$$\int s \, d\mu = \sum_j c_j \mu(E_j)$$

Claim 2.6.3 —

$$s(x) = \sum_{j=1}^n c_j \mathbb{1}_{E_j}(x) = \sum_{k=1}^m d_k \mathbb{1}_{E_k}(x)$$

where $X = \biguplus E_j = \biguplus E_k$. If $x \in E_j \cap E_k$ then $c_j = d_k$.

Proof. We know that $\biguplus_{j,k} E_j \cap E_k = X$ and $E_j = \biguplus_k E_j \cap E_k$

GOAL: $\sum_{j=1}^n c_j \mu(E_j) = \sum_{k=1}^m d_k \mu(E_k)$

$$\begin{aligned}
\text{LHS} &= \sum_{j=1}^n c_j \sum_{k=1}^{\infty} \mu(E_j \cap F_k) = \sum_{k=1}^m \sum_{j=1}^n d_k \mu(E_j \cap F_k) \\
&= \sum_{k=1}^m d_k \mu(F_k)
\end{aligned}$$

□

Lemma 2.6.4

Suppose s, t are simple functions then

$$\int (s + t) \, d\mu = \int s \, d\mu + \int t \, d\mu$$

Remark 2.6.5. Can shortly write

$$\int s + t = \int s + \int t$$

Proof.

$$\begin{aligned}
s &= \sum_{j=1}^n c_j \mathbb{1}_{E_j} = \sum_j \sum_k c_j \mathbb{1}_{E_j \cap F_k} \\
t &= \sum_{k=1}^m d_k \mathbb{1}_{F_k} = \sum_j \sum_k d_k \mathbb{1}_{E_j \cap F_k} \\
s + t &= \sum_{j,k} (c_j + d_k) \mathbb{1}_{E_j \cap F_k}
\end{aligned}$$

$$\begin{aligned}
\int s \, d\mu &= \sum_{j,k} c_j \mu(E_j \cap F_k) \\
\int t \, d\mu &= \sum_{j,k} d_k \mu(E_j \cap F_k) \\
\int (s + t) \, d\mu &= \sum_{j,k} (c_j + d_k) \mu(E_j \cap F_k)
\end{aligned}$$

□

Lemma 2.6.6

$\nu(E) = \int_E s \, d\mu = \int s \mathbb{1}_E \, d\mu = \sum c_j \mu(E_j \cap E)$ this defines a measure on \mathcal{M} (given σ -algebra)

Proof. If E^l is a sequence of pairwise disjoint measurable set, check

$$\nu\left(\biguplus E^l\right) = \sum \nu(E^l)$$

$$\begin{aligned} \nu\left(\biguplus E^l\right) &= \sum c_j \mu(E_j \cap \biguplus E^l) \\ &= \sum_{j=1}^n c_j \sum_l \mu(E_j \cap E^l) \\ &= \sum_l \sum_j c_j \mu(E_j \cap E^l) \\ &= \sum_l \nu(E^l) \end{aligned}$$

□

Definition 2.6.7. For any non-negative f , a measurable function, define

$$\int f \, d\mu = \sup_{s \leq f, s \text{ simple}} \int s \, d\mu$$