

# MATH 629 (Measure Theory) Lecture Notes

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Spring 2024

## Contents

<b>1</b>	<b>From Riemann to Lebesgue</b>	<b>2</b>
1.1	Riemann Integral . . . . .	2
1.2	Lebesgue null sets . . . . .	4
1.3	Oscillation and Discontinuity . . . . .	6
<b>2</b>	<b>Measures</b>	<b>9</b>
2.1	Introduction . . . . .	9
2.2	Construction of Measure . . . . .	10
2.3	$\sigma$ -algebra . . . . .	12
	Generating $\sigma$ -algebra . . . . .	14
2.4	Measures . . . . .	16
2.5	Measurable Functions . . . . .	19
2.6	Integration . . . . .	20

# 1 From Riemann to Lebesgue

## §1.1 Riemann Integral

**Definition 1.1.1.**  $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$  is a partition of  $[a, b]$ .

**Definition 1.1.2.** If  $P, P'$  are partitions of  $[a, b]$  and  $P \subseteq P'$ , then  $P'$  is a refinement of  $P$ .

**Definition 1.1.3.** Given a bounded function  $f : [a, b] \rightarrow \mathbb{R}$  and a partition  $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$  we define

$$m_i(f) = \inf_{t \in [x_{i-1}, x_i]} f(t)$$

$$M_i(f) = \sup_{t \in [x_{i-1}, x_i]} f(t)$$

define the lower sum as

$$L(f, P) = \sum_{i=1}^n m_i(f)(x_i - x_{i-1})$$

and the upper sum as

$$U(f, P) = \sum_{i=1}^n M_i(f)(x_i - x_{i-1})$$

### Lemma 1.1.4

Given a bounded function  $f : [a, b] \rightarrow \mathbb{R}$  and partitions  $P$  of  $[a, b]$ . Suppose that  $P'$  is a refinement of  $P$  then

$$(b - a) \inf_{t \in [a, b]} f(t) \leq L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P) \leq (b - a) \sup_{t \in [a, b]} f(t)$$

### Corollary 1.1.5

Suppose that  $P_1, P_2$  are partitions of  $[a, b]$  then  $L(f, P_1) \leq U(f, P_2)$

*Proof.* Let  $P' = P_1 \cup P_2$  then  $P'$  is a refinement of  $P_1$  and  $P_2$  and use Lemma 1.1.4  $\square$

**Lemma 1.1.6**

Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is bounded. Then

$$(b-a) \inf_{t \in [a, b]} f(t) \leq \sup_P L(f, P) \leq \inf_P U(f, P) \leq (b-a) \sup_{t \in [a, b]} f(t)$$

**Definition 1.1.7.** A function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable if

$$\sup_P L(f, P) = \inf_P U(f, P)$$

and the common value is called the Riemann integral of  $f$  and is denoted by  $\int_a^b f$

**Lemma 1.1.8**

Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is bounded. Then  $f$  is Riemann integrable if and only if for any  $\varepsilon > 0$  there exists a partition  $P$  such that

$$U(f, P) - L(f, P) < \varepsilon$$

*Proof.* ( $\Rightarrow$ ) For any  $\varepsilon > 0$ . Suppose that  $f$  is Riemann integrable. Then there exists  $P_1, P_2$  such that

$$L(f, P_1) \geq \int_a^b f - \frac{\varepsilon}{2}$$

$$U(f, P_2) \leq \int_a^b f + \frac{\varepsilon}{2}$$

let  $P = P_1 \cup P_2$  then

$$U(f, P) - L(f, P) \leq \varepsilon$$

( $\Leftarrow$ ) For any  $\varepsilon > 0$ , there exists  $P_\varepsilon$  such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

since  $\varepsilon$  is arbitrary, we have

$$\sup_P L(f, P) = \inf_P U(f, P)$$

□

**Theorem 1.1.9**

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  then  $f$  is Riemann integrable.

*Proof.*  $f$  is continuous on a compact set, so,  $f$  is uniformly continuous. For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any  $x, y \in [a, b]$  if  $|x - y| < \delta$  then  $|f(x) - f(y)| < \frac{\varepsilon}{(b-a)}$ . Let  $N$  be such that  $\frac{(b-a)}{N} < \delta$  and let  $P = \{x_i := a + \frac{(b-a)i}{N}\}$  then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^N (M_i(f) - m_i(f)) \frac{(b-a)}{N} \\ &\leq \sum_{i=1}^N \frac{\varepsilon}{(b-a)} \frac{(b-a)}{N} \\ &= \varepsilon \end{aligned}$$

□

**Remark 1.1.10.** Let  $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$  defined on the  $[0, 1]$ . Then  $U(f, P) = 1$  and  $L(f, P) = 0$  for any partition  $P$ . So,  $f$  is not Riemann integrable.

## §1.2 Lebesgue null sets

**Definition 1.2.1.** For the closed interval  $I = [a, b]$ , the length of  $I$ , denoted as  $\ell(I)$  is defined as  $\ell(I) = b - a$

**Definition 1.2.2.** A set  $E$  is said to be a Lebesgue null set if for any  $\varepsilon > 0$  there exists a sequence of intervals  $\{I_n\}_{n \in \mathbb{N}}$  such that

$$E \subseteq \bigcup_{n=1}^{\infty} I_n$$

and

$$\sum_{n=1}^{\infty} \ell(I_n) < \varepsilon$$

### Lemma 1.2.3

Countable unions of Lebesgue null sets are Lebesgue null sets.

*Proof.* For any  $\varepsilon > 0$  and for each Lebesgue null sets  $E_n$  there exists  $I_{E_n, i}$  such that

$$E_n \subseteq \bigcup_{i=1}^{\infty} I_{E_n, i}$$

and

$$\sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \frac{\varepsilon}{2^n}$$

then

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \varepsilon$$

□

**Definition 1.2.4.** A set  $E \subseteq [a, b]$  has content zero if for any  $\varepsilon > 0$  there exists  $I_1, I_2, \dots, I_n$  such that

$$E \subseteq \bigcup_{i=1}^n I_i$$

and

$$\sum_{i=1}^n \ell(I_i) < \varepsilon$$

**Lemma 1.2.5**

Suppose that  $E \subseteq [a, b]$  is a compact Lebesgue null set then  $E$  has content zero.

*Proof.* For any  $\varepsilon > 0$  there exists a sequence of interval  $\{I_n\}_{n \in \mathbb{N}}$  such that  $E \subseteq \bigcup I_n$  and  $\sum \ell(I_n) < \frac{\varepsilon}{2}$ . Suppose that  $I_n = [a_n, b_n]$ , then let

$$J_n = \left( a_n - \frac{\varepsilon}{2^{n+3}}, b_n + \frac{\varepsilon}{2^{n+3}} \right) \supseteq I_n$$

then from the compactness of  $E$ , there exists a finite subcover  $J_{n_1}, J_{n_2}, \dots, J_{n_k}$  such that  $E \subseteq \bigcup J_{n_i}$  then we construct a finite closed interval  $K_i$  by

$$K_i = \left[ a_{n_i} - \frac{\varepsilon}{2^{n_i+2}}, b_{n_i} + \frac{\varepsilon}{2^{n_i+2}} \right]$$

then  $E \subseteq \bigcup K_i$  and  $\sum \ell(K_i) < \varepsilon$

□

**Corollary 1.2.6**

if  $a < b$  then  $[a, b]$  is not a Lebesgue null set.

*Proof.* By contradiction, since  $[a, b]$  is compact, then  $[a, b]$  has content zero, but  $[a, b]$  don't have content zero. □

### §1.3 Oscillation and Discontinuity

**Definition 1.3.1.** Suppose that  $X \subseteq \mathbb{R}$ ,  $f : X \rightarrow \mathbb{R}$  for any  $x \in X$  and  $\delta > 0$ , define

$$M_{f,\delta}(x) := \sup\{f(y) : d(x, y) < \delta\}$$

$$m_{f,\delta}(x) := \inf\{f(y) : d(x, y) < \delta\}$$

then we define

$$\text{osc}_f(x) := \lim_{\delta \rightarrow 0^+} M_{f,\delta}(x) - m_{f,\delta}(x)$$

#### Lemma 1.3.2

$f$  is continuous at  $x$  if and only if  $\text{osc}_f(x) = 0$ .

*Proof.* ( $\Rightarrow$ ) Suppose that  $f$  is continuous at  $x$ , then for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $d(x, y) < \delta$  then  $|f(x) - f(y)| < \frac{\varepsilon}{2}$ . Then

$$M_{f,\delta}(x) - m_{f,\delta}(x) \leq \sup\{f(y) : d(x, y) < \delta\} - \inf\{f(y) : d(x, y) < \delta\} < \varepsilon$$

( $\Leftarrow$ ) Suppose that  $\text{osc}_f(x) = 0$ , then for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $M_{f,\delta}(x) - m_{f,\delta}(x) < \varepsilon$ . Then for any  $y \in X$  such that  $d(x, y) < \delta$ , we have  $|f(x) - f(y)| < \varepsilon$  then  $f$  is continuous at  $x$ .  $\square$

Before we prove this theorem, we need to prove the following lemma.

#### Lemma 1.3.3

$\{x \in [a, b] : \text{osc}_f(x) \geq \gamma\}$  is closed.

*Proof.* We need to show that  $\{x : \text{osc}_f(x) < \gamma\}$  is open. Fix  $x$  in that set. Let  $\varepsilon = \gamma - \text{osc}_f(x)$  then

$$\sup_{|w-x|<\delta} f(w) - \inf_{|w-x|<\delta} f(w) < \text{osc}_f(x) < \gamma$$

then for any  $w \in (x - \delta, x + \delta)$  if  $|w - x| < \frac{\delta}{2}$  then

$$\text{osc}(w) \leq \sup_{|y-w|<\frac{\delta}{2}} f(y) - \inf_{|y-w|<\frac{\delta}{2}} f(y) < \gamma$$

So,  $B(x, \frac{\delta}{2}) \subseteq \{x : \text{osc}_f(x) < \gamma\}$   $\square$

we observe that

- (i) If the set of discontinuities is a Lebesgue null set, then  $\{x : \text{osc}_f(x) \geq \gamma\}$  is a set of content zero.
- (ii) If  $\{x : \text{osc}_f(x) \geq \gamma\}$  is a Lebesgue null set, then the set of discontinuities is also a Lebesgue null set.

**Lemma 1.3.4**

Suppose that  $f$  is defined on  $[c, d]$ , assume that  $\text{osc}_f(x) < \gamma$  then we can find a partition

$$U(f, P) - L(f, P) < \gamma(b - a)$$

*Proof.* For every  $x \in [c, d]$ , there exists  $\delta_x > 0$  such that

$$\sup_{|w-x|<\delta_x} f(w) - \inf_{|w-x|<\delta_x} f(x) < \gamma$$

construct a cover by

$$B(x, \delta_x) = \{w \in [c, d] : |w - x| < \delta_x\}$$

since  $[c, d]$  is compact, there exists a finite subcover  $B(p_1, \delta_{p_1}), \dots, B(p_n, \delta_{p_n})$  then let  $\delta_0 = \frac{\min\{\delta_{p_i}\}}{100}$  then we can construct a partition  $P = \{c = x_0 < x_1 < \dots < x_n = d\}$  such that  $|x_i - x_{i-1}| < \delta_0$  then  $M_i - m_i < \gamma$  and

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \\ &< \gamma \sum_{i=1}^n (x_i - x_{i-1}) \\ &= \gamma(d - c) \end{aligned}$$

□

**Theorem 1.3.5**

Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  then  $f \in \mathcal{R}([a, b])$  if and only if  $f$  is bounded and the set of discontinuity of  $f$  is a Lebesgue null set.

*Proof.* ( $\Rightarrow$ ) We want to show that for every  $n \in \mathbb{N}$ ,

$$\mathcal{D}_n = \left\{x : \text{osc}_f(x) \geq \frac{1}{n}\right\}$$

is a Lebesgue null set. For any  $\varepsilon > 0$ , since  $f$  is Riemann integrable, there exists a partition  $P$  of  $[a, b]$  such that

$$U(f, P) - L(f, P) = \sum_{i=1}^n (x_i - x_{i-1})(M_i - m_i) \leq \frac{\varepsilon}{n}$$

where  $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$  and  $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$ . in particular

$$\begin{aligned} \sum_{[x_{i-1}, x_i] \cap \mathcal{D}_n \neq \emptyset} (x_i - x_{i-1})(M_i - m_i) &\leq \frac{\varepsilon}{n} \\ \frac{1}{n} \sum_{[x_{i-1}, x_i] \cap \mathcal{D}_n \neq \emptyset} \ell([x_{i-1}, x_i]) &\leq \frac{\varepsilon}{n} \end{aligned}$$

So, this interval cover the set  $\mathcal{D}_n$

( $\Leftarrow$ ) pick  $\varepsilon_1 \ll \varepsilon$ , consider the set  $D(\varepsilon_1) = \{x \in [a, b] : \text{osc}_f(x) \geq \varepsilon_1\}$  closed set. Since  $D(\varepsilon_1)$  is a Lebesgue null set from the Lemma 1.2.5 it has content zero so we can find  $I_1, \dots, I_n$  such that

$$\sum_{j=1}^n \ell(I_j) < \varepsilon_1 \text{ and } D(\varepsilon_1) \subseteq \bigcup_{j=1}^n I_j$$

We form a partition of  $[a, b]$ ,  $a = x_0 < x_1 < \dots < x_N = b$  from  $I_j$ . There are two cases that we need to consider

- 1) if  $[x_{i-1}, x_i] \subseteq I_j$  for some  $j$  then set  $P_i = [x_{i-1}, x_i]$
- 2) if  $[x_{i-1}, x_i] \cap I_j = \emptyset$  for all  $j$  then  $\text{osc}(x) < \varepsilon_1$  for all  $x \in [x_{i-1}, x_i]$ . We want to partition further the interval  $[x_{i-1}, x_i]$  by partition  $P_i$ . Using Lemma 1.3.4 we can find a partition  $P_i$  of  $[x_{i-1}, x_i]$  such that

$$U(f, P_i) - L(f, P_i) < \varepsilon_1(x_i - x_{i-1})$$

We form a partition  $P = P_1 \cup \dots \cup P_N$  then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^N (U(f, P_i) - L(f, P_i)) \\ &= \sum_{i:\text{case 1}} (U(f, P_i) - L(f, P_i)) + \sum_{i:\text{case 2}} (U(f, P_i) - L(f, P_i)) \\ &\leq 2M \sum_{i:\text{case 1}} (x_i - x_{i-1}) + \varepsilon_1 \sum_{i:\text{case 2}} (x_i - x_{i-1}) \\ &\leq 2M\varepsilon_1 + \varepsilon_1(b - a) \\ &= \varepsilon_1(2M + b - a) \end{aligned}$$

□



# 2 Measures

## §2.1 Introduction

We define the  $\ell([c, d]) = d - c$  and If  $E = [c_1, d_1] \cup [c_2, d_2]$  where  $d_1 < c_2$  then  $\ell(E) = d_1 - c_1 + d_2 - c_2$ . This is consistent with the definition

$$\ell(E) = \int \mathbb{1}_E(x) \, dx$$

where the integral denotes the Riemann integral.

if  $E \subseteq [a, b]$  reference interval is

$$\int_a^b \mathbb{1}_E \, dx$$

**Remark 2.1.1.** The consistency of the definition also works with the set  $(c, d)$ ,  $[c, d)$ , and  $(c, d]$ , where the length of all of them is  $d - c$ .

**Remark 2.1.2.** we denote  $\mathbb{1}_E$  to be

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

### Example 2.1.3

Let  $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$  defined on the  $[0, 1]$ . Then  $U(f, P) = 1$  and  $L(f, P) = 0$  for any partition  $P$ .

Fix the reference interval  $[a, b]$  and consider subset of  $[a, b]$

Let  $\mathcal{A}$  = collection of sets for which  $\int_{[a, b]} \mathbb{1}_E \, dx$  exists.

If  $A_1, \dots, A_n \in \mathcal{A}$ , we can make the set to be mutually disjoint by taking  $E_1 = A_1$ ,  $E_2 = A_2 \setminus A_1$ ,  $E_3 = A_3 \setminus (A_1 \cup A_2)$ , and so on.

### Example 2.1.4

For  $E_1, E_2 \in \mathcal{A}$ , we have

$$\mathbb{1}_{E_1 \cap E_2}(x) = \mathbb{1}_{E_1}(x) \mathbb{1}_{E_2}(x)$$

**Example 2.1.5**

For the Riemann integral, we have

$$\int_a^b f(y) = \int_{a-v}^{b-v} f(v+y)$$

and we want

$$\int \mathbb{1}_E(x) \, dx = \int \mathbb{1}_{v+E}$$

where  $v + E = \{v + x : x \in E\}$

Let  $E = \mathbb{Q} \cap [0, 1]$  countable set, we can enumerate  $r_1, r_2, r_3, \dots$  such that

$$E = \bigcup_{n=1}^{\infty} \{r_n\}$$

and

$$\int \mathbb{1}_{\{r_k\}} = 0$$

$E$  should have length zero but according  $\mathbb{1}_E$  is not Riemann integrable.

**§2.2 Construction of Measure**

Suppose that  $\mathcal{C}$  be a collection of sets.

Can we define on suitable large collection of subset of  $\mathbb{R}$ ?

a set function  $\mu : \mathcal{C} \rightarrow [0, \infty]$  such that if  $\{E_j\}_{j=1}^{\infty}$  is a sequence of disjoint set in  $\mathcal{C}$  then

$$\mu\left(\bigcup_{i=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

$$\mu([a, b]) = b - a, \mu([0, 1)) = 1$$

Can we do this for the collection of all subset of  $\mathbb{R}$ ?

Answer: No, Vitali set.

**Theorem 2.2.1**

We cannot define a measure on the collection of all subset of  $\mathbb{R}$ . i.e., there does not exist a set function  $\mu : \mathfrak{P}(\mathbb{R}) \rightarrow [0, \infty]$  such that

- (i)  $\mu(v + E) = \mu(E)$  for all  $E \subseteq \mathbb{R}$  and  $v \in \mathbb{R}$
- (ii)  $\mu([0, 1]) = 1$
- (iii)  $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$  for all disjoint  $A_j \subseteq \mathbb{R}$

Before we prove that theorem, we need to define something and prove the following lemma.

**Definition 2.2.2.** We define a Vitali set  $V$  from picking an element  $x \in [0, 1)$  from each equivalence class of the relation  $x \sim y$  if  $x - y \in \mathbb{Q}$ . (e.g, pick  $x \in O_x$  for  $O_x \in \mathbb{R}/\mathbb{Q}$ )

**Lemma 2.2.3**

Suppose that  $V$  is a Vitali set then

$$V \cap V + q = \emptyset$$

For all  $q \in \mathbb{Q} \setminus \{0\}$

*Proof.* Suppose not, there exists  $a \in V$  such that  $a \in V + q \implies a - q \in V$  but we only pick 1 element in each equivalence class. contradiction.  $\square$

**Lemma 2.2.4**

Let  $V$  be a Vitali set and let  $W = \{q \in [-1, 1] : q \in \mathbb{Q}\}$  and

$$E = \bigcup_{w \in W} V + w$$

then

$$[0, 1] \subseteq E \subseteq [-1, 2]$$

*Proof.* Consider  $E \subseteq [-1, 2]$ . Since  $V \subseteq [0, 1)$ , then for any  $v \in V$ ,  $v \in [0, 1) \implies v + w \in [-1, 2]$ .

For the  $[0, 1] \subseteq E$ , for any  $x \in [0, 1]$  there exists  $O_x \in \mathbb{R}/\mathbb{Q}$  such that  $x \in O_x$ . then there exists  $v \in C_x$  such that  $v \in [0, 1)$  and  $v \in V$ , since both are from the same equivalence

class, then  $x - v \in \mathbb{Q}$  and  $|x - v| < 1 \implies x - v \in (-1, 1)$ . Hence, there exists  $w \in W$  such that  $w = x - v$  so  $v + w = x$ .  $\square$

*Proof of the theorem.* Suppose that  $\mu$  exists then using the result from Lemma 2.2.4 we get that

$$\mu([0, 1]) \leq \mu(E) \leq \mu([-1, 2])$$

from Lemma 2.2.3 we know that each  $V + w$  is disjoint, so

$$\begin{aligned} \mu([0, 1]) &\leq \sum_{w \in W} \mu(V) \leq \mu([-1, 2]) \\ 1 &\leq \sum_{w \in W} \mu(V) \leq 3 \end{aligned}$$

if  $\mu(V) = 0$  then  $\mu(E) = 0$  and if  $\mu(V) > 0$  then  $\mu(E) = \infty$ . Both are contradiction.  $\square$

## §2.3 $\sigma$ -algebra

**Definition 2.3.1.** Given a reference  $X$ . An **algebra** is a collection of subsets of  $X$ ,  $\mathcal{A}$ , such that

- (i)  $X \in \mathcal{A}$
- (ii) If  $A \in \mathcal{A}$  then the complement  $A^c = X \setminus A \in \mathcal{A}$
- (iii) If  $A, B \in \mathcal{A}$  then  $A \cup B \in \mathcal{A}$

**Remark 2.3.2.** •  $\emptyset \in \mathcal{A}$  because  $\emptyset = X^c$

- $A_1, A_2 \in \mathcal{A}$ ,  $A_1 \setminus A_2 = A_1 \cap A_2^c \in \mathcal{A}$
- Observe that if  $A_1, A_2 \in \mathcal{A}$  then  $A_1 \cap A_2 \in \mathcal{A}$  because  $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$

### Example 2.3.3

$X = [a, b]$  and  $\mathcal{A}$  is the collection of all sets  $E \subseteq [a, b]$  such that the Riemann integral  $\int \mathbb{1}_E(t) dt$  exists

**Definition 2.3.4.** A  $\sigma$ -algebra  $\mathcal{M}$  on  $X$  is

- (i) an algebra of subsets of  $X$
- (ii) If  $A_1, A_2, A_3, \dots$  is a sequence of set in  $\mathcal{M}$  then

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$$

$(X, \mathcal{M})$  is called a “**measurable space**”.

**Remark 2.3.5.**  $\mathcal{M}$  is a  $\sigma$ -algebra on  $X$  then it satisfies

- (i)  $X \in \mathcal{M}$
- (ii) If  $A \in \mathcal{M}$  then  $A^c \in \mathcal{M}$
- (iii) countable union of sets in  $\mathcal{M}$  is in  $\mathcal{M}$

**Definition 2.3.6.** Let  $(X, \mathcal{M})$  be a measurable set. Then a measure  $\mu$  is a set function  $\mu : \mathcal{M} \rightarrow [0, \infty], E \mapsto \mu(E)$  such that

- (i)  $\mu(\emptyset) = 0$
- (ii) If  $E_1, E_2, E_3, \dots$  is a sequence of disjoint set in  $\mathcal{M}$  then

$$\mu \left( \bigcup_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \mu(E_j)$$

called  $\sigma$ -additivity.

$(X, \mathcal{M}, \mu)$  is called a “**measure space**”.

**Remark 2.3.7.**

$$\left( \bigcap_{j=1}^{\infty} A_j \right) = \left( \bigcup_{j=1}^{\infty} A_j^c \right)^c \in \mathcal{M}$$

### Example 2.3.8

examples of  $\sigma$ -algebra

- (i)  $\mathcal{M} = \{\emptyset, X\}$
- (ii)  $\mathcal{M} = \mathfrak{P}(X)$  = collection of all subsets of  $X$   
 $\mathbb{N} = \{1, 2, 3, \dots\}$  and  $\mu(E) = |E|$  (the cardinality of  $E$ ) if  $E$  is finite and  $\mu(E) = \infty$  if  $E$  is infinite.
- (iii)  $X$  write  $X$  as a disjoint (countable) union of sets  $A_j$ . Then  $\mathcal{M}$  = all countable unions of  $A_j$ .
- (iv) Let  $X$  be a set. Let  $\mathcal{M}$  be the collection of all sets  $A$ ,  $A \subseteq X$  such that  $A$  is countable or  $A^c$  is countable.
- (v)  $X = \mathbb{R}$  (or  $\mathbb{R}^n$ ),  $\mathcal{B}_{\mathbb{R}}$  is the smallest  $\sigma$ -algebra containing all open sets.

More generally if  $\mathcal{E}$  is a collection of subsets of  $X$  then  $\mathfrak{M}(\mathcal{E})$  is the smallest  $\sigma$ -algebra that contains all sets in  $\mathcal{E}$ .

If  $\mathcal{M}_1, \mathcal{M}_2$  are two  $\sigma$ -algebras, then  $\mathcal{M}_1 \cap \mathcal{M}_2$  is also a  $\sigma$ -algebra.

If  $\{\mathcal{M}_\alpha\}_{\alpha \in \mathcal{I}}$  is a collection of  $\sigma$ -algebras, their intersection is also a  $\sigma$ -algebra.

### Generating $\sigma$ -algebra

**Definition 2.3.9.**  $\mathfrak{M}(\mathcal{E}) :=$  intersection of all  $\sigma$ -algebra that contain the collection  $\mathcal{E}$ . We call it the  $\sigma$ -algebra generated by  $\mathcal{E}$ .

**Remark 2.3.10.** If  $\mathcal{E} \subset \mathcal{F} \implies \mathfrak{M}(\mathcal{E}) \subset \mathfrak{M}(\mathcal{F})$

#### Lemma 2.3.11

If  $\mathcal{E} \subseteq \mathfrak{M}(\mathcal{F})$  then  $\mathfrak{M}(\mathcal{E}) \subseteq \mathfrak{M}(\mathcal{F})$

*Proof.*  $\mathfrak{M}(\mathcal{F})$  is a  $\sigma$ -algebra that contains  $\mathcal{E}$ . It contains the intersection of all  $\sigma$ -algebras which contain  $\mathcal{E}$ .  $\square$

#### Example 2.3.12

$\mathcal{B}_{\mathbb{R}} = \sigma$ -algebra on  $\mathbb{R}$  containing all open sets  $\mathcal{E}$  a collection of all open intervals,  $\mathcal{E} \subseteq \mathcal{O} =$  collection of all open sets in  $\mathbb{R}$ ,  $\mathcal{B}_{\mathbb{R}} = \mathfrak{M}(\mathcal{O})$ .  $\mathfrak{M}(\mathcal{E}) \subseteq \mathcal{B}_{\mathbb{R}}$ . Each open set is a countable union of open intervals. Each open set is contained in  $\mathfrak{M}(\mathcal{E})$ .

Since  $\mathcal{O} \subseteq \mathfrak{M}(\mathcal{E}) \implies \mathfrak{M}(\mathcal{O}) \subseteq \mathfrak{M}(\mathcal{E})$ . get  $\mathfrak{M}(\mathcal{O}) = \mathfrak{M}(\mathcal{E})$ .

**Definition 2.3.13.** Given  $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2), \dots, (X_n, \mathcal{M}_n)$  measurable spaces. Define a “product  $\sigma$ -algebra” on  $X_1 \times X_2 \times \dots \times X_n$  denoted by

$$\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \dots \oplus \mathcal{M}_n = \bigoplus_{j=1}^n \mathcal{M}_j$$

defined as the  $\sigma$ -algebra generated by the sets  $E_1 \times E_2 \times \dots \times E_n$  where  $E_j \in \mathcal{M}_j$ .

i.e., define  $\mathcal{E} := \{E_1 \times E_2 \times \dots \times E_n : E_j \in \mathcal{M}_j\}$  then

$$\bigoplus_{j=1}^n \mathcal{M}_j := \mathfrak{M}(\mathcal{E})$$

**Remark 2.3.14.** Folland defines it the  $\sigma$ -algebra generated by

$$(X_1 \times X_2 \times \cdots \times X_{n-1} \times E_n)$$

where  $E_n \in \mathcal{M}_n$ ,

$$(X_1 \times X_2 \times \cdots \times E_{n-1} \times X_n)$$

where  $E_{n-1} \in \mathcal{M}_{n-1}$ . and so on. To be clear, let

$$\mathcal{E}' := \bigcup_{j=1}^n \{(X_1 \times \cdots \times X_{j-1} \times E_j \times X_{j+1} \times \cdots \times X_n) : E_j \in \mathcal{M}_j\}$$

then

$$\bigoplus_{j=1}^n \mathcal{M}_j := \mathfrak{M}(\mathcal{E}')$$

**Claim 2.3.15** — Both definitions on product of  $\sigma$ -algebra are equivalent.

*Proof.* The goal is to show that  $\mathfrak{M}(\mathcal{E}) = \mathfrak{M}(\mathcal{E}')$ .

( $\supseteq$ ) Obviously,  $\mathcal{E}' \subseteq \mathcal{E}$  so  $\mathfrak{M}(\mathcal{E}') \subseteq \mathfrak{M}(\mathcal{E})$ .

( $\subseteq$ ) We want to show that  $\mathcal{E} \subseteq \mathfrak{M}(\mathcal{E}')$ . Fix  $(E_1 \times E_2 \times \cdots \times E_n) \in \mathcal{E}$  then from the definition of  $\sigma$ -algebra generated by a collection, which is closed under intersection, so we can pick an element from the construction of  $\mathcal{E}'$  and do the intersection, so  $(E_1 \times E_2 \times \cdots \times E_n) \in \mathfrak{M}(\mathcal{E}')$ .

□

### Theorem 2.3.16

Given  $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2)$  measurable spaces. Assume that  $\mathcal{M}_1$  is generated by a collection  $\mathcal{E}_1$  and  $\mathcal{M}_2$  is generated by a collection  $\mathcal{E}_2$ . Then  $\mathcal{M}_1 \oplus \mathcal{M}_2$  is generated by the sets  $E_1 \times X_2, X_1 \times E_2$ , where  $E_1 \in \mathcal{E}_1$  and  $E_2 \in \mathcal{E}_2$ .

*Proof.* Let  $\mathcal{P} := \{E_1 \times E_2 : E_i \in \mathcal{E}_i\}$ , obviously  $\mathfrak{M}(\mathcal{P}) = \mathfrak{M}(\{E_1 \times X_2 : E_1 \in \mathcal{E}_1\} \cup \{X_1 \times E_2 : E_2 \in \mathcal{E}_2\})$  and  $\mathfrak{M}(\mathcal{P}) \subseteq \mathcal{M}_1 \oplus \mathcal{M}_2$ . We need to show that  $\mathcal{M}_1 \oplus \mathcal{M}_2 \subseteq \mathfrak{M}(\mathcal{P})$ . Define

$$\mathcal{G}_1 = \{E_1 \subseteq X_1 : E_1 \times X_2 \in \mathfrak{M}(\mathcal{P})\}$$

$$\mathcal{G}_2 = \{E_2 \subseteq X_2 : X_1 \times E_2 \in \mathfrak{M}(\mathcal{P})\}$$

then  $\mathcal{G}_1$  is a  $\sigma$ -algebra consisting of subset of  $X_1$  which contains  $\mathcal{E}_1$ ,  $\mathcal{E}_1 \subseteq \mathcal{G}_1$ .  $\mathcal{E}_1$  generates  $\mathcal{M}_1$  so  $\mathfrak{M}(\mathcal{E}_1) = \mathcal{M}_1 \subseteq \mathcal{G}_1$ . So, we have  $E_1 \times X_2 \in \mathfrak{M}(\mathcal{P})$  for all  $E_1 \in \mathcal{M}_1$  and  $X_1 \times E_2 \in \mathfrak{M}(\mathcal{P})$  for all  $E_2 \in \mathcal{M}_2$ . The  $\sigma$ -algebra generated by the sets  $E_1 \times X_2, X_1 \times E_2$  is contained  $\mathcal{M}_1 \oplus \mathcal{M}_2 \in \mathfrak{M}(\mathcal{P})$ . □

**Claim 2.3.17** —  $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$ .

where  $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}}$  is generated by  $E_1 \times E_2$ , where  $E_1, E_2 \in \mathcal{B}_{\mathbb{R}}$ . and  $\mathcal{B}_{\mathbb{R}^2}$  is generated by the open sets in  $\mathbb{R}^2$ .

*Proof.*  $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{B}_{\mathbb{R}^2}$ . Want  $\mathcal{B}_{\mathbb{R}^2} \subseteq \mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}}$ . Consider the collection of all open rectangle of the form  $(a_1, b_1) \times (a_2, b_2)$  such  $a_i, b_i \in \mathbb{Q}$ . which are contained in  $O \subseteq \mathbb{R}^2$   $\square$

**Definition 2.3.18** (The Borel  $\sigma$  algebra on the extended real line). We use the notion  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$ . One possibility to define “ $\mathcal{B}_{\overline{\mathbb{R}}}$ ” is the  $\sigma$ -algebra generated by open sets in  $\mathbb{R}$ ,  $\{\infty\}$ ,  $\{-\infty\}$  open intervals should be  $(a, b)$ ,  $(a, \infty]$ ,  $[-\infty, b)$  for  $-\infty \leq a < b \leq \infty$ . Then define  $d(x, y) = |\arctan(x) - \arctan(y)|$  and  $\arctan(\infty) = \pi/2$ ,  $\arctan(-\infty) = -\pi/2$ .

## §2.4 Measures

**Definition 2.4.1.** Measures are  $\sigma$ -additive set functions,  $\mu(\emptyset) = 0$  and

$$\mu \left( \biguplus_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \mu(E_j)$$

where  $E_1, E_2, \dots$  is a sequence of disjoint sets.

**Remark 2.4.2.** There is some property

$$E \subseteq F \implies \mu(E) \leq \mu(F)$$

$$F = E \uplus (F \setminus E) \implies \mu(F) = \mu(E) + \mu(F \setminus E)$$

$\mu(\bigcup A_j) \leq \sum \mu(A_j)$  we can write  $\bigcup A_j$  as a disjoint union, i.e.,  $E_1 = A_1$ ,  $E_2 = A_2 \setminus A_1$ ,  $E_3 = A_3 \setminus (A_1 \cup A_2)$ , and so on then  $\mu(\bigcup A_j) = \mu(\bigcup E_j) = \sum \mu(E_j) \leq \sum \mu(A_j)$

The monotone convergence theorem for sets (continuity from below)

### Theorem 2.4.3

If  $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$  then

$$\mu \left( \bigcup_{j=1}^{\infty} E_j \right) = \lim_{j \rightarrow \infty} \mu(E_j)$$



*Proof.*

$$\bigcup_{j=1}^{\infty} E_j = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus (E_1 \cup E_2)) \cup \dots$$

So, we define  $B_1 = E_1, B_n = E_n \setminus E_{n-1}$  for  $n \geq 2$  then all  $B_j$  are disjoint.

$$\begin{aligned} \bigcup_{j=1}^{\infty} E_j &= \bigcup_{j=1}^{\infty} B_j \\ \mu \left( \bigcup_{j=1}^{\infty} E_j \right) &= \mu \left( \bigcup_{j=1}^{\infty} B_j \right) \\ &= \sum_{j=1}^{\infty} \mu(B_j) \\ &= \mu(E_1) + \sum_{j=2}^{\infty} \mu(E_j \setminus E_{j-1}) \\ &= \mu(E_1) + \sum_{j=2}^{\infty} \mu(E_j) - \mu(E_{j-1}) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \end{aligned}$$

□

**Remark 2.4.4.** If we prove something for the set then we can prove it for the complement.

$$\mu(A) + \mu(A^c) = \mu(X)$$

#### Theorem 2.4.5

If  $\mu(X) < \infty$  then if  $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots, E_n \supseteq E_{n+1}$  for all  $n$  then

$$\mu \left( \bigcap_{j=1}^{\infty} E_j \right) = \lim_{j \rightarrow \infty} \mu(E_j)$$

*Proof.* Assume  $E_j$  are decreasing, i.e.,

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$$

then  $E_1^c \subseteq E_2^c \subseteq \dots$  then

$$\begin{aligned}\mu\left(\bigcup_{j=1}^{\infty} E_j^c\right) &= \lim_{j \rightarrow \infty} \mu(E_j^c) \\ \mu(X) - \mu\left(\left(\bigcup_{j=1}^{\infty} E_j^c\right)^c\right) &= \lim_{j \rightarrow \infty} (\mu(X) - \mu(E_j)) \\ \mu(X) - \mu\left(\bigcap_{j=1}^{\infty} E_j\right) &= \lim_{j \rightarrow \infty} (\mu(X) - \mu(E_j))\end{aligned}$$

□

#### Example 2.4.6

$\mathbb{N}$  with counting measure,  $E_j = \{j, j+1, j+2, \dots\}$ ,  $\mu(E_j) = \infty$ ,  $\bigcap E_j = \emptyset$  has measure 0.

**Definition 2.4.7.** If  $A_1, A_2, A_3, \dots$  is an arbitrary sequence of measurable sets. We can define

$$\begin{aligned}\limsup A_j &:= \bigcap_{n=1}^{\infty} \bigcup_{j \geq n} A_j = \{x : x \in A_n \text{ for infinitely many } n\} \\ \liminf A_j &:= \bigcup_{n=1}^{\infty} \bigcap_{j \geq n} A_j = \{x : x \text{ belong to all but finitely many}\}\end{aligned}$$

#### Lemma 2.4.8 (Borel-Cantelli Lemma)

If  $\{A_j\}$  is a sequence of measurable sets such that

$$\sum_{j=1}^{\infty} \mu(A_j) < \infty$$

then almost every  $x$  (meaning all  $x$  except in a null set) belong to on  $A_n$  for only finitely many  $n$ . Or equivalently,

$$\mu(\limsup A_n) = 0$$

*Proof.*  $\bigcup_{j \geq n} A_j$  are decreasing. In Borel Cantelli, we have  $\sum \mu(A_j) < \infty$ , so  $\mu(\bigcup A_n) = 0$ .

use “continuity from above”

$$\mu(\limsup A_n) = \lim_{n \rightarrow \infty} \mu \left( \bigcup_{j \geq n} A_j \right)$$

$$\mu \left( \bigcup_{j \geq n} A_j \right) \leq \sum_{j \geq n} \mu(A_j) \rightarrow 0$$

as  $n \rightarrow \infty$ . □

Completion of a  $\sigma$ -algebra (when a measure  $\mu$  is given),  $(X, \mathcal{M}, \mu)$   $\bar{\mathcal{M}}$  consists of all unions  $E \cup F$ , where  $E \in \mathcal{M}$  and  $F \subseteq N \in \mathcal{M}$  for some null set  $N$ ,  $\mu(N) = 0$ .

Define  $\bar{\mu}$  by  $\bar{\mu}(E \cup F) = \mu(E)$ .

## §2.5 Measurable Functions

**Definition 2.5.1.**  $f : X \rightarrow Y$  where  $(X, \mathcal{M})$  and  $(Y, \mathcal{N})$  are measurable spaces.  $f$  is  $(\mathcal{M}, \mathcal{N})$ -measurable if for every  $E \in \mathcal{N}$ ,  $f^{-1}(E) \in \mathcal{M}$ . where  $f^{-1}(E) = \{x \in X : f(x) \in E\}$ .

### Lemma 2.5.2

Let  $\mathcal{E}$  generate  $\mathcal{N}$  (i.e.,  $\mathcal{N} = \mathfrak{M}(\mathcal{E})$ ). Then  $f$  is  $(\mathcal{M}, \mathcal{N})$ -measurable if and only if  $f^{-1}(E) \in \mathcal{M}$  for all  $E \in \mathcal{E}$ .

*Proof.* Define  $\mathcal{C} = \{E : f^{-1}(E) \in \mathcal{M}\}$ , observe that  $\mathcal{C}$  is a  $\sigma$ -algebra. then

$$f(x) \in \bigcup E_j \iff x \in f^{-1}\left(\bigcup E_j\right) \iff x \in \bigcup f^{-1}(E_j) \iff \bigcup \{x : f(x) \in E_j\}$$

□

**Claim 2.5.3** —  $f : X \rightarrow Y$  is  $(\mathcal{M}, \mathcal{N})$ -measurable,  $g : Y \rightarrow Z$  is  $(\mathcal{N}, \mathcal{R})$ -measurable then  $g \circ f : X \rightarrow Z$  is  $(\mathcal{M}, \mathcal{R})$ -measurable.

*Proof.*  $(g \circ f)^{-1}(E) = \{x \in X : g(f(x)) \in E\} = f^{-1}(g^{-1}(E)) = \{x \in X : f(x) \in g^{-1}(E)\}$  □

vector-valued-function  $f : X \rightarrow (Y_1 \times Y_2 \times \cdots \times Y_n)$  and defined by  $x \mapsto (f_1(x), f_2(x), \dots, f_n(x))$  where  $f_j : X \rightarrow Y_j$  is  $(\mathcal{M}, \mathcal{N}_j)$ -measurable.

Then  $f$  is  $(\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_n)$  if and only if  $f_i(\mathcal{M}_i, \mathcal{N}_i)$ -measurable.

$$\begin{aligned} f^{-1}(E_1 \times E_2 \times \cdots \times E_n) &= f_1^{-1}(E_1) \cap f_2^{-1}(E_2) \cap \cdots \cap f_n^{-1}(E_n) \\ &= \bigcap_{j=1}^n f_j^{-1}(E_j) \end{aligned}$$

Let  $f : X \rightarrow \mathbb{R}$ ,  $g : X \rightarrow \mathbb{R}$   $\mathcal{M}$   $\sigma$ -algebra on  $X$ ,  $f, g$  are  $\mathcal{M}$ -measurable What about  $f(x) + g(x)$

$M(x) = \max\{f(x), g(x)\}$ ,  $f, g : X \rightarrow \mathbb{R}$ ,  $\mathcal{M}$ -measurable. The collection  $(a, \infty)$  generates the Borel  $\sigma$ -algebra on  $\mathbb{R}$ .  $(a, \infty) \setminus (b, \infty) = (a, b]$  for  $a < b$  and  $(a, c) = \bigcup_n (a, c - \frac{1}{n}]$   
 $M^{-1}(a, \infty) = \{x : f(x) > a, g(x) > a\} = f^{-1}(a, \infty) \cap g^{-1}(a, \infty)$

$f_n : X \rightarrow \mathbb{R}$ ,  $\mathcal{M}$ -measurable, then  $S = \sup_n f_n$  is  $\mathcal{M}$ -measurable.  $S^{-1}(a, \infty) = \{x : \sup f_n(x) > a\} = \bigcup_n \{x : f_n(x) > a\} = \bigcup f_n^{-1}(a, \infty)$

Similar for min and inf.

**Definition 2.5.4.** If  $f_n : X \rightarrow \mathbb{R}$ ,  $\mathcal{M}$ -measurable then

$$\limsup f_n = \inf_k \sup_{n \geq k} f_n$$

$$\liminf f_n = \sup_k \inf_{n \geq k} f_n$$

**Claim 2.5.5** —  $\limsup f_n$  and  $\liminf f_n$  are  $\mathcal{M}$ -measurable.

*Proof.*  $f_n$  has a pointwise limit if  $\lim_{n \rightarrow \infty} f_n(x)$  exists for all  $x \in X$ .  $\{x \subseteq X : \lim f_n(x) \text{ exists}\} = \{x \subseteq X : \underbrace{\limsup f_n(x) - \liminf f_n(x)}_{D(x)} = 0\} = D^{-1}(\{0\})$   $\square$

## §2.6 Integration

**Definition 2.6.1.** nonnegative simple function are measurable function with finitely many values in  $\mathbb{R}$  (NOT on  $\bar{\mathbb{R}}$ ).  $s : X \rightarrow \mathbb{R}$ ,  $s(x) = \sum z_j \mathbb{1}_{x, s(x)=z_j}(x) = \sum z_j \mathbb{1}_{f^{-1}(z_j)}$  If values of  $s$  are  $\{z_1, \dots, z_n\}$

**Theorem 2.6.2**

Consider nonnegative measurable function  $f$ . There exist a sequence of simple function  $s_n$  such that

- $0 \leq s_n \leq s_{n+1} \leq f$  (i.e,  $s_n(x) \leq s_{n+1}(x)$ )
- $\lim_{n \rightarrow \infty} s_n(x) = f(x)$  for all  $x$
- The convergence is uniform on all sets where  $f$  is bounded. If  $E$  is such that  $|f(x)| \leq M$  for all  $x \in E$  then

$$\sup_{x \in E} f(x) - s_n(x) \rightarrow 0$$

*Proof.*  $s_n$  is defined so that it takes value in  $[0, 2^n]$ . Consider the segment  $\frac{k}{2^n}$  on y-axis, then

$$s_n(x) = \begin{cases} k \cdot 2^{-n} & \text{if } k2^{-n} \leq f(x) < (k+1)2^{-n}, 0 \leq k \leq 4^n - 1 \\ 2^n & \text{if } f(x) \geq 2^n \end{cases}$$

If  $f(x) < 2^n$  then  $0 \leq f(x) - s_n(x) \leq 2^{-n}$ . We can see that  $s_n(x) \leq s_{n+1}(x)$  because each step of  $s_{n+1}$  is a refinement of  $s_n$ .  $\square$

We first define the integral for simple function (in analogy to the definition of Riemann-integral for step functions)

**Definition 2.6.3.** Define  $s(x) = \sum_j c_j \mathbb{1}_{E_j}$  where the  $E_j$  are pairwise disjoint,  $\biguplus E_j = X$ , then

$$\int s \, d\mu = \sum_j c_j \mu(E_j)$$

**Claim 2.6.4 —**

$$s(x) = \sum_{j=1}^n c_j \mathbb{1}_{E_j}(x) = \sum_{k=1}^m d_k \mathbb{1}_{E_k}(x)$$

where  $X = \biguplus E_j = \biguplus E_k$ . If  $x \in E_j \cap E_k$  then  $c_j = d_k$ .

*Proof.* We know that  $\biguplus_{j,k} E_j \cap E_k = X$  and  $E_j = \biguplus_k E_j \cap E_k$

GOAL:  $\sum_{j=1}^n c_j \mu(E_j) = \sum_{k=1}^m d_k \mu(E_k)$

$$\begin{aligned}
\text{LHS} &= \sum_{j=1}^n c_j \sum_{k=1}^{\infty} \mu(E_j \cap F_k) = \sum_{k=1}^m \sum_{j=1}^n d_k \mu(E_j \cap F_k) \\
&= \sum_{k=1}^m d_k \mu(F_k)
\end{aligned}$$

□

**Lemma 2.6.5**

Suppose  $s, t$  are simple functions then

$$\int (s + t) \, d\mu = \int s \, d\mu + \int t \, d\mu$$

**Remark 2.6.6.** Can shortly write

$$\int s + t = \int s + \int t$$

*Proof.*

$$\begin{aligned}
s &= \sum_{j=1}^n c_j \mathbb{1}_{E_j} = \sum_j \sum_k c_j \mathbb{1}_{E_j \cap F_k} \\
t &= \sum_{k=1}^m d_k \mathbb{1}_{F_k} = \sum_j \sum_k d_k \mathbb{1}_{E_j \cap F_k} \\
s + t &= \sum_{j,k} (c_j + d_k) \mathbb{1}_{E_j \cap F_k}
\end{aligned}$$

$$\begin{aligned}
\int s \, d\mu &= \sum_{j,k} c_j \mu(E_j \cap F_k) \\
\int t \, d\mu &= \sum_{j,k} d_k \mu(E_j \cap F_k) \\
\int (s + t) \, d\mu &= \sum_{j,k} (c_j + d_k) \mu(E_j \cap F_k)
\end{aligned}$$

□

**Lemma 2.6.7**

$\nu(E) = \int_E s \, d\mu = \int s \mathbb{1}_E \, d\mu = \sum c_j \mu(E_j \cap E)$  this defines a measure on  $\mathcal{M}$  (given  $\sigma$ -algebra)

*Proof.* If  $E^l$  is a sequence of pairwise disjoint measurable set, check

$$\begin{aligned} \nu\left(\biguplus E^l\right) &= \sum \nu(E^l) \\ \nu\left(\biguplus E^l\right) &= \sum c_j \mu(E_j \cap \biguplus E^l) \\ &= \sum_{j=1}^n c_j \sum_l \mu(E_j \cap E^l) \\ &= \sum_l \sum_j c_j \mu(E_j \cap E^l) \\ &= \sum_l \nu(E^l) \end{aligned}$$

□

**Definition 2.6.8.** For any non-negative  $f$ , a measurable function, define

$$\int f \, d\mu = \sup_{s \leq f, s \text{ simple}} \int s \, d\mu$$

**Remark 2.6.9.** If  $0 \leq f \leq g$  then  $\int f \, d\mu \leq \int g \, d\mu$

**Theorem 2.6.10 (Monotone Convergence Theorem)**

If  $\{f_n\}$  is a sequence of measurable function, and  $0 \leq f_n \leq f_{n+1}$  for all  $n$ . (that means  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ ) Then

$$\int f \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

*Proof.* Since  $f_n \leq f_{n+1} \leq f$  then

$$\lim_{n \rightarrow \infty} \int f_n \, d\mu \leq \int f \, d\mu$$

We need to show that

$$\int f \, d\mu \leq \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

So, it suffices to show that for any  $0 \leq s_{\text{simple}} \leq f$ , that

$$\int s \, d\mu \leq \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

It suffices to show that for any  $\varepsilon > 0$ ,

$$(1 - \varepsilon) \int s \, d\mu \leq \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

Define  $E_n = \{x : (1 - \varepsilon)s(x) \leq f_n(x)\}$ , any  $x$  will be in one of the  $E_n$ . Then for any  $x \in E_n$ ,

$$s(x) \leq \frac{f_n(x)}{1 - \varepsilon}$$

Consider the measure defined by

$$\nu(E) = \int_E s \, d\mu$$

(we already show this is a measure in 2.6.7). We have  $E_n \subseteq E_{n+1}$  and  $E_n \rightarrow X$ . By continuity from below 2.4.3,

$$\lim_{n \rightarrow \infty} \nu(E_n) = \nu(X) = \int s \, d\mu$$

We get that

$$\nu(E_n) = \int_{E_n} s \, d\mu \leq \int_{E_n} \frac{f_n(x)}{1 - \varepsilon} \, d\mu \leq \int \frac{f_n(x)}{1 - \varepsilon} \, d\mu = \frac{1}{1 - \varepsilon} \int f_n(x) \, d\mu$$

Finally, we take limit on both sides and we have

$$\lim_{n \rightarrow \infty} \nu(E_n) = \nu(\mathbb{R}) = \int s \, d\mu \leq \lim_{n \rightarrow \infty} \frac{1}{1 - \varepsilon} \int f_n \, d\mu$$

□

### Lemma 2.6.11

If  $f, g$  are non negative measurable function then

$$\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$$

*Proof.* Now we have a tool



- Monotone Convergence Theorem
- Existence of  $s_n \gg f, t_n \gg g$

$$\begin{aligned}\int (s_n + t_n) \, d\mu &= \int s_n \, d\mu + \int t_n \, d\mu \\ \int (f + g) \, d\mu &= \int f \, d\mu + \int g \, d\mu\end{aligned}$$

□

**Lemma 2.6.12**

$f_k \geq 0$ ,  $f_k$  is measurable

$$\int \sum_{k=1}^{\infty} f_k(x) \, d\mu = \sum_{k=1}^{\infty} \int f_k \, d\mu$$

*Proof.* Just apply the Monotone Convergence Theorem.

$$s_n(x) = \sum_{k=1}^n f_k(x) \rightarrow \sum_{k=1}^{\infty} f_k(x)$$

□

**Remark 2.6.13.** We cannot always interchange integrals and limits (monotonicity is key)  
 $f_n(x) = \frac{1}{n} \mathbb{1}_{[0,n]}$ ,  $\int f_n \, d\mu = 1$  but  $\lim_{n \rightarrow \infty} f_n(x) = 0$ .

$$0 = \int \lim_{n \rightarrow \infty} f_n(x) \, d\mu < \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

Or on  $[0, 1]$ ,  $f_n(x) = n \mathbb{1}_{[0,1/n]}$ ,  $\int f_n \, d\mu = 1$  but  $\lim_{n \rightarrow \infty} f_n(x) = 0$ .

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

**Lemma 2.6.14** (Faton's Lemma)

If  $\{f_j\}$  is a sequence of measurable functions

$$\int \liminf_{j \rightarrow \infty} f_j(x) \, d\mu \leq \liminf_{j \rightarrow \infty} \int f_j \, d\mu$$

meaning

$$\int \lim_{k \rightarrow \infty} \underbrace{\inf_{j \geq k} f_j(x)}_{\text{increasing on } k} \, d\mu \leq \lim_{k \rightarrow \infty} \inf_{j \geq k} \int f_j \, d\mu$$

*Proof.*

$$\int \lim_{k \rightarrow \infty} \inf_{j \geq k} f_j(x) \, d\mu \stackrel{\text{MCT}}{=} \lim_{k \rightarrow \infty} \int \inf_{j \geq k} f_j(x) \, d\mu$$

Take any  $l \geq k$ , then  $\inf_{j \geq k} f_j(x) \leq f_l(x)$ , then for  $l \geq k$

$$\begin{aligned} \int \inf_{j \geq k} f_j(x) \, d\mu &\leq \int f_l(x) \, d\mu \\ \int \inf_{j \geq k} f_j(x) \, d\mu &\leq \inf_{j \geq k} \int f_j(x) \, d\mu \end{aligned}$$

□

Integral for “general” measurable functions.

**Definition 2.6.15.** Given a measurable function  $f$ , we define the **positive part** of  $f$  as

$$f^+(x) = \max\{f(x), 0\}$$

and the **negative part** of  $f$  as

$$f^-(x) = \max\{-f(x), 0\}$$

Then we get that

$$f = f^+ - f^-$$

**Definition 2.6.16.**  $f : X \rightarrow \mathbb{R}$  (or  $\bar{\mathbb{R}}$ ) Suppose that  $f$  is a measurable function, then we define

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$$

provided that at least one of  $\int f^\pm \, d\mu$  is finite

**Definition 2.6.17.**  $f : X \rightarrow \mathbb{R}$  (or  $\bar{\mathbb{R}}$ )  $f$  is **integrable** if  $\int f^+ \, d\mu, \int f^- \, d\mu$  is finite  $\iff \int |f| \, d\mu$  is finite

$\mathcal{L}^1$  is the class of integrable function

**Definition 2.6.18.**  $f : X \rightarrow \mathbb{C}$  is measurable ( $\iff \Re(f)$  and  $\Im(f)$  are measurable)  
Assumeing that  $\Re f \in \mathcal{L}^1$  and  $\Im f \in \mathcal{L}^1$  then

$$\int f \, d\mu = \int \Re f \, d\mu + i \int \Im f \, d\mu$$

**Claim 2.6.19** — Suppose that  $f, g$  are measurable then

$$\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$$

$$\int \alpha f \, d\mu + \alpha \int f \, d\mu$$

**Lemma 2.6.20**

$f : X \rightarrow \mathbb{R}$  is measurable, and  $\int |f| \, d\mu = 0$  then  $f = 0$  almost everywhere.

*Proof.* Assume that  $\int |f| \, d\mu = 0$  Find simple function  $s_n$  of  $f$  then  $\int s_n \, d\mu = 0 \implies s_n(x) = 0$  almost everywhere. we claim that  $s_n \rightarrow f$  everywhere then  $s_n = 0$  almost everywhere then  $f = 0$  almost everywhere. There is a set  $N_n, \mu(N_n) = 0$  so that  $s_n = 0$  in  $N_n^c$ .  $N = \bigcup_{n=1}^{\infty} N_n \implies \mu(N) = 0$  and  $N^c$  we have  $s_n \rightarrow f, s_n = 0$   $\square$

We could redefine the notion of  $f = 0$  (being  $f(x) = 0$  for all  $x \in X$ ) to  $f = 0$  almost everywhere (meaning  $f(x) = 0$  except for all null set).

**Remark 2.6.21.**  $\|f\| = \int |f| \, d\mu$  satisfies

- $\|f + g\| \leq \|f\| + \|g\|$
- $\|cf\| = |c|\|f\|$
- $\|f\| = 0 \iff f = 0$  almost everywhere

**Remark 2.6.22.** Almost everywhere equal is an equivalence relation.

$$f \sim g \stackrel{\text{def}}{\iff} f(x) = g(x) \, \mu\text{-almost everywhere}$$

$N = \{f \in \mathcal{L}^1 : f(x) = 0 \text{ almost everywhere}\}$  is a linear subspace of  $\mathcal{L}^1$  vector.  $\mathcal{L}^1/N$  is the set of equivalence classes of  $\mathcal{L}^1$ .

$f_n \rightarrow f$  almost everywhere,  $f_n \geq 0$ ,  $f_n$  measurable, Can we define  $\int f \, d\mu$ ?  $f$  may not be measurable. This problem is fixed if  $f$  we work in a complete measurable space  $(X, \mathcal{M}, \mu) \rightarrow (X, \bar{\mathcal{M}}, \bar{\mu})$  where

$$\bar{\mathcal{M}} = \{A \cup B : A \in \mathcal{M}, B \text{ a subset of a set of measure } 0\}$$

**Lemma 2.6.23**

$f \in \mathcal{L}^1$ ,  $\int |f| \, d\mu < \infty$ . If  $f$  is real valued  $f = f^+ - f^-$ ,

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$$

*Proof.*

$$\begin{aligned} \left| \int f \, d\mu \right| &= \left| \int f^+ - f^- \, d\mu \right| \\ &\leq \left| \int f^+ \, d\mu \right| + \left| \int f^- \, d\mu \right| \\ &= \int f^+ \, d\mu + \int f^- \, d\mu \\ &= \int |f| \, d\mu \end{aligned}$$

□

**Remark 2.6.24.** If  $f$  is complex valued, then  $|f| = \sqrt{(\Re f)^2 + (\Im f)^2}$ . Then

$$\left| \int \Re f \, d\mu \right| \leq \int |\Re f| \, d\mu \leq \int |f| \, d\mu$$

$$\left| \int \Im f \, d\mu \right| \leq \int |\Im f| \, d\mu \leq \int |f| \, d\mu$$

So,

$$\left| \int f \, d\mu \right| \leq 2 \int |f| \, d\mu$$

**Remark 2.6.25.** Estimate  $\int f \, d\mu = \alpha + i\beta = re^{i\phi}$ , then  $e^{-i\phi} \int f \, d\mu$  is real and nonnegative.

$$\begin{aligned} \left| \int f \, d\mu \right| &= \left| e^{-i\phi} \int f \, d\mu \right| \\ &= \Re \int e^{-i\phi} f \, d\mu \\ &\leq \int |e^{-i\phi} f| \, d\mu \\ &= \int |f| \, d\mu \end{aligned}$$

$f \in \mathcal{L}^+$  means non-negative,  $\nu(E) = \int_E f \, d\mu$  this is measure Check the  $\sigma$ -additivity

$$E = \biguplus_{n=1}^{\infty} E_n,$$

$$\begin{aligned} \nu \left( \biguplus_{n=1}^{\infty} E_n \right) &= \int_{\biguplus E_n} f \, d\mu \\ &= \int f \mathbb{1}_{\biguplus E_n} \, d\mu \\ &= \int f \left( \sum_{n=1}^{\infty} \mathbb{1}_{E_n} \right) \, d\mu \\ &= \sum_{n=1}^{\infty} \int f \mathbb{1}_{E_n} \, d\mu \\ &= \sum_{n=1}^{\infty} \nu(E_n) \end{aligned}$$

**Claim 2.6.26** — If  $f \in \mathcal{L}^1 \cap \mathcal{L}^+$  then  $\nu$  is a finite measure.

If  $\nu(E) = \int_E f \, d\mu$  How does  $\int g \, d\nu$  look like?  $\nu(E) = \int f \, d\mu = \int E \, d\nu$  We want “ $f \, d\mu = d\nu$ ”

**Lemma 2.6.27**

If  $f \in \mathcal{L}^+$  and  $\nu(E) = \int_E f \, d\mu$  then for any  $g \in \mathcal{L}^+$  or  $g \in \mathcal{L}^1$  then,

$$\int g \, d\nu = \int gf \, d\mu$$

*Proof.* • True for characteristic functions of measure set by the definition of  $\nu$

- By linearity of the integral, it is true for simple function
- $s_n \nearrow g$  if  $g \in \mathcal{L}^+$ , by Monotone convergence theorem,

$$\int s_n \xrightarrow{\text{MCT}} \int g$$

$$\begin{aligned} \int s_n \, d\nu &= \int s_n f \, d\mu \\ \int g \, d\nu &= \int gf \, d\mu \end{aligned}$$

Then extend to general  $g$  by linearity

□

**Theorem 2.6.28**

If  $X$  is a finite measure space, if  $f_n$  measurable,  $f_n \in \mathcal{L}^1$  (integrable) and  $f_n \rightarrow f$  uniformly on  $X$ . then

$$\int |f_n - f| \, d\mu \rightarrow 0$$

and

$$\int f_n \, d\mu \rightarrow \int f \, d\mu$$

**Remark 2.6.29.** Uniform convergence means

$$\sup_{x \in X} |f_n(x) - f(x)| \rightarrow 0$$

as  $n \rightarrow \infty$

*Proof.* We can rewrite that term as

$$\begin{aligned} \int |f_n - f| \, d\mu &\leq \int \sup_{x \in X} |f_n - f| \, d\mu \\ &= \mu(X) \sup_{x \in X} |f_n - f| \rightarrow 0 \end{aligned}$$

We can rewrite  $f$  as  $f = (f - f_n) + (f_n)$  since  $f - f_n$  converge and  $f_n$  integrable so  $f$  must be integrable.

$$\begin{aligned} \left| \int f_n - \int f \right| &= \left| \int (f_n - f) \, d\mu \right| \\ &\leq \int |f_n - f| \, d\mu \end{aligned}$$

□

**Definition 2.6.30.** Suppose that  $f_n, f$  are measurable  $f_n \rightarrow f$  almost uniformly if for every  $\varepsilon > 0$  there is a measurable set  $E$  such that  $\mu(E) < \varepsilon$  and  $f_n \rightarrow f$  uniformly on  $E^c$  ( $\sup_{x \in E^c} |f_n(x) - f(x)| \rightarrow 0$ )

**Theorem 2.6.31 (Egorov's Theorem)**

If  $\mu(X) < \infty$  and if  $f_n \rightarrow f$  almost everywhere then  $f_n \rightarrow f$  almost uniformly

**Remark 2.6.32.**  $f_n(x) \rightarrow f(x)$  if for every  $k$  there exists  $n = n(k)$  such that  $|f_m(x) - f(x)| < \frac{1}{k}$  for all  $m \geq n(k)$

*Proof.* Fix  $\varepsilon > 0$ , define

$$\begin{aligned} E_n(k) &:= \left\{ x : |f_m(x) - f(x)| \geq \frac{1}{k} \text{ for some } m \geq n \right\} \\ &= \bigcup_{m \geq n} \left\{ x : |f_m(x) - f(x)| \geq \frac{1}{k} \right\} \end{aligned}$$

(Given  $x$  For sufficiently large  $n$ ,  $x \notin E_n(k)$ ),  $E_n(k) \supseteq E_{n+1}(k) \cap_n E_n(k) = \emptyset$  because of  $f_n \rightarrow f$  everywhere. From the continuity from above 2.4.5, we get that  $\lim_{n \rightarrow \infty} \mu(E_n(k)) = 0$ . Find  $n(k)$  such that  $\mu(E_{n(k)}(k)) < \frac{\varepsilon}{2^k}$ , then  $E = \bigcup_k E_{n(k)}(k)$  has measure  $< \varepsilon$ .

For  $x \in \left( \bigcup_k E_{n(k)}(k) \right)^c = \bigcap_k E_{n(k)}(k)^c$  I have for all  $k$   $|f_m(x) - f(x)| < \frac{1}{k}$  for all  $m \geq n(k)$ . So, we get  $f_n \rightarrow f$  uniformly on  $E^c$ .  $\square$