

MATH 629 (Measure Theory) Lecture Notes

PONGSAPHOL PONGSAWAKUL

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1 From Riemann to Lebesgue

§1.1 Riemann Integral

Definition 1.1.1. $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a, b]$.

Definition 1.1.2. If P, P' are partitions of $[a, b]$ and $P \subseteq P'$, then P' is a refinement of P .

Definition 1.1.3. Given a bounded function $f : [a, b] \rightarrow \mathbb{R}$ and a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ we define

$$m_i(f) = \inf_{t \in [x_{i-1}, x_i]} f(t)$$

$$M_i(f) = \sup_{t \in [x_{i-1}, x_i]} f(t)$$

define the lower sum as

$$L(f, P) = \sum_{i=1}^n m_i(f)(x_i - x_{i-1})$$

and the upper sum as

$$U(f, P) = \sum_{i=1}^n M_i(f)(x_i - x_{i-1})$$

Lemma 1.1.4

Given a bounded function $f : [a, b] \rightarrow \mathbb{R}$ and partitions P of $[a, b]$. Suppose that P' is a refinement of P then

$$(b - a) \inf_{t \in [a, b]} f(t) \leq L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P) \leq (b - a) \sup_{t \in [a, b]} f(t)$$

Corollary 1.1.5

Suppose that P_1, P_2 are partitions of $[a, b]$ then $L(f, P_1) \leq U(f, P_2)$

Proof. Let $P' = P_1 \cup P_2$ then P' is a refinement of P_1 and P_2 and use Lemma 1.1.4 \square

Lemma 1.1.6

Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is bounded. Then

$$(b-a) \inf_{t \in [a, b]} f(t) \leq \sup_P L(f, P) \leq \inf_P U(f, P) \leq (b-a) \sup_{t \in [a, b]} f(t)$$

Definition 1.1.7. A function $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable if

$$\sup_P L(f, P) = \inf_P U(f, P)$$

and the common value is called the Riemann integral of f and is denoted by $\int_a^b f$

Lemma 1.1.8

Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is bounded. Then f is Riemann integrable if and only if for any $\varepsilon > 0$ there exists a partition P such that

$$U(f, P) - L(f, P) < \varepsilon$$

Proof. (\Rightarrow) For any $\varepsilon > 0$. Suppose that f is Riemann integrable. Then there exists P_1, P_2 such that

$$L(f, P_1) \geq \int_a^b f - \frac{\varepsilon}{2}$$

$$U(f, P_2) \leq \int_a^b f + \frac{\varepsilon}{2}$$

let $P = P_1 \cup P_2$ then

$$U(f, P) - L(f, P) \leq \varepsilon$$

(\Leftarrow) For any $\varepsilon > 0$, there exists P_ε such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

since ε is arbitrary, we have

$$\sup_P L(f, P) = \inf_P U(f, P)$$

□

Theorem 1.1.9

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ then f is Riemann integrable.

Proof. f is continuous on a compact set, so, f is uniformly continuous. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in [a, b]$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{(b-a)}$. Let N be such that $\frac{(b-a)}{N} < \delta$ and let $P = \{x_i := a + \frac{(b-a)i}{N}\}$ then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^N (M_i(f) - m_i(f)) \frac{(b-a)}{N} \\ &\leq \sum_{i=1}^N \frac{\varepsilon}{(b-a)} \frac{(b-a)}{N} \\ &= \varepsilon \end{aligned}$$

□

Remark 1.1.10. Let $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ defined on the $[0, 1]$. Then $U(f, P) = 1$ and $L(f, P) = 0$ for any partition P . So, f is not Riemann integrable.

§1.2 Lebesgue null sets

Definition 1.2.1. For the closed interval $I = [a, b]$, the length of I , denoted as $\ell(I)$ is defined as $\ell(I) = b - a$

Definition 1.2.2. A set E is said to be a Lebesgue null set if for any $\varepsilon > 0$ there exists a sequence of intervals $\{I_n\}_{n \in \mathbb{N}}$ such that

$$E \subseteq \bigcup_{n=1}^{\infty} I_n$$

and

$$\sum_{n=1}^{\infty} \ell(I_n) < \varepsilon$$

Lemma 1.2.3

Countable unions of Lebesgue null sets are Lebesgue null sets.

Proof. For any $\varepsilon > 0$ and for each Lebesgue null sets E_n there exists $I_{E_n, i}$ such that

$$E_n \subseteq \bigcup_{i=1}^{\infty} I_{E_n, i}$$

and

$$\sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \frac{\varepsilon}{2^n}$$

then

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \varepsilon$$

□

Definition 1.2.4. A set $E \subseteq [a, b]$ has content zero if for any $\varepsilon > 0$ there exists I_1, I_2, \dots, I_n such that

$$E \subseteq \bigcup_{i=1}^n I_i$$

and

$$\sum_{i=1}^n \ell(I_i) < \varepsilon$$

Lemma 1.2.5

Suppose that $E \subseteq [a, b]$ is a compact Lebesgue null set then E has content zero.

Proof. For any $\varepsilon > 0$ there exists a sequence of interval $\{I_n\}_{n \in \mathbb{N}}$ such that $E \subseteq \bigcup I_n$ and $\sum \ell(I_n) < \frac{\varepsilon}{2}$. Suppose that $I_n = [a_n, b_n]$, then let

$$J_n = \left(a_n - \frac{\varepsilon}{2^{n+3}}, b_n + \frac{\varepsilon}{2^{n+3}} \right) \supseteq I_n$$

then from the compactness of E , there exists a finite subcover $J_{n_1}, J_{n_2}, \dots, J_{n_k}$ such that $E \subseteq \bigcup J_{n_i}$ then we construct a finite closed interval K_i by

$$K_i = \left[a_{n_i} - \frac{\varepsilon}{2^{n_i+2}}, b_{n_i} + \frac{\varepsilon}{2^{n_i+2}} \right]$$

then $E \subseteq \bigcup K_i$ and $\sum \ell(K_i) < \varepsilon$

□

Corollary 1.2.6

if $a < b$ then $[a, b]$ is not a Lebesgue null set.

Proof. By contradiction, since $[a, b]$ is compact, then $[a, b]$ has content zero, but $[a, b]$ don't have content zero. □

§1.3 Oscillation and Discontinuity

Definition 1.3.1. Suppose that $X \subseteq \mathbb{R}$, $f : X \rightarrow \mathbb{R}$ for any $x \in X$ and $\delta > 0$, define

$$M_{f,\delta}(x) := \sup\{f(y) : d(x, y) < \delta\}$$

$$m_{f,\delta}(x) := \inf\{f(y) : d(x, y) < \delta\}$$

then we define

$$\text{osc}_f(x) := \lim_{\delta \rightarrow 0^+} M_{f,\delta}(x) - m_{f,\delta}(x)$$

Lemma 1.3.2

f is continuous at x if and only if $\text{osc}_f(x) = 0$.

Proof. (\Rightarrow) Suppose that f is continuous at x , then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Then

$$M_{f,\delta}(x) - m_{f,\delta}(x) \leq \sup\{f(y) : d(x, y) < \delta\} - \inf\{f(y) : d(x, y) < \delta\} < \varepsilon$$

(\Leftarrow) Suppose that $\text{osc}_f(x) = 0$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $M_{f,\delta}(x) - m_{f,\delta}(x) < \varepsilon$. Then for any $y \in X$ such that $d(x, y) < \delta$, we have $|f(x) - f(y)| < \varepsilon$ then f is continuous at x . \square

Before we prove this theorem, we need to prove the following lemma.

Lemma 1.3.3

$\{x \in [a, b] : \text{osc}_f(x) \geq \gamma\}$ is closed.

Proof. We need to show that $\{x : \text{osc}_f(x) < \gamma\}$ is open. Fix x in that set. Let $\varepsilon = \gamma - \text{osc}_f(x)$ then

$$\sup_{|w-x|<\delta} f(w) - \inf_{|w-x|<\delta} f(w) < \text{osc}_f(x) < \gamma$$

then for any $w \in (x - \delta, x + \delta)$ if $|w - x| < \frac{\delta}{2}$ then

$$\text{osc}(w) \leq \sup_{|y-w|<\frac{\delta}{2}} f(y) - \inf_{|y-w|<\frac{\delta}{2}} f(y) < \gamma$$

So, $B(x, \frac{\delta}{2}) \subseteq \{x : \text{osc}_f(x) < \gamma\}$ \square

we observe that

- (i) If the set of discontinuities is a Lebesgue null set, then $\{x : \text{osc}_f(x) \geq \gamma\}$ is a set of content zero.
- (ii) If $\{x : \text{osc}_f(x) \geq \gamma\}$ is a Lebesgue null set, then the set of discontinuities is also a Lebesgue null set.

Lemma 1.3.4

Suppose that f is defined on $[c, d]$, assume that $\text{osc}_f(x) < \gamma$ then we can find a partition

$$U(f, P) - L(f, P) < \gamma(b - a)$$

Proof. For every $x \in [c, d]$, there exists $\delta_x > 0$ such that

$$\sup_{|w-x|<\delta_x} f(w) - \inf_{|w-x|<\delta_x} f(x) < \gamma$$

construct a cover by

$$B(x, \delta_x) = \{w \in [c, d] : |w - x| < \delta_x\}$$

since $[c, d]$ is compact, there exists a finite subcover $B(p_1, \delta_{p_1}), \dots, B(p_n, \delta_{p_n})$ then let $\delta_0 = \frac{\min\{\delta_{p_i}\}}{100}$ then we can construct a partition $P = \{c = x_0 < x_1 < \dots < x_n = d\}$ such that $|x_i - x_{i-1}| < \delta_0$ then $M_i - m_i < \gamma$ and

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \\ &< \gamma \sum_{i=1}^n (x_i - x_{i-1}) \\ &= \gamma(d - c) \end{aligned}$$

□

Theorem 1.3.5

Suppose that $f : [a, b] \rightarrow \mathbb{R}$ then $f \in \mathcal{R}([a, b])$ if and only if f is bounded and the set of discontinuity of f is a Lebesgue null set.

Proof. (\Rightarrow) We want to show that for every $n \in \mathbb{N}$,

$$\mathcal{D}_n = \left\{x : \text{osc}_f(x) \geq \frac{1}{n}\right\}$$

is a Lebesgue null set. For any $\varepsilon > 0$, since f is Riemann integrable, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) = \sum_{i=1}^n (x_i - x_{i-1})(M_i - m_i) \leq \frac{\varepsilon}{n}$$

where $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ and $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$. in particular

$$\begin{aligned} \sum_{[x_{i-1}, x_i] \cap \mathcal{D}_n \neq \emptyset} (x_i - x_{i-1})(M_i - m_i) &\leq \frac{\varepsilon}{n} \\ \frac{1}{n} \sum_{[x_{i-1}, x_i] \cap \mathcal{D}_n \neq \emptyset} \ell([x_{i-1}, x_i]) &\leq \frac{\varepsilon}{n} \end{aligned}$$

So, this interval cover the set \mathcal{D}_n

(\Leftarrow) pick $\varepsilon_1 \ll \varepsilon$, consider the set $D(\varepsilon_1) = \{x \in [a, b] : \text{osc}_f(x) \geq \varepsilon_1\}$ closed set. Since $D(\varepsilon_1)$ is a Lebesgue null set from the Lemma 1.2.5 it has content zero so we can find I_1, \dots, I_n such that

$$\sum_{j=1}^n \ell(I_j) < \varepsilon_1 \text{ and } D(\varepsilon_1) \subseteq \bigcup_{j=1}^n I_j$$

We form a partition of $[a, b]$, $a = x_0 < x_1 < \dots < x_N = b$ from I_j . There are two cases that we need to consider

- 1) if $[x_{i-1}, x_i] \subseteq I_j$ for some j then set $P_i = [x_{i-1}, x_i]$
- 2) if $[x_{i-1}, x_i] \cap I_j = \emptyset$ for all j then $\text{osc}(x) < \varepsilon_1$ for all $x \in [x_{i-1}, x_i]$. We want to partition further the interval $[x_{i-1}, x_i]$ by partition P_i . Using Lemma 1.3.4 we can find a partition P_i of $[x_{i-1}, x_i]$ such that

$$U(f, P_i) - L(f, P_i) < \varepsilon_1(x_i - x_{i-1})$$

We form a partition $P = P_1 \cup \dots \cup P_N$ then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^N (U(f, P_i) - L(f, P_i)) \\ &= \sum_{i:\text{case 1}} (U(f, P_i) - L(f, P_i)) + \sum_{i:\text{case 2}} (U(f, P_i) - L(f, P_i)) \\ &\leq 2M \sum_{i:\text{case 1}} (x_i - x_{i-1}) + \varepsilon_1 \sum_{i:\text{case 2}} (x_i - x_{i-1}) \\ &\leq 2M\varepsilon_1 + \varepsilon_1(b - a) \\ &= \varepsilon_1(2M + b - a) \end{aligned}$$

□

2 Measures

§2.1 Introduction

We define the $\ell([c, d]) = d - c$ and If $E = [c_1, d_1] \cup [c_2, d_2]$ where $d_1 < c_2$ then $\ell(E) = d_1 - c_1 + d_2 - c_2$. This is consistent with the definition

$$\ell(E) = \int \mathbb{1}_E(x) \, dx$$

where the integral denotes the Riemann integral.

if $E \subseteq [a, b]$ reference interval is

$$\int_a^b \mathbb{1}_E \, dx$$

Remark 2.1.1. The consistency of the definition also works with the set (c, d) , $[c, d)$, and $(c, d]$, where the length of all of them is $d - c$.

Remark 2.1.2. we denote $\mathbb{1}_E$ to be

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Example 2.1.3

Let $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ defined on the $[0, 1]$. Then $U(f, P) = 1$ and $L(f, P) = 0$ for any partition P .

Fix the reference interval $[a, b]$ and consider subset of $[a, b]$

Let $\mathcal{A} =$ collection of sets for which $\int_{[a, b]} \mathbb{1}_E \, dx$ exists.

If $A_1, \dots, A_n \in \mathcal{A}$, we can make the set to be mutually disjoint by taking $E_1 = A_1$, $E_2 = A_2 \setminus A_1$, $E_3 = A_3 \setminus (A_1 \cup A_2)$, and so on.

Example 2.1.4

For $E_1, E_2 \in \mathcal{A}$, we have

$$\mathbb{1}_{E_1 \cap E_2}(x) = \mathbb{1}_{E_1}(x) \mathbb{1}_{E_2}(x)$$

Example 2.1.5

For the Riemann integral, we have

$$\int_a^b f(y) = \int_{a-v}^{b-v} f(v+y)$$

and we want

$$\int \mathbb{1}_E(x) \, dx = \int \mathbb{1}_{v+E}$$

where $v + E = \{v + x : x \in E\}$

Let $E = \mathbb{Q} \cap [0, 1]$ countable set, we can enumerate r_1, r_2, r_3, \dots such that

$$E = \bigcup_{n=1}^{\infty} \{r_n\}$$

and

$$\int \mathbb{1}_{\{r_k\}} = 0$$

E should have length zero but according $\mathbb{1}_E$ is not Riemann integrable.

§2.2 Construction of Measure

Suppose that \mathcal{C} be a collection of sets.

Can we define on suitable large collection of subset of \mathbb{R} ?

a set function $\mu : \mathcal{C} \rightarrow [0, \infty]$ such that if $\{E_j\}_{j=1}^{\infty}$ is a sequence of disjoint set in \mathcal{C} then

$$\mu\left(\bigcup_{i=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

$$\mu([a, b]) = b - a, \mu([0, 1)) = 1$$

Can we do this for the collection of all subset of \mathbb{R} ?

Answer: No, Vitali set.

Theorem 2.2.1

We cannot define a measure on the collection of all subset of \mathbb{R} . i.e., there does not exist a set function $\mu : \mathfrak{P}(\mathbb{R}) \rightarrow [0, \infty]$ such that

- (i) $\mu(v + E) = \mu(E)$ for all $E \subseteq \mathbb{R}$ and $v \in \mathbb{R}$
- (ii) $\mu([0, 1]) = 1$
- (iii) $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$ for all disjoint $A_j \subseteq \mathbb{R}$

Before we prove that theorem, we need to define something and prove the following lemma.

Definition 2.2.2. We define a Vitali set V from picking an element $x \in [0, 1)$ from each equivalence class of the relation $x \sim y$ if $x - y \in \mathbb{Q}$. (e.g, pick $x \in O_x$ for $O_x \in \mathbb{R}/\mathbb{Q}$)

Lemma 2.2.3

Suppose that V is a Vitali set then

$$V \cap V + q = \emptyset$$

For all $q \in \mathbb{Q} \setminus \{0\}$

Proof. Suppose not, there exists $a \in V$ such that $a \in V + q \implies a - q \in V$ but we only pick 1 element in each equivalence class. contradiction. \square

Lemma 2.2.4

Let V be a Vitali set and let $W = \{q \in [-1, 1] : q \in \mathbb{Q}\}$ and

$$E = \bigcup_{w \in W} V + w$$

then

$$[0, 1] \subseteq E \subseteq [-1, 2]$$

Proof. Consider $E \subseteq [-1, 2]$. Since $V \subseteq [0, 1)$, then for any $v \in V$, $v \in [0, 1) \implies v + w \in [-1, 2]$.

For the $[0, 1] \subseteq E$, for any $x \in [0, 1]$ there exists $O_x \in \mathbb{R}/\mathbb{Q}$ such that $x \in O_x$. then there exists $v \in C_x$ such that $v \in [0, 1)$ and $v \in V$, since both are from the same equivalence

class, then $x - v \in \mathbb{Q}$ and $|x - v| < 1 \implies x - v \in (-1, 1)$. Hence, there exists $w \in W$ such that $w = x - v$ so $v + w = x$. \square

Proof of the theorem. Suppose that μ exists then using the result from Lemma 2.2.4 we get that

$$\mu([0, 1]) \leq \mu(E) \leq \mu([-1, 2])$$

from Lemma 2.2.3 we know that each $V + w$ is disjoint, so

$$\begin{aligned} \mu([0, 1]) &\leq \sum_{w \in W} \mu(V) \leq \mu([-1, 2]) \\ 1 &\leq \sum_{w \in W} \mu(V) \leq 3 \end{aligned}$$

if $\mu(V) = 0$ then $\mu(E) = 0$ and if $\mu(V) > 0$ then $\mu(E) = \infty$. Both are contradiction. \square

§2.3 σ -algebra

Definition 2.3.1. Given a reference X . An **algebra** is a collection of subsets of X , \mathcal{A} , such that

- (i) $X \in \mathcal{A}$
- (ii) If $A \in \mathcal{A}$ then the complement $A^c = X \setminus A \in \mathcal{A}$
- (iii) If $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$

Remark 2.3.2. • $\emptyset \in \mathcal{A}$ because $\emptyset = X^c$

- $A_1, A_2 \in \mathcal{A}$, $A_1 \setminus A_2 = A_1 \cap A_2^c \in \mathcal{A}$
- Observe that if $A_1, A_2 \in \mathcal{A}$ then $A_1 \cap A_2 \in \mathcal{A}$ because $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$

Example 2.3.3

$X = [a, b]$ and \mathcal{A} is the collection of all sets $E \subseteq [a, b]$ such that the Riemann integral $\int \mathbb{1}_E(t) dt$ exists

Definition 2.3.4. A σ -algebra \mathcal{M} on X is

- (i) an algebra of subsets of X
- (ii) If A_1, A_2, A_3, \dots is a sequence of set in \mathcal{M} then

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$$

(X, \mathcal{M}) is called a “**measurable space**”.

Remark 2.3.5. \mathcal{M} is a σ -algebra on X then it satisfies

- (i) $X \in \mathcal{M}$
- (ii) If $A \in \mathcal{M}$ then $A^c \in \mathcal{M}$
- (iii) countable union of sets in \mathcal{M} is in \mathcal{M}

Definition 2.3.6. Let (X, \mathcal{M}) be a measurable set. Then a measure μ is a set function $\mu : \mathcal{M} \rightarrow [0, \infty], E \mapsto \mu(E)$ such that

- (i) $\mu(\emptyset) = 0$
- (ii) If E_1, E_2, E_3, \dots is a sequence of disjoint set in \mathcal{M} then

$$\mu \left(\bigcup_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \mu(E_j)$$

called σ -additivity.

(X, \mathcal{M}, μ) is called a “**measure space**”.

Remark 2.3.7.

$$\left(\bigcap_{j=1}^{\infty} A_j \right) = \left(\bigcup_{j=1}^{\infty} A_j^c \right)^c \in \mathcal{M}$$

Example 2.3.8

examples of σ -algebra

- (i) $\mathcal{M} = \{\emptyset, X\}$
- (ii) $\mathcal{M} = \mathfrak{P}(X)$ = collection of all subsets of X
 $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mu(E) = |E|$ (the cardinality of E) if E is finite and $\mu(E) = \infty$ if E is infinite.
- (iii) X write X as a disjoint (countable) union of sets A_j . Then \mathcal{M} = all countable unions of A_j .
- (iv) Let X be a set. Let \mathcal{M} be the collection of all sets A , $A \subseteq X$ such that A is countable or A^c is countable.
- (v) $X = \mathbb{R}$ (or \mathbb{R}^n), $\mathcal{B}_{\mathbb{R}}$ is the smallest σ -algebra containing all open sets.

More generally if \mathcal{E} is a collection of subsets of X then $\mathfrak{M}(\mathcal{E})$ is the smallest σ -algebra that contains all sets in \mathcal{E} .

If $\mathcal{M}_1, \mathcal{M}_2$ are two σ -algebras, then $\mathcal{M}_1 \cap \mathcal{M}_2$ is also a σ -algebra.

If $\{\mathcal{M}_\alpha\}_{\alpha \in \mathcal{I}}$ is a collection of σ -algebras, their intersection is also a σ -algebra.

Generating σ -algebra

Definition 2.3.9. $\mathfrak{M}(\mathcal{E}) :=$ intersection of all σ -algebra that contain the collection \mathcal{E} . We call it the σ -algebra generated by \mathcal{E} . i.e.

$$\mathfrak{M}(\mathcal{E}) = \bigcap_{\substack{\mathcal{F} \in \mathcal{M} \\ \mathcal{E} \subseteq \mathcal{F}}} \mathcal{F}$$

Remark 2.3.10. If $\mathcal{E} \subseteq \mathcal{F} \implies \mathfrak{M}(\mathcal{E}) \subseteq \mathfrak{M}(\mathcal{F})$

Lemma 2.3.11

If $\mathcal{E} \subseteq \mathfrak{M}(\mathcal{F})$ then $\mathfrak{M}(\mathcal{E}) \subseteq \mathfrak{M}(\mathcal{F})$

Proof. $\mathfrak{M}(\mathcal{F})$ is a σ -algebra that contains \mathcal{E} . It contains the intersection of all σ -algebras which contain \mathcal{E} . \square

Example 2.3.12

$\mathcal{B}_{\mathbb{R}} = \sigma$ -algebra on \mathbb{R} containing all open sets \mathcal{E} a collection of all open intervals, $\mathcal{E} \subseteq \mathcal{O} =$ collection of all open sets in \mathbb{R} , $\mathcal{B}_{\mathbb{R}} = \mathfrak{M}(\mathcal{O})$. $\mathfrak{M}(\mathcal{E}) \subseteq \mathcal{B}_{\mathbb{R}}$. Each open set is a countable union of open intervals. Each open set is contained in $\mathfrak{M}(\mathcal{E})$.

Since $\mathcal{O} \subseteq \mathfrak{M}(\mathcal{E}) \implies \mathfrak{M}(\mathcal{O}) \subseteq \mathfrak{M}(\mathcal{E})$. get $\mathfrak{M}(\mathcal{O}) = \mathfrak{M}(\mathcal{E})$.

Definition 2.3.13. Given $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2), \dots, (X_n, \mathcal{M}_n)$ measurable spaces. Define a “product σ -algebra” on $X_1 \times X_2 \times \dots \times X_n$ denoted by

$$\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \dots \oplus \mathcal{M}_n = \bigoplus_{j=1}^n \mathcal{M}_j$$

defined as the σ -algebra generated by the sets $E_1 \times E_2 \times \dots \times E_n$ where $E_j \in \mathcal{M}_j$.

i.e., define $\mathcal{E} := \{(E_1 \times E_2 \times \dots \times E_n) : E_j \in \mathcal{M}_j\}$ then

$$\bigoplus_{j=1}^n \mathcal{M}_j := \mathfrak{M}(\mathcal{E})$$

Remark 2.3.14. Folland defines it the σ -algebra generated by

$$(X_1 \times X_2 \times \cdots \times X_{n-1} \times E_n)$$

where $E_n \in \mathcal{M}_n$,

$$(X_1 \times X_2 \times \cdots \times E_{n-1} \times X_n)$$

where $E_{n-1} \in \mathcal{M}_{n-1}$. and so on. To be clear, let

$$\mathcal{E}' := \bigcup_{j=1}^n \{(X_1 \times \cdots \times X_{j-1} \times E_j \times X_{j+1} \times \cdots \times X_n) : E_j \in \mathcal{M}_j\}$$

then

$$\bigoplus_{j=1}^n \mathcal{M}_j := \mathfrak{M}(\mathcal{E}')$$

Claim 2.3.15 — Both definitions on product of σ -algebra are equivalent.

Proof. The goal is to show that $\mathfrak{M}(\mathcal{E}) = \mathfrak{M}(\mathcal{E}')$.

(\supseteq) Obviously, $\mathcal{E}' \subseteq \mathcal{E}$ so $\mathfrak{M}(\mathcal{E}') \subseteq \mathfrak{M}(\mathcal{E})$.

(\subseteq) We want to show that $\mathcal{E} \subseteq \mathfrak{M}(\mathcal{E}')$. Fix $(E_1 \times E_2 \times \cdots \times E_n) \in \mathcal{E}$ then from the definition of σ -algebra generated by a collection, which is closed under intersection, so we can pick an element from the construction of \mathcal{E}' and do the intersection, so $(E_1 \times E_2 \times \cdots \times E_n) \in \mathfrak{M}(\mathcal{E}')$.

□

Theorem 2.3.16

Given $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2)$ measurable spaces. Assume that \mathcal{M}_1 is generated by a collection \mathcal{E}_1 and \mathcal{M}_2 is generated by a collection \mathcal{E}_2 . Then $\mathcal{M}_1 \oplus \mathcal{M}_2$ is generated by the sets $E_1 \times X_2, X_1 \times E_2$, where $E_1 \in \mathcal{E}_1$ and $E_2 \in \mathcal{E}_2$.

Proof. Let $\mathcal{P} := \{E_1 \times E_2 : E_i \in \mathcal{E}_i\}$, obviously $\mathfrak{M}(\mathcal{P}) = \mathfrak{M}(\{E_1 \times X_2 : E_1 \in \mathcal{E}_1\} \cup \{X_1 \times E_2 : E_2 \in \mathcal{E}_2\})$ and $\mathfrak{M}(\mathcal{P}) \subseteq \mathcal{M}_1 \oplus \mathcal{M}_2$. We need to show that $\mathcal{M}_1 \oplus \mathcal{M}_2 \subseteq \mathfrak{M}(\mathcal{P})$. Define

$$\mathcal{G}_1 = \{E_1 \subseteq X_1 : E_1 \times X_2 \in \mathfrak{M}(\mathcal{P})\}$$

$$\mathcal{G}_2 = \{E_2 \subseteq X_2 : X_1 \times E_2 \in \mathfrak{M}(\mathcal{P})\}$$

then \mathcal{G}_1 is a σ -algebra consisting of subset of X_1 which contains \mathcal{E}_1 , $\mathcal{E}_1 \subseteq \mathcal{G}_1$. \mathcal{E}_1 generates \mathcal{M}_1 so $\mathfrak{M}(\mathcal{E}_1) = \mathcal{M}_1 \subseteq \mathcal{G}_1$. So, we have $E_1 \times X_2 \in \mathfrak{M}(\mathcal{P})$ for all $E_1 \in \mathcal{M}_1$ and $X_1 \times E_2 \in \mathfrak{M}(\mathcal{P})$ for all $E_2 \in \mathcal{M}_2$. The σ -algebra generated by the sets $E_1 \times X_2, X_1 \times E_2$ is contained $\mathcal{M}_1 \oplus \mathcal{M}_2 \in \mathfrak{M}(\mathcal{P})$. □

Claim 2.3.17 — $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$.

where $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}}$ is generated by $E_1 \times E_2$, where $E_1, E_2 \in \mathcal{B}_{\mathbb{R}}$. and $\mathcal{B}_{\mathbb{R}^2}$ is generated by the open sets in \mathbb{R}^2 .

Proof. $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{B}_{\mathbb{R}^2}$. Want $\mathcal{B}_{\mathbb{R}^2} \subseteq \mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}}$. Consider the collection of all open rectangle of the form $(a_1, b_1) \times (a_2, b_2)$ such $a_i, b_i \in \mathbb{Q}$. which are contained in $O \subseteq \mathbb{R}^2$ \square

Definition 2.3.18 (The Borel σ algebra on the extended real line). We use the notion $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$. One possibility to define “ $\mathcal{B}_{\overline{\mathbb{R}}}$ ” is the σ -algebra generated by open sets in \mathbb{R} , $\{\infty\}$, $\{-\infty\}$ open intervals should be (a, b) , $(a, \infty]$, $[-\infty, b)$ for $-\infty \leq a < b \leq \infty$. Then define $d(x, y) = |\arctan(x) - \arctan(y)|$ and $\arctan(\infty) = \pi/2$, $\arctan(-\infty) = -\pi/2$.

§2.4 Measures

Definition 2.4.1. Measures are σ -additive set functions, $\mu(\emptyset) = 0$ and

$$\mu \left(\biguplus_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \mu(E_j)$$

where E_1, E_2, \dots is a sequence of disjoint sets.

Remark 2.4.2. There is some property

$$E \subseteq F \implies \mu(E) \leq \mu(F)$$

$$F = E \uplus (F \setminus E) \implies \mu(F) = \mu(E) + \mu(F \setminus E)$$

$\mu(\bigcup A_j) \leq \sum \mu(A_j)$ we can write $\bigcup A_j$ as a disjoint union, i.e., $E_1 = A_1$, $E_2 = A_2 \setminus A_1$, $E_3 = A_3 \setminus (A_1 \cup A_2)$, and so on then $\mu(\bigcup A_j) = \mu(\bigcup E_j) = \sum \mu(E_j) \leq \sum \mu(A_j)$

The monotone convergence theorem for sets (continuity from below)

Theorem 2.4.3

If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$ then

$$\mu \left(\bigcup_{j=1}^{\infty} E_j \right) = \lim_{j \rightarrow \infty} \mu(E_j)$$

Proof.

$$\bigcup_{j=1}^{\infty} E_j = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus (E_1 \cup E_2)) \cup \dots$$

So, we define $B_1 = E_1, B_n = E_n \setminus E_{n-1}$ for $n \geq 2$ then all B_j are disjoint.

$$\begin{aligned} \bigcup_{j=1}^{\infty} E_j &= \bigcup_{j=1}^{\infty} B_j \\ \mu \left(\bigcup_{j=1}^{\infty} E_j \right) &= \mu \left(\bigcup_{j=1}^{\infty} B_j \right) \\ &= \sum_{j=1}^{\infty} \mu(B_j) \\ &= \mu(E_1) + \sum_{j=2}^{\infty} \mu(E_j \setminus E_{j-1}) \\ &= \mu(E_1) + \sum_{j=2}^{\infty} \mu(E_j) - \mu(E_{j-1}) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \end{aligned}$$

□

Remark 2.4.4. If we prove something for the set then we can prove it for the complement.

$$\mu(A) + \mu(A^c) = \mu(X)$$

Theorem 2.4.5

If $\mu(X) < \infty$ then if $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots, E_n \supseteq E_{n+1}$ for all n then

$$\mu \left(\bigcap_{j=1}^{\infty} E_j \right) = \lim_{j \rightarrow \infty} \mu(E_j)$$

Proof. Assume E_j are decreasing, i.e.,

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$$

then $E_1^c \subseteq E_2^c \subseteq \dots$ then

$$\begin{aligned}\mu\left(\bigcup_{j=1}^{\infty} E_j^c\right) &= \lim_{j \rightarrow \infty} \mu(E_j^c) \\ \mu(X) - \mu\left(\left(\bigcup_{j=1}^{\infty} E_j^c\right)^c\right) &= \lim_{j \rightarrow \infty} (\mu(X) - \mu(E_j)) \\ \mu(X) - \mu\left(\bigcap_{j=1}^{\infty} E_j\right) &= \lim_{j \rightarrow \infty} (\mu(X) - \mu(E_j))\end{aligned}$$

□

Example 2.4.6

\mathbb{N} with counting measure, $E_j = \{j, j+1, j+2, \dots\}$, $\mu(E_j) = \infty$, $\bigcap E_j = \emptyset$ has measure 0.

Definition 2.4.7. If A_1, A_2, A_3, \dots is an arbitrary sequence of measurable sets. We can define

$$\begin{aligned}\limsup A_j &:= \bigcap_{n=1}^{\infty} \bigcup_{j \geq n} A_j = \{x : x \in A_n \text{ for infinitely many } n\} \\ \liminf A_j &:= \bigcup_{n=1}^{\infty} \bigcap_{j \geq n} A_j = \{x : x \text{ belong to all but finitely many}\}\end{aligned}$$

Lemma 2.4.8 (Borel-Cantelli Lemma)

If $\{A_j\}$ is a sequence of measurable sets such that

$$\sum_{j=1}^{\infty} \mu(A_j) < \infty$$

then almost every x (meaning all x except in a null set) belong to on A_n for only finitely many n . Or equivalently,

$$\mu(\limsup A_n) = 0$$

Proof. $\bigcup_{j \geq n} A_j$ are decreasing. In Borel Cantelli, we have $\sum \mu(A_j) < \infty$, so $\mu(\bigcup A_n) = 0$.

use “continuity from above”

$$\mu(\limsup A_n) = \lim_{n \rightarrow \infty} \mu \left(\bigcup_{j \geq n} A_j \right)$$

$$\mu \left(\bigcup_{j \geq n} A_j \right) \leq \sum_{j \geq n} \mu(A_j) \rightarrow 0$$

as $n \rightarrow \infty$. □

Completion of a σ -algebra (when a measure μ is given), $(X, \mathcal{M}, \mu) \overline{\mathcal{M}}$ consists of all unions $E \cup F$, where $E \in \mathcal{M}$ and $F \subseteq N \in \mathcal{M}$ for some null set N , $\mu(N) = 0$.

Define $\bar{\mu}$ by $\bar{\mu}(E \cup F) = \mu(E)$.

§2.5 Measurable Functions

Definition 2.5.1. $f : X \rightarrow Y$ where (X, \mathcal{M}) and (Y, \mathcal{N}) are measurable spaces. f is $(\mathcal{M}, \mathcal{N})$ -measurable if for every $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$. where $f^{-1}(E) = \{x \in X : f(x) \in E\}$.

Lemma 2.5.2

Let \mathcal{E} generate \mathcal{N} (i.e., $\mathcal{N} = \mathfrak{M}(\mathcal{E})$). Then f is $(\mathcal{M}, \mathcal{N})$ -measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

Proof. Define $\mathcal{C} = \{E \in \mathcal{E} : f^{-1}(E) \in \mathcal{M}\}$, observe that \mathcal{C} is a σ -algebra. then

$$f(x) \in \bigcup E_j \iff x \in f^{-1}\left(\bigcup E_j\right) \iff x \in \bigcup f^{-1}(E_j) \iff \bigcup \{x : f(x) \in E_j\}$$

□

Claim 2.5.3 — $f : X \rightarrow Y$ is $(\mathcal{M}, \mathcal{N})$ -measurable, $g : Y \rightarrow Z$ is $(\mathcal{N}, \mathcal{R})$ -measurable then $g \circ f : X \rightarrow Z$ is $(\mathcal{M}, \mathcal{R})$ -measurable.

Proof. $(g \circ f)^{-1}(E) = \{x \in X : g(f(x)) \in E\} = f^{-1}(g^{-1}(E)) = \{x \in X : f(x) \in g^{-1}(E)\}$ □

Claim 2.5.4 — $f : X \rightarrow \mathbb{R}$ is \mathcal{M} -measurable then f^2 is \mathcal{M} -measurable.

Proof. $(f^2)^{-1}(-\infty, a) = \{x : f^2(x) < a\} = \{x : f(x) < \sqrt{a}\} \cup \{x : f(x) > -\sqrt{a}\}$ \square

Claim 2.5.5 — $f : X \rightarrow \mathbb{R}, g : X \rightarrow \mathbb{R}$ are \mathcal{M} -measurable then $f + g$ and $f \cdot g$ are \mathcal{M} -measurable.

Proof.

$$(f + g)^{-1}(-\infty, a) = \bigcup_{r \in \mathbb{Q}} (f^{-1}(-\infty, a + r) \cap g^{-1}(-\infty, r))$$

$$\begin{aligned} (f + g)^2 &= f^2 + 2fg + g^2 \\ fg &= \frac{1}{2} ((f + g)^2 - f^2 - g^2) \end{aligned}$$

\square

Claim 2.5.6 — vector-valued-function $f : X \rightarrow (Y_1 \times Y_2 \times \cdots \times Y_n)$ and defined by $x \mapsto (f_1(x), f_2(x), \dots, f_n(x))$ where $f_j : X \rightarrow Y_j$ is $(\mathcal{M}, \mathcal{N}_j)$ -measurable.

Then f is $(\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_n)$ if and only if $f_i(\mathcal{M}_i, \mathcal{N}_i)$ -measurable.

Proof.

$$\begin{aligned} f^{-1}(E_1 \times E_2 \times \cdots \times E_n) &= f_1^{-1}(E_1) \cap f_2^{-1}(E_2) \cap \cdots \cap f_n^{-1}(E_n) \\ &= \bigcap_{j=1}^n f_j^{-1}(E_j) \end{aligned}$$

\square

Claim 2.5.7 — $M(x) = \max\{f(x), g(x)\}$, $f, g : X \rightarrow \mathbb{R}$, \mathcal{M} -measurable.

Proof. $M^{-1}(-\infty, a) = \{x : M(x) < a\} = \{x : f(x) < a, g(x) < a\} = f^{-1}(-\infty, a) \cap g^{-1}(-\infty, a)$ \square

Claim 2.5.8 — $f_n : X \rightarrow \mathbb{R}$, \mathcal{M} -measurable, then $S(x) = \sup_{n \in \mathbb{N}} f_n$ is \mathcal{M} -measurable.

Proof. $S^{-1}(-\infty, a) = \{x : S(x) < a\} = \{x : \sup f_n(x) < a\} = \bigcap_n \{x : f_n(x) < a\}$ \square

Remark 2.5.9. We use the similar proof for min and inf.

Definition 2.5.10. If $f_n : X \rightarrow \mathbb{R}$, \mathcal{M} -measurable then

$$\limsup f_n = \inf_k \sup_{n \geq k} f_n$$

$$\liminf f_n = \sup_k \inf_{n \geq k} f_n$$

Claim 2.5.11 — $\limsup f_n$ and $\liminf f_n$ are \mathcal{M} -measurable.

Proof. For $\limsup f_n$, fix k then $\sup_{n \geq k} f_n$ is \mathcal{M} -measurable, $\inf_k \sup_{n \geq k} f_n$ is \mathcal{M} -measurable. Similarly for $\liminf f_n$. \square

Theorem 2.5.12

Let (X, \mathcal{M}) be a measurable space, $f_n : X \rightarrow \mathbb{C}$ be \mathcal{M} -measurable functions. Define

$$E_{lim} = \{x \in X : \lim_{n \rightarrow \infty} f_n(x) \text{ exists}\}$$

then $E_{lim} \in \mathcal{M}$.

Proof. We can rewrite E_{lim} as

$$E_{lim} = \{x \in X : \{f_n(x)\}_{n \in \mathbb{N}} \text{ is a Cauchy sequence}\}$$

Define

$$A_{n,m}(k) = \left\{x \in X : |f_n(x) - f_m(x)| < \frac{1}{k}\right\}$$

then $A_{n,m}(k) \in \mathcal{M}$ for all n, m, k . then

$$E_{lim} = \bigcup_{k \geq 1} \bigcap_{N \geq 1} \bigcup_{m \geq N, n \geq N} A_{m,n}(k)$$

\square

3 Integration

§3.1 Simple Functions

Definition 3.1.1. nonnegative simple function are measurable function with finitely many values in \mathbb{R} (NOT on $\overline{\mathbb{R}}$). $s : X \rightarrow \mathbb{R}$, $s(x) = \sum z_j \mathbb{1}_{x, s(z)=z_j}(x) = \sum z_j \mathbb{1}_{f^{-1}(z_j)}$ If values of s are $\{z_1, \dots, z_n\}$

Theorem 3.1.2

Consider nonnegative measurable function f . There exist a sequence of simple function s_n such that

- $0 \leq s_n \leq s_{n+1} \leq f$ (i.e, $s_n(x) \leq s_{n+1}(x)$)
- $\lim_{n \rightarrow \infty} s_n(x) = f(x)$ for all x
- The convergence is uniform on all sets where f is bounded. If E is such that $|f(x)| \leq M$ for all $x \in E$ then

$$\sup_{x \in E} f(x) - s_n(x) \rightarrow 0$$

Proof. s_n is defined so that it takes value in $[0, 2^n)$. Consider the segment $\frac{k}{2^n}$ on y-axis, then

$$s_n(x) = \begin{cases} k \cdot 2^{-n} & \text{if } k2^{-n} \leq f(x) < (k+1)2^{-n}, 0 \leq k \leq 2^n - 1 \\ 2^n & \text{if } f(x) \geq 2^n \end{cases}$$

If $f(x) < 2^n$ then $0 \leq f(x) - s_n(x) < 2^{-n}$. We can see that $s_n(x) \leq s_{n+1}(x)$ because each step of s_{n+1} is a refinement of s_n . \square

We first define the integral for simple function (in analogy to the definition of Riemann-integral for step functions)

Definition 3.1.3. Define $s(x) = \sum_j c_j \mathbb{1}_{E_j}$ where the E_j are pairwise disjoint, $\biguplus E_j = X$, then

$$\int s \, d\mu = \sum_j c_j \mu(E_j)$$

Claim 3.1.4 —

$$s(x) = \sum_{j=1}^n c_j \mathbb{1}_{E_j}(x) = \sum_{k=1}^m d_k \mathbb{1}_{E_k}(x)$$

where $X = \biguplus E_j = \biguplus E_k$. If $x \in E_j \cap E_k$ then $c_j = d_k$.

Proof. We know that $\biguplus_{j,k} E_j \cap E_k = X$ and $E_j = \biguplus_k E_j \cap E_k$

GOAL: $\sum_{j=1}^n c_j \mu(E_j) = \sum_{k=1}^m d_k \mu(F_k)$

$$\begin{aligned} \text{LHS} &= \sum_{j=1}^n c_j \sum_{k=1}^{\infty} \mu(E_j \cap F_k) = \sum_{k=1}^m \sum_{j=1}^n d_k \mu(E_j \cap E_k) \\ &= \sum_{k=1}^m d_k \mu(F_k) \end{aligned}$$

□

Lemma 3.1.5

Suppose s, t are simple functions then

$$\int (s + t) \, d\mu = \int s \, d\mu + \int t \, d\mu$$

Remark 3.1.6. Can shortly write

$$\int s + t = \int s + \int t$$

Proof.

$$\begin{aligned} s &= \sum_{j=1}^n c_j \mathbb{1}_{E_j} = \sum_j \sum_k c_j \mathbb{1}_{E_j \cap F_k} \\ t &= \sum_{k=1}^m d_k \mathbb{1}_{F_k} = \sum_j \sum_k d_k \mathbb{1}_{E_j \cap F_k} \\ s + t &= \sum_{j,k} (c_j + d_k) \mathbb{1}_{E_j \cap F_k} \end{aligned}$$

$$\begin{aligned}
\int s \, d\mu &= \sum_{j,k} c_j \mu(E_j \cap F_k) \\
\int t \, d\mu &= \sum_{j,k} d_k \mu(E_j \cap F_k) \\
\int (s+t) \, d\mu &= \sum_{j,k} (c_j + d_k) \mu(E_j \cap F_k)
\end{aligned}$$

□

Lemma 3.1.7

$\nu(E) = \int_E s \, d\mu = \int s \mathbb{1}_E \, d\mu = \sum c_j \mu(E_j \cap E)$ this defines a measure on \mathcal{M} (given σ -algebra)

Proof. If E^l is a sequence of pairwise disjoint measurable set, check

$$\begin{aligned}
\nu\left(\biguplus E^l\right) &= \sum \nu(E^l) \\
\nu\left(\biguplus E^l\right) &= \sum c_j \mu(E_j \cap \biguplus E^l) \\
&= \sum_{j=1}^n c_j \sum_l \mu(E_j \cap E^l) \\
&= \sum_l \sum_j c_j \mu(E_j \cap E^l) \\
&= \sum_l \nu(E^l)
\end{aligned}$$

□

§3.2 Non-negative Measurable Functions

Definition 3.2.1. For any non-negative f , a measurable function, define

$$\int f \, d\mu = \sup_{\substack{s \leq f \\ s \text{ simple}}} \int s \, d\mu$$

Remark 3.2.2. If $0 \leq f \leq g$ then $\int f \, d\mu \leq \int g \, d\mu$

Theorem 3.2.3 (Monotone Convergence Theorem)

If $\{f_n\}$ is a sequence of measurable function, and $0 \leq f_n \leq f_{n+1}$ for all n . (that means $f(x) = \lim_{n \rightarrow \infty} f_n(x)$) Then

$$\int f \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

Proof. Since $f_n \leq f_{n+1} \leq f$ then

$$\lim_{n \rightarrow \infty} \int f_n \, d\mu \leq \int f \, d\mu$$

We need to show that

$$\int f \, d\mu \leq \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

So, it suffices to show that for any $0 \leq s \leq f$, that

$$\int s \, d\mu \leq \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

It suffices to show that for any $\varepsilon > 0$,

$$(1 - \varepsilon) \int s \, d\mu \leq \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

Define $E_n = \{x : (1 - \varepsilon)s(x) \leq f_n(x)\}$, any x will be in one of the E_n . Then for any $x \in E_n$,

$$s(x) \leq \frac{f_n(x)}{1 - \varepsilon}$$

Consider the measure defined by

$$\nu(E) = \int_E s \, d\mu$$

(we already show this is a measure in 3.1.7). We have $E_n \subseteq E_{n+1}$ and $E_n \rightarrow X$. By continuity from below 2.4.3,

$$\lim_{n \rightarrow \infty} \nu(E_n) = \nu(X) = \int s \, d\mu$$

We get that

$$\nu(E_n) = \int_{E_n} s \, d\mu \leq \int_{E_n} \frac{f_n(x)}{1 - \varepsilon} \, d\mu \leq \int \frac{f_n(x)}{1 - \varepsilon} \, d\mu = \frac{1}{1 - \varepsilon} \int f_n(x) \, d\mu$$

Finally, we take limit on both sides and we have

$$\lim_{n \rightarrow \infty} \nu(E_n) = \nu(\mathbb{R}) = \int s \, d\mu \leq \lim_{n \rightarrow \infty} \frac{1}{1 - \varepsilon} \int f_n \, d\mu$$

□

Lemma 3.2.4

If f, g are non negative measurable function then

$$\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$$

Proof. Now we have a tool

- Monotone Convergence Theorem
- Existence of $s_n \gg f, t_n \gg g$

$$\begin{aligned} \int (s_n + t_n) \, d\mu &= \int s_n \, d\mu + \int t_n \, d\mu \\ \int (f + g) \, d\mu &= \int f \, d\mu + \int g \, d\mu \end{aligned}$$

□

Lemma 3.2.5

$f_k \geq 0$, f_k is measurable

$$\int \sum_{k=1}^{\infty} f_k(x) \, d\mu = \sum_{k=1}^{\infty} \int f_k \, d\mu$$

Proof. Just apply the Monotone Convergence Theorem.

$$s_n(x) = \sum_{k=1}^n f_k(x) \rightarrow \sum_{k=1}^{\infty} f_k(x)$$

□

Remark 3.2.6. We cannot always interchange integrals and limits (monotonicity is key)
 $f_n(x) = \frac{1}{n} \mathbb{1}_{[0,n]}$, $\int f_n \, d\mu = 1$ but $\lim_{n \rightarrow \infty} f_n(x) = 0$.

$$0 = \int \lim_{n \rightarrow \infty} f_n(x) \, d\mu < \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

Or on $[0, 1]$, $f_n(x) = n \mathbb{1}_{[0, 1/n]}$, $\int f_n \, d\mu = 1$ but $\lim_{n \rightarrow \infty} f_n(x) = 0$.

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Lemma 3.2.7 (Fatou's Lemma)

If $\{f_j\}$ is a sequence of measurable functions

$$\int \liminf_{j \rightarrow \infty} f_j(x) \, d\mu \leq \liminf_{j \rightarrow \infty} \int f_j \, d\mu$$

meaning

$$\int \lim_{k \rightarrow \infty} \underbrace{\inf_{j \geq k} f_j(x)}_{\text{increasing on } k} \, d\mu \leq \lim_{k \rightarrow \infty} \inf_{j \geq k} \int f_j \, d\mu$$

Proof.

$$\int \lim_{k \rightarrow \infty} \inf_{j \geq k} f_j(x) \, d\mu \stackrel{\text{MCT}}{=} \lim_{k \rightarrow \infty} \int \inf_{j \geq k} f_j(x) \, d\mu$$

Take any $l \geq k$, then $\inf_{j \geq k} f_j(x) \leq f_l(x)$, then for $l \geq k$

$$\begin{aligned} \int \inf_{j \geq k} f_j(x) \, d\mu &\leq \int f_l(x) \, d\mu \\ \int \inf_{j \geq k} f_j(x) \, d\mu &\leq \inf_{j \geq k} \int f_j(x) \, d\mu \end{aligned}$$

□

§3.3 General Measurable Functions

Integral for “general” measurable functions.

Definition 3.3.1. Given a measurable function f , we define the **positive part** of f as

$$f^+(x) = \max\{f(x), 0\}$$

and the **negative part** of f as

$$f^-(x) = \max\{-f(x), 0\}$$

Then we get that

$$f = f^+ - f^-$$

Definition 3.3.2. $f : X \rightarrow \mathbb{R}$ (or $\overline{\mathbb{R}}$) Suppose that f is a measurable function, then we define

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$$

provided that at least one of $\int f^\pm \, d\mu$ is finite

Definition 3.3.3. $f : X \rightarrow \mathbb{R}$ (or $\overline{\mathbb{R}}$) f is **integrable** if $\int f^+ \, d\mu, \int f^- \, d\mu$ is finite $\iff \int |f| \, d\mu$ is finite

\mathcal{L}^1 is the class of integrable function

Definition 3.3.4. $f : X \rightarrow \mathbb{C}$ is measurable ($\iff \Re(f)$ and $\Im(f)$ are measurable) Assumeing that $\Re f \in \mathcal{L}^1$ and $\Im f \in \mathcal{L}^1$ then

$$\int f \, d\mu = \int \Re f \, d\mu + i \int \Im f \, d\mu$$

Claim 3.3.5 — Suppose that f, g are measurable then

$$\begin{aligned} \int f + g \, d\mu &= \int f \, d\mu + \int g \, d\mu \\ \int \alpha f \, d\mu + \alpha \int f \, d\mu \end{aligned}$$

Lemma 3.3.6

$f : X \rightarrow \overline{\mathbb{R}}$ is measurable, and $\int |f| \, d\mu = 0$ then $f = 0$ almost everywhere.

Proof. Define $E_n = \{x : |f(x)| > \frac{1}{n}\}$ then from continuity from below, we get that

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$$

define $E = \bigcup E_n$ and we can write $E = \{x : |f(x)| > 0\}$ then we know that

$$|f| \geq |f| \mathbb{1}_{E_n} \geq \frac{1}{n} \mathbb{1}_{E_n}$$

then

$$\begin{aligned} \int |f| \, d\mu &\geq \int \frac{1}{n} \mathbb{1}_{E_n} \, d\mu \\ &= \frac{1}{n} \mu(E_n) \end{aligned}$$

we get that $\mu(E_n) = 0$ for all n then $\mu(E) = 0$. Therefore $f = 0$ almost everywhere. \square

Remark 3.3.7. $\|f\| = \int |f| \, d\mu$ satisfies

- $\|f + g\| \leq \|f\| + \|g\|$
- $\|cf\| = |c|\|f\|$
- $\|f\| = 0 \iff f = 0$ almost everywhere

Remark 3.3.8. Almost everywhere equal is an equivalence relation.

$$f \sim g \stackrel{\text{def}}{\iff} f(x) = g(x) \text{ } \mu\text{-almost everywhere}$$

$N = \{f \in \mathcal{L}^1 : f(x) = 0 \text{ almost everywhere}\}$ is a linear subspace of \mathcal{L}^1 vector. \mathcal{L}^1/N is the set of equivalence classes of \mathcal{L}^1 .

$f_n \rightarrow f$ almost everywhere, $f_n \geq 0$, f_n measurable, Can we define $\int f \, d\mu$? f may not be measurable. This problem is fixed if f we work in a complete measurable space $(X, \mathcal{M}, \mu) \rightarrow (X, \overline{\mathcal{M}}, \overline{\mu})$ where

$$\overline{\mathcal{M}} = \{A \cup B : A \in \mathcal{M}, B \text{ a subset of a set of measure } 0\}$$

Lemma 3.3.9

$f \in \mathcal{L}^1$, $\int |f| \, d\mu < \infty$. If f is real valued $f = f^+ - f^-$,

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$$

Proof.

$$\begin{aligned} \left| \int f \, d\mu \right| &= \left| \int f^+ - f^- \, d\mu \right| \\ &\leq \left| \int f^+ \, d\mu \right| + \left| \int f^- \, d\mu \right| \\ &= \int f^+ \, d\mu + \int f^- \, d\mu \\ &= \int |f| \, d\mu \end{aligned}$$

□

Remark 3.3.10. If f is complex valued, then $|f| = \sqrt{(\Re f)^2 + (\Im f)^2}$. Then

$$\left| \int \Re f \right| \leq \int |\Re f| \leq \int |f|$$

So,

$$\left| \int \Im f \right| \leq \int |\Im f| \leq \int |f|$$

$$\left| \int f \, d\mu \right| \leq 2 \int |f|$$

Remark 3.3.11. Estimate $\int f \, d\mu = \alpha + i\beta = re^{i\phi}$, then $e^{-i\phi} \int f \, d\mu$ is real and nonnegative.

$$\begin{aligned} \left| \int f \, d\mu \right| &= \left| e^{-i\phi} \int f \, d\mu \right| \\ &= \Re \int e^{-i\phi} f \, d\mu \\ &\leq \int |e^{-i\phi} f| \, d\mu \\ &= \int |f| \, d\mu \end{aligned}$$

Lemma 3.3.12

$f \in \mathcal{L}^+$ means non-negative, then $\nu(E) = \int_E f \, d\mu$ define measure

Proof. Check the σ -additivity $E = \bigsqcup_{n=1}^{\infty} E_n$,

$$\begin{aligned} \nu \left(\bigsqcup_{n=1}^{\infty} E_n \right) &= \int_{\bigsqcup E_n} f \, d\mu \\ &= \int f \mathbb{1}_{\bigsqcup E_n} \, d\mu \\ &= \int f \left(\sum_{n=1}^{\infty} \mathbb{1}_{E_n} \right) \, d\mu \\ &= \sum_{n=1}^{\infty} \int f \mathbb{1}_{E_n} \, d\mu \\ &= \sum_{n=1}^{\infty} \nu(E_n) \end{aligned}$$

□

Claim 3.3.13 — If $f \in \mathcal{L}^1 \cap \mathcal{L}^+$ then ν is a finite measure.

If $\nu(E) = \int_E f \, d\mu$ How does $\int g \, d\nu$ look like? $\nu(E) = \int f \, d\mu = \int E \, d\nu$ We want “ $f \, d\mu = d\nu$ ”

Lemma 3.3.14

If $f \in \mathcal{L}^+$ and $\nu(E) = \int_E f \, d\mu$ then for any $g \in \mathcal{L}^+$ or $g \in \mathcal{L}^1$ then,

$$\int g \, d\nu = \int gf \, d\mu$$

Proof. • True for characteristic functions of measure set by the definition of ν . Fix $g = \mathbb{1}_E$ for some $E \in \mathcal{M}$

$$\int g \, d\nu = \int \mathbb{1}_E \, d\mu = \nu(E) = \int_E f \, d\mu = \int \mathbb{1}_E f \, d\mu = \int gf \, d\mu$$

• By linearity of the integral, it is true for simple function. Fix $g = \sum_{j=1}^n c_j \mathbb{1}_{E_j}$, then

$$\int g \, d\nu = \sum_{j=1}^n c_j \nu(E_j) = \sum_{j=1}^n c_j \int_{E_j} f \, d\mu = \int gf \, d\mu$$

• $s_n \nearrow g$ if $g \in \mathcal{L}^+$, by Monotone convergence theorem,

$$\int s_n \nearrow_{\text{MCT}} \int g$$

$$\int s_n \, d\nu = \int s_n f \, d\mu$$

$$\int g \, d\nu = \int gf \, d\mu$$

Then extend to general g by linearity

□

Theorem 3.3.15

If X is a finite measure space, if f_n measurable, $f_n \in \mathcal{L}^1$ (integrable) and $f_n \rightarrow f$ uniformly on X . then

$$\int |f_n - f| \, d\mu \rightarrow 0$$

and

$$\int f_n \, d\mu \rightarrow \int f \, d\mu$$

Remark 3.3.16. Uniform convergence means

$$\lim_{n \rightarrow \infty} \sup_{x \in X} |f_n(x) - f(x)| = 0$$

Proof. We can rewrite that term as

$$\begin{aligned} \int |f_n - f| \, d\mu &\leq \int \sup_{x \in X} |f_n - f| \, d\mu \\ &= \mu(X) \sup_{x \in X} |f_n - f| \rightarrow 0 \end{aligned}$$

We can rewrite f as $f = (f - f_n) + (f_n)$ since $f - f_n$ converge and f_n integrable so f must be integrable.

$$\begin{aligned} \left| \int f_n - \int f \right| &= \left| \int (f_n - f) \, d\mu \right| \\ &\leq \int |f_n - f| \, d\mu \end{aligned}$$

□

Definition 3.3.17. Suppose that f_n, f are measurable $f_n \rightarrow f$ almost uniformly if for every $\varepsilon > 0$ there is a measurable set E such that $\mu(E) < \varepsilon$ and $f_n \rightarrow f$ uniformly on E^c ($\sup_{x \in E^c} |f_n(x) - f(x)| \rightarrow 0$)

Theorem 3.3.18 (Egorov's Theorem)

If $\mu(X) < \infty$ and if $f_n \rightarrow f$ almost everywhere then $f_n \rightarrow f$ almost uniformly

Remark 3.3.19. $f_n(x) \rightarrow f(x)$ if for every k there exists $n = n(k)$ such that $|f_m(x) - f(x)| < \frac{1}{k}$ for all $m \geq n(k)$

Proof. Fix $\varepsilon > 0$, define

$$\begin{aligned} E_n(k) &:= \left\{ x : |f_m(x) - f(x)| \geq \frac{1}{k} \text{ for some } m \geq n \right\} \\ &= \bigcup_{m \geq n} \left\{ x : |f_m(x) - f(x)| \geq \frac{1}{k} \right\} \end{aligned}$$

(Given x For sufficiently large n , $x \notin E_n(k)$), $E_n(k) \supseteq E_{n+1}(k) \cap_n E_n(k) = \emptyset$ because of $f_n \rightarrow f$ everywhere. Form the continuity from above 2.4.5, we get that

$\lim_{n \rightarrow \infty} \mu(E_n(k)) = 0$. Find $n(k)$ such that $\mu(E_{n(k)}(k)) < \frac{\varepsilon}{2^k}$, then $E = \bigcup_k E_{n(k)}(k)$ has measure $< \varepsilon$.

For $x \in \left(\bigcup_k E_{n(k)}(k)\right)^c = \bigcap_k E_{n(k)}(k)^c$ I have for all k $|f_m(x) - f(x)| < \frac{1}{k}$ for all $m \geq n(k)$. So, we get $f_n \rightarrow f$ uniformly on E^c . \square

Theorem 3.3.20 (Baby Dominated Convergence Theorem)

Given (X, \mathcal{M}, μ) where μ is a finite measure ($\mu(X) < \infty$). Let $\{f_n\}$ be measurable functions, $f_n \rightarrow f$ everywhere.

$$|f_n| \leq C \implies \int |f_n - f| \, d\mu \rightarrow 0$$

i.e. f_n converges with respect to L^1 -(semi-)norm.

Corollary 3.3.21

$$\int_X f_n \, d\mu \rightarrow \int_X f \, d\mu$$

Proof. Tools:

- (i) If $f_n \rightarrow f$ uniformly then $\int |f_n - f| \, d\mu \rightarrow 0$
- (ii) Egorov's Theorem

$|f(x)| \leq C$, f is measurable. Given any $\varepsilon > 0$, By Egorov's Theorem, find a set of measure E that $\mu(E) < \frac{\varepsilon}{4C}$ such that $f_n \rightarrow f$ uniformly on E^c . Then

$$\int |f_n - f| \, d\mu \leq \int_E |f_n - f| \, d\mu + \int_{E^c} |f_n - f| \, d\mu$$

we know that $|f_n - f| \leq |f_n| + |f| \leq 2C$ then

$$\begin{aligned} \int |f_n - f| \, d\mu &\leq 2C\mu(E) + \int_{E^c} |f_n - f| \, d\mu \\ &\leq \frac{\varepsilon}{2} + \int_{E^c} |f_n - f| \, d\mu \end{aligned}$$

so for large n , the second term will be $< \frac{\varepsilon}{2}$. \square

Theorem 3.3.22 (Dominated Convergence Theorem)

Given (X, \mathcal{M}, μ) where μ is a finite measure ($\mu(X) < \infty$). Let $\{f_n\}$ be measurable functions, $f_n \rightarrow f$ almost everywhere.

$$\sup_n |f_n| \in \mathcal{L}^1 \implies \int |f_n - f| \, d\mu \rightarrow 0$$

Proof. Define $g(x) = \sup_n |f_n(x)|$ The trick is

$$|f_n - f| = \begin{cases} \frac{|f_n - f|}{g} g & \text{if } g > 0 \\ 0 & \text{if } g = 0 \end{cases}$$

define a new measure $\nu(E) = \int_E g \, d\mu$. Then ν is a finite measure, and

$$\begin{aligned} g \, d\mu &= d\nu \\ \int h \, d\nu &= \int hg \, d\mu \end{aligned}$$

then define

$$h_n = \begin{cases} \frac{|f_n - f|}{g} & \text{if } g > 0 \\ 0 & \text{if } g = 0 \end{cases} \implies \begin{aligned} |h_n(x)| &\leq 1 \\ h_n(x) &\rightarrow 0 \end{aligned}$$

Then

$$\begin{aligned} \int |f_n - f| \, d\mu &= \int h_n g \, d\mu \\ &= \int h_n \, d\nu \rightarrow 0 \end{aligned}$$

By Baby Dominated Convergence Theorem □

§3.4 Integration from Riemann to Lebesgue**Theorem 3.4.1**

If f is Riemann integrable on $[a, b]$ then f is Lebesgue integrable.

$$\int_a^b f(x) \, dx = \int_{[a,b]} f \, d\mu$$

where μ is Lebesgue measure.

Proof. Define

$$U_P f(x) = \begin{cases} M_j & \text{if } x \in [x_{j-1}, x_j) \\ M_n & \text{if } x \in [x_{n-1}, x_n] \end{cases}$$

Similarly for the lower sum $L_P f(x)$. If P' is a refinement of P then $U_{P'} f(x) \leq U_P f(x)$ and $L_{P'} f(x) \geq L_P f(x)$. Since f is Riemann integrable, then

$$\inf_P U(f, P) =: \bar{\mathcal{I}}_a^b(f) = \underline{\mathcal{I}}_a^b(f) := \sup_P L(f, P)$$

Choose a sequence of partitions P_n such that

$$\begin{aligned} \int U_{P_n} f &\rightarrow \bar{\mathcal{I}}_a^b(f) \\ \int L_{P_n} f &\rightarrow \underline{\mathcal{I}}_a^b(f) \end{aligned}$$

Since that P_{n+1} is a refinement of the P_n then $U_{P_n} f \searrow U(x)$ and $L_{P_n} f \nearrow L(x)$ and $L(x) = U(x)$. Notice that from Riemann integrable, $|f| < C$, then

$$\begin{aligned} \int_{[a,b]} U_{P_n} f &\rightarrow \bar{\mathcal{I}}_a^b(f) = \int_{[a,b]} U(x) \, dm \\ \int_{[a,b]} L_{P_n} f &\rightarrow \underline{\mathcal{I}}_a^b(f) = \int_{[a,b]} L(x) \, dm \end{aligned}$$

If f is Riemann integrable,

$$\int U \, dm = \int L \, dm = \int_a^b f(x) \, dx$$

and $U \geq L$ then

$$\int (U - L) \, dm = 0 \implies U(x) = L(x)$$

almost everywhere, $L(x) \leq f(x) \leq U(x) \implies f = L$ almost everywhere and $f = U$ almost everywhere. Then f is Lebesgue integrable and

$$\int f \, dm = \int L \, dm = \int U \, dm$$

□

Definition 3.4.2 (Improper Riemann integrals).

$$\int_0^\infty f(x) \, dx, \int_1^\infty f(x) \, dx, \int_0^1 f(x) \, dx$$

if f is not Riemann-integrable on the domain but on every compact subinterval. We can define as

$$\int_1^\infty f(x) \, dx = \lim_{R \rightarrow \infty} \int_1^R f(x) \, dx$$

Example 3.4.3

$$\int_1^\infty \frac{\sin x}{x} dx = \lim_{R \rightarrow \infty} \int_1^R \frac{\sin x}{x} dx$$

$I_k = [2k\pi + \frac{\pi}{4}, 2k\pi + \frac{3\pi}{4}]$, $\sin x \geq \frac{1}{\sqrt{2}}$, so, $\frac{\sin x}{x} \geq \frac{1}{x\sqrt{2}} \cdot \frac{1}{2k\pi + \frac{3\pi}{4}}$. We can do integration by parts

$$\int_1^R \frac{\sin x}{x} dx = -\frac{\cos x}{x} \Big|_1^R - \int_1^R -\frac{\cos x}{x^2} dx$$

Example 3.4.4

$$\int_0^\infty \sin(x^2) dx$$

consider $\sin(x^2)$

$$\begin{aligned} \sqrt{2k\pi + \frac{\pi}{2}} &\leq \sqrt{x^2} \leq \sqrt{2k\pi + \frac{3\pi}{4}} \\ \sqrt{2k\pi + \frac{3\pi}{4}} - \sqrt{2k\pi + \frac{\pi}{4}} &\approx \frac{1}{\sqrt{k}} \end{aligned}$$

Lemma 3.4.5

Suppose that if $\int_1^\infty |f(x)| dx < \infty$ then $f \in \mathcal{L}^1$.

Proof.

$$\begin{aligned} \int_1^\infty |f(x)| dx &= \int_1^\infty \lim_{n \rightarrow \infty} |f(x)| \mathbb{1}_{[1,n]}(x) dx \\ &= \lim_{n \rightarrow \infty} \int_1^n |f(x)| dx \end{aligned}$$

□

Theorem 3.4.6

If f is integrable on \mathbb{R} ,

$$\int_{\mathbb{R}} |f(x)| \, dx < \infty$$

$f \in \mathcal{L}^1$ then for every $\varepsilon > 0$, there is a continuous function (C^∞) g , vanishes off a compact set,

$$\int |f - g| \, dm < \varepsilon$$

§3.5 Introduction to Outer Measures

Definition 3.5.1. In our axiomatic theorem on the Lebesgue measure, $m(I) = \ell(I)$, $m((a, b]) = b - a$ and for a general Borel set on \mathbb{R} , m is given by the **outer measure** induced by the collection of intervals

$$\varrho(E) = \inf \sum_{k=1}^{\infty} \ell(I_k)$$

where the inf is taken over collections $\{I_k\}$, such that $E \subseteq \bigcup_{k=1}^{\infty} I_k$

Remark 3.5.2. $\tilde{\varrho}$ defined similarly but we only admit open intervals in the infimum. Obviously, $\tilde{\varrho}(E) \geq \varrho(E)$. Need to show that $\tilde{\varrho}(E) \leq \varrho(E)$ we may assume that $\varrho(E) < \infty$, show $\tilde{\varrho}(E) \leq \varrho(E) + \varepsilon$. There is a collection of intervals I_k such that

$$\sum_k \ell(I_k) < \varrho(E) + \frac{\varepsilon}{2}$$

If $I_k = [a_k, b_k]$, then define $J_k = (a_k - \frac{\varepsilon}{2^{k+2}}, b_k + \frac{\varepsilon}{2^{k+2}})$. Then $\ell(J_k) = \ell(I_k) + \frac{\varepsilon}{2^{k+1}}$ then

$$\begin{aligned} \tilde{\varrho}(E) &\leq \sum_{k=1}^{\infty} \ell(J_k) \leq \sum_{k=1}^{\infty} \ell(I_k) + \varepsilon \sum_{k=1}^{\infty} 2^{-k-1} \\ &\leq \varrho(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

Lemma 3.5.3

$$m(E) = \sup\{m(K) : K \subseteq E, K \text{ compact}\}$$

Proof. Case where $E = \overline{E}$ and E is bounded, there is nothing to show

Assume that E is bounded, GOAL: find $K \subseteq E$ such that $m(E \setminus K) < \varepsilon$. Consider $\overline{E} \setminus E$, find $O \supseteq \overline{E} \setminus E$, $m(O \setminus (\overline{E} \setminus E)) < \varepsilon$. then $O^c \cap \overline{E} \subseteq E$ because if $x \in O^c$, either $x \in \overline{E}$ or

$x \in E$. $E \setminus K = E \cap K^c \subseteq O \cup \overline{E}^c$. Since $E \subseteq \overline{E}$ then $E \setminus K \subseteq O$ and $E \setminus K \subseteq O \setminus (\overline{E} \setminus E)$ has measure $< \varepsilon$. \square

Theorem 3.5.4

For every Borel set E , $m(E) < \infty$, there is an open set $O \supseteq E$ such that $m(O \setminus E) < \varepsilon$. where $m(E) = \inf \sum \ell(I_n)$ where inf take over I_k , I_k are open, $E \subseteq \bigcup I_k$

Proof. Define $E_n = E \cap \overline{B}(0, n)$. Find compact set $K_n \subseteq E_n \setminus E_{n-1}$ then $m((E_n \setminus E_{n-1}) \setminus K_n) < \varepsilon 2^{-n-1}$. The set $H_l = K_1 \cup \dots \cup K_l$ is compact and increasing, $H_l \subseteq E_l$ and

$$m(E_l) - \varepsilon \leq m(H_l) \leq m(E_l) \rightarrow m(E)$$

\square

Theorem 3.5.5

Given an open set O , we can decompose O as a disjoint union of “dyadic cubes”

Theorem 3.5.6

We can choose the cubes a dyadic cubes such that if $O \neq \mathbb{R}^n$ such that

$$\text{diam}(Q) < \text{dist}(Q, O^c) \leq 4\text{diam}(Q)$$

Remark 3.5.7. If side length of Q is 2^{-k} then the diameter is $\sqrt{n}2^{-k}$.

Theorem 3.5.8 (Whitney decomposition theorem)

Given Ω open set in \mathbb{R}^n , $\Omega \neq \mathbb{R}^n$, there is a family \mathcal{F} of dyadic cubes such that

- they are disjoint
- $\bigsqcup_{Q \in \mathcal{F}} Q = \Omega$
- For every $Q \in \mathcal{F}$, $C\text{diam}(Q) < \text{dist}(Q, \Omega^c) \leq (2C + 2)\text{diam}(Q)$

Proof. Define for each k a family of dyadic cubes \mathcal{F}_k of side length 2^{-k} (i.e., the diameter is $\sqrt{n}2^{-k}$) intersecting the region

$$\Omega_k = \{x : A2^{-k}\sqrt{n} \leq \text{dist}(x, \Omega^c) \leq 2A \cdot 2^{-k}\sqrt{n}\}$$

Pick a cube in \mathcal{F}_k , Q_1 it contains an $x_Q \in \Omega_k$

$$\text{dist}(Q, \Omega^{\mathbb{L}}) \leq \text{dist}(x_Q, \Omega^{\mathbb{L}}) \leq 2A \cdot 2^{-k} \sqrt{n} - 2A \text{diam}(Q)$$

$$\begin{aligned} \text{dist}(Q, \Omega^{\mathbb{L}}) &\geq \text{dist}(x_Q, \Omega^{\mathbb{L}}) - \text{diam}(Q) \\ &\geq A2^{-k} \sqrt{n} - 2A \text{diam}(Q) \\ &= (A - 1) \cdot \text{diam}(Q) \end{aligned}$$

Then $\mathcal{F}_T = \bigcup \mathcal{F}_k$ and finally \mathcal{F} = collection of all maximal (with respect to inclusion) cubes in \mathcal{F}_T . Fix Q, Q' and assume that $Q \subseteq Q'$ then

$$(A - 1) \text{diam} Q' \leq \text{dist}(Q', \Omega^{\mathbb{L}}) \leq \text{dist}(Q, \Omega^{\mathbb{L}}) \leq 2A \cdot \text{diam}(Q)$$

$$\text{then } \text{diam}(Q') \leq \frac{2A}{A-1} \text{diam}(Q)$$

□

we know that for every $\varepsilon_1 > 0$ we can find a simple function s such that $\int |f - s| \, dm < \varepsilon_1$. (For non-negative f use MCT $s_n \nearrow f$ and $s_n \leq f$ so $\int s_n \nearrow \int f \implies \int f - s_n \rightarrow 0$ and then we use $f = f_+ - f_-$)

$$s = \sum c_j \mathbb{1}_{E_j}, \quad E_j \subseteq O_j, \quad m(O_j \setminus E_j) < \varepsilon \quad \tilde{s} = \sum c_j \mathbb{1}_{O_j}$$

$$\begin{aligned} \int \tilde{s} - s \, dm &= \left| \int \sum c_j \mathbb{1}_{O_j} - \mathbb{1}_{E_j} \, dm \right| \\ &\leq \sum_{j=1}^N |c_j| |m(O_j \setminus E_j)| \\ &\leq \varepsilon_2 \end{aligned}$$

Then $\mathbb{1}_{O_j} = \sum_{\nu} \mathbb{1}_{Q_{\nu}}$ where $\{Q_{\nu}\}$ are the Whitney cubes in Whitney Theorem. $|O_j| = \sum_{\nu \in I} m(Q_{\nu})$ There is a finite \tilde{I}_j such that

$$\int \left| \mathbb{1}_{O_j} - \sum_{\nu \in \tilde{I}_j} \mathbb{1}_{Q_{\nu}} \right| < \varepsilon_3$$

replace $\sum_{j=1}^N c_j \mathbb{1}_{O_j}$ by $\sum_{j=1}^N c_j \mathbb{1}_{\bigcup_{\nu \in \tilde{I}_j} Q_{\nu}}$

then

$$\mathbb{1}_{Q_{\nu}}(x_1, \dots, x_n) = \prod_{i=1}^n \mathbb{1}_{\nu, i}(x_i)$$

Lemma 3.5.9

For any $f \in L^1$ there exists s a step function such that $\int |f - s| \, dm < \varepsilon$

Proof. Suppose that $f \in L^1$ I want to show that there exists s a step function such that $\int |f - s| \, dm < \varepsilon$ for any $\varepsilon > 0$. Since $f = f^+ - f^-$, WLOG, $f \geq 0$ (otherwise we can do each positive and negative part and do the sum of both step functions with $\frac{\varepsilon}{2}$ bound). Given any $\varepsilon > 0$, there exists s' a simple function such that $\int |f - s'| \, dm < \frac{\varepsilon}{2}$. Then we can write $s' = \sum_{j=1}^N c_j \mathbb{1}_{E_j}$, then there exists O_j open set such that $E_j \subseteq O_j$ and $m(O_j \setminus E_j) < \frac{\varepsilon}{4|c_j|N}$. Since O_j is an open set, then $O_j = \bigcup (a_i, b_i)$ then define $K_n = \bigcup_{i=1}^n (a_i, b_i)$ from continuity from below, there exists n' such that $m(K_{n'}) > m(O_j) - \frac{\varepsilon}{4|c_j|N}$ and $K_{n'}$ contain finite interval, then we define $O'_j := K_{n'}$. Define $s = \sum_{j=1}^N c_j \mathbb{1}_{O'_j}$ then

$$\begin{aligned}
 \int |f - s| \, dm &\leq \int |f - s'| \, dm + \int |s' - s| \, dm \\
 &\leq \frac{\varepsilon}{2} + \int \left| \sum_{j=1}^N c_j (\mathbb{1}_{E_j} - \mathbb{1}_{O'_j}) \right| \, dm \\
 &\leq \frac{\varepsilon}{2} + \sum_{j=1}^N |c_j| \int |\mathbb{1}_{E_j} - \mathbb{1}_{O'_j}| \, dm \\
 &= \frac{\varepsilon}{2} + \sum_{j=1}^N |c_j| (m(E_j \setminus O'_j) + m(O'_j \setminus E_j)) \\
 &\leq \frac{\varepsilon}{2} + \sum_{j=1}^N |c_j| (m(O_j \setminus O'_j) + m(O_j \setminus E_j)) \\
 &\leq \frac{\varepsilon}{2} + \sum_{j=1}^N |c_j| \left(\frac{\varepsilon}{4|c_j|N} + \frac{\varepsilon}{4|c_j|N} \right) \\
 &< \varepsilon
 \end{aligned}$$

□

4 L^p Spaces

§4.1 normed spaces

Remark 4.1.1. If $f_n \rightarrow f$ almost everywhere, do we have $\int |f_n - f| d\mu \rightarrow 0$?

- No, if $f_n = \mathbb{1}_{[n, n+1]}$ then $f_n \rightarrow 0$ almost everywhere but $\int |f_n - 0| d\mu = 1$ and $\int |f_n - f_m| d\mu = 2$.
- No, if $f_n = \mathbb{1}_{[0, \frac{1}{n}]}$

Remark 4.1.2. Convergence in L^1 implies convergence almost everywhere? No, If $2^k \leq n \leq 2^{k+1}$ where $n = 2^k + j$, $j = 0, \dots, 2^k - 1$ $f_{2^k+1} = \mathbb{1}_{[i2^{-k}, (i+1)2^{-k}]}$ for $i = 0, \dots, 2^k - 1$. For $2^k \leq n \leq 2^{k+1}$, $\|f_n\|_{L^1} = 2^{-k}$

Claim 4.1.3 — If $f_n \rightarrow f$ in L^1 ($\int |f_n - f| d\mu \rightarrow 0$) then there is a subsequence $f_{n_k} \rightarrow f$ almost everywhere.

Proof. Consider the normed space L^1 space of semi-normed space \mathcal{L}^1 . (define as a equivalence class of almost everywhere where $f \underset{\text{a.e.}}{\sim} g$ if $f = g$ almost everywhere) Construct a convergence subsequence (a.e. and also in Norm) Choose $\varepsilon = \frac{1}{2^k}$ there exists number $N(k)$ such that $\|f_l - f_m\| < \frac{1}{2^k}$ for $l, m \geq N(k)$ for $l, m \geq N(k)$ then $\|f_{N(k)} - f_{N(k+1)}\| \leq \frac{1}{2^k}$ Define

$$G(x) = |f_{N(1)}(x)| + \sum_{k=1}^{\infty} |f_{N(k+1)}(x) - f_{N(k)}(x)|$$

then

$$\int G(x) d\mu = \int |f_{N(1)}(x)| d\mu + \sum_{k=1}^{\infty} \int |f_{N(k+1)}(x) - f_{N(k)}(x)| d\mu \leq \|f_{N(1)}\|_1 + \sum_{k=1}^{\infty} \frac{1}{2^k}$$

So, G is integrable, $\int |G(x)| d\mu < \infty$ then $G(x) < \infty$ almost everywhere. We see that for almost everywhere,

$$f_{N_1}(x) + \sum_{k=1}^{\infty} f_{N(k+1)} - f_{N(k)}(x)$$

converges for almost everywhere x , define

$$s_M(x) = f_{N(1)}(x) + f_{N(2)}(x) - f_{N(1)}(x) + \dots + f_{N(M+1)}(x) - f_{N(M)}(x) = f_{N(M+1)}(x)$$

then $s_{M-1}(x) = f_{N(M)}(x)$ and as $M \rightarrow \infty$, this converges for almost everywhere x .

$$f(x) = \lim_{M \rightarrow \infty} f_{N(M)}(x)$$

$$\begin{aligned} f(x) &= f_{N_1}(x) + \sum_{M=1}^{\infty} f_{N(M+1)}(x) - f_{N(M)}(x) \\ \int |f(x) - f_{N_1}(x)| \, d\mu &= \int \sum_{M=1}^{\infty} f_{N(M+1)}(x) - f_{N(M)}(x) \, d\mu \\ &\leq \int |f_{N(M+1)}(x) - f_{N(M)}(x)| + \int |f_{N(M+2)} - f_{N(M+1)}| + \dots \\ &= \sum_{k=M}^{\infty} \int |f_{N(k+1)}(x) - f_{N(k)}(x)| \, d\mu \\ &\leq 2^{1-M} \end{aligned}$$

This shows convergence of $f_{N(M)} \rightarrow f$ in L^1 . What happens with $l \geq N(k)$,

$$\|f_l - f\| \leq \|f_l - f_{N(k)}\| + \|f_{N(k)} - f\| \leq \frac{1}{2^k}, \rightarrow 0$$

□

L^1 or (\mathcal{L}^1) are complete, in the sense that every Cauchy sequence converges. $\{f_n\}$ is Cauchy. For every $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for $l, m \geq N(\varepsilon)$ then $\|f_l - f_m\| < \varepsilon$

§4.2 Function space

Definition 4.2.1.

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{\frac{1}{p}}$$

where L^p is space of equivalence class and \mathcal{L}^p is space of functions, $f \in \mathcal{L}^p$ if $\|f\|_p < \infty$

Theorem 4.2.2

$\|f\|_p$ is a norm on L^p , if $p \geq 1$ (not a norm if $p < 1$ because triangle inequality fails)

Proof. for any $f, g \in L^p$,

$$\begin{aligned} \int |f + g|^p \, d\mu &\leq \int (2 \max(|f|, |g|))^p \, d\mu \\ &= 2 \left(\int \max(|f|^p, |g|^p) \, d\mu \right) \\ &\leq 2 \int (|f|^p + |g|^p) \, d\mu \end{aligned}$$

□

Remark 4.2.3. $\|f + g\|_p \leq 2^{\frac{1}{p}}(\|f\|_p + \|g\|_p)$

Theorem 4.2.4

For $p < 1$ we have inequality

$$\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$$

Proof. we claim that

$$\int |f + g|^p d\mu \leq \int |f|^p d\mu + \int |g|^p d\mu$$

for $a, b \in [0, \infty)$, $(a + b)^p \leq a^p + b^p$ WLOG $b \leq a$ $f(x) = 1 + x^p - (1 - x)^p$, $f'(x) \geq 0 \implies (1 + x)^p \leq 1 + x^p$ for $0 \leq x \leq 1$ □

Remark 4.2.5. For $p < 1$ we do not get $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ for $x, y \in \mathbb{R}^2$ want to disprove $\|x + y\|_p \leq \|x\|_p + \|y\|_p$, $p < 1$

$$2^{\frac{1}{p}} = (1^p + 1^p)^{\frac{1}{p}}$$

(it is because failure of convexity of the norm $p < 1$)

Claim 4.2.6 — For $0 < \theta < 1$, $a, b \geq 0$, then $a^{1-\theta}b^\theta \leq (1 - \theta)a + \theta b$

Proof. Generalized AM-GM inequality ($\sqrt[p]{ab} \leq \frac{a+b}{2}$) then put for $0 < \theta < 1$ then $a^{1-\theta}b^\theta \leq (1 - \theta)a + \theta b$ WLOG $b \leq a$ then

$$\left(\frac{b}{a}\right)^\theta \leq 1 - \theta + \theta \frac{b}{a}$$

let $x = \frac{b}{a}$ for $0 \leq x \leq 1$ we need to show that $g(x) = 1 - \theta + \theta x - x^\theta \geq 0$ then $g'(x) = -1 + \theta - \theta x^{\theta-1} \leq 0$ (because $0 \leq \theta \leq 1$) □

Claim 4.2.7 (Holder's inequality) — Given $p > 1$, q to be such that

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \left(q = \frac{p}{p-1}\right)$$

for $f \in L^p, g \in L^q$, then $fg \in L^1$ and

$$\int |fg| \, d\mu \leq \|f\|_p \|g\|_q$$

Proof. Rewrite AM-GM (generalized) as “Young’s inequality” substitute $a = u^p, 1 - \theta = \frac{1}{p}, b = v^q, \theta = \frac{1}{q}$ then we get

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$$

apply $f(x)g(x)$

$$\int |f(x)| |g(x)| \, d\mu \leq \int \frac{|f(x)|^p}{p} \, d\mu + \int \frac{|g(x)|^q}{q} \, d\mu = \frac{\|f\|_p^p}{p} + \frac{\|g\|_q^q}{q}$$

(This is Holder when two norms are normalized $\|f\|_p = 1 = \|g\|_q$)

Then $\frac{f(x)}{\|f\|_p}$ has “p-norm” equal to 1 because

$$\left(\int \left| \frac{f(x)}{\|f\|_p} \right|^p \, d\mu \right)^{\frac{1}{p}} = \frac{1}{\|f\|_p} \left(\int |f(x)|^p \, d\mu \right)^{\frac{1}{p}}$$

Substitute $f = \frac{f}{\|f\|_p}$ and $g = \frac{g}{\|g\|_q}$ then we get

$$\begin{aligned} \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \, d\mu &\leq \frac{1}{p} + \frac{1}{q} = 1 \\ \int |fg| \, d\mu &\leq \|f\|_p \|g\|_q \end{aligned}$$

□

Theorem 4.2.8 (Minkowski’s inequality)

$p \geq 1$ We do have a triangle inequality $\|f + g\|_p \leq \|f\|_p + \|g\|_p$

$$\left(\int |f + g|^p \, d\mu \right)^{\frac{1}{p}} \leq \left(\int |f|^p \, d\mu \right)^{\frac{1}{p}} + \left(\int |g|^p \, d\mu \right)^{\frac{1}{p}}$$

Proof. It is enough to show that

$$\|f + g\|_p^p \leq (\|f\|_p + \|g\|_p) \|f + g\|_p^{p-1}$$

$$\begin{aligned}
\int |f+g|^{p-1+1} d\mu &= \int |f+g|^{p-1} |f| d\mu + \int |f+g|^{p-1} |g| d\mu \\
&\leq \left(\int |f|^p d\mu \right)^{\frac{1}{p}} \left(\int |f+g|^p d\mu \right)^{\frac{p-1}{p}} + \left(\int |g|^p d\mu \right)^{\frac{1}{p}} \left(\int |f+g|^p d\mu \right)^{\frac{p-1}{p}} \\
&= (\|f\|_p + \|g\|_p) \|f+g\|_p^{p-1}
\end{aligned}$$

□

Remark 4.2.9. Holder's inequality holds

$$\int fg d\mu \leq \|f\|_p \|g\|_q$$

where $\frac{1}{p} + \frac{1}{q} = 1$ can be generalized to several factors

$$\int f_1 f_2 \cdots f_n d\mu \leq \prod_{j=1}^n \|f_j\|_{p_j}$$

where $\sum_{j=1}^n \frac{1}{p_j} = 1$

§4.3 Application

Lemma 4.3.1 (Shebyshev's inequality)

This is an inequality for the distribution function (given a measure space (X, \mathcal{M}, μ)) $\mu_f(\alpha) = \mu(\{x : |f(x)| \geq \alpha\})$ The Shebyshev's inequality is

$$\mu_f(\alpha) \leq \frac{\|f\|_p^p}{\alpha^p}$$

Proof.

$$\begin{aligned}
\mu_f(\alpha) &= \int_{E_\alpha} 1 d\mu \\
&\leq \int_{E_\alpha} \frac{|f(x)|^p}{\alpha^p} d\mu \\
&\leq \frac{1}{\alpha^p} \int |f(x)|^p d\mu
\end{aligned}$$

(Probabilitiy may call this Markov's inequality)

□

$$\mu_{cf}(\alpha) = \mu(\{x : |cf(x)| \geq \alpha\}) = \mu(\{x : |f(x)| \geq \frac{\alpha}{|c|}\})$$

$$\alpha \mu_{cf}(\alpha)^{\frac{1}{p}} = |c| \left| \frac{\alpha}{|c|} \right| \mu_f \left(\frac{\alpha}{|c|} \right)^{\frac{1}{p}}$$

Claim 4.3.2 — If $\delta_0 + \delta_1 = 1$, $\delta_0, \delta_1 \geq 0$, then

$$E_\alpha(f + g) \subseteq E_{\alpha\delta_0}(f) \cup E_{\alpha\delta_1}(f)$$

Theorem 4.3.3

If $\mu(X) < \infty$ then $L^q \subseteq L^p$ for $p \leq q$

Proof. We need an inequality $\|f\|_p \leq C\|f\|_q$

$$\left(\int 1|f(x)|^p \, d\mu \right)^{\frac{1}{p}} \leq C \left(\int |f(x)|^q \, d\mu \right)^{\frac{1}{q}}$$

Apply Holder with exponent $\frac{q}{p} > 1$, $\left(\frac{q}{p}\right)'$ where

$$\frac{1}{\left(\frac{q}{p}\right)} + \frac{1}{\left(\frac{q}{p}\right)'} = 1$$

$$\begin{aligned} \int |f|^p \cdot 1 \, d\mu &\leq \left(\int (|f|^p)^{\frac{q}{p}} \, d\mu \right)^{\frac{p}{q}} \left(\int_X 1^{\left(\frac{q}{p}\right)'} \, d\mu \right)^{\frac{1}{\left(\frac{q}{p}\right)'}} \\ &= \|f\|_q^p \mu(X)^{\frac{1}{\left(\frac{q}{p}\right)'}} \\ &= \left[\|f\|_q \mu(X)^{\frac{1}{\left(\frac{q}{p}\right)'}} \right]^p \end{aligned}$$

□

Another extreme case would be \mathbb{N} with counting measure In this $L^p(\mathbb{N}, \mu)$ is denoted by $\ell^p(\mathbb{N})$

Theorem 4.3.4

For $p \geq 1$, $\ell^p \subseteq \ell^q$ for $p \leq q$

Proof. We want to prove $\|f\|_{\ell^q} \leq C\|f\|_{\ell^p}$. If $\|f\|_{\ell^p} < 1$, this means

$$\sum_{n=1}^{\infty} |f(n)|^p \leq 1$$

$\implies |f(n)|^p \leq 1$ for all n

$$\sum_{n=1}^{\infty} |f(n)|^q \leq \sum_{n=1}^{\infty} |f(n)|^p$$

provided that $|f(n)|^q \leq |f(n)|^p$

For $f \in \ell^p$, $\frac{f}{\|f\|_p}$ has ℓ^p norm equal to 1 therefore $\left\| \frac{f}{\|f\|_p} \right\|_q \leq 1$ then $\|f\|_q \leq \|f\|_p$ \square

Theorem 4.3.5 (Littlewood Theorem)

Every measurable function is nearly continuous. i.e., $f \in L^1(A)$ there exists g continuous such that

$$\int |f(x) - g(x)| \, dm < \varepsilon$$

Theorem 4.3.6 (Lusin's Theorem)

$f : [a, b] \rightarrow \mathbb{C}$ (or \mathbb{R}) almost everywhere, then there is a compact set $K \subseteq [a, b]$ such that $f|_K$ is continuous and $\mu([a, b] \setminus K) = 0$

Example 4.3.7

$f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ let $\{r_k\}$ be enumeration of rational number in $[a, b]$. Define

$$O = \bigcup_{k=1}^{\infty} \left(r_k - \frac{\varepsilon}{2^{k+2}}, r_k + \frac{\varepsilon}{2^{k+2}} \right)$$

then $m(O) < \varepsilon$, $K = [a, b] \setminus O$, $f|_K = 1$ is continuous.

Example 4.3.8

$$\sum_{k=1}^{\infty} \frac{1}{|x - r_k|^{10}} 2^{-k} \mathbb{1}_{[a, b]}$$

(this function is in L^p , if $p < \frac{1}{10}$). The Challenging part is to find a K as in Lusin's theorem such that $f = f|_K$ is continuous.

Proof. $f : A \rightarrow \mathbb{R}$ (or \mathbb{C}), $m(A) < \infty$ The for every $\varepsilon > 0$, there is a compact set $K \subseteq A$, such that $m(A \setminus K) < \varepsilon$ and $f|_K$ is continuous.

1. We can find a set E_1 such that $m(A \setminus E_1) < \frac{\varepsilon}{3}$ and $f|_{E_1}$ is bounded.

$$S_\alpha = \{x \in A : |f(x)| > \alpha\}$$

Then $\bigcup S_\alpha = \bigcup S_{2^M}$ has measure zero by the assumption. From continuity from above, we get

$$m(S_\alpha) = m\left(\bigcup S_{2^M}\right) = \lim_{M \rightarrow \infty} m(S_{2^M})$$

because $m(A) < \infty$. For large M , $m(S_M) < \frac{\varepsilon}{3}$, let $A \setminus E_1 = S_{2^M}$ (M large)

2. We know that f is bounded on E_1 . Can find a sequence g_n of continuous functions $g_n \rightarrow f$ almost everywhere on E_1 , $m(E_1) < \infty$.
3. Using Egorov's theorem: $g_n \rightarrow f$ almost uniformly on E_1 . Find $E_2 \subseteq E_1$ such that $m(E_1 \setminus E_2) < \frac{\varepsilon}{3}$ and $g_n \rightarrow f$ uniformly on E_2 . $f|_{E_2}$ is continuous on E_2 , we can find $E_3 = K$ compact, $E_3 \subseteq E_2$, $m(E_2 \setminus E_3) < \frac{\varepsilon}{3}$ and $f|_K$ is continuous.

$$m(A \setminus K) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

□

5 Construction of Measures

§5.1 Abstract Outer Measure

Definition 5.1.1. Given $X, \mathcal{E} \subseteq \mathfrak{P}(X)$ where $\emptyset \in \mathcal{E}$ and $X \in \mathcal{E}$, $u : \mathcal{E} \rightarrow [0, \infty]$. For any set

$$u^*(E) = \inf_{\bigcup_{j=1}^{\infty} E_j \supseteq E} \sum_{j=1}^{\infty} u(E_j)$$

where $E_j \in \mathcal{E}$, called the (concrete) outer measure induced by \mathcal{E} . ($u^*(E) = \infty$ if we cannot cover E with a countable collection of sets in \mathcal{E}).

Remark 5.1.2. It is not necessary to assume that $X \in \mathcal{E}$. In this case, $u^*(E) = \infty$ if we cannot cover E with a countable collection of sets in \mathcal{E} .

Lemma 5.1.3

An outer measure u^* induced by the collection \mathcal{E} satisfies

- (i) $u^*(\emptyset) = 0$
- (ii) If $A \subseteq B$ then $u^*(A) \leq u^*(B)$
- (iii) $u^*(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} u^*(E_j)$

E.g., A (concrete) outer measure is an abstract outer measure

Proof. $u^*(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} u^*(E_j)$. WLOG, LHS $< \infty$, for each j find a collection $\{E_k^j\}_{k \in \mathbb{N}}$ such that

$$\sum u(E_k^j) < u^*(A_j) + \varepsilon 2^{-j-1}$$

$\{E_k^j\}_{j,k=1}^{\infty}$ covers $\bigcup A_j$

$$u^*\left(\bigcup A_j\right) \leq \sum_j \sum_k u(E_k^j) \leq \sum_j u^*(A_j) + \varepsilon \sum_j 2^{-j-1}$$

□

Definition 5.1.4 (Abstract Outer Measure). A set function $\varrho : \mathfrak{P}(X) \rightarrow [0, \infty]$ is an abstract outer measure if

- (i) $\varrho(\emptyset) = 0$
- (ii) Monotonicity: $A \subseteq B \implies \varrho(A) \leq \varrho(B)$
- (iii) Subadditivity: $\varrho\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \varrho(E_j)$

§5.2 Caratheodory's Construction

Definition 5.2.1. Given ϱ abstract outer measure, a set A is Caratheodory measure or ϱ -measurable (in the sence of) if **for all** $E \in \mathfrak{P}(X)$,

$$\varrho(E) = \varrho(E \cap A) + \varrho(E \cap A^c)$$

Remark 5.2.2. It is trivial that

$$\varrho(E) \leq \varrho(E \cap A) + \varrho(E \cap A^c)$$

By subaddictivity, so we only need

$$\varrho(E) \geq \varrho(E \cap A) + \varrho(E \cap A^c)$$

to show the equality of the outer measure.

Theorem 5.2.3 (Caratheodory's Theorem)

The collection of ϱ -measurable sets is a σ -algebra \mathcal{M} ,

$$\mathcal{M} = \{A \subseteq X : \forall E \subseteq X, \varrho(E) = \varrho(E \cap A) + \varrho(E \cap A^c)\}$$

$\varrho|_{\mathcal{M}}$ is a measure (in fact, a complete measure)

Proof. First, I will show that \mathcal{M} is an algebra.

- (i) $\varrho(E) = \varrho(E \cap X) + \varrho(E \cap X^c)$, so $X \in \mathcal{M}$.
- (ii) If $A \in \mathcal{M}$, then for any $E \in \mathfrak{P}(X)$,

$$\begin{aligned} \varrho(E) &= \varrho(E \cap A) + \varrho(E \cap A^c) \\ &= \varrho(E \cap A^c) + \varrho(E \cap (A^c)^c) \end{aligned}$$

so $A^c \in \mathcal{M}$.

(iii) For $A, B \in \mathcal{M}$, it is enough to show that $A \cup B \in \mathcal{M}$. For any $E \in \mathfrak{P}(X)$,

$$\begin{aligned}\varrho(E) &= \varrho(E \cap A) + \varrho(E \cap A^c) \\ &= \varrho(E \cap A \cap B) + \varrho(E \cap A \cap B^c) + \varrho(E \cap A^c \cap B) + \varrho(E \cap A^c \cap B^c)\end{aligned}$$

We know that $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, so from the subadditivity of ϱ , we get that

$$\varrho(E \cap (A \cup B)) \leq \varrho(E \cap (A \cap B)) + \varrho(E \cap (A \cap B^c)) + \varrho(E \cap (A^c \cap B))$$

hence,

$$\begin{aligned}\varrho(E) &\geq \varrho(E \cap (A \cup B)) + \varrho(E \cap A^c \cap B^c) \\ &= \varrho(E \cap (A \cup B)) + \varrho(E \cap (A \cup B)^c)\end{aligned}$$

Now, we can conclude that \mathcal{M} is an algebra and we know that for any $A, B \in \mathcal{M}$, if $A \cap B = \emptyset$ then

$$\varrho(A \uplus B) = \varrho((A \uplus B) \cap A) + \varrho((A \uplus B) \cap A^c) = \varrho(A) + \varrho(B)$$

so ϱ is finite additive on \mathcal{M} .

To show that \mathcal{M} is a σ -algebra, it is enough to show that \mathcal{M} is closed under countable disjoint unions. Considering $\{A_j\}_{j=1}^\infty$ pairwise disjoint, $A_j \in \mathcal{M}$, define $B_n = \biguplus_{j=1}^n A_j$ and $B = \biguplus_{j=1}^\infty A_j$. For any $E \in \mathfrak{P}(X)$,

$$\varrho(E \cap B_n) = \varrho(E \cap B_n \cap A_n) + \varrho(E \cap B_n \cap A_n^c) = \varrho(E \cap A_n) + \varrho(E \cap B_{n-1})$$

then from iteration, we get that

$$\varrho(E \cap B_n) = \sum_{j=1}^n \varrho(E \cap A_j)$$

and

$$\varrho(E) = \varrho(E \cap B_n) + \varrho(E \cap B_n^c) \geq \sum_{j=1}^n \varrho(E \cap A_j) + \varrho(E \cap B^c)$$

then $n \rightarrow \infty$ gives that

$$\begin{aligned}\varrho(E) &\geq \sum_{j=1}^\infty \varrho(E \cap A_j) + \varrho(E \cap B^c) \geq \varrho\left(\biguplus_{j=1}^\infty E \cap A_j\right) + \varrho(E \cap B^c) \\ &= \varrho(E \cap B) + \varrho(E \cap B^c) \geq \varrho(E)\end{aligned}$$

Therefore, $B \in \mathcal{M}$ and finally we get that

$$\varrho\left(\biguplus_{j=1}^\infty A_j\right) = \sum_{j=1}^\infty \varrho(A_j)$$

Now, we can conclude that $\mu := \varrho|_{\mathcal{M}}$ is a measure on \mathcal{M} . then we need to show that μ is complete. Let $A \in \mathcal{M}$ such that $\mu(A) = 0$. For any $E \in \mathfrak{P}(X)$, we have that

$$\varrho(E) \leq \varrho(E \cap A) + \varrho(E \cap A^c) = \varrho(E \cap A^c) \leq \varrho(E)$$

Therefore, μ is complete. \square

Definition 5.2.4. ϱ is complete if for $A \in \mathcal{M}$, if $\varrho(A) = 0$ then $\varrho(N) = 0$ for $N \subseteq A$.

Example 5.2.5

For any X there exists a trivial σ -algebra $\mathcal{M} = \{\emptyset, X\}$ satisfies ϱ -measurable sets. Fix $c > 0$, define

$$\varrho(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ c & \text{if } E \neq \emptyset \end{cases}$$

§5.3 Rings and semirings

“Intervals” or “I-cells” half open intervals of the form $(a, b]$

$$(a, b] \uplus (b, c] = (a, c]$$

n-dimensional analogy n-cells

$$(a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n]$$

Definition 5.3.1 (Semiring). Given X , a collection of subsets \mathcal{S} of X is a semiring if

- (i) $\emptyset \in \mathcal{S}$
- (ii) $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$
- (iii) $A, B \in \mathcal{S} \implies A \setminus B = \biguplus_{j=1}^n C_j$ where $C_j \in \mathcal{S}$

Example 5.3.2

$\mathcal{S} = \{(a, b] \mid a, b \in \mathbb{R}\}$ is a semiring.

Definition 5.3.3 (Ring). A collection of subsets \mathcal{R} is a ring if

- (i) $\emptyset \in \mathcal{R}$
- (ii) $A, B \in \mathcal{R} \implies A \setminus B \in \mathcal{R}$
- (iii) $A, B \in \mathcal{R} \implies A \cup B \in \mathcal{R}$

Remark 5.3.4. $A \cap B = A \setminus (A \setminus B)$, $A \triangle B = (A \setminus B) \cup (B \setminus A)$

The ring is equivalent to for $A, B \in \mathcal{R}$

(i) $\emptyset \in \mathcal{R}$

(ii) $A \cap B \in \mathcal{R}$

(iii) $A \triangle B \in \mathcal{R}$

where \cap is multiplication and \triangle is addition.

Finite \implies unions of disjoint sets in \mathcal{R} are in \mathcal{R} .

$$A \cup B = (A \cap B) \uplus (A \triangle B)$$

$$(A \triangle B) \cap A = (A \setminus B \cup B \setminus A) \cap A = A \setminus B$$

Definition 5.3.5. \mathcal{E} a collection. $\mathfrak{R}(\mathcal{E})$ is the smallest ring that contains the collection.

$$\mathfrak{R}(\mathcal{E}) = \bigcap_{\substack{\mathcal{E} \subseteq \mathcal{R} \\ \mathcal{R} \text{ ring}}} \mathcal{R}$$

Lemma 5.3.6

Suppose that \mathcal{E} any collection and \mathcal{R} is a ring. If $\mathcal{E} \subseteq \mathcal{R}$ then $\mathfrak{R}(\mathcal{E}) \subseteq \mathcal{R}$

Proof. Obvious □

Theorem 5.3.7

Let \mathcal{S} be a semiring (of subsets of X). Then $\mathfrak{R}(\mathcal{S})$ is a ring generated by \mathcal{S} , is the collection of finite disjoint unions of sets in \mathcal{S} .

Proof. GOAL: $A \setminus B \in \mathfrak{R}(\mathcal{S})$.

$$\bigsqcup A_j \setminus \bigsqcup B_k = \bigsqcup_j (A_j \setminus \bigsqcup B_k)$$

Need to check that for each j , $(A_j \setminus \bigsqcup B_k)$ is a disjoint union of \mathcal{S} . Take $A \in \mathcal{S}$, $B_1, \dots, B_n \in \mathcal{S}$, B_i are disjoint

Claim _{n} : $A \setminus \bigsqcup B_k \in$ disjoint union of sets in \mathcal{S} . By induction $n = 1$, by definition (iii)

then $\text{Claim}_{n-1} \implies \text{Claim}_n$. Assume

$$\begin{aligned} A \setminus \biguplus_{k=1}^{n-1} B_k &= \biguplus_{l=1}^M C_l \\ A \setminus \left(\biguplus_{k=1}^{n-1} B_k \right) \setminus B_n &= \bigcup_{l=1}^M (C_l \setminus B_n) \\ A \setminus \left(\biguplus_{k=1}^n B_k \right) &= \biguplus_{l=1}^M \biguplus_{j=1}^{M(l,n)} C_{l,n,j} \end{aligned}$$

The hard part is to show that $\mathcal{R}(\mathcal{S})$ is a ring. $A = \biguplus A_j, B = \biguplus B_k$ then

$$A \cup B = \biguplus_{j,k} \underbrace{(A_j \cap B_k)}_{\in \mathcal{S}}$$

□

Lemma 5.3.8

If $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n$ are semirings, then the collection $(A_1 \times A_2 \times \dots \times A_n), A_i \in \mathcal{S}_i$ form a semiring

Proof. By induction, it suffices to check this for $n = 2$.

- $\emptyset \times \emptyset \in \mathcal{S}_1 \times \mathcal{S}_2$
- $A_1 \times A_2 \cap B_1 \times B_2 = (A_1 \cap B_1) \times (A_2 \cap B_2) \in \mathcal{S}_1 \times \mathcal{S}_2$
-

$$\begin{aligned} (A_1 \times A_2) \setminus (B_1 \times B_2) &= (A_1 \setminus B_1) \times A_2 \uplus (A_1 \cap B_1) \times (A_2 \setminus B_2) \\ &= \biguplus_{k=1}^{M_1} (C_{1,k} \times A_2) \uplus \biguplus_{l=1}^{M_2} (A_1 \cap B_1) \times (C_{2,l}) \end{aligned}$$

□

§5.4 Contents, premeasures, and their extensions

Definition 5.4.1. A **content** on a semiring \mathcal{S} (ring) is $\varrho : \mathcal{S} \rightarrow [0, \infty]$

- (i) $\varrho(\emptyset) = 0$
- (ii) If $\{A_k\}_{k=1}^N$ disjoint, $A_k \in \mathcal{S}$ and if $\biguplus_{k=1}^N A_k \in \mathcal{S}$ then $\varrho\left(\biguplus_{k=1}^N A_k\right) = \sum_{k=1}^N \varrho(A_k)$

Remark 5.4.2. In a ring \mathcal{R} same definition but we have $\biguplus_{k=1}^N A_j \in \mathcal{R}$ if $A_j \in \mathcal{R}$

Definition 5.4.3. A **premeasure** on a semiring (ring) is $\nu : \mathcal{S} \rightarrow [0, \infty]$ such that

- (i) $\nu(\emptyset) = 0$
- (ii) If $\{A_k\}_{k=1}^\infty$ disjoint, $A_k \in \mathcal{S}$ and if $\biguplus_{k=1}^N A_k \in \mathcal{S}$ then $\nu\left(\biguplus_{k=1}^N A_k\right) = \sum_{k=1}^N \nu(A_k)$

Example 5.4.4

S_1 = intervals of the form $(a, b]$ for $a, b \in \mathbb{R}$ $\varrho((a, b]) = b - a$ is a premeasure on S . First check ϱ is a content on S_1 . $(a, b] = \bigcup_{j=1}^M (a_j, a_{j+1}]$ ordered $a_1 < a_2 < \dots < a_{M+1} = b$
 $\sum a_{j+1} - a_j = b - a$

Theorem 5.4.5

A content on a semiring extends (uniquely) to the $\mathfrak{R}(\mathcal{S})$.

$$\varrho\left(\biguplus_{j=1}^N A_j\right) = \sum_{j=1}^N \varrho(A_j)$$

Proof. Have to check well-defined. Let $A \in \mathfrak{R}(\mathcal{S})$

$$A = \biguplus_{j=1}^{M_1} A_j = \biguplus_{k=1}^{M_2} B_k$$

Want to show that

$$\sum_{j=1}^{M_1} \varrho(A_j) = \sum_{k=1}^{M_2} \varrho(B_k)$$

$$\begin{aligned}
A_j &= \bigcup_{k=1}^{M_2} (A_j \cap B_k) \\
B_k &= \bigcup_{j=1}^{M_1} (A_j \cap B_k) \\
\sum_{j=1}^{M_1} \varrho(A_j) &= \sum_{j=1}^{M_2} \sum_{k=1}^{M_2} \varrho(A_j \cap B_k) \\
\sum_{k=1}^{M_2} \varrho(B_k) &= \sum_{k=1}^{M_2} \sum_{j=1}^{M_1} \varrho(A_j \cap B_k)
\end{aligned}$$

□

Claim 5.4.6 — Half open interval length is a premeasure.

Proof. Preliminary consideration: If $(a_j, b_j], (a, b] \subseteq \bigcup (a_j, b_j]$ then $b - a \leq \sum (b_j - a_j)$

Now $(a, b] = \bigcup_{j=1}^{\infty} (a_j, b_j]$, $a_j < b_j$ then

Claim:

$$\sum_{j=1}^N (b_j - a_j) \leq b - a$$

for all N . From the monotonicity property of a content.

Now show

$$\sum (b_j - a_j) \geq b - a - C\varepsilon$$

$[a + \varepsilon, b]$ this is covered by $(a_j, b_j]$ and in fact by the open set $(a_j, b_j + \varepsilon 2^{-j-1})$ There is a finite subcover of $[a + \varepsilon, b]$ by $(a_{j_i}, b_{j_i} + \varepsilon 2^{-j_i-1})$ for $i = 1, \dots, M$

$$\sum_{i=1}^M (b_{j_i} + \varepsilon 2^{-j_i-1} - a_{j_i}) \leq \sum_j (b_j - a_j) + \sum_{j=1}^{\infty} \varepsilon 2^{-j-1}$$

□

§5.5 Extend premeasures to measure on a σ -algebra

Idea: use the outer measure ν^*

$$\nu^*(E) = \inf \sum_{A_j \in \mathcal{S}, \bigcup A_j \supseteq E} \nu(A_j)$$

use the Caratheodory construction to produce a measure ν^* on some σ -algebra \mathcal{M}^* .

$\nu^{**}(E)$ = same except S is replaced by $\mathcal{R}(S)$

Claim 5.5.1 — $\nu^{**}(E) = \nu^*(E)$ if ν is a content.

Proof. Show \leq , WLOG $\nu^{**}(E) < \infty$. Given $\varepsilon > 0$ fin $A_j \in \mathcal{R}(S)$ such that $\sum \nu(A_j) < \nu^{**}(A) + \varepsilon$

$$A_j = \biguplus_{k=1}^{M(j)} C_{j,k}, \nu(A) = \sum_{n=1}^{M(j)} \nu(C_{j,k})$$

then

$$\sum_j \sum_{k=1}^{M(j)} \nu(C_{j,k}) < \nu^{**}(A) + \varepsilon \implies \nu^*(A) \leq \nu^{**}(A) + \varepsilon$$

□

Theorem 5.5.2 (Hahn-Kolmogorov-Caratheodory-Frechet) (i) If ν is a content on S (semiring) over X , (and therefore on $\mathcal{R}(S)$). Then the Caratheodory σ -algebra \mathcal{M}^* of ν^* -measurable set contains S and $\mathcal{R}(S)$ and $\mathfrak{M}(S) \subseteq \overline{\mathfrak{M}}$

(ii) If ν is a premeasure then $\nu^*|_S = \nu$ then ν^* is a measure (on $\mathcal{M}^* \supseteq \overline{\mathfrak{M}(S)}$)

ν^* -measurable if for all $E \subseteq X$, $\nu^*(E) \geq \nu^*(E \cap A) + \nu^*(E \cap A^c)$ (for another direction is obvious, so, we need to only check one direction)

Proof. Show that $S \subseteq \mathcal{M}^*$. Let $A \subseteq S$. Fix $E \subseteq X$, work with an ε -efficient cover, i.e., find a collection $\{A_j\}_{j=1}^\infty$ such that $A_j \in S, \bigcup A_j \supseteq E : \sum \nu(A_j) \leq \nu^*(E) + \varepsilon$ We get that $A_j = (A_j \cap A) \uplus (A_j \cap A^c)$ $\nu(A_j) = \nu(A_j \cap A) + \nu(A_j \cap A^c)$

$$\nu^*(E) + \varepsilon \geq \sum \nu(A_j) = \sum \nu(A_j \cap A) + \sum \nu(A_j \cap A^c)$$

We know that $E \cap A \subseteq \bigcup (A_j \cap A)$ and $E \cap A^c \subseteq \bigcup (A_j \cap A^c)$ then

$$\nu^*(E) + \varepsilon \geq \nu^*(E \cap A) + \nu^{**}(E \cap A^c) = \nu^*(E \cap A) + \nu^*(E \cap A^c)$$

For $A \in S$ show $\nu^*(A) = \nu(A)$.

It is easy to show that $\nu^*(A) \leq \nu(A)$ since $\{A\}$ is a cover of A .

We need $\nu(A) \leq \nu^*(A) + \varepsilon$ (for all $\varepsilon > 0$)

Pick $\{A_j\}$, $\bigcup A_j \supseteq A$, $\sum \nu(A_j) < \nu^*(A) + \varepsilon = \nu^{**}(A) + \varepsilon$ Define $B_1 = A_1, B_2 = A_2 \setminus A_1, B_3 = A_3 \setminus (A_1 \cup A_2), \dots$ B_j is disjoint and $B_n \in \mathcal{R}(S)$ (but might not in the semiring) We claim that premeasure assumption We know that $\bigcup_n (A \cap B_n) = A$

$$\nu(A) = \sum_n \nu(A \cap B_n) \leq \sum_n \nu(A_n) \leq \nu^{**}(E) + \varepsilon$$

□

Example 5.5.3

$E \subseteq \mathbb{N}$

$$\varrho(E) = \limsup_{n \rightarrow \infty} \frac{|E \cap [1, n]|}{n}$$

If E is finite $\varrho(E) = 0$, if $E = \mathbb{N}$ then $\varrho(E) = 1$ But

$$\varrho(\mathbb{N}) \not\leq \sum_{n=1}^{\infty} \varrho(\{n\})$$

So, Subadditivity fails, not a premeasure

Extension of a premeasure ν on a (semi) ring \mathcal{S} to a measure μ on the σ -algebra generated by \mathcal{S} .

1. If $\tilde{\mu}$ is an extension of ν and $\mu = \nu^*|_{\mathcal{M}}$ then $\tilde{\mu}(E) \leq \mu(E)$.

If $E \in \mathcal{M}$ is such $\mu(E) < \infty \implies \mu(E) = \tilde{\mu}(E)$.

If ν is a σ -finite premeasure on \mathcal{S} then $\mu = \tilde{\mu}$ $X = \bigcup_{j=1}^{\infty} X_j, X_j \in \mathcal{S}, \nu(X_j) < \infty$.

Proof. 1. WLOG $\mu(E) < \infty$, $\mu(E) = \nu^*(E) = \inf_{A_K \in \mathcal{S}, \bigcup A_K \supseteq E} \sum_{k=1}^{\infty} \nu(A_k)$ or ∞ Given $\varepsilon > 0$ we find a $\{A_k\}$ of set in \mathcal{S}

$$\nu^*(E) = \sum_{k=1}^{\infty} \nu(A_k) < \nu^*(E) + \varepsilon$$

replace A_k by $A_k \setminus (A_1 \cup \dots \cup A_{k-1})$ to get a disjoint collection (May assume that the A_k are disjoint)

$$\tilde{\mu}(E) = \tilde{\mu}\left(\biguplus A_k\right) = \sum \tilde{\mu}(A_k) = \sum \nu(A_k) \leq \nu^*(E) + \varepsilon$$

□

On complete union of set $\mathcal{R}(\mathcal{S})$, μ and $\tilde{\mu}$ coincide.

Take $E \in \mathcal{M}$, $\mu(E) = \nu^*(E) < \infty$, show that $\tilde{\mu}(E) = \mu(E)$

Find a countable cover of E with set $B_k \in \mathcal{R}(\mathcal{S})$ such that

$$\sum \nu(B_k) < \nu^*(E) + \varepsilon, B = \bigcup_{k=1}^{\infty} B_k, \mu(B) = \tilde{\mu}(B)$$

$$\mu(E) \leq \mu(B) = \tilde{\mu}(B) = \tilde{\mu}(E) + \tilde{\mu}(B \setminus E) \leq \tilde{\mu}(E) + \mu(B \setminus E)$$

Theorem 5.5.4 (i)

$$\nu^*|_{\mathfrak{M}}(E) = \begin{cases} \bar{\mu}(E) & \text{if } E \in \overline{\mathcal{M}} \\ \infty & \text{if } E \in \mathfrak{M}^* \setminus \overline{\mathcal{M}} \end{cases}$$

(ii) $E \in \mathfrak{M}^*$ if and only if E is locally $\overline{\mathcal{M}}$ -measurable. i.e. for every set $A \in \overline{\mathcal{M}}$ such $\bar{\mu}(A) < \infty$, we have $E \cap A \in \overline{\mathcal{M}}$ (If ν is σ -finite $\overline{\mathcal{M}} = \mathfrak{M}^*$)

6 Product measure

§6.1 Introduction

Given two (or finitely many) measure space $(X_1, \mathcal{M}_1, \mu_1), (X_2, \mathcal{M}_2, \mu_2)$ $X = X_1 \times X_2$, On X there is a natural σ -algebra $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2 =:$ σ -algebra generated by set of the form. $E_1 \times E_2, E_1 \in \mathcal{M}_1, E_2 \in \mathcal{M}_2$ and $\mu(E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2)$

Extend the set function μ , defined on the general rectangles to all of $\mathcal{M}_1 \otimes \mathcal{M}_2$.

Claim 6.1.1 — The rectangles form a semiring (use cartesian product $\mathcal{M}_1 \times \mathcal{M}_2$ of semirings are semiring)

Theorem 6.1.2

Given $(X_1, \mathcal{M}_1, \mu_1), (X_2, \mathcal{M}_2, \mu_2)$, σ -finite measure spaces, there exists a unique measure μ on $\mathcal{M}_1 \otimes \mathcal{M}_2$ such that $\mu(E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2)$ for all $E_1 \in \mathcal{M}_1, E_2 \in \mathcal{M}_2$

Proof. $\mathcal{S}_{\text{rect}} = \mathcal{M}_1 \times \mathcal{M}_2$ semiring of rectangles. Need to show that $\nu : \mathcal{S}_{\text{rect}} \rightarrow [0, \infty)$ is a premeasure verify σ -additivity on $\mathcal{S}_{\text{rect}}$ If

$$E_1 \times E_2 = \biguplus_{j=1}^{\infty} E_{j,1} \times E_{j,2}$$

then

$$\nu(E_1 \times E_2) = \sum_{j=1}^{\infty} \nu(E_{j,1} \times E_{j,2})$$

$$\mathbb{1}_{E_1 \times E_2}(x_1, x_2) = \sum_{j=1}^{\infty} \mathbb{1}_{E_{j,1} \times E_{j,2}}(x_1, x_2) \iff \mathbb{1}_{E_1}(x_1)\mathbb{1}_{E_2}(x_2) = \sum_{j=1}^{\infty} \mathbb{1}_{E_{j,1}}(x_1)\mathbb{1}_{E_{j,2}}(x_2)$$

$$\begin{aligned}
\mathbb{1}_{E_1}(x_1)\mathbb{1}_{E_2}(x_2) &= \sum_{j=1}^{\infty} \mathbb{1}_{E_{j,1}}(x_1)\mathbb{1}_{E_{j,2}}(x_2) \\
\int \mathbb{1}_{E_1}(x_1)\mathbb{1}_{E_2}(x_2) \, d\mu_2 &= \int \sum_{j=1}^{\infty} \mathbb{1}_{E_{j,1}}(x_1)\mathbb{1}_{E_{j,2}}(x_2) \, d\mu_2 \\
\mathbb{1}_{E_1}(x_1)\mu_2(E_2) &= \sum_{j=1}^{\infty} \int \mathbb{1}_{E_{j,1}}(x_1)\mathbb{1}_{E_{j,2}}(x_2) \, d\mu_2 \\
&= \sum_{j=1}^{\infty} \mathbb{1}_{E_{j,1}}(x_1)\mu_2(E_{j,2}) \\
\int \mathbb{1}_{E_1}(x_1)\mu_2(E_2) \, d\mu_1 &= \int \sum_{j=1}^{\infty} \mathbb{1}_{E_{j,1}}(x_1)\mu_2(E_{j,2}) \, d\mu_1 \\
\mu_1(E_1)\mu_2(E_2) &= \sum_{j=1}^{\infty} \mu_1(E_{j,1})\mu_2(E_{j,2}) \\
\nu(E_1 \times E_2) &= \sum_{j=1}^{\infty} \nu(E_{j,1} \times E_{j,2})
\end{aligned}$$

□

Lemma 6.1.3

If μ_1 is σ -finite and μ_2 is σ -finite, then ν is σ -finite.

Proof. $X_1 = \biguplus_{j=1}^{\infty} X_{1,j}$, $X_2 = \biguplus_{k=1}^{\infty} X_{2,k}$, $\mu_1(X_{1,j}) < \infty$, $\mu_2(X_{2,k}) < \infty$, $X_1 \times X_2 = \biguplus_{j,k} X_{1,j} \times X_{2,k}$, $\nu(X_{1,j} \times X_{2,k}) = \mu_1(X_{1,j})\mu_2(X_{2,k}) < \infty$ □

Remark 6.1.4. In Summary, we have proved that the product pre-measure extends uniquely to $\mathcal{M}_1 \otimes \mathcal{M}_2$. (under the assumption of σ -finiteness of μ_1, μ_2)

Remark 6.1.5. Constructed a product measure on $\mathcal{M}_1 \otimes \mathcal{M}_2$ by $\mu(E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2)$ (this is unique if $\mathcal{M}_1, \mathcal{M}_2$ are σ -finite)

Lemma 6.1.6

Let $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$, Define

$$sl_{x_1}(E) = \{x_2 : (x_1, x_2) \in E\}$$

$$sl^{x_2}(E) = \{x_1 : (x_1, x_2) \in E\}$$

Then $sl_{x_1}(E) \in \mathcal{M}_2$, $sl^{x_2}(E) \in \mathcal{M}_1$

Proof. Considering the rectangle case, this is “easy”

$$sl_{x_1}(E_1 \times E_2) = \{x_2 : (x_1, x_2) \in E_1 \times E_2\} = \begin{cases} \emptyset, & x_1 \notin E_1 \\ E_2, & x_1 \in E_1 \end{cases}$$

Define

$$\mathfrak{C}_{x_1} = \{E \in \mathfrak{P}(X_1 \times X_2) : sl_{x_1}(E) \in \mathcal{M}_2\}$$

we know that $\mathcal{S}_{\text{rect}} \subseteq \mathfrak{C}_{x_1}$

Check \mathfrak{C}_{x_1} is a σ -algebra (If so, \mathfrak{C}_{x_1} contains $\mathfrak{M}(\mathcal{S}_{\text{rect}}) = \mathcal{M}_1 \otimes \mathcal{M}_2$)

(i) $sl_{x_1}(X_1 \times X_2) = X_2 \in \mathcal{M}_2$, so, $X_1 \times X_2 \in \mathfrak{C}_{x_1}$

(ii) Let $E \in \mathfrak{C}_{x_1}$, we know that $sl_{x_1}(E) \in \mathcal{M}_2$, consider $E^c = X \setminus E$

$$sl_{x_1}(E^c) = \{x_2 : (x_1, x_2) \notin E\} = \{x_2 : x_2 \notin sl_{x_1}(E)\} = (sl_{x_1}(E))^c \in \mathcal{M}_2$$

(iii) For $E_j \in \mathfrak{C}_{x_1}$, we know that $sl_{x_1}(E_j) \in \mathcal{M}_2$, then

$$sl_{x_1}\left(\bigcup_{j=1}^{\infty} E_j\right) = \bigcup_{j=1}^{\infty} sl_{x_1}(E_j) \in \mathcal{M}_2$$

So, \mathfrak{C}_{x_1} is a σ -algebra, and \mathfrak{C}_{x_1} contains $\mathcal{S}_{\text{rect}}$, so \mathfrak{C}_{x_1} contains $\mathcal{M}_1 \otimes \mathcal{M}_2$. Hence, for any $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$, $sl_{x_1}(E) \in \mathcal{M}_2$. \square

Lemma 6.1.7

Let f be a $(\mathcal{M}_1 \otimes \mathcal{M}_2)$ -measurable. Then $f_{x_1}(x_2) = f(x_1, x_2)$ is an \mathcal{M}_2 -measurable, and $f^{x_2}(x_1) = f(x_1, x_2)$ is an \mathcal{M}_1 -measurable

Proof. Check: $f_{x_1}^{-1}(I) \in \mathcal{M}_2$, for any Borel set I

$$\begin{aligned} f_{x_1}^{-1}(I) &= \{x_2 : f_{x_1}(x_2) = f(x_1, x_2) \in I\} \\ &= sl_{x_1}(\{(x_1, x_2) : f(x_1, x_2) \in I\}) \\ &= sl_{x_1}(\underbrace{f^{-1}(I)}_{\in \mathcal{M}_1 \otimes \mathcal{M}_2}) \end{aligned}$$

□

§6.2 Extension of Measure on Integral

Theorem 6.2.1 (Cavalieri's Principle)

Let $(X_1, \mathcal{M}_1, \mu_1), (X_2, \mathcal{M}_2, \mu_2)$ be σ -finite and let μ be a product measure of two measure space. For any $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$, then

(i) $g_E : x_1 \mapsto \mu_2(sl_{x_1}(E))$ is \mathcal{M}_1 -measurable

(ii) $h_E : x_2 \mapsto \mu_1(sl^{x_2}(E))$ is \mathcal{M}_2 -measurable

and

(i) $\mu(E) = \int g_E \, d\mu_1 = \int \mu_2(sl_{x_1}(E)) \, d\mu_1$

(ii) $\mu(E) = \int h_E \, d\mu_2 = \int \mu_1(sl^{x_2}(E)) \, d\mu_2$

Proof. First, consider the case that $\mu_1(X_1) < \infty$ and $\mu_2(X_2) < \infty$. Similarly, considering the rectangle case, for any $E \in \mathcal{S}_{\text{rect}}$,

$$g_E(x_1) = \begin{cases} \mu_2(E_2) & \text{if } x_1 \in E_1 \\ 0 & \text{if } x_1 \notin E_1 \end{cases}$$

So, g_E is \mathcal{M}_1 -measurable. Moreover,

$$\int g_E \, d\mu_1 = \int_{E_1} \mu_2(E_2) \, d\mu_1 = \mu_1(E_1)\mu_2(E_2) = \mu(E)$$

proved the theorem for the rectangle case.

Next, define

$$\mathfrak{C} = \left\{ E \in \mathfrak{P}(X) : g_E \text{ is } \mathcal{M}_1\text{-measurable, } \mu(E) = \int g_E \, d\mu_1 \right\}$$

I will show that \mathfrak{C} is a Dynkin system. For the case \mathcal{M}_1 -measurable,

(i) $g_X(x_1) = \mu_2(sl_{x_1}(X)) = \mu_2(X_2)$ is a constant function, so it is \mathcal{M}_1 -measurable.

(ii) For $E \in \mathfrak{C}$, $g_{E^c}(x_1) = \mu_2(sl_{x_1}(E^c)) = \mu_2(X_2 \setminus sl_{x_1}(E)) = \mu_2(X_2) - \mu_2(sl_{x_1}(E))$ is \mathcal{M}_1 -measurable.

(iii) For $E_j \in \mathfrak{C}$, where E_j are disjoint,

$$g_{\biguplus_{j=1}^{\infty} E_j}(x_1) = \mu_2 \left(sl_{x_1} \left(\biguplus_{j=1}^{\infty} E_j \right) \right) = \sum_{j=1}^{\infty} \mu_2(sl_{x_1}(E_j))$$

is \mathcal{M}_1 -measurable.

For the case of $\mu(E) = \int g_E \, d\mu_1$,

(i)

$$\int g_X \, d\mu_1 = \int \mu_2(sl_{x_1}(X)) \, d\mu_1 = \mu_1(X_1)\mu_2(X_2) = \mu(X)$$

(ii) For $E \in \mathfrak{C}$,

$$\begin{aligned} \int g_{E^c} \, d\mu &= \int \mu_2(sl_{x_1}(E^c)) \, d\mu = \int \mu_2(X_2) - \mu_2(sl_{x_1}(E)) \, d\mu \\ &= \mu(X) - \mu(E) = \mu(E^c) \end{aligned}$$

(iii) For $E_j \in \mathfrak{C}$, where E_j are disjoint,

$$\begin{aligned} \int g_{\biguplus_{j=1}^{\infty} E_j} \, d\mu &= \int \sum_{j=1}^{\infty} \mu_2(sl_{x_1}(E_j)) \, d\mu = \sum_{j=1}^{\infty} \int \mu_2(sl_{x_1}(E_j)) \, d\mu \\ &= \sum_{j=1}^{\infty} \mu(E_j) = \mu \left(\biguplus_{j=1}^{\infty} E_j \right) \end{aligned}$$

We have a theorem stating that if \mathcal{E} is \cap -stable then $\mathcal{D}(\mathcal{E}) = \mathfrak{M}(\mathcal{E})$. I will show that \mathfrak{C} is a Dynkin system containing $\mathcal{S}_{\text{rect}}$ and $\mathfrak{M}(\mathcal{S}_{\text{rect}}) \subseteq \mathfrak{C}$. Since the rectangle cases are \mathcal{M}_1 -measurable, \mathfrak{C} contains $\mathcal{S}_{\text{rect}}$. Moreover, since $\mathcal{S}_{\text{rect}}$ is semiring, so it is \cap -stable, and from the definition of Dynkin system, $\mathfrak{M}(\mathcal{S}_{\text{rect}}) = \mathcal{D}(\mathcal{S}_{\text{rect}}) \subseteq \mathfrak{C}$. Hence, for any $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$, g_E is \mathcal{M}_1 -measurable and $\mu(E) = \int g_E \, d\mu_1$.

□

Theorem 6.2.2 (Tonelli's Theorem)

Let $(X_i, \mathcal{M}_i, \mu_i), i = 1, 2$ σ -finite measure spaces, $\mu = \mu_1 \times \mu_2$ be a product measure on $\mathcal{M}_1 \otimes \mathcal{M}_2$. Let $f \in \mathcal{L}^+(X, \mu)$ then

- $x_1 \mapsto \int f_{x_1} d\mu_2$ belongs to $\mathcal{L}^+(X_1, \mu_1)$
- $x_2 \mapsto \int f^{x_2} d\mu_1$ belongs to $\mathcal{L}^+(X_2, \mu_2)$

and

$$\int f d\mu = \int \int f_{x_1} d\mu_2 d\mu_1$$

Proof. We have proved this theorem for simple functions / indicator functions of measurable sets. Let $\{s_n\}_{n=1}^\infty$ be a sequence of simple functions such that $s_n(x) \nearrow f(x)$ for all x . define

$$g_n(x_1) = \int s_n(x_1, x_2) d\mu_2, \quad h_n(x_2) = \int s_n(x_1, x_2) d\mu_1$$

$$g(x_1) = \int f(x_1, x_2) d\mu_2, \quad h(x_2) = \int f(x_1, x_2) d\mu_1$$

By Monotone Convergence Theorem, we have $g_n \nearrow g$, $h_n \nearrow h$ and limit of measurable functions is measurable. By the fact that $g_n \nearrow g$,

$$\lim_{n \rightarrow \infty} \int g_n d\mu_1 = \int g d\mu_1 = \int \int f(x_1, x_2) d\mu_2 d\mu_1$$

$$\lim_{n \rightarrow \infty} \int g_n d\mu_1 = \lim_{n \rightarrow \infty} \int \int s_n(x_1, x_2) d\mu_2 d\mu_1 \stackrel{\text{Cavalieri}}{=} \lim_{n \rightarrow \infty} \int s_n d\mu \stackrel{\text{def}}{=} \int f d\mu$$

□

Theorem 6.2.3 (Fubini's Theorem)

$f \in \mathcal{L}^1(X, \mu)$ then $f_{x_1} \in \mathcal{L}^1(X_2, \mu_2)$ for μ_1 almost every x_1 and $f^{x_2} \in \mathcal{L}^1(X_1, \mu_1)$ for μ_2 almost every x_2 and

$$\int f d\mu = \int \int f_{x_1} d\mu_2 d\mu_1 = \int \int f^{x_2} d\mu_1 d\mu_2$$

§6.3 Distribution Function

Definition 6.3.1. Given (X, \mathcal{M}, μ) , given measurable function f , define

$$\alpha \mapsto \mu_f(\alpha) = \mu(\{x : |f| > \alpha\})$$

$\alpha > 0$

Theorem 6.3.2

Given (X, \mathcal{M}, μ) ,

$$\int |f|^p \, d\mu = \int_0^\infty p\alpha^{p-1} \mu_f(\alpha) \, d\alpha$$

Example 6.3.3

example where μ_f shows

$$\mu_f(\alpha) = \frac{\|f\|_p^p}{\alpha^p}$$

Chebyshev's inequality

$$L^{p,\infty} = \text{"weake type p-spaces"} \iff f \text{ measurable} \iff \sup_\alpha \alpha^p \mu_f(\alpha) < \infty$$

Proof. We first assume μ is σ -finite. Considering Right-hand-side

$$\begin{aligned} \int_0^\infty p\alpha^{p-1} \mu_f(\alpha) \, d\alpha &= \int_0^\infty p\alpha^{p-1} \int_{|f(x)| > \alpha} d\mu \, d\alpha \\ &= \int_0^\infty \int_X p\alpha^{p-1} \mathbb{1}_{\{|f(x)| > \alpha\}}(x, \alpha) \, d\mu \, d\alpha \end{aligned}$$

$x \mapsto f(x)$ is measurable as a function on X , and as a function on $X \times [0, \infty)$. define $g : \mathbb{R}^2 \rightarrow \mathbb{R}, (t, \alpha) \mapsto (|t| - \alpha) \mathbb{1}_{\alpha > 0}$ then $g \circ f$ is \mathcal{M} -measurable because $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ then

$$\int_0^\infty p\alpha^{p-1} \mu_f(\alpha) \, d\alpha = \int_X \underbrace{\int_{\alpha=0}^{|f(x)|} p\alpha^{p-1} \, d\alpha}_{|f(x)|^p} \, d\mu$$

This proves the formula for σ -finite X .

For the general cases, let assume that there exists α such that $\mu_f(\alpha) = \infty, \mu(\{x : |f(x)| > \alpha\}) = \infty \implies \mu_f(\beta) = \infty$ for $0 < \beta < \alpha$. In this case, we have $\infty = \infty$

Assume $\mu_f(\alpha) < \infty$ for all $\alpha > 0$

claim: $\{x : |f(x)| > 0\} = \bigcup_n \{x : |f(x)| > \frac{1}{n}\}$, σ -finite because $\mu(\{x : |f(x)| > \frac{1}{n}\}) = \mu_f(\frac{1}{n}) < \infty$ \square

Example 6.3.4

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

equivalent to

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Proof. Let

$$F(x, y) = \begin{cases} ye^{-y^2(1+x^2)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int F dm(x, y) = \int_{y=0}^{\infty} e^{-y^2} \int_{x=0}^{\infty} e^{-y^2 x^2} y dx dy$$

By changing variable, let $w = yx, dw = y dx$ then

$$\int_{y=0}^{\infty} e^{-y^2} \int_{x=0}^{\infty} e^{-y^2 x^2} y dx dy = \int_{y=0}^{\infty} e^{-y^2} dy \int_{w=0}^{\infty} e^{-w^2} dw$$

Considering LHS

$$\int F dm = \int_{x=0}^{\infty} \int_y y \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} e^{-y^2(1+x^2)} dy dx$$

let $w = y\sqrt{1+x^2}, dw = \sqrt{1+x^2} dy$ then

$$\begin{aligned} \int F dm &= \int_0^{\infty} \frac{1}{1+x^2} \int_0^{\infty} \frac{2w}{2} e^{-w^2} dw dx \\ &= \frac{1}{2} \int_0^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{4} \end{aligned}$$

Considering the first integration and the second integration, we get that

$$\begin{aligned} \left(\int_0^{\infty} e^{-s^2} ds \right)^2 &= \frac{\pi}{4} \\ \int_0^{\infty} e^{-s^2} ds &= \frac{\sqrt{\pi}}{2} \end{aligned}$$

□

Example 6.3.5

$$\int_0^\infty e^{-ax} (\sin x)^2 \frac{dx}{x}$$

Hint: To apply Fubini-Tonelli,

$$\iint_{[0,\infty] \times [0,1]} e^{-ax} \sin(2xy) \, dm(x, y)$$

(value $1/4 \ln(1 + 4/a^2)$)

§6.4 Linear Change of Variable**Theorem 6.4.1**

Let $f \in L^1(\mathbb{R}^n)$ or $(L^+(\mathbb{R}^n))$ and let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an invertible $n \times n$ matrix. Then

$$|\det A| \int f(Ax) \, dm(x) = \int f(x) \, dm(x)$$

Moreover, $m(AE) = |\det A| m(E)$ for every Lebesgue measurable set E .

Proof. We first assume that f is Borel measurable, then $x \mapsto f(Ax)$ is Borel measurable. (since $x \mapsto Ax$ is continuous) We consider a number of special cases. In what follows write $\mathbb{R}^n \ni (x_1, x')$ where $x_1 \in \mathbb{R}, x' \in \mathbb{R}^{n-1}$.

Case 1:

$$A = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$\lambda \neq 0$ then $\det A = \lambda$. then

$$\begin{aligned} |\lambda| \int f(Ax) \, dm(x) &= |\lambda| \int_{\mathbb{R}^{d-1}} \left[\int_{\mathbb{R}} f(\lambda x, x') \, dm_1(x_1) \right] dm_{n-1}(x') \\ \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} f(cx, x') \, dm_1(x_1) \right] dm_{n-1}(x') &= \int f(x) \, dm(x) \end{aligned}$$

Case 2:

$$A = \begin{bmatrix} 1 & \lambda & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Then $\det A = 1$

$$\begin{aligned} 1 \int f(Ax) \, dm(x) &= \int_{\mathbb{R}^{d-1}} \left[\int_{\mathbb{R}} f(x_1 + \lambda x_2, x') \, dm_1(x_1) \right] dm_{n-1}(x') \\ &= \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} f(x_1 + \lambda x_2, x') \, dm_1(x_1) \right] dm_{n-1}(x') = \int f(x) \, dm(x) \end{aligned}$$

Case 3: A permutes two coordinates: there exist $j \neq k$ where $A(e_i) = e_i$ if $i \notin \{j, k\}$, $A(e_j) = e_k$, $A(e_k) = e_j$. Then $\det A = -1$.

$$1 \int f(Ax) \, dm(x) = \int f(x) \, dm(x)$$

General case: In general we can write $A = E_1 \dots E_n$ where each E_j is in Case 1, 2, or 3.

$$\begin{aligned} |\det A| \int f(Ax) \, dx &= |\det(E_1 \dots E_{n-1})| |\det E_n| \int f(f \circ E_1 \dots E_{n-1})(E_n x) \, dx \\ &= |\det(E_1 \dots E_{n-1})| \int f(E_1 \dots E_{n-1} x) \, dx \\ &= \dots \int f(x) \, dx \end{aligned}$$

This proves the theorem when f is Borel measurable. In particular, if E is Borel measurable then

$$\begin{aligned} *m(AE) &= \int \mathbb{1}_{AE}(x) \, dm(x) \\ &= |\det A| \int \mathbb{1}_{AE}(Ax) \, dm(x) \\ &= |\det A| \int \mathbb{1}_E(x) \, dm(x) \\ &= |\det A| m(E) \end{aligned}$$

Finally consider the case when f is Lebesgue measurable. Then there is a Borel measurable function g and a Borel measurable set E with $m(E) = 0$ and $f = g$ on $\mathbb{R}^d \setminus E$ (so $f = g$ almost everywhere). Then $f \circ A = g \circ A$ on $\mathbb{R}^d \setminus A^{-1}(E)$. By $*m(A^{-1}E) = 0$ So $f \circ A$ is Lebesgue measurable and $f \circ A = g \circ A$ almost everywhere. So

$$\begin{aligned} |\det A| \int f(Ax) \, dm(x) &= |\det A| \int g(Ax) \, dm(x) \\ &= \int g(x) \, dm(x) = \int f(x) \, dm(x) \end{aligned}$$

Also, arguing exactly as in $*$ we have $m(AE) = |\det A| m(E)$ for every Lebesgue measurable set E . \square

Theorem 6.4.2

Assume $f, g : [a, b] \rightarrow \mathbb{R}$ are Lebesgue integrable. Define

$$F(x) = \int_{[a,x]} f \, dm$$

$$G(x) = \int_{[a,x]} g \, dm$$

then

$$\int_{[a,b]} Fg \, dm = F(b)G(b) - \int_{[a,b]} fG \, dm$$

Remark 6.4.3. if $u, v : [a, b] \rightarrow \mathbb{R}$ are continuously differentiable with $u(a) = v(a) = 0$, then

$$u(x) = \int_{[a,x]} u' \, dm$$

$$v(x) = \int_{[a,x]} v' \, dm$$

So, the above theorem implies

$$\int_{[a,b]} uv' \, dm = u(b)v(b) - \int_{[a,b]} u'v \, dm$$

Proof. Let $T = \{(x, t) : a \leq x \leq b, a \leq t \leq x\} = \{(x, t) : a \leq t \leq b, t \leq x \leq b\}$. Then T is a Lebesgue measurable set in \mathbb{R}^2 and $(x, t) \mapsto f(t)g(x)\mathbb{1}_T(x, t)$ is Lebesgue measurable. Also by Tonelli's Theorem,

$$\begin{aligned} \int |f(t)g(x)\mathbb{1}_T(x, t)| \, dm_2(x, t) &= \int_{[a,b]} |g(x)| \left[\int_{[a,x]} |f(t)| \, dt \right] dx \\ &\leq \int_{[a,b]} |g| \, dm \int_{[a,b]} |f| \, dm < \infty \end{aligned}$$

So, by Fubini's Theorem,

$$\begin{aligned} \int f(t)g(x)\mathbb{1}_T(x, t) \, dm_2(x, t) &= \int_{[a,b]} g(x) \left[\int_{[a,x]} f(t) \, dt \right] dx \\ &= \int_{[a,b]} gF \, dm \end{aligned}$$

and

$$\begin{aligned}
 \int f(t)g(x)\mathbb{1}_T(x,t) \, dm_2(x,t) &= \int_{[a,b]} f(t) \left[\int_{[t,b]} g(x) \, dx \right] dt \\
 &= \int_{[a,b]} f(t) \left[\int_{[a,b]} g(x) \, dx - \int_{[a,t]} g(x) \, dx \right] dt \\
 &= \int_{[a,b]} f(t) \left[\int_{[a,b]} g(x) \, dx \right] dt \\
 &\quad - \int_{[a,b]} f(t) \left[\int_{[a,t]} g(x) \, dx \right] dt \\
 &= F(b)G(a) - \int_{[a,b]} fG \, dm
 \end{aligned}$$

So,

$$\int_{[a,b]} Fg \, dm = F(b)G(b) - \int_{[a,b]} fG \, dm$$

□

§6.5 Change of Variable

We know $A \in GL(n, \mathbb{R})$ A is $n \times n$ matrix, A is invertible $\iff \det A \neq 0$. If $f \in \mathcal{L}^+$ or $f \in \mathcal{L}^1$ then

$$\begin{aligned}
 \int f(x) \, dm(x) &= \int f(Ax) |\det A| \, dm(x) \\
 A\Omega &= \{Ax : x \in \Omega\}
 \end{aligned}$$

The step is to replace linear transformation to non-linear transformation.

$\phi(x) = \phi(x_1, \dots, x_n)$, ϕ , \mathbb{R}^n -valued, $\phi = (\phi_1, \dots, \phi_n)$. we know that ϕ is differentiable on Ω . if $\frac{\partial \phi_j}{\partial x_k}$ exist and are continuous.

$$\phi'(x) = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \dots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \dots & \frac{\partial \phi_n}{\partial x_n} \end{pmatrix}$$

$$\phi(x) = Ax, \phi'(x) = A$$

Definition 6.5.1. Let $\Omega_1 \subseteq \mathbb{R}^n$, $\Omega_2 \subseteq \mathbb{R}^n$. We say that $\phi : \Omega_1 \rightarrow \Omega_2$ is a C^1 -diffeomorphism. if

- (i) ϕ is bijective

- (ii) $\phi \in C^1$ (or $\frac{\partial \phi_j}{\partial x_k}$ exist and are continuous)
- (iii) $\det \phi'(x) \neq 0$ (i.e., $\phi'(x)$ is invertible)

Remark 6.5.2. We say that ϕ_1 and ϕ_2 are diffeomorphically equivalent.

1. $\Omega_1 \sim \Omega_1$, $\phi(x) = x$, $\phi'(x) = I$
2. $\phi : \Omega_1 \rightarrow \Omega_2$, C^1 -diffeomorphism, $\phi^{-1}(x)$ is also a C^1 -diffeomorphism. (use the inverse function theorem which proves the differentiability of ϕ^{-1}) $(\phi^{-1})'(y) = (\phi'(\phi^{-1}(y)))^{-1} = (\phi'(x))^{-1}$ where $x = \phi^{-1}(y)$
- 3.

Example 6.5.3

$\phi : \underbrace{(0, 2) \times (0, \pi/4)}_{\Omega_1} \rightarrow \Omega_2$ defined by $\phi(x_1, x_2) = (x_1 \sin x_2, x_1 \cos x_2)$ $\phi(r, \theta) = (r \sin \theta, r \cos \theta)$

$$\phi'(x) = \begin{pmatrix} \sin x_2 & x_1 \cos x_2 \\ \cos x_2 & -x_1 \sin x_2 \end{pmatrix}$$

$$\det \phi'(x) = -x_1 = -r$$

Theorem 6.5.4

Let Ω_1, Ω_2 are open sets in \mathbb{R}^n , $\phi : \Omega_1 \rightarrow \Omega_2$ is a C_1 -diffeomorphism. $f \in \mathcal{L}^1(\Omega_2)$ or $f \in \mathcal{L}^+(\Omega_2)$ then

$$\int_{\Omega_2} f(y) \, dm(y) = \int_{\Omega_1} f(\phi(x)) |\det \phi'(x)| \, dm(x)$$

Remark 6.5.5. In calculus, we have substitution rule

$$\int_{\phi(a)}^{\phi(b)} f(y) \, dy = \int_a^b f(\phi(x)) \phi'(x) \, dx$$

Proof. It is enough to show that

$$\int_{\Omega_2} f(y) \, dm(y) \leq \int_{\Omega_1} f(\phi(x)) |\det \phi'(x)| \, dm(x)$$

Then we may apply this inequality for the diffeomorphism $\phi^{-1} : \Omega_2 \rightarrow \Omega_1$ to get the reverse inequality.

$$\int_{\Omega_1} g(x) \, dm(x) \leq \int_{\Omega_2} g(\phi^{-1}(y)) |\det(\phi^{-1})'(y)| \, dm(y)$$

Apply

$$g(x) = f(\phi(x)) |\det \phi'(x)|$$

$$\int_{\Omega_1} g(x) \, dm(x) \leq \int_{\Omega_2} f(\phi(\phi^{-1}(y))) |\det \phi'(\phi^{-1}(y))| |\det(\phi^{-1})'(y)| \, dm(y)$$

we know that $\phi(\phi^{-1}(y)) = y$, $\phi^{-1}(\phi'(x)) = x$ then take the differentiable, we get

$$(\phi^{-1})'(\phi(x)) \phi'(x) = I$$

$$\phi'(\phi^{-1}(y)) (\phi^{-1})'(y) = I$$

If we can prove $f \in L^+(\Omega_2)$ we can also prove it for general $f \in L^1$

Noted that it suffices to prove the \leq

First step: $f = \mathbb{1}_{\phi(E)}$ we need to prove

$$m(\phi(E))^1 \leq \int_E |\det \phi'(x)| \, dx$$

Assume that E is a cube Q away from the boundary $\text{dist}(Q, \Omega_1^c) > 0$ For any $\varepsilon > 0$, let $\delta > 0$,

$$\|\phi'(\tilde{x})^{-1} \phi'(x) - I\|_{\infty-\infty} < \varepsilon, \quad \text{where } |x - \tilde{x}|_\infty < \delta$$

where $|x - \tilde{x}|_\infty = \max_{1 \leq j \leq n} |x_j - \tilde{x}_j|$ and $\|A\|_{\infty-\infty} = \max_i \sum_{k=1}^n |a_{ik}|$ we decompose $Q = \bigsqcup Q^\nu$ where $\text{diam} Q^\nu < \delta$ then $|\det \phi'(x) - \det \phi'(\tilde{x})| < \varepsilon$ if $|x - \tilde{x}| < \delta$ Now, we get that $\phi(Q_\nu)$ are all disjoint then

$$m(\phi(Q)) = \sum_{\nu} m(\phi(Q_\nu))$$

$$\phi(Q_\nu) = \phi'(x_\nu) \phi'(x_\nu)^{-1} \phi$$

then

$$\Psi^\nu = x \mapsto \phi'(x_\nu)^{-1} \phi(x)$$

for $x \in Q^\nu$, $\|(\Psi^\nu)' - I\|_{\infty-\infty} < \varepsilon$

Claim: $\Psi^\nu(Q^\nu)$ contain in a cube centered at $\Psi^\nu(x^\nu)$ with side length $= (1 + \varepsilon) \text{side length}(Q_\nu)$. Thus $m(\Psi^\nu(Q^\nu)) \leq (1 + \varepsilon)^n m(Q_\nu)$. (Need to estimate $\Psi^\nu(x) - \Psi^\nu(x^\nu)$ for $x \in Q_\nu$)

Assuming the claim.

$$m(\phi(Q_\nu)) = |\det \phi'(x_\nu)| m(\Psi^\nu(Q))$$

¹Note that $\mathbb{1}_{\phi(E)}(\phi(x)) = \mathbb{1}_E(x)$

by the Corollary for linear differentiation then

$$\begin{aligned}
m(\phi(Q_\nu)) &\leq |\det \phi'(x_\nu)|(1+\varepsilon)^n \int_{Q_\nu} dm(x) \\
&= (1+\varepsilon)^n \sum_\nu \int_{Q_\nu} \underbrace{|\det \phi'(x) - (\det \phi'(x) - \det \phi'(x''))|}_{\leq |\det \phi'(x)| + \varepsilon} dm(x) \\
&\leq (1+\varepsilon)^n \sum_\nu \int_{Q_\nu} |\det \phi'(x)| + (1+\varepsilon)^n \sum_\nu \int_{Q_\nu} \varepsilon \\
&\leq (1+\varepsilon)^n \int_Q |\det \phi'(x)| dm(x) + (1+\varepsilon)^n \varepsilon m(Q)
\end{aligned}$$

For the **Claim**: we need to check

$$\begin{aligned}
|\Psi_j^\nu(x) - \Psi_j^\nu(x^\nu)| &\leq (1+\varepsilon) \max_{1 \leq k \leq n} |x_k - x_k^\nu| \\
|\Psi_j^\nu(x) - \Psi_j^\nu(x^\nu)| &= \left| \int_{s=0}^1 \sum_{k=1}^n (x_k - x_k^\nu) \frac{\partial \Psi_j^\nu}{\partial x_k}(x^\nu + s(x - x^\nu)) ds \right| \\
&= \left| (x_j - x_j^\nu) \int_0^1 \sum_{k=1}^n (x_k - x_k^\nu) \left(\frac{\partial \Psi_j^\nu}{\partial x_k} - I_{jk} \right) (x^\nu + s(x - x^\nu)) ds \right| \\
&\leq |x_j - x_j^\nu| + \int_0^1 \|x - x^\nu\|_\infty \sum_k \underbrace{\left| \frac{\partial \Psi_j^\nu}{\partial x_k} - I_{jk} \right|}_{\leq \varepsilon} (x^\nu + s(x - x^\nu)) ds \\
&\leq |x_j - x_j^\nu| + \varepsilon \|x - x^\nu\|_\infty
\end{aligned}$$

So, $\|\Psi^\nu(x) - \Psi^\nu(x^\nu)\|_\infty \leq (1+\varepsilon)\|x - x^\nu\|_\infty$

If E is an open set $\mathcal{O} \subseteq \Omega_1$ then by Whitney theorem, we can decompose

$$\mathcal{O} = \biguplus_{l=1}^{\infty} Q_l$$

where the cubes Q_l are disjoint and always stay from the boundary of Ω_1 . then

$$m(\phi(\mathcal{O})) = \sum m(\phi(Q_l)) \leq \sum_l \int_{Q_l} |\det \phi'(x)| dm = \int_{\mathcal{O}} |\det \phi'(x)| dx$$

Let $R \in \mathbb{N}$ then

$$\Omega_{1,R}\{x : |x| \leq R, \text{dist}(x_1, \Omega_1^c) \geq \frac{1}{R}\}$$

and we get that $\overline{\Omega_{1,R}}$ is compact and

$$\int_{\Omega_{1,R}} |\det \phi'(x)| < \infty$$

Let $E \subseteq \Omega_{1,R}$ then find an $\mathcal{O}_k, m(\mathcal{O}_k) \leq m(E) + 2^{-k}$. We can construct such that $\mathcal{O}_1 \supseteq \mathcal{O}_2 \supseteq \dots$. we can use continuity from above or dominated convergence theorem to get

$$m(E) \leq \int_{\mathcal{O}_k} |\det \phi'(x)| \, dm(x) \rightarrow \int_E |\det \phi'(x)| \, dm(x)$$

Main Step: (special case of $f = \mathbb{1}_E$)

$$m(\phi(E)) \leq \int_E |\det \phi'| \, dm$$

for $E \subseteq \Omega_1$ open set. replace $\phi(E)$ by A and E by $\phi^{-1}(A)$. A_1, \dots, A_j are given, c_1, \dots, c_j , $c_j > 0$, let

$$s(x) = \sum_{i=1}^j c_i \mathbb{1}_{A_i}$$

then

$$\sum_{i=1}^j c_i \int \mathbb{1}_{A_i} \, dm \leq \sum_{i=1}^j c_i \int_{\phi^{-1}(A_i)} |\det \phi'(x)| \, dm$$

$$x \in \phi^{-1}(A_i) \iff \phi(x) \in A_i$$

then $\mathbb{1}_{\phi^{-1}(A_i)} = \mathbb{1}_{A_i}(\phi(x))$ so

$$\int s(y) \, dm(y) \leq \int \sum c_i \mathbb{1}_{\phi^{-1}(A_i)}(x) |\det \phi'(x)| \, dm(x) \leq \int s(\phi(x)) |\det \phi'(x)| \, dm(x)$$

then $s_n \nearrow f$ apply MCT given the formula for $f \in \mathcal{L}^+$

$$f \in L^1 \implies f = (\Re f)_+ - (\Re f)_- + i((\Im f)_+ - (\Im f)_-)$$

□

§6.6 Lebesgue Differentiation Theorem

In Calculus, $f \in C[a, b]$ we have theorem that

$$F(x) = \int_a^x f(t) \, dt$$

then F is differentiable at all $x \in [a, b]$ and $F' = f$ we use the prove that

$$\lim_{h \rightarrow 0^+} \frac{F(x+h) - F(x)}{h} = f(x), x < b$$

$$\lim_{h \rightarrow 0^-} \frac{F(x+h) - F(x)}{h} = f(x), x > a$$

rewrite the limit as

$$\frac{F(x+h) - f(x)}{h} = \frac{1}{h} \int_x^{x+h} (f(t) - f(x) + f(x)) \, dt = f(x) + \frac{1}{h} \int_x^{x+h} f(t) - f(x) \, dt$$

For t close to x , use $|f(t) - f(x)| < \varepsilon$, $|t| < \delta$ for given similarly for negative

Theorem 6.6.1 (Lebesgue Differentiation Theorem)

If $f \in L^1([a, b])$,

$$F(x) = \int_{[a, x]} f \, dm$$

then F is differentiable almost everywhere and $F' = f$ almost everywhere.

Definition 6.6.2. In \mathbb{R}^n averages over ball $B(x, r) = \{y : |y - x| < r\}$

$$\mathcal{A}_r f(x) = \frac{1}{m(B(x, r))} \int_{B(x, r)} |f| \, dm$$

Definition 6.6.3.

$$M_{HL}f(x) = \sup_{r>0} \mathcal{A}_r |f|$$

Remark 6.6.4. Note the function $x \mapsto \mathcal{A}_r f(x)$ is continuous

$$\left| \int_{B(x, r)} |f| - \int_{B(\tilde{x}, r)} |f| \right| \leq \int_{B(x, r) \triangle B(\tilde{x}, r)} |f| \rightarrow 0$$

as $x \rightarrow \tilde{x}$ Observe that

$$\Omega_\alpha = \left\{ x : \sup_{r>0} \mathcal{A}_r f(x) > \alpha \right\}$$

is an open set because union of open sets are open set

Theorem 6.6.5 (Hardy and Littlewood)

M satisfies the inequality

$$(i) \quad m(\{x : M_{HL}f(x) > \alpha\}) \leq C_n \frac{\|f\|_1}{\alpha}$$

$$(ii) \quad \text{If } p > 1, \|M_{HL}f\|_p \leq C_{p,n} \|f\|_p$$

(iii) is clearly true for $p = \infty$ (for simplicity for continuous f)

In general part (ii) is an instance of an interpolation.

We say that a constant A essentially bounds a function f if $|f(x)| \leq A$ for almost every x .

$$\text{esssup } |f(x)| = \inf\{A : A \text{ essentially bounds } |f|\}$$

$$|f(x)| \leq B + \frac{1}{n} \text{ for } x \in X \setminus N_n \implies |f(x)| < B \text{ for } x \in X \setminus \bigcup N_n$$

Proof. $\Omega_\alpha = \{x : M_{HL}f(x) > \alpha\}$, $m(\Omega_\alpha) = \sup_{K \subseteq \Omega_\alpha} m(K)$. It suffices to show that for every compact set K contained in Ω_α , $m(K) \leq 3^n \frac{\|f\|_1}{\alpha}$ fix $x \in K$ then there exists a ball $B(x, r(x))$ such that

$$\frac{1}{m(B(x, r(x)))} \int_{B(x, r(x))} |f| \, dm > \alpha \implies m(B(x, r(x))) < \frac{1}{\alpha} \int_{B(x, r(x))} |f| \, dm$$

there is a finite family of ball $B(x_i, r(x_i))$, for $i = 1, \dots, L$ such that

$$K \subseteq \bigcup_{B \in \mathfrak{B}} B$$

K is compact

Covering Lemma Given ball $\mathfrak{B} = \{B_1, \dots, B_L\}$ there exists subcollection $\tilde{\mathfrak{B}} \subseteq \mathfrak{B}$ of ball $\{B : B \in \tilde{\mathfrak{B}}\}$ which is pairwise disjoint such that the three fold balls $\{3B : B \in \tilde{\mathfrak{B}}\}$ contain

$$\bigcup_{B \in \mathfrak{B}} B$$

(union of the original ball) then Notice that $B^* \equiv 3B \equiv$ ball of radius 3 radius of B centered at the center of B . $(x_B + 3(B - x_B))$

Proof by induction on the cardinality of \mathfrak{B} . $|\mathfrak{B}| = N > 1$, find ball $B_1 \in \mathfrak{B}$ with the maximal radius. Observe that for any $B \in \mathfrak{B}$, if $B \cap B_1 \neq \emptyset$ then $B \subseteq B_1^*$. Consider the collection $\mathfrak{B}_2 = \{\text{all ball in } \mathfrak{B} \text{ which are disjoint from } B_1\}$ By the induction hypothesis (and $|\mathfrak{B}_2| < |\mathfrak{B}| = N$) there are finitely many balls B_2, \dots, B_M disjoint and

$$\bigcup_{j=2}^M B_j^* \supseteq \bigcup_{B \in \mathfrak{B}_2} B$$

All balls in \mathfrak{B} , that intersect B_1 are contained in B_1^* .

$$m(B(x, r(x_j))) \leq \frac{1}{\alpha} \int_{B(x_j, r(x_j))} |f|$$

Apply the covering lemma, we get balls

$$B_1 = B(x_1, r(x_1)), \dots, B_M = B(x_M, r(x_M))$$

disjoint, and

$$\bigcup B_i^* \supseteq K$$

then

$$\begin{aligned} m(K) &\leq m\left(\bigcup_{i=1}^M B_i^*\right) \leq \sum_{i=1}^M m(B_i^*) = 3^n \sum_{i=1}^M m(B_i) \\ &\leq 3^n \sum_{i=1}^n \frac{1}{\alpha} \int_{B_i} |f| \, dm \leq \frac{3^n}{\alpha} \int |f| \, dm = 3^n \alpha^{-1} \|f\|_1 \end{aligned}$$

□

Claim in the Lebesgue differentiation theorem Let $f \in L^1$ then for almost every where $x \in \mathbb{R}^n$

$$\frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) \, dm(y) = f(x)$$

slightly strong statement

$$\frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dm \rightarrow 0$$

for almost everywhere x (If this is true for a specific $x \in \mathbb{R}^n$ then we call x a Lebesgue point)

A Practice Exam

§A.1 Practice Exam 1

Problem A.1.1. Let E_n be Lebesgue measurable subsets of $[0, 1]$ such that $E_{n+1} \subseteq E_n$. What can you say about the Lebesgue measure of $\bigcap_n E_n$? Does your answer necessarily hold when $[0, 1]$ is replaced by $[0, \infty)$?

solution. We can use continuity from above because $\mu([0, 1]) < \infty$. We can say that

$$\mu\left(\bigcap_n E_n\right) = \lim_{n \rightarrow \infty} \mu(E_n)$$

In case of $[0, \infty)$, we can't use continuity from below because if $E_n = [n, n+1)$ then $\mu(E_n) = 1$ but $\bigcap_n E_n = \emptyset$, so, $\lim_{n \rightarrow \infty} \mu(E_n) = 1$ but $\mu(\bigcap_n E_n) = 0$. \square

Problem A.1.2. Let

$$\begin{aligned} E = \{x = (x_1, x_2) \in \mathbb{R}^2 : \\ \frac{1}{1 + (x_1 - x_2)^3} \leq e^{\sin x_1} \text{ if } x_1^{23} < 3|\cos(x_1 + x_2)|, \\ \sqrt{1 + e^{|x_2| + |x_1|}} > e^{x_1^2} \text{ if } x_1^{23} > 3|\cos(x_1 + x_2)| \text{ and } x_2 \in \mathbb{R} \setminus \mathbb{Q}, \\ \sqrt{\cos(|x_1 x_2|)} \sin(x_1 x_2) > 0 \text{ if } x_1^{23} > 3|\cos(x_1 + x_2)| \text{ and } x_2 \in \mathbb{Q}\} \end{aligned}$$

- (i) Is the characteristic function of E Borel measurable?
- (ii) If \mathcal{M} denote the σ -algebra of Lebesgue measurable subsets of \mathbb{R} does E belong to $\mathcal{M} \oplus \mathcal{M}$?

solution. (i) Let $f_1(x_1, x_2) = \frac{1}{1 + (x_1 - x_2)^3} - e^{\sin x_1}$, $f_2(x_1, x_2) = \sqrt{1 + e^{|x_2| + |x_1|}} - e^{x_1^2}$, $f_3(x_1, x_2) = \sqrt{\cos(|x_1 x_2|)} \sin(x_1 x_2)$ and $g(x_1, x_2) = x_1^{23} - 3|\cos(x_1 + x_2)|$. Then $E = \{x : f_1(x) \leq 0 \text{ if } g(x) < 0, f_2(x) > 0 \text{ if } g(x) > 0 \text{ and } x_2 \in \mathbb{R} \setminus \mathbb{Q} \dots\}$. $E = (f_1^{-1}(-\infty, 0] \cap g^{-1}(-\infty, 0)) \cup (f_2^{-1}(0, \infty) \cap g^{-1}((0, \infty)) \cap \mathbb{R} \times (\mathbb{R} \setminus \mathbb{Q})) \cup \dots$. Then f_1, f_2, f_3, g are Borel measurable (because it is continuous) functions then E is Borel measurable.

- (ii)

\square

Problem A.1.3. For each of the statements give a proof or find a counterexample.

- (i) For each $r \in \mathbb{R}$ let $f_r : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Define $G(x) = \sup_{r \in \mathbb{R}} f_r(x)$. Is G a Borel measurable function?
- (ii) For each $r \in \mathbb{R}$ let $f_r : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function. Define $G(x) = \sup_{r \in \mathbb{R}} f_r(x)$. Is G a Borel measurable function?
- (iii) For each $r \in \mathbb{R}$ let $f_r : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function. Define $G(x) = \sup_{r \in \mathbb{Q}} f_r(x)$. Is G a Borel measurable function?
- (iv) For each $r \in \mathbb{Q}$ let $f_r : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function. Define $S(x) = \sum_{r \in \mathbb{Q}} f_r(x)$. Is S a Borel measurable function?
- (v) What happens if you replace Borel measurable by Lebesgue measurable in the above statements?

solution. (i) $G^{-1}((-\infty, x]) = \bigcap_{r \in \mathbb{R}} f_r^{-1}((-\infty, x])$ and f_r is continuous so, G is Borel measurable.

- (ii) No, $f_r = \mathbb{1}_{\{c_r\}}$ where $c_r \in \mathbb{Q} + r$ and $c_r \in [0, 1)$ then $G^{-1}(\{1\})$ is Vitali set.
- (iii) For $E \in \mathcal{M}$, $G^{-1}(E) = \bigcap_{r \in \mathbb{Q}} f_r^{-1}(E)$ and f_r is Borel measurable so, G is Borel measurable.
- (iv) Suppose that $\{q_r\}$ is an enumeration of \mathbb{Q} then define $g_n = \sum_{r=1}^n f_{q_r}$ then suppose that limit exists then $\limsup g_n$ and $\liminf g_n$ are Borel measurable then S is Borel measurable.

□

Problem A.1.4. Assume that f_n is a sequence of $L^1(\mu)$ functions.

- (i) If in addition $\mu(X) < \infty$ and

$$\lim_{n \rightarrow \infty} \sup_{x \in X} |f_n(x) - f(x)| = 0$$

then you have learned that $f \in L^1(\mu)$ and that $\lim_{n \rightarrow \infty} \int f_n \, d\mu = \int f \, d\mu$. Review the proof.

- (ii) Show that if $\mu(X) = \infty$ both conclusions may fail in (i).
- (iii) If $\mu(X) < \infty$ if $f_n \rightarrow f$ converges just pointwise show that for every $\varepsilon > 0$ there exists a set A of measure $< \varepsilon$ such that

$$\int_A |f_n - f| \, d\mu < \varepsilon$$

solution. (i) Suppose that $\lim_{n \rightarrow \infty} \|f_n - f\|_{\sup} \rightarrow 0$ then

$$\lim_{n \rightarrow \infty} \int |f_n - f| \, d\mu \leq \lim_{n \rightarrow \infty} \mu(X) \|f_n - f\|_{\sup} = 0$$

and we know that $f = (f - f_n) + (f_n)$ then

$$\int |f| \, d\mu = \int |f - f_n + f_n| \, d\mu \leq \int |f - f_n| \, d\mu + \int |f_n| \, d\mu$$

so, $f \in L^1$ and

$$\left| \int f_n - f \, d\mu \right| \leq \int |f_n - f| \, d\mu$$

So,

$$\lim_{n \rightarrow \infty} \int f_n \, d\mu = \int f \, d\mu$$

(ii) Let $f_n(x) = \frac{1}{x} \mathbb{1}_{[1,n]}$ then $f_n \in L^1$ but $f_n \rightarrow f = \frac{1}{x} \mathbb{1}_{[1,\infty)}$ and $f \notin L^1$.

(iii) By Egorov's theorem, for any $\varepsilon > 0$ there exists $A \in \mathcal{M}$ such that $\mu(A) < \varepsilon$ and $f_n \rightarrow f$ uniformly on A^c . Then select n such that $\|f_n - f\| < \frac{\varepsilon}{\mu(X)}$ then

$$\int_{A^c} |f_n - f| \, d\mu < \varepsilon$$

□

Problem A.1.5. Let \mathcal{M} be a σ -algebra on X .

(i) Consider a function $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : X \rightarrow \mathbb{R}^2$ such that for all rational number r_1, r_2 the sets

$$\{x \in X : r_1 < f_1(x), f_2(x) < r_2\}$$

belong to \mathcal{M} . Is f $(\mathcal{M}, \mathcal{B}(\mathbb{R}^2))$ -measurable?

(ii) Let $f_n : X \rightarrow \overline{\mathbb{R}}$ a sequence of measurable functions. Show that the set

$$E = \{x \in X : \{f_n(x)\}_{n=1}^\infty \text{ is a nondecreasing sequence}\}$$

is measurable.

solution. (i) $\{x : r_1 < f_1(x), f_2(x) < r_2\} = f_1^{-1}((r_1, \infty)) \cap f_2^{-1}((-\infty, r_2))$ It is enough to show that for any $p \in \mathbb{R}$, $f_1^{-1}(p, \infty) \in \mathcal{M}$ and $f_2^{-1}(-\infty, p) \in \mathcal{M}$.

- Fix p , there exists a sequence $\{x_n\}, \{y_n\}$ of \mathbb{Q} such that $x_n \rightarrow p^+$ and $y_n \rightarrow \infty$. Then

$$f_1^{-1}(p, \infty) = \bigcup_{n=1}^{\infty} f_1^{-1}((x_n, \infty)) \cap f_2^{-1}((-\infty, y_n)) \in \mathcal{M}$$

- Similarly, there exists a sequence $\{x_n\}, \{y_n\}$ of \mathbb{Q} such that $x_n \rightarrow -\infty$ and $y_n \rightarrow p^-$. Then

$$f_2^{-1}(-\infty, p) = \bigcup_{n=1}^{\infty} f_1^{-1}((x_n, \infty)) \cap f_2^{-1}((-\infty, y_n)) \in \mathcal{M}$$

- (ii) Let $g_k = f_k - f_{k+1}$ then g_k is measurable. So,

$$E = \bigcap_{k=1}^{\infty} \{x \in X : g_k(x) \geq 0\}$$

□

Problem A.1.6. Let f be a Lebesgue measurable function on \mathbb{R}^n . Let m denote Lebesgue measure on \mathbb{R}^n .

Show that the following three statements are equivalent:

- (a) f is integrable (i.e. belong to $L^1(\mathbb{R}^n)$).
- (b) $\sum_{k \in \mathbb{Z}} 2^k m(\{x : |f(x)| > 2^k\}) < \infty$.
- (c) $\sum_{k \in \mathbb{Z}} 2^k m(\{x : 2^k \leq |f(x)| < 2^{k+1}\}) < \infty$.

solution. • (a) \Rightarrow (c):

$$\sum_{k \in \mathbb{Z}} 2^k m(\{x : 2^k \leq |f(x)| < 2^{k+1}\}) \leq \int |f| \, dm < \infty$$

- (b) \Rightarrow (c):

$$\sum_{k \in \mathbb{Z}} 2^k m(\{x : 2^k \leq |f(x)| < 2^{k+1}\}) < \sum_{k \in \mathbb{Z}} 2^k m(\{x : |f(x)| > 2^{k-1}\}) < \infty$$

- (c) \Rightarrow (a):

$$\int |f| \, dm \leq \sum_{k \in \mathbb{Z}} 2^{k+1} m(\{x : 2^k \leq |f(x)| < 2^{k+1}\}) < \infty$$

- (c) \Rightarrow (b): Let $E_k = \{x : 2^k \leq |f(x)| < 2^{k+1}\}$ then

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} 2^k m(\{x : |f(x)| > 2^k\}) &\leq \sum_{k \in \mathbb{Z}} 2^k \sum_{j=k-1}^{\infty} E_j \\
&= \left(\sum_{k \in \mathbb{Z}} 2^k E_{k-1} + E_k \right) + \left(\sum_{k \in \mathbb{Z}} 2^k \sum_{j=k+1}^{\infty} E_j \right) \\
&= 3 \sum_{k \in \mathbb{Z}} 2^k E_k + \left(\sum_{j \in \mathbb{Z}} \sum_{k=-\infty}^{j-1} 2^k E_j \right) \\
&\leq 3 \sum_{k \in \mathbb{Z}} 2^k E_k + \left(\sum_{j \in \mathbb{Z}} 2^j E_j \right) \\
&= 4 \sum_{k \in \mathbb{Z}} 2^k E_k
\end{aligned}$$

□

Problem A.1.7. Let m be Lebesgue measure on \mathbb{R} and f be a Lebesgue measurable function with $\int |f| \, dm < \infty$. Define $G(x) = \int_{-\infty}^x f \, dm$. Prove that G is uniformly continuous on \mathbb{R} .

solution. I want to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $E \in \mathcal{M}$ with $m(E) < \delta$, implies that $\int_E |f| \, dm < \varepsilon$. Define $E_n = \{x : |f(x)| > n\}$. Define $f_n = 1_{E_n} |f|$ then $|f_n| \leq |f|$. Then by Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int f_n \, dm = 0$$

There exists k such that $\int f_k \, dm < \varepsilon/2$, select $\delta = \frac{\varepsilon}{2k}$. Given any $E \in \mathcal{M}$ then

$$\begin{aligned}
\int_E |f| \, dm &= \int_{E \cap E_n} |f| \, dm + \int_{E \cap E_n^c} |f| \, dm \\
&\leq \frac{\varepsilon}{2} + n \mu(E) \\
&\leq \varepsilon
\end{aligned}$$

For any $\varepsilon > 0$, pick δ to be in this theorem, then if $|x - y| < \delta$ then $|G(x) - G(y)| < \varepsilon$. □

Problem A.1.8. Determine the limits

- (i) $\lim_{n \rightarrow \infty} \int_0^{n^{99/100}} (x/n)^n \, dx$
- (ii) $\lim_{n \rightarrow \infty} \int_0^n (1 - \frac{x}{n})^n e^{-2x} \, dx$

and, in both cases carefully justify your computation.

solution. (i) define $f_n = \mathbb{1}_{[0, n^{99/100}]}(x/n)^n$ and $f =$ then $|f_n| \leq |f| \in L^1([0, \infty))$ then by Dominated Convergence Theorem,

□

Problem A.1.9. Let $f \in \mathcal{L}^1(\mathbb{R}^n)$. Let m be Lebesgue measure in \mathbb{R}^n . Prove that for $t > 0$

$$t^n \int f(tx) \, dm = \int f(x) \, dx$$

Hint: First prove this for indicator functions of cubes, then for indicator functions of sets of finite measure.

solution. Suppose that $f = \mathbb{1}_E$, then

$$\mathbb{1}_E(tx) = \begin{cases} 1 & \text{if } tx \in E \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } x \in \frac{1}{t}E \\ 0 & \text{otherwise} \end{cases} = \mathbb{1}_{\frac{1}{t}E}(x)$$

and $m(\frac{1}{t}E) = t^{-n}m(E)$

□

Problem A.1.10. Let $f \in \mathcal{L}^1(\mathbb{R}^n)$. Then

- (i) $\lim_{|h| \rightarrow 0} \int |f(x+h) - f(x)| \, dm = 0$.
- (ii) $\lim_{t \rightarrow 1} \int |f(tx) - f(x)| \, dm = 0$.
- (iii) Can the Lebesgue dominated convergence be used for the proof of (i) or (ii)?

solution. (i) there exists g continuous function with compact support such that $\int |f - g| < \frac{\varepsilon}{3}$ then

$$|f(x+h) - f(x)| \leq |f(x+h) - g(x+h)| + |g(x+h) - g(x)| + |g(x) - f(x)|$$

Obviously, $\int |f(x+h) - g(x+h)| \, dm < \frac{\varepsilon}{3}$ and $\int |g(x) - f(x)| \, dm < \frac{\varepsilon}{3}$ then

$$\int |g(x+h) - g(x)| \, dm \leq \varepsilon \cdot 2\mu(K)$$

(ii)

□

Problem A.1.11. Let $I = [a, b]$, $f \in \mathcal{L}^1(I)$. Show that

$$\lim_{n \rightarrow \infty} \int_I f(x) \sin(nx) \, dm(x) = 0$$

solution. There exists g step such that $\int |f - g| \, dm < \varepsilon$ then

$$\begin{aligned} \left| \int f(x) \sin(nx) - g(x) \sin(nx) \, dm \right| &\leq \int |f(x) - g(x)| |\sin(nx)| \, dm \\ \left| \int f(x) \sin(nx) \, dm \right| - \left| \int g(x) \sin(nx) \, dm \right| &\leq \varepsilon \end{aligned}$$

then fix some interval c then

$$\begin{aligned} \left| \int c \sin(nx) \, dm \right| &= \frac{1}{n} \left| \int c \sin(x) \, dm \right| \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. □

Problem A.1.12. Recall the monotone convergence theorem and Fatou's lemma.

- (i) Show that Fatou's lemma implies the monotone convergence theorem.
- (ii) Show that the monotone convergence theorem implies Fatou's lemma.

solution. (i) $f_n \leq f \implies \int f_n \leq \int f$ So, $\lim_{n \rightarrow \infty} \int f_n \leq \int f$. Then from Fatou's lemma, we have

$$\int \liminf_{k \rightarrow \infty} f_k \, d\mu \leq \liminf_{k \rightarrow \infty} \int f_k \, d\mu$$

Since f_n is non-decreasing, so, $\int f_n$ is also non-decreasing. So, we have $\liminf_{k \rightarrow \infty} f_k = \lim_{k \rightarrow \infty} f_k$ then

$$\int \lim_{k \rightarrow \infty} f_k \, d\mu \leq \lim_{k \rightarrow \infty} \int f_k \, d\mu$$

- (ii) we want to show that

$$\int \liminf_{k \rightarrow \infty} f_k \, d\mu \leq \liminf_{k \rightarrow \infty} \int f_k \, d\mu$$

It is enough to show that

$$\int \lim_{n \rightarrow \infty} \inf_{m \geq n} f_m \, d\mu \leq \lim_{n \rightarrow \infty} \inf_{m \geq n} \int f_m \, d\mu$$

From the monotone convergence theorem, we have

$$\int \lim_{n \rightarrow \infty} \inf_{m \geq n} f_m \, d\mu \stackrel{\text{MCT}}{=} \lim_{n \rightarrow \infty} \int \inf_{m \geq n} f_m \, d\mu$$

Then for any $k \geq n$, we have $\inf_{m \geq n} f_m \leq f_k$ then

$$\int \inf_{m \geq n} f_m \, d\mu \leq \int f_k \, d\mu$$

So, we have

$$\int \lim_{n \rightarrow \infty} \inf_{m \geq n} f_m \, d\mu \leq \lim_{n \rightarrow \infty} \inf_{m \geq n} \int f_m \, d\mu$$

□

Problem A.1.13. Determine

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{2n \sin(x/n)}{x(1+x^2)} \, dx$$

Provide justifications.

solution. We know that $|n/x \sin(x/n)| \leq 1$ then define

$$f_n = \frac{2n \sin(x/n)}{x(1+x^2)}$$

Then $|f_n| \leq \frac{2n}{x(1+x^2)}$ and $\int \frac{2n}{x(1+x^2)} \, dx < \infty$ then by Dominated Convergence Theorem,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty \frac{2n \sin(x/n)}{x(1+x^2)} \, dx &= \int_0^\infty \lim_{n \rightarrow \infty} \frac{2n \sin(x/n)}{x(1+x^2)} \, dx \\ &= \int_0^\infty \frac{2}{1+x^2} \, dx \\ &= \pi \end{aligned}$$

□

Problem A.1.14. Let $f(x) = \sin(x^2)$ on the measure space $X = [1, \infty)$ (with Lebesgue measure m). Prove:

- (i) $\int_{[1, \infty)} |f| \, dm = \infty$
- (ii) $\lim_{R \rightarrow \infty} \int_{[0, R]} f \, dm$ exists (and is finite).

Hint: For part (ii) use that $2x \sin(x^2)$ is the derivative of $-\cos(x^2)$.

solution. (i) Constructing triangles under the curve, we have

$$\begin{aligned} \int_{[0, \infty)} |f| \, dm &\geq \sum_{k=1}^{\infty} \frac{1}{2} \sqrt{(k+1)\pi} - \sqrt{k\pi} \\ &= \frac{1}{2} \sqrt{\pi} \sum_{k=1}^{\infty} \sqrt{k+1} - \sqrt{k} \\ &= \frac{1}{2} \sqrt{\pi} \left(\lim_{k \rightarrow \infty} \sqrt{k} - \sqrt{1} \right) \\ &= \infty \end{aligned}$$

- (ii) Define $u = 1/2x$ then $dv = 2x \sin(x^2) dx$ then $v = -\cos(x^2)$ and $du = -1/2x^2 dx$ then

$$\begin{aligned} \int_0^R f dx &= uv \Big|_0^R - \int_0^R v du \\ &= -\frac{1}{2} \cos(R^2) + \frac{1}{2} \cos(0) - \int_0^R -\cos(x^2) du \end{aligned}$$

□

Problem A.1.15. Let $f : X \rightarrow \overline{\mathbb{R}}$ be a nonnegative measurable function on the measure space (X, \mathcal{M}, μ) and assume $\mu(X) < \infty$.

- (i) Let $E_R = \{x \in X : |f(x)| > R\}$. Prove: If $|f(x)| < \infty$ for almost every $x \in X$ then $\lim_{R \rightarrow \infty} \mu(E_R) = 0$.
- (ii) Is the conclusion in (i) still valid if we drop the assumption of finite measure space? Give a proof or counterexample.

solution. (i) Using continuity from above, we have

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu \left(\bigcap_{n=1}^{\infty} E_n \right)$$

and $|f(x)| < \infty$ for almost every $x \in X$ then $\mu \left(\bigcap_{n=1}^{\infty} E_n \right) = 0$, so $\lim_{R \rightarrow \infty} \mu(E_R) = 0$.

- (ii) $f(x) = x$

□

Problem A.1.16. Let $p > 0$. For $x \in \mathbb{R}^n$ let $|x|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. Let $\Omega = \{x \in \mathbb{R}^n : |x|_p > 3\}$. Show that

$$\int_{\Omega} |x|_p^{-\alpha} dm < \infty$$

if and only if $\alpha > n$. What is the result if you replace Ω by Ω^c ?

solution. Define $E_k = \{x \in \mathbb{R}^n : 3^k \leq |x|_p < 3^{k+1}\}$ then

$$\begin{aligned}
 \int_{\Omega} |x|_p^{-\alpha} \, dm &= \sum_{k=1}^{\infty} \int_{E_k} |x|_p^{-\alpha} \, dm \\
 &\leq \sum_{k=1}^{\infty} c_1 3^{-\alpha k} \mu(E_k) \\
 &\leq \sum_{k=1}^{\infty} c_1 3^{-\alpha k} c_2 3^{kn} \\
 &= c_1 c_2 \sum_{k=1}^{\infty} 3^{k(n-\alpha)}
 \end{aligned}$$

So, $n - \alpha < 0$ then $\alpha > n$. For the converse use the same bound (but lower bound)

For Ω^c , define $E_k = \{x \in \mathbb{R}^n : 3^{-k} \leq |x|_p < 3^{-k+1}\}$ then

$$\begin{aligned}
 \int_{\Omega^c} |x|_p^{-\alpha} \, dm &= \sum_{k=1}^{\infty} \int_{E_k} |x|_p^{-\alpha} \, dm \\
 &\leq \sum_{k=1}^{\infty} c_1 3^{\alpha k} \mu(E_k) \\
 &\leq \sum_{k=1}^{\infty} c_1 3^{\alpha k} c_2 3^{-kn} \\
 &= c_1 c_2 \sum_{k=1}^{\infty} 3^{k(\alpha-n)}
 \end{aligned}$$

So, $\alpha - n < 0$ then $\alpha < n$. □