# MATH 629 (Measure Theory) Lecture Notes

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Spring 2024

## Contents

# 1 From Riemann to Lebesgue

## §1.1 Riemann Integral

**Definition 1.1.1.** $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a, b]$.

**Definition 1.1.2.** If $P, P'$ are partitions of $[a, b]$ and $P \subseteq P'$, then $P'$ is a refinement of $P$.

**Definition 1.1.3.** Given a bounded function $f : [a, b] \to \mathbb{R}$ and a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ we define

$$m_i(f) = \inf_{t \in [x_{i-1}, x_i]} f(t)$$

$$M_i(f) = \sup_{t \in [x_{i-1}, x_i]} f(t)$$

define the lower sum as

$$L(f, P) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1})$$

and the upper sum as

$$U(f, P) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$$

**Lemma 1.1.4**

Given a bounded function $f : [a, b] \to \mathbb{R}$ and partitons $P$ of $[a, b]$. Suppose that $P'$ is a refinement of $P$ then

$$(b - a) \inf_{t \in [a,b]} f(t) \leq L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P) \leq (b - a) \sup_{t \in [a,b]} f(t)$$

**Corollary 1.1.5**

Suppose that $P_1, P_2$ are partitions of $[a, b]$ then $L(f, P_1) \leq U(f, P_2)$

*Proof.* Let $P' = P_1 \cup P_2$ then $P'$ is a refinement of $P_1$ and $P_2$ and use Lemma 1.1.4 □

**Lemma 1.1.6**

Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Then

$$(b-a) \inf_{t\in[a,b]} f(t) \leq \sup_P L(f,P) \leq \inf_P U(f,P) \leq (b-a) \sup_{t\in[a,b]} f(t)$$

**Definition 1.1.7.** A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if

$$\sup_P L(f,P) = \inf_P U(f,P)$$

and the common value is called the Riemann integral of $f$ and is denoted by $\int_a^b f$

**Lemma 1.1.8**

Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Then $f$ is Riemann integrable if and only if for any $\varepsilon > 0$ there exists a partition $P$ such that

$$U(f,P) - L(f,P) < \varepsilon$$

*Proof.* ($\Rightarrow$) For any $\varepsilon > 0$. Suppose that $f$ is Riemann integrable. Then there exists $P_1, P_2$ such that

$$L(f,P_1) \geq \int_a^b f - \frac{\varepsilon}{2}$$

$$U(f,P_2) \leq \int_a^b f + \frac{\varepsilon}{2}$$

let $P = P_1 \cup P_2$ then

$$U(f,P) - L(f,P) \leq \varepsilon$$

($\Leftarrow$) For any $\varepsilon > 0$, there exists $P_\varepsilon$ such that

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$$

since $\varepsilon$ is arbitrary, we have

$$\sup_P L(f,P) = \inf_P U(f,P)$$

□

**Theorem 1.1.9**

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ then $f$ is Riemann integrable.

*Proof.* $f$ is continuous on a compact set, so, $f$ is uniformly continous. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in [a, b]$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{(b-a)}$. Let $N$ be such that $\frac{(b-a)}{N} < \delta$ and let $P = \{x_i := a + \frac{(b-a)i}{N}\}$ then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{N} (M_i(f) - m_i(f)) \frac{(b-a)}{N} \\ &\leq \sum_{i=1}^{N} \frac{\varepsilon}{(b-a)} \frac{(b-a)}{N} \\ &= \varepsilon \end{aligned}$$

□

**Remark 1.1.10.** Let $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ defined on the $[0, 1]$. Then $U(f, P) = 1$ and $L(f, P) = 0$ for any partition $P$. So, $f$ is not Riemann integrable.

## §1.2 Lebesgue null sets

**Definition 1.2.1.** For the closed interval $I = [a, b]$, the length of $I$, denoted as $\ell(I)$ is defined as $\ell(I) = b - a$

**Definition 1.2.2.** A set $E$ is said to be a Lebesgue null set if for any $\varepsilon > 0$ there exists a sequence of intervals $\{I_n\}_{n \in \mathbb{N}}$ such that

$$E \subseteq \bigcup_{n=1}^{\infty} I_n$$

and

$$\sum_{n=1}^{\infty} \ell(I_n) < \varepsilon$$

**Lemma 1.2.3**

Countable unions of Lebesgue null sets are Lebesgue null sets.

*Proof.* For any $\varepsilon > 0$ and for each Lebesgue null sets $E_n$ there exists $I_{E_n,i}$ such that

$$E_n \subseteq \bigcup_{i=1}^{\infty} I_{E_n,i}$$

and

$$\sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \frac{\varepsilon}{2^n}$$

then

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \varepsilon$$

□

**Definition 1.2.4.** A set $E \subseteq [a,b]$ has content zero if for any $\varepsilon > 0$ there exists $I_1, I_2, \ldots, I_n$ such that

$$E \subseteq \bigcup_{i=1}^{n} I_i$$

and

$$\sum_{i=1}^{n} \ell(I_i) < \varepsilon$$

**Lemma 1.2.5**

Suppose that $E \subseteq [a,b]$ is a compact Lebesgue null set then $E$ has content zero.

*Proof.* For any $\varepsilon > 0$ there exists a sequnce of interval $\{I_n\}_{n\in\mathbb{N}}$ such that $E \subseteq \bigcup I_n$ and $\sum \ell(I_n) < \frac{\varepsilon}{2}$. Suppose that $I_n = [a_n, b_n]$, then let

$$J_n = \left(a_n - \frac{\varepsilon}{2^{n+3}}, b_n + \frac{\varepsilon}{2^{n+3}}\right) \supseteq E_n$$

then from the compactness of $E$, there exists a finite subcover $J_{n_1}, J_{n_2}, \ldots, J_{n_k}$ such that $E \subseteq \bigcup J_{n_i}$ then we construct a finite closed interval $K_i$ by

$$K_i = \left[a_{n_i} - \frac{\varepsilon}{2^{n_i+2}}, b_{n_i} + \frac{\varepsilon}{2^{n_i+2}}\right]$$

then $E \subseteq \bigcup K_i$ and $\sum \ell(K_i) < \varepsilon$ □

**Corollary 1.2.6**

if $a < b$ then $[a,b]$ is not a Lebesgue null set.

*Proof.* By contradiction, since $[a,b]$ is compact, then $[a,b]$ has content zero, but $[a,b]$ don't have content zero. □

## §1.3 Oscillation and Discontinuity

**Definition 1.3.1.** Suppose that $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ for any $x \in X$ and $\delta > 0$, define

$$M_{f,\delta}(x) := \sup\{f(y) : d(x,y) < \delta\}$$

$$m_{f,\delta}(x) := \inf\{f(y) : d(x,y) < \delta\}$$

then we define

$$\operatorname{osc}_f(x) := \lim_{\delta \to 0+} M_{f,\delta}(x) - m_{f,\delta}(x)$$

**Lemma 1.3.2**

$f$ is continuous at $x$ if and only if $\operatorname{osc}_f(x) = 0$.

*Proof.* ($\Rightarrow$) Suppose that $f$ is continuous at $x$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x,y) < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Then

$$M_{f,\delta}(x) - m_{f,\delta}(x) \leq \sup\{f(y) : d(x,y) < \delta\} - \inf\{f(y) : d(x,y) < \delta\} < \varepsilon$$

($\Leftarrow$) Suppose that $\operatorname{osc}_f(x) = 0$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $M_{f,\delta}(x) - m_{f,\delta}(x) < \varepsilon$. Then for any $y \in X$ such that $d(x,y) < \delta$, we have $|f(x) - f(y)| < \varepsilon$ then $f$ is continuous at $x$. □

Before we prove this theorem, we need to prove the following lemma.

**Lemma 1.3.3**

$\{x \in [a,b] : \operatorname{osc}_f(x) \geq \gamma\}$ is closed.

*Proof.* We need to show that $\{x : \operatorname{osc}_f(x) < \gamma\}$ is open. Fix $x$ in that set. Let $\varepsilon = \gamma - \operatorname{osc}_f(x)$ then

$$\sup_{|w-x|<\delta} f(w) - \inf_{|w-x|<\delta} f(w) < \operatorname{osc}_f(x) < \gamma$$

then for any $w \in (x - \delta, x + \delta)$ if $|w - x| < \frac{\delta}{2}$ then

$$\operatorname{osc}(w) \leq \sup_{|y-w|<\frac{\delta}{2}} f(y) - \inf_{|y-w|<\frac{\delta}{2}} f(y) < \gamma$$

So, $B\left(x, \frac{\delta}{2}\right) \subseteq \{x : \operatorname{osc}_f(x) < \gamma\}$ □

we observe that

(i) If the set of discontinuities is a Lebesque null set, then $\{x : \operatorname{osc}_f(x) \geq \gamma\}$ is a set of content zero.

(ii) If $\{x : \operatorname{osc}_f(x) \geq \gamma\}$ is a Lebesgue null set, then the set of discontinuities is also a Lebesgue null set.

**Lemma 1.3.4**

Suppose that $f$ is defined on $[c, d]$, assume that $\operatorname{osc}_f(x) < \gamma$ then we can find a partition

$$U(f, P) - L(f, P) < \gamma(b - a)$$

*Proof.* For every $x \in [c, d]$, there exists $\delta_x > 0$ such that

$$\sup_{|w-x|<\delta_x} f(w) - \inf_{|w-x|<\delta_x} f(x) < \gamma$$

construct a cover by

$$B(x, \delta_x) = \{w \in [c, d] : |w - x| < \delta_x\}$$

since $[c, d]$ is compact, there exists a finite subcover $B(p_1, \delta_{p_1}), \ldots, B(p_n, \delta_{p_n})$ then let $\delta_0 = \frac{\min\{\delta_{p_i}\}}{100}$ then we can construct a partition $P = \{c = x_0 < x_1 < \cdots < x_n = d\}$ such that $|x_i - x_{i-1}| < \delta_0$ then $M_i - m_i < \gamma$ and

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &< \gamma \sum_{i=1}^{n} (x_i - x_{i-1}) \\ &= \gamma(d - c) \end{aligned}$$

□

**Theorem 1.3.5**

Suppose that $f : [a, b] \to \mathbb{R}$ then $f \in \mathcal{R}([a, b])$ if and only if $f$ is bounded and the set of discontinuity of $f$ is a Lebesgue null set.

*Proof.* ($\Rightarrow$) We want to show that for every $n \in \mathbb{N}$,

$$\mathcal{D}_n = \left\{x : \operatorname{osc}_f(x) \geq \frac{1}{n}\right\}$$

is a Lebesgue null set. For any $\varepsilon > 0$, since $f$ is Riemann integrable, there exists a partition $P$ of $[a, b]$ such that

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1})(M_i - m_i) \leq \frac{\varepsilon}{n}$$

where $M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$ and $m_i = \inf_{x\in[x_{i-1},x_i]} f(x)$. in particular

$$\sum_{[x_{i-1},x_i]\cap\mathcal{D}_n\neq\emptyset} (x_i - x_{i-1})(M_i - m_i) \leq \frac{\varepsilon}{n}$$
$$\frac{1}{n}\sum_{[x_{i-1},x_i]\cap\mathcal{D}_n\neq\emptyset} \ell([x_{i-1},x_i]) \leq \frac{\varepsilon}{n}$$

So, this interval cover the set $\mathcal{D}_n$

($\Leftarrow$) pick $\varepsilon_1 \ll \varepsilon$, consider the set $D(\varepsilon_1) = \{x \in [a,b] : \operatorname{osc}_f(x) \geq \varepsilon_1\}$ closed set. Since $D(\varepsilon_1)$ is a Lebesgue null set from the Lemma 1.2.5 it has content zero so we can find $I_1,\ldots,I_n$ such that

$$\sum_{j=1}^{n} \ell(I_j) < \varepsilon_1 \text{ and } D(\varepsilon_1) \subseteq \bigcup_{j=1}^{n} I_j$$

We form a partition of $[a,b]$, $a = x_0 < x_1 < \cdots < x_N = b$ from $I_j$. There are two cases that we need to consider

1) if $[x_{i-1},x_i] \subseteq I_j$ for some $j$ then set $P_i = [x_{i-1},x_i]$
2) if $[x_{i-1},x_i] \cap I_j = \emptyset$ for all $j$ then $\operatorname{osc}(x) < \varepsilon_1$ for all $x \in [x_{i-1},x_i]$. We want to partition further the interval $[x_{i-1},x_i]$ by partition $P_i$. Using Lemma 1.3.4 we can find a partition $P_i$ of $[x_{i-1},x_i]$ such that

$$U(f,P_i) - L(f,P_i) < \varepsilon_1(x_i - x_{i-1})$$

We form a partition $P = P_1 \cup \cdots \cup P_N$ then

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i=1}^{N}(U(f,P_i) - L(f,P_i)) \\
&= \sum_{i:\text{case 1}} (U(f,P_i) - L(f,P_i)) + \sum_{i:\text{case 2}} (U(f,P_i) - L(f,P_i)) \\
&\leq 2M \sum_{i:\text{case 1}} (x_i - x_{i-1}) + \varepsilon_1 \sum_{i:\text{case 2}} (x_i - x_{i-1}) \\
&\leq 2M\varepsilon_1 + \varepsilon_1(b-a) \\
&= \varepsilon_1(2M + b - a)
\end{aligned}$$

$\square$

# 2 Measures

## §2.1 Introduction

We define the $\ell([c,d]) = d - c$ and If $E = [c_1, d_1] \cup [c_2, d_2]$ where $d_1 < c_2$ then $\ell(E) = d_1 - c_1 + d_2 - c_2$. This is consistent with the definition

$$\ell(E) = \int \mathbb{1}_E(x) \, \mathrm{d}x$$

where the integral denotes the Riemann integral.

if $E \subseteq [a,b]$ reference interval is

$$\int_a^b \mathbb{1}_E \, \mathrm{d}x$$

**Remark 2.1.1.** The consistency of the definition also works with the set $(c,d)$, $[c,d)$, and $(c,d]$, where the length of all of them is $d - c$.

**Remark 2.1.2.** we defnote $\mathbb{1}_E$ to be

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

**Example 2.1.3**

Let $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ defined on the $[0,1]$. Then $U(f,P) = 1$ and $L(f,P) = 0$ for any partition $P$.

Fix the reference interval $[a,b]$ and consider subset of $[a,b]$

Let $\mathcal{A} =$ collection of sets for which $\int_{[a,b]} \mathbb{1}_E \, \mathrm{d}x$ exists.

If $A_1, \ldots, A_n \in \mathcal{A}$, we can make the set to be mutually disjoint by taking $E_1 = A_1$, $E_2 = A_2 \setminus A_1$, $E_3 = A_3 \setminus (A_1 \cup A_2)$, and so on.

**Example 2.1.4**

For $E_1, E_2 \in \mathcal{A}$, we have

$$\mathbb{1}_{E_1 \cap E_2}(x) = \mathbb{1}_{E_1}(x) \mathbb{1}_{E_2}(x)$$

**Example 2.1.5**

For the Riemann integral, we have

$$\int_a^b f(y) = \int_{a-v}^{b-v} f(v+y)$$

and we want

$$\int \mathbb{1}_E(x)\ \mathrm{d}x = \int \mathbb{1}_{v+E}$$

where $v + E = \{v + x : x \in E\}$

Let $E = \mathbb{Q} \cap [0,1]$ countable set, we can enumerate $r_1, r_2, r_3, \ldots$ such that

$$E = \bigcup_{n=1}^{\infty} \{r_n\}$$

and

$$\int \mathbb{1}_{\{r_k\}} = 0$$

$E$ should have length zero but according $\mathbb{1}_E$ is not Riemann integrable.

## §2.2 Construction of Measure

Suppose that $\mathcal{C}$ be a collection of sets.

Can we define on suitable large collection of subset of $\mathbb{R}$?

a set function $\mu : \mathcal{C} \to [0,\infty]$ such that if $\{E_j\}_{j=1}^{\infty}$ is a sequence of disjoint set in $\mathcal{C}$ then

$$\mu\left(\bigcup_{i=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

$\mu([a,b]) = b - a$, $\mu([0,1)) = 1$

Can we do this for the collection of all subset of $\mathbb{R}$?

Answer: No, Vitali set.

**Theorem 2.2.1**

We cannot define a measure on the collection of all subset of $\mathbb{R}$. i.e., there does not exist a set function $\mu : \mathfrak{P}(\mathbb{R}) \to [0, \infty]$ such that

(i) $\mu(v + E) = \mu(E)$ for all $E \subseteq \mathbb{R}$ and $v \in \mathbb{R}$

(ii) $\mu([0, 1]) = 1$

(iii) $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$ for all disjoint $A_j \subseteq \mathbb{R}$

Before we prove that theorem, we need to define something and prove the following lemma.

**Definition 2.2.2.** We define a Vitali set $V$ from picking an element $x \in [0, 1)$ from each equivalence class of the relation $x \sim y$ if $x - y \in \mathbb{Q}$. (e.g, pick $x \in O_x$ for $O_x \in \mathbb{R}/\mathbb{Q}$)

**Lemma 2.2.3**

Suppose that $V$ is a Vitali set then

$$V \cap V + q = \emptyset$$

For all $q \in \mathbb{Q} \setminus \{0\}$

*Proof.* Suppose not, there exists $a \in V$ such that $a \in V + q \implies a - q \in V$ but we only pick 1 element in each equivalence class. contradiction. $\square$

**Lemma 2.2.4**

Let $V$ be a Vitali set and let $W = \{q \in [-1, 1] : q \in \mathbb{Q}\}$ and

$$E = \bigcup_{w \in W} V + w$$

then

$$[0, 1] \subseteq E \subseteq [-1, 2]$$

*Proof.* Consider $E \subseteq [-1, 2]$. Since $V \subseteq [0, 1)$, then for any $v \in V$, $v \in [0, 1) \implies v + w \in [-1, 2]$.

For the $[0, 1] \subseteq E$, for any $x \in [0, 1]$ there exists $O_x \in \mathbb{R}/\mathbb{Q}$ such that $x \in O_x$. then there exists $v \in C_x$ such that $v \in [0, 1)$ and $v \in V$, since both are from the same equivalence class, then $x - v \in \mathbb{Q}$ and $|x - v| < 1 \implies x - v \in (-1, 1)$. Hence, there exists $w \in W$ such that $w = x - v$ so $v + w = x$. $\square$

*Proof of the theorem.* Suppose that $\mu$ exists then using the result from Lemma 2.2.4 we get that

$$\mu([0,1]) \leq \mu(E) \leq \mu([-1,2])$$

from Lemma 2.2.3 we know that each $V + w$ is disjoint, so

$$\mu([0,1]) \leq \sum_{w \in W} \mu(V) \leq \mu([-1,2])$$
$$1 \leq \sum_{w \in W} \mu(V) \leq 3$$

if $\mu(V) = 0$ then $\mu(E) = 0$ and if $\mu(V) > 0$ then $\mu(E) = \infty$. Both are contradiction. $\square$

## §2.3 $\sigma$-algebra

**Definition 2.3.1.** Given a reference $X$. An **algebra** is a collection of subsets of $X$, $\mathcal{A}$, such that

(i) $X \in \mathcal{A}$

(ii) If $A \in \mathcal{A}$ then the complement $A^{\complement} = X \setminus A \in \mathcal{A}$

(iii) If $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$

**Remark 2.3.2.**
- $\emptyset \in \mathcal{A}$ because $\emptyset = X^{\complement}$
- $A_1, A_2 \in \mathcal{A}$, $A_1 \setminus A_2 = A_1 \cap A_2^{\complement} \in \mathcal{A}$
- Observe that if $A_1, A_2 \in \mathcal{A}$ then $A_1 \cap A_2 \in \mathcal{A}$ because $(A_1 \cap A_2)^{\complement} = A_1^{\complement} \cup A_2^{\complement}$

**Example 2.3.3**

$X = [a, b]$ and $\mathcal{A}$ is the collection of all sets $E \subseteq [a, b]$ such that the Riemann integral $\int \mathbb{1}_E(t)\ dt$ exists

**Definition 2.3.4.** A $\sigma$-algebra $\mathcal{M}$ on $X$ is

(i) an algebra of subsets of $X$

(ii) If $A_1, A_2, A_3, \ldots$ is a sequence of set in $\mathcal{M}$ then

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$$

$(X, \mathcal{M})$ is called a "**measurable space**".

> **Remark 2.3.5.** $\mathcal{M}$ is a $\sigma$-algebra on $X$ then it satisfies
>
> (i) $X \in \mathcal{M}$
>
> (ii) If $A \in \mathcal{M}$ then $A^{\complement} \in \mathcal{M}$
>
> (iii) countable union of sets in $\mathcal{M}$ is in $\mathcal{M}$

**Definition 2.3.6.** Let $(X, \mathcal{M})$ be a measurable set. Then a measure $\mu$ is a set function $\mu : \mathcal{M} \to [0, \infty], E \mapsto \mu(E)$ such that

(i) $\mu(\emptyset) = 0$

(ii) If $E_1, E_2, E_3, \dots$ is a sequence of disjoint set in $\mathcal{M}$ then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

called $\sigma$-additivity.

$(X, \mathcal{M}, \mu)$ is called a "**measure space**".

> **Remark 2.3.7.**
>
> $$\left(\bigcap_{j=1}^{\infty} A_j\right) = \left(\bigcup_{j=1}^{\infty} A_j^{\complement}\right)^{\complement} \in \mathcal{M}$$

**Example 2.3.8**

examples of $\sigma$-algebra

(i) $\mathcal{M} = \{\emptyset, X\}$

(ii) $\mathcal{M} = \mathfrak{P}(X) =$ collection of all subsets of $X$

$\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mu(E) = |E|$ (the cardinality of $E$) if $E$ is finite and $\mu(E) = \infty$ if $E$ is infinite.

(iii) $X$ write $X$ as a disjoint (countable) union of sets $A_j$. Then $\mathcal{M} =$ all countable unions of $A_j$.

(iv) Let $X$ be a set. Let $\mathcal{M}$ be the collection of all sets $A$, $A \subseteq X$ such that $A$ is countable or $A^{\complement}$ is countable.

(v) $X = \mathbb{R}$ (or $\mathbb{R}^n$), $\mathcal{B}_{\mathbb{R}}$ is the smallest $\sigma$-algebra containing all open sets.

More generally if $\mathcal{E}$ is a collection of subsets of $X$ then $\mathfrak{M}(\mathcal{E})$ is the smallest $\sigma$-algebra that contains all sets in $\mathcal{E}$.

If $\mathcal{M}_1, \mathcal{M}_2$ are two $\sigma$-algebras, then $\mathcal{M}_1 \cap \mathcal{M}_2$ is also a $\sigma$-algebra.

If $\{\mathcal{M}_\alpha\}_{\alpha \in \mathcal{I}}$ is a collection of $\sigma$-algebras, their intersection is also a $\sigma$-algebra.

### Generating $\sigma$-algebra

**Definition 2.3.9.** $\mathfrak{M}(\mathcal{E}) :=$ intersection of all $\sigma$-algebra that contain the collection $\mathcal{E}$ We call it the $\sigma$-algebra generated by $\mathcal{E}$. i.e.

$$\mathfrak{M}(\mathcal{E}) = \bigcap_{\substack{\mathcal{F} \in \mathcal{M} \\ \mathcal{E} \subseteq \mathcal{F}}} \mathcal{F}$$

**Remark 2.3.10.** If $\mathcal{E} \subseteq \mathcal{F} \implies \mathfrak{M}(\mathcal{E}) \subseteq \mathfrak{M}(\mathcal{F})$

**Lemma 2.3.11**

If $\mathcal{E} \subseteq \mathfrak{M}(\mathcal{F})$ then $\mathfrak{M}(\mathcal{E}) \subseteq \mathfrak{M}(\mathcal{F})$

*Proof.* $\mathfrak{M}(\mathcal{F})$ is a $\sigma$-algebra that contains $\mathcal{E}$ It contains the intersection of all $\sigma$-algebras which contain $\mathcal{E}$ □

**Example 2.3.12**

$\mathcal{B}_\mathbb{R} = \sigma$-algebra on $\mathbb{R}$ containing all open sets $\mathcal{E}$ a collection of all open intervals, $\mathcal{E} \subseteq \mathcal{O} =$ collection of all open sets in $\mathbb{R}$, $\mathcal{B}_\mathbb{R} = \mathfrak{M}(\mathcal{O})$. $\mathfrak{M}(\mathcal{E}) \subseteq \mathcal{B}_\mathbb{R}$. Each open set is a countable union of open intervals. Each open set is contained in $\mathfrak{M}(\mathcal{E})$.

Since $\mathcal{O} \subseteq \mathfrak{M}(\mathcal{E}) \implies \mathfrak{M}(\mathcal{O}) \subseteq \mathfrak{M}(\mathcal{E})$. get $\mathfrak{M}(\mathcal{O}) = \mathfrak{M}(\mathcal{E})$.

**Definition 2.3.13.** Given $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2), \ldots, (X_n, \mathcal{M}_n)$ measurable spaces. Define a "product $\sigma$-algebra" on $X_1 \times X_2 \times \cdots \times X_n$ denoted by

$$\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_n = \bigoplus_{j=1}^{n} \mathcal{M}_j$$

defined as the $\sigma$-algebra generated by the sets $E_1 \times E_2 \times \cdots \times E_n$ where $E_j \in \mathcal{M}_j$.

i.e., define $\mathcal{E} := \{(E_1 \times E_2 \times \cdots \times E_n) : E_j \in \mathcal{M}_j\}$ then

$$\bigoplus_{j=1}^{n} \mathcal{M}_j := \mathfrak{M}(\mathcal{E})$$

**Remark 2.3.14.** Folland defines it the $\sigma$-algebra generated by

$$(X_1 \times X_2 \times \cdots \times X_{n-1} \times E_n)$$

where $E_n \in \mathcal{M}_n$,

$$(X_1 \times X_2 \times \cdots E_{n-1} \times X_n)$$

where $E_{n-1} \in \mathcal{M}_{n-1}$. and so on. To be clear, let

$$\mathcal{E}' := \bigcup_{j=1}^{n} \{(X_1 \times \cdots \times X_{j-1} \times E_j \times X_{j+1} \times \cdots \times X_n) : E_j \in \mathcal{M}_j\}$$

then

$$\bigoplus_{j=1}^{n} \mathcal{M}_j := \mathfrak{M}(\mathcal{E}')$$

**Claim 2.3.15** — Both defintions on product of $\sigma$-algebra are equivalent.

*Proof.* The goal is to show that $\mathfrak{M}(\mathcal{E}) = \mathfrak{M}(\mathcal{E}')$.

($\supseteq$) Obviously, $\mathcal{E}' \subseteq \mathcal{E}$ so $\mathfrak{M}(\mathcal{E}') \subseteq \mathfrak{M}(\mathcal{E})$.

($\subseteq$) We want to show that $\mathcal{E} \subseteq \mathfrak{M}(\mathcal{E}')$. Fix $(E_1 \times E_2 \times \cdots \times E_n) \in \mathcal{E}$ then from the definition of $\sigma$-algebra generated by a collection, which is closed under intersection, so we can pick an element from the construction of $\mathcal{E}'$ and do the intersection, so $(E_1 \times E_2 \cdots \times E_n) \in \mathfrak{M}(\mathcal{E})$.

$\square$

**Theorem 2.3.16**

Given $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2)$ measurable spaces. Assume that $\mathcal{M}_1$ is generated by a collection $\mathcal{E}_1$ and $\mathcal{M}_2$ is generated by a collection $\mathcal{E}_2$. Then $\mathcal{M}_1 \oplus \mathcal{M}_2$ is generated by the sets $E_1 \times X_2$, $X_1 \times E_2$, where $E_1 \in \mathcal{E}_1$ and $E_2 \in \mathcal{E}_2$.

*Proof.* Let $\mathcal{P} := \{E_1 \times E_2 : E_i \in \mathcal{E}_i\}$, obviously $\mathfrak{M}(\mathcal{P}) = \mathfrak{M}(\{E_1 \times X_2 : E_1 \in \mathcal{E}_1\} \cup \{X_1 \times E_2 : E_2 \in \mathcal{E}_2\})$ and $\mathfrak{M}(\mathcal{P}) \subseteq \mathcal{M}_1 \oplus \mathcal{M}_2$. We need to show that $\mathcal{M}_1 \oplus \mathcal{M}_2 \subseteq \mathfrak{M}(\mathcal{P})$. Define

$$\mathcal{G}_1 = \{E_1 \subseteq X_1 : E_1 \times X_2 \in \mathfrak{M}(\mathcal{P})\}$$
$$\mathcal{G}_2 = \{E_2 \subseteq X_2 : X_1 \times E_2 \in \mathfrak{M}(\mathcal{P})\}$$

then $\mathcal{G}_1$ is a $\sigma$-algebra consisting of subset of $X_1$ which contains $\mathcal{E}_1$, $\mathcal{E}_1 \subseteq \mathcal{G}_1$. $\mathcal{E}_1$ generates $\mathcal{M}_1$ so $\mathfrak{M}(\mathcal{E}_1) = \mathcal{M}_1 \subseteq \mathcal{G}_1$. So, we have $E_1 \times X_2 \in \mathfrak{M}(\mathcal{P})$ for all $E_1 \in \mathcal{M}_1$ and $X_1 \times E_2 \in \mathfrak{M}(\mathcal{P})$ for all $E_2 \in \mathcal{M}_2$. The $\sigma$-algebra generated by the sets $E_1 \times X_2$, $X_1 \times E_2$ is contained $\mathcal{M}_1 \oplus \mathcal{M}_2 \in \mathfrak{M}(\mathcal{P})$. $\square$

**Claim 2.3.17** — $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$.

where $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}}$ is generated by $E_1 \times E_2$, where $E_1, E_2 \in \mathcal{B}_{\mathbb{R}}$. and $\mathcal{B}_{\mathbb{R}^2}$ is generated by the open sets in $\mathbb{R}^2$.

*Proof.* $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{B}_{\mathbb{R}^2}$. Want $\mathcal{B}_{\mathbb{R}^2} \subseteq \mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}}$. Consider the collection of all open rectangle of the form $(a_1, b_1) \times (a_2, b_2)$ such $a_i, b_i \in \mathbb{Q}$. which are contained in $O \subseteq \mathbb{R}^2$ $\square$

**Definition 2.3.18** (The Borel $\sigma$ algebra on the extended real line)**.** We use the notion $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$. One possibility to define "$\mathcal{B}_{\overline{\mathbb{R}}}$" is the $\sigma$-algebra generated by open sets in $\mathbb{R}, \{\infty\}, \{-\infty\}$ open intervals should be $(a, b), (a, \infty], [-\infty, b)$ for $-\infty \leq a < b \leq \infty$. Then define $d(x, y) = |\arctan(x) - \arctan(y)|$ and $\arctan(\infty) = \pi/2$, $\arctan(-\infty) = -\pi/2$.

## §2.4 Measures

**Definition 2.4.1.** Measures are $\sigma$-additive set functions, $\mu(\emptyset) = 0$ and

$$\mu\left(\biguplus_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

where $E_1, E_2, \dots$ is a sequence of disjoint sets.

**Remark 2.4.2.** There is some property

$E \subseteq F \implies \mu(E) \leq \mu(F)$

$F = E \uplus (F \setminus E) \implies \mu(F) = \mu(E) + \mu(F \setminus E)$

$\mu(\bigcup A_j) \leq \sum \mu(A_j)$ we can write $\bigcup A_j$ as a disjoint union, i.e., $E_1 = A_1$, $E_2 = A_2 \setminus A_1$, $E_3 = A_3 \setminus (A_1 \cup A_2)$, and so on then $\mu(\bigcup A_j) = \mu(\bigcup E_j) = \sum \mu(E_j) \leq \mu(A_j)$

The monotone convergence theorem for sets (continuity from below)

**Theorem 2.4.3**
If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j \to \infty} \mu(E_j)$$

*Proof.*

$$\bigcup_{j=1}^{\infty} E_j = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus (E_1 \cup E_2)) \cup \cdots$$

So, we define $B_1 = E_1, B_n = E_n \setminus E_{n-1}$ for $n \geq 2$ then all $B_j$ are disjoint.

$$\begin{aligned}
\bigcup_{j=1}^{\infty} E_j &= \bigcup_{j=1}^{\infty} B_j \\
\mu\left(\bigcup_{j=1}^{\infty} E_j\right) &= \mu\left(\bigcup_{j=1}^{\infty} B_j\right) \\
&= \sum_{j=1}^{\infty} \mu(B_j) \\
&= \mu(E_1) + \sum_{j=2}^{\infty} \mu(E_j \setminus E_{j-1}) \\
&= \mu(E_1) + \sum_{j=2}^{\infty} \mu(E_j) - \mu(E_{j-1}) \\
&= \lim_{n\to\infty} \mu(E_n)
\end{aligned}$$

$\square$

**Remark 2.4.4.** If we prove something for the set then we can prove it for the complement.

$$\mu(A) + \mu(A^{\complement}) = \mu(X)$$

**Theorem 2.4.5**

If $\mu(X) < \infty$ then if $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$, $E_n \supseteq E_{n+1}$ for all $n$ then

$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$$

*Proof.* Assume $E_j$ are decreasing, i.e.,

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$$

then $E_1^\complement \subseteq E_2^\complement \subseteq \cdots$ then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j^\complement\right) = \lim_{j\to\infty} \mu(E_j^\complement)$$

$$\mu(X) - \mu\left(\left(\bigcup_{j=1}^{\infty} E_j^\complement\right)^\complement\right) = \lim_{j\to\infty} (\mu(X) - \mu(E_j))$$

$$\mu(X) - \mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} (\mu(X) - \mu(E_j))$$

□

**Example 2.4.6**

$\mathbb{N}$ with counting measure, $E_j = \{j, j+1, j+2, \ldots\}$, $\mu(E_j) = \infty$, $\bigcap E_j = \emptyset$ has measure 0.

**Definition 2.4.7.** If $A_1, A_2, A_3, \ldots$ is an arbitrary sequence of measurable sets. We can define

$$\limsup A_j := \bigcap_{n=1}^{\infty} \bigcup_{j\geq n} A_j = \{x : x \in A_n \text{ for infinitely many } n\}$$

$$\liminf A_j := \bigcup_{n=1}^{\infty} \bigcap_{j\geq n} A_j = \{x : x \text{ belong to all but finitely many}\}$$

**Lemma 2.4.8** (Borel-Cantelli Lemma)

If $\{A_j\}$ is a sequence of measurable sets such that

$$\sum_{j=1}^{\infty} \mu(A_j) < \infty$$

then almost every $x$ (meaning all $x$ except in a null set) belong to on $A_n$ for only finitely many $n$. Or equivalently,

$$\mu\left(\limsup A_n\right) = 0$$

*Proof.* $\bigcup_{j\geq n} A_j$ are decreasing. In Borel Cantelli, we have $\sum \mu(A_j) < \infty$, so $\mu\left(\bigcup A_n\right) = 0$.

use "continuity from above"

$$\mu(\limsup A_n) = \lim_{n\to\infty} \mu\left(\bigcup_{j\geq n} A_j\right)$$

$$\mu\left(\bigcup_{j\geq n} A_j\right) \leq \sum_{j\geq n} \mu(A_j) \to 0$$

as $n \to \infty$. $\square$

Completion of a $\sigma$-algebra (when a measure $\mu$ is given), $(X, \mathcal{M}, \mu)$ $\overline{\mathcal{M}}$ consists of all unions $E \cup F$, where $E \in \mathcal{M}$ and $F \subseteq N \in \mathcal{M}$ for some null set $N$, $\mu(N) = 0$.

Define $\overline{\mu}$ by $\overline{\mu}(E \cup F) = \mu(E)$.

## §2.5 Measurable Functions

**Definition 2.5.1.** $f : X \to Y$ where $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces. $f$ is $(\mathcal{M}, \mathcal{N})$-measurable if for every $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$. where $f^{-1}(E) = \{x \in X : f(x) \in E\}$.

**Lemma 2.5.2**

Let $\mathcal{E}$ generate $\mathcal{N}$ (i.e., $\mathcal{N} = \mathfrak{M}(\mathcal{E})$). Then $f$ is $(\mathcal{M}, \mathcal{N})$-measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

*Proof.* Define $\mathcal{C} = \{E : f^{-1}(E) \in \mathcal{M}\}$, observe that $\mathcal{C}$ is a $\sigma$-algebra. then

$$f(x) = \bigcup E_j \iff x \in f^{-1}\left(\bigcup E_j\right) \iff x \in \bigcup f^{-1}(E_j) \iff \bigcup\{x : f(x) \in E_j\}$$

$\square$

**Claim 2.5.3** — $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable, $g : Y \to Z$ is $(\mathcal{N}, \mathcal{R})$-measurable then $g \circ f : X \to Z$ is $(\mathcal{M}, \mathcal{R})$-measurable.

*Proof.* $(g \circ f)^{-1}(E) = \{x \in X : g(f(x)) \in E\} = f^{-1}(g^{-1}(E)) = \{x \in X : f(x) \in g^{-1}(E)\}$ $\square$

**Claim 2.5.4** — $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable then $f^2$ is $\mathcal{M}$-measurable.

*Proof.* $(f^2)^{-1}(-\infty, a) = \{x : f^2(x) < a\} = \{x : f(x) < \sqrt{a}\} \cup \{x : f(x) > -\sqrt{a}\}$ □

**Claim 2.5.5** — $f : X \to \mathbb{R}, g : X \to \mathbb{R}$ are $\mathcal{M}$-measurable then $f + g$ and $f \cdot g$ are $\mathcal{M}$-measurable.

*Proof.*

$$(f + g)^{-1}(-\infty, a) = \bigcup_{r \in \mathbb{Q}} \left(f^{-1}(-\infty, a + r) \cap g^{-1}(-\infty, r)\right)$$

$$\begin{aligned}(f + g)^2 &= f^2 + 2fg + g^2 \\ fg &= \frac{1}{2}\left((f + g)^2 - f^2 - g^2\right)\end{aligned}$$

□

**Claim 2.5.6** — vector-valued-function $f : X \to (Y_1 \times Y_2 \times \cdots \times Y_n)$ and defined by $x \mapsto (f_1(x), f_2(x), \ldots, f_n(x))$ where $f_j : X \to Y_j$ is $(\mathcal{M}, \mathcal{N}_j)$-measurable.

Then $f$ is $(\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_n)$ if and only if $f_i(\mathcal{M}_i, \mathcal{N}_i)$-measurable.

*Proof.*

$$\begin{aligned}f^{-1}(E_1 \times E_2 \times \cdots \times E_n) &= f_1^{-1}(E_1) \cap f_2^{-1}(E_2) \cap \cdots \cap f_n^{-1}(E_n) \\ &= \bigcap_{j=1}^{n} f_j^{-1}(E_j)\end{aligned}$$

□

**Claim 2.5.7** — $M(x) = \max\{f(x), g(x)\}$, $f, g : X \to \mathbb{R}$, $\mathcal{M}$-measurable.

*Proof.* $M^{-1}(-\infty, a) = \{x : M(x) < a\} = \{x : f(x) < a, g(x) < a\} = f^{-1}(-\infty, a) \cap g^{-1}(-\infty, a)$ □

**Claim 2.5.8** — $f_n : X \to \mathbb{R}$, $\mathcal{M}$-measurable, then $S(x) = \sup_{n \in \mathbb{N}} f_n$ is $\mathcal{M}$-measurable.

*Proof.* $S^{-1}(-\infty, a) = \{x : S(x) < a\} = \{x : \sup f_n(x) < a\} = \bigcap_n \{x : f_n(x) < a\}$ □

**Remark 2.5.9.** We use the similar proof for min and inf.

**Definition 2.5.10.** If $f_n : X \to \mathbb{R}$, $\mathcal{M}$-measurable then

$$\limsup f_n = \inf_k \sup_{n \geq k} f_n$$

$$\liminf f_n = \sup_k \inf_{n \geq k} f_n$$

**Claim 2.5.11** — $\limsup f_n$ and $\liminf f_n$ are $\mathcal{M}$-measurable.

*Proof.* For $\limsup f_n$, fix $k$ then $\sup_{n \geq k} f_n$ is $\mathcal{M}$-measurable, $\inf_k \sup_{n \geq k} f_n$ is $\mathcal{M}$-measurable. Similarly for $\liminf f_n$. □

**Theorem 2.5.12**

Let $(X, \mathcal{M})$ be a measurable space, $f_n : X \to \mathbb{C}$ be $\mathcal{M}$-measurable functions. Define

$$E_{lim} = \{x \in X : \lim_{n \to \infty} f_n(x) \text{ exists}\}$$

then $E_{lim} \in \mathcal{M}$.

*Proof.* We can rewrite $E_{lim}$ as

$$E_{lim} = \{x \in X : \{f_n(x)\}_{n \in \mathbb{N}} \text{ is a Cauchy sequence}\}$$

Define

$$A_{n,m}(k) = \left\{ x \in X : |f_n(x) - f_m(x)| < \frac{1}{k} \right\}$$

then $A_{n,m}(k) \in \mathcal{M}$ for all $n, m, k$. then

$$E_{lim} = \bigcup_{k \geq 1} \bigcap_{N \geq 1} \bigcup_{m \geq N, n \geq N} A_{m,n}(k)$$

□

# 3 Integration

## §3.1 Simple Functions

**Definition 3.1.1.** nonnegative simple function are measurable function with finitely many values in $\mathbb{R}$ (NOT on $\overline{\mathbb{R}}$). $s : X \to \mathbb{R}$, $s(x) = \sum z_j \mathbb{1}_{x,s(z)=z_j}(x) = \sum z_j \mathbb{1}_{f^{-1}(z_j)}$ If values of $s$ are $\{z_1, \ldots, z_n\}$

**Theorem 3.1.2**

Consider nonnegative measurable function $f$. There exist a sequence of simple function $s_n$ such that

- $0 \leq s_n \leq s_{n+1} \leq f$ (i.e, $s_n(x) \leq s_{n+1}(x)$)
- $\lim\limits_{n\to\infty} s_n(x) = f(x)$ for all $x$
- The convergence is uniform on all sets where $f$ is bounded. If $E$ is such that $|f(x)| \leq M$ for all $x \in E$ then

$$\sup_{x\in E} f(x) - s_n(x) \to 0$$

*Proof.* $s_n$ is defined so that if takes value in $[0, 2^n)$. Consider the segment $\frac{k}{2^n}$ on y-axis, then

$$s_n(x) = \begin{cases} k \cdot 2^{-n} & \text{if } k2^{-n} \leq f(x) < (k+1)2^{-n}, 0 \leq k \leq 4^n - 1 \\ 2^n & \text{if } f(x) \geq 2^n \end{cases}$$

If $f(x) < 2^n$ then $0 \leq f(x) - s_n(x) \leq 2^{-n}$. We can see that $s_n(x) \leq s_{n+1}(x)$ because each step of $s_{n+1}$ is a refinement of $s_n$. □

We first define the integral for simple function (in analogy to the definition of Riemann-integral for stop functions)

**Definition 3.1.3.** Define $s(x) = \sum_j c_j \mathbb{1}_{E_j}$ where the $E_j$ are pairwise disjoint, $\biguplus E_j = X$, then

$$\int s \, \mathrm{d}\mu = \sum_j c_j \mu(E_j)$$

**Claim 3.1.4** —

$$s(x) = \sum_{j=1}^{n} c_j \mathbb{1}_{E_j}(x) = \sum_{k=1}^{m} d_k \mathbb{1}_{E_k}(x)$$

where $X = \biguplus E_j = \biguplus E_k$. If $x \in E_j \cap E_k$ then $c_j = d_k$.

*Proof.* We know that $\biguplus_{j,k} E_j \cap E_k = X$ and $E_j = \biguplus_k E_j \cap E_k$

GOAL: $\sum_{j=1}^{n} c_j \mu(E_j) = \sum_{k=1}^{m} d_k \mu(F_k)$

$$\begin{aligned} \text{LHS} = \sum_{j=1}^{n} c_j \sum_{k=1}^{\infty} \mu(E_j \cap F_k) &= \sum_{k=1}^{m} \sum_{j=1}^{n} d_k \mu(E_j \cap E_k) \\ &= \sum_{k=1}^{m} d_k \mu(F_k) \end{aligned}$$

□

**Lemma 3.1.5**

Suppose $s, t$ are simple functions then

$$\int (s+t) \, \mathrm{d}\mu = \int s \, \mathrm{d}\mu + \int t \, \mathrm{d}\mu$$

**Remark 3.1.6.** Can shortly write

$$\int s + t = \int s + \int t$$

*Proof.*

$$\begin{aligned} s &= \sum_{j=1}^{n} c_j \mathbb{1}_{E_j} = \sum_j \sum_k c_j \mathbb{1}_{E_j \cap F_k} \\ t &= \sum_{k=1}^{m} d_k \mathbb{1}_{F_k} = \sum_j \sum_k d_k \mathbb{1}_{E_j \cap F_k} \\ s + t &= \sum_{j,k} (c_j + d_k) \mathbb{1}_{E_j \cap F_k} \end{aligned}$$

$$\int s \, \mathrm{d}\mu = \sum_{j,k} c_j \mu(E_j \cap F_k)$$

$$\int t \, \mathrm{d}\mu = \sum_{j,k} d_k \mu(E_j \cap F_k)$$

$$\int (s+t) \, \mathrm{d}\mu = \sum_{j,k} (c_j + d_k) \mu(E_j \cap F_k)$$

□

**Lemma 3.1.7**

$\nu(E) = \int_E s \, \mathrm{d}\mu = \int s \mathbb{1}_E \, \mathrm{d}\mu = \sum c_j \mu(E_j \cap E)$ this defines a measure on $\mathcal{M}$ (given $\sigma$-algebra)

*Proof.* If $E^l$ is a sequence of pairwise disjoint measureable set, check

$$\nu\left(\biguplus E^l\right) = \sum \nu(E^l)$$

$$\begin{aligned}
\nu\left(\biguplus E^l\right) &= \sum c_j \mu(E_j \cap \biguplus E^l) \\
&= \sum_{j=1}^{n} c_j \sum_l \mu(E_j \cap E^l) \\
&= \sum_l \sum_j c_j \mu(E_j \cap E^l) \\
&= \sum_l \nu(E^l)
\end{aligned}$$

□

## §3.2 Non-negative Measurable Functions

**Definition 3.2.1.** For any non-negative $f$, a measurable function, define

$$\int f \, \mathrm{d}\mu = \sup_{\substack{s \le f \\ s \text{ simple}}} \int s \, \mathrm{d}\mu$$

**Remark 3.2.2.** If $0 \le f \le g$ then $\int f \, \mathrm{d}\mu \le \int g \, \mathrm{d}\mu$

**Theorem 3.2.3** (Monotone Convergence Theorem)
If $\{f_n\}$ is a sequence of measurable function, and $0 \le f_n \le f_{n+1}$ for all $n$. (that means $f(x) = \lim_{n\to\infty} f_n(x)$)Then

$$\int f \, \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu$$

*Proof.* Since $f_n \le f_{n+1} \le f$ then

$$\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu \le \int f \, \mathrm{d}\mu$$

We need to show that

$$\int f \, \mathrm{d}\mu \le \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu$$

So, it suffices to show that for any $0 \le \underset{\text{simple}}{s} \le f$, that

$$\int s \, \mathrm{d}\mu \le \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu$$

It suffices to show that for any $\varepsilon > 0$,

$$(1-\varepsilon) \int s \, \mathrm{d}\mu \le \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu$$

Define $E_n = \{x : (1-\varepsilon)s(x) \le f_n(x)\}$, any $x$ will be in one of the $E_n$. Then for any $x \in E_n$,

$$s(x) \le \frac{f_n(x)}{1-\varepsilon}$$

Consider the measure defined by

$$\nu(E) = \int_E s \, \mathrm{d}\mu$$

(we already show this is a measure in 3.1.7). We have $E_n \subseteq E_{n+1}$ and $E_n \to X$. By continuity from below 2.4.3,

$$\lim_{n\to\infty} \nu(E_n) = \nu(X) = \int s \, \mathrm{d}\mu$$

We get that

$$\nu(E_n) = \int_{E_n} s \, \mathrm{d}\mu \le \int_{E_n} \frac{f_n(x)}{1-\varepsilon} \, \mathrm{d}\mu \le \int \frac{f_n(x)}{1-\varepsilon} \, \mathrm{d}\mu = \frac{1}{1-\varepsilon} \int f_n(x) \, \mathrm{d}\mu$$

Finally, we take limit on both sides and we have

$$\lim_{n\to\infty} \nu(E_n) = \nu(\mathbb{R}) = \int s \,\mathrm{d}\mu \le \lim_{n\to\infty} \frac{1}{1-\varepsilon} \int f_n \,\mathrm{d}\mu$$

□

**Lemma 3.2.4**

If $f, g$ are non negative measurable function then

$$\int (f+g) \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu$$

*Proof.* Now we have a tool

- Monotone Convergence Theorem
- Existence of $s_n \gg f, t_n \gg g$

$$\int (s_n + t_n) \,\mathrm{d}\mu = \int s_n \,\mathrm{d}\mu + \int t_n \,\mathrm{d}\mu$$
$$\int (f+g) \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu$$

□

**Lemma 3.2.5**

$f_k \ge 0$, $f_k$ is measurable

$$\int \sum_{k=1}^{\infty} f_k(x) \,\mathrm{d}\mu = \sum_{k=1}^{\infty} \int f_k \,\mathrm{d}\mu$$

*Proof.* Just apply the Monotone Convergence Theorem.

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \to \sum_{k=1}^{\infty} f_k(x)$$

□

**Remark 3.2.6.** We cannot always interchange integrals and limits (monotonicity is key) $f_n(x) = \frac{1}{n}\mathbb{1}_{[0,n]}, \int f_n \,\mathrm{d}\mu = 1$ but $\lim_{n\to\infty} f_n(x) = 0$.

$$0 = \int \lim_{n\to\infty} f_n(x) \,\mathrm{d}\mu < \lim_{n\to\infty} \int f_n \,\mathrm{d}\mu$$

Or on $[0,1]$, $f_n(x) = n\mathbb{1}_{[0,1/n]}$, $\int f_n \, \mathrm{d}\mu = 1$ but $\lim_{n\to\infty} f_n(x) = 0$.

$$\lim_{n\to\infty} f_n(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

**Lemma 3.2.7** (Fatou's Lemma)
If $\{f_j\}$ is a sequnce of measurable functions

$$\int \liminf_{j\to\infty} f_j(x) \, \mathrm{d}\mu \le \liminf_{j\to\infty} \int f_j \, \mathrm{d}\mu$$

meaning

$$\int \lim_{k\to\infty} \underbrace{\inf_{j\ge k} f_j(x) \, \mathrm{d}\mu}_{\text{increasing on } k} \le \lim_{k\to\infty} \inf_{j\ge k} \int f_j \, \mathrm{d}\mu$$

*Proof.*

$$\int \lim_{k\to\infty} \inf_{j\ge k} f_j(x) \, \mathrm{d}\mu \underset{\text{MCT}}{=} \lim_{k\to\infty} \int \inf_{j\ge k} f_j(x) \, \mathrm{d}\mu$$

Take any $l \ge k$, then $\inf_{j\ge k} f_j(x) \le f_l(x)$, then for $l \ge k$

$$\int \inf_{j\ge k} f_j(x) \, \mathrm{d}\mu \le \int f_l(x) \, \mathrm{d}\mu$$

$$\int \inf_{j\ge k} f_j(x) \, \mathrm{d}\mu \le \inf_{j\ge k} \int f_j(x) \, \mathrm{d}\mu$$

□

## §3.3 General Measurable Functions

Integral for "general" measurable functions.

**Definition 3.3.1.** Given a measurable function $f$, we define the **positive part** of $f$ as

$$f^+(x) = \max\{f(x), 0\}$$

and the **negative part** of $f$ as

$$f^-(x) = \max\{-f(x), 0\}$$

Then we get that

$$f = f^+ - f^-$$

**Definition 3.3.2.** $f : X \to \mathbb{R}$ (or $\overline{\mathbb{R}}$) Suppose that $f$ is a measurable function, then we define

$$\int f \, \mathrm{d}\mu = \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu$$

provided that at least one of $\int f^{\pm} \, \mathrm{d}\mu$ is finite

**Definition 3.3.3.** $f : X \to \mathbb{R}$ (or $\overline{\mathbb{R}}$) $f$ is **integrable** if $\int f^+ \, \mathrm{d}\mu$, $\int f^- \, \mathrm{d}\mu$ is finite $\iff$ $\int |f| \, \mathrm{d}\mu$ is finite

$\mathcal{L}^1$ is the class of integrable function

**Definition 3.3.4.** $f : X \to \mathbb{C}$ is measureable ( $\iff$ $\Re(f)$ and $\Im(f)$ are measurable) Assumeing that $\Re f \in \mathcal{L}^1$ and $\Im f \in \mathcal{L}^1$ then

$$\int f \, \mathrm{d}\mu = \int \Re f \, \mathrm{d}\mu + i \int \Im f \, \mathrm{d}\mu$$

**Claim 3.3.5** — Suppose that $f, g$ are measurable then

$$\int f + g \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$$

$$\int \alpha f \, \mathrm{d}\mu + \alpha \int f \, \mathrm{d}\mu$$

**Lemma 3.3.6**

$f : X \to \overline{\mathbb{R}}$ is measurable, and $\int |f| \, \mathrm{d}\mu = 0$ then $f = 0$ almost everywhere.

*Proof.* Define $E_n = \left\{x : |f(x)| > \frac{1}{n}\right\}$ then from continuity from below, we get that

$$\lim_{n \to \infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$$

define $E = \bigcup E_n$ and we can write $E = \{x : |f(x)| > 0\}$then we know that

$$|f| \geq |f| \mathbb{1}_{E_n} \geq \frac{1}{n} \mathbb{1}_{E_n}$$

then

$$\begin{aligned} \int |f| \, \mathrm{d}\mu &\geq \int \frac{1}{n} \mathbb{1}_{E_n} \, \mathrm{d}\mu \\ &= \frac{1}{n} \mu(E_n) \end{aligned}$$

we get that $\mu(E_n) = 0$ for all $n$ then $\mu(E) = 0$. Therefore $f = 0$ almost everywhere. $\square$

**Remark 3.3.7.** $\|f\| = \int |f| \, \mathrm{d}\mu$ satisfies

- $\|f+g\| \leq \|f\| + \|g\|$
- $\|cf\| = |c|\|f\|$
- $\|f\| = 0 \iff f = 0$ almost everywhere

**Remark 3.3.8.** Almost everywhere equal is an equivalence relation.

$$f \sim g \underset{\text{def}}{\iff} f(x) = g(x) \ \mu\text{-almost everywhere}$$

$N = \{f \in \mathcal{L}^1 : f(x) = 0 \text{ almost everywhere}\}$ is a linear subspace of $\mathcal{L}^1$ vector. $\mathcal{L}^1/N$ is the set of equivalence classes of $\mathcal{L}^1$.

$f_n \to f$ almost everywhere, $f_n \geq 0$, $f_n$ measurable, Can we define $\int f \, \mathrm{d}\mu$? $f$ may not be measurable. This problem is fixed if $f$ we work in a complete measurable space $(X, \mathcal{M}, \mu) \to (X, \overline{\mathcal{M}}, \overline{\mu})$ where

$$\overline{\mathcal{M}} = \{A \cup B : A \in \mathcal{M}, B \text{ a subset of a set of measure } 0\}$$

**Lemma 3.3.9**

$f \in \mathcal{L}^1$, $\int |f| \, \mathrm{d}\mu < \infty$. If $f$ is real valued $f = f^+ - f^-$,

$$\left| \int f \, \mathrm{d}\mu \right| \leq \int |f| \, \mathrm{d}\mu$$

*Proof.*

$$\begin{aligned} \left| \int f \, \mathrm{d}\mu \right| &= \left| \int f^+ - f^- \, \mathrm{d}\mu \right| \\ &\leq \left| \int f^+ \, \mathrm{d}\mu \right| + \left| \int f^- \, \mathrm{d}\mu \right| \\ &= \int f^+ \, \mathrm{d}\mu + \int f^- \, \mathrm{d}\mu \\ &= \int |f| \, \mathrm{d}\mu \end{aligned}$$

□

**Remark 3.3.10.** If $f$ is complex valued, then $|f| = \sqrt{(\Re f)^2 + (\Im f)^2}$. Then

$$\left| \int \Re f \right| \leq \int |\Re f| \leq \int |f|$$

$$\left|\int \Im f\right| \leq \int |\Im f| \leq \int |f|$$

So,

$$\left|\int f \, \mathrm{d}\mu\right| \leq 2\int |f|$$

**Remark 3.3.11.** Estimate $\int f \, \mathrm{d}\mu = \alpha + i\beta = re^{i\phi}$, then $e^{-i\phi}\int f \, \mathrm{d}\mu$ is real and nonnegative.

$$\begin{aligned}\left|\int f \, \mathrm{d}\mu\right| &= \left|e^{-i\phi}\int f \, \mathrm{d}\mu\right| \\ &= \Re\int e^{-i\phi} f \, \mathrm{d}\mu \\ &\leq \int |e^{-i\phi} f| \, \mathrm{d}\mu \\ &= \int |f| \, \mathrm{d}\mu\end{aligned}$$

**Lemma 3.3.12**

$f \in \mathcal{L}^+$ means non-negative, then $\nu(E) = \int_E f \, \mathrm{d}\mu$ define measure

*Proof.* Check the $\sigma$-additivity $E = \biguplus_{n=1}^{\infty} E_n$,

$$\begin{aligned}\nu\left(\biguplus_{n=1}^{\infty} E_n\right) &= \int_{\biguplus E_n} f \, \mathrm{d}\mu \\ &= \int f \mathbb{1}_{\biguplus E_n} \, \mathrm{d}\mu \\ &= \int f\left(\sum_{n=1}^{\infty} \mathbb{1}_{E_n}\right) \, \mathrm{d}\mu \\ &= \sum_{n=1}^{\infty}\int f\mathbb{1}_{E_n} \, \mathrm{d}\mu \\ &= \sum_{n=1}^{\infty} \nu(E_n)\end{aligned}$$

□

**Claim 3.3.13** — If $f \in \mathcal{L}^1 \cap \mathcal{L}^+$ then $\nu$ is a finite measure.

If $\nu(E) = intf \, \mathrm{d}\mu$ How does $\int g \, \mathrm{d}\nu$ look like? $\nu(E) = \int f \, \mathrm{d}\mu = \int E \, \mathrm{d}\nu$ We want "$f \, \mathrm{d}\mu = \mathrm{d}\nu$"

**Lemma 3.3.14**

If $f \in \mathcal{L}^+$ and $\nu(E) = \int_E f \, \mathrm{d}\mu$ then for any $g \in \mathcal{L}^+$ or $g \in \mathcal{L}^1$ then,

$$\int g \, \mathrm{d}\nu = \int gf \, \mathrm{d}\mu$$

*Proof.*

- True for characteristic functions of measure set by the definition of $\nu$. Fix $g = \mathbb{1}_E$ for some $E \in \mathcal{M}$

$$\int g \, \mathrm{d}\nu = \int \mathbb{1}_E \, \mathrm{d}\mu = \nu(E) = \int_E f \, \mathrm{d}\mu = \int \mathbb{1}_E f \, \mathrm{d}\mu = \int gf \, \mathrm{d}\mu$$

- By linearity of the integral, it is true for simple function. Fix $g = \sum_{j=1}^n c_j \mathbb{1}_{E_j}$, then

$$\int g \, \mathrm{d}\nu = \sum_{j=1}^n c_j \nu(E_j) = \sum_{j=1}^n c_j \int_{E_j} f \, \mathrm{d}\mu = \int gf \, \mathrm{d}\mu$$

- $s_n \nearrow g$ if $g \in \mathcal{L}^+$, by Monotone convergence theorem,

$$\int s_n \underset{\text{MCT}}{\nearrow} \int g$$

$$\int s_n \, \mathrm{d}\nu = \int s_n f \, \mathrm{d}\mu$$
$$\int g \, \mathrm{d}\nu = \int gf \, \mathrm{d}\mu$$

Then extend to general g by linearity

$\square$

**Theorem 3.3.15**

If $X$ is a finite measure space, if $f_n$ measurable, $f_n \in \mathcal{L}^1$ (integrable) and $f_n \to f$ uniformly on $X$. then

$$\int |f_n - f| \, \mathrm{d}\mu \to 0$$

and

$$\int f_n \, \mathrm{d}\mu \to \int f \, \mathrm{d}\mu$$

**Remark 3.3.16.** Uniform convergence means

$$\lim_{n\to\infty} \sup_{x\in X} |f_n(x) - f(x)| = 0$$

*Proof.* We can rewrite that term as

$$\begin{aligned}\int |f_n - f| \, \mathrm{d}\mu &\le \int \sup_{x\in X} |f_n - f| \, \mathrm{d}\mu \\ &= \mu(X) \sup_{x\in X} |f_n - f| \to 0\end{aligned}$$

We can rewrite $f$ as $f = (f - f_n) + (f_n)$ since $f - f_n$ converge and $f_n$ integrable so $f$ must be integrable.

$$\begin{aligned}\left| \int f_n - \int f \right| &= \left| \int (f_n - f) \, \mathrm{d}\mu \right| \\ &\le \int |f_n - f| \, \mathrm{d}\mu\end{aligned}$$

□

**Definition 3.3.17.** Suppose that $f_n, f$ are measurable $f_n \to f$ almost uniformly if for every $\varepsilon > 0$ there is a measurable set $E$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on $E^\complement$ ($\sup_{x\in E^\complement} |f_n(x) - f(x)| \to 0$)

**Theorem 3.3.18** (Egorov's Theorem)
If $\mu(X) < \infty$ and if $f_n \to f$ almost everywhere then $f_n \to f$ almost uniformly

**Remark 3.3.19.** $f_n(x) \to f(x)$ if for every $k$ there exists $n = n(k)$ such that $|f_m(x) - f(x)| < \frac{1}{k}$ for all $m \ge n(k)$

*Proof.* Fix $\varepsilon > 0$, define

$$\begin{aligned}E_n(k) &:= \left\{ x : |f_m(x) - f(x)| \ge \frac{1}{k} \text{ for some } m \ge n \right\} \\ &= \bigcup_{m\ge n} \left\{ x : |f_m(x) - f(x)| \ge \frac{1}{k} \right\}\end{aligned}$$

(Given $x$ For sufficiently large $n$, $x \notin E_n(k)$), $E_n(k) \supseteq E_{n+1}(k)$ $\bigcap_n E_n(k) = \emptyset$ because of $f_n \to f$ everywhere. Form the continuity from above 2.4.5, we get that

$\lim_{n\to\infty} \mu(E_n(k)) = 0$. Find $n(k)$ such that $\mu(E_{n(k)}(k)) < \frac{\varepsilon}{2^k}$, then $E = \bigcup_k E_{n(k)}(k)$ has measure $< \varepsilon$.

For $x \in \left(\bigcup_k E_{n(k)}(k)\right)^\complement = \bigcap_k E_{n(k)}(k)^\complement$ I have for all $k$ $|f_m(x) - f(x)| < \frac{1}{k}$ for all $m \geq n(k)$. So, we get $f_n \to f$ uniformly on $E^\complement$. $\square$

**Theorem 3.3.20** (Baby Dominated Convergence Theorem)
Given $(X, \mathcal{M}, \mu)$ where $\mu$ is a finite measure ($\mu(X) < \infty$). Let $\{f_n\}$ be measurable functions, $f_n \to f$ everywhere.

$$|f_n| \leq C \implies \int |f_n - f| \, \mathrm{d}\mu \to 0$$

i.e. $f_n$ converges with respect to $L^1$-(semi-)norm.

**Corollary 3.3.21**

$$\int_X f_n \, \mathrm{d}\mu \to \int_X f \, \mathrm{d}\mu$$

*Proof.* Tools:

(i) If $f_n \to f$ uniformly then $\int |f_n - f| \, \mathrm{d}\mu \to 0$

(ii) Egorov's Theorem

$|f(x)| \leq C$, $f$ is measurable. Given any $\varepsilon > 0$, By Egorov's Theorem, find a set of measure $E$ that $\mu(E) < \frac{\varepsilon}{4C}$ such that $f_n \to f$ uniformly on $E^\complement$. Then

$$\int |f_n - f| \, \mathrm{d}\mu \leq \int_E |f_n - f| \, \mathrm{d}\mu + \int_{E^\complement} |f_n - f| \, \mathrm{d}\mu$$

we know that $|f_n - f| \leq |f_n| + |f| \leq 2C$ then

$$\begin{aligned} \int |f_n - f| \, \mathrm{d}\mu &\leq 2C\mu(E) + \int_{E^\complement} |f_n - f| \, \mathrm{d}\mu \\ &\leq \frac{\varepsilon}{2} + \int_{E^\complement} |f_n - f| \, \mathrm{d}\mu \end{aligned}$$

so for large $n$, the second term will be $< \frac{\varepsilon}{2}$. $\square$

**Theorem 3.3.22** (Dominated Convergence Theorem)

Given $(X, \mathcal{M}, \mu)$ where $\mu$ is a finite measure ($\mu(X) < \infty$). Let $\{f_n\}$ be measurable functions, $f_n \to f$ almost everywhere.

$$\sup_n |f_n| \in \mathcal{L}^1 \implies \int |f_n - f| \, \mathrm{d}\mu \to 0$$

*Proof.* Define $g(x) = \sup_n |f_n(x)|$ The trick is

$$|f_n - f| = \begin{cases} \frac{|f_n - f|}{g} g & \text{if } g > 0 \\ 0 & \text{if } g = 0 \end{cases}$$

define a new measure $\nu(E) = \int_E g \, \mathrm{d}\mu$. Then $\nu$ is a finite measure, and

$$g \, \mathrm{d}\mu = \mathrm{d}\nu$$
$$\int h \, \mathrm{d}\nu = \int hg \, \mathrm{d}\mu$$

then define

$$h_n = \begin{cases} \frac{|f_n - f|}{g} & \text{if } g > 0 \implies |h_n(x)| \leq 2 \\ 0 & \text{if } g = 0 \implies h_n(x) \to 0 \end{cases}$$

Then

$$\begin{aligned} \int |f_n - f| \, \mathrm{d}\mu &= \int h_n g \, \mathrm{d}\mu \\ &= \int h_n \, \mathrm{d}\nu \to 0 \end{aligned}$$

By Baby Dominated Convergence Theorem □

## §3.4 Integration from Riemann to Lebesgue

**Theorem 3.4.1**

If $f$ is Riemann integrable on $[a, b]$ then $f$ is Lebesgue integrable.

$$\int_a^b f(x) \, \mathrm{d}x = \int_{[a,b]} f \, \mathrm{d}\mu$$

where $\mu$ is Lebesgue measure.

*Proof.* Define

$$U_P f(x) = \begin{cases} M_j & \text{if } x \in [x_{j-1}, x_j) \\ M_n & \text{if } x \in [x_{n-1}, x_n] \end{cases}$$

Similarly for the lower sum $L_P f(x)$. If $P'$ is a refinement of $P$ then $U_{P'} f(x) \le U_P f(x)$ and $L_{P'} f(x) \ge L_P f(x)$. Since $f$ is Riemann integrable, then

$$\inf_P U(f, P) =: \overline{\mathcal{I}}_a^b(f) = \underline{\mathcal{I}}_a^b(f) := \sup_P L(f, P)$$

Choose a sequence of partitions $P_n$ such that

$$\int U_{P_n} f \to \overline{\mathcal{I}}_a^b(f)$$
$$\int L_{P_n} f \to \underline{\mathcal{I}}_a^b(f)$$

Since that $P_{n+1}$ is a refinement of the $P_n$ then $U_{P_n} f \searrow U(x)$ and $L_{P_n} f \nearrow L(x)$ and $L(x) = U(x)$. Notice that from Riemann integrable, $|f| < C$, then

$$\int_{[a,b]} U_{P_n} f \to \overline{\mathcal{I}}_a^b(f) = \int_{[a,b]} U(x) \ \mathrm{d}m$$
$$\int_{[a,b]} L_{P_n} f \to \underline{\mathcal{I}}_a^b(f) = \int_{[a,b]} L(x) \ \mathrm{d}m$$

If $f$ is Riemann integrable,

$$\int U \ \mathrm{d}m = \int L \ \mathrm{d}m = \int_a^b f(x) \ \mathrm{d}x$$

and $U \ge L$ then

$$\int (U - L) \ \mathrm{d}m = 0 \implies U(x) = L(x)$$

almose everywhere, $L(x) \le f(x) \le U(x) \implies f = L$ almost everywhere and $f = U$ almost everywhere. Then $f$ is Lebesgue integrable and

$$\int f \ \mathrm{d}m = \int L \ \mathrm{d}m = \int U \ \mathrm{d}m$$

□

**Definition 3.4.2** (Improper Riemann integrals)**.**

$$\int_0^\infty f(x) \ \mathrm{d}x, \int_1^\infty f(x) \ \mathrm{d}x, \int_0^1 f(x) \ \mathrm{d}x$$

if $f$ is not Riemann-integrable on the domain but on every compact subinterval. We can define as

$$\int_1^\infty f(x) \ \mathrm{d}x = \lim_{R \to \infty} \int_1^R f(x) \ \mathrm{d}x$$

**Example 3.4.3**

$$\int_1^\infty \frac{\sin x}{x}\ \mathrm{d}x = \lim_{R\to\infty}\int_1^R \frac{\sin x}{x}\ \mathrm{d}x$$

$I_k = [2k\pi + \frac{\pi}{4}, 2k\pi + \frac{3\pi}{4}]$, $\sin x \geq \frac{1}{\sqrt{2}}$, so, $\frac{\sin x}{x} \geq \frac{1}{x\sqrt{2}} \cdot \frac{1}{2k\pi + \frac{3\pi}{4}}$. We can do integration by parts

$$\int_1^R \frac{\sin x}{x}\ \mathrm{d}x = -\frac{\cos x}{x}\Big|_1^R - \int_1^R -\frac{\cos x}{x^2}\ \mathrm{d}x$$

**Example 3.4.4**

$$\int_0^\infty \sin(x^2)\ \mathrm{d}x$$

consider $\sin(x^2)$

$$\sqrt{2k\pi + \frac{\pi}{2}} \leq \sqrt{x^2} \leq \sqrt{2k\pi + \frac{3\pi}{4}}$$

$$\sqrt{2k\pi + \frac{3\pi}{4}} - \sqrt{2k\pi + \frac{\pi}{4}} \approx \frac{1}{\sqrt{k}}$$

**Lemma 3.4.5**

Suppose that if $\int_1^\infty |f(x)|\ \mathrm{d}x < \infty$ then $f \in \mathcal{L}^1$.

*Proof.*

$$\begin{aligned}\int_1^\infty |f(x)|\ \mathrm{d}x &= \int_1^\infty \lim_{n\to\infty} |f(x)|\mathbb{1}_{[1,n]}(x)\ \mathrm{d}x \\ &= \lim_{n\to\infty}\int_1^n |f(x)|\ \mathrm{d}x\end{aligned}$$

□

**Theorem 3.4.6**

If $f$ is integrable on $\mathbb{R}$,

$$\int_{\mathbb{R}} |f(x)| \, \mathrm{d}x < \infty$$

$f \in \mathcal{L}^1$ then for every $\varepsilon > 0$, there is a continuous function ($C^\infty$) $g$, vanishes off a compact set,

$$\int |f - g| \, \mathrm{d}m < \varepsilon$$

## §3.5 Outer Measures

**Definition 3.5.1.** In our axiometic theorem on the Lebesgue measure, $m(I) = \ell(I)$, $m((a, b]) = b - a$ and for a general Borel set on $\mathbb{R}$, $m$ is given by the **outer measure** induced by the collection of intervals

$$\varrho(E) = \inf \sum_{k=1}^{\infty} \ell(I_k)$$

where the inf is taken over collections $\{I_k\}$, such that $E \subseteq \bigcup_{k=1}^{\infty} I_k$

**Remark 3.5.2.** $\widetilde{\varrho}$ defined similarly but we only admit open intervals in the infimum. Obviously, $\widetilde{\varrho}(E) \geq \varrho(E)$ Need to show that $\widetilde{\varrho}(E) \leq \varrho(E)$ we may assume that $\varrho(E) < \infty$, show $\widetilde{\varrho}(E) \leq \varrho(E) + \varepsilon$. There is a collection of intervals $I_k$ such that

$$\sum_k \ell(I_n) < \varrho(E) + \frac{\varepsilon}{2}$$

If $I_k = [a_k, b_k]$, then define $J_k = (a_k - \frac{\varepsilon}{2^{k+2}}, b_k + \frac{\varepsilon}{2^{k+2}})$ Then $\ell(J_k) = \ell(I_k) + \frac{\varepsilon}{2^{k+1}}$ then

$$\begin{aligned} \widetilde{\varrho}(E) \leq \sum_{k=1}^{\infty} \ell(J_k) &\leq \sum_{k=1}^{\infty} \ell(J_k) + \varepsilon 2^{-k-1} \\ &\leq \varrho(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

**Lemma 3.5.3**

$m(E) = \sup\{m(K) : K \subseteq E, K \text{ compact}\}$

*Proof.* Case where $E = \overline{E}$ and $E$ is bounded, there is nothing to show

Assume that $E$ is bounded, GOAL: find $K \subseteq E$ such that $m(E \setminus K) < \varepsilon$. Consider $\overline{E} \setminus E$, find $O \supseteq \overline{E} \setminus E$, $m(O \setminus (\overline{E} \setminus E)) < \varepsilon$. then $O^\complement \cap \overline{E} \subseteq E$ because if $x \in O^\complement$, either $x \in \overline{E}$ or

$x \in E$. $E \setminus K = E \cap K^{\complement} \subseteq O \cup \overline{E}^{\complement}$. Since $E \subseteq \overline{E}$ then $E \setminus K \subseteq O$ and $E \setminus K \subseteq O \setminus (\overline{E} \setminus E)$ has measure $< \varepsilon$. □

**Theorem 3.5.4**

For every Borel set $E$, $m(E) < \infty$, there is an open set $O \supseteq E$ such that $m(O \setminus E) < \varepsilon$. where $m(E) = \inf \sum \ell(I_n)$ where inf take over $I_k$, $I_k$ are open, $E \subseteq \bigcup I_k$

*Proof.* Define $E_n = E \cap \overline{B}(0, n)$. Find compact set $K_n \subseteq E_n \setminus E_{n-1}$ then $m((E_n \setminus E_{n-1}) \setminus K_n) < \varepsilon 2^{-n-1}$. The set $H_l = K_1 \cup \cdots \cup K_l$ is compact and increasing, $H_l \subseteq E_l$ and

$$m(E_l) - \varepsilon \leq m(H_l) \leq m(E_l) \to m(E)$$

□

**Theorem 3.5.5**

Given an open set $O$, we can decompose $O$ as a disjoint union of "dyadic cubes"

**Theorem 3.5.6**

We can choose the cubes a dyadic cubes such that if $O \neq \mathbb{R}^n$ such that

$$\operatorname{diam}(Q) < \operatorname{dist}(Q, O^C) \leq 4\operatorname{diam}(Q)$$

**Remark 3.5.7.** If side length of $Q$ is $2^{-k}$ then the diameter is $\sqrt{n}2^{-k}$.

**Theorem 3.5.8** (Whitney decomposition theorem)

Given $\Omega$ open set in $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, there is a family $\mathcal{F}$ of dyadic cubes such that

- they are disjoint
- $\biguplus_{Q \in \mathcal{F}} Q = \Omega$
- For every $Q \in \mathcal{F}$, $C\operatorname{diam}(Q) < \operatorname{dist}(Q, \Omega^C) \leq (2C+2)\operatorname{diam}(Q)$

*Proof.* Define for each $k$ a family of dyadic cubes $\mathcal{F}_k$ of side length $2^{-k}$ (i.e., the diameter is $\sqrt{n}2^{-k}$) intersecting the region

$$\Omega_k = \{x : A2^{-k}\sqrt{n} \leq \operatorname{dist}(x, \Omega^{\complement}) \leq 2A \cdot 2^{-k}\sqrt{n}\}$$

Pick a cube in $\mathcal{F}_k$, $Q_1$ it contains an $x_Q \in \Omega_k$

$$\operatorname{dist}(Q, \Omega^{\complement}) \le \operatorname{dist}(x_Q, \Omega^{\complement}) \le 2A \cdot 2^{-k}\sqrt{n} - 2A\operatorname{diam}(Q)$$

$$\begin{aligned}\operatorname{dist}(Q, \Omega^{\complement}) &\ge \operatorname{dist}(x_Q, \Omega^{\complement}) - \operatorname{diam}(Q)\\ &\ge A2^{-k}\sqrt{n} - 2A\operatorname{diam}(Q)\\ &= (A-1)\cdot \operatorname{diam}(Q)\end{aligned}$$

Then $\mathcal{F}_T = \bigcup \mathcal{F}_k$ and finally $\mathcal{F}$ = collection of all maximal (with respect to inclusion) cubes in $\mathcal{F}_T$. Fix $Q, Q'$ and assume that $Q \subseteq Q'$ then

$$(A-1)\operatorname{diam}Q' \le \operatorname{dist}(Q', \Omega^{\complement}) \le \operatorname{dist}(Q, \Omega^{\complement}) \le 2A \cdot \operatorname{diam}(Q)$$

then $\operatorname{diam}(Q') \le \frac{2A}{A-1}\operatorname{diam}(Q)$ $\square$

we know that for every $\varepsilon_1 > 0$ we can find a simple function $s$ such that $\int |f - s| \, \mathrm{d}m < \varepsilon_1$. (For non-negative $f$ use MCT $s_n \nearrow f$ and $s_n \le f$ so $\int s_n \nearrow \int f \implies \int f - s_n \to 0$ and then we use $f = f_+ - f_-$)

$s = \sum c_j \mathbb{1}_{E_j}$, $E_j \subseteq O_j$, $m(O_j \setminus E_j) < \varepsilon$ $\widetilde{s} = \sum c_j \mathbb{1}_{O_j}$

$$\begin{aligned}\int \widetilde{s} - s \, \mathrm{d}m &= \left|\int \sum c_j \mathbb{1}_{O_j} - \mathbb{1}_{E_j} \, \mathrm{d}m\right|\\ &\le \sum_{j=1}^{N} |c_j||m(O_j \setminus E_j)|\\ &\le \varepsilon_2\end{aligned}$$

Then $\mathbb{1}_{O_j} = \sum_\nu \mathbb{1}_{Q_\nu}$ where $\{Q_\nu\}$ are the Whitney cubes in Whitney Theorem. $|O_j| = \sum_{\nu \in I} m(Q_\nu)$ There is a finite $\widetilde{I}_j$ such that

$$\int \left|\mathbb{1}_{O_j} - \sum_{\nu \in \widetilde{I}_j} \mathbb{1}_{Q_\nu}\right| < \varepsilon_3$$

replace $\sum_{j=1}^N c_j \mathbb{1}_{O_j}$ by $\sum_{j=1}^N c_j \mathbb{1}_{\bigcup_{\nu \in \widetilde{I}_j} Q_\nu}$

then

$$\mathbb{1}_{Q_\nu}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \mathbb{I}_{\nu, \mathbb{i}}(x_i)$$

**Lemma 3.5.9**

For any $f \in L^1$ there exists $s$ a step function such that $\int |f - s| \, \mathrm{d}m < \varepsilon$

*Proof.* Suppose that $f \in L^1$ I want to show that there exists $s$ a step function such that $\int |f - s| \, \mathrm{d}m < \varepsilon$ for any $\varepsilon > 0$. Since $f = f^+ - f^-$, WLOG, $f \geq 0$ (otherwise we can do each positive and negative part and do the sum of both step functions with $\frac{\varepsilon}{2}$ bound). Given any $\varepsilon > 0$, there exists $s'$ a simple function such that $\int |f - s'| \, \mathrm{d}m < \frac{\varepsilon}{2}$. Then we can write $s' = \sum_{j=1}^{N} c_j \mathbb{1}_{E_j}$, then there exists $O_j$ open set such that $E_j \subseteq O_j$ and $m(O_j \setminus E_j) < \frac{\varepsilon}{4|c_j|N}$. Since $O_j$ is an open set, then $O_j = \bigcup (a_i, b_i)$ then define $K_n = \bigcup_{i=1}^{n} (a_i, b_i)$ from continuity from below, there exists $n'$ such that $m(K_{n'}) > m(O_j) - \frac{\varepsilon}{4|c_j|N}$ and $K_{n'}$ contain finite interval, then we define $O'_j := K_{n'}$. Define $s = \sum_{j=1}^{N} c_j \mathbb{1}_{O'_j}$ then

$$\begin{aligned}
\int |f - s| \, \mathrm{d}m &\leq \int |f - s'| \, \mathrm{d}m + \int |s' - s| \, \mathrm{d}m \\
&\leq \frac{\varepsilon}{2} + \int \left| \sum_{j=1}^{N} c_j (\mathbb{1}_{E_j} - \mathbb{1}_{O'_j}) \right| \, \mathrm{d}m \\
&\leq \frac{\varepsilon}{2} + \sum_{j=1}^{N} |c_j| \int \left| \mathbb{1}_{E_j} - \mathbb{1}_{O'_j} \right| \, \mathrm{d}m \\
&= \frac{\varepsilon}{2} + \sum_{j=1}^{N} |c_j| (m(E_j \setminus O'_j) + m(O'_j \setminus E_j)) \\
&\leq \frac{\varepsilon}{2} + \sum_{j=1}^{N} |c_j| (m(O_j \setminus O'_j) + m(O_j \setminus E_j)) \\
&\leq \frac{\varepsilon}{2} + \sum_{j=1}^{N} |c_j| \left( \frac{\varepsilon}{4|c_j|N} + \frac{\varepsilon}{4|c_j|N} \right) \\
&< \varepsilon
\end{aligned}$$

$\square$

# 4 $L^p$ Spaces

## §4.1 normed spaces

**Remark 4.1.1.** If $f_n \to f$ almost everywhere, do we have $\int |f_n - f| \, \mathrm{d}\mu \to 0$?

- No, if $f_n = \mathbb{1}_{[n,n+1]}$ then $f_n \to 0$ almost everywhere but $\int |f_n - 0| \, \mathrm{d}\mu = 1$ and $\int |f_n - f_m| \, \mathrm{d}\mu = 2$.
- No, if $f_n = \mathbb{1}_{[0,\frac{1}{n}]}$

**Remark 4.1.2.** Convergence in $L^1$ implies convergence almost everywhere? No, If $2^k \leq n \leq 2^{k+1}$ where $n = 2^k + j$, $j = 0, \ldots, 2^k - 1$ $f_{2^k+1} = \mathbb{1}_{[i2^{-k},(i+1)2^{-k}]}$ for $i = 0, \ldots, 2^k - 1$. For $2^k \leq n \leq 2^{k+1}$, $\|f_n\|_{L^1} = 2^{-k}$

**Claim 4.1.3** — If $f_n \to f$ in $L^1$ ($\int |f_n - f| \, \mathrm{d}\mu \to 0$) then there is a subsequence $f_{n_k} \to f$ almost everywhere.

*Proof.* Consider the normed space $L^1$ space of semi-nromed space $\mathcal{L}^1$. (define as a equivalence class of almost everywhere where $f \underset{\text{a.e.}}{\sim} g$ if $f = g$ almost everywhere) Construct a convergence subsequence (a.e. and also in Norm) Choose $\varepsilon = \frac{1}{2^k}$ there exists number $N(k)$ such that $\|f_l - f_m\| < \frac{1}{2^k}$ for $l, m \geq N(k)$ for $l, m \geq N(k)$ then $\|f_{N(k)} - f_{N(k+1)}\| \leq \frac{1}{2^k}\|$ Define

$$G(x) = |f_{N(1)}(x)| + \sum_{k=1}^{\infty} |f_{N(k+1)}(x) - f_{N(k)}(x)|$$

then

$$\int G(x) \, \mathrm{d}\mu = \int |f_{N(1)}(x)| \, \mathrm{d}\mu + \sum_{k=1}^{\infty} \int |f_{N(k+1)}(x) - f_{N(k)}(x)| \, \mathrm{d}\mu \quad \leq \|f_{N(1)}\|_1 + \sum_{k=1}^{\infty} \frac{1}{2^k}$$

So, $G$ is integrable, $\int |G(x)| \, \mathrm{d}\mu < \infty$ then $G(x) < \infty$ almost everywhere. We se that for almost everywhere,

$$f_{N_1}(x) + \sum_{k=1}^{\infty} f_{N(k+1)} - f_{N(k)}(x)$$

converges for almost everywhere $x$, define

$$s_M(x) = f_{N(1)}(x) + f_{N(2)}(x) - f_{N(1)}(x) + \ldots + f_{N(M+1)}(x) - f_{N(M)}(x) = f_{N(M+1)}(x)$$

then $s_{M-1}(x) = f_{N(M)}(x)$ and as $M \to \infty$, this is converges for almost everywhere $x$.

$f(x) = \lim_{M\to\infty} f_{N(M)}$

$$f(x) = f_{N_1}(x) + \sum_{M=1}^{\infty} f_{N(M+1)}(x) - f_{N(M)}(x)$$

$$\begin{aligned}\int |f(x) - f_{N_1}(x)| \, \mathrm{d}\mu &= \int \sum_{M=1}^{\infty} f_{N(M+1)}(x) - f_{N(M)}(x) \, \mathrm{d}\mu \\ &\leq \int |f_{N(M+1)}(x) - f_{N(M)}(x)| + \int |f_{N(M+2)} - f_{N(M+1)}| + \dots \\ &= \sum_{k=M}^{\infty} \int |f_{N(k+1)}(x) - f_{N(k)}(x)| \, \mathrm{d}\mu \\ &\leq 2^{1-M}\end{aligned}$$

This shows convergence of $f_{N(M)} \to f$ in $L^1$. What happens with $l \geq N(k)$,

$$\|f_l - f\| \leq \|f_l - f_{N(k)}\| + \|f_{N(k)} - f\| \leq \frac{1}{2^k}, \to 0$$

□

$L^1$ or ($\mathcal{L}^1$) are complete, in the sense that every Cauchy sequence converges. $\{f_n\}$ cxauchy, For every $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for $l, m \geq N(\varepsilon)$ then $\|f_l - f_m\| < \varepsilon$

**Definition 4.1.4.**

$$\|f\|_p = \left(\int |f|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}}$$

where $L^P$ is space of equivalence class and $\mathcal{L}^p$ is space of functions, $f \in \mathcal{L}^p$ if $\|f\|_p < \infty$

**Theorem 4.1.5**

$\|f\|_p$ is a norm on $L^p$, if $p \geq 1$ (not a nrom if $p < 1$ because triangle inquality fails)

*Proof.* for any $f, g \in L^p$,

$$\begin{aligned}\int |f+g|^p \, \mathrm{d}\mu &\leq \int (2 \max |f|, |g|)^p \, \mathrm{d}\mu \\ &= 2\left(\int \max |f|^p, |g|^p\right) \, \mathrm{d}\mu \\ &\leq 2 \int |f|^p + |g|^p \, \mathrm{d}\mu\end{aligned}$$

□

**Remark 4.1.6.** $\|f+g\|_p \le 2^{\frac{1}{p}}(\|f\|_p + \|g\|_p)$

**Theorem 4.1.7**

For $p < 1$ we have inequality

$$\|f+g\|_p^p \le \|f\|_p^p + \|g\|_p^p$$

*Proof.* we claim that

$$\int |f+g|^p \, \mathrm{d}\mu \le \int |f|^p \, \mathrm{d}\mu + \int |g|^p \, \mathrm{d}\mu$$

for $a, b \in [0, \infty)$, $(a+b)^p \le a^p + b^p$ WLOG $b \le a$ $f(x) = 1 + x^p - (1-x)^p$, $f'(x) \ge 0 \implies (1+x)^p \le 1 + x^p$ for $0 \le x \le 1$ $\square$

**Remark 4.1.8.** For $p < 1$ we do not get$\|f+g\|_p \le \|f\|_p + \|g\|_p$ for $x, y \in \mathbb{R}^2$ want to disprove $\|x+y\|_p \le \|x\|_p + \|y\|_p, p < 1$

$$2^{\frac{1}{p}} = (1^p + 1^p)^{\frac{1}{p}}$$

(it is because failure of convexity of the norm $p < 1$)

**Claim 4.1.9** — For $0 < \theta < 1$, $a, b \ge 0$, then $a^{1-\theta} b^\theta \le (1-\theta)a + \theta b$

*Proof.* Generalized AM-GM inequality ($\sqrt{ab} \le \frac{a+b}{2}$) then put for $0 < \theta < 1$ then $a^{1-\theta} b^\theta \le (1-\theta)a + \theta b$ WLOG $b \le a$ then

$$\left(\frac{b}{a}\right)^\theta \le 1 - \theta + \theta \frac{b}{a}$$

let $x = \frac{b}{a}$ for $0 \le x \le 1$ we need to show that $g(x) = 1 - \theta + \theta x - x^\theta \ge 0$ then $g'(x) = -1 + \theta - \theta x^{\theta - 1} \le 0$ (because $0 \le \theta \le 1$) $\square$

**Claim 4.1.10** (Holder's inequality) — Given $p > 1$, $p'$ to be such that

$$\frac{1}{p'} + \frac{1}{p} = 1 \quad \left(p' = \frac{p}{p-1}\right)$$

for $f \in L^p, g \in L^{p'}$, then $fg \in L^1$ and

$$\int |fg| \, \mathrm{d}\mu \leq \|f\|_p \|g\|_{p'}$$

*Proof.* Rewrite AM-GM (generalized) as "Young's inquality" substitute $a = u^p, 1 - \theta = \frac{1}{p}, b = v^{p'}, \theta = \frac{1}{p'}$ then we get

$$uv \leq \frac{1}{p} u^p + \frac{1}{p'} v^{p'}$$

apply $f(x)g(x)$

$$\int |f(x)||g(x)| \, \mathrm{d}\mu \leq \int \frac{|f(x)|^p}{p} \, \mathrm{d}\mu + \int \frac{|g(x)|^{p'}}{p'} \, \mathrm{d}\mu = \frac{\|f\|_p^p}{p} + \frac{\|g\|_{p'}^{p'}}{p'}$$

(This is Holder when two norms are normalized $\|f\|_p = 1 = \|g\|_{p'}$)

Then $\frac{f(x)}{\|f\|_p}$ has "p-norm" equal to 1 because

$$\left( \int \left| \frac{f(x)}{\|f\|_p} \right|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} = \frac{1}{\|f\|_p} \left( \int |f(x)|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

So

$$\int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_{p'}} \leq 1$$

□

**Theorem 4.1.11** (Minkowski's inquality)

$p \geq 1$ We do have a triangle inquality $\|f + g\|_p \leq \|f\|_p + \|g\|_p$

$$\left( \int |f + g|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \leq \left( \int |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} + \left( \int |g|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

*Proof.* It is enough to show that

$$\|f + g\|_p^p \leq (\|f\|_p + \|g\|_p) \|f + g\|_p^{p-1}$$

$$\begin{aligned}
\int |f+g|^{p-1+1} \, \mathrm{d}\mu &= \int |f + g|^{p-1} |f| \, \mathrm{d}\mu + \int |f + g|^{p-1} |g| \, \mathrm{d}\mu \\
&\leq \left( \int |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \left( \int |f + g|^p \, \mathrm{d}\mu \right)^{\frac{p-1}{p}} + \left( \int |g|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \left( \int |f + g|^p \, \mathrm{d}\mu \right)^{\frac{p-1}{p}} \\
&= (\|f\|_p + \|g\|_p) \|f + g\|_p^{p-1}
\end{aligned}$$

□

**Remark 4.1.12.** Holder's inequality holds

$$\int fg \, \mathrm{d}\mu \le \|f\|_p \|g\|_{p'}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ can be generalized to several factors

$$\int f_1 f_2 \cdots f_n \, \mathrm{d}\mu = \prod_{j=1}^{n} \|f\|_{p_j}$$

where $\sum_{j=1}^{n} \frac{1}{p_j} = 1$

**Lemma 4.1.13** (Shebyshev's inequality)

This is an inequality for the distribution function (given a measure space $(X, \mathcal{M}, \mu)$) $\mu_f(\alpha) = \mu(\{x : |f(x)| \ge \alpha\})$ The Shebyshev's inequality is

$$\mu_f(\alpha) \le \frac{\|f\|_p^p}{\alpha^p}$$

*Proof.*

$$\begin{aligned} \mu_f(\alpha) &= \int_{E_\alpha} \mathbb{1} \, \mathrm{d}\mu \\ &\le \int_{E_\alpha} \frac{|f(x)|^p}{\alpha^p} \, \mathrm{d}\mu \\ &\le \frac{1}{\alpha^p} \int |f(x)|^p \, \mathrm{d}\mu \end{aligned}$$

(Probabilitiy may call this Markov's inequality) □

$\mu_{cf}(\alpha) = \mu(\{x : |cf(x)| \ge \alpha\}) = \mu(\{x : |f(x)| \ge \frac{\alpha}{|c|}\})$

$\alpha \mu_{cf}(\alpha)^{\frac{1}{p}} = |c| \frac{\alpha}{|c|} \mu_f \left(\frac{\alpha}{|c|}\right)^{\frac{1}{p}}$

**Claim 4.1.14** — If $\delta_0 + \delta_1 = 1$, $\delta_0, \delta_1 \ge 0$, then

$$E_\alpha(f+g) \subseteq E_{\alpha\delta_0}(f) \cup E_{\alpha\delta_1}(f)$$

**Theorem 4.1.15**

If $\mu(X) < \infty$ then $L^q \subseteq L^p$ for $p \le q$

*Proof.* We need an inequality $\|f\|_p \leq C\|f\|_q$

$$\left(\int 1|f(x)|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} \leq C\left(\int |f(x)|^q \, \mathrm{d}\mu\right)^{\frac{1}{q}}$$

Apply Holder with exponent $\frac{q}{p} > 1$, $\left(\frac{q}{p}\right)'$ where

$$\frac{1}{\left(\frac{q}{p}\right)} + \frac{1}{\left(\frac{q}{p}\right)'} = 1$$

$$\begin{aligned}
\int |f|^p \cdot 1 \, \mathrm{d}\mu &\leq \left(\int (|f|^p)^{\frac{q}{p}} \, \mathrm{d}\mu\right)^{\frac{p}{q}} \left(\int_X 1^{\left(\frac{q}{p}\right)'} \, \mathrm{d}\mu\right)^{\frac{1}{\left(\frac{q}{p}\right)'}} \\
&= \|f\|_q^p \mu(X)^{\frac{1}{\left(\frac{q}{p}\right)'}} \\
&= \left[\|f\|_q \mu(X)^{\frac{1}{\left(\frac{q}{p}\right)'}}\right]^p
\end{aligned}$$

□

Another extreme case would be $\mathbb{N}$ with counting measure In this $L^p(\mathbb{N}, \mu)$ is denoted by $\ell^p(\mathbb{N})$

**Theorem 4.1.16**

For $p \geq 1$, $\ell^p \subseteq \ell^q$ for $p \leq q$

*Proof.* We wan to prove $\|f\|_{\ell^q} \leq C\|f\|_{\ell^p}$ If $\|f\|_{\ell^p} < 1$ , this means

$$\sum_{n=1}^{\infty} |f(n)|^p \leq 1$$

$\implies |f(n)|^p \leq 1$ for all $n$

$$\sum_{n=1}^{\infty} |f(n)|^q \leq \sum_{n=1}^{\infty} |f(n)|^p$$

provided that $|f(n)|^q \leq |f(n)|^p$

For $f \in \ell^p$, $\frac{f}{\|f\|_p}$ has $\ell^p$ norm equal to 1 therefore $\left\|\frac{f}{\|f\|_p}\right\|_q \leq 1$ then $\|f\|_q \leq \|f\|_p$ □

**Theorem 4.1.17** (Littlewood Theorem)

Every measurable function is nearly continuous. i.e., $f \in L^1(A)$ there exists $g$ continuous such that

$$\int |f(x) - g(x)| \, \mathrm{d}m < \varepsilon$$

**Theorem 4.1.18** (Lusin's Theorem)

$f : [a, b] \to \mathbb{C}$ (or $\mathbb{R}$) almost everywhere, then there is a compact set $K \subseteq [a, b]$ such that $f|_K$ is continuous and $\mu([a, b] \setminus K) = 0$

**Example 4.1.19**

$f(x) = \mathbb{1}_{\mathbb{Q}'}(x)$ let $\{r_k\}$ be enumeration of rational number in $[a, b]$. Define

$$O = \bigcup_{k=1}^{\infty} \left( r_k - \frac{\varepsilon}{2^{k+2}}, r_k + \frac{\varepsilon}{2^{k+2}} \right)$$

then $m(O) < \varepsilon$, $K = [a, b] \setminus O$, $f|_K = 1$ is continuous.

**Example 4.1.20**

$$\sum_{k=1}^{\infty} \frac{1}{|x - r_k|^1 0} 2^{-k} \mathbb{1}_{[a,b]}$$

(this function is in $L^p$, if $p < \frac{1}{10}$). The Challenging part is to find a $K$ as in Lusin's theorem such that $f = f|_K$ is continuous.

*Proof.* $f : A \to \mathbb{R}$ (or $\mathbb{C}$), $m(A) < \infty$ The for every $\varepsilon > 0$, there is a compact set $K \subseteq A$, such that $m(A \setminus K) < \varepsilon$ and $f|_K$ is continuous.

1. We can find a set $E_1$ such that $m(A \setminus E_1) < \frac{\varepsilon}{3}$ and $f|_{E_1}$ is bounded.

$$S_\alpha = \{x \in A : |f(x)| > \alpha\}$$

Then $\bigcup S_\alpha = \bigcup S_{2^M}$ has measure zero by the assumption. From continuity from above, we get

$$m(S_\alpha) = m\left(\bigcup S_{2^M}\right) = \lim_{M \to \infty} m(S_{2^M})$$

because $m(A) < \infty$. For large $M$, $m(S_M) < \frac{\varepsilon}{3}$, let $A \setminus E_1 = S_{2^M}$ ($M$ large)

2. We know that $f$ is bounded on $E_1$ Can find a sequence $g_n$ of continuous functions $g_n \to f$ almost everywhere on $E_1$, $m(E_1) < \infty$.

3. Using Egorov's theorem: $g_n \to f$ almost uniformly on $E_1$. Find $E_2 \subseteq E_1$ such that $m(E_1 \setminus E_2) < \frac{\varepsilon}{3}$ and $g_n \to f$ uniformly on $E_2$. $f|_{E_2}$ is continuous on $E_2$, we can find $E_3 = K$ compact, $E_3 \subseteq E_2$, $m(E_2 \setminus E_3) < \frac{\varepsilon}{3}$ and $f|_K$ is continuous.

$$m(A \setminus K) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$\square$

## §4.2 Outer Measure

**Definition 4.2.1.** Given $X$, $\mathcal{E} \subseteq \mathfrak{P}(X)$ where $\emptyset \in \mathcal{E}$ and $X \in \mathcal{E}$, $u : \mathcal{E} \to [0, \infty]$. For any set

$$u^*(E) = \inf_{\bigcup E_j \supseteq E} \sum_{j=1}^{\infty} u(E_j)$$

where $E_j \in \mathcal{E}$, called the outer measure induced by $\mathcal{E}$.

> **Remark 4.2.2.** It is not necessary to assume that $X \in \mathcal{E}$. In this case, $u * (E) = \infty$ if we cannot cover $E$ with a countable collection of sets in $\mathcal{E}$.

**Lemma 4.2.3**

An outer measure $u^*$ incuced by the collection $\mathcal{E}$ satisfies

(i) $u^*(\emptyset) = 0$

(ii) For $A \subseteq B$, $u^*(A) \leq u^*(B)$ Monotonicity

(iii) $u^*(\bigcup E_j) \leq \sum u^*(E_j)$ Subaddictivity

# A Practice Exam

## §A.1 Practice Exam 1

**Problem A.1.1.** Let $E_n$ be Lebesgue measurable subsets of $[0,1]$ such that $E_{n+1} \subseteq E_n$. What can you say about the Lebesgue measure of $\bigcap_n E_n$? Does your answer necessarily hold when $[0,1]$ is replaced by $[0,\infty)$?

*solution.* We can use continuity from above because $\mu([0,1]) < \infty$. We can say that

$$\mu\left(\bigcap_n E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

In case of $[0,\infty)$, we can't use continuity from below becuase if $E_n = [n, n+1)$ then $\mu(E_n) = 1$ but $\bigcap_n E_n = \emptyset$, so, $\lim_{n\to\infty} \mu(E_n) = 1$ but $\mu\left(\bigcap_n E_n\right) = 0$. $\square$

**Problem A.1.2.**