MATH 629 Lecture Notes

Pongsaphol Pongsawakul

Spring 2024

Contents

1 From Riemann to Lebesgue

§1.1 Riemann Integral

Definition 1.1.1. $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a, b]$.

Definition 1.1.2. If P, P' are partitions of $[a, b]$ and $P \subseteq P'$, then P' is a refinement of P.

Definition 1.1.3. Given a bounded function $f : [a, b] \to \mathbb{R}$ and a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ we define

$$m_i(f) = \inf_{t \in [x_{i-1}, x_i]} f(t)$$

$$M_i(f) = \sup_{t \in [x_{i-1}, x_i]} f(t)$$

define the lower sum as

$$L(f, P) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1})$$

and the upper sum as

$$U(f, P) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$$

Lemma 1.1.4

Given a bounded function $f : [a, b] \to \mathbb{R}$ and partitons P of $[a, b]$. Suppose that P' is a refinement of P then

$$(b - a) \inf_{t \in [a,b]} f(t) \leq L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P) \leq (b - a) \sup_{t \in [a,b]} f(t)$$

Corollary 1.1.5

Suppose that P_1, P_2 are partitions of $[a, b]$ then $L(f, P_1) \leq U(f, P_2)$

Proof. Let $P' = P_1 \cup P_2$ then P' is a refinement of P_1 and P_2 and use Lemma 1.1.4 □

Lemma 1.1.6

Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Then

$$(b-a) \inf_{t\in[a,b]} f(t) \leq \sup_P L(f, P) \leq \inf_P U(f, P) \leq (b-a) \sup_{t\in[a,b]} f(t)$$

Definition 1.1.7. A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if

$$\sup_P L(f, P) = \inf_P U(f, P)$$

and the common value is called the Riemann integral of f and is denoted by $\int_a^b f$

Lemma 1.1.8

Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Then f is Riemann integrable if and only if for any $\varepsilon > 0$ there exists a partition P such that

$$U(f, P) - L(f, P) < \varepsilon$$

Proof. ($\Rightarrow$) For any $\varepsilon > 0$. Suppose that f is Riemann integrable. Then there exists P_1, P_2 such that

$$L(f, P_1) \geq \int_a^b f - \frac{\varepsilon}{2}$$

$$U(f, P_2) \leq \int_a^b f + \frac{\varepsilon}{2}$$

let $P = P_1 \cup P_2$ then

$$U(f, P) - L(f, P) \leq \varepsilon$$

($\Leftarrow$) For any $\varepsilon > 0$, there exists P_ε such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

since ε is arbitrary, we have

$$\sup_P L(f, P) = \inf_P U(f, P)$$

$\square$

Theorem 1.1.9

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ then f is Riemann integrable.

Proof. f is continuous on a compact set, so, f is uniformly continous. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in [a, b]$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{(b-a)}$. Let N be such that $\frac{(b-a)}{N} < \delta$ and let $P = \{x_i := a + \frac{(b-a)i}{N}\}$ then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{N} (M_i(f) - m_i(f)) \frac{(b-a)}{N} \\ &\leq \sum_{i=1}^{N} \frac{\varepsilon}{(b-a)} \frac{(b-a)}{N} \\ &= \varepsilon \end{aligned}$$

□

Remark 1.1.10. Let $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ defined on the $[0, 1]$. Then $U(f, P) = 1$ and $L(f, P) = 0$ for any partition P. So, f is not Riemann integrable.

§1.2 Lebesgue null sets

Definition 1.2.1. For the closed interval $I = [a, b]$, the length of I, denoted as $\ell(I)$ is defined as $\ell(I) = b - a$

Definition 1.2.2. A set E is said to be a Lebesgue null set if for any $\varepsilon > 0$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E \subseteq \bigcup_{n=1}^{\infty} I_n$$

and

$$\sum_{n=1}^{\infty} \ell(I_n) < \varepsilon$$

Lemma 1.2.3

Countable unions of Lebesgue null sets are Lebesgue null sets.

Proof. For any $\varepsilon > 0$ and for each Lebesgue null sets E_n there exists $I_{E_n,i}$ such that

$$E_n \subseteq \bigcup_{i=1}^{\infty} I_{E_n,i}$$

and

$$\sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \frac{\varepsilon}{2^n}$$

then

$$\sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \varepsilon$$

□

Definition 1.2.4. A set $E \subseteq [a,b]$ has content zero if for any $\varepsilon > 0$ there exists $I_1, I_2, \dots, I_n$ such that

$$E \subseteq \bigcup_{i=1}^{n} I_i$$

and

$$\sum_{i=1}^{n} \ell(I_i) < \varepsilon$$

Lemma 1.2.5

Suppose that $E \subseteq [a,b]$ is a compact Lebesgue null set then E has content zero.

Proof. For any $\varepsilon > 0$ there exists a sequnce of interval $\{I_n\}_{n\in\mathbb{N}}$ such that $E \subseteq \bigcup I_n$ and $\sum \ell(I_n) < \frac{\varepsilon}{2}$. Suppose that $I_n = [a_n, b_n]$, then let

$$J_n = \left(a_n - \frac{\varepsilon}{2^{n+3}}, b_n + \frac{\varepsilon}{2^{n+3}}\right) \supseteq E_n$$

then from the compactness of E, there exists a finite subcover $J_{n_1}, J_{n_2}, \dots, J_{n_k}$ such that $E \subseteq \bigcup J_{n_i}$ then we construct a finite closed interval K_i by

$$K_i = \left[a_{n_i} - \frac{\varepsilon}{2^{n_i+2}}, b_{n_i} + \frac{\varepsilon}{2^{n_i+2}}\right]$$

then $E \subseteq \bigcup K_i$ and $\sum \ell(K_i) < \varepsilon$ □

Corollary 1.2.6

if $a < b$ then $[a,b]$ is not a Lebesgue null set.

Proof. By contradiction, since $[a,b]$ is compact, then $[a,b]$ has content zero, but $[a,b]$ don't have content zero. □

§1.3 Oscillation and Discontinuity

Definition 1.3.1. Suppose that $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ for any $x \in X$ and $\delta > 0$, define

$$M_{f,\delta}(x) := \sup\{f(y) : d(x, y) < \delta\}$$

$$m_{f,\delta}(x) := \inf\{f(y) : d(x, y) < \delta\}$$

then we define

$$\operatorname{osc}_f(x) := \lim_{\delta \to 0+} M_{f,\delta}(x) - m_{f,\delta}(x)$$

Lemma 1.3.2

f is continuous at x if and only if $\operatorname{osc}_f(x) = 0$.

Proof. ($\Rightarrow$) Suppose that f is continuous at x, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Then

$$M_{f,\delta}(x) - m_{f,\delta}(x) \leq \sup\{f(y) : d(x, y) < \delta\} - \inf\{f(y) : d(x, y) < \delta\} < \varepsilon$$

($\Leftarrow$) Suppose that $\operatorname{osc}_f(x) = 0$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $M_{f,\delta}(x) - m_{f,\delta}(x) < \varepsilon$. Then for any $y \in X$ such that $d(x, y) < \delta$, we have $|f(x) - f(y)| < \varepsilon$ then f is continuous at x. $\square$

Theorem 1.3.3

Suppose that $f : [a, b] \to \mathbb{R}$ then $f \in \mathcal{R}([a, b])$ if and only if f is bounded and the set of discontinuity of f is a Lebesgue null set.

Before we prove this theorem, we need to prove the following lemma.

Lemma 1.3.4

$\{x \in [a, b] : \operatorname{osc}_f(x) \geq \gamma\}$ is closed.

Proof. We need to show that $\{x : \operatorname{osc}_f(x) < \gamma\}$ is open. Fix x in that set. Let $\varepsilon = \gamma - \operatorname{osc}_f(x)$ then

$$\sup_{|w-x|<\delta} f(w) - \inf_{|w-x|<\delta} f(w) < \operatorname{osc}_f(x) < \gamma$$

then for any $w \in (x-\delta, x+\delta)$ if $|w-x| < \frac{\delta}{2}$ then

$$\operatorname{osc}(w) \leq \sup_{|y-w|<\frac{\delta}{2}} f(y) - \inf_{|y-w|<\frac{\delta}{2}} f(y) < \gamma$$

So, $B\left(x, \frac{\delta}{2}\right) \subseteq \{x : \operatorname{osc}_f(x) < \gamma\}$ □

we observe that

(i) If the set of discontinuities is a Lebesque null set, then $\{x : \operatorname{osc}_f(x) \geq \gamma\}$ is a set of content zero.

(ii) If $\{x : \operatorname{osc}_f(x) \geq \gamma\}$ is a Lebesgue null set, then the set of discontinuities is also a Lebesque null set.

Proof of Theorem. ($\Rightarrow$) We want to show that for every $n \in \mathbb{N}$,

$$\mathcal{D}_n = \left\{x : \operatorname{osc}_f(x) \geq \frac{1}{n}\right\}$$

is a Lebesque null set. For any $\varepsilon > 0$, since f is Riemann integrable, there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (x_i - x_{i-1})(M_i - m_i) \leq \frac{\varepsilon}{n}$$

where $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ and $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$. in particular

$$\sum_{[x_{i-1},x_i] \cap \mathcal{D}_n \neq \emptyset} (x_i - x_{i-1})(M_i - m_i) \leq \frac{\varepsilon}{n}$$
$$\frac{1}{n} \sum_{[x_{i-1},x_i] \cap \mathcal{D}_n \neq \emptyset} \ell([x_{i-1}, x_i]) \leq \frac{\varepsilon}{n}$$

So, this interval cover the set $\mathcal{D}_n$ □

For the other direction, we need some lemma in order to prove the theorem

Lemma 1.3.5

Suppose that f is defined on $[c,d]$, assume that $\operatorname{osc}_f(x) < \gamma$ then we can find a partition

$$U(f,P) - L(f,P) < \gamma(b-a)$$

Proof. For every $x \in [c, d]$, there exists $\delta_x > 0$ such that

$$\sup_{|w-x|<\delta_x} f(w) - \inf_{|w-x|<\delta_x} f(x) < \gamma$$

construct a cover by

$$B(x, \delta_x) = \{w \in [c, d] : |w - x| < \delta_x\}$$

since $[c, d]$ is compact, there exists a finite subcover $B(p_1, \delta_{p_1}), \ldots, B(p_n, \delta_{p_n})$ then let $\delta_0 = \frac{\min\{\delta_{p_i}\}}{100}$ then we can construct a partition $P = \{c = x_0 < x_1 < \cdots < x_n = d\}$ such that $|x_i - x_{i-1}| < \delta_0$ then $M_i - m_i < \gamma$ and

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\
&< \gamma \sum_{i=1}^{n} (x_i - x_{i-1}) \\
&= \gamma(d - c)
\end{aligned}$$

□

2 Measures

§2.1 Introduction

We define the $\ell([c,d]) = d - c$ and If $E = [c_1, d_1] \cup [c_2, d_2]$ where $d_1 < c_2$ then $\ell(E) = d_1 - c_1 + d_2 - c_2$. This is consistent with the definition

$$\ell(E) = \int \mathbb{1}_E(x)\ \mathrm{d}x$$

where the integral denotes the Riemann integral.

if $E \subseteq [a,b]$ reference interval is

$$\int_a^b \mathbb{1}_E\ \mathrm{d}x$$

Remark 2.1.1. The consistency of the definition also works with the set (c,d), $[c,d)$, and $(c,d]$, where the length of all of them is $d - c$.

Remark 2.1.2. we defnote $\mathbb{1}_E$ to be such that

$$\mathbb{1}_E = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Example 2.1.3

Let $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ defined on the $[0,1]$. Then $U(f,P) = 1$ and $L(f,P) = 0$ for any partition P.

Fix the reference interval $[a,b]$ and consider subset of $[a,b]$

Let $\mathcal{A}$ = collection of sets for which $\int_{[a,b]} \mathbb{1}_E\ \mathrm{d}x$ exists.

If $A_1, \ldots, A_n \in \mathcal{A}$, we can make the set to be mutually disjoint by taking $E_1 = A_1$, $E_2 = A_2 \setminus A_1$, $E_3 = A_3 \setminus (A_1 \cup A_2)$, and so on.

Example 2.1.4

For $E_1, E_2 \in \mathcal{A}$, we have

$$\mathbb{1}_{E_1 \cap E_2}(x) = \mathbb{1}_{E_1}(x)\mathbb{1}_{E_2}(x)$$

Example 2.1.5

For the Riemann integral, we have

$$\int_a^b f(y) = \int_{a-v}^{b-v} f(v+y)$$

and we want

$$\int \mathbb{1}_E(x)\ \mathrm{d}x = \int \mathbb{1}_{v+E}$$

where $v + E = \{v + x : x \in E\}$

Let $E = \mathbb{Q} \cap [0,1]$ countable set, we can enumerate $r_1, r_2, r_3, \ldots$ such that

$$E = \bigcup_{n=1}^{\infty} \{r_n\}$$

and

$$\int \mathbb{1}_{\{r_k\}} = 0$$

E should have length zero but according $\mathbb{1}_E$ is not Riemann integrable.

§2.2 Construction of Measure

Suppose that $\mathcal{C}$ be a collection of sets.

Can we define on suitable large collection of subset of $\mathbb{R}$?

a set sunction $\mu : \mathcal{C} \to [0,\infty) \cup \{\infty\}$ such that if $\{E_j\}_{j=1}^{\infty}$ is a sequence of disjoint set in $\mathcal{C}$ then

$$\bigcup E_j = \mathcal{C}$$

$$\mu\left(\bigcup_{i=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

$\mu([a,b]) = b - a$, $\mu([0,1]) = 1$

Can we do this for the collection of all subset of $\mathbb{R}$?

Answer: No, Vitali set.

Theorem 2.2.1

We cannot define a measure on the collection of all subset of $\mathbb{R}$.

Before we prove that theorem, we need to define something and prove the following lemma.

Definition 2.2.2. We define a Vitali set V from picking an element $x \in [0, 1)$ from each equivalence class of the relation $x \sim y$ if $x - y \in \mathbb{Q}$. (e.g, pick $x \in O_x$ for $O_x \in \mathbb{R}/\mathbb{Q}$)

Lemma 2.2.3

Suppose that V is a Vitali set then

$$V \cap V + q = \emptyset$$

For all $q \in \mathbb{Q} \setminus \{0\}$

Proof. Suppose not, there exists $a \in V$ such that $a \in V + q \implies a - q \in V$ but we only pick 1 element in each equivalence class. contradiction. ☐

Lemma 2.2.4

Let V be a Vitali set and let $W = \{q \in [-1, 1] : q \in \mathbb{Q}\}$ and

$$E = \bigcup_{w \in W} V + w$$

then

$$[0, 1] \subseteq E \subseteq [-1, 2]$$

Proof. Consider $E \subseteq [-1, 2]$. Since $V \subseteq [0, 1)$, then for any $v \in V$, $v \in [0, 1) \implies v + w \in [-1, 2]$.

For the $[0, 1] \subseteq E$, for any $x \in [0, 1]$ there exists $O_x \in \mathbb{R}/\mathbb{Q}$ such that $x \in O_x$. then there exists $v \in C_x$ such that $v \in [0, 1)$ and $v \in V$, since both are from the same equivalence class, then $x - v \in \mathbb{Q}$ and $|x - v| < 1 \implies x - v \in (-1, 1)$. Hence, there exists $w \in W$ such that $w = x - v$ so $v + w = x$. ☐

Proof of the theorem. Suppose that μ exists then using the result from Lemma 2.2.4 we get that

$$\mu([0, 1]) \leq \mu(E) \leq \mu([-1, 2])$$

from Lemma 2.2.3 we know that each $V + w$ is disjoint, so

$$\mu([0,1]) \leq \sum_{w \in W} \mu(V) \leq \mu([-1,2])$$
$$1 \leq \sum_{w \in W} \mu(V) \leq 3$$

if $\mu(V) = 0$ then $\mu(E) = 0$ and if $\mu(V) > 0$ then $\mu(E) = \infty$. Both are contradiction. $\square$