

# MATH 629 (Measure Theory) Lecture Notes

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# 1 From Riemann to Lebesgue

## §1.1 Riemann Integral

**Definition 1.1.1.**  $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$  is a partition of  $[a, b]$ .

**Definition 1.1.2.** If  $P, P'$  are partitions of  $[a, b]$  and  $P \subseteq P'$ , then  $P'$  is a refinement of  $P$ .

**Definition 1.1.3.** Given a bounded function  $f : [a, b] \rightarrow \mathbb{R}$  and a partition  $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$  we define

$$m_i(f) = \inf_{t \in [x_{i-1}, x_i]} f(t)$$

$$M_i(f) = \sup_{t \in [x_{i-1}, x_i]} f(t)$$

define the lower sum as

$$L(f, P) = \sum_{i=1}^n m_i(f)(x_i - x_{i-1})$$

and the upper sum as

$$U(f, P) = \sum_{i=1}^n M_i(f)(x_i - x_{i-1})$$

### Lemma 1.1.4

Given a bounded function  $f : [a, b] \rightarrow \mathbb{R}$  and partitions  $P$  of  $[a, b]$ . Suppose that  $P'$  is a refinement of  $P$  then

$$(b - a) \inf_{t \in [a, b]} f(t) \leq L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P) \leq (b - a) \sup_{t \in [a, b]} f(t)$$

### Corollary 1.1.5

Suppose that  $P_1, P_2$  are partitions of  $[a, b]$  then  $L(f, P_1) \leq U(f, P_2)$

*Proof.* Let  $P' = P_1 \cup P_2$  then  $P'$  is a refinement of  $P_1$  and  $P_2$  and use Lemma 1.1.4  $\square$

**Lemma 1.1.6**

Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is bounded. Then

$$(b - a) \inf_{t \in [a, b]} f(t) \leq \sup_P L(f, P) \leq \inf_P U(f, P) \leq (b - a) \sup_{t \in [a, b]} f(t)$$

**Definition 1.1.7.** A function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable if

$$\sup_P L(f, P) = \inf_P U(f, P)$$

and the common value is called the Riemann integral of  $f$  and is denoted by  $\int_a^b f$

**Lemma 1.1.8**

Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is bounded. Then  $f$  is Riemann integrable if and only if for any  $\varepsilon > 0$  there exists a partition  $P$  such that

$$U(f, P) - L(f, P) < \varepsilon$$

*Proof.* ( $\Rightarrow$ ) For any  $\varepsilon > 0$ . Suppose that  $f$  is Riemann integrable. Then there exists  $P_1, P_2$  such that

$$L(f, P_1) \geq \int_a^b f - \frac{\varepsilon}{2}$$

$$U(f, P_2) \leq \int_a^b f + \frac{\varepsilon}{2}$$

let  $P = P_1 \cup P_2$  then

$$U(f, P) - L(f, P) \leq \varepsilon$$

( $\Leftarrow$ ) For any  $\varepsilon > 0$ , there exists  $P_\varepsilon$  such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

since  $\varepsilon$  is arbitrary, we have

$$\sup_P L(f, P) = \inf_P U(f, P)$$

□

**Theorem 1.1.9**

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  then  $f$  is Riemann integrable.

*Proof.*  $f$  is continuous on a compact set, so,  $f$  is uniformly continuous. For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any  $x, y \in [a, b]$  if  $|x - y| < \delta$  then  $|f(x) - f(y)| < \frac{\varepsilon}{(b-a)}$ . Let  $N$  be such that  $\frac{(b-a)}{N} < \delta$  and let  $P = \{x_i := a + \frac{(b-a)i}{N}\}$  then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^N (M_i(f) - m_i(f)) \frac{(b-a)}{N} \\ &\leq \sum_{i=1}^N \frac{\varepsilon}{(b-a)} \frac{(b-a)}{N} \\ &= \varepsilon \end{aligned}$$

□

**Remark 1.1.10.** Let  $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$  defined on the  $[0, 1]$ . Then  $U(f, P) = 1$  and  $L(f, P) = 0$  for any partition  $P$ . So,  $f$  is not Riemann integrable.

## §1.2 Lebesgue null sets

**Definition 1.2.1.** For the closed interval  $I = [a, b]$ , the length of  $I$ , denoted as  $\ell(I)$  is defined as  $\ell(I) = b - a$

**Definition 1.2.2.** A set  $E$  is said to be a Lebesgue null set if for any  $\varepsilon > 0$  there exists a sequence of intervals  $\{I_n\}_{n \in \mathbb{N}}$  such that

$$E \subseteq \bigcup_{n=1}^{\infty} I_n$$

and

$$\sum_{n=1}^{\infty} \ell(I_n) < \varepsilon$$

### Lemma 1.2.3

Countable unions of Lebesgue null sets are Lebesgue null sets.

*Proof.* For any  $\varepsilon > 0$  and for each Lebesgue null sets  $E_n$  there exists  $I_{E_n, i}$  such that

$$E_n \subseteq \bigcup_{i=1}^{\infty} I_{E_n, i}$$

and

$$\sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \frac{\varepsilon}{2^n}$$

then

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{E_n,i}) < \varepsilon$$

□

**Definition 1.2.4.** A set  $E \subseteq [a, b]$  has content zero if for any  $\varepsilon > 0$  there exists  $I_1, I_2, \dots, I_n$  such that

$$E \subseteq \bigcup_{i=1}^n I_i$$

and

$$\sum_{i=1}^n \ell(I_i) < \varepsilon$$

**Lemma 1.2.5**

Suppose that  $E \subseteq [a, b]$  is a compact Lebesgue null set then  $E$  has content zero.

*Proof.* For any  $\varepsilon > 0$  there exists a sequence of interval  $\{I_n\}_{n \in \mathbb{N}}$  such that  $E \subseteq \bigcup I_n$  and  $\sum \ell(I_n) < \frac{\varepsilon}{2}$ . Suppose that  $I_n = [a_n, b_n]$ , then let

$$J_n = \left( a_n - \frac{\varepsilon}{2^{n+3}}, b_n + \frac{\varepsilon}{2^{n+3}} \right) \supseteq I_n$$

then from the compactness of  $E$ , there exists a finite subcover  $J_{n_1}, J_{n_2}, \dots, J_{n_k}$  such that  $E \subseteq \bigcup J_{n_i}$  then we construct a finite closed interval  $K_i$  by

$$K_i = \left[ a_{n_i} - \frac{\varepsilon}{2^{n_i+2}}, b_{n_i} + \frac{\varepsilon}{2^{n_i+2}} \right]$$

then  $E \subseteq \bigcup K_i$  and  $\sum \ell(K_i) < \varepsilon$

□

**Corollary 1.2.6**

if  $a < b$  then  $[a, b]$  is not a Lebesgue null set.

*Proof.* By contradiction, since  $[a, b]$  is compact, then  $[a, b]$  has content zero, but  $[a, b]$  don't have content zero. □

### §1.3 Oscillation and Discontinuity

**Definition 1.3.1.** Suppose that  $X \subseteq \mathbb{R}$ ,  $f : X \rightarrow \mathbb{R}$  for any  $x \in X$  and  $\delta > 0$ , define

$$M_{f,\delta}(x) := \sup\{f(y) : d(x, y) < \delta\}$$

$$m_{f,\delta}(x) := \inf\{f(y) : d(x, y) < \delta\}$$

then we define

$$\text{osc}_f(x) := \lim_{\delta \rightarrow 0^+} M_{f,\delta}(x) - m_{f,\delta}(x)$$

#### Lemma 1.3.2

$f$  is continuous at  $x$  if and only if  $\text{osc}_f(x) = 0$ .

*Proof.* ( $\Rightarrow$ ) Suppose that  $f$  is continuous at  $x$ , then for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $d(x, y) < \delta$  then  $|f(x) - f(y)| < \frac{\varepsilon}{2}$ . Then

$$M_{f,\delta}(x) - m_{f,\delta}(x) \leq \sup\{f(y) : d(x, y) < \delta\} - \inf\{f(y) : d(x, y) < \delta\} < \varepsilon$$

( $\Leftarrow$ ) Suppose that  $\text{osc}_f(x) = 0$ , then for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $M_{f,\delta}(x) - m_{f,\delta}(x) < \varepsilon$ . Then for any  $y \in X$  such that  $d(x, y) < \delta$ , we have  $|f(x) - f(y)| < \varepsilon$  then  $f$  is continuous at  $x$ .  $\square$

Before we prove this theorem, we need to prove the following lemma.

#### Lemma 1.3.3

$\{x \in [a, b] : \text{osc}_f(x) \geq \gamma\}$  is closed.

*Proof.* We need to show that  $\{x : \text{osc}_f(x) < \gamma\}$  is open. Fix  $x$  in that set. Let  $\varepsilon = \gamma - \text{osc}_f(x)$  then

$$\sup_{|w-x|<\delta} f(w) - \inf_{|w-x|<\delta} f(w) < \text{osc}_f(x) < \gamma$$

then for any  $w \in (x - \delta, x + \delta)$  if  $|w - x| < \frac{\delta}{2}$  then

$$\text{osc}(w) \leq \sup_{|y-w|<\frac{\delta}{2}} f(y) - \inf_{|y-w|<\frac{\delta}{2}} f(y) < \gamma$$

So,  $B(x, \frac{\delta}{2}) \subseteq \{x : \text{osc}_f(x) < \gamma\}$   $\square$

we observe that

- (i) If the set of discontinuities is a Lebesgue null set, then  $\{x : \text{osc}_f(x) \geq \gamma\}$  is a set of content zero.
- (ii) If  $\{x : \text{osc}_f(x) \geq \gamma\}$  is a Lebesgue null set, then the set of discontinuities is also a Lebesgue null set.

**Lemma 1.3.4**

Suppose that  $f$  is defined on  $[c, d]$ , assume that  $\text{osc}_f(x) < \gamma$  then we can find a partition

$$U(f, P) - L(f, P) < \gamma(b - a)$$

*Proof.* For every  $x \in [c, d]$ , there exists  $\delta_x > 0$  such that

$$\sup_{|w-x|<\delta_x} f(w) - \inf_{|w-x|<\delta_x} f(x) < \gamma$$

construct a cover by

$$B(x, \delta_x) = \{w \in [c, d] : |w - x| < \delta_x\}$$

since  $[c, d]$  is compact, there exists a finite subcover  $B(p_1, \delta_{p_1}), \dots, B(p_n, \delta_{p_n})$  then let  $\delta_0 = \frac{\min\{\delta_{p_i}\}}{100}$  then we can construct a partition  $P = \{c = x_0 < x_1 < \dots < x_n = d\}$  such that  $|x_i - x_{i-1}| < \delta_0$  then  $M_i - m_i < \gamma$  and

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \\ &< \gamma \sum_{i=1}^n (x_i - x_{i-1}) \\ &= \gamma(d - c) \end{aligned}$$

□

**Theorem 1.3.5**

Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  then  $f \in \mathcal{R}([a, b])$  if and only if  $f$  is bounded and the set of discontinuity of  $f$  is a Lebesgue null set.

*Proof.* ( $\Rightarrow$ ) We want to show that for every  $n \in \mathbb{N}$ ,

$$\mathcal{D}_n = \left\{x : \text{osc}_f(x) \geq \frac{1}{n}\right\}$$

is a Lebesgue null set. For any  $\varepsilon > 0$ , since  $f$  is Riemann integrable, there exists a partition  $P$  of  $[a, b]$  such that

$$U(f, P) - L(f, P) = \sum_{i=1}^n (x_i - x_{i-1})(M_i - m_i) \leq \frac{\varepsilon}{n}$$

where  $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$  and  $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$ . in particular

$$\begin{aligned} \sum_{[x_{i-1}, x_i] \cap \mathcal{D}_n \neq \emptyset} (x_i - x_{i-1})(M_i - m_i) &\leq \frac{\varepsilon}{n} \\ \frac{1}{n} \sum_{[x_{i-1}, x_i] \cap \mathcal{D}_n \neq \emptyset} \ell([x_{i-1}, x_i]) &\leq \frac{\varepsilon}{n} \end{aligned}$$

So, this interval cover the set  $\mathcal{D}_n$

( $\Leftarrow$ ) pick  $\varepsilon_1 \ll \varepsilon$ , consider the set  $D(\varepsilon_1) = \{x \in [a, b] : \text{osc}_f(x) \geq \varepsilon_1\}$  closed set. Since  $D(\varepsilon_1)$  is a Lebesgue null set from the Lemma 1.2.5 it has content zero so we can find  $I_1, \dots, I_n$  such that

$$\sum_{j=1}^n \ell(I_j) < \varepsilon_1 \text{ and } D(\varepsilon_1) \subseteq \bigcup_{j=1}^n I_j$$

We form a partition of  $[a, b]$ ,  $a = x_0 < x_1 < \dots < x_N = b$  from  $I_j$ . There are two cases that we need to consider

- 1) if  $[x_{i-1}, x_i] \subseteq I_j$  for some  $j$  then set  $P_i = [x_{i-1}, x_i]$
- 2) if  $[x_{i-1}, x_i] \cap I_j = \emptyset$  for all  $j$  then  $\text{osc}(x) < \varepsilon_1$  for all  $x \in [x_{i-1}, x_i]$ . We want to partition further the interval  $[x_{i-1}, x_i]$  by partition  $P_i$ . Using Lemma 1.3.4 we can find a partition  $P_i$  of  $[x_{i-1}, x_i]$  such that

$$U(f, P_i) - L(f, P_i) < \varepsilon_1(x_i - x_{i-1})$$

We form a partition  $P = P_1 \cup \dots \cup P_N$  then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^N (U(f, P_i) - L(f, P_i)) \\ &= \sum_{i:\text{case 1}} (U(f, P_i) - L(f, P_i)) + \sum_{i:\text{case 2}} (U(f, P_i) - L(f, P_i)) \\ &\leq 2M \sum_{i:\text{case 1}} (x_i - x_{i-1}) + \varepsilon_1 \sum_{i:\text{case 2}} (x_i - x_{i-1}) \\ &\leq 2M\varepsilon_1 + \varepsilon_1(b - a) \\ &= \varepsilon_1(2M + b - a) \end{aligned}$$

□



# 2 Measures

## §2.1 Introduction

We define the  $\ell([c, d]) = d - c$  and If  $E = [c_1, d_1] \cup [c_2, d_2]$  where  $d_1 < c_2$  then  $\ell(E) = d_1 - c_1 + d_2 - c_2$ . This is consistent with the definition

$$\ell(E) = \int \mathbb{1}_E(x) \, dx$$

where the integral denotes the Riemann integral.

if  $E \subseteq [a, b]$  reference interval is

$$\int_a^b \mathbb{1}_E \, dx$$

**Remark 2.1.1.** The consistency of the definition also works with the set  $(c, d)$ ,  $[c, d)$ , and  $(c, d]$ , where the length of all of them is  $d - c$ .

**Remark 2.1.2.** we denote  $\mathbb{1}_E$  to be

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

### Example 2.1.3

Let  $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$  defined on the  $[0, 1]$ . Then  $U(f, P) = 1$  and  $L(f, P) = 0$  for any partition  $P$ .

Fix the reference interval  $[a, b]$  and consider subset of  $[a, b]$

Let  $\mathcal{A} =$  collection of sets for which  $\int_{[a, b]} \mathbb{1}_E \, dx$  exists.

If  $A_1, \dots, A_n \in \mathcal{A}$ , we can make the set to be mutually disjoint by taking  $E_1 = A_1$ ,  $E_2 = A_2 \setminus A_1$ ,  $E_3 = A_3 \setminus (A_1 \cup A_2)$ , and so on.

### Example 2.1.4

For  $E_1, E_2 \in \mathcal{A}$ , we have

$$\mathbb{1}_{E_1 \cap E_2}(x) = \mathbb{1}_{E_1}(x) \mathbb{1}_{E_2}(x)$$

**Example 2.1.5**

For the Riemann integral, we have

$$\int_a^b f(y) = \int_{a-v}^{b-v} f(v+y)$$

and we want

$$\int \mathbb{1}_E(x) \, dx = \int \mathbb{1}_{v+E}$$

where  $v + E = \{v + x : x \in E\}$

Let  $E = \mathbb{Q} \cap [0, 1]$  countable set, we can enumerate  $r_1, r_2, r_3, \dots$  such that

$$E = \bigcup_{n=1}^{\infty} \{r_n\}$$

and

$$\int \mathbb{1}_{\{r_k\}} = 0$$

$E$  should have length zero but according  $\mathbb{1}_E$  is not Riemann integrable.

**§2.2 Construction of Measure**

Suppose that  $\mathcal{C}$  be a collection of sets.

Can we define on suitable large collection of subset of  $\mathbb{R}$ ?

a set function  $\mu : \mathcal{C} \rightarrow [0, \infty]$  such that if  $\{E_j\}_{j=1}^{\infty}$  is a sequence of disjoint set in  $\mathcal{C}$  then

$$\mu\left(\bigcup_{i=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

$$\mu([a, b]) = b - a, \mu([0, 1)) = 1$$

Can we do this for the collection of all subset of  $\mathbb{R}$ ?

Answer: No, Vitali set.

**Theorem 2.2.1**

We cannot define a measure on the collection of all subset of  $\mathbb{R}$ . i.e., there does not exist a set function  $\mu : \mathfrak{P}(\mathbb{R}) \rightarrow [0, \infty]$  such that

- (i)  $\mu(v + E) = \mu(E)$  for all  $E \subseteq \mathbb{R}$  and  $v \in \mathbb{R}$
- (ii)  $\mu([0, 1]) = 1$
- (iii)  $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$  for all disjoint  $A_j \subseteq \mathbb{R}$

Before we prove that theorem, we need to define something and prove the following lemma.

**Definition 2.2.2.** We define a Vitali set  $V$  from picking an element  $x \in [0, 1)$  from each equivalence class of the relation  $x \sim y$  if  $x - y \in \mathbb{Q}$ . (e.g, pick  $x \in O_x$  for  $O_x \in \mathbb{R}/\mathbb{Q}$ )

**Lemma 2.2.3**

Suppose that  $V$  is a Vitali set then

$$V \cap V + q = \emptyset$$

For all  $q \in \mathbb{Q} \setminus \{0\}$

*Proof.* Suppose not, there exists  $a \in V$  such that  $a \in V + q \implies a - q \in V$  but we only pick 1 element in each equivalence class. contradiction.  $\square$

**Lemma 2.2.4**

Let  $V$  be a Vitali set and let  $W = \{q \in [-1, 1] : q \in \mathbb{Q}\}$  and

$$E = \bigcup_{w \in W} V + w$$

then

$$[0, 1] \subseteq E \subseteq [-1, 2]$$

*Proof.* Consider  $E \subseteq [-1, 2]$ . Since  $V \subseteq [0, 1)$ , then for any  $v \in V$ ,  $v \in [0, 1) \implies v + w \in [-1, 2]$ .

For the  $[0, 1] \subseteq E$ , for any  $x \in [0, 1]$  there exists  $O_x \in \mathbb{R}/\mathbb{Q}$  such that  $x \in O_x$ . then there exists  $v \in C_x$  such that  $v \in [0, 1)$  and  $v \in V$ , since both are from the same equivalence

class, then  $x - v \in \mathbb{Q}$  and  $|x - v| < 1 \implies x - v \in (-1, 1)$ . Hence, there exists  $w \in W$  such that  $w = x - v$  so  $v + w = x$ .  $\square$

*Proof of the theorem.* Suppose that  $\mu$  exists then using the result from Lemma 2.2.4 we get that

$$\mu([0, 1]) \leq \mu(E) \leq \mu([-1, 2])$$

from Lemma 2.2.3 we know that each  $V + w$  is disjoint, so

$$\begin{aligned} \mu([0, 1]) &\leq \sum_{w \in W} \mu(V) \leq \mu([-1, 2]) \\ 1 &\leq \sum_{w \in W} \mu(V) \leq 3 \end{aligned}$$

if  $\mu(V) = 0$  then  $\mu(E) = 0$  and if  $\mu(V) > 0$  then  $\mu(E) = \infty$ . Both are contradiction.  $\square$

## §2.3 $\sigma$ -algebra

**Definition 2.3.1.** Given a reference  $X$ . An **algebra** is a collection of subsets of  $X$ ,  $\mathcal{A}$ , such that

- (i)  $X \in \mathcal{A}$
- (ii) If  $A \in \mathcal{A}$  then the complement  $A^c = X \setminus A \in \mathcal{A}$
- (iii) If  $A, B \in \mathcal{A}$  then  $A \cup B \in \mathcal{A}$

**Remark 2.3.2.** •  $\emptyset \in \mathcal{A}$  because  $\emptyset = X^c$

- $A_1, A_2 \in \mathcal{A}$ ,  $A_1 \setminus A_2 = A_1 \cap A_2^c \in \mathcal{A}$
- Observe that if  $A_1, A_2 \in \mathcal{A}$  then  $A_1 \cap A_2 \in \mathcal{A}$  because  $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$

### Example 2.3.3

$X = [a, b]$  and  $\mathcal{A}$  is the collection of all sets  $E \subseteq [a, b]$  such that the Riemann integral  $\int \mathbb{1}_E(t) dt$  exists

**Definition 2.3.4.** A  $\sigma$ -algebra  $\mathcal{M}$  on  $X$  is

- (i) an algebra of subsets of  $X$
- (ii) If  $A_1, A_2, A_3, \dots$  is a sequence of set in  $\mathcal{M}$  then

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$$

$(X, \mathcal{M})$  is called a “**measurable space**”.

**Remark 2.3.5.**  $\mathcal{M}$  is a  $\sigma$ -algebra on  $X$  then it satisfies

- (i)  $X \in \mathcal{M}$
- (ii) If  $A \in \mathcal{M}$  then  $A^c \in \mathcal{M}$
- (iii) countable union of sets in  $\mathcal{M}$  is in  $\mathcal{M}$

**Definition 2.3.6.** Let  $(X, \mathcal{M})$  be a measurable set. Then a measure  $\mu$  is a set function  $\mu : \mathcal{M} \rightarrow [0, \infty]$ ,  $E \mapsto \mu(E)$  such that

- (i)  $\mu(\emptyset) = 0$
- (ii) If  $E_1, E_2, E_3, \dots$  is a sequence of disjoint set in  $\mathcal{M}$  then

$$\mu \left( \bigcup_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \mu(E_j)$$

called  $\sigma$ -additivity.

$(X, \mathcal{M}, \mu)$  is called a “**measure space**”.

**Remark 2.3.7.**

$$\left( \bigcap_{j=1}^{\infty} A_j \right) = \left( \bigcup_{j=1}^{\infty} A_j^c \right)^c \in \mathcal{M}$$

### Example 2.3.8

examples of  $\sigma$ -algebra

- (i)  $\mathcal{M} = \{\emptyset, X\}$
- (ii)  $\mathcal{M} = \mathfrak{P}(X)$  = collection of all subsets of  $X$   
 $\mathbb{N} = \{1, 2, 3, \dots\}$  and  $\mu(E) = |E|$  (the cardinality of  $E$ ) if  $E$  is finite and  $\mu(E) = \infty$  if  $E$  is infinite.
- (iii)  $X$  write  $X$  as a disjoint (countable) union of sets  $A_j$ . Then  $\mathcal{M}$  = all countable unions of  $A_j$ .
- (iv) Let  $X$  be a set. Let  $\mathcal{M}$  be the collection of all sets  $A$ ,  $A \subseteq X$  such that  $A$  is countable or  $A^c$  is countable.
- (v)  $X = \mathbb{R}$  (or  $\mathbb{R}^n$ ),  $\mathcal{B}_{\mathbb{R}}$  is the smallest  $\sigma$ -algebra containing all open sets.

More generally if  $\mathcal{E}$  is a collection of subsets of  $X$  then  $\mathfrak{M}(\mathcal{E})$  is the smallest  $\sigma$ -algebra that contains all sets in  $\mathcal{E}$ .

If  $\mathcal{M}_1, \mathcal{M}_2$  are two  $\sigma$ -algebras, then  $\mathcal{M}_1 \cap \mathcal{M}_2$  is also a  $\sigma$ -algebra.

If  $\{\mathcal{M}_\alpha\}_{\alpha \in \mathcal{I}}$  is a collection of  $\sigma$ -algebras, their intersection is also a  $\sigma$ -algebra.

### Generating $\sigma$ -algebra

**Definition 2.3.9.**  $\mathfrak{M}(\mathcal{E}) :=$  intersection of all  $\sigma$ -algebra that contain the collection  $\mathcal{E}$ . We call it the  $\sigma$ -algebra generated by  $\mathcal{E}$ . i.e.

$$\mathfrak{M}(\mathcal{E}) = \bigcap_{\substack{\mathcal{F} \in \mathcal{M} \\ \mathcal{E} \subseteq \mathcal{F}}} \mathcal{F}$$

**Remark 2.3.10.** If  $\mathcal{E} \subseteq \mathcal{F} \implies \mathfrak{M}(\mathcal{E}) \subseteq \mathfrak{M}(\mathcal{F})$

#### Lemma 2.3.11

If  $\mathcal{E} \subseteq \mathfrak{M}(\mathcal{F})$  then  $\mathfrak{M}(\mathcal{E}) \subseteq \mathfrak{M}(\mathcal{F})$

*Proof.*  $\mathfrak{M}(\mathcal{F})$  is a  $\sigma$ -algebra that contains  $\mathcal{E}$ . It contains the intersection of all  $\sigma$ -algebras which contain  $\mathcal{E}$ .  $\square$

#### Example 2.3.12

$\mathcal{B}_{\mathbb{R}} = \sigma$ -algebra on  $\mathbb{R}$  containing all open sets  $\mathcal{E}$  a collection of all open intervals,  $\mathcal{E} \subseteq \mathcal{O} =$  collection of all open sets in  $\mathbb{R}$ ,  $\mathcal{B}_{\mathbb{R}} = \mathfrak{M}(\mathcal{O})$ .  $\mathfrak{M}(\mathcal{E}) \subseteq \mathcal{B}_{\mathbb{R}}$ . Each open set is a countable union of open intervals. Each open set is contained in  $\mathfrak{M}(\mathcal{E})$ .

Since  $\mathcal{O} \subseteq \mathfrak{M}(\mathcal{E}) \implies \mathfrak{M}(\mathcal{O}) \subseteq \mathfrak{M}(\mathcal{E})$ . get  $\mathfrak{M}(\mathcal{O}) = \mathfrak{M}(\mathcal{E})$ .

**Definition 2.3.13.** Given  $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2), \dots, (X_n, \mathcal{M}_n)$  measurable spaces. Define a “product  $\sigma$ -algebra” on  $X_1 \times X_2 \times \dots \times X_n$  denoted by

$$\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \dots \oplus \mathcal{M}_n = \bigoplus_{j=1}^n \mathcal{M}_j$$

defined as the  $\sigma$ -algebra generated by the sets  $E_1 \times E_2 \times \dots \times E_n$  where  $E_j \in \mathcal{M}_j$ .

i.e., define  $\mathcal{E} := \{(E_1 \times E_2 \times \dots \times E_n) : E_j \in \mathcal{M}_j\}$  then

$$\bigoplus_{j=1}^n \mathcal{M}_j := \mathfrak{M}(\mathcal{E})$$

**Remark 2.3.14.** Folland defines it the  $\sigma$ -algebra generated by

$$(X_1 \times X_2 \times \cdots \times X_{n-1} \times E_n)$$

where  $E_n \in \mathcal{M}_n$ ,

$$(X_1 \times X_2 \times \cdots \times E_{n-1} \times X_n)$$

where  $E_{n-1} \in \mathcal{M}_{n-1}$ . and so on. To be clear, let

$$\mathcal{E}' := \bigcup_{j=1}^n \{(X_1 \times \cdots \times X_{j-1} \times E_j \times X_{j+1} \times \cdots \times X_n) : E_j \in \mathcal{M}_j\}$$

then

$$\bigoplus_{j=1}^n \mathcal{M}_j := \mathfrak{M}(\mathcal{E}')$$

**Claim 2.3.15** — Both definitions on product of  $\sigma$ -algebra are equivalent.

*Proof.* The goal is to show that  $\mathfrak{M}(\mathcal{E}) = \mathfrak{M}(\mathcal{E}')$ .

( $\supseteq$ ) Obviously,  $\mathcal{E}' \subseteq \mathcal{E}$  so  $\mathfrak{M}(\mathcal{E}') \subseteq \mathfrak{M}(\mathcal{E})$ .

( $\subseteq$ ) We want to show that  $\mathcal{E} \subseteq \mathfrak{M}(\mathcal{E}')$ . Fix  $(E_1 \times E_2 \times \cdots \times E_n) \in \mathcal{E}$  then from the definition of  $\sigma$ -algebra generated by a collection, which is closed under intersection, so we can pick an element from the construction of  $\mathcal{E}'$  and do the intersection, so  $(E_1 \times E_2 \times \cdots \times E_n) \in \mathfrak{M}(\mathcal{E}')$ .

□

### Theorem 2.3.16

Given  $(X_1, \mathcal{M}_1), (X_2, \mathcal{M}_2)$  measurable spaces. Assume that  $\mathcal{M}_1$  is generated by a collection  $\mathcal{E}_1$  and  $\mathcal{M}_2$  is generated by a collection  $\mathcal{E}_2$ . Then  $\mathcal{M}_1 \oplus \mathcal{M}_2$  is generated by the sets  $E_1 \times X_2, X_1 \times E_2$ , where  $E_1 \in \mathcal{E}_1$  and  $E_2 \in \mathcal{E}_2$ .

*Proof.* Let  $\mathcal{P} := \{E_1 \times E_2 : E_i \in \mathcal{E}_i\}$ , obviously  $\mathfrak{M}(\mathcal{P}) = \mathfrak{M}(\{E_1 \times X_2 : E_1 \in \mathcal{E}_1\} \cup \{X_1 \times E_2 : E_2 \in \mathcal{E}_2\})$  and  $\mathfrak{M}(\mathcal{P}) \subseteq \mathcal{M}_1 \oplus \mathcal{M}_2$ . We need to show that  $\mathcal{M}_1 \oplus \mathcal{M}_2 \subseteq \mathfrak{M}(\mathcal{P})$ . Define

$$\mathcal{G}_1 = \{E_1 \subseteq X_1 : E_1 \times X_2 \in \mathfrak{M}(\mathcal{P})\}$$

$$\mathcal{G}_2 = \{E_2 \subseteq X_2 : X_1 \times E_2 \in \mathfrak{M}(\mathcal{P})\}$$

then  $\mathcal{G}_1$  is a  $\sigma$ -algebra consisting of subset of  $X_1$  which contains  $\mathcal{E}_1$ ,  $\mathcal{E}_1 \subseteq \mathcal{G}_1$ .  $\mathcal{E}_1$  generates  $\mathcal{M}_1$  so  $\mathfrak{M}(\mathcal{E}_1) = \mathcal{M}_1 \subseteq \mathcal{G}_1$ . So, we have  $E_1 \times X_2 \in \mathfrak{M}(\mathcal{P})$  for all  $E_1 \in \mathcal{M}_1$  and  $X_1 \times E_2 \in \mathfrak{M}(\mathcal{P})$  for all  $E_2 \in \mathcal{M}_2$ . The  $\sigma$ -algebra generated by the sets  $E_1 \times X_2, X_1 \times E_2$  is contained  $\mathcal{M}_1 \oplus \mathcal{M}_2 \in \mathfrak{M}(\mathcal{P})$ . □

**Claim 2.3.17** —  $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$ .

where  $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}}$  is generated by  $E_1 \times E_2$ , where  $E_1, E_2 \in \mathcal{B}_{\mathbb{R}}$ . and  $\mathcal{B}_{\mathbb{R}^2}$  is generated by the open sets in  $\mathbb{R}^2$ .

*Proof.*  $\mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{B}_{\mathbb{R}^2}$ . Want  $\mathcal{B}_{\mathbb{R}^2} \subseteq \mathcal{B}_{\mathbb{R}} \oplus \mathcal{B}_{\mathbb{R}}$ . Consider the collection of all open rectangle of the form  $(a_1, b_1) \times (a_2, b_2)$  such  $a_i, b_i \in \mathbb{Q}$ . which are contained in  $O \subseteq \mathbb{R}^2$   $\square$

**Definition 2.3.18** (The Borel  $\sigma$  algebra on the extended real line). We use the notion  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$ . One possibility to define “ $\mathcal{B}_{\overline{\mathbb{R}}}$ ” is the  $\sigma$ -algebra generated by open sets in  $\mathbb{R}$ ,  $\{\infty\}$ ,  $\{-\infty\}$  open intervals should be  $(a, b)$ ,  $(a, \infty]$ ,  $[-\infty, b)$  for  $-\infty \leq a < b \leq \infty$ . Then define  $d(x, y) = |\arctan(x) - \arctan(y)|$  and  $\arctan(\infty) = \pi/2$ ,  $\arctan(-\infty) = -\pi/2$ .

## §2.4 Measures

**Definition 2.4.1.** Measures are  $\sigma$ -additive set functions,  $\mu(\emptyset) = 0$  and

$$\mu \left( \biguplus_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \mu(E_j)$$

where  $E_1, E_2, \dots$  is a sequence of disjoint sets.

**Remark 2.4.2.** There is some property

$$E \subseteq F \implies \mu(E) \leq \mu(F)$$

$$F = E \uplus (F \setminus E) \implies \mu(F) = \mu(E) + \mu(F \setminus E)$$

$\mu(\bigcup A_j) \leq \sum \mu(A_j)$  we can write  $\bigcup A_j$  as a disjoint union, i.e.,  $E_1 = A_1$ ,  $E_2 = A_2 \setminus A_1$ ,  $E_3 = A_3 \setminus (A_1 \cup A_2)$ , and so on then  $\mu(\bigcup A_j) = \mu(\bigcup E_j) = \sum \mu(E_j) \leq \sum \mu(A_j)$

The monotone convergence theorem for sets (continuity from below)

### Theorem 2.4.3

If  $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$  then

$$\mu \left( \bigcup_{j=1}^{\infty} E_j \right) = \lim_{j \rightarrow \infty} \mu(E_j)$$



*Proof.*

$$\bigcup_{j=1}^{\infty} E_j = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus (E_1 \cup E_2)) \cup \dots$$

So, we define  $B_1 = E_1, B_n = E_n \setminus E_{n-1}$  for  $n \geq 2$  then all  $B_j$  are disjoint.

$$\begin{aligned} \bigcup_{j=1}^{\infty} E_j &= \bigcup_{j=1}^{\infty} B_j \\ \mu \left( \bigcup_{j=1}^{\infty} E_j \right) &= \mu \left( \bigcup_{j=1}^{\infty} B_j \right) \\ &= \sum_{j=1}^{\infty} \mu(B_j) \\ &= \mu(E_1) + \sum_{j=2}^{\infty} \mu(E_j \setminus E_{j-1}) \\ &= \mu(E_1) + \sum_{j=2}^{\infty} \mu(E_j) - \mu(E_{j-1}) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \end{aligned}$$

□

**Remark 2.4.4.** If we prove something for the set then we can prove it for the complement.

$$\mu(A) + \mu(A^c) = \mu(X)$$

#### Theorem 2.4.5

If  $\mu(X) < \infty$  then if  $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots, E_n \supseteq E_{n+1}$  for all  $n$  then

$$\mu \left( \bigcap_{j=1}^{\infty} E_j \right) = \lim_{j \rightarrow \infty} \mu(E_j)$$

*Proof.* Assume  $E_j$  are decreasing, i.e.,

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$$

then  $E_1^c \subseteq E_2^c \subseteq \dots$  then

$$\begin{aligned}\mu\left(\bigcup_{j=1}^{\infty} E_j^c\right) &= \lim_{j \rightarrow \infty} \mu(E_j^c) \\ \mu(X) - \mu\left(\left(\bigcup_{j=1}^{\infty} E_j^c\right)^c\right) &= \lim_{j \rightarrow \infty} (\mu(X) - \mu(E_j)) \\ \mu(X) - \mu\left(\bigcap_{j=1}^{\infty} E_j\right) &= \lim_{j \rightarrow \infty} (\mu(X) - \mu(E_j))\end{aligned}$$

□

#### Example 2.4.6

$\mathbb{N}$  with counting measure,  $E_j = \{j, j+1, j+2, \dots\}$ ,  $\mu(E_j) = \infty$ ,  $\bigcap E_j = \emptyset$  has measure 0.

**Definition 2.4.7.** If  $A_1, A_2, A_3, \dots$  is an arbitrary sequence of measurable sets. We can define

$$\begin{aligned}\limsup A_j &:= \bigcap_{n=1}^{\infty} \bigcup_{j \geq n} A_j = \{x : x \in A_n \text{ for infinitely many } n\} \\ \liminf A_j &:= \bigcup_{n=1}^{\infty} \bigcap_{j \geq n} A_j = \{x : x \text{ belong to all but finitely many}\}\end{aligned}$$

#### Lemma 2.4.8 (Borel-Cantelli Lemma)

If  $\{A_j\}$  is a sequence of measurable sets such that

$$\sum_{j=1}^{\infty} \mu(A_j) < \infty$$

then almost every  $x$  (meaning all  $x$  except in a null set) belong to on  $A_n$  for only finitely many  $n$ . Or equivalently,

$$\mu(\limsup A_n) = 0$$

*Proof.*  $\bigcup_{j \geq n} A_j$  are decreasing. In Borel Cantelli, we have  $\sum \mu(A_j) < \infty$ , so  $\mu(\bigcup A_n) = 0$ .

use “continuity from above”

$$\mu(\limsup A_n) = \lim_{n \rightarrow \infty} \mu \left( \bigcup_{j \geq n} A_j \right)$$

$$\mu \left( \bigcup_{j \geq n} A_j \right) \leq \sum_{j \geq n} \mu(A_j) \rightarrow 0$$

as  $n \rightarrow \infty$ . □

Completion of a  $\sigma$ -algebra (when a measure  $\mu$  is given),  $(X, \mathcal{M}, \mu) \overline{\mathcal{M}}$  consists of all unions  $E \cup F$ , where  $E \in \mathcal{M}$  and  $F \subseteq N \in \mathcal{M}$  for some null set  $N$ ,  $\mu(N) = 0$ .

Define  $\bar{\mu}$  by  $\bar{\mu}(E \cup F) = \mu(E)$ .

## §2.5 Measurable Functions

**Definition 2.5.1.**  $f : X \rightarrow Y$  where  $(X, \mathcal{M})$  and  $(Y, \mathcal{N})$  are measurable spaces.  $f$  is  $(\mathcal{M}, \mathcal{N})$ -measurable if for every  $E \in \mathcal{N}$ ,  $f^{-1}(E) \in \mathcal{M}$ . where  $f^{-1}(E) = \{x \in X : f(x) \in E\}$ .

### Lemma 2.5.2

Let  $\mathcal{E}$  generate  $\mathcal{N}$  (i.e.,  $\mathcal{N} = \mathfrak{M}(\mathcal{E})$ ). Then  $f$  is  $(\mathcal{M}, \mathcal{N})$ -measurable if and only if  $f^{-1}(E) \in \mathcal{M}$  for all  $E \in \mathcal{E}$ .

*Proof.* Define  $\mathcal{C} = \{E : f^{-1}(E) \in \mathcal{M}\}$ , observe that  $\mathcal{C}$  is a  $\sigma$ -algebra. then

$$f(x) \in \bigcup E_j \iff x \in f^{-1}\left(\bigcup E_j\right) \iff x \in \bigcup f^{-1}(E_j) \iff \bigcup \{x : f(x) \in E_j\}$$

□

**Claim 2.5.3** —  $f : X \rightarrow Y$  is  $(\mathcal{M}, \mathcal{N})$ -measurable,  $g : Y \rightarrow Z$  is  $(\mathcal{N}, \mathcal{R})$ -measurable then  $g \circ f : X \rightarrow Z$  is  $(\mathcal{M}, \mathcal{R})$ -measurable.

*Proof.*  $(g \circ f)^{-1}(E) = \{x \in X : g(f(x)) \in E\} = f^{-1}(g^{-1}(E)) = \{x \in X : f(x) \in g^{-1}(E)\}$  □

**Claim 2.5.4** —  $f : X \rightarrow \mathbb{R}$  is  $\mathcal{M}$ -measurable then  $f^2$  is  $\mathcal{M}$ -measurable.

*Proof.*  $(f^2)^{-1}(-\infty, a) = \{x : f^2(x) < a\} = \{x : f(x) < \sqrt{a}\} \cup \{x : f(x) > -\sqrt{a}\}$   $\square$

**Claim 2.5.5** —  $f : X \rightarrow \mathbb{R}, g : X \rightarrow \mathbb{R}$  are  $\mathcal{M}$ -measurable then  $f + g$  and  $f \cdot g$  are  $\mathcal{M}$ -measurable.

*Proof.*

$$(f + g)^{-1}(-\infty, a) = \bigcup_{r \in \mathbb{Q}} (f^{-1}(-\infty, a + r) \cap g^{-1}(-\infty, r))$$

$$\begin{aligned} (f + g)^2 &= f^2 + 2fg + g^2 \\ fg &= \frac{1}{2} ((f + g)^2 - f^2 - g^2) \end{aligned}$$

$\square$

**Claim 2.5.6** — vector-valued-function  $f : X \rightarrow (Y_1 \times Y_2 \times \cdots \times Y_n)$  and defined by  $x \mapsto (f_1(x), f_2(x), \dots, f_n(x))$  where  $f_j : X \rightarrow Y_j$  is  $(\mathcal{M}, \mathcal{N}_j)$ -measurable.

Then  $f$  is  $(\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_n)$  if and only if  $f_i(\mathcal{M}_i, \mathcal{N}_i)$ -measurable.

*Proof.*

$$\begin{aligned} f^{-1}(E_1 \times E_2 \times \cdots \times E_n) &= f_1^{-1}(E_1) \cap f_2^{-1}(E_2) \cap \cdots \cap f_n^{-1}(E_n) \\ &= \bigcap_{j=1}^n f_j^{-1}(E_j) \end{aligned}$$

$\square$

**Claim 2.5.7** —  $M(x) = \max\{f(x), g(x)\}$ ,  $f, g : X \rightarrow \mathbb{R}$ ,  $\mathcal{M}$ -measurable.

*Proof.*  $M^{-1}(-\infty, a) = \{x : M(x) < a\} = \{x : f(x) < a, g(x) < a\} = f^{-1}(-\infty, a) \cap g^{-1}(-\infty, a)$   $\square$

**Claim 2.5.8** —  $f_n : X \rightarrow \mathbb{R}$ ,  $\mathcal{M}$ -measurable, then  $S(x) = \sup_{n \in \mathbb{N}} f_n$  is  $\mathcal{M}$ -measurable.

*Proof.*  $S^{-1}(-\infty, a) = \{x : S(x) < a\} = \{x : \sup f_n(x) < a\} = \bigcap_n \{x : f_n(x) < a\}$   $\square$

**Remark 2.5.9.** We use the similar proof for min and inf.

**Definition 2.5.10.** If  $f_n : X \rightarrow \mathbb{R}$ ,  $\mathcal{M}$ -measurable then

$$\limsup f_n = \inf_k \sup_{n \geq k} f_n$$

$$\liminf f_n = \sup_k \inf_{n \geq k} f_n$$

**Claim 2.5.11** —  $\limsup f_n$  and  $\liminf f_n$  are  $\mathcal{M}$ -measurable.

*Proof.* For  $\limsup f_n$ , fix  $k$  then  $\sup_{n \geq k} f_n$  is  $\mathcal{M}$ -measurable,  $\inf_k \sup_{n \geq k} f_n$  is  $\mathcal{M}$ -measurable. Similarly for  $\liminf f_n$ .  $\square$

**Theorem 2.5.12**

Let  $(X, \mathcal{M})$  be a measurable space,  $f_n : X \rightarrow \mathbb{C}$  be  $\mathcal{M}$ -measurable functions. Define

$$E_{lim} = \{x \in X : \lim_{n \rightarrow \infty} f_n(x) \text{ exists}\}$$

then  $E_{lim} \in \mathcal{M}$ .

*Proof.* We can rewrite  $E_{lim}$  as

$$E_{lim} = \{x \in X : \{f_n(x)\}_{n \in \mathbb{N}} \text{ is a Cauchy sequence}\}$$

Define

$$A_{n,m}(k) = \left\{x \in X : |f_n(x) - f_m(x)| < \frac{1}{k}\right\}$$

then  $A_{n,m}(k) \in \mathcal{M}$  for all  $n, m, k$ . then

$$E_{lim} = \bigcup_{k \geq 1} \bigcap_{N \geq 1} \bigcup_{m \geq N, n \geq N} A_{m,n}(k)$$

$\square$

# 3 Integration

## §3.1 Simple Functions

**Definition 3.1.1.** nonnegative simple function are measurable function with finitely many values in  $\mathbb{R}$  (NOT on  $\overline{\mathbb{R}}$ ).  $s : X \rightarrow \mathbb{R}$ ,  $s(x) = \sum z_j \mathbb{1}_{x, s(z)=z_j}(x) = \sum z_j \mathbb{1}_{f^{-1}(z_j)}$  If values of  $s$  are  $\{z_1, \dots, z_n\}$

### Theorem 3.1.2

Consider nonnegative measurable function  $f$ . There exist a sequence of simple function  $s_n$  such that

- $0 \leq s_n \leq s_{n+1} \leq f$  (i.e,  $s_n(x) \leq s_{n+1}(x)$ )
- $\lim_{n \rightarrow \infty} s_n(x) = f(x)$  for all  $x$
- The convergence is uniform on all sets where  $f$  is bounded. If  $E$  is such that  $|f(x)| \leq M$  for all  $x \in E$  then

$$\sup_{x \in E} f(x) - s_n(x) \rightarrow 0$$

*Proof.*  $s_n$  is defined so that it takes value in  $[0, 2^n)$ . Consider the segment  $\frac{k}{2^n}$  on y-axis, then

$$s_n(x) = \begin{cases} k \cdot 2^{-n} & \text{if } k2^{-n} \leq f(x) < (k+1)2^{-n}, 0 \leq k \leq 2^n - 1 \\ 2^n & \text{if } f(x) \geq 2^n \end{cases}$$

If  $f(x) < 2^n$  then  $0 \leq f(x) - s_n(x) < 2^{-n}$ . We can see that  $s_n(x) \leq s_{n+1}(x)$  because each step of  $s_{n+1}$  is a refinement of  $s_n$ .  $\square$

We first define the integral for simple function (in analogy to the definition of Riemann-integral for step functions)

**Definition 3.1.3.** Define  $s(x) = \sum_j c_j \mathbb{1}_{E_j}$  where the  $E_j$  are pairwise disjoint,  $\biguplus E_j = X$ , then

$$\int s \, d\mu = \sum_j c_j \mu(E_j)$$

**Claim 3.1.4 —**

$$s(x) = \sum_{j=1}^n c_j \mathbb{1}_{E_j}(x) = \sum_{k=1}^m d_k \mathbb{1}_{E_k}(x)$$

where  $X = \biguplus E_j = \biguplus E_k$ . If  $x \in E_j \cap E_k$  then  $c_j = d_k$ .

*Proof.* We know that  $\biguplus_{j,k} E_j \cap E_k = X$  and  $E_j = \biguplus_k E_j \cap E_k$

GOAL:  $\sum_{j=1}^n c_j \mu(E_j) = \sum_{k=1}^m d_k \mu(F_k)$

$$\begin{aligned} \text{LHS} &= \sum_{j=1}^n c_j \sum_{k=1}^{\infty} \mu(E_j \cap F_k) = \sum_{k=1}^m \sum_{j=1}^n d_k \mu(E_j \cap E_k) \\ &= \sum_{k=1}^m d_k \mu(F_k) \end{aligned}$$

□

**Lemma 3.1.5**

Suppose  $s, t$  are simple functions then

$$\int (s + t) \, d\mu = \int s \, d\mu + \int t \, d\mu$$

**Remark 3.1.6.** Can shortly write

$$\int s + t = \int s + \int t$$

*Proof.*

$$\begin{aligned} s &= \sum_{j=1}^n c_j \mathbb{1}_{E_j} = \sum_j \sum_k c_j \mathbb{1}_{E_j \cap F_k} \\ t &= \sum_{k=1}^m d_k \mathbb{1}_{F_k} = \sum_j \sum_k d_k \mathbb{1}_{E_j \cap F_k} \\ s + t &= \sum_{j,k} (c_j + d_k) \mathbb{1}_{E_j \cap F_k} \end{aligned}$$

$$\begin{aligned}
\int s \, d\mu &= \sum_{j,k} c_j \mu(E_j \cap F_k) \\
\int t \, d\mu &= \sum_{j,k} d_k \mu(E_j \cap F_k) \\
\int (s+t) \, d\mu &= \sum_{j,k} (c_j + d_k) \mu(E_j \cap F_k)
\end{aligned}$$

□

**Lemma 3.1.7**

$\nu(E) = \int_E s \, d\mu = \int s \mathbb{1}_E \, d\mu = \sum c_j \mu(E_j \cap E)$  this defines a measure on  $\mathcal{M}$  (given  $\sigma$ -algebra)

*Proof.* If  $E^l$  is a sequence of pairwise disjoint measurable set, check

$$\begin{aligned}
\nu\left(\biguplus E^l\right) &= \sum \nu(E^l) \\
\nu\left(\biguplus E^l\right) &= \sum c_j \mu(E_j \cap \biguplus E^l) \\
&= \sum_{j=1}^n c_j \sum_l \mu(E_j \cap E^l) \\
&= \sum_l \sum_j c_j \mu(E_j \cap E^l) \\
&= \sum_l \nu(E^l)
\end{aligned}$$

□

**§3.2 Non-negative Measurable Functions**

**Definition 3.2.1.** For any non-negative  $f$ , a measurable function, define

$$\int f \, d\mu = \sup_{\substack{s \leq f \\ s \text{ simple}}} \int s \, d\mu$$



**Remark 3.2.2.** If  $0 \leq f \leq g$  then  $\int f \, d\mu \leq \int g \, d\mu$

**Theorem 3.2.3 (Monotone Convergence Theorem)**

If  $\{f_n\}$  is a sequence of measurable function, and  $0 \leq f_n \leq f_{n+1}$  for all  $n$ . (that means  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ ) Then

$$\int f \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

*Proof.* Since  $f_n \leq f_{n+1} \leq f$  then

$$\lim_{n \rightarrow \infty} \int f_n \, d\mu \leq \int f \, d\mu$$

We need to show that

$$\int f \, d\mu \leq \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

So, it suffices to show that for any  $0 \leq s \leq f$ , that

$$\int s \, d\mu \leq \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

It suffices to show that for any  $\varepsilon > 0$ ,

$$(1 - \varepsilon) \int s \, d\mu \leq \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

Define  $E_n = \{x : (1 - \varepsilon)s(x) \leq f_n(x)\}$ , any  $x$  will be in one of the  $E_n$ . Then for any  $x \in E_n$ ,

$$s(x) \leq \frac{f_n(x)}{1 - \varepsilon}$$

Consider the measure defined by

$$\nu(E) = \int_E s \, d\mu$$

(we already show this is a measure in 3.1.7). We have  $E_n \subseteq E_{n+1}$  and  $E_n \rightarrow X$ . By continuity from below 2.4.3,

$$\lim_{n \rightarrow \infty} \nu(E_n) = \nu(X) = \int s \, d\mu$$

We get that

$$\nu(E_n) = \int_{E_n} s \, d\mu \leq \int_{E_n} \frac{f_n(x)}{1 - \varepsilon} \, d\mu \leq \int \frac{f_n(x)}{1 - \varepsilon} \, d\mu = \frac{1}{1 - \varepsilon} \int f_n(x) \, d\mu$$

Finally, we take limit on both sides and we have

$$\lim_{n \rightarrow \infty} \nu(E_n) = \nu(\mathbb{R}) = \int s \, d\mu \leq \lim_{n \rightarrow \infty} \frac{1}{1 - \varepsilon} \int f_n \, d\mu$$

□

### Lemma 3.2.4

If  $f, g$  are non negative measurable function then

$$\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$$

*Proof.* Now we have a tool

- Monotone Convergence Theorem
- Existence of  $s_n \gg f, t_n \gg g$

$$\begin{aligned} \int (s_n + t_n) \, d\mu &= \int s_n \, d\mu + \int t_n \, d\mu \\ \int (f + g) \, d\mu &= \int f \, d\mu + \int g \, d\mu \end{aligned}$$

□

### Lemma 3.2.5

$f_k \geq 0$ ,  $f_k$  is measurable

$$\int \sum_{k=1}^{\infty} f_k(x) \, d\mu = \sum_{k=1}^{\infty} \int f_k \, d\mu$$

*Proof.* Just apply the Monotone Convergence Theorem.

$$s_n(x) = \sum_{k=1}^n f_k(x) \rightarrow \sum_{k=1}^{\infty} f_k(x)$$

□

**Remark 3.2.6.** We cannot always interchange integrals and limits (monotonicity is key)  
 $f_n(x) = \frac{1}{n} \mathbb{1}_{[0,n]}$ ,  $\int f_n \, d\mu = 1$  but  $\lim_{n \rightarrow \infty} f_n(x) = 0$ .

$$0 = \int \lim_{n \rightarrow \infty} f_n(x) \, d\mu < \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

Or on  $[0, 1]$ ,  $f_n(x) = n \mathbb{1}_{[0, 1/n]}$ ,  $\int f_n \, d\mu = 1$  but  $\lim_{n \rightarrow \infty} f_n(x) = 0$ .

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

### Lemma 3.2.7 (Fatou's Lemma)

If  $\{f_j\}$  is a sequence of measurable functions

$$\int \liminf_{j \rightarrow \infty} f_j(x) \, d\mu \leq \liminf_{j \rightarrow \infty} \int f_j \, d\mu$$

meaning

$$\int \lim_{k \rightarrow \infty} \underbrace{\inf_{j \geq k} f_j(x)}_{\text{increasing on } k} \, d\mu \leq \lim_{k \rightarrow \infty} \inf_{j \geq k} \int f_j \, d\mu$$

*Proof.*

$$\int \lim_{k \rightarrow \infty} \inf_{j \geq k} f_j(x) \, d\mu \stackrel{\text{MCT}}{=} \lim_{k \rightarrow \infty} \int \inf_{j \geq k} f_j(x) \, d\mu$$

Take any  $l \geq k$ , then  $\inf_{j \geq k} f_j(x) \leq f_l(x)$ , then for  $l \geq k$

$$\begin{aligned} \int \inf_{j \geq k} f_j(x) \, d\mu &\leq \int f_l(x) \, d\mu \\ \int \inf_{j \geq k} f_j(x) \, d\mu &\leq \inf_{j \geq k} \int f_j(x) \, d\mu \end{aligned}$$

□

## §3.3 General Measurable Functions

Integral for “general” measurable functions.

**Definition 3.3.1.** Given a measurable function  $f$ , we define the **positive part** of  $f$  as

$$f^+(x) = \max\{f(x), 0\}$$

and the **negative part** of  $f$  as

$$f^-(x) = \max\{-f(x), 0\}$$

Then we get that

$$f = f^+ - f^-$$

**Definition 3.3.2.**  $f : X \rightarrow \mathbb{R}$  (or  $\overline{\mathbb{R}}$ ) Suppose that  $f$  is a measurable function, then we define

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$$

provided that at least one of  $\int f^\pm \, d\mu$  is finite

**Definition 3.3.3.**  $f : X \rightarrow \mathbb{R}$  (or  $\overline{\mathbb{R}}$ )  $f$  is **integrable** if  $\int f^+ \, d\mu, \int f^- \, d\mu$  is finite  $\iff \int |f| \, d\mu$  is finite

$\mathcal{L}^1$  is the class of integrable function

**Definition 3.3.4.**  $f : X \rightarrow \mathbb{C}$  is measurable ( $\iff \Re(f)$  and  $\Im(f)$  are measurable) Assumeing that  $\Re f \in \mathcal{L}^1$  and  $\Im f \in \mathcal{L}^1$  then

$$\int f \, d\mu = \int \Re f \, d\mu + i \int \Im f \, d\mu$$

**Claim 3.3.5 —** Suppose that  $f, g$  are measurable then

$$\begin{aligned} \int f + g \, d\mu &= \int f \, d\mu + \int g \, d\mu \\ \int \alpha f \, d\mu + \alpha \int f \, d\mu \end{aligned}$$

**Lemma 3.3.6**

$f : X \rightarrow \overline{\mathbb{R}}$  is measurable, and  $\int |f| \, d\mu = 0$  then  $f = 0$  almost everywhere.

*Proof.* Define  $E_n = \{x : |f(x)| > \frac{1}{n}\}$  then from continuity from below, we get that

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$$

define  $E = \bigcup E_n$  and we can write  $E = \{x : |f(x)| > 0\}$  then we know that

$$|f| \geq |f| \mathbb{1}_{E_n} \geq \frac{1}{n} \mathbb{1}_{E_n}$$

then

$$\begin{aligned} \int |f| \, d\mu &\geq \int \frac{1}{n} \mathbb{1}_{E_n} \, d\mu \\ &= \frac{1}{n} \mu(E_n) \end{aligned}$$

we get that  $\mu(E_n) = 0$  for all  $n$  then  $\mu(E) = 0$ . Therefore  $f = 0$  almost everywhere.  $\square$

**Remark 3.3.7.**  $\|f\| = \int |f| \, d\mu$  satisfies

- $\|f + g\| \leq \|f\| + \|g\|$
- $\|cf\| = |c|\|f\|$
- $\|f\| = 0 \iff f = 0$  almost everywhere

**Remark 3.3.8.** Almost everywhere equal is an equivalence relation.

$$f \sim g \stackrel{\text{def}}{\iff} f(x) = g(x) \text{ } \mu\text{-almost everywhere}$$

$N = \{f \in \mathcal{L}^1 : f(x) = 0 \text{ almost everywhere}\}$  is a linear subspace of  $\mathcal{L}^1$  vector.  $\mathcal{L}^1/N$  is the set of equivalence classes of  $\mathcal{L}^1$ .

$f_n \rightarrow f$  almost everywhere,  $f_n \geq 0$ ,  $f_n$  measurable, Can we define  $\int f \, d\mu$ ?  $f$  may not be measurable. This problem is fixed if  $f$  we work in a complete measurable space  $(X, \mathcal{M}, \mu) \rightarrow (X, \overline{\mathcal{M}}, \overline{\mu})$  where

$$\overline{\mathcal{M}} = \{A \cup B : A \in \mathcal{M}, B \text{ a subset of a set of measure } 0\}$$

**Lemma 3.3.9**

$f \in \mathcal{L}^1$ ,  $\int |f| \, d\mu < \infty$ . If  $f$  is real valued  $f = f^+ - f^-$ ,

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$$

*Proof.*

$$\begin{aligned} \left| \int f \, d\mu \right| &= \left| \int f^+ - f^- \, d\mu \right| \\ &\leq \left| \int f^+ \, d\mu \right| + \left| \int f^- \, d\mu \right| \\ &= \int f^+ \, d\mu + \int f^- \, d\mu \\ &= \int |f| \, d\mu \end{aligned}$$

□

**Remark 3.3.10.** If  $f$  is complex valued, then  $|f| = \sqrt{(\Re f)^2 + (\Im f)^2}$ . Then

$$\left| \int \Re f \right| \leq \int |\Re f| \leq \int |f|$$

So,

$$\left| \int \Im f \right| \leq \int |\Im f| \leq \int |f|$$

$$\left| \int f \, d\mu \right| \leq 2 \int |f|$$

**Remark 3.3.11.** Estimate  $\int f \, d\mu = \alpha + i\beta = re^{i\phi}$ , then  $e^{-i\phi} \int f \, d\mu$  is real and nonnegative.

$$\begin{aligned} \left| \int f \, d\mu \right| &= \left| e^{-i\phi} \int f \, d\mu \right| \\ &= \Re \int e^{-i\phi} f \, d\mu \\ &\leq \int |e^{-i\phi} f| \, d\mu \\ &= \int |f| \, d\mu \end{aligned}$$

### Lemma 3.3.12

$f \in \mathcal{L}^+$  means non-negative, then  $\nu(E) = \int_E f \, d\mu$  define measure

*Proof.* Check the  $\sigma$ -additivity  $E = \bigsqcup_{n=1}^{\infty} E_n$ ,

$$\begin{aligned} \nu \left( \bigsqcup_{n=1}^{\infty} E_n \right) &= \int_{\bigsqcup E_n} f \, d\mu \\ &= \int f \mathbb{1}_{\bigsqcup E_n} \, d\mu \\ &= \int f \left( \sum_{n=1}^{\infty} \mathbb{1}_{E_n} \right) \, d\mu \\ &= \sum_{n=1}^{\infty} \int f \mathbb{1}_{E_n} \, d\mu \\ &= \sum_{n=1}^{\infty} \nu(E_n) \end{aligned}$$

□

**Claim 3.3.13** — If  $f \in \mathcal{L}^1 \cap \mathcal{L}^+$  then  $\nu$  is a finite measure.

If  $\nu(E) = \int_E f \, d\mu$  How does  $\int g \, d\nu$  look like?  $\nu(E) = \int f \, d\mu = \int_E f \, d\mu$  We want “ $f \, d\mu = d\nu$ ”

**Lemma 3.3.14**

If  $f \in \mathcal{L}^+$  and  $\nu(E) = \int_E f \, d\mu$  then for any  $g \in \mathcal{L}^+$  or  $g \in \mathcal{L}^1$  then,

$$\int g \, d\nu = \int gf \, d\mu$$

*Proof.* • True for characteristic functions of measure set by the definition of  $\nu$ . Fix  $g = \mathbb{1}_E$  for some  $E \in \mathcal{M}$

$$\int g \, d\nu = \int \mathbb{1}_E \, d\mu = \nu(E) = \int_E f \, d\mu = \int \mathbb{1}_E f \, d\mu = \int gf \, d\mu$$

• By linearity of the integral, it is true for simple function. Fix  $g = \sum_{j=1}^n c_j \mathbb{1}_{E_j}$ , then

$$\int g \, d\nu = \sum_{j=1}^n c_j \nu(E_j) = \sum_{j=1}^n c_j \int_{E_j} f \, d\mu = \int gf \, d\mu$$

•  $s_n \nearrow g$  if  $g \in \mathcal{L}^+$ , by Monotone convergence theorem,

$$\int s_n \nearrow_{\text{MCT}} \int g$$

$$\begin{aligned} \int s_n \, d\nu &= \int s_n f \, d\mu \\ \int g \, d\nu &= \int gf \, d\mu \end{aligned}$$

Then extend to general  $g$  by linearity

□

**Theorem 3.3.15**

If  $X$  is a finite measure space, if  $f_n$  measurable,  $f_n \in \mathcal{L}^1$  (integrable) and  $f_n \rightarrow f$  uniformly on  $X$ . then

$$\int |f_n - f| \, d\mu \rightarrow 0$$

and

$$\int f_n \, d\mu \rightarrow \int f \, d\mu$$

**Remark 3.3.16.** Uniform convergence means

$$\lim_{n \rightarrow \infty} \sup_{x \in X} |f_n(x) - f(x)| = 0$$

*Proof.* We can rewrite that term as

$$\begin{aligned} \int |f_n - f| \, d\mu &\leq \int \sup_{x \in X} |f_n - f| \, d\mu \\ &= \mu(X) \sup_{x \in X} |f_n - f| \rightarrow 0 \end{aligned}$$

We can rewrite  $f$  as  $f = (f - f_n) + (f_n)$  since  $f - f_n$  converge and  $f_n$  integrable so  $f$  must be integrable.

$$\begin{aligned} \left| \int f_n - \int f \right| &= \left| \int (f_n - f) \, d\mu \right| \\ &\leq \int |f_n - f| \, d\mu \end{aligned}$$

□

**Definition 3.3.17.** Suppose that  $f_n, f$  are measurable  $f_n \rightarrow f$  almost uniformly if for every  $\varepsilon > 0$  there is a measurable set  $E$  such that  $\mu(E) < \varepsilon$  and  $f_n \rightarrow f$  uniformly on  $E^c$  ( $\sup_{x \in E^c} |f_n(x) - f(x)| \rightarrow 0$ )

**Theorem 3.3.18** (Egorov's Theorem)

If  $\mu(X) < \infty$  and if  $f_n \rightarrow f$  almost everywhere then  $f_n \rightarrow f$  almost uniformly

**Remark 3.3.19.**  $f_n(x) \rightarrow f(x)$  if for every  $k$  there exists  $n = n(k)$  such that  $|f_m(x) - f(x)| < \frac{1}{k}$  for all  $m \geq n(k)$

*Proof.* Fix  $\varepsilon > 0$ , define

$$\begin{aligned} E_n(k) &:= \left\{ x : |f_m(x) - f(x)| \geq \frac{1}{k} \text{ for some } m \geq n \right\} \\ &= \bigcup_{m \geq n} \left\{ x : |f_m(x) - f(x)| \geq \frac{1}{k} \right\} \end{aligned}$$

(Given  $x$  For sufficiently large  $n$ ,  $x \notin E_n(k)$ ),  $E_n(k) \supseteq E_{n+1}(k) \cap_n E_n(k) = \emptyset$  because of  $f_n \rightarrow f$  everywhere. Form the continuity from above 2.4.5, we get that



$\lim_{n \rightarrow \infty} \mu(E_n(k)) = 0$ . Find  $n(k)$  such that  $\mu(E_{n(k)}(k)) < \frac{\varepsilon}{2^k}$ , then  $E = \bigcup_k E_{n(k)}(k)$  has measure  $< \varepsilon$ .

For  $x \in \left(\bigcup_k E_{n(k)}(k)\right)^c = \bigcap_k E_{n(k)}(k)^c$  I have for all  $k$   $|f_m(x) - f(x)| < \frac{1}{k}$  for all  $m \geq n(k)$ . So, we get  $f_n \rightarrow f$  uniformly on  $E^c$ .  $\square$

**Theorem 3.3.20 (Baby Dominated Convergence Theorem)**

Given  $(X, \mathcal{M}, \mu)$  where  $\mu$  is a finite measure ( $\mu(X) < \infty$ ). Let  $\{f_n\}$  be measurable functions,  $f_n \rightarrow f$  everywhere.

$$|f_n| \leq C \implies \int |f_n - f| \, d\mu \rightarrow 0$$

i.e.  $f_n$  converges with respect to  $L^1$ -(semi-)norm.

**Corollary 3.3.21**

$$\int_X f_n \, d\mu \rightarrow \int_X f \, d\mu$$

*Proof.* Tools:

- (i) If  $f_n \rightarrow f$  uniformly then  $\int |f_n - f| \, d\mu \rightarrow 0$
- (ii) Egorov's Theorem

$|f(x)| \leq C$ ,  $f$  is measurable. Given any  $\varepsilon > 0$ , By Egorov's Theorem, find a set of measure  $E$  that  $\mu(E) < \frac{\varepsilon}{4C}$  such that  $f_n \rightarrow f$  uniformly on  $E^c$ . Then

$$\int |f_n - f| \, d\mu \leq \int_E |f_n - f| \, d\mu + \int_{E^c} |f_n - f| \, d\mu$$

we know that  $|f_n - f| \leq |f_n| + |f| \leq 2C$  then

$$\begin{aligned} \int |f_n - f| \, d\mu &\leq 2C\mu(E) + \int_{E^c} |f_n - f| \, d\mu \\ &\leq \frac{\varepsilon}{2} + \int_{E^c} |f_n - f| \, d\mu \end{aligned}$$

so for large  $n$ , the second term will be  $< \frac{\varepsilon}{2}$ .  $\square$

**Theorem 3.3.22 (Dominated Convergence Theorem)**

Given  $(X, \mathcal{M}, \mu)$  where  $\mu$  is a finite measure ( $\mu(X) < \infty$ ). Let  $\{f_n\}$  be measurable functions,  $f_n \rightarrow f$  almost everywhere.

$$\sup_n |f_n| \in \mathcal{L}^1 \implies \int |f_n - f| \, d\mu \rightarrow 0$$

*Proof.* Define  $g(x) = \sup_n |f_n(x)|$  The trick is

$$|f_n - f| = \begin{cases} \frac{|f_n - f|}{g} g & \text{if } g > 0 \\ 0 & \text{if } g = 0 \end{cases}$$

define a new measure  $\nu(E) = \int_E g \, d\mu$ . Then  $\nu$  is a finite measure, and

$$\begin{aligned} g \, d\mu &= d\nu \\ \int h \, d\nu &= \int hg \, d\mu \end{aligned}$$

then define

$$h_n = \begin{cases} \frac{|f_n - f|}{g} & \text{if } g > 0 \\ 0 & \text{if } g = 0 \end{cases} \implies \begin{aligned} |h_n(x)| &\leq 1 \\ h_n(x) &\rightarrow 0 \end{aligned}$$

Then

$$\begin{aligned} \int |f_n - f| \, d\mu &= \int h_n g \, d\mu \\ &= \int h_n \, d\nu \rightarrow 0 \end{aligned}$$

By Baby Dominated Convergence Theorem □

**§3.4 Integration from Riemann to Lebesgue****Theorem 3.4.1**

If  $f$  is Riemann integrable on  $[a, b]$  then  $f$  is Lebesgue integrable.

$$\int_a^b f(x) \, dx = \int_{[a,b]} f \, d\mu$$

where  $\mu$  is Lebesgue measure.

*Proof.* Define

$$U_P f(x) = \begin{cases} M_j & \text{if } x \in [x_{j-1}, x_j) \\ M_n & \text{if } x \in [x_{n-1}, x_n] \end{cases}$$

Similarly for the lower sum  $L_P f(x)$ . If  $P'$  is a refinement of  $P$  then  $U_{P'} f(x) \leq U_P f(x)$  and  $L_{P'} f(x) \geq L_P f(x)$ . Since  $f$  is Riemann integrable, then

$$\inf_P U(f, P) =: \bar{\mathcal{I}}_a^b(f) = \mathcal{I}_a^b(f) := \sup_P L(f, P)$$

Choose a sequence of partitions  $P_n$  such that

$$\begin{aligned} \int U_{P_n} f &\rightarrow \bar{\mathcal{I}}_a^b(f) \\ \int L_{P_n} f &\rightarrow \mathcal{I}_a^b(f) \end{aligned}$$

Since that  $P_{n+1}$  is a refinement of the  $P_n$  then  $U_{P_n} f \searrow U(x)$  and  $L_{P_n} f \nearrow L(x)$  and  $L(x) = U(x)$ . Notice that from Riemann integrable,  $|f| < C$ , then

$$\begin{aligned} \int_{[a,b]} U_{P_n} f &\rightarrow \bar{\mathcal{I}}_a^b(f) = \int_{[a,b]} U(x) \, dm \\ \int_{[a,b]} L_{P_n} f &\rightarrow \mathcal{I}_a^b(f) = \int_{[a,b]} L(x) \, dm \end{aligned}$$

If  $f$  is Riemann integrable,

$$\int U \, dm = \int L \, dm = \int_a^b f(x) \, dx$$

and  $U \geq L$  then

$$\int (U - L) \, dm = 0 \implies U(x) = L(x)$$

almost everywhere,  $L(x) \leq f(x) \leq U(x) \implies f = L$  almost everywhere and  $f = U$  almost everywhere. Then  $f$  is Lebesgue integrable and

$$\int f \, dm = \int L \, dm = \int U \, dm$$

□

**Definition 3.4.2** (Improper Riemann integrals).

$$\int_0^\infty f(x) \, dx, \int_1^\infty f(x) \, dx, \int_0^1 f(x) \, dx$$

if  $f$  is not Riemann-integrable on the domain but on every compact subinterval. We can define as

$$\int_1^\infty f(x) \, dx = \lim_{R \rightarrow \infty} \int_1^R f(x) \, dx$$

**Example 3.4.3**

$$\int_1^\infty \frac{\sin x}{x} dx = \lim_{R \rightarrow \infty} \int_1^R \frac{\sin x}{x} dx$$

$I_k = [2k\pi + \frac{\pi}{4}, 2k\pi + \frac{3\pi}{4}]$ ,  $\sin x \geq \frac{1}{\sqrt{2}}$ , so,  $\frac{\sin x}{x} \geq \frac{1}{x\sqrt{2}} \cdot \frac{1}{2k\pi + \frac{3\pi}{4}}$ . We can do integration by parts

$$\int_1^R \frac{\sin x}{x} dx = -\frac{\cos x}{x} \Big|_1^R - \int_1^R -\frac{\cos x}{x^2} dx$$

**Example 3.4.4**

$$\int_0^\infty \sin(x^2) dx$$

consider  $\sin(x^2)$

$$\begin{aligned} \sqrt{2k\pi + \frac{\pi}{2}} &\leq \sqrt{x^2} \leq \sqrt{2k\pi + \frac{3\pi}{4}} \\ \sqrt{2k\pi + \frac{3\pi}{4}} - \sqrt{2k\pi + \frac{\pi}{4}} &\approx \frac{1}{\sqrt{k}} \end{aligned}$$

**Lemma 3.4.5**

Suppose that if  $\int_1^\infty |f(x)| dx < \infty$  then  $f \in \mathcal{L}^1$ .

*Proof.*

$$\begin{aligned} \int_1^\infty |f(x)| dx &= \int_1^\infty \lim_{n \rightarrow \infty} |f(x)| \mathbb{1}_{[1,n]}(x) dx \\ &= \lim_{n \rightarrow \infty} \int_1^n |f(x)| dx \end{aligned}$$

□

**Theorem 3.4.6**

If  $f$  is integrable on  $\mathbb{R}$ ,

$$\int_{\mathbb{R}} |f(x)| \, dx < \infty$$

$f \in \mathcal{L}^1$  then for every  $\varepsilon > 0$ , there is a continuous function ( $C^\infty$ )  $g$ , vanishes off a compact set,

$$\int |f - g| \, dm < \varepsilon$$

**§3.5 Outer Measures**

**Definition 3.5.1.** In our axiomatic theorem on the Lebesgue measure,  $m(I) = \ell(I)$ ,  $m((a, b]) = b - a$  and for a general Borel set on  $\mathbb{R}$ ,  $m$  is given by the **outer measure** induced by the collection of intervals

$$\varrho(E) = \inf \sum_{k=1}^{\infty} \ell(I_k)$$

where the inf is taken over collections  $\{I_k\}$ , such that  $E \subseteq \bigcup_{k=1}^{\infty} I_k$

**Remark 3.5.2.**  $\tilde{\varrho}$  defined similarly but we only admit open intervals in the infimum. Obviously,  $\tilde{\varrho}(E) \geq \varrho(E)$ . Need to show that  $\tilde{\varrho}(E) \leq \varrho(E)$  we may assume that  $\varrho(E) < \infty$ , show  $\tilde{\varrho}(E) \leq \varrho(E) + \varepsilon$ . There is a collection of intervals  $I_k$  such that

$$\sum_k \ell(I_k) < \varrho(E) + \frac{\varepsilon}{2}$$

If  $I_k = [a_k, b_k]$ , then define  $J_k = (a_k - \frac{\varepsilon}{2^{k+2}}, b_k + \frac{\varepsilon}{2^{k+2}})$ . Then  $\ell(J_k) = \ell(I_k) + \frac{\varepsilon}{2^{k+1}}$  then

$$\begin{aligned} \tilde{\varrho}(E) &\leq \sum_{k=1}^{\infty} \ell(J_k) \leq \sum_{k=1}^{\infty} \ell(I_k) + \varepsilon \sum_{k=1}^{\infty} 2^{-k-1} \\ &\leq \varrho(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

**Lemma 3.5.3**

$$m(E) = \sup\{m(K) : K \subseteq E, K \text{ compact}\}$$

*Proof.* Case where  $E = \overline{E}$  and  $E$  is bounded, there is nothing to show

Assume that  $E$  is bounded, GOAL: find  $K \subseteq E$  such that  $m(E \setminus K) < \varepsilon$ . Consider  $\overline{E} \setminus E$ , find  $O \supseteq \overline{E} \setminus E$ ,  $m(O \setminus (\overline{E} \setminus E)) < \varepsilon$ . then  $O^c \cap \overline{E} \subseteq E$  because if  $x \in O^c$ , either  $x \in \overline{E}$  or

$x \in E$ .  $E \setminus K = E \cap K^c \subseteq O \cup \overline{E}^c$ . Since  $E \subseteq \overline{E}$  then  $E \setminus K \subseteq O$  and  $E \setminus K \subseteq O \setminus (\overline{E} \setminus E)$  has measure  $< \varepsilon$ .  $\square$

#### Theorem 3.5.4

For every Borel set  $E$ ,  $m(E) < \infty$ , there is an open set  $O \supseteq E$  such that  $m(O \setminus E) < \varepsilon$ . where  $m(E) = \inf \sum \ell(I_n)$  where inf take over  $I_k$ ,  $I_k$  are open,  $E \subseteq \bigcup I_k$

*Proof.* Define  $E_n = E \cap \overline{B}(0, n)$ . Find compact set  $K_n \subseteq E_n \setminus E_{n-1}$  then  $m((E_n \setminus E_{n-1}) \setminus K_n) < \varepsilon 2^{-n-1}$ . The set  $H_l = K_1 \cup \dots \cup K_l$  is compact and increasing,  $H_l \subseteq E_l$  and

$$m(E_l) - \varepsilon \leq m(H_l) \leq m(E_l) \rightarrow m(E)$$

$\square$

#### Theorem 3.5.5

Given an open set  $O$ , we can decompose  $O$  as a disjoint union of “dyadic cubes”

#### Theorem 3.5.6

We can choose the cubes a dyadic cubes such that if  $O \neq \mathbb{R}^n$  such that

$$\text{diam}(Q) < \text{dist}(Q, O^c) \leq 4\text{diam}(Q)$$

**Remark 3.5.7.** If side length of  $Q$  is  $2^{-k}$  then the diameter is  $\sqrt{n}2^{-k}$ .

#### Theorem 3.5.8 (Whitney decomposition theorem)

Given  $\Omega$  open set in  $\mathbb{R}^n$ ,  $\Omega \neq \mathbb{R}^n$ , there is a family  $\mathcal{F}$  of dyadic cubes such that

- they are disjoint
- $\bigsqcup_{Q \in \mathcal{F}} Q = \Omega$
- For every  $Q \in \mathcal{F}$ ,  $C\text{diam}(Q) < \text{dist}(Q, \Omega^c) \leq (2C + 2)\text{diam}(Q)$

*Proof.* Define for each  $k$  a family of dyadic cubes  $\mathcal{F}_k$  of side length  $2^{-k}$  (i.e., the diameter is  $\sqrt{n}2^{-k}$ ) intersecting the region

$$\Omega_k = \{x : A2^{-k}\sqrt{n} \leq \text{dist}(x, \Omega^c) \leq 2A \cdot 2^{-k}\sqrt{n}\}$$

Pick a cube in  $\mathcal{F}_k$ ,  $Q_1$  it contains an  $x_Q \in \Omega_k$

$$\text{dist}(Q, \Omega^{\mathbb{L}}) \leq \text{dist}(x_Q, \Omega^{\mathbb{L}}) \leq 2A \cdot 2^{-k} \sqrt{n} - 2A \text{diam}(Q)$$

$$\begin{aligned} \text{dist}(Q, \Omega^{\mathbb{L}}) &\geq \text{dist}(x_Q, \Omega^{\mathbb{L}}) - \text{diam}(Q) \\ &\geq A2^{-k} \sqrt{n} - 2A \text{diam}(Q) \\ &= (A - 1) \cdot \text{diam}(Q) \end{aligned}$$

Then  $\mathcal{F}_T = \bigcup \mathcal{F}_k$  and finally  $\mathcal{F}$  = collection of all maximal (with respect to inclusion) cubes in  $\mathcal{F}_T$ . Fix  $Q, Q'$  and assume that  $Q \subseteq Q'$  then

$$(A - 1) \text{diam} Q' \leq \text{dist}(Q', \Omega^{\mathbb{L}}) \leq \text{dist}(Q, \Omega^{\mathbb{L}}) \leq 2A \cdot \text{diam}(Q)$$

$$\text{then } \text{diam}(Q') \leq \frac{2A}{A-1} \text{diam}(Q)$$

□

we know that for every  $\varepsilon_1 > 0$  we can find a simple function  $s$  such that  $\int |f - s| \, dm < \varepsilon_1$ . (For non-negative  $f$  use MCT  $s_n \nearrow f$  and  $s_n \leq f$  so  $\int s_n \nearrow \int f \implies \int f - s_n \rightarrow 0$  and then we use  $f = f_+ - f_-$ )

$$s = \sum c_j \mathbb{1}_{E_j}, \quad E_j \subseteq O_j, \quad m(O_j \setminus E_j) < \varepsilon \quad \tilde{s} = \sum c_j \mathbb{1}_{O_j}$$

$$\begin{aligned} \int \tilde{s} - s \, dm &= \left| \int \sum c_j \mathbb{1}_{O_j} - \mathbb{1}_{E_j} \, dm \right| \\ &\leq \sum_{j=1}^N |c_j| |m(O_j \setminus E_j)| \\ &\leq \varepsilon_2 \end{aligned}$$

Then  $\mathbb{1}_{O_j} = \sum_{\nu} \mathbb{1}_{Q_{\nu}}$  where  $\{Q_{\nu}\}$  are the Whitney cubes in Whitney Theorem.  $|O_j| = \sum_{\nu \in I} m(Q_{\nu})$  There is a finite  $\tilde{I}_j$  such that

$$\int \left| \mathbb{1}_{O_j} - \sum_{\nu \in \tilde{I}_j} \mathbb{1}_{Q_{\nu}} \right| < \varepsilon_3$$

replace  $\sum_{j=1}^N c_j \mathbb{1}_{O_j}$  by  $\sum_{j=1}^N c_j \mathbb{1}_{\bigcup_{\nu \in \tilde{I}_j} Q_{\nu}}$

then

$$\mathbb{1}_{Q_{\nu}}(x_1, \dots, x_n) = \prod_{i=1}^n \mathbb{1}_{\nu, i}(x_i)$$

**Lemma 3.5.9**

For any  $f \in L^1$  there exists  $s$  a step function such that  $\int |f - s| \, dm < \varepsilon$

*Proof.* Suppose that  $f \in L^1$  I want to show that there exists  $s$  a step function such that  $\int |f - s| \, dm < \varepsilon$  for any  $\varepsilon > 0$ . Since  $f = f^+ - f^-$ , WLOG,  $f \geq 0$  (otherwise we can do each positive and negative part and do the sum of both step functions with  $\frac{\varepsilon}{2}$  bound). Given any  $\varepsilon > 0$ , there exists  $s'$  a simple function such that  $\int |f - s'| \, dm < \frac{\varepsilon}{2}$ . Then we can write  $s' = \sum_{j=1}^N c_j \mathbb{1}_{E_j}$ , then there exists  $O_j$  open set such that  $E_j \subseteq O_j$  and  $m(O_j \setminus E_j) < \frac{\varepsilon}{4|c_j|N}$ . Since  $O_j$  is an open set, then  $O_j = \bigcup (a_i, b_i)$  then define  $K_n = \bigcup_{i=1}^n (a_i, b_i)$  from continuity from below, there exists  $n'$  such that  $m(K_{n'}) > m(O_j) - \frac{\varepsilon}{4|c_j|N}$  and  $K_{n'}$  contain finite interval, then we define  $O'_j := K_{n'}$ . Define  $s = \sum_{j=1}^N c_j \mathbb{1}_{O'_j}$  then

$$\begin{aligned}
 \int |f - s| \, dm &\leq \int |f - s'| \, dm + \int |s' - s| \, dm \\
 &\leq \frac{\varepsilon}{2} + \int \left| \sum_{j=1}^N c_j (\mathbb{1}_{E_j} - \mathbb{1}_{O'_j}) \right| \, dm \\
 &\leq \frac{\varepsilon}{2} + \sum_{j=1}^N |c_j| \int |\mathbb{1}_{E_j} - \mathbb{1}_{O'_j}| \, dm \\
 &= \frac{\varepsilon}{2} + \sum_{j=1}^N |c_j| (m(E_j \setminus O'_j) + m(O'_j \setminus E_j)) \\
 &\leq \frac{\varepsilon}{2} + \sum_{j=1}^N |c_j| (m(O_j \setminus O'_j) + m(O_j \setminus E_j)) \\
 &\leq \frac{\varepsilon}{2} + \sum_{j=1}^N |c_j| \left( \frac{\varepsilon}{4|c_j|N} + \frac{\varepsilon}{4|c_j|N} \right) \\
 &< \varepsilon
 \end{aligned}$$

□



# 4 $L^p$ Spaces

## §4.1 normed spaces

**Remark 4.1.1.** If  $f_n \rightarrow f$  almost everywhere, do we have  $\int |f_n - f| d\mu \rightarrow 0$ ?

- No, if  $f_n = \mathbb{1}_{[n, n+1]}$  then  $f_n \rightarrow 0$  almost everywhere but  $\int |f_n - 0| d\mu = 1$  and  $\int |f_n - f_m| d\mu = 2$ .
- No, if  $f_n = \mathbb{1}_{[0, \frac{1}{n}]}$

**Remark 4.1.2.** Convergence in  $L^1$  implies convergence almost everywhere? No, If  $2^k \leq n \leq 2^{k+1}$  where  $n = 2^k + j$ ,  $j = 0, \dots, 2^k - 1$   $f_{2^k+1} = \mathbb{1}_{[i2^{-k}, (i+1)2^{-k}]}$  for  $i = 0, \dots, 2^k - 1$ . For  $2^k \leq n \leq 2^{k+1}$ ,  $\|f_n\|_{L^1} = 2^{-k}$

**Claim 4.1.3** — If  $f_n \rightarrow f$  in  $L^1$  ( $\int |f_n - f| d\mu \rightarrow 0$ ) then there is a subsequence  $f_{n_k} \rightarrow f$  almost everywhere.

*Proof.* Consider the normed space  $L^1$  space of semi-normed space  $\mathcal{L}^1$ . (define as a equivalence class of almost everywhere where  $f \underset{\text{a.e.}}{\sim} g$  if  $f = g$  almost everywhere) Construct a convergence subsequence (a.e. and also in Norm) Choose  $\varepsilon = \frac{1}{2^k}$  there exists number  $N(k)$  such that  $\|f_l - f_m\| < \frac{1}{2^k}$  for  $l, m \geq N(k)$  for  $l, m \geq N(k)$  then  $\|f_{N(k)} - f_{N(k+1)}\| \leq \frac{1}{2^k}$  Define

$$G(x) = |f_{N(1)}(x)| + \sum_{k=1}^{\infty} |f_{N(k+1)}(x) - f_{N(k)}(x)|$$

then

$$\int G(x) d\mu = \int |f_{N(1)}(x)| d\mu + \sum_{k=1}^{\infty} \int |f_{N(k+1)}(x) - f_{N(k)}(x)| d\mu \leq \|f_{N(1)}\|_1 + \sum_{k=1}^{\infty} \frac{1}{2^k}$$

So,  $G$  is integrable,  $\int |G(x)| d\mu < \infty$  then  $G(x) < \infty$  almost everywhere. We see that for almost everywhere,

$$f_{N_1}(x) + \sum_{k=1}^{\infty} f_{N(k+1)} - f_{N(k)}(x)$$

converges for almost everywhere  $x$ , define

$$s_M(x) = f_{N(1)}(x) + f_{N(2)}(x) - f_{N(1)}(x) + \dots + f_{N(M+1)}(x) - f_{N(M)}(x) = f_{N(M+1)}(x)$$

then  $s_{M-1}(x) = f_{N(M)}(x)$  and as  $M \rightarrow \infty$ , this converges for almost everywhere  $x$ .

$$f(x) = \lim_{M \rightarrow \infty} f_{N(M)}(x)$$

$$\begin{aligned} f(x) &= f_{N_1}(x) + \sum_{M=1}^{\infty} f_{N(M+1)}(x) - f_{N(M)}(x) \\ \int |f(x) - f_{N_1}(x)| \, d\mu &= \int \sum_{M=1}^{\infty} f_{N(M+1)}(x) - f_{N(M)}(x) \, d\mu \\ &\leq \int |f_{N(M+1)}(x) - f_{N(M)}(x)| + \int |f_{N(M+2)} - f_{N(M+1)}| + \dots \\ &= \sum_{k=M}^{\infty} \int |f_{N(k+1)}(x) - f_{N(k)}(x)| \, d\mu \\ &\leq 2^{1-M} \end{aligned}$$

This shows convergence of  $f_{N(M)} \rightarrow f$  in  $L^1$ . What happens with  $l \geq N(k)$ ,

$$\|f_l - f\| \leq \|f_l - f_{N(k)}\| + \|f_{N(k)} - f\| \leq \frac{1}{2^k}, \rightarrow 0$$

□

$L^1$  or  $(\mathcal{L}^1)$  are complete, in the sense that every Cauchy sequence converges.  $\{f_n\}$  cxauchy, For every  $\varepsilon > 0$ , there exists  $N(\varepsilon)$  such that for  $l, m \geq N(\varepsilon)$  then  $\|f_l - f_m\| < \varepsilon$

**Definition 4.1.4.**

$$\|f\|_p = \left( \int |f|^p \, d\mu \right)^{\frac{1}{p}}$$

where  $L^p$  is space of equivalence class and  $\mathcal{L}^p$  is space of functions,  $f \in \mathcal{L}^p$  if  $\|f\|_p < \infty$

**Theorem 4.1.5**

$\|f\|_p$  is a norm on  $L^p$ , if  $p \geq 1$  (not a norm if  $p < 1$  because triangle inequality fails)

*Proof.* for any  $f, g \in L^p$ ,

$$\begin{aligned} \int |f + g|^p \, d\mu &\leq \int (2 \max(|f|, |g|))^p \, d\mu \\ &= 2 \left( \int \max(|f|^p, |g|^p) \, d\mu \right) \\ &\leq 2 \int |f|^p + |g|^p \, d\mu \end{aligned}$$

□

**Remark 4.1.6.**  $\|f + g\|_p \leq 2^{\frac{1}{p}}(\|f\|_p + \|g\|_p)$

**Theorem 4.1.7**

For  $p < 1$  we have inequality

$$\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$$

*Proof.* we claim that

$$\int |f + g|^p d\mu \leq \int |f|^p d\mu + \int |g|^p d\mu$$

for  $a, b \in [0, \infty)$ ,  $(a + b)^p \leq a^p + b^p$  WLOG  $b \leq a$   $f(x) = 1 + x^p - (1 - x)^p$ ,  $f'(x) \geq 0 \implies (1 + x)^p \leq 1 + x^p$  for  $0 \leq x \leq 1$   $\square$

**Remark 4.1.8.** For  $p < 1$  we do not get  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$  for  $x, y \in \mathbb{R}^2$  want to disprove  $\|x + y\|_p \leq \|x\|_p + \|y\|_p, p < 1$

$$2^{\frac{1}{p}} = (1^p + 1^p)^{\frac{1}{p}}$$

(it is because failure of convexity of the norm  $p < 1$ )

**Claim 4.1.9** — For  $0 < \theta < 1$ ,  $a, b \geq 0$ , then  $a^{1-\theta}b^\theta \leq (1 - \theta)a + \theta b$

*Proof.* Generalized AM-GM inequality ( $\sqrt[\theta]{ab} \leq \frac{a+b}{2}$ ) then put for  $0 < \theta < 1$  then  $a^{1-\theta}b^\theta \leq (1 - \theta)a + \theta b$  WLOG  $b \leq a$  then

$$\left(\frac{b}{a}\right)^\theta \leq 1 - \theta + \theta \frac{b}{a}$$

let  $x = \frac{b}{a}$  for  $0 \leq x \leq 1$  we need to show that  $g(x) = 1 - \theta + \theta x - x^\theta \geq 0$  then  $g'(x) = -1 + \theta - \theta x^{\theta-1} \leq 0$  (because  $0 \leq \theta \leq 1$ )  $\square$

**Claim 4.1.10 (Holder's inequality)** — Given  $p > 1$ ,  $q$  to be such that

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \left(q = \frac{p}{p-1}\right)$$

for  $f \in L^p, g \in L^q$ , then  $fg \in L^1$  and

$$\int |fg| \, d\mu \leq \|f\|_p \|g\|_q$$

*Proof.* Rewrite AM-GM (generalized) as “Young’s inequality” substitute  $a = u^p, 1 - \theta = \frac{1}{p}, b = v^q, \theta = \frac{1}{q}$  then we get

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$$

apply  $f(x)g(x)$

$$\int |f(x)g(x)| \, d\mu \leq \int \frac{|f(x)|^p}{p} \, d\mu + \int \frac{|g(x)|^q}{q} \, d\mu = \frac{\|f\|_p^p}{p} + \frac{\|g\|_q^q}{q}$$

(This is Holder when two norms are normalized  $\|f\|_p = 1 = \|g\|_q$ )

Then  $\frac{f(x)}{\|f\|_p}$  has “p-norm” equal to 1 because

$$\left( \int \left| \frac{f(x)}{\|f\|_p} \right|^p \, d\mu \right)^{\frac{1}{p}} = \frac{1}{\|f\|_p} \left( \int |f(x)|^p \, d\mu \right)^{\frac{1}{p}}$$

Substitute  $f = \frac{f}{\|f\|_p}$  and  $g = \frac{g}{\|g\|_q}$  then we get

$$\begin{aligned} \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \, d\mu &\leq \frac{1}{p} + \frac{1}{q} = 1 \\ \int |fg| \, d\mu &\leq \|f\|_p \|g\|_q \end{aligned}$$

□

**Theorem 4.1.11** (Minkowski’s inequality)

$p \geq 1$  We do have a triangle inequality  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$

$$\left( \int |f + g|^p \, d\mu \right)^{\frac{1}{p}} \leq \left( \int |f|^p \, d\mu \right)^{\frac{1}{p}} + \left( \int |g|^p \, d\mu \right)^{\frac{1}{p}}$$

*Proof.* It is enough to show that

$$\|f + g\|_p^p \leq (\|f\|_p + \|g\|_p) \|f + g\|_p^{p-1}$$

$$\begin{aligned}
\int |f+g|^{p-1+1} d\mu &= \int |f+g|^{p-1} |f| d\mu + \int |f+g|^{p-1} |g| d\mu \\
&\leq \left( \int |f|^p d\mu \right)^{\frac{1}{p}} \left( \int |f+g|^p d\mu \right)^{\frac{p-1}{p}} + \left( \int |g|^p d\mu \right)^{\frac{1}{p}} \left( \int |f+g|^p d\mu \right)^{\frac{p-1}{p}} \\
&= (\|f\|_p + \|g\|_p) \|f+g\|_p^{p-1}
\end{aligned}$$

□

**Remark 4.1.12.** Holder's inequality holds

$$\int fg d\mu \leq \|f\|_p \|g\|_q$$

where  $\frac{1}{p} + \frac{1}{q} = 1$  can be generalized to several factors

$$\int f_1 f_2 \cdots f_n d\mu \leq \prod_{j=1}^n \|f_j\|_{p_j}$$

where  $\sum_{j=1}^n \frac{1}{p_j} = 1$

**Lemma 4.1.13** (Shebyshev's inequality)

This is an inequality for the distribution function (given a measure space  $(X, \mathcal{M}, \mu)$ )  $\mu_f(\alpha) = \mu(\{x : |f(x)| \geq \alpha\})$  The Shebyshev's inequality is

$$\mu_f(\alpha) \leq \frac{\|f\|_p^p}{\alpha^p}$$

*Proof.*

$$\begin{aligned}
\mu_f(\alpha) &= \int_{E_\alpha} 1 d\mu \\
&\leq \int_{E_\alpha} \frac{|f(x)|^p}{\alpha^p} d\mu \\
&\leq \frac{1}{\alpha^p} \int |f(x)|^p d\mu
\end{aligned}$$

(Probabilitiy may call this Markov's inequality)

□

$$\mu_{cf}(\alpha) = \mu(\{x : |cf(x)| \geq \alpha\}) = \mu(\{x : |f(x)| \geq \frac{\alpha}{|c|}\})$$

$$\alpha \mu_{cf}(\alpha)^{\frac{1}{p}} = |c| \frac{\alpha}{|c|} \mu_f\left(\frac{\alpha}{|c|}\right)^{\frac{1}{p}}$$

**Claim 4.1.14** — If  $\delta_0 + \delta_1 = 1$ ,  $\delta_0, \delta_1 \geq 0$ , then

$$E_\alpha(f + g) \subseteq E_{\alpha\delta_0}(f) \cup E_{\alpha\delta_1}(f)$$

**Theorem 4.1.15**

If  $\mu(X) < \infty$  then  $L^q \subseteq L^p$  for  $p \leq q$

*Proof.* We need an inequality  $\|f\|_p \leq C\|f\|_q$

$$\left( \int 1|f(x)|^p \, d\mu \right)^{\frac{1}{p}} \leq C \left( \int |f(x)|^q \, d\mu \right)^{\frac{1}{q}}$$

Apply Holder with exponent  $\frac{q}{p} > 1$ ,  $\left(\frac{q}{p}\right)'$  where

$$\frac{1}{\left(\frac{q}{p}\right)} + \frac{1}{\left(\frac{q}{p}\right)'} = 1$$

$$\begin{aligned} \int |f|^p \cdot 1 \, d\mu &\leq \left( \int (|f|^p)^{\frac{q}{p}} \, d\mu \right)^{\frac{p}{q}} \left( \int_X 1^{\left(\frac{q}{p}\right)'} \, d\mu \right)^{\frac{1}{\left(\frac{q}{p}\right)'}} \\ &= \|f\|_q^p \mu(X)^{\frac{1}{\left(\frac{q}{p}\right)'}} \\ &= \left[ \|f\|_q \mu(X)^{\frac{1}{\left(\frac{q}{p}\right)'}} \right]^p \end{aligned}$$

□

Another extreme case would be  $\mathbb{N}$  with counting measure In this  $L^p(\mathbb{N}, \mu)$  is denoted by  $\ell^p(\mathbb{N})$

**Theorem 4.1.16**

For  $p \geq 1$ ,  $\ell^p \subseteq \ell^q$  for  $p \leq q$

*Proof.* We want to prove  $\|f\|_{\ell^q} \leq C\|f\|_{\ell^p}$  If  $\|f\|_{\ell^p} < 1$ , this means

$$\sum_{n=1}^{\infty} |f(n)|^p \leq 1$$

$\implies |f(n)|^p \leq 1$  for all  $n$

$$\sum_{n=1}^{\infty} |f(n)|^q \leq \sum_{n=1}^{\infty} |f(n)|^p$$

provided that  $|f(n)|^q \leq |f(n)|^p$

For  $f \in \ell^p$ ,  $\frac{f}{\|f\|_p}$  has  $\ell^p$  norm equal to 1 therefore  $\left\| \frac{f}{\|f\|_p} \right\|_q \leq 1$  then  $\|f\|_q \leq \|f\|_p$   $\square$

#### Theorem 4.1.17 (Littlewood Theorem)

Every measurable function is nearly continuous. i.e.,  $f \in L^1(A)$  there exists  $g$  continuous such that

$$\int |f(x) - g(x)| \, dm < \varepsilon$$

#### Theorem 4.1.18 (Lusin's Theorem)

$f : [a, b] \rightarrow \mathbb{C}$  (or  $\mathbb{R}$ ) almost everywhere, then there is a compact set  $K \subseteq [a, b]$  such that  $f|_K$  is continuous and  $\mu([a, b] \setminus K) = 0$

#### Example 4.1.19

$f(x) = \mathbb{1}_{\mathbb{Q}}(x)$  let  $\{r_k\}$  be enumeration of rational number in  $[a, b]$ . Define

$$O = \bigcup_{k=1}^{\infty} \left( r_k - \frac{\varepsilon}{2^{k+2}}, r_k + \frac{\varepsilon}{2^{k+2}} \right)$$

then  $m(O) < \varepsilon$ ,  $K = [a, b] \setminus O$ ,  $f|_K = 1$  is continuous.

#### Example 4.1.20

$$\sum_{k=1}^{\infty} \frac{1}{|x - r_k|^{10}} 2^{-k} \mathbb{1}_{[a, b]}$$

(this function is in  $L^p$ , if  $p < \frac{1}{10}$ ). The Challenging part is to find a  $K$  as in Lusin's theorem such that  $f = f|_K$  is continuous.

*Proof.*  $f : A \rightarrow \mathbb{R}$  (or  $\mathbb{C}$ ),  $m(A) < \infty$  The for every  $\varepsilon > 0$ , there is a compact set  $K \subseteq A$ , such that  $m(A \setminus K) < \varepsilon$  and  $f|_K$  is continuous.

1. We can find a set  $E_1$  such that  $m(A \setminus E_1) < \frac{\varepsilon}{3}$  and  $f|_{E_1}$  is bounded.

$$S_\alpha = \{x \in A : |f(x)| > \alpha\}$$

Then  $\bigcup S_\alpha = \bigcup S_{2^M}$  has measure zero by the assumption. From continuity from above, we get

$$m(S_\alpha) = m\left(\bigcup S_{2^M}\right) = \lim_{M \rightarrow \infty} m(S_{2^M})$$

because  $m(A) < \infty$ . For large  $M$ ,  $m(S_M) < \frac{\varepsilon}{3}$ , let  $A \setminus E_1 = S_{2^M}$  ( $M$  large)

2. We know that  $f$  is bounded on  $E_1$ . Can find a sequence  $g_n$  of continuous functions  $g_n \rightarrow f$  almost everywhere on  $E_1$ ,  $m(E_1) < \infty$ .
3. Using Egorov's theorem:  $g_n \rightarrow f$  almost uniformly on  $E_1$ . Find  $E_2 \subseteq E_1$  such that  $m(E_1 \setminus E_2) < \frac{\varepsilon}{3}$  and  $g_n \rightarrow f$  uniformly on  $E_2$ .  $f|_{E_2}$  is continuous on  $E_2$ , we can find  $E_3 = K$  compact,  $E_3 \subseteq E_2$ ,  $m(E_2 \setminus E_3) < \frac{\varepsilon}{3}$  and  $f|_K$  is continuous.

$$m(A \setminus K) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

□

## §4.2 Outer Measure

**Definition 4.2.1.** Abstract outer measure,

$$\varrho : \mathfrak{P}(X) \rightarrow [0, \infty]$$

$\varrho(\emptyset) = 0$ , which have

- Monotonicity  $A \subseteq B$ ,  $\varrho(A) \leq \varrho(B)$
- $\sigma$ -Subadditivity  $\varrho\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \varrho(E_j)$

**Definition 4.2.2.** Given  $X$ ,  $\mathcal{E} \subseteq \mathfrak{P}(X)$  where  $\emptyset \in \mathcal{E}$  and  $X \in \mathcal{E}$ ,  $u : \mathcal{E} \rightarrow [0, \infty]$ . For any set

$$u^*(E) = \inf_{\bigcup E_j \supseteq E} \sum_{j=1}^{\infty} u(E_j)$$

where  $E_j \in \mathcal{E}$ , called the (concrete) outer measure induced by  $\mathcal{E}$ . ( $u^*(E) = \infty$  if we cannot cover  $E$  with a countable collection of sets in  $\mathcal{E}$ )

**Remark 4.2.3.** It is not necessary to assume that  $X \in \mathcal{E}$ . In this case,  $u^*(E) = \infty$  if we cannot cover  $E$  with a countable collection of sets in  $\mathcal{E}$ .



**Lemma 4.2.4**

An outer measure  $u^*$  incuded by the collection  $\mathcal{E}$  satisfies

- (i)  $u^*(\emptyset) = 0$
- (ii) For  $A \subseteq B$ ,  $u^*(A) \leq u^*(B)$  Monotonicity
- (iii)  $u^*(\bigcup E_j) \leq \sum u^*(E_j)$  Subaddictivity

E.g., A (concrete) outer measure is an abstract outer measure

*Proof.*  $u^*(\bigcup A_j) \leq \sum u^*(A_j)$  WLOG,  $LHS < \infty$ , for each  $j$  find a collection  $\{E_k^j\}_{k \in \mathbb{N}}$

$$\sum u(E_k^j) < u^*(A_j) + \varepsilon 2^{-j-1}$$

$\{E_k^j\}_{j,k=1}^\infty$  covers  $\bigcup A_j$

$$u^*\left(\bigcup A_j\right) \leq \sum_j \sum_k u(E_k^j) \leq \sum_j u^*(A_j) + \varepsilon \sum_j 2^{-j-1}$$

□

**§4.3 Caratheodory's Construction**

**Definition 4.3.1.** Given  $\varrho$  abstract outer measure, a set  $A$  is Caratheodory measure or  $\varrho$ -measurable (in the sence of) if **for all**  $E \in \mathfrak{P}(X)$ ,

$$\varrho(E) = \varrho(E \cap A) + \varrho(E \cap A^c)$$

**Remark 4.3.2.** It is trivial that

$$\varrho(E) \leq \varrho(E \cap A) + \varrho(E \cap A^c)$$

So, we only need

$$\varrho(E) \geq \varrho(E \cap A) + \varrho(E \cap A^c)$$

to show the equality of the outer measure.

**Theorem 4.3.3 (Caratheodory's Theorem)**

The collection of  $\varrho$ -measurable sets is a  $\sigma$ -algebra  $\mathcal{M}$ ,  $\varrho|_{\mathcal{M}}$  is a measure (in fact, a complete measure)

*Proof.* for any set  $\varrho(\emptyset) = 0$ . For  $A \neq \emptyset$ ,  $\varrho(A) = c \neq 0$ ,  $\varrho(E) = \varrho(E \cap A) + \varrho(E \cap A^c)$

1. want to show that for any  $A, B \in \mathcal{M}$ ,  $A \cap B \in \mathcal{M}$

$$\begin{aligned} \varrho(E) &\geq \varrho(E \cap A) + \varrho(E \cap A^c) \\ &= \varrho(E \cap A \cap B) + \varrho(E \cap A \cap B^c) + \varrho(E \cap A^c \cap B) + \varrho(E \cap A^c \cap B^c) \\ &\geq \varrho(E \cap (A \cap B)) \end{aligned}$$

This shos that  $\mathcal{M}$  is an algebra. A finite union of sets in  $\mathcal{M}$  can be written as a finite disjoint union. We also get the addicitivity of  $\varrho$  when applied to sets in  $\mathcal{M}$  let  $E = A \uplus B$  then

$$\varrho(A \uplus B) = \varrho((A \uplus B) \cap A) + \varrho((A \uplus B) \cap A^c) = \varrho(A) + \varrho(B)$$

2.  $A_1, \dots, A_n$  pairwise **disjoint**, let  $u_n = \biguplus_{j=1}^n A_j$ . We claim that

$$\varrho(u_n \cap E) = \sum_{j=1}^n \varrho(A_j \cap E)$$

(for all  $E$ ) Observe that  $u_n = A_n \uplus u_{n-1}$ ,  $A_n, u_{n-1} \in \mathcal{M}$  then

$$\begin{aligned} \varrho(u_n \cap E) &= \varrho(A_n \cap u_n \cap E) + \varrho(A_n^c \cap u_n \cap E) \\ &= \varrho(A_n \cap E) + \varrho(u_{n-1} \cap E) \end{aligned}$$

Iterate and we get the claim.

Set  $U = \bigcup_{j=1}^{\infty} A_j$ , GOAL: is to show  $\varrho(E) = \varrho(E \cap U) + \varrho(E \cap U^c)$ .

$$\begin{aligned} \varrho(E) &\geq \varrho(u_n \cap E) + \varrho(u_n^c \cap E) \\ &\geq \varrho(u_n \cap E) + \varrho(U^c \cap E) \\ &= \sum_{j=1}^n \varrho(A_j \cap E) + \varrho(U^c \cap E) \\ &\geq \varrho(U \cap E) + \varrho(U^c \cap E) \end{aligned}$$

So  $\varrho$  is  $\sigma$ -addictive on  $\mathcal{M}$ .

□

**Definition 4.3.4.**  $\varrho$  is complete if for  $A \in \mathcal{M}$ , if  $\varrho(A) = 0$  then  $\varrho(N) = 0$  for  $N \subseteq A$ .

# A Practice Exam

## §A.1 Practice Exam 1

**Problem A.1.1.** Let  $E_n$  be Lebesgue measurable subsets of  $[0, 1]$  such that  $E_{n+1} \subseteq E_n$ . What can you say about the Lebesgue measure of  $\bigcap_n E_n$ ? Does your answer necessarily hold when  $[0, 1]$  is replaced by  $[0, \infty)$ ?

*solution.* We can use continuity from above because  $\mu([0, 1]) < \infty$ . We can say that

$$\mu\left(\bigcap_n E_n\right) = \lim_{n \rightarrow \infty} \mu(E_n)$$

In case of  $[0, \infty)$ , we can't use continuity from below because if  $E_n = [n, n+1)$  then  $\mu(E_n) = 1$  but  $\bigcap_n E_n = \emptyset$ , so,  $\lim_{n \rightarrow \infty} \mu(E_n) = 1$  but  $\mu(\bigcap_n E_n) = 0$ .  $\square$

**Problem A.1.2.** Let

$$\begin{aligned} E = \{x = (x_1, x_2) \in \mathbb{R}^2 : \\ \frac{1}{1 + (x_1 - x_2)^3} \leq e^{\sin x} \text{ if } x_1^{23} < 3|\cos(x_1 + x_2)|, \\ \sqrt{1 + e^{|x_2| + |x_1|}} > e^{x_1^2} \text{ if } x_1^{23} > 3|\cos(x_1 + x_2)| \text{ and } x_2 \in \mathbb{R} \setminus \mathbb{Q}, \\ \sqrt{\cos(|x_1 x_2|)} \sin(x_1 x_2) > 0 \text{ if } x_1^{23} > 3|\cos(x_1 + x_2)| \text{ and } x_2 \in \mathbb{Q}\} \end{aligned}$$

- (i) Is the characteristic function of  $E$  Borel measurable?
- (ii) If  $\mathcal{M}$  denote the  $\sigma$ -algebra of Lebesgue measurable subsets of  $\mathbb{R}$  does  $E$  belong to  $\mathcal{M} \oplus \mathcal{M}$ ?

**Problem A.1.3.** For each of the statements give a proof or find a counterexample.

- (i) For each  $r \in \mathbb{R}$  let  $f_r : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Define  $G(x) = \sup_{r \in \mathbb{R}} f_r(x)$ . Is  $G$  a Borel measurable function?
- (ii) For each  $r \in \mathbb{R}$  let  $f_r : \mathbb{R} \rightarrow \mathbb{R}$  be a Borel measurable function. Define  $G(x) = \sup_{r \in \mathbb{R}} f_r(x)$ . Is  $G$  a Borel measurable function?
- (iii) For each  $r \in \mathbb{R}$  let  $f_r : \mathbb{R} \rightarrow \mathbb{R}$  be a Borel measurable function. Define  $G(x) = \sup_{r \in \mathbb{Q}} f_r(x)$ . Is  $G$  a Borel measurable function?

- (iv) For each  $r \in \mathbb{Q}$  let  $f_r : \mathbb{R} \rightarrow \mathbb{R}$  be a Borel measurable function. Define  $S(x) = \sum_{r \in \mathbb{Q}} f_r(x)$ . Is  $S$  a Borel measurable function?
- (v) What happens if you replace Borel measurable by Lebesgue measurable in the above statements?

**Problem A.1.4.** Assume that  $f_n$  is a sequence of  $L^1(\mu)$  functions.

- (i) If in addition  $\mu(X) < \infty$  and

$$\lim_{n \rightarrow \infty} \sup_{x \in X} |f_n(x) - f(x)| = 0$$

then you have learned that  $f \in L^1(\mu)$  and that  $\lim_{n \rightarrow \infty} \int f_n \, d\mu = \int f \, d\mu$ . Review the proof.

- (ii) Show that if  $\mu(X) = \infty$  both conclusions may fail in (i).
- (iii) If  $\mu(X) < \infty$  if  $f_n \rightarrow f$  converges just pointwise show that for every  $\varepsilon > 0$  there exists a set  $A$  of measure  $< \varepsilon$  such that

$$\int_A |f_n - f| \, d\mu < \varepsilon$$

*solution.* (i) Suppose that  $\lim_{n \rightarrow \infty} \|f_n - f\|_{\sup} \rightarrow 0$  then

$$\lim_{n \rightarrow \infty} \int |f_n - f| \, d\mu \leq \lim_{n \rightarrow \infty} \mu(X) \|f_n - f\|_{\sup} = 0$$

and we know that  $f = (f - f_n) + (f_n)$  then

$$\int |f| \, d\mu = \int |f - f_n + f_n| \, d\mu \leq \int |f - f_n| \, d\mu + \int |f_n| \, d\mu$$

so,  $f \in L^1$  and

$$\left| \int f_n - f \, d\mu \right| \leq \int |f_n - f| \, d\mu$$

So,

$$\lim_{n \rightarrow \infty} \int f_n \, d\mu = \int f \, d\mu$$

- (ii) Let  $f_n(x) = \frac{1}{x} \mathbb{1}_{[1, n]}$  then  $f_n \in L^1$  but  $f_n \rightarrow f = \frac{1}{x} \mathbb{1}_{[1, \infty)}$  and  $f \notin L^1$ .
- (iii) Define  $E_n(k) = \{x : \exists m \geq n, |f_m(x) - f(x)| > 1/k\}$  fix  $k$ , then

$$E_n(k) \supseteq E_{n+1}(k) \text{ and } \bigcup_{n \in \mathbb{N}} E_n(k) = \emptyset$$

From continuity from above, there exists  $n_k$  such that  $\mu(E_{n_k}(k)) < \frac{\varepsilon}{2^k}$ . Define

$$A = \bigcup_{k \in \mathbb{N}} E_{n_k}(k), \mu(A) < \varepsilon$$

and for all  $x \in A^c$ , there exists  $k$  such that  $\frac{1}{k} < \varepsilon$  for  $m \geq n_k$ ,

$$|f_m(x) - f(x)| < \varepsilon$$

Then substitute  $\varepsilon = \varepsilon/\mu(X)$  and we get the result.

□

**Problem A.1.5.** Let  $\mathcal{M}$  be a  $\sigma$ -algebra on  $X$ .

- (i) Consider a function  $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : X \rightarrow \mathbb{R}^2$  such that for all rational number  $r_1, r_2$  the sets

$$\{x \in X : r_1 < f_1(x), f_2(x) < r_2\}$$

belong to  $\mathcal{M}$ . Is  $f$   $(\mathcal{M}, \mathcal{B}(\mathbb{R}^2))$ -measurable?

- (ii) Let  $f_n : X \rightarrow \overline{\mathbb{R}}$  a sequence of measurable functions. Show that the set

$$E = \{x \in X : \{f_n(x)\}_{n=1}^\infty \text{ is a nondecreasing sequence}\}$$

is measurable.

**Problem A.1.6.** Let  $f$  be a Lebesgue measurable function on  $\mathbb{R}^n$ . Let  $m$  denote Lebesgue measure on  $\mathbb{R}^n$ .

Show that the following three statements are equivalent:

- (a)  $f$  is integrable (i.e. belong to  $L^1(\mathbb{R}^n)$ ).
- (b)  $\sum_{k \in \mathbb{Z}} 2^k m(\{x : |f(x)| < 2^k\}) < \infty$ .
- (c)  $\sum_{k \in \mathbb{Z}} 2^k m(\{x : 2^k \leq |f(x)| < 2^{k+1}\}) < \infty$ .

**Problem A.1.7.** Let  $m$  be Lebesgue measure on  $\mathbb{R}$  and  $f$  be a Lebesgue measurable function with  $\int |f| dm < \infty$ . Define  $G(x) = \int_{-\infty}^x f dm$ . Prove that  $G$  is uniformly continuous on  $\mathbb{R}$ .

**Problem A.1.8.** Determine the limits

- (i)  $\lim_{n \rightarrow \infty} \int_0^n (1 - \frac{x}{n})^n e^{x/2} dx$
- (ii)  $\lim_{n \rightarrow \infty} \int_0^n (1 + \frac{x}{n})^n e^{x/2} dx$

and, in both cases carefully justify your computation.

*solution.* (i) define

$$f_n = \mathbb{1}_{[0,n]} \left(1 - \frac{x}{n}\right)^n e^{x/2}$$

and  $f = e^{-x}e^{x/2}$  then  $|f_n| \leq |f| \in L^1([0, \infty))$  then by Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n \, dx = \int_0^\infty \lim_{n \rightarrow \infty} f_n \, dx = \int_0^\infty e^{-x/2} \, dx = 2$$

□

**Problem A.1.9.** Let  $f \in \mathcal{L}^1(\mathbb{R}^n)$ . Let  $m$  be Lebesgue measure in  $\mathbb{R}^n$ . Prove that for  $t > 0$

$$t^n \int f(tx) \, dm = \int f(x) \, dx$$

Hint: First prove this for indicator functions of cubes, then for indicator functions of sets of finite measure.

*solution.* Suppose that  $f = \mathbb{1}_E$  where  $E$  is a cube, then

□

**Problem A.1.10.** Let  $f \in \mathcal{L}^1(\mathbb{R}^n)$ . Then

- (i)  $\lim_{|h| \rightarrow 0} \int |f(x+h) - f(x)| \, dm = 0$ .
- (ii)  $\lim_{t \rightarrow 1} \int |f(tx) - f(x)| \, dm = 0$ .
- (iii) Can the Lebesgue dominated convergence be used for the proof of (i) or (ii)?

**Problem A.1.11.** Let  $I = [a, b]$ ,  $f \in \mathcal{L}^1(I)$ . Show that

$$\lim_{n \rightarrow \infty} \int_I f(x) \sin(nx) \, dm(x) = 0$$

**Problem A.1.12.** Recall the monotone convergence theorem and Fatou's lemma.

- (i) Show that Fatou's lemma implies the monotone convergence theorem.
- (ii) Show that the monotone convergence theorem implies Fatou's lemma.

**Problem A.1.13.** Determine

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{2n \sin(x/n)}{x(1+x^2)} dx$$

Provide justifications.

**Problem A.1.14.** Let  $f(x) = \sin(x^2)$  on the measure space  $X = [1, \infty)$  (with Lebesgue measure  $m$ ). Prove:

- (i)  $\int_{[1, \infty)} |f| dm = \infty$
- (ii)  $\lim_{R \rightarrow \infty} \int_{[0, R]} f dm$  exists (and is finite).

Hint: For part (ii) use that  $2x \sin(x^2)$  is the derivative of  $-\cos(x^2)$ .

**Problem A.1.15.** Let  $f : X \rightarrow \overline{\mathbb{R}}$  be a nonnegative measurable function on the measure space  $(X, \mathcal{M}, \mu)$  and assume  $\mu(X) < \infty$ .

- (i) Let  $E_R = \{x \in X : |f(x)| > R\}$ . Prove: If  $|f(x)| < \infty$  for almost every  $x \in X$  then  $\lim_{R \rightarrow \infty} \mu(E_R) = 0$ .
- (ii) Is the conclusion in (i) still valid if we drop the assumption of finite measure space? Give a proof or counterexample.

**Problem A.1.16.** Let  $p > 0$ . For  $x \in \mathbb{R}^n$  let  $|x|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ . Let  $\Omega = \{x \in \mathbb{R}^n : |x|_p > 3\}$ . Show that

$$\int_{\Omega} |x|_p^{-\alpha} dm < \infty$$

if and only if  $\alpha > n$ . What is the result if you replace  $\Omega$  by  $\Omega^c$ ?