

Probing EFX via PMMS: (Non-)Existence Results in Discrete Fair Division

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Fair Division

Fairly divide **resources** among a group of **agents**.

Fairly divide an **inheritance** among a group of **siblings**.

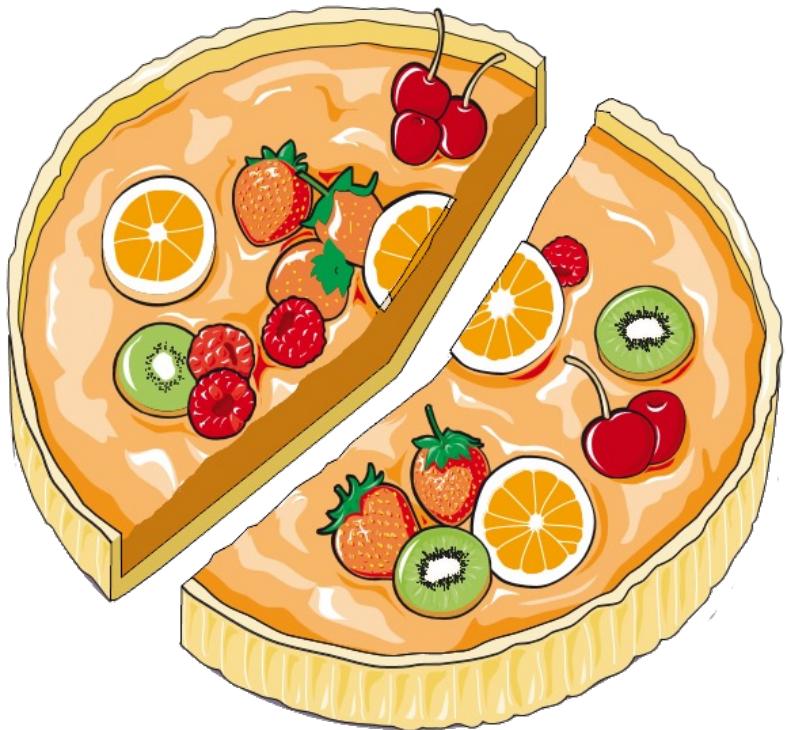


Fairly divide **tasks** among a group of **employees**.



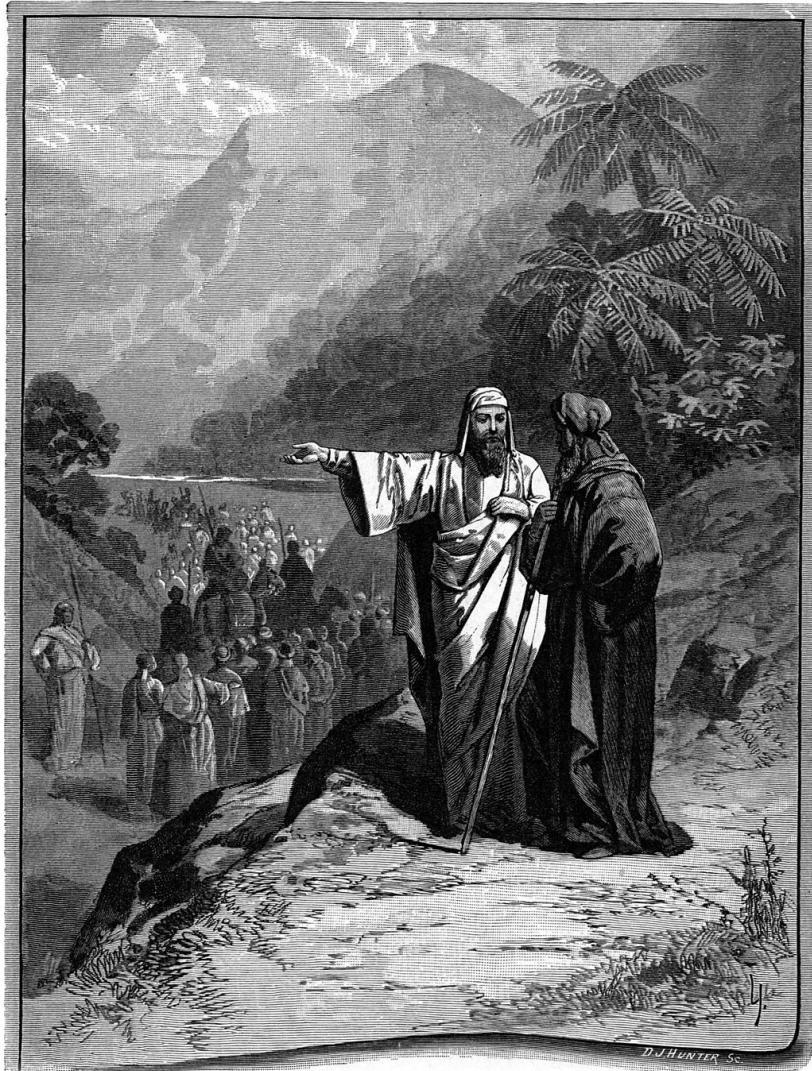
Question: What is the **strongest** fairness notion possible in the **worst case**?

Two Agents: Cut And Choose



For **two agents**, the ideal method is **cut and choose**.

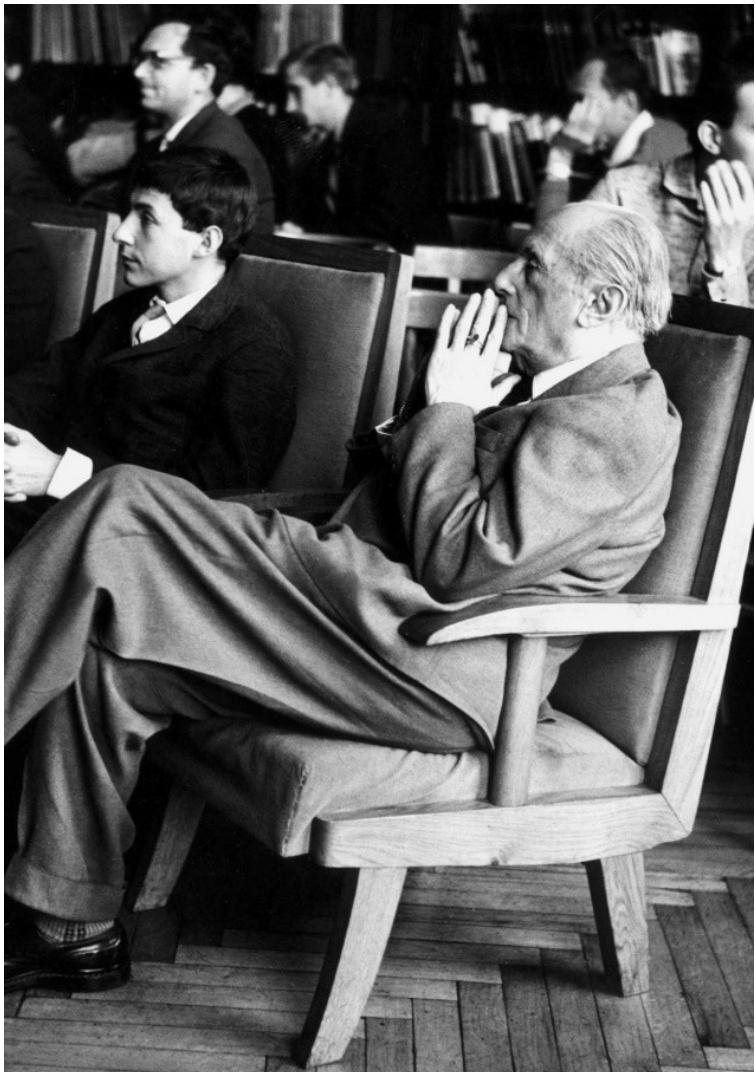
Two Agents: Cut And Choose



For **two agents**, the ideal method is **cut and choose**.

A **cuts** resources into two parts
B **chooses** the one it prefers

Two Agents: Cut And Choose



For **two agents**, the ideal method is **cut and choose**.

A **cuts** resources into two parts
B **chooses** the one it prefers

Question: How can **cut and choose** be generalized to **more agents**?

The Model

	item a	item b	item c	item d
agent 1	3	1	1	1
agent 2	2	3	1	1
agent 3	1	1	1	1

For now, assume **additive** valuations: $v_i(X_i) = \sum_{g \in X_i} v_i(\{g\})$
We also consider **monotone** valuations $v_i : 2^{[m]} \rightarrow \mathbb{R}$.

Items are **indivisible** and have to be **fully allocated**
in a **deterministic** way.

Agents have **subjective preferences** and **equal entitlements**.

Fairness Notions

Allocation is **envy-free (EF)** if $v_i(X_i) \geq v_i(X_j)$ for all i, j .

agent i	item a	item b
	4	3

Agent i does not satisfy EF 

agent i	item a	item b
	4	3

agent i	item a	item b
	4	3

Agent i satisfies EF 

EF allocations **exist** for **divisible items**.
EF allocations might **not exist** for **indivisible items**.

Fairness Notions

Allocation is **envy-free up to some good (EF1)**
if for all i, j , $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for some $g \in X_j$.

agent i	item a	item b	item c
	3	3	5

Agent i does not satisfy EF

Agent i satisfies EF1

Theorem [LMMS'2001]:
EF1 allocations exist for **indivisible** items.

Fairness Notions

Allocation is **envy-free up to any good (EFX)**
if for all i, j , $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for any $g \in X_j$.

agent i	item a	item b	item c
	4	2	3

Agent i does not satisfy EFX

Agent i satisfies EF1

Open Problem: Does EFX **exist** for **indivisible items**?
Theorem [CGM'2020]: EFX **exists** for **three agents**.

Fairness Notions

The **maximin share**: $\mu_i(S) = \max_{X \cup Y = S} \min \{v_i(X), v_i(Y)\}$.

	a	b	c	d
<i>i</i>	2	4	5	6

The maximin share is 8.

Allocation is **pairwise-maximin-share-fair (PMMS)**

if for all i, j , $v_i(X_i) \geq \mu_i(X_i \cup X_j)$.

	a	b	c	d
<i>i</i>	2	4	5	6

Agent *i* does not satisfy PMMS 
Agent *i* satisfies EFX 

	a	b	c	d
<i>i</i>	2	4	5	6

Agent *i* satisfies PMMS 
Agent *i* satisfies EFX 

Fairness Notions

PMMS: $v_i(X_i) \geq \mu_i(X_i \cup X_j)$

where $\mu_i(S) = \max_{X \cup Y = S} \min \{v_i(X), v_i(Y)\}$

	item a	item b	item c	item d
agent 1	3	1	1	1
agent 2	2	3	1	1
agent 3	1	1	1	1

Comparison-based fairness (this talk):

For each pair of agents i and j , the condition depends on X_i and X_j .

Share-based fairness (not in this talk):

For each agent i , the condition depends on X_i and number of agents.

Fairness Notions

Question: What is the **strongest** fairness notion possible in the **worst case**?

	definition	status
EF	$v_i(X_i) \geq v_i(X_j)$	does not exist
PMMS	$v_i(X_i) \geq \mu_i(X_i \cup X_j)$	exists for 2 agents open for 3 agents
EFX	$v_i(X_i) \geq v_i(X_j \setminus \{g\}) \quad \forall g \in X_j$	exists for 3 agents open for 4 agents
EF1	$v_i(X_i) \geq v_i(X_j \setminus \{g\}) \quad \exists g \in X_j$	exists

Multiplicative approximations of fairness (not in this talk):
0.781-PMMS exists [K'2017] and 0.618-EFX exists [AMN'2020].

EFX vs. PMMS

EFX and PMMS were both introduced in 2016.

The Unreasonable Fairness of Maximum Nash Welfare

IOANNIS CARAGIANNIS, University of Patras

DAVID KUROKAWA, Carnegie Mellon University

HERVÉ MOULIN, University of Glasgow and Higher School of Economics, St Petersburg

ARIEL D. PROCACCIA, Carnegie Mellon University

NISARG SHAH, Carnegie Mellon University

JUNXING WANG, Carnegie Mellon University

The *maximum Nash welfare (MNW)* solution — which selects an allocation that maximizes the product of utilities — is known to provide outstanding fairness guarantees when allocating divisible goods. And while it seems to lose its luster when applied to indivisible goods, we show that, in fact, the MNW solution is unexpectedly, strikingly fair even in that setting. In particular, we prove that it selects allocations that are envy free up to one good — a compelling notion that is quite elusive when coupled with economic efficiency. We also establish that the MNW solution provides a good approximation to another popular (yet possibly infeasible) fairness property, the maximin share guarantee, in theory and — even more so — in practice. While finding the MNW solution is computationally hard, we develop a nontrivial implementation, and demonstrate that it scales well on real data. These results lead us to believe that MNW is the ultimate solution for allocating indivisible goods, and underlie its deployment on a popular fair division website.

CCS Concepts: •**Theory of computation**→ **Algorithmic mechanism design**; •**Applied computing**→ **Economics**;

Additional Key Words and Phrases: Fair division, Resource allocation, Nash welfare

EFX vs. PMMS

EFX got a lot more attention since then.

EFX: 21,500 results

Google Scholar "EFX" OR "envy-freeness up to any good" 

Articles About 21,500 results (0.18 sec)

Any time Computing Envy-Free up to Any Good (EFX) Allocations via Local Search S Brânzei - arXiv preprint arXiv:2510.05429, 2025 - arxiv.org ... EFX (envy-free up to any good) allocations of m indivisible goods among n agents with additive valuations. EFX ... Our algorithm employs simulated annealing with the total number of EFX ...    All 2 versions 

EFX exists for three agents BR Chaudhury, J Garg, K Mehlhorn - Journal of the ACM, 2024 - dl.acm.org ... envy-freeness up to any good (EFX). Despite significant efforts by many researchers for several years, the existence of EFX ... In this article, we show constructively that an EFX allocation ...   Cited by 211 Related articles All 13 versions

Improving envy freeness up to any good guarantees through rainbow cycle number BR Chaudhury, J Garg, K Mehlhorn... - Mathematics of ..., 2024 - pubsonline.informs.org ... In particular, it is known that 0.618-EFX allocations exist and that EFX ... ϵ)-EFX allocation with sublinear number of unallocated goods and high Nash welfare. For this, we reduce the EFX ...   Cited by 2 Related articles All 7 versions

include patents include citations 

PMMS: 38 results

Google Scholar "PMMS" or "pairwise maximin share" 

Articles About 38 results (0.08 sec)

Any time Exact and approximation algorithms for PMMS under identical constraints S Dai, G Gao, X Guo, Y Zhang - ... Conference on Theory and Applications of ..., 2022 - Springer ... Nash Social Welfare allocation is always PMMS and EFX if the valuation ... pairwise maximin share allocations for identical variants. PMMS graph is introduced to help us find the PMMS ...   Cited by 5 Related articles All 3 versions

Groupwise maximin fair allocation of indivisible goods S Barman, A Biswas, S Krishnamurthy... - Proceedings of the AAAI ..., 2018 - ojs.aaai.org ... pairwise maximin share guarantee (PMMS), a notion defined by (Caragiannis et al. 2016). In PMMS, ... that GMMS is a strict generalization of PMMS and MMS. The relevance of GMMS is ...   Cited by 88 Related articles All 7 versions 

Exact and approximate maximin share allocations in multi-graphs G Christodoulou, S Mastrakoulis - arXiv preprint arXiv:2506.20317, 2025 - arxiv.org ... Pairwise maximin share and α -PMMS Another fairness notion which we will consider in this work, is the pairwise maximin share (PMMS) ... if we can guarantee exact PMMS value for all ...   Cited by 3 Related articles All 2 versions 

include patents include citations 

About PMMS

PMMS: $v_i(X_i) \geq \mu_i(X_i \cup X_j)$

where $\mu_i(S) = \max_{X \cup Y = S} \min \{v_i(X), v_i(Y)\}$

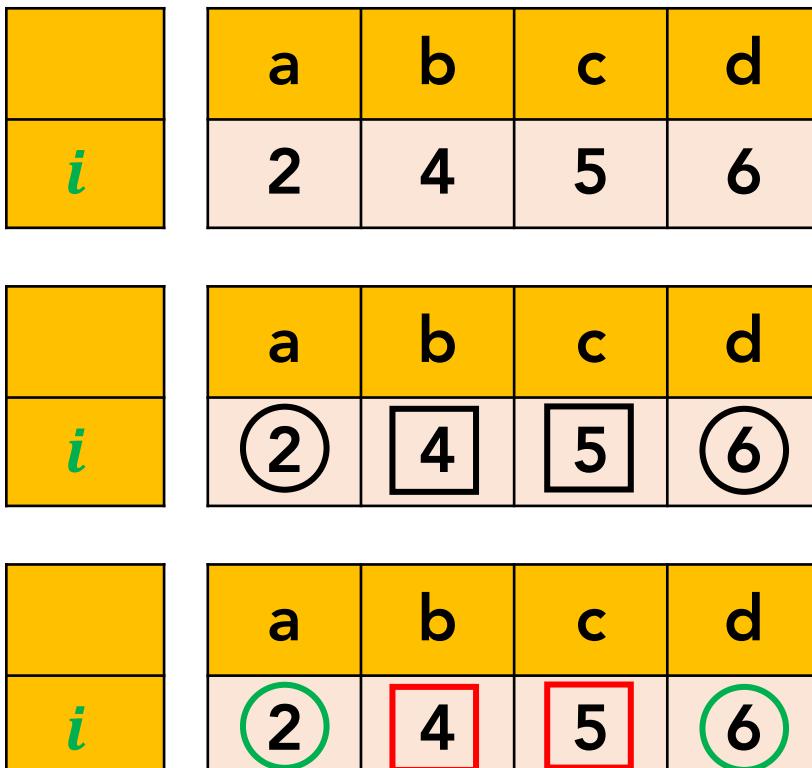
1. PMMS is exactly the guarantee the cutter gets in cut and choose.
2. PMMS extends to chores and mixed manna, unlike EFX.
3. PMMS is stronger than EFX in non-degenerate instances.
4. PMMS is computationally harder than EFX.

About PMMS

1. PMMS is exactly the guarantee the **cutter** gets in **cut and choose**.

PMMS: $v_i(X_i) \geq \mu_i(X_i \cup X_j)$

where $\mu_i(S) = \max_{X \cup Y = S} \min \{v_i(X), v_i(Y)\}$



	a	b	c	d
<i>i</i>	2	4	5	6



Cutter cuts into two parts.
Guarantees **PMMS**.

	a	b	c	d
<i>i</i>	(2)	4	5	(6)

Chooser chooses one.
Guarantees **EF**.

About PMMS

2. PMMS extends cleanly to **chores** and **mixed manna**, unlike EFX.

PMMS for goods, chores, and mixed manna: $v_i(X_i) \geq \mu_i(X_i \cup X_j)$
where $\mu_i(S) = \max_{X \cup Y = S} \min \{v_i(X), v_i(Y)\}$

<i>i</i>

a	b	c	d
2	4	5	6

Agent *i* satisfies PMMS 
MMS share is 8.

<i>i</i>

a	b	c	d
-2	-4	-5	-6

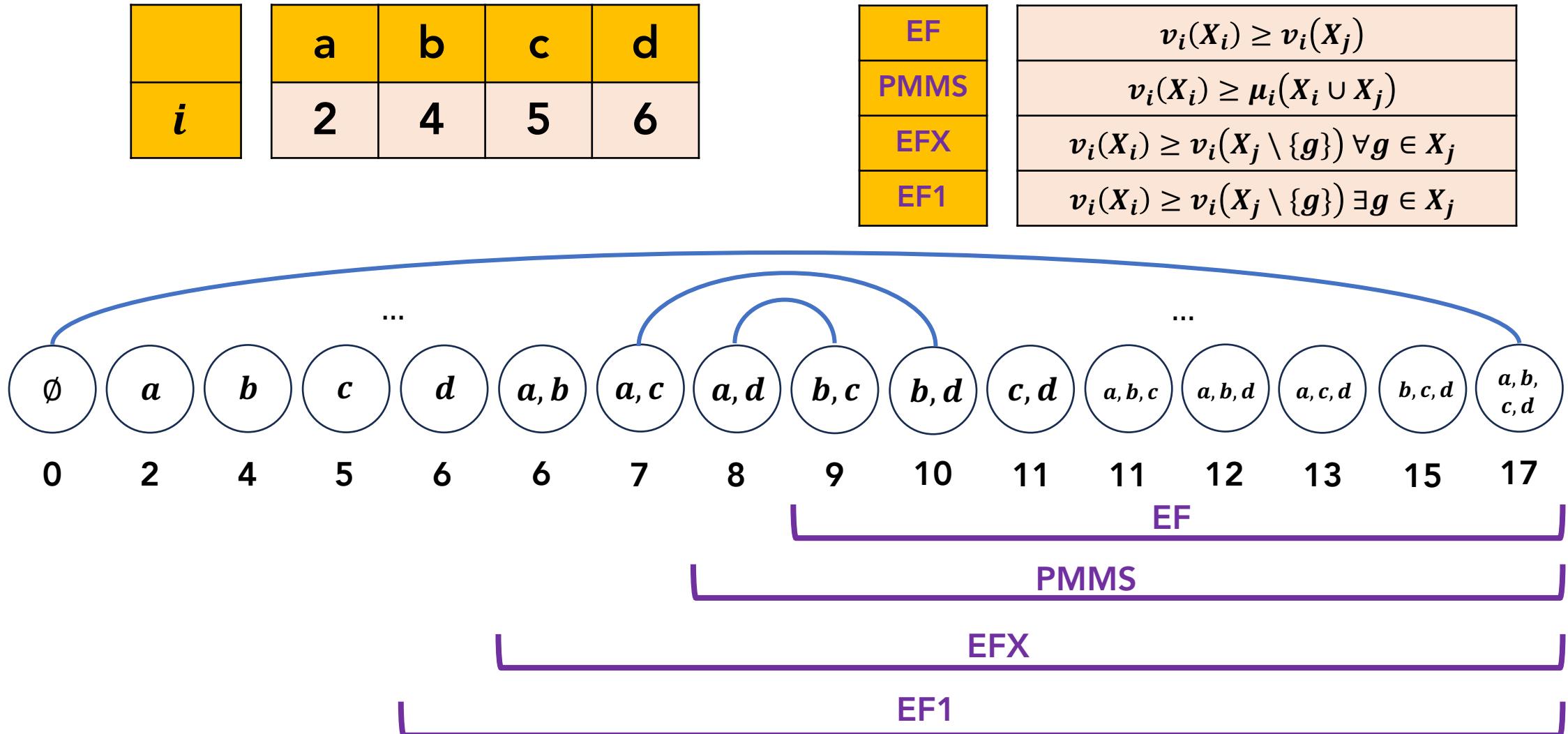
Agent *i* satisfies PMMS 
MMS share is -9.

EFX for goods: $v_i(X_i) \geq v_i(X_j \setminus \{g\}) \forall g \in X_j$

EFX for chores: $v_i(X_i \setminus \{g\}) \geq v_i(X_j) \forall g \in X_i$

About PMMS

3. PMMS is **stronger** than EFX in **non-degenerate** instances.



About PMMS

4. PMMS is computationally harder than EFX.

Theorem: Finding a PMMS allocation for two identical additive agents is NP-hard.

Proof: Reduction from the Partition problem.

Theorem [GHH'2023]:
Finding an EFX allocation for two additive agents is in P.

Research Direction

Big Open Problems:

Does EFX exist for additive valuations?

Does PMMS exist for additive valuations?

Our Goals:

Show impossibility for weaker assumptions.

Show possibility for stronger assumptions.

Negative Results

Theorem (Negative): For any n , there is an n -agent instance with n monotone valuations that admits no PMMS allocation.

Extending the 2-agent impossibility to n agents is non-trivial.

Theorem (Negative): There is an instance with 2 monotone and 1 additive valuation that admits no PMMS allocation.

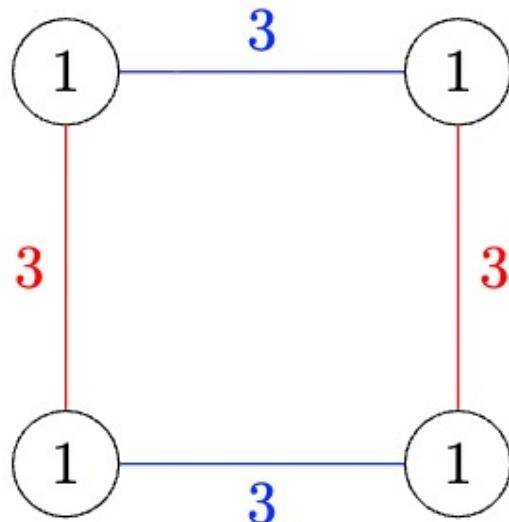
EFX known to exists for 2 monotone and 1 additive valuations.
Thus, PMMS for 3 agents requires new techniques.

Monotone Valuations

Theorem (Negative): For any n , there is an n -agent instance with n monotone valuations that admits no PMMS allocation.

PMMS: $v_i(X_i) \geq \mu_i(X_i \cup X_j)$ where $\mu_i(S) = \max_{X \cup Y = S} \min \{v_i(X), v_i(Y)\}$

Two agents (red and blue)



Nodes show values for single items.
Edges show values for pairs of items.

The MMS shares are $\mu_1 = \mu_2 = 3$.
No allocation satisfies
 $v_1(X_1) \geq 3$ and $v_2(X_2) \geq 3$.

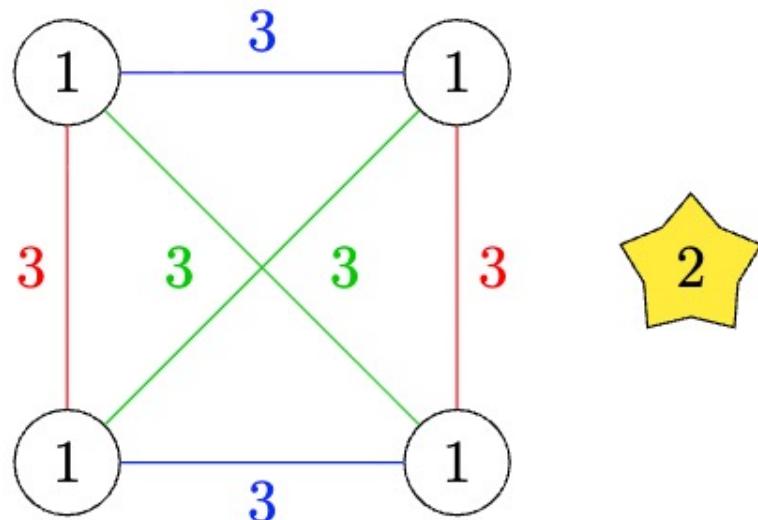
How to extend this to 3 agents?

Monotone Valuations

Theorem (Negative): For any n , there is an n -agent instance with n monotone valuations that admits no PMMS allocation.

PMMS: $v_i(X_i) \geq \mu_i(X_i \cup X_j)$ where $\mu_i(S) = \max_{X \cup Y = S} \min \{v_i(X), v_i(Y)\}$

Three agents (red, blue, green)



Nodes show values for **single** items.
Edges show values for **pairs** of items.

One agent gets only the **star**.
The remaining items **cannot** be divided between the other two agents.

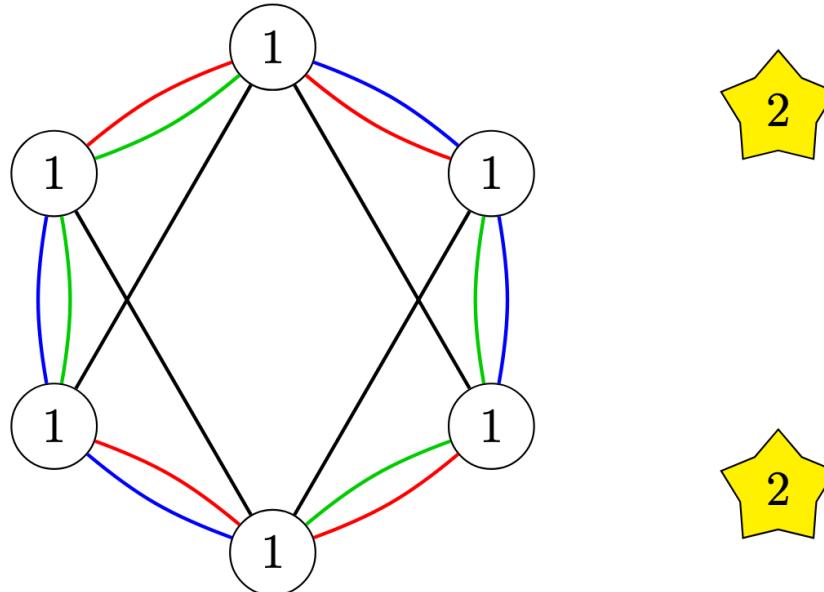
How to extend this to 4 agents?

Monotone Valuations

Theorem (Negative): For any n , there is an n -agent instance with n monotone valuations that admits no PMMS allocation.

PMMS: $v_i(X_i) \geq \mu_i(X_i \cup X_j)$ where $\mu_i(S) = \max_{X \cup Y = S} \min \{v_i(X), v_i(Y)\}$

Four agents
(red, blue, green, black)



Nodes show values for **single** items.
Edges show values for **triplets** of items.

For n agents, our construction
uses $(n - 2) + O(\log n)$ items.

Open Problem:
Could $(n - 2) + O(1)$ be enough?

MMS-Feasible Valuations

Theorem (Negative): For any n , there is an n -agent instance with n monotone valuations that admits no PMMS allocation.

Question: Can extra assumptions on valuations help?
We consider MMS-feasible vals, a generalization of additive.

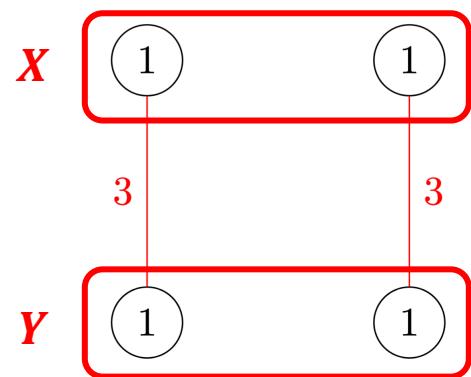
Open Problem: Does PMMS exist for MMS-feasible vals?

Open Problem: Does PMMS exist for additive valuations?

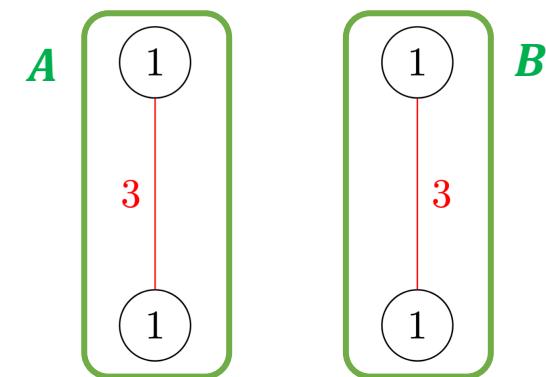
MMS-Feasible Valuations

Valuation is **MMS-feasible** if $\max\{v_i(X), v_i(Y)\} \geq \min\{v_i(A), v_i(B)\}$ for any $S \subseteq [m]$ and any two partitions $X \cup Y = S$ and $A \cup B = S$

Observation 1: Our earlier construction **violates MMS-feasibility**.



$$\max\{v_i(X), v_i(Y)\} = 1$$



$$\min\{v_i(A), v_i(B)\} = 3$$

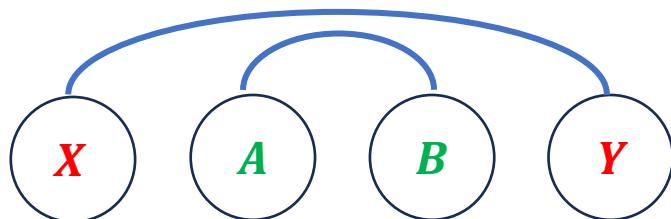
Observation 2: Additive valuations **satisfy MMS-feasibility**.

Proof: $\max\{v_i(X), v_i(Y)\} \geq (1/2) \cdot v_i(S) \geq \min\{v_i(A), v_i(B)\}$

MMS-Feasible Valuations

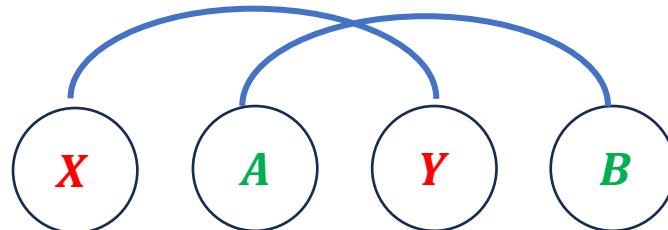
Valuation is **MMS-feasible** if $\max\{v_i(X), v_i(Y)\} \geq \min\{v_i(A), v_i(B)\}$ for any $S \subseteq [m]$ and any two partitions $X \cup Y = S$ and $A \cup B = S$

Recall: We order bundles from least to most valuable and draw arcs between complements.



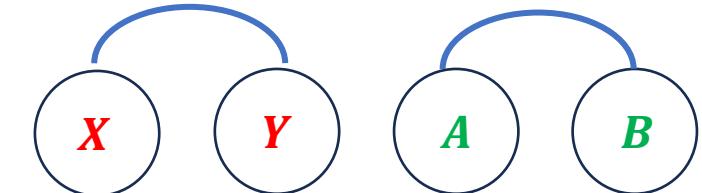
MMS-feasible

Additive



MMS-feasible

Additive



Not MMS-feasible

Additive

MMS-Feasible Valuations

Valuation is **MMS-feasible** if $\max\{v_i(X), v_i(Y)\} \geq \min\{v_i(A), v_i(B)\}$ for any $S \subseteq [m]$ and any two partitions $X \cup Y = S$ and $A \cup B = S$

Theorem (Positive for Two Agents):
PMMS exists for 1 monotone and 1 MMS-feasible valuation.

Theorem (Negative for Three Agents): There is an instance with 2 monotone and 1 MMS-feasible valuation that admits no PMMS allocation.

Open Problems:
Does PMMS exist for 1 monotone and 2 MMS-feasible?
Does PMMS exist for 3 MMS-feasible?

MMS-Feasible Valuations

MMS-feasible: $\max\{v_i(X), v_i(Y)\} \geq \min\{v_i(A), v_i(B)\}.$

Assumptions for 2 agents
2 monotone
1 monotone + 1 MMS-feasible
2 MMS-feasible

EFX	PMMS
exists	does not exist
exists	exists
exists	exists

Assumptions for 3 agents
3 monotone
2 monotone + 1 MMS-feasible
1 monotone + 2 MMS-feasible
3 MMS-feasible

EFX	PMMS
open	does not exist
exists	does not exist
exists	open
exists	open

Positive Results

Searching over instances requires assigning values to all bundles:
There are essentially $(2^m)!$ valuations.

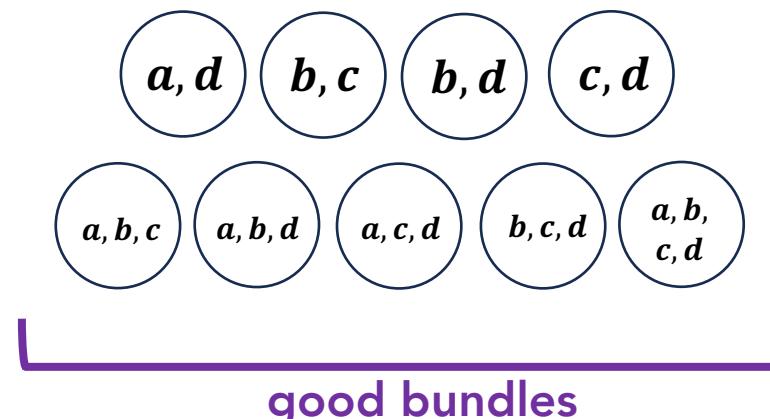
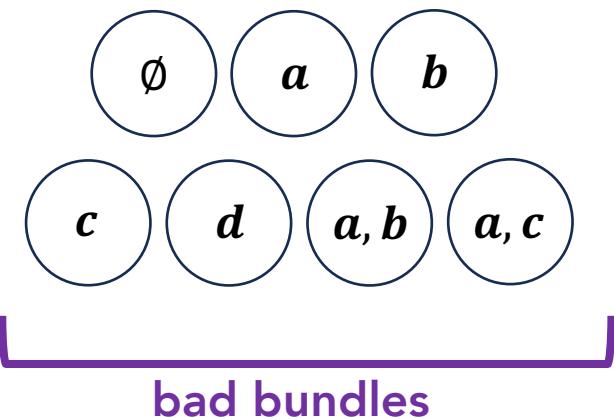
We could restrict to valuations with few possible values:
Binary-valued (or **dichotomous**): $v_i(X) \in \{0, 1\}$.
There are 2^{2^m} such valuations.

We could restrict to valuations with few valuable bundles:
Pair-demand: $v_i(X) = \max_{T \subseteq X, |T| \leq 2} v_i(T)$.
There are $(m^2)!$ such valuations.

Positive Results

Theorem (Positive): PMMS exists for binary-valued MMS-feasible valuations.

Binary-valued (or dichotomous): $v_i(X) \in \{0, 1\}$.



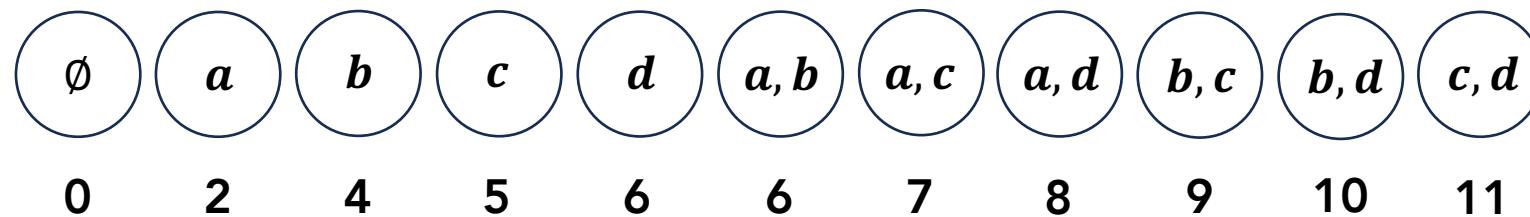
Does not exist for two binary-valued valuations without MMS-feasibility.

No monotonicity required! Applies to goods, chores, and mixed manna.

Positive Results

Theorem (Positive): PMMS exists for additive pair-demand valuations.

Additive pair-demand: $v_i(X) = \max_{T \subseteq X, |T| \leq 2} \sum_{g \in T} v_i(\{g\})$.



Does not exist for two general pair-demand valuations.

Open Problem: Does it exist for MMS-feasible pair-demand valuations?

Summary

Big Open Problems:

Does EFX exist for additive valuations?

Does PMMS exist for additive valuations?

We can make progress by showing
impossibility for weaker assumptions and
possibility for stronger assumptions.

PMMS is an interesting and stronger variant of EFX.

PMMS requires new techniques, possibly useful for EFX.