

# Probing EFX via PMMS: (Non-)Existence Results in Discrete Fair Division

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University of Wrocław

# Fair Division

Fairly divide **resources** among a group of **agents**.

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Fairly divide an **inheritance**  
among a group of **siblings**.



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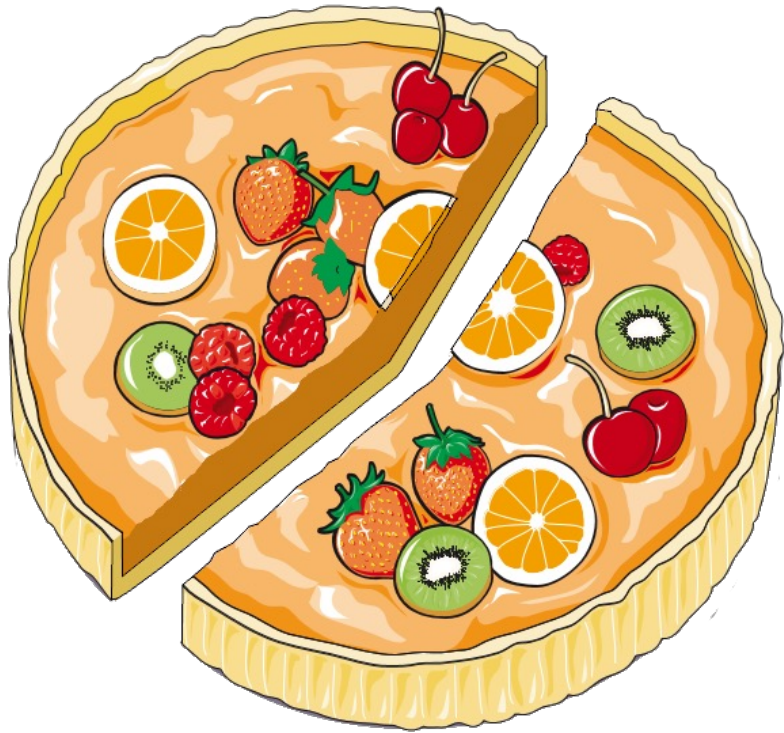


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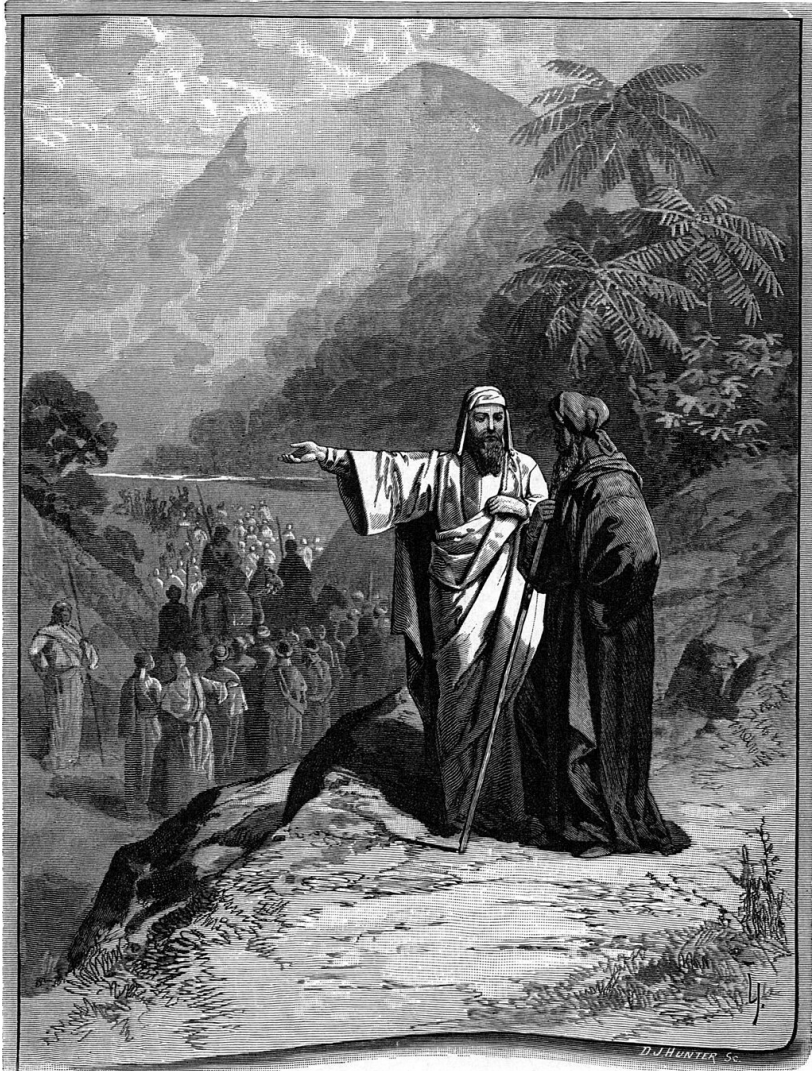
Question: What is the **strongest** fairness notion possible in the **worst case**?

# Two Agents: Cut And Choose



For **two agents**, the ideal method is **cut and choose**.

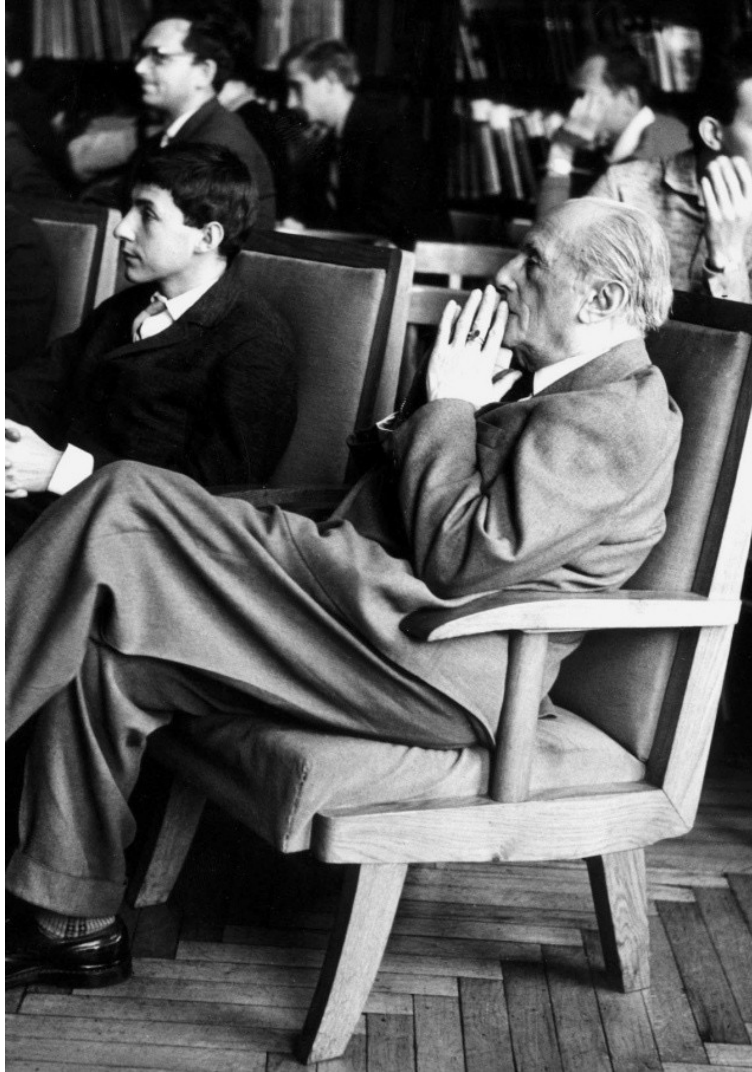
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Question: How can **cut and choose** be generalized to **more agents**?

# The Model

|         | item a | item b | item c | item d |
|---------|--------|--------|--------|--------|
| agent 1 | 3      | 1      | 1      | 1      |
| agent 2 | 2      | 3      | 1      | 1      |
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For now, assume **additive** valuations:  $v_i(X_i) = \sum_{g \in X_i} v_i(\{g\})$   
We also consider **monotone** valuations  $v_i : 2^{[m]} \rightarrow \mathbb{R}$ .

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Items are **indivisible** and have to be **fully allocated**  
in a **deterministic** way.

Agents have **subjective preferences** and **equal entitlements**.

# Fairness Notions

Allocation is **envy-free** (EF) if  $v_i(X_i) \geq v_i(X_j)$  for all  $i, j$ .

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EF allocations **exist** for **divisible** items.  
EF allocations might **not exist** for **indivisible** items.

# Fairness Notions

Allocation is **envy-free up to some good** (EF1)  
if for all  $i, j$ ,  $v_i(X_i) \geq v_i(X_j \setminus \{g\})$  for some  $g \in X_j$ .

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Theorem [LMMS'2001]:  
EF1 allocations **exist** for **indivisible** items.

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Allocation is **envy-free up to any good (EFX)**  
if for all  $i, j$ ,  $v_i(X_i) \geq v_i(X_j \setminus \{g\})$  for **any**  $g \in X_j$ .

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Open Problem: Does EFX **exist** for **indivisible** items?  
Theorem [CGM'2020]: EFX **exists** for **three agents**.

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The **maximin share**:  $\mu_i(S) = \max_{X \uplus Y = S} \min \{v_i(X), v_i(Y)\}$ .

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Allocation is **pairwise-maximin-share-fair** (PMMS)  
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Agent *i* satisfies EFX ✅

|          | a | b | c | d |
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# Fairness Notions

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**Comparison-based** fairness (**this talk**):

For each pair of agents  $i$  and  $j$ , the condition depends on  $X_i$  and  $X_j$ .

**Share-based** fairness (**not in this talk**):

For each agent  $i$ , the condition depends on  $X_i$  and number of agents.

# Fairness Notions

Question: What is the **strongest** fairness notion possible in the **worst case**?

|      | definition   | status                                   |
|------|--|--|
| EF   | $v_i(X_i) \geq v_i(X_j)$                                   | does not exist                           |
| PMMS | $v_i(X_i) \geq \mu_i(X_i \cup X_j)$                        | exists for 2 agents<br>open for 3 agents |
| EFX  | $v_i(X_i) \geq v_i(X_j \setminus \{g\}) \forall g \in X_j$ | exists for 3 agents<br>open for 4 agents |
| EF1  | $v_i(X_i) \geq v_i(X_j \setminus \{g\}) \exists g \in X_j$ | exists                                   |

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**Multiplicative approximations** of fairness (**not in this talk**):  
**0.781-PMMS** exists [K'2017] and **0.618-EFX** exists [AMN'2020].

# EFX vs. PMMS

EFX and PMMS were both introduced in 2016.

## The Unreasonable Fairness of Maximum Nash Welfare

IOANNIS CARAGIANNIS, University of Patras

DAVID KUROKAWA, Carnegie Mellon University

HERVÉ MOULIN, University of Glasgow and Higher School of Economics, St Petersburg

ARIEL D. PROCACCIA, Carnegie Mellon University

NISARG SHAH, Carnegie Mellon University

JUNXING WANG, Carnegie Mellon University

The *maximum Nash welfare (MNW)* solution — which selects an allocation that maximizes the product of utilities — is known to provide outstanding fairness guarantees when allocating divisible goods. And while it seems to lose its luster when applied to indivisible goods, we show that, in fact, the MNW solution is unexpectedly, strikingly fair even in that setting. In particular, we prove that it selects allocations that are envy free up to one good — a compelling notion that is quite elusive when coupled with economic efficiency. We also establish that the MNW solution provides a good approximation to another popular (yet possibly infeasible) fairness property, the maximin share guarantee, in theory and — even more so — in practice. While finding the MNW solution is computationally hard, we develop a nontrivial implementation, and demonstrate that it scales well on real data. These results lead us to believe that MNW is the ultimate solution for allocating indivisible goods, and underlie its deployment on a popular fair division website.


CCS Concepts: •**Theory of computation**→ **Algorithmic mechanism design**; •**Applied computing**→ **Economics**;


Additional Key Words and Phrases: Fair division, Resource allocation, Nash welfare

# EFX vs. PMMS

EFX got a lot more attention since then.

## EFX: 21,500 results

 Scholar

"EFX" OR "envy-freeness up to any good" 

Articles About 21,500 results (0.18 sec)

Any time

Since 2025

Since 2024

Since 2021

Custom range...

Sort by relevance


Sort by date

Any type

Review articles

☐ include patents

☒ include citations

 Create alert

Computing Envy-Free up to Any Good (EFX) Allocations via Local Search

[S Brânzei](#) - arXiv preprint arXiv:2510.05429, 2025 - arxiv.org

... **EFX** (envy-free up to any good) allocations of  $m$  indivisible goods among  $n$  agents with additive valuations. **EFX** ... Our algorithm employs simulated annealing with the total number of **EFX** ...

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EFX exists for three agents

[BR Chaudhury, J Garg, K Mehlhorn](#) - Journal of the ACM, 2024 - dl.acm.org

... **envy-freeness up to any good (EFX)**. Despite significant efforts by many researchers for several years, the existence of **EFX** ... In this article, we show constructively that an **EFX** allocation ...

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Improving envy freeness up to any good guarantees through rainbow cycle number


[BR Chaudhury, J Garg, K Mehlhorn](#)... - Mathematics of ..., 2024 - pubsonline.informs.org

... In particular, it is known that  $0.618$ -**EFX** allocations exist and that **EFX** ...  $\epsilon$  ) -**EFX** allocation with sublinear number of unallocated goods and high Nash welfare. For this, we reduce the **EFX** ...

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## PMMS: 38 results

 Scholar

"PMMS" or "pairwise maximin share" 

Articles About 38 results (0.08 sec)

Any time

Since 2025

Since 2024

Since 2021

Custom range...

Sort by relevance


Sort by date

Any type

Review articles

☐ include patents

☒ include citations

 Create alert

Exact and approximation algorithms for PMMS under identical constraints

[S Dai, G Gao, X Guo, Y Zhang](#) - ... Conference on Theory and Applications of ..., 2022 - Springer

... Nash Social Welfare allocation is always **PMMS** and EFX if the valuation ... **pairwise maximin share** allocations for identical variants. **PMMS** graph is introduced to help us find the **PMMS** ...

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Groupwise maximin fair allocation of indivisible goods

[S Barman, A Biswas, S Krishnamurthy](#)... - Proceedings of the AAAI ..., 2018 - ojs.aaai.org

... **pairwise maximin share** guarantee (**PMMS**), a notion defined by (Caragiannis et al. 2016). In **PMMS**, ... that GMMS is a strict generalization of **PMMS** and MMS. The relevance of GMMS is ...

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Exact and approximate maximin share allocations in multi-graphs

[G Christodoulou, S Mastrakoulis](#) - arXiv preprint arXiv:2506.20317, 2025 - arxiv.org

... **Pairwise maximin share** and  $\alpha$ -**PMMS** Another fairness notion which we will consider in this work, is the **pairwise maximin share (PMMS)** ... if we can guarantee exact **PMMS** value for all ...

[☆ Save](#) [🔗 Cite](#) [Cited by 3](#) [Related articles](#) [All 2 versions](#) [🔗](#)

# About PMMS

**PMMS:**  $v_i(X_i) \geq \mu_i(X_i \cup X_j)$   
where  $\mu_i(S) = \max_{X \sqcup Y = S} \min \{v_i(X), v_i(Y)\}$

1. **PMMS** is exactly the guarantee the **cutter** gets in **cut and choose**.
2. **PMMS** extends to **chores** and **mixed manna**, unlike **EFX**.
3. **PMMS** is **stronger** than **EFX** in **non-degenerate** instances.
4. **PMMS** is **computationally harder** than **EFX**.

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|----------|---|---|---|---|
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**Cutter** cuts into two parts.  
Guarantees **PMMS**.

**Chooser** chooses one.  
Guarantees **EF**.

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Agent *i* satisfies PMMS ✓  
MMS share is 8.

|          | a  | b  | c  | d  |
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# About PMMS

3. PMMS is stronger than EFX in non-degenerate instances.

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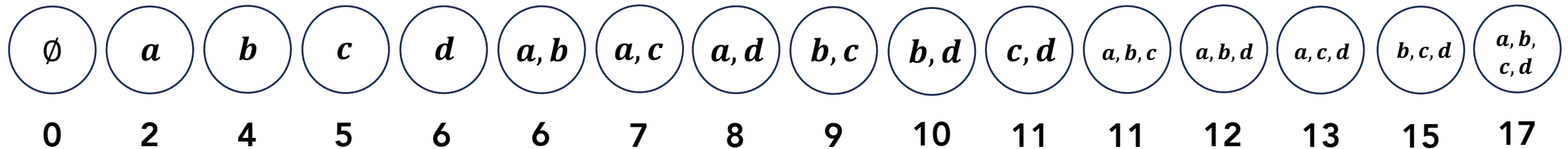
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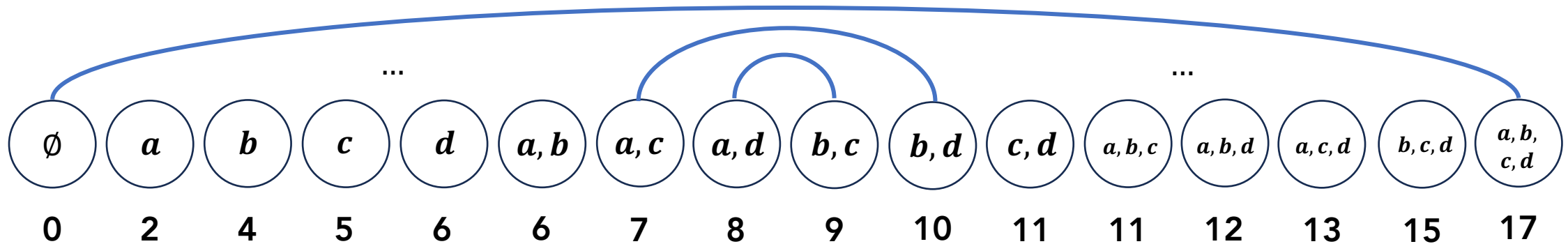
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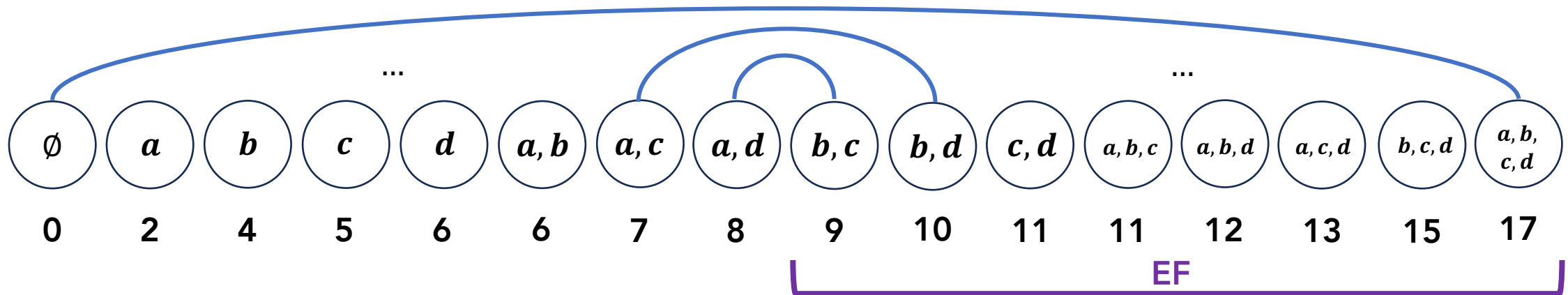
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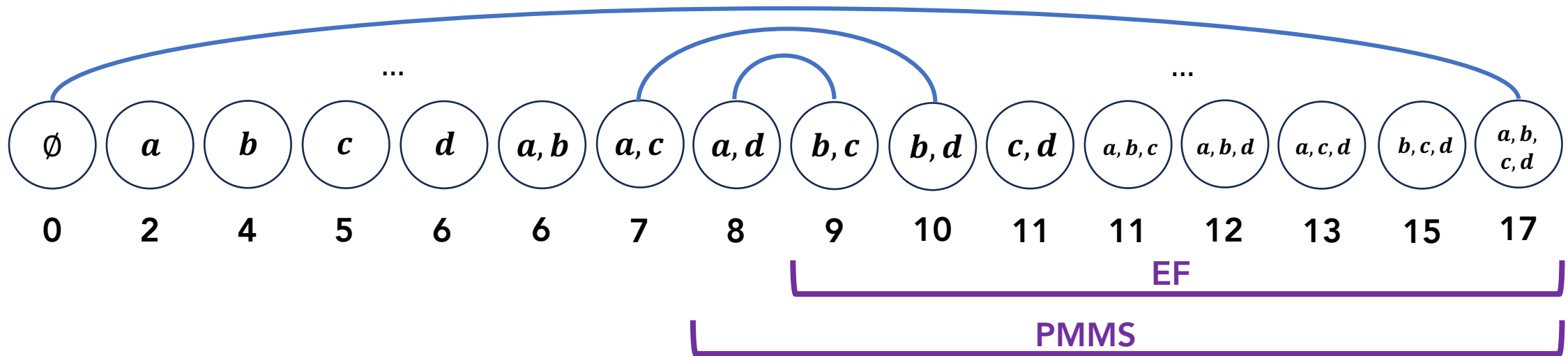
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|  |
|--|
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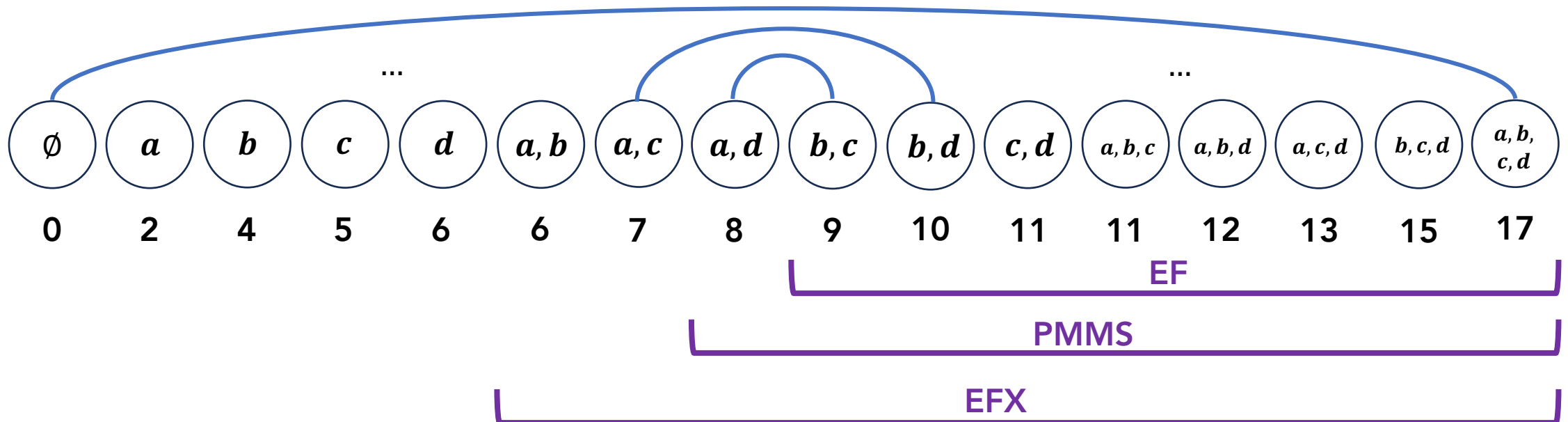
# About PMMS

3. PMMS is stronger than EFX in non-degenerate instances.

|     | a | b | c | d |
|-----|---|---|---|---|
| $i$ | 2 | 4 | 5 | 6 |

|      |
|------|
| EF   |
| PMMS |
| EFX  |
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|  |
|--|
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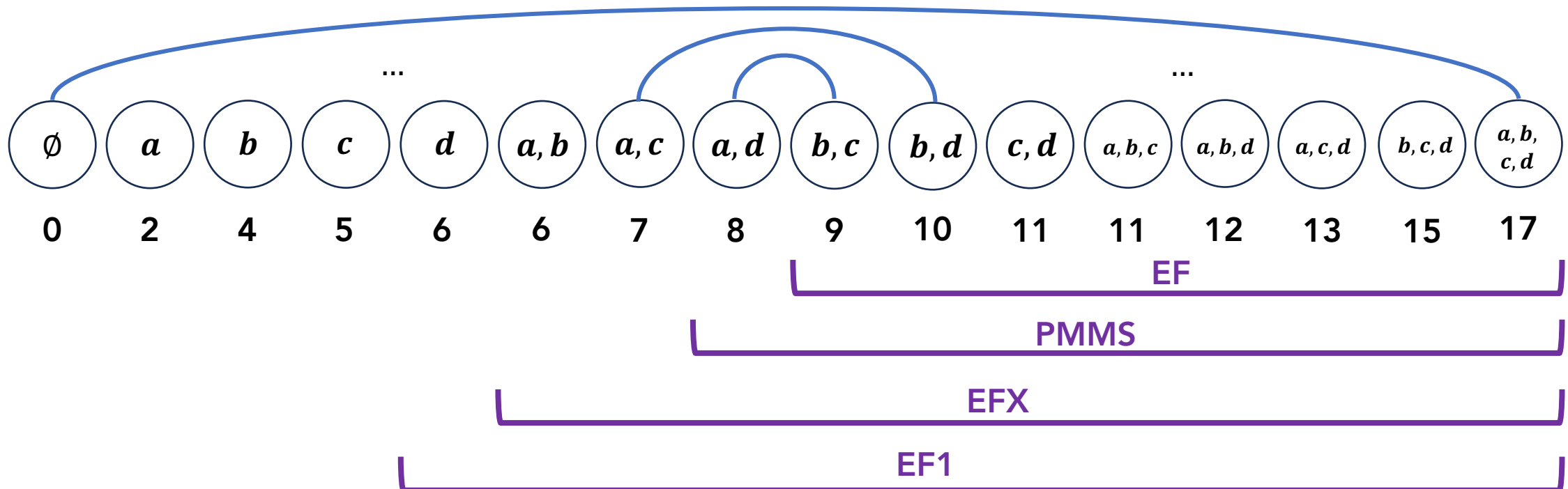
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# About PMMS

4. PMMS is computationally harder than EFX.

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Theorem: Finding a PMMS allocation for two identical additive agents is NP-hard.

Proof: Reduction from the Partition problem.

Theorem [GHH'2023]:  
Finding an EFX allocation for two additive agents is in P.

# Research Direction

## Big Open Problems:

Does **EFX** exist for **additive valuations**?  
Does **PMMS** exist for **additive valuations**?

## Our Goals:

Show **impossibility** for **weaker** assumptions.  
Show **possibility** for **stronger** assumptions.

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Theorem (**Negative**): There is an instance with **2 monotone** and **1 additive** valuation that admits no **PMMS** allocation.

**EFX** known to exist for **2 monotone** and **1 additive** valuations.  
Thus, **PMMS** for 3 agents requires new techniques.

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Theorem (**Negative**): For any  $n$ , there is an  $n$ -agent instance with  $n$  monotone valuations that admits no PMMS allocation.

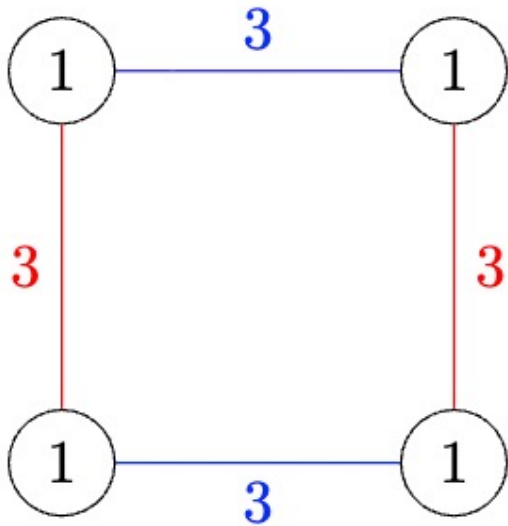
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Two agents (**red** and **blue**)

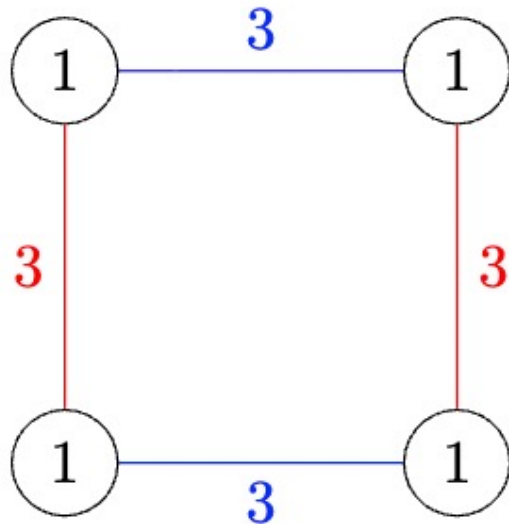


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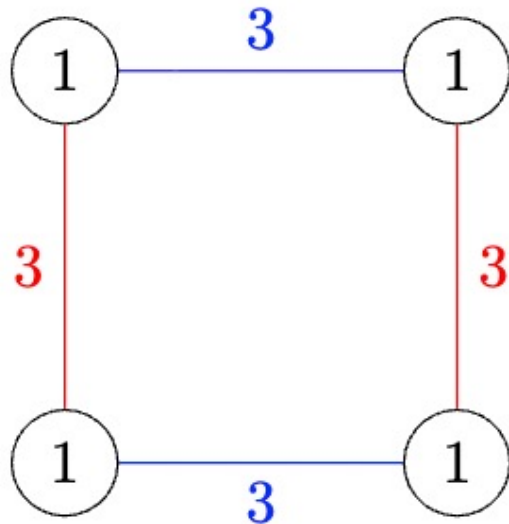
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**Nodes** show values for **single** items.  
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The MMS shares are  $\mu_1 = \mu_2 = 3$ .

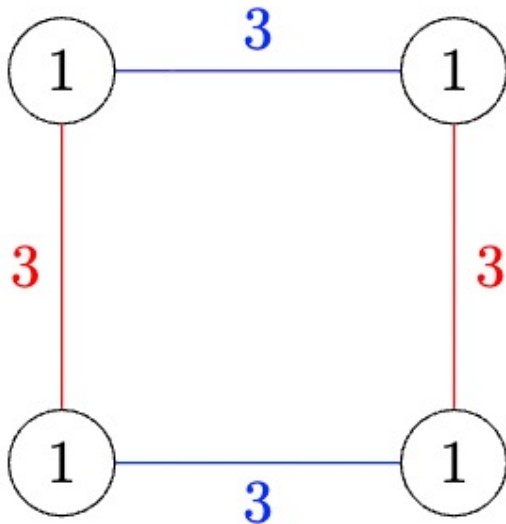
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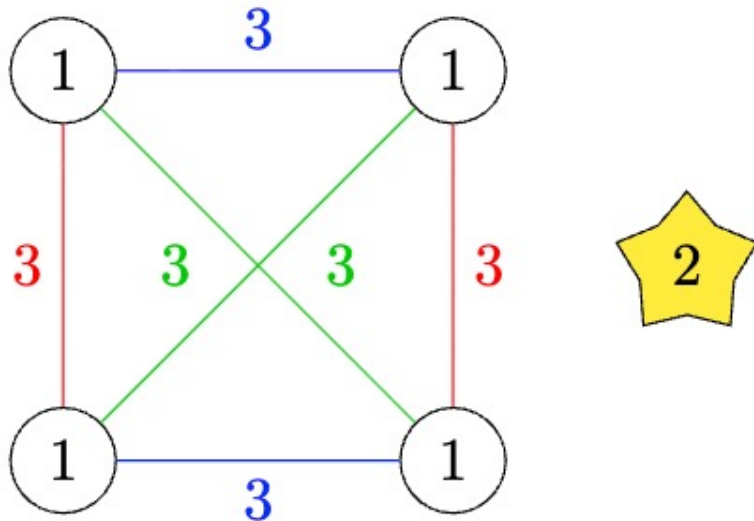
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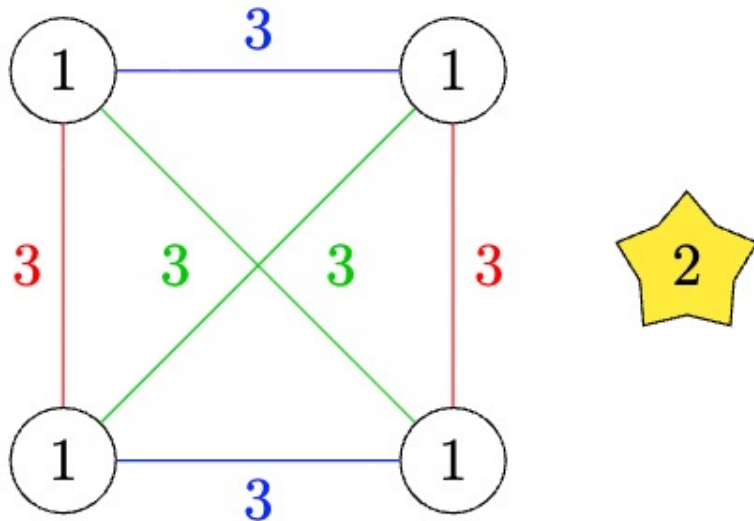
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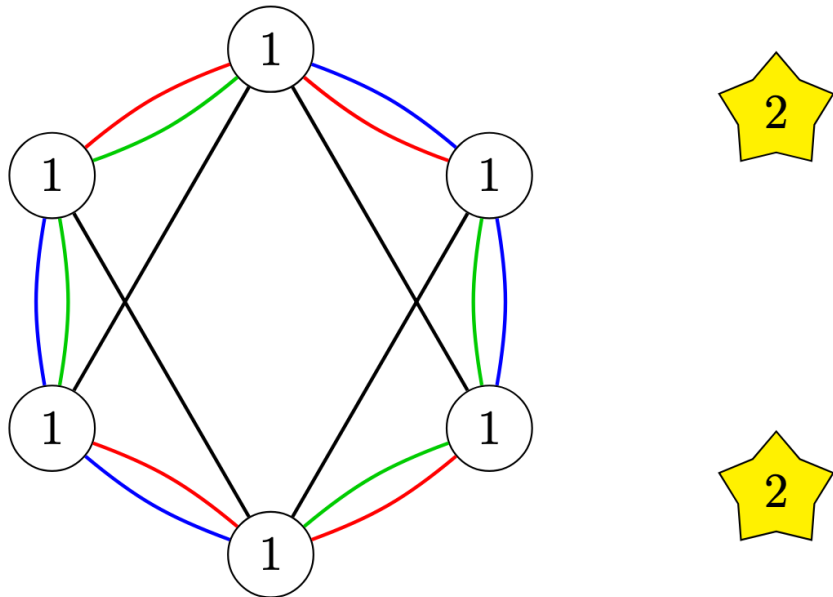
How to extend this to **4 agents**?

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Four agents  
(red, blue, green, black)



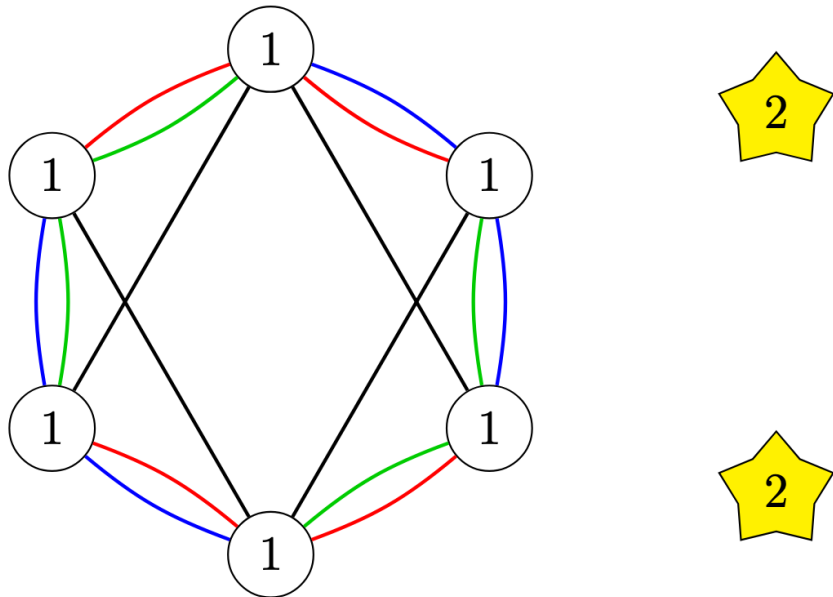
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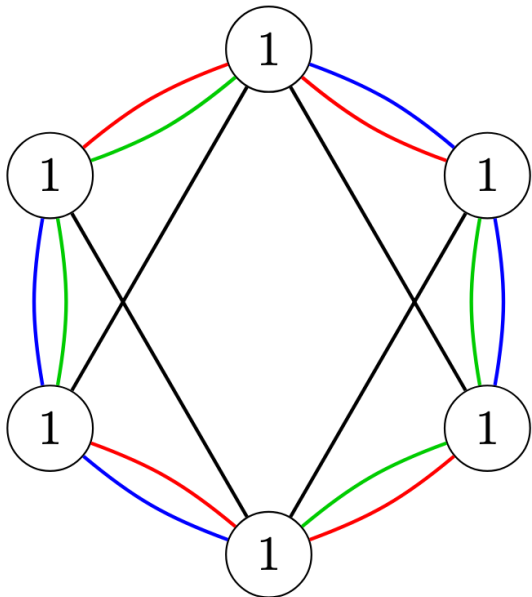
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Open Problem:  
Could  $(n - 2) + O(1)$  be enough?

# MMS-Feasible Valuations

Theorem (**Negative**): For any  $n$ , there is an  $n$ -agent instance with  $n$  **monotone** valuations that admits no **PMMS** allocation.

Question: Can extra assumptions on valuations help?  
We consider **MMS-feasible** vals, a generalization of **additive**.

Open Problem: Does **PMMS** exist for **MMS-feasible** vals?  
Open Problem: Does **PMMS** exist for **additive** valuations?

# MMS-Feasible Valuations

Valuation is **MMS-feasible** if  $\max\{v_i(X), v_i(Y)\} \geq \min\{v_i(A), v_i(B)\}$   
for any  $S \subseteq [m]$  and any two partitions  $X \uplus Y = S$  and  $A \uplus B = S$

# MMS-Feasible Valuations

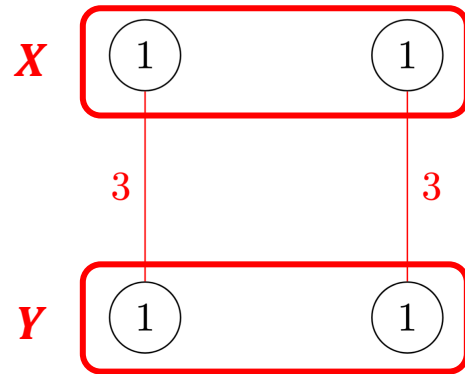
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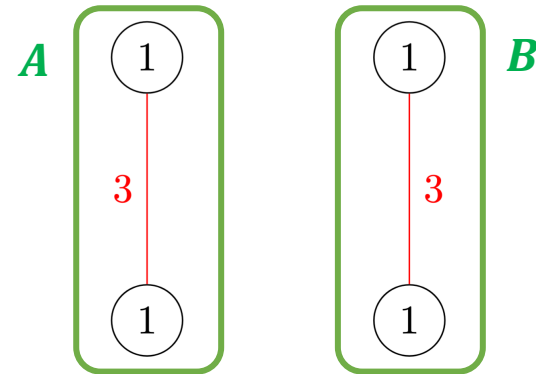
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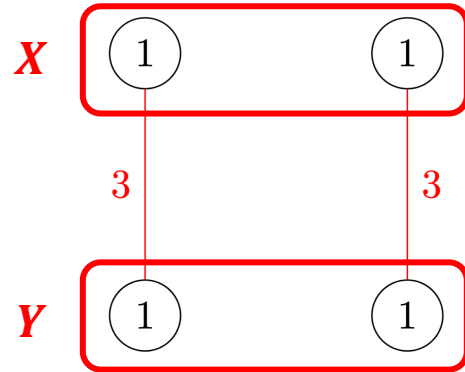


$$\min\{v_i(A), v_i(B)\} = 3$$

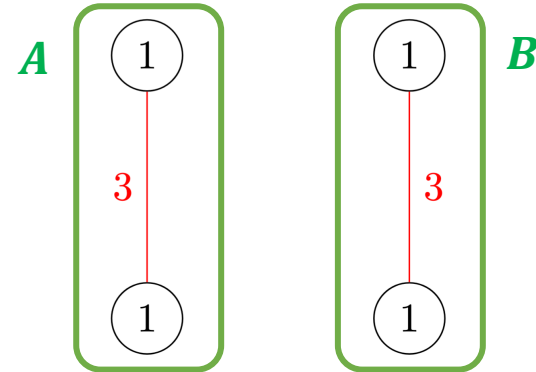
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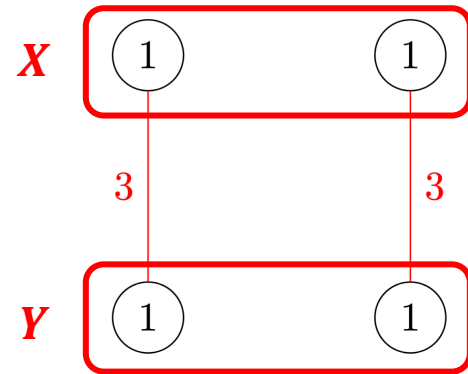
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Observation 2: Additive valuations **satisfy MMS-feasibility**.

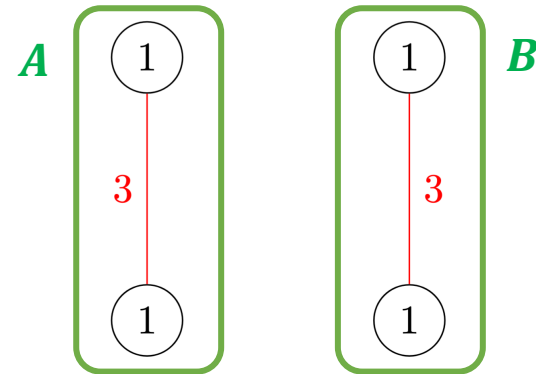
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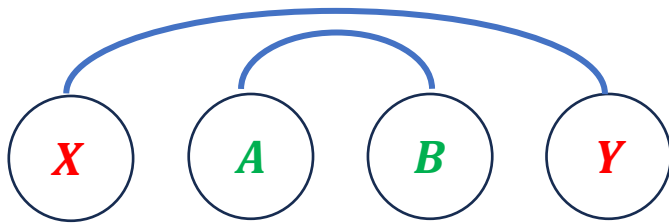
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Proof:  $\max\{v_i(X), v_i(Y)\} \geq (1/2) \cdot v_i(S) \geq \min\{v_i(A), v_i(B)\}$

# MMS-Feasible Valuations

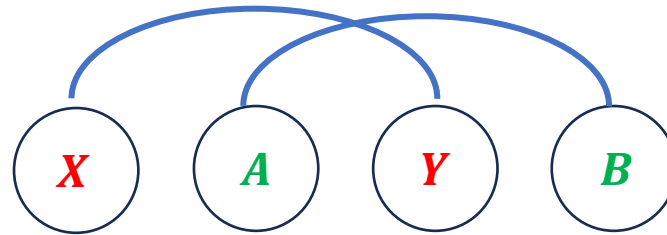
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Recall: We order bundles from least to most valuable and draw arcs between complements.



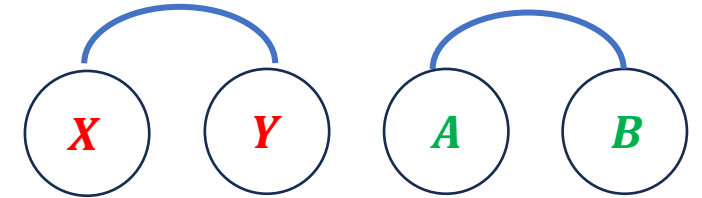
MMS-feasible ✓

Additive ✓



MMS-feasible ✓

Additive ✗



Not MMS-feasible ✗

Additive ✗

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Open Problems:  
Does **PMMS** exist for 1 monotone and 2 MMS-feasible?  
Does **PMMS** exist for 3 MMS-feasible?

# MMS-Feasible Valuations

**MMS-feasible:**  $\max\{v_i(X), v_i(Y)\} \geq \min\{v_i(A), v_i(B)\}$ .

| Assumptions for 2 agents    |
|-----------------------------|
| 2 monotone                  |
| 1 monotone + 1 MMS-feasible |
| 2 MMS-feasible              |

| EFX    | PMMS           |
|--------|----------------|
| exists | does not exist |
| exists | exists         |
| exists | exists         |

| Assumptions for 3 agents    |
|-----------------------------|
| 3 monotone                  |
| 2 monotone + 1 MMS-feasible |
| 1 monotone + 2 MMS-feasible |
| 3 MMS-feasible              |

| EFX    | PMMS           |
|--------|----------------|
| open   | does not exist |
| exists | does not exist |
| exists | open           |
| exists | open           |

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There are essentially  $(2^m)!$  valuations.

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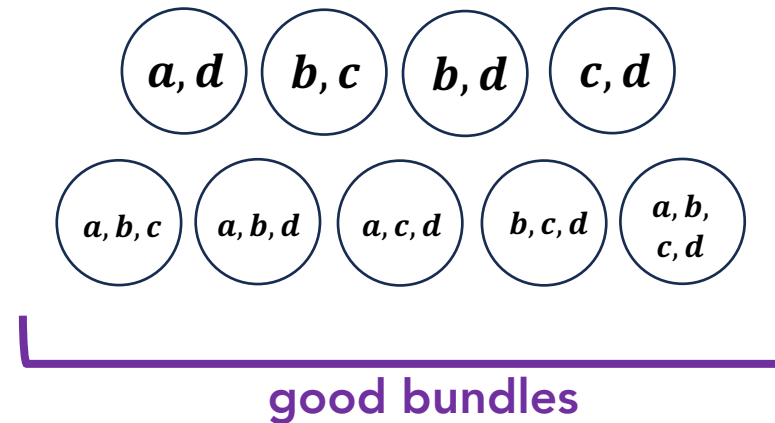
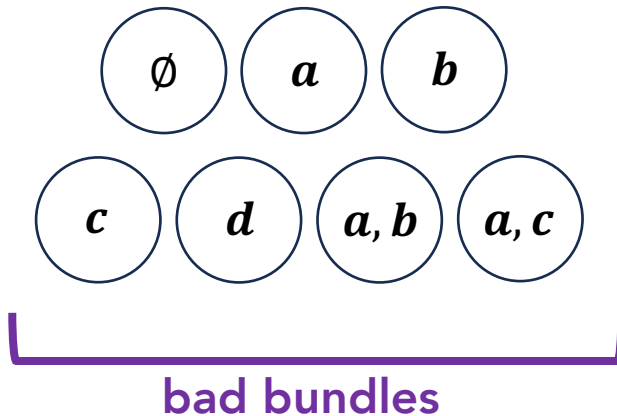
**Pair-demand**:  $v_i(X) = \max_{T \subseteq X, |T| \leq 2} v_i(T)$ .

There are  $(m^2)!$  such valuations.

# Positive Results

Theorem (**Positive**): **PMMS** exists for **binary-valued MMS-feasible** valuations.

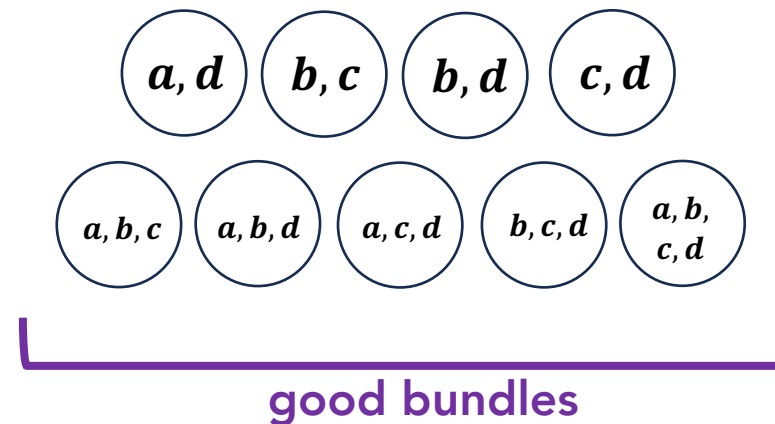
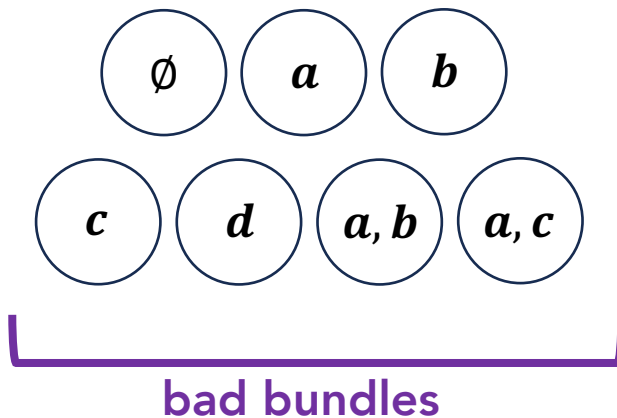
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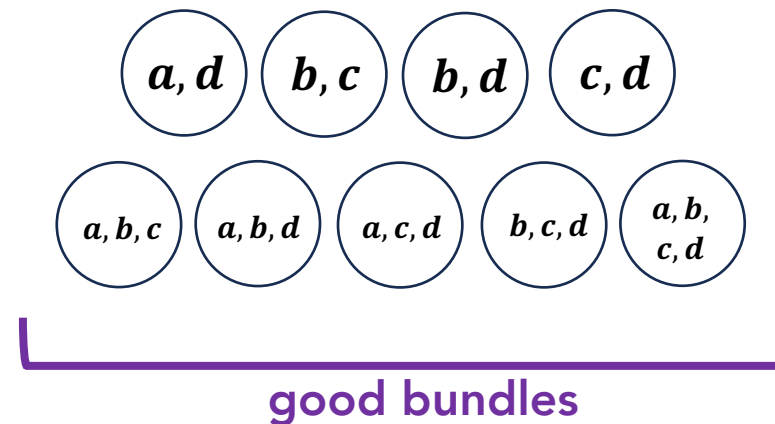
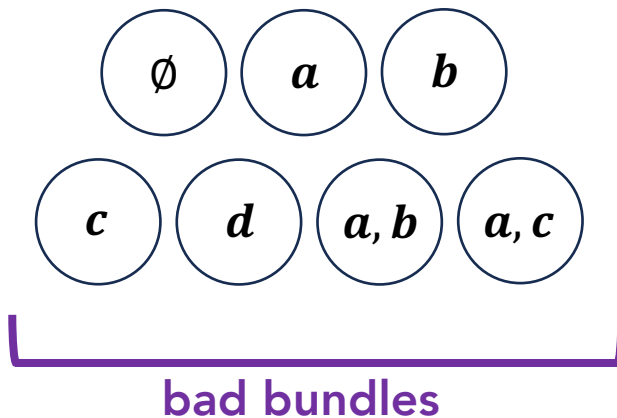


**Does not exist** for two **binary-valued** valuations without **MMS-feasibility**.

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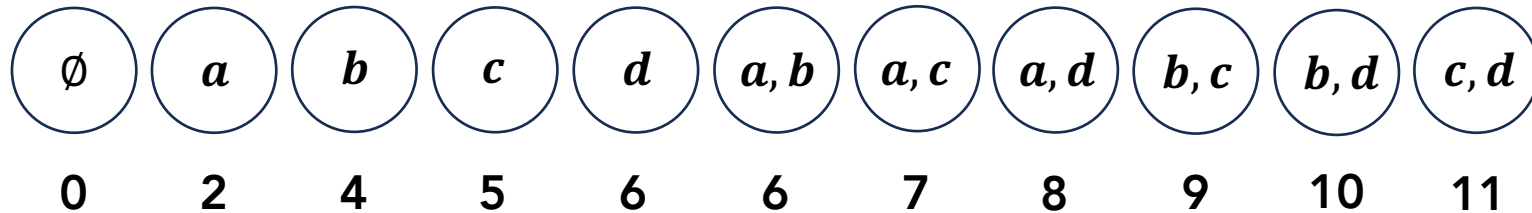
**Does not exist** for two **binary-valued** valuations without **MMS-feasibility**.

No monotonicity required! Applies to **goods**, **chores**, and **mixed manna**.

# Positive Results

Theorem (**Positive**): **PMMS** exists for additive pair-demand valuations.

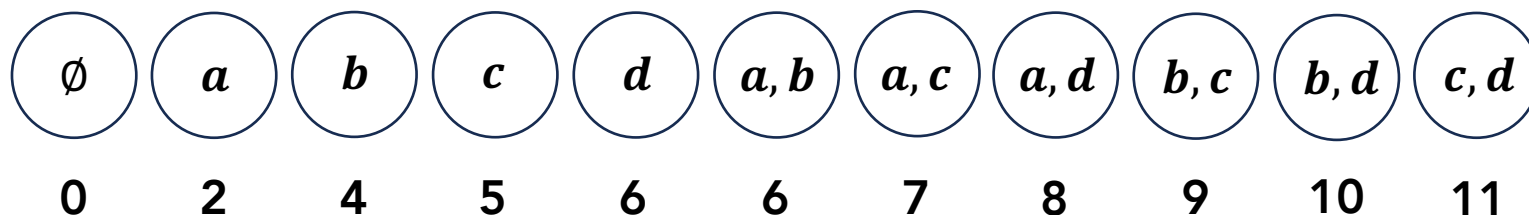
Additive pair-demand:  $v_i(X) = \max_{T \subseteq X, |T| \leq 2} \sum_{g \in T} v_i(\{g\})$ .



# Positive Results

Theorem (Positive): PMMS exists for additive pair-demand valuations.

Additive pair-demand:  $v_i(X) = \max_{T \subseteq X, |T| \leq 2} \sum_{g \in T} v_i(\{g\})$ .



Does not exist for two general pair-demand valuations.

Open Problem: Does it exist for MMS-feasible pair-demand valuations?

# Summary

Big Open Problems:

Does **EFX** exist for **additive valuations**?  
Does **PMMS** exist for **additive valuations**?

We can make progress by showing  
**impossibility** for **weaker** assumptions and  
**possibility** for **stronger** assumptions.

**PMMS** is an **interesting** and **stronger** variant of **EFX**.

**PMMS** requires **new techniques**, possibly useful for **EFX**.