##### Stability of stationary equilibrium solutions of a diffuse interface electrical breakdown model

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The aim of the present work is to study qualitative characteristics and to perform a numerical analysis of a diffuse interface model describing the development of an electrical breakdown channel in a solid dielectric. Stability of the system equilibrium positions is analysed. Conditions for the breakdown channel development from small perturbations of the intact medium are found. A differential scheme for the problem is constructed and investigated, an informative estimate of its stability is given. The obtained theoretical results are validated by a computer simulation.

Key words and phrases: diffuse interface model, phase field, stability, electrical breakdown.

## 1 Introduction

Electrical breakdown of a solid dielectric is a rapid process, which involves a variety of mutually interrelated physical mechanisms [1]. At present, it is practically impossible to identify and characterize them at the level acceptable for predictive first-principal modelling and simulation. Therefore, a promising approach to the electrical breakdown modelling is to utalize reasonably complex phenomenological models suitable for practical settings analysis.

Among the variety of approaches suggested to describe electrical breakdown, a particular place is occupied by the phase field (or diffuse interface) model originally introduced in [2]. According to the phase field modelling framework, the essentially one dimensional breakdown channel is described using a scalar phase field, i.e., a smooth function with values within the interval. It is assumed that the channel occupies a spatial domain where , while in completely undamaged medium . The spatial domain with separating damaged and undamaged medium is considered to be a "diffuse" boundary of finite width. The process of the breakdown channel development ("growth") is described as evolution of over time , governed by a physically motivated evolutionary PDE. The damaged and undamaged states of the medium are treated as two different phases. The process itself is described as phase transition between them. The transition occurs under specified conditions. Diffuse interface models are mainly phenomenological ones. In other words, they are mostly based on certain macroscopic laws and fundamental assumptions and do not rely on first-principal physics and elementary mechanisms related to the particular macroscopic problem under consideration.

Today diffuse interface models provide a solid and widely used framework to describe multi-phase phenomena in hydrodynamics [3–5], solid and fracture mechanics [6], material science [7], solidification and phase transition problems [8–10], phase field crystals [11–13].

The model suggested in [2] can be considered as a generalization of widely known diffuse interface models in solid mechanics. The model is investigated and further generalized in [14–16].

The main aim of this paper is to analyse stability of stationary equilibrium solutions of the phase-field model for the electrical breakdown development. In fact, it is shown that the development of the electrical breakdown channel in the considered model is closely related to the stability loss of the equilibrium solutions.

The model is considered in its simplest form, as it is presented in [2]. Moreover, a spatially one-dimensional setting is used to perform theoretical analysis. As a final result, a stability condition is formulated in terms of the model parameters. Whenever the condition is violated, the intact medium loses stability and small perturbations cause it to evolve into a breakdown-channel-like structure.

To confirm the obtained theoretical results, we present a simple explicit finite-difference scheme allowing to analyse the breakdown process numerically. Since the goal of our computations is to analyse the instability development of in the initial phase field distribution, a careful and comprehensive analysis of stability for the scheme is performed. The analysis is essential to clearly distinguish between numerical artifacts and the development of "physical" instabilities in the solution. The performed simulation completely confirms the obtained theoretical results.

The structure of the paper is as follows. In section 2 we present the mathematical model considered in the rest of the paper. Section 3 is devoted to the stability analysis of equilibrium solutions. In section 4 we analyse the finite-difference scheme. Finally, in section 5 we present numerical results confirming the theoretical findings. In conclusion, we outline and summarize the main results of the paper.

## 2 The mathematical model and problem statement

### 2.1 The mathematical model

Let us discuss briefly the phase-field mathematical model describing the electrical breakdown channel propagation, according to [2]. The model assumes that the electrical breakdown evolution is described as phase transition from the initial, undamaged, state of the medium to its damaged state. The spatial domain occupied by the damaged phase is considered as the breakdown channel. The breakdown channel development is described as a process of formation of the domains occupied by the damaged phase.

We assume that the medium occupies a bounded spatial domain . The state of the medium is given by two scalar-valued fields: , — the phase field and — the electric potential.

It is assumed that is continuous and suffucuently smooth with values . The value corresponds to the initial, undamaged state of the medium, and the value — to the damaged phase, which is related to the state of the medium in the breakdown channel.

The only property of the medium is its electric permittivity . Its value depends on the state of the medium and is given by:

Here is the permittivity of the undamaged medium; is the so-called interpolation function smoothly connects the and values. It is assumed that , , ; is a small regularizing parameter. Note that at we have (which corresponds to the undamaged phase) and at we have (which corresponds to the damaged phase, assumed to be an ideal conductor).

The following expression for the Helmholtz free energy of the system is postulated:

Here and are parameters; can be interpreted as the energy needed to create a breakdown channel of unit length and is the characteristic width of the diffuse interface.

Evolution of fields and is governed by the following system of equation:

Here is the so-called mobility. It has the meaning of the change speed of under a unit "force" applied. According to , the fields evolve in a way to minimize .

The explicit form of with given by is:

where and is the dot product. The first equation, , in this system is linear in . The second one, , is a nonlinear Allen-Cahn type equation.

### 2.2 One-dimensional problem

Consider a one-dimensional form of and . Let the closed domain be , with and being a closed interval. Let and also , — i.e., the distribution of electric permittivity and the initial condition for depend only on the -coordinate. We also assume that the following boundary conditions are defined at : , and also at the faces of perpendicular to the and axes. For we set: , , where , and also at the faces of perpendicular to the and axes. Here denotes the directional derivative along the outward unit normal to .

Taking the formulated assumptions into account, the solution of , has the form , , — i.e., does not depend on and , — on and .

Therefore, can be reduced to:

since as does not depend on and . The solution of satisfying the boundary conditions is .

Substituting the obtained solution for into we obtain:

The solution of is defined in the spatially one-dimensional domain . For further convenience we write as:

Note that now the parameters , and do not enter the considered equation explicitly.

Equation is supplemented by the initial condition

and also by the boundary conditions

For simplicity of further analysis we also assumed that .

So, the pair of functions and , where is the solution of the problem ,, and is given by , satisfies , under the provided assumptions.

## 3 Stability analysis of equilibrium solutions

Under certain conditions, the electrical breakdown can develop form small perturbations of the undamaged medium properties. To clarify these conditions in this section we study stability of constant solutions , of the equation .

First, one has to find stationary constant solutions of . From the definition follows the expressions for the derivatives of :

Substituting into and taking into account, one has:

First, consider the case , which leads to . Hence, and are equilibrium solutions.

Second, let . Then

Note that and, moreover, is monotonically increasing. Therefore, in case the equation has a solution given by

Otherwise the equation has only two solutions.

So, the number of constant equilibrium solutions depends on the following condition being satisfied:

It will be shown later how the condition is connected with the stability properties of the equilibrium solutions and the equation itself.

Let us now procees to the stability analysis of the equilibrium solutions.

Let be a solution of , be its perturbation. Writing down the equation for the perturbed solution , after linearizing we obtain the following equation for :

For further analysis it is convenient to write as:

where and are the respective parameters.

Choosing , one obtains from the following relation for the parameters of the perturbation:

from where follows:

Now it is easy to see that, depending on the value of the coefficient

three cases arise: [label=.]

1. . In this case, from it follows that , i.e., there exists a perturbation growing in time. Hence, the equilibrium solution is unstable.

2. . Then, for an arbitrary remains . Next, any perturbation in the interval can be represented as Fourier integral over harmonics decreasing at least as the harmonic for . Hence, the equilibrium solution is stable.

3. . Repeating the same reasoning as in the case , one can observe that there exist arbitrarily slowly decreasing harmonics (i.e., harmonics with arbitrarily small values of ). This case corresponds to neutral stability of the equilibrium solution, and linear analysis does not provide complete information. This case will be considered in more details later.

We now proceed to the discussion of the particular equilibrium states.

Consider the equilibrium solution . One has , (see ), which leads to . As it was noted before, this case requires an elaborate analysis, which will be performed later.

Consider the equilibrium solution . In this case , (see ). As a result, we obtain:

The equilibrium state is stable if , i.e., as:

For this case of an unstable equilibrium, let us find such that increasing harmonics are replaced by the decreasing ones. To do this, consider with , and given above to obtain:

from where follows:

Note that the condition is exactly the right-hand side of the inequality . To explain this and to form a complete picture of what is happening, let us look at the equilibrium solutions from a slightly different perspective.

Solving the equation , we were finding the zeros of the function

Hence, each equilibrium solution uniquely corresponds to a zero of the function . From the derivation of the equation for the perturbation it follows that in its right-hand side the coefficient at is . Later, analyzing the equation for an equilibrium solution , we considered several cases depending on the sign of the coefficient , which turns out to be exactly .

Summing up the results, one can state the following. The function defined by is smooth on and always has zeros and . The third zero exists under the condition . Each equilibrium solution uniquely corresponds to a zero of the function . Their stability properties are described in terms of the sign of at the zeros: positive values of correspond to the unstable solution and negative ones — to the stable one.

It is also clear that in the case of vanishing (as for ) the linear analysis is not enough — it is necessary to analyse the sign of the first higher-order non-vanishing derivative of — the equilibrium solution is stable if this derivative is negative and unstable if it is positive.

Finally we show that has a non-vanishing derivative at its zero (if the latter exists). Indeed, one has:

Taking into account that , one obtains:

Then:

Now it is possible to provide a comprehensive analysis of the behavior of at its zeros. As it can be seen from the conditions and , its behavior is governed by the value of the parameter

First, consider the case . The zeros of are and ; , . The qualitative behavior of is shown schematically on Fig. 1. It can be seen that the equilibrium solution is unstable and is stable. Such case can be conventionally called the case of "weak electric field". This means that with all the parameters except the electric field being fixed, the latter is so small that even an almost completely damaged medium with is "healed" over time and evolves to the completely undamaged state .

Second, consider the case . The zeros of are , (see ) and ; (since is smooth). The behavior of in this case is shown on Fig. 2. The equilibrium solutions are: — stable one, — unstable one, and — also stable. Such case can be conventionally called the case of the "medium electric field". This means that as the values of are sufficiently close to , the damage increases, i.e., tends to zero; as the values of are sufficiently close to , the damage decreases, i.e., tends to one; at certain intermediate values the equilibrium is unstable.

Finally, consider the case . The zeros of are and ; , . The qualitative behavior of is schematically shown on Fig. 3. The equilibrium solutions are: — the stable one, — the unstable one. This case can be conventionally called the case of "strong electric field". This means that the electric field is sufficiently strong and any state arbitrarily close to the completely undamaged one (i.e., any state close to ) evolves towards the completely damaged state . Essentially this is the case where the completely damaged state develops from arbitrarily small perturbations of the completely undamaged equilibrium solution.

In all the three cases stability of the equilibrium solution is defined by the higher order derivatives of .

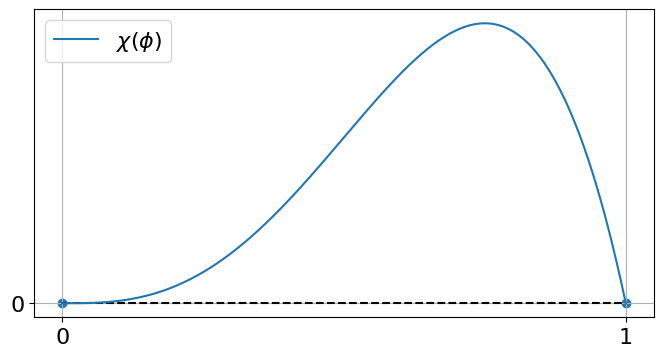


Figure 1: Characteristic behavior of , "weak electric field" case.

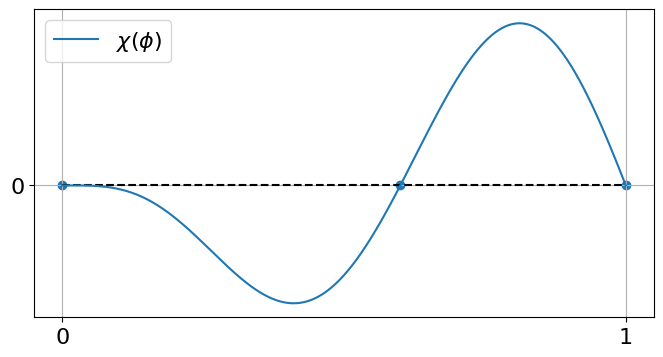


Figure 2: Characteristic behavior of , "medium electric field" case.

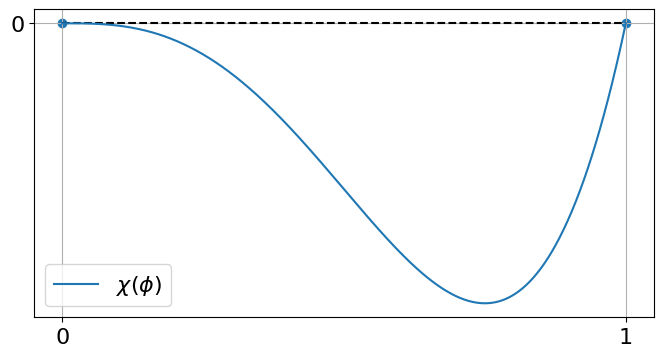


Figure 3: Characteristic behavior of , "strong electric field" case.

## 4 The finite-difference scheme

In this section, we present a finite-difference scheme for solving the equation in the domain . The equation is subjected to initial conditions and boundary conditions .

Consider a regular mesh with a time step  and spatial step . Let  with  being the number of nodes. The nodes of the spatiotemporal grid are given by , , . Define by  the value of a mesh function  at the node . Then the finite-difference approximations read



or, in the explicit form,





It is easy to see that the scheme has the first order of approximation in time and the second order of approximation in spatial terms.

To study properties of the scheme , , the linear theory can be used (see, e.g., ?, Chapter~10 or ?, Chapter~IX). The central result of the theory states, in a somewhat simplified form, that if a finite-difference scheme is stable and approximates a continuous problem then the solution of the finite-dimensional problem converges to the solution of the continuous one with order not lower then the order of approximation.

To apply this result for the nonlinear setting , , we proceed as follows: (i) linearize the equation for a fixed  and then (ii) apply the spectral stability argument ? to the derived linearized equation. As the stability criteria are satisfied for the linearized equation, stability should be expected for the complete, nonlinear, problem. In this case, convergence of the approximate solution should be expected as well — since the finite-difference problem is stable and approximates the continuous one. The results of such non-rigorous analysis will be further confirmed by numerical computations in the fully nonlinear setting.

### 4.1 Stability estimate

In this section we derive a stability condition for the finite-difference scheme , using the so-called principal of "frozen coefficients" (see, e.g., ?). Let  and  be solutions of the finite-difference equation . Substitute  into to obtain:





Linearizing this equation around , assuming that perturbations  are small and taking into account that  is a solution of the finite-difference problem, we obtain:



We now apply spectral stability analysis to the derived equation for perturbations. Let , . Substituting this representation into one obtains:



or



According to the spectral stability argument, a time step  provides stability of the scheme in the domain  with  as  if there exists  such that for an arbitrary  it holds . Note that here it is also possible to use more strict condition . If for an arbitrary  it holds , then stability will be provided also for an unbounded time interval, i.e., for . Strictly speaking, the spectral argument does not provide a sufficient stability condition; however, stability should be expected in practice.

First, consider the expression for . We have , , and the equation takes the form of



Hence, for an arbitrary  it holds  if and only if



As the condition is satisfied, one can expect stability of the scheme when the solution describes an almost completely damaged state  in the domain .

Note that under the condition one also can expect stable computations for  for an arbitrary value . In this case the following is true:



Hence, there exists  such that  holds, — since  and  are continuous on . It should be noted that, despite such versatility, the estimate is poorly applicable in practice and requires clarification, which will be done later.

We now consider the expression at the value . Note that , . We see that for  it is possible to achieve  with demanded sufficiently small values of  and the condition , similar to the one for . Substituting  (see ), we obtain



So, under the condition , it is expected that there exist such values of  and  that the difference scheme is stable for  and . Naturally the condition is equivalent to the stability condition for the equilibrium state  of the equation .

### 4.2 Improved stability estimate

In the previous section form the analysis of equation it was derived stability condition for finite-difference scheme and for . The assumption of its usefulness is based on the fact that typical "'natural" solution of the model will has a form of the transition process from the undamaged state  to the completely damaged state  occurring in a finite time interval and then infinitely long staying in the damaged state .

However the performed analysis of the equation is not sufficient at . Indeed, it was used that at ,  (see expression ), — but it was not accounted that  growth fast and reaches large values for small values of , see Fig. 4. This means that the equations of the model are stable at , but can be unstable in the small neighbourhood of . Such situation is not satisfactory and we now try to improve the obtained stability estimates.

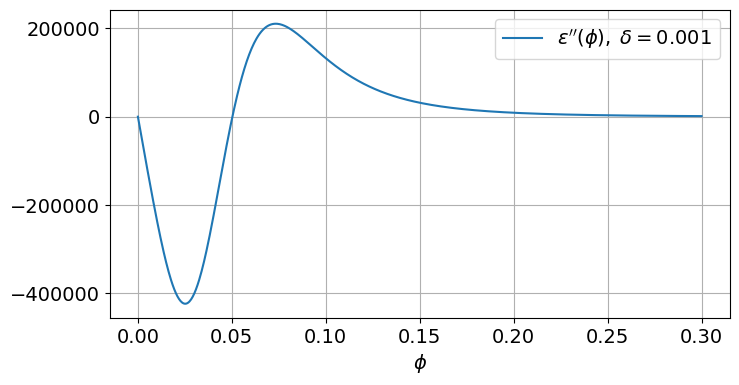


Figure 4: Typical behavior of  in the vicinity of .

To proceed let us estimate extremums of  in the neighbourhood of . First, find zeros of . We have



form where:



or, taking into account:



Let  and  such that  is bounded. Then:





Hence, a sequence  has not more than two partial limits  and  — which are zeros of the equation . To the first zero  it corresponds



to the second zero  it corresponds



From here it can be seen that for  the function  has two zeros in the neighbourhood of :



We now estimate  at  for . Let , . Then:



and:



The derived estimates are shown as black dashed lines on Fig. 5.

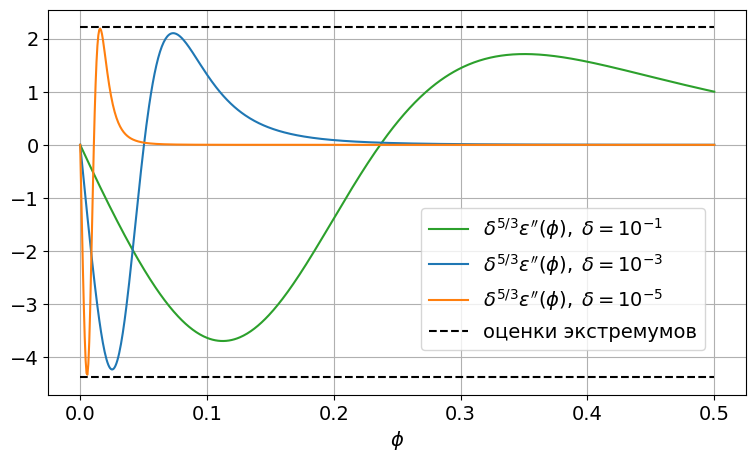


Figure 5: Qualitative behavior of  for small values of .

Now, to derive new stability estimate we consider equation at . Note that . The term inside braces in is negative since  is small and  is negative and large in its absolute value. Therefore  can be estimated as  — such estimate makes inequality stronger. Then from inequality it follows that



Condition  is satisfied for an arbitrary , if and only if



Numerical experiments described in the next sections indicates that more strong version of the estimate is also valid (note the doubled denominator):



Finally, more simple estimate not weaker then is:



Note that the derived stability estimate for finite-difference scheme , includes all the parameters of the equation , except . Notably, this is the only parameter of the model which has somehow artificial nature and can not be related directly to the underlying physics.

## 5 Numerical studies

In this section we present some numerical results obtained using finite-difference scheme ,, presented above. The main goal of the simulations is to confirm theoretical stability estimates derived for the model itself and its approximations, i.e., — convergence and stability of the finite-difference scheme (see section ) and stability properties of the equilibrium solutions described in section 3. Besides this we will check temporal behaviour of the total free energy of the system.

### 5.1 Typical solution

Let us set the parameters of the equation as follows:



Note that this set of parameters corresponds to the "strong electric field" case, see .

The equation is solved in spatiotemporal domain



Boundary conditions are defined as follows:



Note that  is a twice differentiable function everywhere except finite number of points; more over, its second derivative is bounded.

Let  be the number of grid steps on  (the number of nodes is, respectively, );  — be the number of mesh steps on . Spatial and temporal mesh step sizes are given by .

On Fig. 6 the typical solution is presented. It easy to see gradual evolution of the breakdown channel (identified as a spatial domain where medium is damaged) staring from the sufficiently small perturbation of the initial, completely undamaged, state. Approximately at time  the medium in the breakdown channel becomes completely damaged: the values of  in the neighborhood of  is closed to zero value. Note that for  the breakdown channel (identified as a spatial domain where  essentially differs from 1) practically does not change its width which is equal approximately to , as supposed by the model. In turn, for  when  reaches its minimal values, it start to grow wide with almost constant velocity.

### 5.2 Evolution of free energy

In the model under consideration, the free energy of the system is given by with its density given by . In finite-dimensional setting, the terms in are approximated in the obvious way. The only essential addition we need to mention is approximation of first derivative, i.e., . Further the simplest approach is considered which uses standard 1st-order approximation of .

To simulate solution of the system, we use the same finite-difference scheme, initial and boundary conditions as in the previous section.

The results are given in Fig. 7 where dependence  is shown. The colored vertical dashed lines correspond to the same moments of time as on Fig. 6. It is interesting to note that for  free energy  is almost constant; further, for  it starts to decrease sufficiently fast until . Later, after  the energy decreases linearly, until the complete damage of the medium.

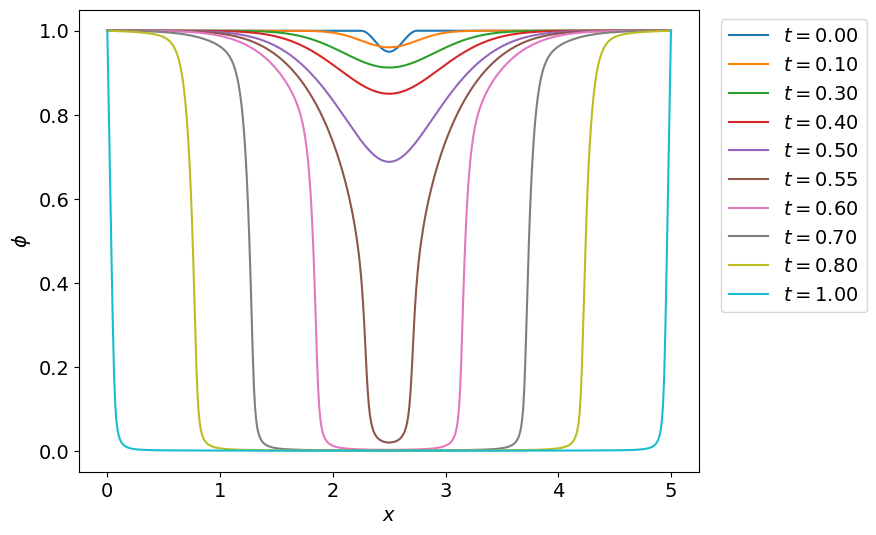


Figure 6: Typical solution of the problem, , .

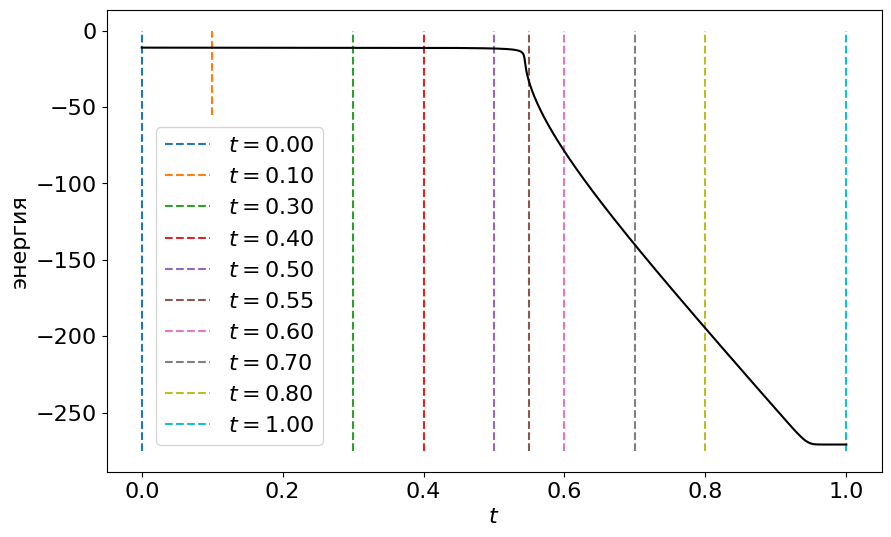


Figure 7: Free energy evolution.

It can be seen from the derivation of and , that during its evolution, the system tends to minimize its free energy. Due to this, the correct monotone evolution of free energy is of the crucial importance for simulation to be qualitatively correct. We do not provide here the strict proof of the correct behavior of the scheme (which is an important, but standalone task), but rather check it in the performed simulations. This means that under given parameters of the model and discretization, the scheme is gradient-stable (or, which is same, energy stable).

### 5.3 Stability of the scheme

In this section we check stability estimate of the finite-difference scheme.

Let the model parameters be defined according to and , boundary conditions are given by . We assume by convention that the scheme is unstable as soon as the simulation finishes abnormally with floating point overflow due to division by zero (e.g., in expression for  at ) or values of  go to infinity (as in overflow of variables of the type double). We then increase  and , keeping information about pairs of values at which stable approximation turns into unstable one. Plotting these values we can depict stability region of the scheme on the – plane. Such plot is presented on Fig. 8 together with theoretical boundary given by the stability estimate .

Numerical experiment demonstrates that the estimate is successful and sufficiently sharp: theoretical and experimental stability boundaries follows each other and are relatively closed. Moreover, the theoretical curve lies above the experimental one which means that theoretical estimate is more strict then the experimental one. Actually this was a reason divide right-hand side of the original estimate by two.

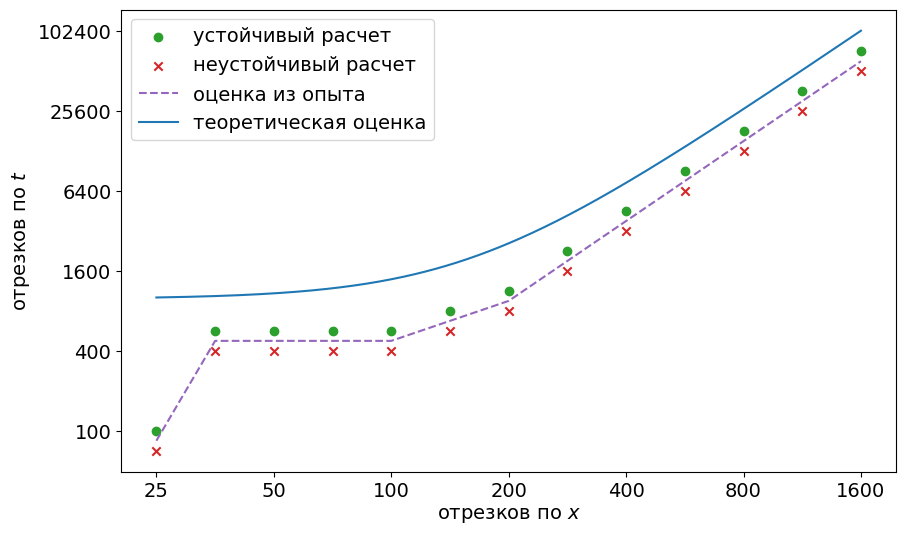


Figure 8: Theoretical and experimental stability boundaries.

### 5.4 Convergence

Approximation property of the finite-difference scheme , for solution of , and is obvious. Stability of the scheme is provided by the theoretical estimate , which was verified against numerical experiments. We now check convergence of the finite-difference solution.

Define  and  norms as



on the space  of twice differentiable functions in the closed spatiotemporal domain .

Assume that computational mesh  together with some dependence . Restricting functions from  on the mesh , we obtain a space of mesh functions . Functional norms in the space  can be defined as



Now consider the results of the numerical experiments. Convergence will be analysed using norms  и  defined on the space of mesh functions defined above. Since analytical solution of the problem is not known, comparison will be performed using a sequence of approximate solutions against the solution obtained using the finest mesh. To compute difference between two finite-difference solutions, we restrict solution obtained using more fine grid on the more coarse grid, ignoring fine solution values at the respective mesh nodes.

To proceed we set the parameters of the model as described above, see , . Let  be the number of mesh steps in spatial domain,  be the number of mesh steps in time domain. In all simulations stability condition is satisfied.

First, we set  and perform a sequence of simulations with gradually increasing , doubling it at each step. Comparison of the obtained solutions with the one obtained for  is shown in Fig. 9. Simulated results clearly shows that the scheme produces solutions with first order convergence rate to the "exact" one in time  which confirms theoretical estimates; the scheme has  accuracy.

Second, we set  perform a sequence of simulations with increasing , again, doubling it at each step.

Comparison of the obtained solutions with the one obtained for  is shown in Fig. 10. Simulated results clearly shows that the scheme produces solutions with second order convergence rate to the "'exact" one in spatial variable  which confirms theoretical estimates; the scheme has  accuracy.

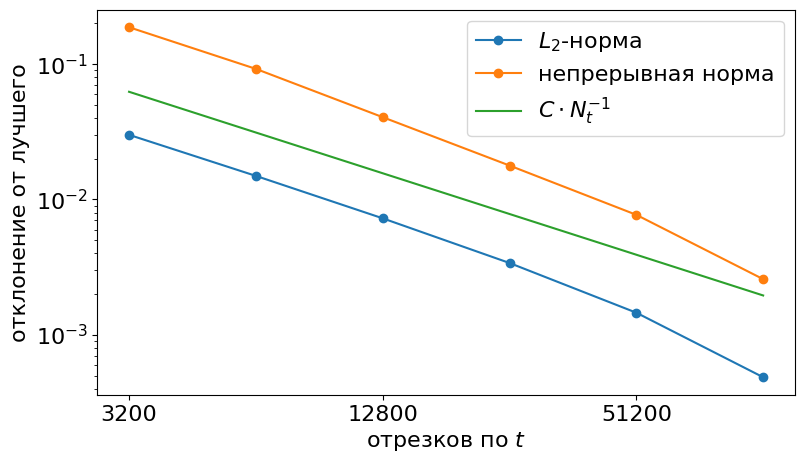


Figure 9: Error norm for fixed .

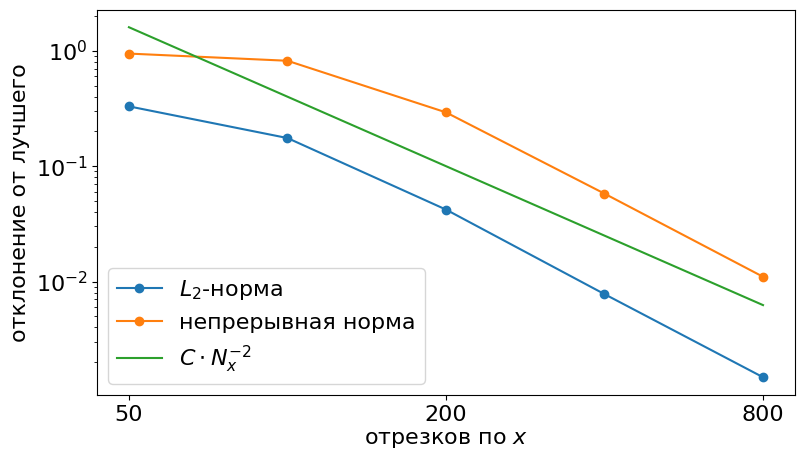


Figure 10: Error norm for fixed .

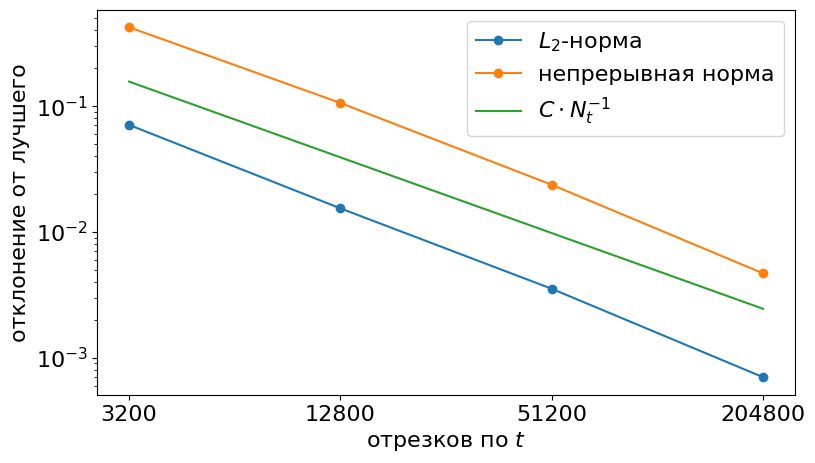


Figure 11: Error norm for .

We now refine both  and  in such a way that for the given values of  and   it holds stability condition . For the parameters of the model defined above, one can set . Computations of the errors between a sequence of approximate solutions and the reference one is performed in the same way as earlier; the results are presented on Fig. 11. As it is expected, the convergence order is  as soon as spatial and temporal discretization parameters are related as described above.

Note that in the first two experiments, the limiting solution was not actual solution of the differential problem since only one discretization parameter was tending to its limiting value. The second one had the fixed value and, therefore, reference numerical solution always had irremovable approximation error. In the third experiment, under assumption that the scheme is stable, numerical solutions converges to the solution of the initial differential problem , , .

### 5.5 Stability of the equilibrium states

Earlier we studied equilibrium solutions  of the equation . Their number and stability type are defined by the value of the parameter  given by .

Define parameters of the model according to and , — except the value of  which will be defined later. As the initial condition, we set perturbation of the constant equilibrium state: . The amplitude  is considered to be relatively small, i.e., . A number of computational steps is given by  and .

If the equilibrium state is stable, then for an arbitrary  perturbations in the initial condition decay; if the equilibrium state is unstable, then there exists some , such that for any  perturbations increase in time.

Let us set , , . For , as it was defined above, one has , , .

First, consider ,  — which is the case of "weak electric field". In this case the system has two equilibrium states:  — the unstable one and  — the stable one. On the Figs. 12 and 13 it is easy to see theoretically predicted evolution of the solution: for  perturbations growth, for  — perturbation decreases. For value  derivative of the function  (see ) vanishes, therefore, to observe growth of perturbations, it is necessary to define sufficiently small values of , which, in turn, provides sufficiently small value of . Note that in the experiment with , we take , in order to keep the values of  below .

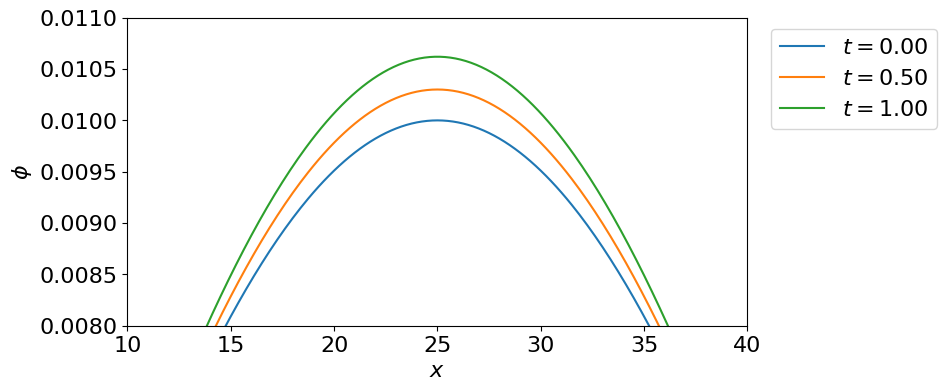


Figure 12: "Weak electric field" case: perturbed equilibrium state , unstable.

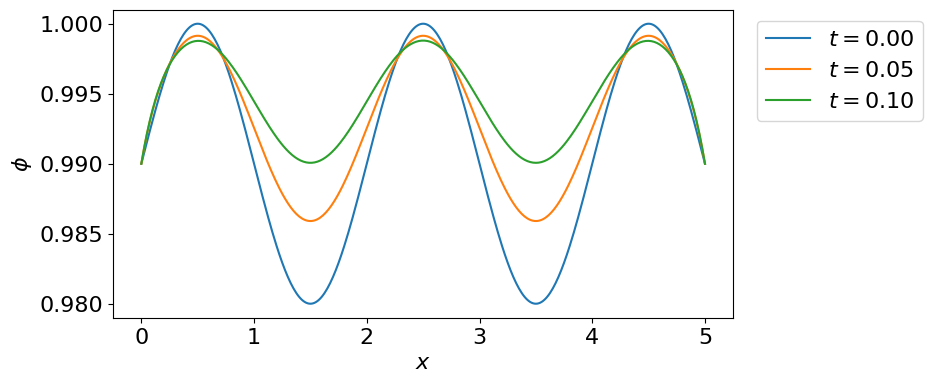


Figure 13: "Weak electric field" case: perturbed equilibrium state , unstable.

Consider now ,  — i.e., the case of "medium electric field". In this case the system has three equilibrium states:  — stable,  — unstable ( is a zero of  in the interval ),  — stable. Evolution of the perturbed solution is shown on Figs. 14, 15 and 16. As it can be seen, the observed behavior is in accordance with the theoretically predicted one.

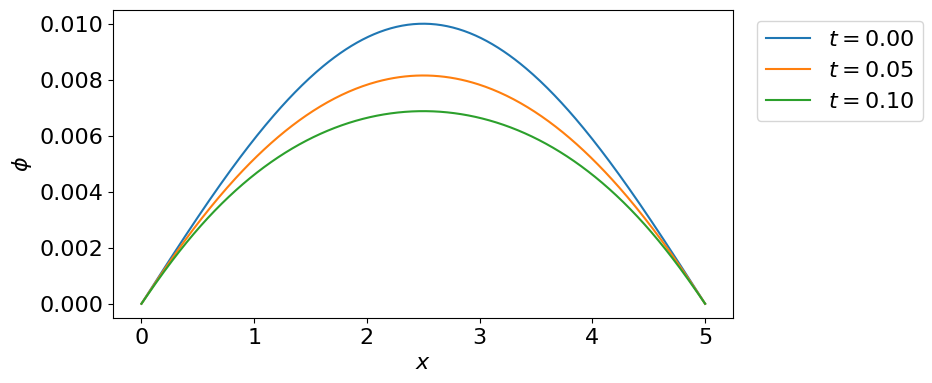


Figure 14: "Medium electric field" case: perturbed equilibrium state , stable.

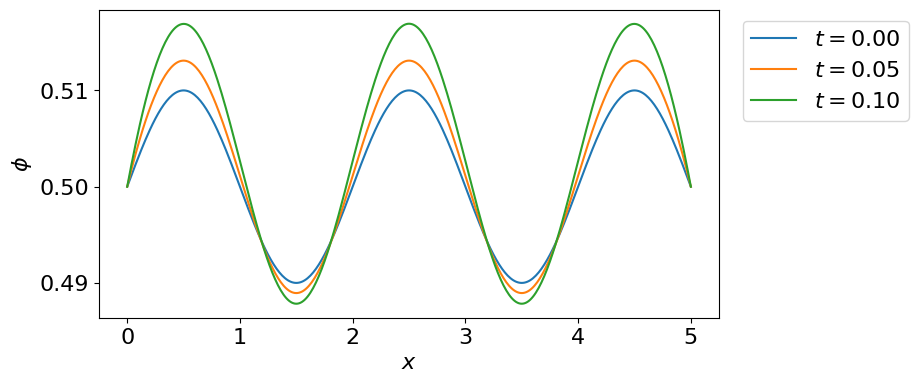


Figure 15: "Medium electric field" case: perturbed equilibrium , unstable.

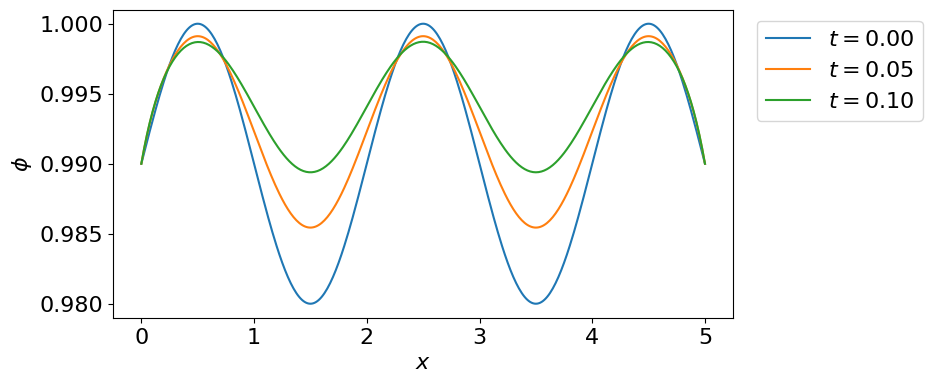


Figure 16: "Medium electric field" case: perturbed equilibrium state , stable.

Finally, consider the case of ,  — which correspond to the "strong electric field" case. The system has two equilibrium states: , stable, and , unstable. Evolution of the perturbed solution is shown on Figs. 17 and 18. As in other considered cases, the results coincide with the theoretical predictions.

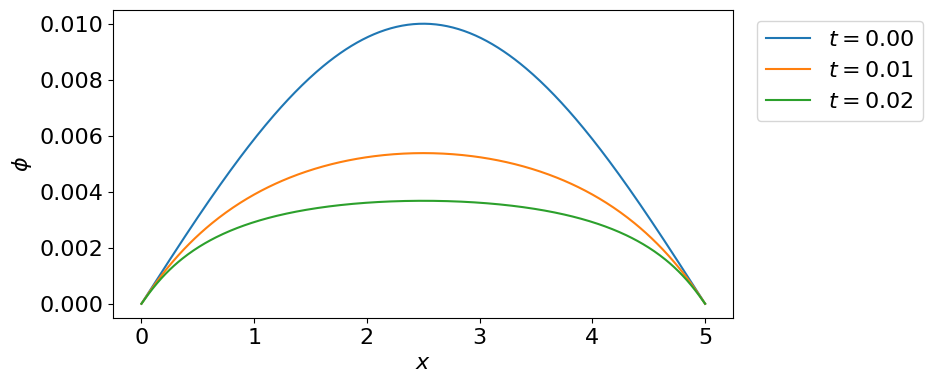


Figure 17: "Strong electric field" case: perturbed equilibrium state , stable.

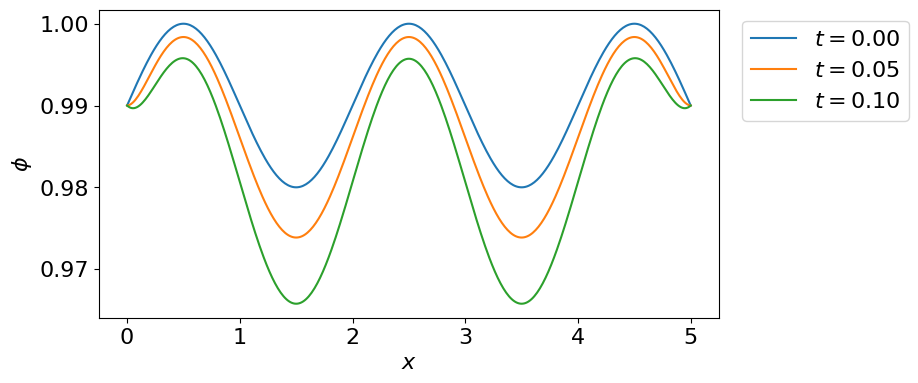


Figure 18: "Strong electric field" case: perturbed equilibrium state , unstable.

## 6 Conclusions

In this paper we study stability properties of the phase-field model for electrical breakdown channel evolution. The central result is a classification of the equilibrium solutions of the model and their stability. From practical point of view, these results allows to make meaningful conclusions regarding qualitative and quantitative properties of the model. Particularly it was shown under which conditions small perturbations of the equilibrium solutions develop into channel-like structure typical for of electrical breakdown process.

Besides this, a simple explicit finite-difference scheme for solution of the model in spatially one-dimensional setting is considered. The main question addressed here are stability conditions which guaranties correctness of the simulations. Deep connections between stability conditions of the model and the one of the finite-difference scheme are shown. The presented results of the numerical simulations confirms predictions of the theoretical analysis of the model.