Markov-Functional Interest Rate Models*

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Abstract We introduce a general class of interest rate models in which the value of pure discount bonds can be expressed as a functional of some (low-dimensional) Markov process. At the abstract level this class includes all current models of practical importance. By specifying these models in Markov-functional form, we obtain a specification which is efficient to implement. An additional advantage of Markov-functional models is the fact that the specification of the model can be such that the forward rate distribution implied by market option prices can be fitted exactly, which makes these models particularly suited for derivatives pricing. We give examples of Markov-functional models that are fitted to market prices of caps/floors and swaptions.

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1. Introduction

Amongst practitioners in the interest rate derivatives market a consensus is starting to emerge as to the desirable and most important properties of an interest rate pricing model. These properties stem from the way these models are used in practice. To determine the prices of exotic derivatives, pricing models are used as 'extrapolation tools'. First the model parameters are chosen so that the model values of relevant liquid instruments agree with market prices, then the model is used to price the exotic. From this perspective, the properties of a good pricing model for derivatives can be summarised as follows. The model should

a) be arbitrage-free;
b) be well-calibrated, correctly pricing as many relevant liquid instruments as possible without overfitting;
c) be realistic and transparent in its properties;
d) allow an efficient implementation.

So far, models proposed in the literature have not been able to combine all four requirements. The contribution of this paper is to present a modelling framework within which one can develop models possessing all four attributes.

The *conditio sine qua non* for all models is, of course, freedom from arbitrage. The general framework for arbitrage-free interest rate models was laid out by Heath, Jarrow and Morton (1992), HJM hereafter. All models we will consider are arbitrage-free interest rate models and representatives of the class of HJM models.

The traditional approach for specifying an interest rate model is to take one or more mathematically convenient underlying variables and to make certain distributional assumptions to reflect the future uncertainty of these variables. One possible choice for the underlying variables are the instantaneous forward rates. This is the setup for the model proposed by HJM. Unfortunately, for most distributional assumptions one can make for the forward rates (in the HJM framework this is known as 'specifying the volatility function'), the dynamics of the forward rates become path-dependent which makes numerical implementations of the model very cumbersome. An interesting sub-class was proposed independently by Cheyette (1992) and Ritchken and Sankarasubramanian (1995). They find restrictions on the volatility functions such that the path-dependency can be reflected by one additional state-variable, which makes reasonably efficient numerical implementation possible. Instead of explicitly restricting the volatility functions one can also choose not to model the complete forward curve directly, but to focus attention on a single rate, the instantaneous short rate. A few of the best known examples of this approach are the models proposed by Hull-White (1994), Cox-Ingersoll-Ross (1985) or Black-Derman-Toy (1990) / Black-Karasinski (1991). These models assume that the short rate has a normal, chi-square or log-normal distribution respectively. The attractive feature of these models is their simple and efficient numerical implementation.

A general problem with models from the traditional approach is that neither the instantaneous forward rates, nor the instantaneous short rate can be observed in the markets. To make matters worse, prices for instruments which can be observed in the markets such as caps, floors and swaptions often depend in complicated non-linear ways on the underlying model parameters. Hence, to replicate market prices one chooses a set of reference instruments and solves for the 'correct' parameter values by solving a non-linear optimization problem such that the pricing error of the model is minimised. These procedures can be numerically quite intensive and are known to be plagued by numerical instabilities.

A radical departure from the traditional approach has emerged recently in the literature. Instead of using some unobserved rates as the underlying variables, these new models take rates which are traded in the markets such as LIBOR or swap rates as the underlying variables. Consequently, these new models have become known as 'Market Models'; they were introduced independently by Miltersen, Sandmann and Sondermann (1997) and Brace, Gatarek and Musiela (1997), and have been extended by Jamshidian (1998). In the case when market option prices are given by some simple formula such as Black's formula it is easy to find specifications of these models such that market prices can be fitted exactly and model parametrization is therefore trivial. However, the dynamics of forward rates are, just like in the HJM models, path-dependent which makes efficient numerical implementation very difficult, especially for American-style products. Furthermore, when market prices are not given by Black's formula, as is the case in currencies such as Yen where interest rates are very low, model parametrization is no easier than for the earlier models in the literature (see, for example, Andersen and Andreasen, 1998).

In this paper we propose models that can fit the observed prices of liquid instruments in a similar fashion to the Market Models, but which also have the advantage that prices can be calculated just as efficiently as in the short-rate models of the traditional approach. To achieve this we consider the general class of *Markov-functional interest rate models*, the defining characteristic of which is that pure discount bond prices are at any time a function of some low-dimensional process which is Markovian in some martingale measure. This ensures that implementation is efficient since it is only necessary to track the driving Markov process. Market Models do not possess this property (for a *low*-dimensional Markov process) and this is the impediment to their efficient implementation. The freedom to choose the functional form is what permits accurate calibration of Markov-functional models to relevant market prices, a property not possessed by short-rate models. The remaining freedom to specify the law of the driving Markov process is what allows us to make the model realistic.

The idea of describing interest rate models in terms of functional forms of an underlying Markov process has also been proposed in the papers by Hagan and Woodward (1997) and Schmidt (1997). The focus of these papers is primarily on the form of the Markov process. By contrast, our focus is on the functional forms and showing how they can be chosen such that the Markov-functional model has certain pre-specified properties. In particular, we show how to generate Markov-functional models that replicate market prices for LIBOR- and swap-based instruments. In a forthcoming paper Balland and Hughston (1999) present a model for LIBOR rates which was derived independently from our work and employs similar ideas.

This paper is organised as follows. We begin in Section 2 with some notation and a brief summary of the derivative pricing results that we shall need. Section 3 contains a description of a general Markov-functional interest rate model and examples within this class. It concludes by examining the relationship of this class to existing Market Models. In Section 4, we discuss the properties of the driving Markov process which will ensure the resulting Markov-functional model is realistic and suitable for the pricing problem at hand and in Section 5 we present some numerical results to compare this model with some other standard models currently in use. Finally, we conclude in Section 6.

2. Preliminaries

The results and ideas presented in this paper rely on a knowledge of the general $\mathcal{L}^1$ theory of option pricing. To present this theory in any detail here would be onerous. Instead we will very briefly summarise the key ideas that we need, both without proof and without including all the technical regularity conditions on the economy required to make these results hold true. A full and rigorous treatment of the background theory can be found in Hunt and Kennedy (2000).

Throughout we will be working in a single currency economy $\mathcal{E}$ in which the underlying assets are a set of pure discount bonds. We denote by D_{tT} the value at time t of the bond which matures and pays unity at T, and thus in particular $D_{TT} = 1$. For the presentation here it will be enough to consider an economy comprising only a finite number of these bonds, $T \in \mathcal{T}$ where $\mathcal{T} = \{T_i, i = 1, 2, \ldots, n\}$, but the results can be extended to a continuum $\mathcal{T} = \Re_+$. We denote by $\mathcal{F}_t$ the information available at time t from observing the prices of these assets, $\mathcal{F}_t = \sigma(D_{uT} : u \leq t, T \in \mathcal{T})$. We will allow trading in this economy, buying and selling of the assets throughout time, but will proclude the injection of external funds into the economy—all trading strategies must be self-financing. The value of a portfolio generated in this way by trading in the assets of the economy will be called a *price process* and any price process that is almost surely positive is a *numeraire*.

We shall be interested in the problem of finding the value of a derivative within this economy. We suppose that there is some time ∂^* on which the value of the derivative, V_{∂^*}, will have been determined from the evolution of the asset prices, and thus we need only consider the prior evolution of the economy up until the time ∂^*. We assume that the derivative in question can be replicated and that the economy admits a numeraire pair $(N, \boldsymbol{N})$, meaning a numeraire N and a measure $\boldsymbol{N}$ equivalent to the original measure $\boldsymbol{P}$ (with respect to which the economy is defined) such that the vector process $(D_{tT}/N_t, T \in \mathcal{T})$ is an $\{\mathcal{F}_t\}$ martingale. The measure $\boldsymbol{N}$ is called a martingale measure. It then follows (see for example Hunt and Kennedy (2000), Musiela and Rutkowski (1997)) that $\mathcal{E}$ is arbitrage-free and the value of the derivative at any time t prior to ∂^* is given by

$$\begin{aligned} V_t &= N_t \boldsymbol{E}_{\boldsymbol{N}}[V_{\partial^*} N_{\partial^*}^{-1} | \mathcal{F}_t] \\ &= N_t \boldsymbol{E}_{\boldsymbol{N}}[V_T N_T^{-1} | \mathcal{F}_t] \end{aligned} \tag{1}$$

for any $t \leq T \leq \partial^*$.

3. Markov-Functional Interest Rate Models

The class of models with which we shall work we refer to as *Markov-functional Interest Rate Models* (M-F models). The assumptions we make here are motivated by two key issues: first, the need for a model to be well-calibrated to market prices of relevant standard market instruments and, secondly, the requirement that the model can be efficiently implemented.

Central to the approach is the assumption that the uncertainty in the economy can be captured via some low dimensional (time-inhomogeneous) Markov process $\{x_t : 0 \le t \le \partial^*\}$ in that, for any t, the state of the economy at t is summarised via x_t. This is clearly the defining property of a model that can be implemented in practice. It is true for all classical spot rate models, in which case $x_t = r_t$. The examples of Constantinides (1992), generated by directly modelling the state price density (for more on this in a general non-Markovian setting, see Jin and Glasserman (1997)), also share this property, as does the Rational Log-normal Model of Flesaker and Hughston (1996). Models not possessing this property are Market Models, where the dimension of the Markov process x is much higher, and, of course, general HJM models.

With the exception of Market Models, which suffer from high dimensionality, existing models fail to calibrate well to the distribution of relevant market rates. The key to achieving this, without the extra dimensionality, is first to define a process x of low dimension and then to define its relationship to the assets in the economy in a way which yields the desired distributions. In applications there will usually be an obvious and natural choice for the process x, and typically it will be Gaussian which is particularly easy to implement. In practice x will be one- or at most two-dimensional, the examples in this paper all being one-dimensional.

3.1. Definition

We now describe a general Markov-functional interest rate model. Let $(N, \boldsymbol{N})$ denote a numeraire pair for the economy $\mathcal{E}$. We make the following assumptions

(i) the process x_t is a (time-inhomogeneous) Markov process under the measure $\boldsymbol{N}$,

(ii) the pure discount bond prices are of the form

$$D_{tS} = D_{tS}(x_t), \qquad 0 \le t \le \partial_S \le S\,,$$

for some boundary curve $\partial_S : [0, \partial^*] \to [0, \partial^*]$,

(iii) the numeraire N, itself a price process, is of the form

$$N_t = N_t(x_t) \qquad 0 \le t \le \partial^* .$$

The *boundary curve* $\partial_S : [0, \partial^*] \to [0, \partial^*]$ will be chosen to be appropriate for the particular pricing problem under consideration. In the examples we discuss later the products of interest share some common terminal time T and the curve ∂_S is taken to be

$$\partial_S = \begin{cases} S, & \text{if } S \le T, \\ T, & \text{if } S > T. \end{cases} \tag{2}$$

(see Figure 1). For all practical applications, M-F models will have a boundary curve given by (2).

Insert Figure 1 here

Figure 1 Boundary Curve

Note that if $\partial_S = S$ then Assumption (ii) follows from Assumption (iii) via the valuation formula (1). Conversely, Assumption (iii) regarding the numeraire is no additional assumption over (ii) if the numeraire is one of the pure discount bonds, or a linear combination thereof. However, this assumption does rule out working in the risk-neutral measure with the cash account as numeraire.

The Markovian assumption on x_t is a natural one. For suppose we consider a product which makes payment at some time T of an amount which is determined by the asset prices over the time interval $[t, T]$. If we insist for all such products that the value at t is of the form $V_t(x_t)$ (i.e. a function of (x_t, t)), then it follows that the process x_t must be Markovian in any martingale measure $\boldsymbol{M}$ corresponding to a numeraire M of the form $M_t(x_t)$.

With these assumptions, to completely specify the model it is sufficient to know

(P.i) the law of the process x under $\boldsymbol{N}$,

(P.ii) the functional form of the discount factors on the boundary ∂_S, i.e. $D_{\partial_S S}(x_{\partial_S})$ for $S \in [0, \partial^*]$,

(P.iii) the functional form of the numeraire $N_t(x_t)$ for $0 \leq t \leq \partial^*$.

From this we can recover the discount factors on the interior of the region bounded by ∂_S via the martingale property for numeraire-rebased assets under $\boldsymbol{N}$,

$$D_{tS}(x_t) = N_t(x_t)\boldsymbol{E}_{\boldsymbol{N}}\left[\frac{D_{\partial_S S}(x_{\partial_S})}{N_{\partial_S}(x_{\partial_S})}\Big|\mathcal{F}_t\right]. \tag{3}$$

The M-F models we will develop in this paper will be suitable for pricing multi-temporal (such as American-style) products. It is also possible to construct M-F models which are suited for pricing European-style products and the interested reader is referred to Hunt-Kennedy-Scott (1996). For pricing multi-temporal products, we need to fit the joint distribution of a set of swap-rates at different points in time. As we shall see, given a one-dimensional Markov process, it is possible to fit the *marginal* distributions. However, for successfully pricing the product it is important to capture the *joint* distribution of the swap-rates under consideration. We can affect this joint distribution through the choice of the underlying Markov process. We will discuss this important point further in Section 4.

3.2. Implying the Functional Form of the Numeraire from Swaption Prices

Consider an exotic product which depends on a set of forward swap rates or forward LIBOR's each observed at a distinct time. Denote these rates by $\{y_t^{(i)}, i = 1, \ldots, n\}$ and let T_i, $i = 1, \ldots, n$, denote the setting dates for these par swap rates. We make the assumption that for each of the payment dates $S_j^{(i)}$, $j = 1, 2, \ldots, n_i$, either $S_j^{(i)} > T_n$ or $S_j^{(i)} = T_k$, some $k > i$. If this assumption is not valid it can be made to hold by introducing auxillary swap rates $y_t^{(k)}$ as necessary.

Our aim in this subsection is to construct a one-dimensional M-F model which correctly prices options on the swaps associated with these forward rates.

We will assume that

(i) we are given the law of the process x under $\boldsymbol{N}$, and

(ii) the functional forms $D_{T_n S}(x_{T_n})$ are known for $S \geq T_n$.

From the discussion above it is clear that to completely specify our model it remains to specify the functional form of the numeraire. We will make two further assumptions:

(iii) our choice of numeraire is such that $N_{T_n}(x_{T_n})$ can be inferred from the functional forms of the discount factors at the boundary, i.e. those in (ii) above, and

(iv) the ith forward rate at time T_i, $y^{(i)}_{T_i}$, is a monotonic increasing function of the variable x_{T_i}.

In the remainder of this subsection we show how market prices of the calibrating vanilla swaptions can be used to imply, numerically at least, the functional forms $N_{T_i}(x_{T_i})$ for $i = 1, \ldots, n-1$.

Equivalent to calibrating the model to the vanilla swaptions is to calibrate it to the inferred market prices of digital swaptions, see e.g. Dupire (1994). The digital swaption corresponding to $y^{(i)}_t$ having strike K has payoff at time T_i of

$$\widetilde{V}^{(i)}_{T_i}(K) = P^{(i)}_{T_i} \mathbf{1}(y^{(i)}_{T_i} > K)\,,$$

where $P^{(i)}_{T_i}$ denotes the accrual factor for the payoff.

Applying (1), the value of this option at time zero is given by

$$\widetilde{V}^{(i)}_0(K) = N_0(x_0) \boldsymbol{E}_{\boldsymbol{N}} \Big[\hat{P}^{(i)}_{T_i}(x_{T_i}) \mathbf{1}(y^{(i)}_{T_i}(x_{T_i}) > K) \Big] \tag{4}$$

where

$$\hat{P}^{(i)}_{T_i}(x_{T_i}) = \frac{P^{(i)}_{T_i}(x_{T_i})}{N_{T_i}(x_{T_i})}.$$

To determine the functional forms of the numeraire $N_{T_i}(x_{T_i})$ we work back iteratively from the terminal time T_n. Consider the ith step in this procedure. Assume that $N_{T_k}(x_{T_k}), k = i+1, \ldots, n$, have already been determined. We can also assume

$$\hat{D}_{T_i S}(x_{T_i}) = \frac{D_{T_i S}(x_{T_i})}{N_{T_i}(x_{T_i})}$$

for relevant $S > T_i$ are known (and hence so is $\hat{P}^{(i)}_{T_i}$) having been determined using (3) and the known (conditional) distributions of x_{T_k}, $k = i, \ldots, n$.

Now consider $y^{(i)}_{T_i}$ which can be written as

$$y^{(i)}_{T_i} = \frac{N^{-1}_{T_i} - D_{T_i S^{(i)}_{n_i}} N^{-1}_{T_i}}{P^{(i)}_{T_i} N^{-1}_{T_i}} \,. \tag{5}$$

Rearranging equation (5) we have that

$$N_{T_i}(x_{T_i}) = \frac{1}{\hat{P}^{(i)}_{T_i}(x_{T_i}) y^{(i)}_{T_i}(x_{T_i}) + \hat{D}_{T_i S^{(i)}_{n_i}}(x_{T_i})} \,. \tag{6}$$

Thus to determine $N_{T_i}(x_{T_i})$ it is sufficient to find the functional form $y^{(i)}_{T_i}(x_{T_i})$.

By assumption (iv) there exists a unique value of K, say $K^{(i)}(x^*)$, such that the set identity

$$\{x_{T_i} > x^*\} = \{y^{(i)}_{T_i} > K^{(i)}(x^*)\} \tag{7}$$

holds almost surely. Now define

$$J^{(i)}_0(x^*) = N_0(x_0) \boldsymbol{E}_{\boldsymbol{N}} \Big[\hat{P}^{(i)}_{T_i}(x_{T_i}) \mathbf{1}(x_{T_i} > x^*) \Big] \,. \tag{8}$$

For any given x^* we can calculate the value of $J^{(i)}_0(x^*)$ using the known distribution of x_{T_i} under $\boldsymbol{N}$. Further, using market prices we can then find the value of K such that

$$J^{(i)}_0(x^*) = \widetilde{V}^{(i)}_0(K) \,. \tag{9}$$

Comparing (4) and (8) we see that the value of K satisfying (9) is precisely $K^{(i)}(x^*)$. Clearly, from (7), knowing $K^{(i)}(x^*)$ for any x^* is equivalent to knowing the functional form $y^{(i)}_{T_i}(x_{T_i})$ and we are done.

It is common market practice to use Black's formula to determine the swaption prices $V^{(i)}_0(K)$. Observe, however, that the techniques here apply more generally. In particular, if the currency concerned is one for which there is a large volatility skew, meaning the volatility used as input to Black's formula is highly dependent on the level of the strike K, these techniques can still be applied. This is particularly important in currencies such as Yen in which it is not reasonable to model rates via a log-normal process. The ability of M-F models to 'adapt' to all different markets is one of their major advantages over other models.

3.3. LIBOR Model

In this subsection and the next we introduce two example models which can be used to price LIBOR and swap based interest rate derivatives.

The set of market rates that concern us here are the forward LIBOR's $L_t^{(i)}$ for $i = 1, 2, \ldots, n$. We assume that the swaption measure $\boldsymbol{S}^{(n)}$ exists corresponding to the numeraire $D_{tT_{n+1}}$, a measure under which the $D_{tT_{n+1}}$-rebased assets $D_{tS}/D_{tT_{n+1}}$ are martingales. Note, that in the case of LIBOR rates the measure $\boldsymbol{S}^{(n)}$ is often called a *forward measure.*

As described in Section 3.1, an M-F model is specified by Properties (P.i)–(P.iii). We will be consistent with Black's formula for caplets on $L_t^{(n)}$ if we assume that $L_t^{(n)}$ is a log-normal martingale under $\boldsymbol{S}^{(n)}$, i.e.

$$dL_t^{(n)} = \sigma_t^{(n)} L_t^{(n)} dW_t \tag{10}$$

where W_t is a standard Brownian motion under $\boldsymbol{S}^{(n)}$. In applications we will often take $\sigma_t^{(n)} = \sigma e^{at}$, some σ and some *mean reversion parameter* a, for reasons explained in Section 4. It follows from (10) that we may write

$$L_t^{(n)} = L_0^{(n)} \exp\Big(-\tfrac{1}{2}\int_0^t (\sigma_u^{(n)})^2\, du + x_t\Big)\,,$$

where x_t, a deterministic time-change of a Brownian motion, satisfies

$$dx_t = \sigma_t^{(n)} dW_t\,. \tag{11}$$

We take x as the driving Markov process for our model, and this completes the specification of (P.i).

The boundary curve ∂_S for this problem is exactly that of Equation (2). For this application the only functional forms needed on the boundary are $D_{T_iT_i}(x_{T_i})$ for $i = 1, 2, \ldots, n$, trivially the unit map, and $D_{T_nT_{n+1}}(x_{T_n})$. This latter form follows from the relationship

$$D_{T_nT_{n+1}} = \frac{1}{1 + \alpha_n L_{T_n}^{(n)}}\,,$$

which yields

$$D_{T_nT_{n+1}} = \frac{1}{1 + \alpha_n L_0^{(n)} \exp(-\frac{1}{2}\int_0^{T_n} (\sigma_u^{(n)})^2\, du + x_{T_n})}\,.$$

This completes (P.ii).

It remains to find the functional form $N_{T_i}(x_{T_i})$, in our case $D_{T_iT_{n+1}}(x_{T_i})$, and for this we apply the techniques of Section 3.2. Take as calibrating instruments the caplets on the forward LIBOR's $L_t^{(i)}$. The value of the ith corresponding digital option of strike K is given by

$$\widetilde{V}_0^{(i)}(K) = D_{0T_{n+1}}(x_0)\boldsymbol{E}_{\boldsymbol{S}^{(n)}}\left[\frac{D_{T_iT_{i+1}}(x_{T_i})}{D_{T_iT_{n+1}}(x_{T_i})}\mathbf{1}(L_{T_i}^{(i)}(x_{T_i}) > K)\right].$$

If we assume the market value is given by Black's formula with volatility $\tilde{\sigma}^{(i)}$, the price at time zero for this digital is

$$\widetilde{V}_0^{(i)}(K) = D_{0T_{i+1}}(x_0)\Phi(d_2^{(i)}) \tag{12}$$

where

$$d_2^{(i)} = \frac{\log(L_0^{(i)}/K)}{\tilde{\sigma}^{(i)}\sqrt{T_i}} - \tfrac{1}{2}\tilde{\sigma}^{(i)}\sqrt{T_i}\,,$$

and Φ denotes the cumulative normal distribution function. (Observe that Black's formula implies that the marginal distributions of the $L_{T_i}^{(i)}$ are log-normal in their respective swaption measures.)

To determine the functional form $D_{T_iT_{n+1}}(x_{T_i})$ for $i < n$ we proceed as in Section 3.2. Suppose we choose some $x^* \in \Re$. Evaluate by numerical integration

$$\begin{aligned}
J_0^{(i)}(x^*) &= D_{0T_{n+1}}(x_0)\boldsymbol{E}_{\boldsymbol{S}^{(n)}}\left[\frac{D_{T_iT_{i+1}}(x_{T_i})}{D_{T_iT_{n+1}}(x_{T_i})}\mathbf{1}(x_{T_i} > x^*)\right] \\
&= D_{0T_{n+1}}(x_0)\boldsymbol{E}_{\boldsymbol{S}^{(n)}}\left[\boldsymbol{E}_{\boldsymbol{S}^{(n)}}\left[\frac{D_{T_{i+1}T_{i+1}}(x_{T_{i+1}})}{D_{T_{i+1}T_{n+1}}(x_{T_{i+1}})}\Big|\mathcal{F}_{T_i}\right]\mathbf{1}(x_{T_i} > x^*)\right] \\
&= D_{0T_{n+1}}(x_0)\int_{x^*}^{\infty}\left[\int_{-\infty}^{\infty}\frac{1}{D_{T_{i+1}T_{n+1}}(u)}\phi_{x_{T_{i+1}}|x_{T_i}}(u)\,du\right]\phi_{x_{T_i}}(v)\,dv
\end{aligned} \tag{13}$$

where $\phi_{x_{T_i}}$ denotes the transition density function of x_{T_i} and $\phi_{x_{T_{i+1}}|x_{T_i}}$ the density of $x_{T_{i+1}}$ given x_{T_i}. Note from (11) that $\phi_{x_{T_{i+1}}|x_{T_i}}$ is a normal density function with mean x_{T_i} and variance $\int_{T_i}^{T_{i+1}}(\sigma_u^{(n)})^2\,du$. Finally note the integrand in (13) only depends on $D_{T_{i+1}T_{n+1}}(x_{T_{i+1}})$ which has already been determined in the previous iteration at T_{i+1}.

The value of $D_{T_iT_{n+1}}(x^*)$ can now be determined as follows. Recall from Equation (6) that to determine $D_{T_iT_{n+1}}(x^*)$ it is sufficient to find the functional form $L_{T_i}^{(i)}(x^*)$. From (7) and (9)

$$L_{T_i}^{(i)}(x^*) = K^{(i)}(x^*)$$

where $K^{(i)}(x^*)$ solves

$$J_0^{(i)}(x^*) = \widetilde{V}_0^{(i)}(K^{(i)}(x^*)) \,. \tag{14}$$

We have just evaluated the LHS of (14) numerically and $K^{(i)}(x^*)$ can thus be found from (12) using some simple algorithm. Formally,

$$L_{T_i}^{(i)}(x^*) = L_0^{(i)} \exp\left[-\tfrac{1}{2}(\tilde{\sigma}^{(i)})^2 T_i - \tilde{\sigma}^{(i)}\sqrt{T_i}\Phi^{-1}\left(\frac{J_0^{(i)}(x^*)}{D_{0T_{i+1}(x^*)}}\right)\right].$$

Finally, to obtain the value of $D_{T_iT_{n+1}}(x^*)$ we use (6).

An example of the type of product we might wish to model with a LIBOR M-F model are flexible caps. A *flexible cap* is defined by a finite sequence of dates T_i, $i = 1, 2, \ldots, n+1$, a set of strikes K_i, $i = 1, 2, \ldots, n$, and a limit number m. The holder of a flexible cap has the right to choose when to exercise up to a maximum of m caplets. To determine the optimal exercise dates for the m caplets, the holder has to solve simultaneously m optimization problems. For a more detailed analysis of flexible caps we refer to Pedersen and Sidenius (1997).

3.4. Swap Model

For the construction of a swap M-F model we consider the special case of a cancellable swap for which the ith forward par swap rate $y_t^{(i)}$, which sets on date T_i, has coupons precisely at dates $T_{i+1}, \ldots, T_{n+1}$. For this case the last par swap rate $y_t^{(n)}$ is just the forward LIBOR, $L_t^{(n)}$, for the period $[T_n, T_{n+1}]$. As in the above example, we take $D_{tT_{n+1}}$ as our numeraire and assume that the swaption measure $\boldsymbol{S}^{(n)}$ exists. Further, we specify properties (P.i) and (P.ii) exactly as for the LIBOR model. However (P.iii), the functional form for the numeraire $D_{tT_{n+1}}$ at times T_i, $i = 1, \ldots, n-1$, will need to be determined.

For this new model the value of the digital swaption having strike K and corresponding to $y_t^{(i)}$ is given by

$$\widetilde{V}_0^{(i)}(K) = D_{0T_{n+1}}(x_0)\boldsymbol{E}_{\boldsymbol{S}^{(n)}}\left[\frac{P_{T_i}^{(i)}(x_{T_i})}{D_{T_iT_{n+1}}(x_{T_i})}\mathbf{1}(y_{T_i}^{(i)}(x_{T_i}) > K)\right],$$

where

$$P_t^{(i)}(x_t) = \sum_{j=i}^{n} \alpha_j D_{tT_{j+1}}(x_t) \,.$$

If we assume that the market value is given by Black's formula then the price at time zero of this digital swaption has the form

$$\widetilde{V}_0^{(i)}(K) = P_0^{(i)}(x_0)\Phi(d_2^{(i)}) \tag{15}$$

where

$$d_2^{(i)} = \frac{\log(y_0^{(i)}/K)}{\tilde{\sigma}^{(i)}\sqrt{T_i}} - \tfrac{1}{2}\tilde{\sigma}^{(i)}\sqrt{T_i}\,.$$

Here Black's formula implies that the marginal distribution of the $y_{T_i}^{(i)}$ are log-normally distributed in their respective swaption measures.

Next suppose we choose some $x^* \in \Re$ and evaluate by numerical integration

$$\begin{aligned} J_0^{(i)}(x^*) &= D_{0T_{n+1}}(x_0)\boldsymbol{E}_{\boldsymbol{S}^{(n)}}\left[\frac{P_{T_i}^{(i)}(x_{T_i})}{D_{T_iT_{n+1}}(x_{T_i})}\mathbf{1}(x_{T_i} > x^*)\right] \\ &= D_{0T_{n+1}}(x_0)\boldsymbol{E}_{\boldsymbol{S}^{(n)}}\left[\boldsymbol{E}_{\boldsymbol{S}^{(n)}}\left[\frac{P_{T_{i+1}}^{(i)}(x_{T_{i+1}})}{D_{T_{i+1}T_{n+1}}(x_{T_{i+1}})}\middle|\mathcal{F}_{T_i}\right]\mathbf{1}(x_{T_i} > x^*)\right] \quad . \\ &= D_{0T_{n+1}}(x_0)\int_{x^*}^{\infty}\left[\int_{-\infty}^{\infty}\frac{P_{T_{i+1}}^{(i)}(u)}{D_{T_{i+1}T_{n+1}}(u)}\phi_{x_{T_{i+1}}|x_{T_i}}(u)\,du\right]\phi_{x_{T_i}}(v)\,dv \end{aligned}$$

Note to calculate a value for $J_0^{(i)}(x^*)$ we need to know $D_{T_{i+1}T_j}(x_{T_{i+1}})$, $j > i$. These will have already been determined in the previous iteration.

Now, as in Section 3.3,

$$y_{T_i}^{(i)}(x^*) = K^{(i)}(x^*)$$

where $K^{(i)}(x^*)$ solves

$$J_0^{(i)}(x^*) = \widetilde{V}_0^{(i)}(K^{(i)}(x^*))\,. \tag{16}$$

Having evaluated the LHS of (16) numerically, $K^{(i)}(x^*)$ can be recovered from (15). Formally we have

$$y_{T_i}^{(i)}(x^*) = y_0^{(i)}\exp\left[-\tfrac{1}{2}(\tilde{\sigma}^{(i)})^2T_i - \tilde{\sigma}^{(i)}\sqrt{T_i}\Phi^{-1}\left(\frac{J_0^{(i)}(x^*)}{P_0^{(i)}(x^*)}\right)\right].$$

The value of $D_{T_iT_{n+1}}(x^*)$ can now be calculated using (6).

An example of a product we might wish to model with a swap M-F model is a *Bermudan swaption*. Once again this product is specified via a finite number of dates, T_i, $S_j^{(i)}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n_i$, and a set of strikes K_i, $i = 1, 2, \ldots, n$. The holder of this option is free to exercise on any of the dates T_i and upon exercise enters a swap with corresponding par swap rate $y_t^{(i)}$ at t and which on date T_i has value

$$P_{T_i}^{(i)}(y_{T_i}^{(i)} - K_i)\,,$$

where

$$P_t^{(i)} = \sum_{j=1}^{n_i} \alpha_j^{(i)} D_{tS_j^{(i)}}\,,$$

the $\alpha_j^{(i)}$ being daycount fractions.

3.5. Relationship to Market Models

Recently several authors including Miltersen, Sandmann and Sonderman (1997) and Brace, Gatarek and Musiela (1997) have studied a class of interest rates models parametrized by forward LIBOR's. Jamshidian (1998) has extended this to models parametrized by general swap rates and they are collectively known as 'Market Models'. When these market rates are taken to be log-normal martingales in their respective swaption measures ($dy_t^{(i)} = \sigma_t^{(i)} y_t^{(i)} dW_t^{(i)}$ where $W^{(i)}$ is a standard Brownian motion under $\boldsymbol{S}^{(i)}$ and $\sigma^{(i)}$ is a deterministic function of time) the models yield market prices for standard swaption products. The obvious question is how does our approach relate to that of BGM and Jamshidian.

Market Models present a general framework for modelling interest rate derivatives and as such M-F models are a subset thereof, just as Market Models fit within the HJM framework. At this level Market Models are just an alternative model parametrization. However, when Market Models make the additional assumption that forward par swap rates be log-normal martingales in their associated swaption measures, this assumption is much stronger and more restrictive than ours—we only assume the martingale property and a log-normal distribution for swap rates on their respective fixing dates. This additional restriction in the Market Models is precisely what makes those models difficult to use for American-style products in practice because they cannot be characterised by a low-dimensional Markov process. The following result, which can be extended to include the more general framework of Jamshidian, makes this statement precise for the LIBOR Market Models.

Theorem 1 *Let $L = (L_t^{(1)}, \ldots, L_t^{(n)})$, $n > 1$, be a non-trivial log-normal* LIBOR *Market Model, where $L_t^{(i)}$ denotes a forward* LIBOR *rate. Then there exists no one-dimensional process x such that*

(i) $L_t^{(i)} = L_t^{(i)}(x_t) \in C^{2,1}(\Re, \Re_+)$ *for* $i = 1, 2, \ldots, n$,

(ii) each $L_t^{(i)}(x_t)$ *is strictly monotone in* x_t.

That is, L is not a one-dimensional M-F model.

Remark: We believe this result extends to hold for any process x of dimension less than n and any functional forms $L_t^{(i)}(x_t)$.

Proof: Suppose such a process x exists. Then it follows from the invertibility of the map $x_t \to L_t^{(n)}(x_t)$ that we can write

$$L_t^{(i)} = L_t^{(i)}(L_t^{(n)}), \qquad i = 1, 2, \ldots, n\,. \tag{17}$$

Since L is a log-normal LIBOR Market Model it follows (Brace, Gatarek and Musiela (1997)) that it satisfies an SDE under $\boldsymbol{S}^{(n)}$ of the form

$$dL_t^{(i)} = \mu_t^{(i)}\, dt + \sigma_t^{(i)} L_t^{(i)} dW_t^{(i)} \tag{18}$$

for some n-dimensional Brownian motion $W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$ and some process $\mu_t = (\mu_t^{(1)}, \ldots, \mu_t^{(n-1)}, 0)$. On the other hand, if we apply Itô's formula to (17), we obtain

$$\begin{aligned} dL_t^{(i)} &= \left[\frac{\partial L_t^{(i)}}{\partial t} + \tfrac{1}{2} (\sigma_t^{(n)} L_t^{(n)})^2 \frac{\partial^2 L_t^{(i)}}{\partial (L_t^{(n)})^2} \right] dt + \frac{\partial L_t^{(i)}}{\partial L_t^{(n)}} dL_t^{(n)} \\ &= \left[\frac{\partial L_t^{(i)}}{\partial t} + \tfrac{1}{2} (\sigma_t^{(n)} L_t^{(n)})^2 \frac{\partial^2 L_t^{(i)}}{\partial (L_t^{(n)})^2} \right] dt + \frac{\partial L_t^{(i)}}{\partial L_t^{(n)}} \sigma_t^{(n)} L_t^{(n)} dW_t^{(n)}\,. \end{aligned} \tag{19}$$

Note, that $\mu_t^{(n)} = 0$ since $L_t^{(n)}$ is a martingale under $\boldsymbol{S}^{(n)}$. Equating the local martingale terms in (18) and (19) yields $W^{(i)} \equiv W^{(n)}$, all i, and

$$\frac{\partial L_t^{(i)}}{\partial L_t^{(n)}} = \frac{\sigma_t^{(i)} L_t^{(i)}}{\sigma_t^{(n)} L_t^{(n)}}\,. \tag{20}$$

Solving (20) we find

$$L_t^{(i)} = c_i(t) (L_t^{(n)})^{\beta_i(t)} \tag{21}$$

where $c_i(t)$ is some function of t and $\beta_i(t) = \sigma_t^{(i)} / \sigma_t^{(n)}$.

Having concluded that $W^{(i)} = W^{(n)}$, all i, Equation (18) reduces (Jamshidian, 1998) to the form

$$dL_t^{(i)} = -\Big(\sum_{j=i+1}^{n} \frac{\alpha_j \sigma_t^{(j)} L_t^{(j)}}{1+\alpha_j L_t^{(j)}}\Big)\sigma_t^{(i)} L_t^{(i)}\, dt + \sigma_t^{(i)} L_t^{(i)} dW_t\,. \tag{22}$$

Substituting (21) back into (19) and equating finite variation terms in (19) and (22) now gives (after some rearrangement)

$$\frac{1}{\sigma_t^{(i)}}\Big(\frac{c_i'(t)}{c_i(t)} + \beta_i'(t)\log L_t^{(n)}\Big) + \tfrac{1}{2}\sigma_t^{(n)}(\beta_i(t)-1) = -\sum_{j=i+1}^{n} \frac{\alpha_j \sigma_t^{(j)} L_t^{(j)}}{1+\alpha_j L_t^{(j)}}\,.$$

In particular, taking $i = n-1$, yields $\sigma_t^{(n)} = 0$ and thus the model is degenerate. ■

4. Mean reversion, forward volatilities and correlation

4.1. Mean reversion and correlation

Mean reversion of interest rates is considered a desirable property of a model because it is perceived that interest rates tend to trade within a fairly tightly defined range. This is indeed true, but when pricing exotic derivatives it is the effect of mean reversion on the correlation of interest rates at different times that is more important. We illustrate this for the Hull-White (1994) (HW) model because it is particularly tractable.

In the standard HW model the short-rate process r_t solves the SDE

$$dr_t = (\theta_t - a_t r_t)dt + \sigma_t dW_t\,, \tag{23}$$

where a_t, θ_t and σ_t are deterministic functions of t and W_t is a standard Brownian motion. For the purposes of this discussion we take $a_t \equiv a$, some constant.

Suppose we fit the HW model to market prices. The best we can do is to fit the model so that it correctly values one caplet for each T_i, the one having strike K_i say. It turns out that, given the initial discount curve $\{D_{0T}, T > 0\}$ and a, the market cap prices determine $v_i = \mathrm{Var}(r_{T_i})$, each i. The general correlation structure $\rho(r_{T_i}, r_{T_j})$ depends on a, via the relationship

$$\rho(r_{T_i}, r_{T_j}) = e^{-a(T_j - T_i)}\sqrt{\frac{v_i}{v_j}}\,.$$

Note that, given the market cap prices, the ratio v_i/v_j is independent of a. Thus, increasing a has the effect of reducing the correlation between the short rate at different times, hence also the covariance of spot LIBOR at different times. This is important for pricing path dependent and American options whose values depend on the joint distribution $(r_{T_i}, i = 1, 2, \ldots, n)$.

For other models, including M-F models, the analytic formulæ above will not hold, of course, but the general principle does. Given the marginal distributions for a set of spot interest rates $\{y^{(i)}_{T_i},\ i = 1, 2, \ldots, n\}$, a higher mean reversion for spot interest rates leads to a lower correlation between the $\{y^{(i)}_{T_i},\ i = 1, 2, \ldots, n\}$.

4.2. Mean reversion and forward volatilities

In the models of Section 3 we have not explicitly presented an SDE for spot LIBOR or spot par swap rates. We therefore need to work a little harder to understand how to introduce mean reversion within these models. To do this we consider the HW model once again.

We have at Equation (10) parametrized our M-F example in terms of a *forward* LIBOR process $L^{(n)}$. If we can understand the effect of mean reversion within a model such as HW on $L^{(n)}$ we can apply the same principles to a more general M-F model.

An analysis of the HW model defined via (23) shows that

$$L^{(n)}_t = X_t - \alpha_n^{-1}$$

where

$$dX_t = \Big(\frac{e^{-aT_n} - e^{-aT_{n+1}}}{a}\Big)\sigma e^{at} X_t d\widetilde{W}_t\,, \tag{24}$$

$\widetilde{W}$ is a standard Brownian motion under the measure $\boldsymbol{S}^{(n)}$. The forward LIBOR is a log-normal martingale minus a constant. Notice the dependence on time t of the diffusion coefficient of the martingale term: the volatility is of the form *constant* $\times$ X_t $\times$ e^{at}. This is the motivation for the volatility structure we chose for $L^{(n)}$ in the M-F model definition at Equation (10). What is important is not so much the exact functional form but the fact that the volatility increases through time. The faster the increase, the lower the correlation between spot interest rates which set at different times.

4.3. Mean reversion within the M-F LIBOR model

To conclude this discussion of mean reversion we return to the M-F LIBOR model of Section 3.3 and show how taking $\sigma^{(n)}_t = \sigma e^{at}$ in (10) leads to mean reversion of spot

LIBOR. Suppose the market cap prices are such that the implied volatilities for $L^{(1)}$ and $L^{(n)}$ are the same, $\hat{\sigma}$ say, and all initial forward values are the same, $L_0^{(i)} = L_0$, all i.

Insert Figure 2 here

Figure 2 Mean Reversion

Figure 2 shows the evolution of $L^{(1)}$ and $L^{(n)}$ in the situation when the driving Markov process x has increased (significantly), $x_{T_1} > x_0$. Both $L^{(1)}$ and $L^{(n)}$ have increased over the interval $[0, T_1]$ but $L^{(1)}$ has increased by more. The reason for this is as follows. Over $[0, T_1]$, $L^{(1)}$ has (root mean square) volatility $\hat{\sigma}$. By comparison, $L^{(n)}$ has (root mean square) volatility $\hat{\sigma}$ *over the whole interval* $[0, T_n]$, but its volatility is increasing exponentially and thus its (root mean square) volatility over $[0, T_1]$ is less than $\hat{\sigma}$. Since $L^{(n)}$ is a martingale under $\boldsymbol{S}^{(n)}$, it follows that

$$\boldsymbol{E}_{\boldsymbol{S}^{(n)}}\big[L_{T_n}^{(n)} | \mathcal{F}_{T_1}\big] = L_{T_1}^{(n)} < L_{T_1}^{(1)} .$$

That is, in Figure 2 when spot LIBOR has moved up, from its initial value L_0 at time zero to its value at T_1, $L_{T_1}^{(1)}$, the expected value of spot LIBOR at T_n is less than $L_{T_1}^{(1)}$. Conversely, when spot LIBOR moves down by time T_1 ($x_{T_1} < x_0$) the expected value of spot LIBOR at T_n is greater than its value at T_1. This is mean reversion.

5. Numerical Results

For a 30 year DEM Bermudan, which is exercisable every five years we have compared in Table 1 below, for different levels of mean-reversion, the prices calculated by three different models: Black and Karasinski (1991) (BK), MF and the Hull-White (1994) (HW) model. Note that our implementation of the Hull-White model prohibits negative mean-reversions so we have not been able to include these results.

For every level of mean-reversion, we have given the prices of the embedded European swaptions (5×25, 10×20, 15×15, 20×10, 25×5). Since all three models are calibrated to these prices, all models should agree exactly on these prices. The differences reported in the table are due to numerical errors. We adopted the following approach. First we chose a level for the BK mean reversion parameter and reasonable levels for the BK volatilities. We then used these parameters to generate the prices, using the BK model, of the underlying European swaptions. We then calibrated the other, HW and MF, models to these prices. The resultant implied volatilities used for each case are reported in the second column. At the bottom of each block in the table, we report the value of the Bermudan swaption as calculated by each of the three models.

From the table we see that the mean-reversion parameter has a significant impact on the price of the Bermudan swaption. It is intuitively clear why this is the case. The reason a Bermudan option has more value than the maximum of the embedded European option prices is the freedom it offers to delay or advance the exercise decision of the underlying swap during the life of the contract to a date when it is most profitable. The relative value between exercising 'now' or 'later' depends very much on the correlation of the underlying swap-rates between different time points. This correlation structure is exactly what is being controlled by the mean-reversion parameter. By contrast, the effect of changing the model is considerably less, and what difference there is will be due in part to the fact that the mean-reversion paramter has a slightly different meaning for each model. We conclude that the precise (marginal) distributional assumptions made have a secondary role in determining prices for exotic options relative to the joint distributions as captured by the mean-reversion parameter.

6. Conclusion

We have introduced a new class of models, motivated very much by the practicalities that a model both be an accurate reflection of market prices and be one which can be implemented efficiently. The examples of Section 3, which can easily be generalised to model markets for which option prices exhibit significant 'volatility skew', illustrate the ease with which the models can be implemented and how well tailored they are to practical derivative pricing.

The ideas in this paper require further study. The next step is to develop a practical extension to multi-dimensional processes (two dimensions being the most important case), and to gain a deeper understanding of how different examples of these models behave.

In terms of fitting market prices, the generality sacrificed by restricting to Markovian models is more than redressed by the extra flexibility offered by only fitting distributions at terminal dates. This offers the ability, as demonstrated in the one-dimensional case in Hunt and Kennedy (1998), to create models which fit market swaption and cap prices simultaneously, in direct contrast to the existing Market Models.

Table 1: Value of 30 year Bermudan, exercisable every 5 years

Strike: 0.0624, Currency: DEM, Valuation Date: 11-feb-98

Mean Reversion = -0.05							
European		Receivers			Payers		
Mat	ImVol	BK	MF	HW	BK	MF	HW
05 × 25	8.17	447.7	445.9	–	456.6	456.6	–
10 × 20	7.74	365.2	363.8	–	448.2	448.6	–
15 × 15	7.90	285.8	284.7	–	350.8	351.4	–
20 × 10	8.29	191.8	190.8	–	241.6	242.1	–
25 × 05	8.68	91.0	90.9	–	126.2	126.1	–
Bermudan		510.0	502.7	–	572.2	566.9	–
Mean Reversion = 0.06							
European		Receivers			Payers		
Mat	ImVol	BK	MF	HW	BK	MF	HW
05 × 25	8.46	463.7	462.8	464.2	472.6	472.0	473.3
10 × 20	7.91	374.0	373.5	374.2	457.0	456.9	457.5
15 × 15	7.80	281.8	281.5	281.9	346.8	346.8	347.1
20 × 10	7.81	179.5	179.4	179.6	229.4	229.3	229.5
25 × 05	8.16	84.8	84.7	84.7	119.9	120.0	120.1
Bermudan		606.1	602.9	608.2	743.6	717.7	727.3
Mean Reversion = 0.20							
European		Receivers			Payers		
Mat	ImVol	BK	MF	HW	BK	MF	HW
05 × 25	8.37	458.7	457.6	459.3	467.5	466.8	468.0
10 × 20	6.99	326.2	325.4	326.2	409.1	408.8	409.3
15 × 15	6.67	237.1	236.4	237.3	302.1	301.7	302.4
20 × 10	7.32	166.7	166.4	166.6	216.6	216.4	216.6
25 × 05	9.40	99.8	99.5	99.6	134.9	134.8	134.8
Bermudan		647.4	665.8	673.9	885.5	814.7	813.6

Mat denotes maturity and tenor of embedded European option.
ImVol denotes implied volatility of embedded European option.
BK are prices calculated with Black-Karasinski model.
MF are prices calculated with Markov-Functional model.
HW are prices calculated with Hull-White model.

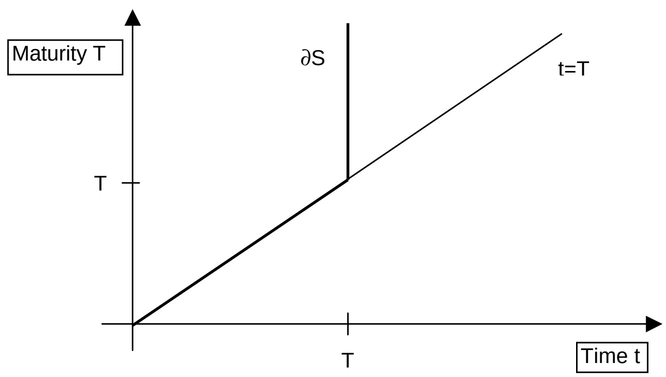

Figure 1

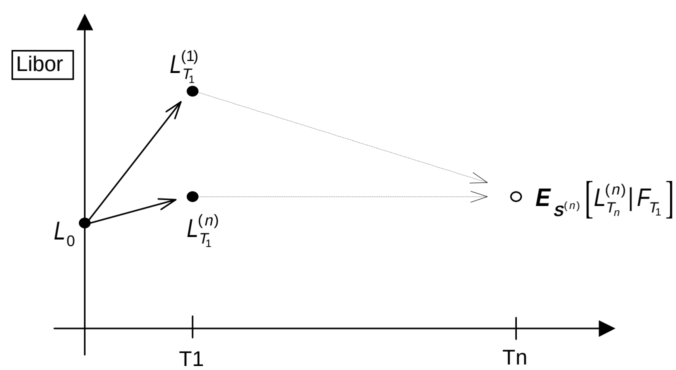

Figure 2

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