Complex Functions and Laplace Transforms

Chapter 1: Analytic Functions

1. Necessary condition for a function to be analytic

= Cauchy - Riemann Equations

$$u_x = v_y$$
$$v_x = -u_y$$

Derivation: $f(z) = u + iv$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$z + \Delta z = (x + \Delta x) + i(y + \Delta y)$$

substitute

Case 1 : let $\Delta y = 0$, $\Delta x \to 0$ $\Rightarrow$ wholly real

Case 2 : let $\Delta x = 0$, $\Delta y \to 0$ $\rightarrow$ wholly imaginary

equate both cases

2. Sufficient condition for analyticity = existence of $f'(z)$

Derivation

$$z = x + iy$$
$$\Delta z = \Delta x + i\Delta y$$

find $z + \Delta z$

Find the Taylor's series expansion

$$f(z+\Delta z) = \left\{ u(x,y) + \frac{\partial u}{\partial x}\cdot\Delta x + \frac{\partial u}{\partial y}\cdot\Delta y \right\} + i\left\{ v(x,y) + \frac{\partial v}{\partial x}\cdot\Delta x + \frac{\partial v}{\partial y}\cdot\Delta y \right\}$$

group $\Delta x, \Delta y$ terms

apply C-R eqns

find $\dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

3. C-R Equations in polar form

$$u_r = \frac{1}{r} v_\theta$$

$$v_r = \frac{-1}{r} u_\theta$$

Derivation : $z = re^{i\theta}$ $\quad (z = x+iy$
$= r\cos\theta + i\sin\theta)$

$$u+iv = f(z)$$

$$u+iv = f(re^{i\theta})$$

diff. w.r.t r

diff. w.r.t θ

equate real and imaginary parts

4. To find the derivative of an analytic function

in cartesian form : $f'(z) = u_x + iv_x$

in polar form : $f'(z) = e^{-i\theta}(u_r + iv_r)$

5. Relation Between complex trigonometric functions and hyperbolic functions

→ $\cos ix = \cosh x$

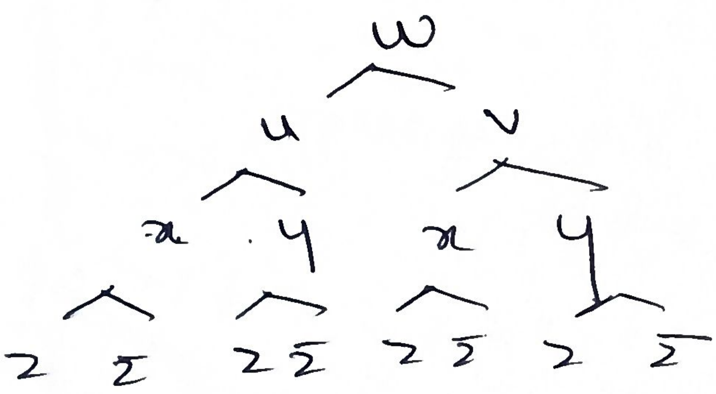
→ $\sin ix = i\sinh x$

$\boxed{\text{Do Ex 7}}$

6. Every analytic function is a function of z alone

<u>Proof</u> $\quad x = \dfrac{z+\bar{z}}{2} \qquad y = \dfrac{z-\bar{z}}{2i}$

$$w = f(z) = u+iv$$



$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}}$$

$$= \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right)$$

$$+ i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right)$$

substitute values

7. If $f(z) = u + iv$, then u & v are harmonic functions

i.e $u_{xx} + u_{yy} = 0$

$v_{xx} + v_{yy} = 0$

Derivation: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ → 1 & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ → 2

diff 1 w.r.t x & 2 w.r.t y

& diff 1 w.r.t y & 2 w.r.t x

8. To find the conjugate harmonic function

$du = u_x\,dx + u_y\,dy$

$dv = v_x\,dx + v_y\,dy$

} sub C-R as needed

9. If $f(z) = u + iv$ is analytic, $u(x,y) = c_1$ & $v(x,y) = c_2$ ~~then~~ are orthogonal

Derivation $u(x,y) = c_1$ — 1

$v(x,y) = c_2$ — 2

Diff 1 w.r.t x ⇒ $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$

Find $\frac{dy}{dx} = m_1$

Diff 2 w.r.t x ⇒ $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dx} = 0$

Find $\frac{dy}{dx} = m_2$

Find $m_1 \times m_2$, sub CR equations

10. An analytic function with constant real part is constant

Derivation $\dfrac{\partial u}{\partial x} = 0$, $\dfrac{\partial u}{\partial y} = 0$

u = constant

$\dfrac{\partial u}{\partial x} = 0 \Rightarrow \dfrac{\partial v}{\partial y} = 0$ & $\dfrac{\partial v}{\partial x} = 0$

$f(z) = c_1 + ic_2$

11. An analytic function w/ a constant modulus is constant.

$|f(z)| = \text{constant}$

$u^2 + v^2 = c^2$

diff. w.r.t x , diff. w.r.t y

$2uu_x + 2vv_x = 0$ & $2uu_y + 2vv_y = 0$

make all terms derivatives of x

square & add.

Do Ex 3, Ex 7 ***

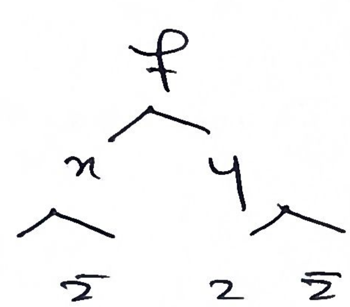
12. $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = \dfrac{4\partial^2}{\partial z \partial \bar{z}}$

i.e $\nabla^2 f(z) = \dfrac{4\partial^2}{\partial z \partial \bar{z}}$

Laplacian operator

Derivation $x = \dfrac{z + \bar{z}}{2}$ $y = \dfrac{z - \bar{z}}{2i}$

Find $\dfrac{\partial x}{\partial z}$, $\dfrac{\partial x}{\partial \bar{z}}$, $\dfrac{\partial y}{\partial z}$, $\dfrac{\partial y}{\partial \bar{z}}$



Find $\dfrac{\partial}{\partial z}$: $\dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial z} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial z}$ (1)

Do Ex 12

Find $\dfrac{\partial}{\partial \bar{z}}$: $\dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial \bar{z}} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial \bar{z}}$ (2)

multiply (1) & (2)

13. Construction of analytic functions using the Milne – Thomson method.

Case 1: when the real part is given

$f'(z) = u_x(z, 0) - i u_y(z, 0)$

Case 2: when the img. part is given

$f'(z) = v_y(z, 0) + i v_x(z, 0)$

Case 3: a combination of both

$u + v \Rightarrow F(z) = (1 - i) f(z)$

$3u + v \Rightarrow F(z) = (3 - i) f(z)$

Do Ex 1, Ex 11

14. Construction of an analytic fn. when the fn. is is polar form

$f'(z) = e^{-i\theta}(u_r + i v_r)$

integrate

15. Laplacian eqn. in polar form

$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$

16. Confomal Mappings

(i) $w = \frac{1}{z}$ $\quad x = \frac{u}{u^2 + v^2}$ $\quad y = \frac{-v}{u^2 + v^2}$

(ii) $w = z^2$ $\quad u = x^2 - y^2$ $\quad v = 2xy$

(iii) in polar form

$w = Re^{i\phi}$ $\quad w = z^2$

$z = re^{i\theta}$ $\quad R = r^2 \quad \phi = 2\theta$

***

Do ex 8, 151

~~ax^2+~~

$a(x^2+y^2) + bx + cy + d = 0$

Do 2nd ...

7. Bilinear Transformation

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Take out factor if ∞.

$\boxed{\text{Do Ex 1, 4, 6}}$

Chapter 2: Complex Integration

1. Line Integrals

$$\int f(z)\,dz$$

$$\int f(z)(dx + i\,dy)$$ write in terms of a single var.

$\boxed{\text{Do Ex 1, Exercise 1, 9}}$

2. Cauchy's Integral Theorem

Statement: If $f(z)$ is an analytic function of z & $f'(z)$ is continuous at all points inside and on a simple closed curve c, then $\oint f(z)\,dz = 0$

3. Cauchy's Integral Formula

Statement: If $f(z)$ is analytic within & on a simple closed curve c, and if a is any pt in c, then

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$\boxed{\text{Do Ex 2, Ex 5, Ex 10, Ex 11, Ex 12}}$ ***

4. Cauchy's Integral Formula for Derivatives

: If $f(z)$ is analytic inside and on a simple closed curve c & $z=a$ is any pt in the region enclosed by c, then

$$\int_c \frac{f(z)}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^n(a)$$

$\boxed{\text{Do Ex 3, 4, 5}}$

5. Taylor's Series

$$f(z) = f(a) + \frac{f'(a)(z-a)}{1!} + \frac{f''(a)(z-a)^2}{2!} + \frac{f'''(a)(z-a)^3}{3!} + \dots\dots$$

Do Ex 2, Ex 6

6. Laurent Series

(a) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots\dots \quad |x| < 1$

(b) $(1-x)^{-1} = 1 + x + x^2 + \dots\dots \quad |x| < 1$

(c) $e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots\dots$

(d) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \dots\dots$

Do Ex 2, Ex 4

(e) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\dots$

(f) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\dots$

7. Zeroes, Poles, Singularities & Residues

Methods of finding residues

(a) for simple poles: $\left(\frac{p(z)}{q'(z)}\right)_{z=a}$.

(b) $\lim_{z \to z_0} (z - z_0)^n f(z)$

(c) use the Laurent series expansion, coeff. of $\frac{1}{z}$

(d) pole of nth order: $\frac{1}{(n-1)!} \lim_{z \to a} \frac{\partial^{n-1}}{\partial z^{n-1}} \left\{ (z-a)^n f(z) \right\}$

8. Cauchy's Residue Theorem

$$\int_C f(z) = 2\pi i \sum R$$

$\boxed{\text{Do ex 2, 3, 6, 8, 10,}}$

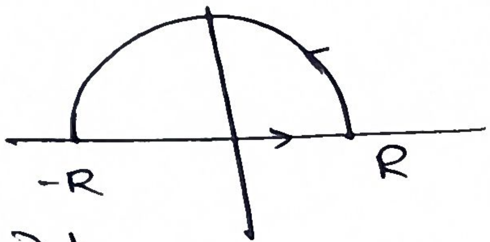
9. Evaluation of Real definite integrals

Type 1: $\displaystyle\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$

put $\cos\theta = \dfrac{z^2+1}{2z}$, $\sin\theta = \dfrac{z^2-1}{2iz}$, $d\theta = \dfrac{dz}{iz}$

Do Ex 8, 7, 6, 5, 4, 3, 2.

Type 2: $\displaystyle\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx.$



$$\int_C f(z)\, dz = \int_{-R}^{R} f(z)\, dz + \int_\Gamma f(z)\, dz$$

Consider the integral $\int_C f(z) dz$, where C is the closed contour consisting of the semicircle Γ of Radius R, large enough to include all the poles above the x axis and the real axis from $-R \to R$

as $R \to \infty$ $\displaystyle\int_\Gamma f(z)\, dz \to 0$

$\boxed{\text{Do Ex 2, 3, 4}}$

Type 3: $\displaystyle\int_{-\infty}^{\infty} \frac{p(x) \sin mx}{q(x)}$ or $\displaystyle\int_{-\infty}^{\infty} \frac{p(x) \cos mx}{q(x)}$

Let $f(z) = \dfrac{e^{imz}}{x^2 \text{ [illegible] fn}}$

$\boxed{\text{Do eq. 1, 2, 3, 4}}$

find poles

find $\displaystyle\int_{-\infty}^{\infty} f(z)\, dz$ equate real & img. parts

First shifting Theorem

$\mathcal{L}\{e^{at} f(t)\} = \mathcal{L}\{f(t)\}_{s \to s-a}$

Second shifting Theorem

$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t)\}$

Change of scale property

$\mathcal{L}\{f(at)\} = \frac{1}{a}\mathcal{L}\{f(t)\}_{s \to s/a}$

Laplace Transforms of derivatives

$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

Laplace Transforms of integrals

$\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \frac{\mathcal{L}\{f(t)\}}{s}$

Multiplication by t

$\mathcal{L}\{t^n f(t)\} = \frac{(-1)^n d^n}{ds^n}\mathcal{L}\{f(t)\}$

Division by t

$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \mathcal{L}\{f(t)\}\,ds$

Initial value Theorem

$f(0) = \lim_{t \to 0} f(t) = \lim_{s \to \infty} s\mathcal{L}\{f(t)\}$

Final value Theorem

$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} s\mathcal{L}\{f(t)\}$

LTs of periodic functions

$f(t+p) = f(t)$

$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt$

Inverse Laplace Transforms

First shifting theorem

$\mathcal{L}^{-1}(F(s-a)) = e^{at}\mathcal{L}^{-1}\{F(s)\}$

of derivatives

$\mathcal{L}^{-1}\{F'(s)\} = -t\,\mathcal{L}^{-1}\{F(s)\}$

use when $\mathcal{L}^{-1}\left\{\frac{\text{linear}}{(\text{quad term})^2}\right\}$

or $\mathcal{L}^{-1}$ of $\tan^{-1}$, log, $\cot^{-1}$

multiplication by s

$\mathcal{L}^{-1}(sF(s)) = \frac{d}{dt}\mathcal{L}^{-1}(F(s))$

use when degree of dr > degree of nr.

division by s

$\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t \mathcal{L}^{-1}F(s)\,dt$

change of scale

$\mathcal{L}^{-1}(F(as)) = \frac{1}{a}\mathcal{L}^{-1}(F(s))_{t \to t/a}$

second shifting theorem

~~$\mathcal{L}^{-1}(F(s-a)$~~

$\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)\,u(t-a)$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$

- $\cosh x = \frac{e^{x} + e^{-x}}{2}$
- $\sinh x = \frac{e^{x} - e^{-x}}{2}$
- $\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

1. $\mathcal{L}(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$

2. $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

3. $\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$

4. $\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}$

5. $\mathcal{L}(\sinh at) = \frac{a}{s^2-a^2}$

6. $\mathcal{L}(\cosh at) = \frac{s}{s^2-a^2}$

<u>Unit step function</u> / <u>Heavyside step function</u>

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

<u>Unit impulse function</u> / <u>Dirac delta function</u>

$$\delta(t-a) = \lim_{h \to 0} I(h, t-a)$$

$$I(h, t-a) = \begin{cases} \frac{1}{h}, & a \le t \le a+h \\ 0, & \text{otherwise} \end{cases}$$