Mathematics for Machine Learning

Unit 2

Linear Transformations

Linear transformations - one to one & onto linear transformations - null space, range space, dimension theorem (statement only) - matrix representation of linear transformations - eigen values, eigenvectors, diagonalization

* **Linear Transformation** : Let V and W be vector spaces over the same field F. Let T be the linear transformation from V to W denoted by $T: V \to W$. It is a linear transformation if:

(i) $T(x+y) = T(x) + T(y)$

(ii) $T(cx) = cT(x)$

P.T that $T(c_1x + c_2y) = c_1T(x) + c_2T(y)$

Zero Transformation : $T_0(v) = 0$

Identity Transformation: $I(v) = v$

One-One - If $T(x_1) = T(x_2)$
$\Rightarrow x_1 = x_2$

Onto : For $w \in W$, if there exists a $v \in V$, such that $T(v) = w$, then T is onto

Bijective : Both one-one and onto

* Null space and range space

Let V, W be vector spaces over the same field F. Let T be a transformation

$T: V \to W$

Then the set of all vectors $v \in V$ such that $T(v) = 0$ is called the null space. The set of all $T(v)$, $v \in V$ is called the range space / image space

* Dimension Theorem

- If V and W are vector spaces over the field F and if $T: V \to W$ is a linear transformation and if V is finite dimensional then

$$\boxed{\text{Rank}(T) + \text{Nullity}(T) = \dim(V)}$$

1. If T is linear, then p.t $T(x-y) = T(x) - T(y)$

Ans Since T is linear: $T(c_1x + c_2y) = c_1T(x) + c_2T(y)$

when $c_1 = 1$, $c_2 = -1$ $= T((1)(x) + (-1)(y))$

$$\boxed{T(x-y) = T(x) - T(y)}$$

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2. Let P be a fixed $m \times m$ matrix with entries over F & Q be a fixed $n \times n$ matrix over F. If T is a transformation defined on $F^{(m \times n)}$ by $T(A) = PAQ$, then p.t T is linear.

Ans $P = m \times m$

$\boxed{A = m \times n}$

$Q = n \times n$ Let $A, B \in F^{(m \times n)}$

$T(c_1A + c_2B) = P(c_1A + c_2B)Q$

$= (c_1PA + c_2PB)Q$

$= c_1PAQ + c_2PBQ$

$= c_1T(A) + c_2T(B)$

$\therefore$ is a linear transformation

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3. Let $T: R^2 \to R^2$ be defined by $T(a,b) = (2a+b, a)$. Show that T is linear.

Ans Let $(a_1,b_1), (a_2,b_2) \in R^2$

$T(c_1(a_1,b_1) + c_2(a_2,b_2)) = T(c_1a_1 + c_2a_2, c_1b_1 + c_2b_2)$

$= (2(c_1a_1 + c_2a_2) + c_1b_1 + c_2b_2,$

$c_1b_1 + c_2b_2)$

$= (2c_1a_1 + 2c_1b_1 + 2c_2a_2 + c_2b_2,\ c_1a_1 + c_2a_2)$

$= c_1(2a_1+b_1, a_1) + c_2(2a_2+b_2, a_2)$

$= c_1T(a_1,b_1) + c_2T(a_2,b_2)$

$\therefore$ is a linear transformation.

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(4) Check whether $T: R^2 \rightarrow R^2$ defined by $T(x,y) = (\sin x, y)$ is a linear transformation.

Ans Let $(x_1,y_1), (x_2,y_2) \in R^2$

$T(c_1(x_1,y_1) + c_2(x_2,y_2)) =$

$T(c_1x_1 + c_2x_2,\ c_1y_1 + c_2y_2)$

$= (\sin(c_1x_1 + c_2x_2),\ c_1y_1 + c_2y_2)$ — (1)

However:

$c_1T(x_1,y_1) + c_2T(x_2,y_2)$

$= c_1(\sin x_1, y_1) + c_2(\sin x_2, y_2)$ — (2)

(1) $\neq$ (2)

$\Rightarrow$ not linear

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(5) Is there a linear transformation: $T: R^3 \rightarrow R^2$, such that $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$?

Ans $T(-2,0,-6) = -2T(1,0,3)$

$= -2(1,1)$

$= (-2,-2)$ but $T(-2,0,6) = (2,1)$

$\Rightarrow$ Transformation does not exist.

6) P.T over the set of all polynomials of degree less than or equal to n, the transformation defined by:

$T(a_0 + a_1x + \dots a_nx^n) = (a_0, a_1 \dots a_n)$ is linear.

Ans Let $f = a_0 + a_1x + \dots a_nx^n$

$g = b_0 + b_1x + \dots b_nx^n$

$T[c_1f + c_2g] = T[c_1(a_0 + a_1x + \dots a_nx^n) + c_2(b_0 + b_1x + \dots b_nx^n)]$

$= T[c_1(a_0+b_0) + c_2(a_1x + b_1x) + \dots]$

$= (c_1a_0, c_1a_1, \dots c_1x^n) + (c_2b_0, c_2b_1, \dots c_2b_nx^n)$

$= c_1T(a_0 + a_1x + \dots a_nx^n) + c_2T(b_0 + b_1x + \dots b^n)$

$= c_1T(f) + c_2T(g)$

$\therefore$ is a linear transformation.

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7) Show that the transformation $T: R^3 \rightarrow R^2$ defined by $T(x, y, z) = (z, x+y)$ is linear.

Let $v = (x_1, y_1, z_1)$

$w = (x_2, y_2, z_2)$

$T(c_1v + c_2w) = T(c_1(x_1, y_1, z_1) + c_2(x_2, y_2, z_2))$

$= T(c_1x_1 + c_2x_2, c_1y_1 + c_2y_2, c_1z_1 + c_2z_2)$

$= (c_1z_1 + c_2z_2, c_1x_1 + c_2x_2 + c_1y_1 + c_2y_2)$

$= c_1(z_1, x_1 + y_1) + c_2(z_2, x_2 + y_2) = c_1T(x_1, y_1, z_1) + c_2T(x_2, y_2, z_2)$

$= c_1 T(v) + c_2 T(w)$

$\therefore$ is linear

(8) Find whether the transformation $T: R^2 \to R^3$ defined by

$T(x,y) = (x+1, 2y, x+y)$ is linear

Ans

Let $v = (x_1, y_1)$

$w = (x_2, y_2)$

$T(c_1 v + c_2 w) = T\left(c_1(x_1, y_1) + c_2(x_2, y_2)\right)$

$= T\left(c_1 x_1 + c_1 y_1 + c_2 x_2 + c_2 y_2\right)$

$= T\left(c_1 x_1 + c_2 x_2, \ c_1 y_1 + c_2 y_2\right)$

$= \left(c_1 x_1 + c_2 x_2 + 1, \ 2c_1 y_1 + 2c_2 y_2, \ c_1 x_1 + c_2 x_2 + c_1 y_1 + c_2 y_2\right)$

$\cancel{= c_1(x_1, 2y_1, x_1 + y_1)}$

$\neq c_1 T(v) + c_2 T(w)$ not linear

(9) If V is the vector space of all $n \times n$ matrices over F and if B is an arbitrary matrix in V, show that the transformation $T: V \to V$ show that the transformation $T: V \to V$ defined by $T(A) = AB - BA$ is linear. Show that $T(A) = A + B$ is not linear unless $B = 0$.

Let $v = A$

$w = A'$

$T(c_1 v + c_2 w) = T(c_1 A + c_2 A')$

$= (c_1 A + c_2 A')B - B(c_1 A + c_2 A')$

$= (c_1A + c_2A')B - B(c_1A + c_2A')$

$= c_1AB + c_2A'B - c_1BA + c_2BA'$

$= c_1(AB - BA) + c_2(A'B - BA') \quad -(1)$

$c_1T(v) + c_2T(w) = c_1(AB - BA) + c_2(A'B - BA') \quad (2)$

$(1) = (2)$

$\therefore$ is linear.

(ii)

$T(A) = A + B$

Let $v = A$

$w = A'$

$T(c_1v + c_2w) = T(c_1A + c_2A')$

$= c_1A + c_2A' + B \quad -(1)$

However: $c_1T(v) + c_2T(w) = c_1(A+B) + c_2(A' + B)$.

$= c_1A + c_2A' + 2B \quad -(2)$

$(1) \neq (2)$ unless $B = 0$

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(10) Find $T(1,0)$, where $T: R^2 \to R^3$ is defined by $T(1,2) = (3,2,1)$ and $T(3,4) = (6,5,4)$.

Ans $T(1,2) = (3,2,1)$

$T(3,4) = (6,5,4)$

$(1,0) = a(1,2) + b(3,4)$

$1 = a + 3b \qquad a = -2$

$0 = 2a + 4b \qquad b = 1$

$\Rightarrow T(1,0) = -2T(1,2) + T(3,4)$

$= -2(3,2,1) + 1(6,5,4)$

$= (0,1,2)$ //

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(11) Find a basis and dimension for R_T and N_T for the linear transformation $T: R^3 \to R^3$, defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3)$$

(X) write vertically

$\underline{\underline{R_T}}$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix}$$

convert to row echelon form:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & -4 & 4 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 + R_1$

$R_3 \leftrightarrow R_3 - 2R_1$

$2 + 2$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

~~$R_3 \leftrightarrow R_3 + 2R_1$~~

$R_3 \leftrightarrow R_3 + R_2$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis $= \{(1,2,-1), (0,1,-1)\}$

Dimension : $\dim(R_T) = 2$ //

N_T $\quad x_1 - x_2 + 2x_3 = 0$

$2x_1 + x_2 = 0$ $\quad$ write horizontally

$-x_1 - 2x_2 + 2x_3 = 0$

$$B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix}$$

convert to row echelon form:

$$B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix}$$ $R_2 \leftrightarrow R_2 - 2R_1$

$R_3 \leftrightarrow R_3 + R_1$ $\quad 1+2$

$$B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} //$$ $R_3 \leftrightarrow R_3 + R_2$

$\dim(N_T)$ = no. of unknowns - no. of non-zero rows

$= 3 - 2$

$= 1$

To Find the basis:

$x_1 - x_2 + 2x_3 = 0$

$3x_2 - 4x_3 = 0$

x_3 = free

when $x_3 = 3$

$x_1 - x_2 + 6 = 0$

$3x_2 - 12 = 0$

$\boxed{x_2 = 4}$ $\quad \boxed{x_1 = -2}$

The basis is $(-2, 4, 3)$ //

(12) Find a basis and dimension of R_T and N_T for the linear transformation $T: R^4 \to R^3$ defined by $T(x_1, x_2, x_3, x_4) =$

$(x_1 - x_2 + x_3 + x_4,\ x_1 + 2x_3 - x_4,\ x_1 + x_2 + 3x_3 - 3x_4)$

Ans R_T

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix}$$

converting to row echelon form :

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \quad \begin{array}{l} R_2 \leftrightarrow R_2 + R_1 \\ R_3 \leftrightarrow R_3 - R_1 \\ R_4 \leftrightarrow R_4 - R_1 \end{array}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_3 \leftrightarrow R_3 - R_2 \\ R_4 \leftrightarrow R_4 + 2R_2 \end{array}$$

basis of $R_T = \{(1,1,1), (0,1,2)\}$

$\dim(R_T) = 2$

N_T $$B = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix}$$

Converting to row echelon form

$$B = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 - R_1$

$R_3 \leftrightarrow R_3 - R_1$

$$B = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - R_2$

$$B = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim(N_T) = 4 - 2 = 2$

basis:

$x_1 - x_2 + x_3 + x_4 = 0$

$x_2 + x_3 - 2x_4 = 0$

Let $x_3 = 0 \quad x_2 = 2 \quad x_4 = 1 \quad x_1 = 1$

$2 + 0 -$

$x_3 = 2, \; x_2 = 0 \quad x_4 = 1 \quad x_1 = 1$ check

(13) If V is the vector space of 2×2 matrices, if $M = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$

and if $T: V \to V$ be the linear transformation defined by

$T(A) = MA$, Find the basis & dimension of N_T and R_T

Ans $M = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$

Find images of standard basis.

$$= \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$$

write each of these as a row vector

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$

convert to row echelon form

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix} \quad R_3 \leftrightarrow R_3 + R_1$$

R3

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \leftrightarrow R_4 + R_2$$

$\dim(R_T) = 2$ basis $= \left\{ \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \right\}$

N_T

Let $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ be an element of N_T

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \Rightarrow \left.\begin{matrix} x_1 - x_3 = 0 \\ x_2 - x_4 = 0 \end{matrix}\right\}$$

$-x_1 + x_3 = 0$ ×

$\dim(N_T) = 2$ $\quad -x_2 + x_4 = 0$

Let x_3, x_4 be free variables

when $x_3 = 0 \quad x_4 = 1$

$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad x_1 = 0$, $x_2 = 1$

when $x_3 = 1 \quad x_4 = 0$ $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

(14) Find a linear transformation $R^3 \to R^3$ whose range space R_T is generated by (1,2,3) and (4,5,6).

Ans

$T(x,y,z) = ?$

$T(1,0,0) = (1,2,3)$

$T(0,1,0) = (4,5,6)$

$T(0,0,1) = (0,0,0)$

$T(x,y,z) = (x+4y, 2x+5y, 3x+6y)$

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(15) Find a linear transformation $T: R^4 \to R^3$ whose null space N_T is generated by (1,2,3,4) and (0,1,1,1)

Ans

make sets of equations

$a_1 + 2b_1 + 3c_1 + 4d_1 = 0$ (1) & $0a_1 + b_1 + c_1 + d_1 = 0$ (2)

$a_2 + 2b_2 + 3c_2 + 4d_2 = 0$ (3) & $0a_2 + b_2 + c_2 + d_2 = 0$ (4)

$a_3 + 2b_3 + 3c_3 + 4d_3 = 0$ (5) & $b_3 + c_3 + d_3 = 0$ (6)

A. in (1), (2) $d_1 = 0$

$\Rightarrow a_1 + 2b + 3c_1 = 0$ — A

$b_1 + c_1 + 0 = 0$ — B ~~A-B ⇒ $a_1 + b_1 + 2c_1 = 0$~~

A+B

$\Rightarrow a_1 + 3b_1 + 4c_1 = 0$

If $a_1 = -1$

$b_1 = -1$ $(-1, -1, 1)$

$c_1 = 1$

B. In (3) (4), take $c_2 = 0$

$a_2 + 2b_2 + 4d_2 = 0$ — C

$b_2 + d_2 = 0$ — D

$a_2 + 3b_2 + 5d_2 = 0$

If $b_2 = -1$

$d_2 = 1$ $\quad (-2, -1, 1)$

$a_2 = -2$

C. In (5) (6) take $b_3 = 0$

$a_3 + 3c_3 + 4d_3 = 0$

$c_3 + d_3 = 0$

$a_3 + 4c_3 + 5d_3 = 0$

If $c_3 = -1$

$d_3 = 1$ $\quad (-1, -1, 1)$

$a_3 = -1$

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(16) Suppose $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ then $T: R^2 \to R^2$ is a linear transformation.

(i) Is T onto?

(ii) Is T one to one?

Let $\begin{pmatrix} a \\ b \end{pmatrix} \in R^2$

Assume $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{matrix} x + y = a \\ x + 2y = b \end{matrix}$$

Consider the augmented matrix

$$\begin{bmatrix} 1 & 1 & a \\ 1 & 2 & b \end{bmatrix}$$

convert to row echelon form

$$\begin{bmatrix} 1 & 1 & a \\ 0 & 1 & b-a \end{bmatrix}$$

$x+y=a$

$\boxed{y=b-a}$

$x+b-a=a$

$\boxed{x=2a-b}$

The system is consistent $\Rightarrow$ it is onto

one - one

Assume $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x+y=0$

$x+2y=0$

if $x=0, y=0$

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow$ it is one-one.

(17) Let $T: R^4 \to R^2$ be a linear transformation defined by

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+d \\ b+c \end{pmatrix}$$

Let $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2$

Let $\begin{pmatrix} x \\ y \\ 0 \\ 0 \end{pmatrix} \in R^4$

then $T\begin{pmatrix} x \\ y \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x+0 \\ y+0 \end{pmatrix} \in R^2$

$\Rightarrow$ It is onto

<u>one - one</u>

Let $\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \in R^4$

then $T\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{bmatrix} 1-1 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow$ <u>not one-one</u>

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18 ★★★

Q. If $T: R^3 \to R^2$ is the linear transformation defined by $T(x,y,z) = (2x+y-z,\ 3x-2y+4z)$, find the matrix of T relative to the bases e and f where $\{e_1 = (1,1,1),\ e_2 = (1,1,0)$ and $e_3 = (1,0,0)\}$ and $\{f_1 = (1,3),\ f_2 = (1,4)\}$. Also verify that $[T(v)]_f = [T]_e^f [v]_e$

<u>Ans</u> $T(x,y,z) = (2x+y-z,\ 3x-2y+4z)$

$[T]_e^f = [T(e_1)\ T(e_2)\ T(e_3)]$

$(a,b) = c_1 f_1 + c_2 f_2$

$(a,b) = c_1(1,3) + c_2(1,4)$

$$\boxed{\begin{aligned} c_1 + c_2 &= a \\ 3c_1 + 4c_2 &= b \end{aligned}}$$

$c_1 + c_2 = a$ $\times 3$

$3a + 4c_2 = b$

$4c_1 + 4c_2 = 4a$

$3c_1 + 4c_2 = b$

$+c_1 = 4a - b$

$\boxed{-b + 4a = c_1}$

$3c_1 + 3c_2 = 3a$

$3c_1 + 4c_2 = b$ $(-)$

$-c_2 = 3a - b$

$\boxed{c_2 = b - 3a}$

To find $[T]_e^f$

$(a,b) = (4a-b)f_1 + (b-3a)f_2$ — *

$T(x,y,z) = (2x+y-z,\ 3x-2y+4z)$

$T(e_1) = T(1,1,1) = (2,5) = 3f_1 - f_2$

$T(e_2) = T(1,1,0) = (3,1) = 11f_1 - 8f_2$

$T(e_3) = T(1,0,0) = (2,3) = 5f_1 - 3f_2$

$\therefore [T]_e^f = \begin{bmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{bmatrix}$ write coefficients vertically

To find $[v]_e$

$(a,b,c) = c_1 e_1 + c_2 e_2 + c_3 e_3$

$= c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$

$\boxed{a = c_1 + c_2 + c_3}$ $\boxed{b = c_1 + c_2}$ $\boxed{c = c_1}$

Ans:

(i) $T(1,0,0) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$T(0,1,0) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$

$T(0,0,1) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$

$\therefore [T]_e^f = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} = A$ //

(ii) ~~$T(a,b,c) = c_1e_1 + c_2e_2 + c_2e_3$~~

$T(a,b) = c_1f_1 + c_2f_2$

$T(a,b) = c_1(1,3) + c_2(2,5)$

$a = c_1 + 2c_2 \quad \times 3$ ~~$5a = 5c_1 + 10c_2$~~

$b = 3c_1 + 5c_2$ ~~$b = 3c_1 + 5c_2$~~

$3a = 3c_1 + 6c_2$ ~~$5a - b = 2c_1 - 3$~~

$b = 3c_1 + 5c_2$ ~~$c_1 = \frac{5a-b}{2}$~~

$3a - b = c_2$

$\boxed{c_2 = \frac{-b+3a}{}}$ $\boxed{c_1 = -5a + 2b}$

$$\Rightarrow T(a,b) = \left(\frac{a+b}{3}\right)e_1 + \left(\frac{2a-b}{3}\right)e_2$$

$$[T]_e : T(e_1) = (-2,1) = \frac{-1}{3}e_1 + \frac{-5}{3}e_2$$

$$T(e_2) = (1,1) = \frac{2}{3}e_1 + \frac{1}{3}e_2$$

$$[T]_e = \begin{bmatrix} -1/3 & 2/3 \\ -5/3 & 1/3 \end{bmatrix}$$

---

(20) The matrix $A = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}$ determines a linear transformation

$T: R^3 \to R^2$, defined by $T(v) = Av$, where v is a column vector

(i) Show that the matrix representation of T relative to the usual bases of R^3 and R^2 is A itself.

(ii) Find the matrix representation of T relative to the following bases of R^3 and R^2

$$e = \{e_1 = (1,1,1),\ e_2 = (1,1,0),\ e_3 = (1,0,0)\} \text{ and}$$

$$f = \{f_1 = (1,3),\ f_2 = (2,5)\}$$

RHS $[T]_e^f [v]_e$

$$= \begin{bmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{bmatrix} \begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix}$$

$$= \begin{bmatrix} 5a+6b-8c \\ -3a-5b+7c \end{bmatrix} = [T(v)]_f = LHS$$

LHS = RHS

hence proved.

(19) If T is the linear operator on R^2 defined by $T(x_1, x_2) = (-x_2, x_1)$, find the matrix of the basis $e = \{e_1 = (1,2), e_2 = (1,-1)\}$

Ans To find : $[T]_e$

$T(a_1, a_2) = c_1 e_1 + c_2 e_2$

$= c_1(1,2) + c_2(1,-1)$

$\boxed{a = c_1 + c_2}$ $\quad a+b = 3c_1$

$\boxed{b = 2c_1 - c_2}$ $\quad \boxed{c_1 = \frac{a+b}{3}}$

$2a = 2c_1 + 2c_2$

$b = 2c_1 - c_2$ $\quad (-)$

$2a - b = 3c_2$ $\quad \boxed{c_2 = \frac{2a-b}{3}}$

$\boxed{c_1 = c}$ $\quad b = c_1 + c_2$ $\quad \boxed{c_2 = b - c}$

$a = b + c_3$ $\quad \boxed{c_3 = a - b}$

$\therefore (a, b, c) = c e_1 + (b-c) e_2 + (a-b) e_3$

$$[v]_e = \begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix}$$

To prove: $[T(v)]_f = [T]_e^f [v]_e$

$T(v) = T(a, b, c) = (2a + b - c,\ 3a - 2b + 4c)$

From *

$[T(v)]_f = \left[4(2a+b-c) - (3a-2b+4c)\right] f_1$

$+ \left[(3a - 2b + 4c) - 3(2a + b - c)\right] f_2$

$= \left[8a + 4b - 4c \quad - 3a + 2b - 4c\right] f_1 +$

$\left[3a - 2b + 4c - 6a - 3b + 3c\right] f_2$

$= (5a + 6b - 8c) f_1 + (-3a - 5b + 7c) f_2$

$$\therefore [T(v)]_f = \begin{bmatrix} 5a + 6b - 8c \\ -3a - 5b + 7c \end{bmatrix}$$ //

$T(a,b) = \cancel{a}(3a-b)f_2 + (-5a+2b)f_1$

$[T]_e^f = T(e_1) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 8f_2 + -12f_1$

$-12f_1 + 8f_2$

$T(e_2) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 7 \\ -3 \end{pmatrix} = -41f_1 + 24f_2$

$T(e_3) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -8f_1 + 5f_2$

$\therefore [T]_e^f = \begin{bmatrix} -12 & -41 & -8 \\ 8 & 24 & 5 \end{bmatrix}$ //

---

(21) If $T(x,y) = (2x-3y, x+y)$, find $[T]_e$, where $e = f = \{e_1 = (1,2), e_2 (2,3)\}$. Verify also that $[T]_e [v]_e = [T(v)]_e$ for any $v \in \mathbb{R}^2$.

Ans find $[T]_e$

$T(a,b) = c_1e_1 + c_2e_2$

$T(a,b) = c_1(1,2) + c_2(2,3) \Rightarrow$

$a = c_1 + 2c_2 \quad \times 2$

$b = 2c_1 + 3c_2$

$2a = 2c_1 + 4c_2$

$b = 2c_1 + 3c_2$

$c_2 = 2a - b$

$c_1 = -3a + 2b$

$T(a,b) = (-3a+b)e_1 + (2a-b)e_2$ *

$[T]_e = T(e_1) = T(1,2) = (-4,3) = 15e_1 - 11e_2$

$= T(e_2) = T(2,3) = (-5,5) = 20e_1 - 15e_2$

$$[T]_e = \begin{bmatrix} 15 & 20 \\ -11 & -15 \end{bmatrix}$$ //

To find $[v]_e$

$T(a,b) = c_1 e_1 + c_2 e_2$

$c_1 = -3a + 2b$

$c_2 = 2a - b$

$$[v]_e = \begin{bmatrix} -3a+2b \\ 2a-b \end{bmatrix}$$

To find $[T(v)]_e$

$T(v) = (\overset{a}{2a-3b}, \overset{b}{a+b})$

from *

$=$

$\cancel{2(-3a+b) - (2a-b)}$

$= [-3(2a-3b) + 1(a+b)]e_1 + [2(2a-3b) - (a+b)]e_2$

$= (-6a+9b+a+b)e_1 + (4a-6b-a-b)e_2$

$= (-5a+10b)e_1 + (3a-7b)e_2$

$$[T]_e[v]_e = \begin{bmatrix} 15 & 20 \\ -11 & -15 \end{bmatrix} \begin{bmatrix} -3a + 2b \\ 2a - b \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{-45a + 40a} \overset{30b}{} - 60a \\ -45a + 30b + 40a - 20b \\ 33a - 22b \quad -30a + 15b \end{bmatrix} = \begin{bmatrix} -5a + 10b \\ 3a - 7b \end{bmatrix}$$

$\therefore [T(v)]_e = [T]_e [v]_e$ → just a coeff. matrix

find coeff

sub in condition

put q given values and sub eqn from $[T]_e$

---

(22) If V is the vector space of 2×2 matrices over R and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find the matrix of the linear operator on V in the usual basis when

(i) $T(A) = MA$

(ii) $T(A) = AM$

(iii) $T(A) = MA - AM$

Ans $T(A) = MA$

$\Rightarrow E_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$

$E_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$

$E_3 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}$

$E_4 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$

$$\therefore [T]_e = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$$

(ii) $T(A) = MA$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$$

$$[T]_E = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{bmatrix}$$

(iii) $T(A) = MA - AM$

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$$

$\vdots$

$$\cancel{\phi}[E]_E = \begin{bmatrix} 0 & -c & b & 0 \\ -b & (a-d) & 0 & b \\ c & 0 & (d-a) & -c \\ 0 & c & -b & 0 \end{bmatrix}$$

23) In the vector space of polynomials in x of degree ≤ 3 if (27)

$D: V \to V$ is the differential operator defined by $Df(x) = \frac{d}{dx} f(x)$.

Find the matrix of D in the basis $(1, x, x^2, x^3)$.

Verify that $[D]_e [f(x)]_e = [Df(x)]_e$

where $f(x) = a + bx + cx_2 + dx_3$

<u>Ans</u> $[D]_e = \begin{bmatrix} \text{matrix} \\ \text{with differentiated} \\ \text{eqns} \end{bmatrix}$

~~$D(e_1) = \frac{d}{dx} = a + b + c + d = 0$~~

constant $= e_1$

$x = e_2$

$x^2 = e_3$

$x^3 = e_4$

~~$D(e_2) = \frac{d}{dx} = a + bx + cx + dx =$~~

$D(e_1) = \frac{d}{dx}(1) = 0 = 0 + 0x + 0x^2 + 0x^3$

$D(e_2) = \frac{d}{dx}(x) = 1 = 1 + 0x + 0x^2 + 0x^3$

$D(e_2) = \frac{d}{dx}(x^2) = 2x = 0 + 2x + 0x^2 + 0x^3$

$D(e_3) = \frac{d}{dx}(x^3) = 3x^2 = 0 + 0x + 3x^2 + 0x^3$

$[D]_e = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$[f(x)]_e = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$[D(f(x))]_e = \frac{d}{dx}(a + bx + cx^2 + dx^3)$$

$$= b + 2cx + 3dx^2$$

$$= \begin{bmatrix} b \\ 2c \\ 3d \\ 0 \end{bmatrix}$$

$$LHS = [D]_e [f(x)]_e$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ 2b \\ 3c \\ 0 \end{bmatrix}$$

$$= [D(f(x))]_e$$

LHS = RHS

hence proved

24) If V is a 2 dimensional vector space over R and if T is a linear operator on V such that its matrix rep. in the usual basis is $[T]_e = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, prove that

$T^2 - (a+d)T + (ad-bc)I = 0.$

Ans $T(x,y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$= (ax+by,\ cx+dy)$

$T^2(x,y) = T(T(x,y)) = T(\overset{x}{ax+by},\ \overset{y?}{cx+dy})$

~~$= ((ax+by)x + b(cx+dy)y,$~~

$= (a(ax+by)+b(cx+dy),\ c(ax+by)+d(cx+dy))$

$= (a^2x+aby+bcx+bdy,\ acx+bcy+cdx+d^2y)$ — (1)

$-(a+d)T = -(a+d)(ax+by,\ cx+dy)$

$= -(a^2x+aby+adx+dby,\ acx+ady+d^2y)$ — (2)

$(ad-bc)I(x,y) = (ab-bc)(x,y)$

$= (abx, -byc)$ — (3)

(1) + (2) + (3) $= 0$

25) Find the eigenvalues and eigen vectors of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

Verify that their sum and product are equal to the sum 2 trace of A and $|A|$ respectively.

Ans $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

characteristic equation:

$$\lambda^3 - (\text{sum of diag})\lambda^2 + \left(\begin{array}{c}\text{sum of minors} \\ \text{of diag elements}\end{array}\right)\lambda - |A| = 0$$

Trace(A) $= 7$

$$\text{sum. of minors} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix}.$$

$$= 8 + 3 + 3$$

$$= 14$$

$|A| = 8$

$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

$$\boxed{\begin{array}{l} \lambda_1 = 4 \\ \lambda_2 = 1 \\ \lambda_3 = 2 \end{array}}$$

when $\boxed{\lambda = 1}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 2 & -1 & 0 & 2 \\ -1 & 2 & 1. & -1 \end{array}$$

$$\frac{x_1}{3} = \frac{x_2}{-1} = \frac{x_3}{-2}$$

$\underline{(3, -1, -2)}$

when $\boxed{\lambda = 2}$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & -1 & & 0 & & 1 \\ -1 & & 1 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$\boxed{(0,-1,-1)}$

when $\boxed{\lambda = 4}$

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -1 & & -1 & & 0 & & -1 \\ -1 & & -1 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{1} \qquad (0,-1,1)$$

sum of eigenvalues = $1+2+4 = 7$ = trace

product of eigen values $= 8 = |A|$ //

(26) Verify that the eigen values of A^2 and A^{-1} are respectively the squares and reciprocals of the eigen values of A, given that

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Ans A is a Δ^{er} matrix

$\Rightarrow$ eigen values are main diagonal values

$$\boxed{\begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= 2 \\ \lambda_3 &= 5 \end{aligned}}$$

$$A^2 = \begin{bmatrix} 9 & 5 & 38 \\ 0 & 4 & 42 \\ 0 & 0 & 25 \end{bmatrix}$$

A^2 is also a right Δ^{er} matrix $\Rightarrow$ eigen values are

$\lambda_1 = 9, \lambda_2 = 4, \lambda_3 = 25$

= square of eigen values of A

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

$$= \begin{bmatrix} 1/3 & -1/6 & -1/15 \\ 0 & 1/2 & -3/5 \\ 0 & 0 & 1/5 \end{bmatrix}$$

eigen values = $1/3, 1/2, 1/5$

= reciprocal of eigen values of A.

(27) Verify that the eigen vectors of the following real symmetric matrix are orthogonal in pairs.

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

<u>Ans</u> characteristic equation

$$\lambda^3 - 11\lambda^2 + (\text{sum of minors of diag})\lambda - |A| = 0$$

$|A| = 36$

$$\text{sum of minors} = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= 14 + 8 + 14$$

$$= 36$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\boxed{\begin{array}{l} \lambda_1 = 6 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}}$$

when $\boxed{\lambda = 2}$ $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

3-2 = 1
5-2 = 3
3-2 = 1

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & & \\ -1 & & 1 & & 1 & & -1 & \\ 3 & & -1 & & -1 & & 3 & \end{array}$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$

$$\boxed{X_1 = (-1, 0, 1)}$$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

when $\boxed{\lambda = 3}$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -1 & 1 & 0 & -1 \\ 2 & -1 & -1 & 2 \end{array}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\boxed{X_2 = (1,1,1)}$$

when $\boxed{\lambda = 6}$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -1 & 1 & -3 & -1 \\ -1 & -1 & -1 & -1 \end{array}$$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}$$

$$\boxed{X_3 = (1,-2,1)}$$

$-1 \neq 3$

$3 - 1$

$X_1 X_2^T = 0$

$X_2 X_3^T = 0$

$X_3 X_1^T = 0$

$\therefore$ The eigenvectors are orthogonal in pairs

28 Diagonalize the matrix $A = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$ using similarity transformation and hence find A^4

$$A = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

similarity transformation

$P^{-1} A P = D$

Characteristic equation

$\lambda^3 - 0\lambda^2 + (\text{sum of minors}) \cancel{\lambda} \mp |A| = 0$

$|A| = -12$

sum of minor elements $= \cancel{\lambda^3} - \cancel{0\lambda}^2$

$$\begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} 2 & -7 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}$$

$= -5 - 6 + -2$

$= -13$

$\lambda^3 - 0\lambda^2 - 13\lambda^2 + 12 = 0$

$$\boxed{\begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= -4 \\ \lambda_3 &= 1 \end{aligned}}$$

when $\boxed{\lambda = 1}$

$$\begin{bmatrix} 1 & 2 & -7 \\ 2 & 0 & 2 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ 2 & & -7 & & 1 & & 2 \\ 0 & & 2 & & 2 & & 0 \end{array}$$

$$\frac{x_1}{4} = \frac{x_2}{-16} = \frac{x_3}{-4}$$

$$\boxed{X_1 = (1, -4, -1)}$$

when $\boxed{\lambda = 3}$

$$A = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -7 \\ 2 & -2 & 2 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{array}{cccccccccc} & x_1 & & & x_2 & & & x_3 & \\ 2 & & -7 & & & -1 & & & 2 \\ -2 & & 2 & & & 2 & & & -2 \end{array}$$

$$\frac{x_1}{-10} = \frac{x_2}{-12} = \frac{x_3}{-2}$$

$-14+2$

$2-4$

$$\boxed{X_2 = (5, 6, 1)}$$

when $\boxed{\lambda = -4}$

$$\begin{bmatrix} 6 & 2 & -7 \\ 2 & 5 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & & \\ 2 & & -7 & & 6 & & 2 \\ 5 & & 2 & & 2 & & 5 \end{array}$$

$-3+4$

$4+12$

$-14-12$
$30-4$

$$\frac{x_1}{39} = \frac{x_2}{-26} = \frac{x_3}{26}$$

$-14-12$

$30-4$

$$\boxed{X_3 = (3, -2, 2)}$$

$$P = \begin{bmatrix} 1 & 5 & 3 \\ -4 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} \operatorname{adj} P$$

$$P^{-1} = \begin{bmatrix} 1/5 & -1/10 & -2/5 \\ 1/7 & 1/14 & -1/7 \\ 1/35 & -3/35 & 13/35 \end{bmatrix}$$

Taking LCM

$$P^{-1} = \frac{1}{70} \begin{bmatrix} 14 & -7 & -28 \\ 10 & 5 & -10 \\ 2 & -6 & 26 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{70} \begin{bmatrix} 14 & -7 & -28 \\ 10 & 5 & -10 \\ 2 & -6 & 26 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}}_{\text{check}} \begin{bmatrix} 1 & 5 & 3 \\ -4 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

$$\frac{1}{70} \begin{bmatrix} 14 & -7 & 28 \\ 10 & 5 & -10 \\ 2 & -6 & 26 \end{bmatrix} \begin{bmatrix} 1 & 15 & -12 \\ -4 & 18 & 8 \\ -1 & 3 & -8 \end{bmatrix}$$

$$= \frac{1}{70} \begin{bmatrix} 70 & 0 & 0 \\ 0 & 210 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= D$$

To find A^4

$$A^4 = P D^4 P^{-1}$$

$$= \frac{1}{70}\begin{bmatrix} 1 & 5 & 3 \\ -4 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 256 \end{bmatrix}\begin{bmatrix} 14 & -7 & -28 \\ 10 & 5 & -10 \\ 2 & -6 & 26 \end{bmatrix}$$

$$= \frac{1}{70}\begin{bmatrix} 5600 & -2590 & 15890 \\ 3780 & 5530 & -18060 \\ 1820 & -2660 & 12530 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & -37 & 227 \\ 54 & 79 & -258 \\ 26 & -38 & -179 \end{bmatrix}$$

(29) Find the matrix P that diagonalizes the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ by means of similarity transformation.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$|A| = 5$

sum of minors $= \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$

$= 4 + 3 + 4 = 11$

characteristic equation $= \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$$\boxed{\lambda_1 = 5,\ \lambda_2 = 1,\ \lambda_3 = 1}$$

when $\boxed{\lambda = 5}$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & & \\ 2 & & 1 & & -3 & & 2 & \\ -2 & & 1 & & 1 & & -2 & \end{array}$$

$$\frac{x_1}{4} = \frac{x_2}{\cancel{5}4} = \frac{x_3}{4}$$

$$\boxed{X_1 = (1,1,1)}$$

when $\boxed{\lambda = 1}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1 + 2x_2 + x_3 = 0$

IF $x_1 = 1$ $\quad X_2 = (1, 0, -1)$

$x_2 = 0$

$x_3 = -1$ $\quad X_3 = (2, -1, 0)$

IF $x_1 = 2$

$x_2 = -1$

$x_3 = 0$

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

//

(30) Diagonalize the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by means of an orthogonal transformation.

orthogonal transformation
$Q^T A Q = D$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$$

$|A| = -4$

sum of minors $= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$

$= -3 + 1 + 1$

$= \underline{\underline{-1}}$

$1 - 4$

$2 - 1$

characteristic equation

$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$$\boxed{\begin{aligned} \lambda_1 &= 4 \\ \lambda_2 &= -1 \\ \lambda_3 &= 1 \end{aligned}}$$

when $\boxed{\lambda_1 = 4}$

$$\begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

	x_1		x_2		x_3	
1		-1		-2		1
-3		-2		1		-3

$-2 - 3$

$-1 - 4$

$+6 - 1$

$$\frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{5}$$

$\boxed{X_1 = (-1, -1, 1)}$

when $\boxed{\lambda_2 = -1}$

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{array}{ccccccc} x_1 & & x_2 & & x_3 & & \\ 1 & & -1 & & 3 & & 1 \\ 2 & & -2 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5} \qquad \boxed{X_2 = (0, 1, 1)}$$

$-2+2$

$-1+6$

when $\boxed{\lambda_3 = 1}$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{array}{ccccccc} x_1 & & x_2 & & x_3 & & \\ 1 & & -1 & & 1 & & 1 \\ 0 & & -2 & & 1 & & 0 \end{array}$$

$-1+2$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1} \qquad \boxed{X_3 = (-2, 1, -1)}$$

Modal matrix $= \begin{bmatrix} -1 & 0 & -2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$Q = \begin{bmatrix} -1 & 0 & -2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \qquad Q^T = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ -2 & 1 & -1 \end{bmatrix}$$

$$Q^T A Q = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 0 & -2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} -1/\sqrt{3} & 0 & -2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \qquad Q^T = \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$$

$$Q^T A Q = \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 0 & -2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

$$= D$$

(31) Diagonalize the matrix $\begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$ by means of an orthogonal transformation. (43)

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

$|A| = -72$

sum of minors $= \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix}$

$= \cancel{12} + -\cancel{12} \quad 12$

$= 12$

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characteristic eqn.

$\lambda^3 - 10\lambda^2 + 12\lambda^2 + 72 = 0$

eigen values : $\boxed{\lambda_1 = -2 \quad \lambda_2 = 6 \quad \lambda_3 = 6}$

when $\boxed{\lambda_1 = -2}$

$$\begin{bmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

x_1		x_2		x_3	
0	4	4	0		
8	8	0	8		

$$\frac{x_1}{-32} = \frac{x_2}{0} = \frac{x_3}{32}$$

$\boxed{X_1 = (+1, 0, -1)}$

when $\boxed{\lambda = 6}$

$$\begin{bmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{bmatrix}$$

$$-x_1 + 0x_2 + x_3 = 0$$

choose such that the vectors are orthogonal to each othor

let $x_2 = 0$

$$\boxed{X_2 = (-1, 0, 1)}$$

Let $X_3 = (a, b, c)$

$$-a + c = 0$$

and $a - c = 0$

$\Rightarrow a = c = 0$

$$\boxed{X_3 = (0, 1, 0)}$$

The modal matrix is: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$ $Q^T = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$

P.T $Q^T A Q = D(-2, 6, 6)$ //