Mathematics for Machine Learning

Unit 4

Matrix Decomposition

Cholesky Decomposition method - QR decomposition - generalized inverse of a matrix - singular value decomposition

* Cholesky Decomposition

A is a real system and positive definite

$A = LL^T$, where L is a lower triangular matrix

$$l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2}$$

$$l_{ij} = \frac{1}{l_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk} \right) \quad (i > j)$$

* QR Decomposition (Modified Gram - Schmidt Process)

For every $m \times n$ matrix A - there exists a factorization as the product of Q having orthonormal vectors for its columns (unitary matrix) and an upper triangular matrix R.

Q matrix - Gram - Schmidt Orthonormalization

R matrix - $r_{ii} = \|u_i\|$ $\quad r_{ij} = \langle w_i, v_j \rangle$

orthonormal vector column

* Generalized Inverse of a Matrix

also called Pseudo inverse

moore - penrose generalized inverse.

denoted by $A^+ = V \Sigma_1 U^H$

Σ_1 = diagonal matrix of order $n \times m$ (only has value in the top left corner)

U & V are unitary matrices.

* Singular Value Decomposition.

Here $A = U \Sigma V^T$

Steps:
(i) find $A^T A$

(ii) compute eigen values & eigen vectors of $A^T A$

(iii) normalize vectors

(iv) construct $\Sigma = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(v) compute U, using the relation $Av = \begin{cases} \sigma_i u_i, & \text{if } 1 \le i \le r \\ 0, & \text{if } i > r \end{cases}$

1. Find the cholesky decomposition of $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 5 \end{bmatrix}$

$A = LL^T$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 5 \end{bmatrix}$$

for each row in L, go through each column in L^T

$l_{11}^2 = 3$

$\boxed{l_{11} = \sqrt{3}}$

$l_{21} \cdot l_{11} + 0 \cdot l_2 + 0 = 1$

$l_{21} \cdot \sqrt{3} = 1$

$\boxed{l_{21} = \frac{1}{\sqrt{3}}}$

$l_{11} \cdot l_{31} = -1$

$\sqrt{3} \cdot l_{31} = -1$

$\boxed{l_{31} = \frac{-1}{\sqrt{3}}}$

$l_{21} \cdot l_{11} = 1$

$l_{21} \cdot \sqrt{3} = 1$

$l_{21} = \frac{1}{\sqrt{3}}$

$l_{21} \cdot l_{21} + l_{22} \cdot l_{22} = 3$

$l_{21}^2 + l_{22}^2 = 3$

$\frac{1}{3} + l_{22}^2 = 3$

$l_{22}^2 = 3 - 1/3 = \frac{8}{3}$

$\boxed{l_{22} = \sqrt{\frac{8}{3}}}$

$l_{21} \cdot l_{31} + l_{22} \cdot l_{32} = 1$

$\frac{1}{\sqrt{3}} \cdot \frac{-1}{\sqrt{3}} + \sqrt{\frac{8}{3}} \cdot l_{32} = 1$

$\frac{-1}{3} + \sqrt{\frac{8}{3}}\, l_{32} = 1$

$\sqrt{\frac{8}{3}} \cdot l_{32} = \frac{4}{3}$

$\frac{\cancel{8}^{2}}{\cancel{3}_{1}} l_{32}^2 = \frac{16\,\cancel{4}^{4}}{\cancel{9}_{3}}^{2}$

$\boxed{l_{32} = \sqrt{\frac{2}{3}}}$

$\cancel{l_{31} \cdot l_{31} + l_{32}}$

$l_{31} \cdot l_{11} = -1$

$l_{31} \cdot \sqrt{3} = -1$

$$\boxed{l_{31} = \frac{-1}{\sqrt{3}}}$$

$l_{31} \cdot l_{21} + l_{32} \cdot l_{22} = 1$

$\frac{-1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + l_{32} \cdot \sqrt{\frac{8}{3}} = 1$

$l_{32} = \frac{\sqrt{2}}{\sqrt{3}}$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 5$

$\frac{1}{3} + \frac{2}{3} + l_{33}^2 = 5$

$l_{33}^2 = 4$

$$\boxed{l_{33} = 2}$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & \sqrt{8/3} & 0 \\ -1/\sqrt{3} & \sqrt{2}/\sqrt{3} & 2 \end{bmatrix}$$

(2) Find the Cholesky decomposition of $A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$ (3)

Ans

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$$

$l_{11}^2 = 4$

$\boxed{l_{11} = 2}$

$l_{11} . l_{21} = 12$

$\boxed{l_{21} = 6}$

$l_{11} . l_{31} = -16$

$\boxed{l_{31} = -8}$

$l_{21}^2 + l_{22}^2 = 37$

$36 + l_{22}^2 = 37$

$\boxed{l_{22} = 1}$

$l_{21} . l_{31} + l_{22} . l_{32} = -43$

$(6 . -8) + (1 . l_{32}) = -43$

$-48 + l_{32} = -43$

$\boxed{l_{32} = 5}$

$l_{31} . l_{21} + l_{32} . l_{22} = -43$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 98$

$64 + 25 + l_{33}^2 = 98$

$\boxed{l_{33} = 3}$

$L = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix}$ //

(3) Obtain the QR decomposition for the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$$

Ans

~~use~~

$$V_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$v_1 = (3,4) \qquad v_2 = (2,5)$

$u_1 = v_1 = \langle 3,4 \rangle$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} \cdot u_1$$

~~$\langle 3,4 \rangle - \langle (3,4) \rangle$~~

$$= \langle 2,5 \rangle - \frac{\langle (2,5),(3,4) \rangle}{25} \cdot \langle 3,4 \rangle$$

$$= \langle 2,5 \rangle - \frac{26}{25} \langle 3,4 \rangle$$

$$= \langle 2,5 \rangle - \left\langle \frac{78}{25}, \frac{104}{25} \right\rangle$$

$$u_2 = \left\langle -\frac{28}{25}, \frac{21}{25} \right\rangle$$

orthogonal vectors

$$\boxed{u_1 = \langle 3,4 \rangle \qquad u_2 = \left\langle \frac{-28}{25}, \frac{21}{25} \right\rangle}$$

orthonormal vectors

$$u_1' = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \qquad w_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

$$u_2' = \left\langle \frac{-28/25}{7/5} \quad \frac{+21/25}{7/5} \right\rangle \qquad w_2 = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$$

$$\underline{\underline{Q \text{ matrix}}} = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

$\underline{\underline{\text{Rmatrix}}}$ = upper Δ^{lr} matrix

$$R = \begin{bmatrix} r_{11} & r_{12} \\ & r_{22} \end{bmatrix}$$

$$r_{11} = \|u_1\|^{\cancel{2}} = 5$$

$$r_{22} = \|u_2\|^{\cancel{2}} = 7/5$$

$$r_{12} = \langle w_i , V_j \rangle$$

$$= \langle w_1 , V_2 \rangle$$

$$= \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$= \frac{6}{5} + \frac{20}{5} = \frac{26}{5}$$

$$R = \begin{bmatrix} 5 & 26/5 \\ 0 & 7/5 \end{bmatrix} //$$

$\therefore$ The QR decomposition is:

$$A = QR$$

$$= \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 5 & 26/5 \\ 0 & 7/5 \end{bmatrix}$$

(4) Obtain the QR decomposition of the following matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad V_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \quad V_3 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

$v_1 = \langle 1, 1, 2 \rangle \quad v_2 = \langle 0, 2, 2 \rangle \quad v_3 = \langle -1, 1, 3 \rangle$

By Gram-Schmidt orthogonalization:

$$\boxed{u_1 = v_1 = \langle 1, 1, 2 \rangle}$$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} \cdot u_1$$

$$u_2 = \langle 0, 2, 2 \rangle - \frac{(\langle 0, 2, 2 \rangle, \langle 1, 1, 2 \rangle)}{6} \cdot \langle 1, 1, 2 \rangle$$

$$= \langle 0, 2, 2 \rangle - \frac{6}{6} \cdot \langle 1, 1, 2 \rangle$$

$$\boxed{u_2 = \langle -1, 1, 0 \rangle}$$

$$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} \cdot u_1 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} \cdot u_2$$

$$u_3 = \langle -1,1,3\rangle - \left(\frac{\langle -1,1,3\rangle , \langle 1,1,2\rangle}{6}\right) \cdot \langle 1,1,2\rangle$$

$$- \left(\frac{\langle -1,1,3\rangle \langle -1,1,0\rangle}{2}\right) \cdot \langle -1,1,0\rangle$$

$$\cancel{u_3 = \langle -1,1,3\rangle - \left(\frac{6}{2}\right) \langle -1,1,2\rangle}$$

$-1+1+6$

$$\cancel{\langle -1,1,3}$$

$-1+1+6$

$$u_3 = \langle -1,1,3\rangle - \left(\frac{6}{6}\right) \langle 1,1,2\rangle$$

$$- \left(\frac{2}{2}\right) \langle -1,1,0\rangle$$

$1+1$

$$= \langle -1,1,3\rangle - \langle 1,1,2\rangle - \langle -1,1,0\rangle$$

$$\boxed{u_3 = \langle -1, -1, 1\rangle}$$

To find the orthonormal vectors

$u_1' = \sqrt{6}$

$u_2' = \sqrt{2}$

$u_3' = \sqrt{3}$

$$Q = \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix} \end{array}$$

<u>R matrix</u> $= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$

$r_{11} = \|u_1\| = \sqrt{6}$

$r_{22} = \|u_2\| = \sqrt{2}$

$r_{33} = \|u_3\| = \sqrt{3}$

$r_{12} = \langle w_1 , V_2 \rangle$

$= \langle (1/\sqrt{6}, 1/\sqrt{6}, 2/\sqrt{6}), (0, 2, 2) \rangle$

$= \frac{2}{\sqrt{6}} + \frac{4}{\sqrt{6}} = \frac{6}{\sqrt{6}} \cdot \frac{\times \sqrt{6}}{\sqrt{6}} = \underline{\underline{\sqrt{6}}}$

$r_{13} = \langle w_1 , V_3 \rangle$

$= \langle (1/\sqrt{6}, 1/\sqrt{6}, 2/\sqrt{6}), (-1, 1, 3) \rangle$

$= \frac{-1}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{6}{\sqrt{6}}$

$= \frac{6}{\sqrt{6}} = \underline{\underline{\sqrt{6}}}$

$r_{23} = \langle w_2 , V_3 \rangle$

$= \langle (-1/\sqrt{2}, 1/\sqrt{2}, 0), (-1, 1, 3) \rangle$

$= 1/\sqrt{2} + 1/\sqrt{2} = 2/\sqrt{2} = \sqrt{2}$

$\therefore$ The R matrix is $\begin{bmatrix} \sqrt{6} & \sqrt{6} & \sqrt{6} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{3} \end{bmatrix}$

The QR decomposition is :

$$\begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{6} & \sqrt{6} & \sqrt{6} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

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(5) Find the generalized inverse of $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix}$

$$A^+ = V \Sigma_1 U^H$$

Step 1 Find V

V = normalized eigenvectors of $A^H A$ $\longrightarrow$ hermitian matrix

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix} \qquad A^H = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix}$$

$$A^H A = B = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix} \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 12 & 26 & -20 \\ 12 & 12 & -20 \\ -20 & -20 & 44 \end{bmatrix}$$

$$B = \begin{bmatrix} 12 & 12 & -20 \\ 12 & 12 & -20 \\ -20 & -20 & 44 \end{bmatrix}$$

Find eigen values

$|B| = 0$

sum of minors $= \begin{vmatrix} 12 & -20 \\ -20 & 44 \end{vmatrix} + \begin{vmatrix} 12 & -20 \\ -20 & 44 \end{vmatrix} + \begin{vmatrix} 12 & 12 \\ 12 & 12 \end{vmatrix}$

$= 128 + 128$

$= 256$

Characteristic equation $= \lambda^3 - 68\lambda^2 + 256\lambda = 0$

$\lambda_1 = 64$

$\lambda_2 = 4$

$\lambda_3 \quad 0$

Find the corresponding eigen vectors

when $\lambda = 0$ $\begin{bmatrix} 12 & 12 & -20 \\ 12 & 12 & -20 \\ -20 & -20 & 44 \end{bmatrix}$

$x_1 \qquad x_2 \qquad x_3$

$$\begin{matrix} 12 & & -20 & & 12 & & 12 \\ & \times & & \times & & \times & \\ -20 & & 44 & & -20 & & -20 \end{matrix}$$

$$\frac{x_1}{128} = \frac{x_2}{-128} = \frac{x_3}{0}$$

$\langle 1, -1, 0 \rangle$

$$\boxed{X_1 = \langle -1, 1, 0 \rangle}$$

when $\lambda = 4$ $\begin{bmatrix} 8 & 12 & -20 \\ 12 & 8 & -20 \\ -20 & -20 & 36 \end{bmatrix}$

$x_1 \qquad x_2 \qquad x_3$

$\begin{matrix} 12 & & -20 & & 8 & & 12 \\ & \times & & \times & & \times & \\ 8 & & -20 & & 12 & & 8 \end{matrix}$

$-240+160$

$64-144$

$$\frac{x_1}{-80} = \frac{x_2}{-80} = \frac{x_3}{-80}$$

$$\boxed{X_2 = \langle 1,1,1 \rangle}$$

when $\lambda = 64$ $\begin{bmatrix} -52 & 12 & -20 \\ 12 & -52 & -20 \\ -20 & -20 & -20 \end{bmatrix}$

$x_1 \qquad x_2 \qquad x_3$

$\begin{matrix} 12 & & -20 & & -52 & & 12 \\ & \times & & \times & & \times & \\ -20 & & -20 & & -20 & & -20 \end{matrix}$

$-360+400$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$\boxed{X_3 = \langle 1,1,-2 \rangle}$$

$$V = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \end{bmatrix}$$

Step 2 : To find $\Sigma_1 = D^{-1}$ padded

The eigen values are 64, 4, 0

singular values are 8, 2

$$D = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1/8 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$\Sigma_1 = \begin{bmatrix} 1/8 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ //

Step 3 To find U

$$AA^H = A^HA \Rightarrow U = V = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \end{bmatrix}$$

$$A^+ = V\Sigma_1 U^H$$

$$= \begin{bmatrix} 3/16 & 3/16 & 1/8 \\ 3/16 & 3/16 & 1/8 \\ 1/8 & 1/8 & 1/4 \end{bmatrix}$$

6 Find a SVD of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find eigen values and eigen vectors

$|B| = 0$

sum of minors $= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$

$= 1 + 1 + 0 = 2$

<u>characteristic equation</u>: $\lambda^3 - 2\lambda^2 + 2\lambda - 0 = 0$

$\lambda = 2, 1, 0$

<u>find eigen vectors</u> =
$X_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$
$X_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$
$X_3 \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$

$$V = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

The singular values are $\sqrt{2}, \sqrt{1}, 0$

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 \qquad u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The decomposition is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

(7) Find the SVD of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ans

Find A^TA

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^TA = B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Find eigen values and eigen vectors

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\boxed{\begin{matrix} \lambda = 1 \\ \lambda = 3 \end{matrix}}$$

Find eigen vectors : when $\lambda = 1$ $\quad X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

when $\lambda = 3$ $\quad X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$V = \begin{bmatrix} +1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} //$$

Construction of Σ

$\sigma_1 = \sqrt{3}$ $\qquad \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} //$

$\sigma_2 = 1$

Construction of U

$$u_1 = \frac{1}{\sigma_1} AV_1$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} +1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2$$

$$u_2 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Find u_3

* make 2 equations from u_1 & u_2 & solve

$$u_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$U = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

The SVD is :

$$\begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & +1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} //$$