Mathematics for Machine Learning

Unit - 1

Vector Spaces

Vector spaces - subspaces - linear combinations and systems of linear equations - linear independence and linear dependence - bases and dimensions

* **Definition for vector space:** A vector space over a field F is a non-empty set of objects (called vectors) for which the following axioms hold true:

(i) $a+(b+c) = (a+b)+c, \quad \forall a,b,c \in V$

(ii) $\exists\, 0 \in V$ such that $a+0 = 0+a = a \quad \forall a \in V$

(iii) $\exists\, a \in V$, such that $a+a^{-1} = a^{-1}+a = 0$ and $\exists\, a^{-1} \in V$

$\left.\right\}$ closure + Group

(iv) $a+b = b+a$ (Abelian)

(v) $1.a = a \quad \forall a \in V, 1 \in F$

(vi) $\alpha(a+b) = \alpha a + \alpha b \quad \forall a,b \in V, \alpha \in F$

(vii) $(\alpha+\beta)a = \alpha a + \beta a \quad \forall a \in V, \alpha, \beta \in F$

(viii) $(\alpha\beta).a = \alpha(\beta a) \quad \forall a \in V, \alpha, \beta \in F$

* **Field** - with respect to both operations - '.' & '+', if the set V is abelian, then V is a field.

Elements of fields are scalars.

* **Subspaces** - A nonempty subset W of vector space V over a field F is called a subspace of V if W itself is a vector space with respect to the operations defined on V.

Criterion for Subspace : (i) $0 \in W$

(ii) $x+y \in W \quad \forall x, y \in W$

(iii) $\alpha x \in W \quad \forall x \in W, \alpha \in F$

(It is enough to show that $0 \in W$ and $\alpha(x+y) = \alpha x + \alpha y$ $\forall x, y \in W, \alpha \in F$)

* **Linear Combination and System of Linear Equations** - Let V be a vector space over a field F and let $x_1, x_2 \ldots x_n \in V$, then an element of the form $a_1x_1 + a_2x_2 + \ldots a_nx_n \in V$. where $a_i \in F$ is called the linear combination of $x_1, x_2 \ldots x_n$.

* **Linear Span** - Let S be a non-empty subset of a vectorspace V. Then the set of all linear combinations of the elements of S is called the linear span of S and is denoted by:

$$L[S] = \{a_1x_1 + a_2x_2 + \ldots a_nx_n \mid a_i \in F, x_i \in S\}$$

* **Linearly Independent / Dependent Set of Vectors** - A set of vectors $\{x_1, x_2 \ldots x_n\}$ is said to be linearly dependent if there exists scalars $a_1, a_2 \ldots a_n$, not all zero, such that $a_1x_1 + a_2x_2 + \ldots a_nx_n = 0$

If ~~not~~ all the scalars are zero $\Rightarrow$ linearly independent.

* **Finite Dimension** - A vector space V is said to be finite dimensional (or $\dim(V) = n$), if there exists a linearly independent set of vectors $S = \{e_1, e_2, \ldots e_n\}$ such that S spans V.

* **Basis** : The set $S = \{e_1, e_2 \ldots e_n\}$ is called a basis of V and the no. of elements in a basis is called the dimension of V

eg. $(1,0,0), (0,1,0), (0,0,1)$ = std. basis of R^3

Problems

① Show that $F^n = \{(a_1, a_2 \ldots a_n) \mid a_i \in F\}$ is a vector space over the field F with respect to addition and scalar multiplication, defined component-wise.

Ans Let $a = (a_1, a_2 \ldots a_n)$

$b = (b_1, b_2 \ldots b_n)$

$c = (c_1, c_2 \ldots c_n)$

(i)

$a + (b+c) = a_1 + a_2 + \ldots a_n + (b_1 + b_2 + \ldots b_n + c_1 + c_2 + \ldots c_n)$

$= a_1 + (b_1 + c_1) + a_2 + (b_2 + c_2) + \ldots a_n + (b_n + c_n)$

$= (a_1 + b_1) + c_1 + (a_2 + b_2) + c_2 + \ldots (a_n + b_n) + c_n$

$= (a+b) + c$

$\Rightarrow$ Associative property holds true.

(ii) Cleary $0 = (0, 0 \ldots \ldots 0)$ n - zeros

0 is the identity element

$0 + a = a + 0 = a \quad \forall a \in F^n$

(iii) For $a = (a_1, a_2 \ldots \ldots a_n)$, $a^{-1} = (-a_1, -a_2 + \ldots -a_n)$

$\Rightarrow a + a^{-1} = (a_1 - a_1) + (a_2 - a_2) + \ldots (a_n - a_n)$

$= 0$

$\Rightarrow$ An inverse element exists

(iv) $a = (a_1, a_2 \ldots a_n)$

$b = (b_1, b_2 \ldots \ldots b_n)$

$a + b = a_1 + a_2 + \ldots a_n + b_1 + b_2 + \ldots \ldots b_n$

$= b_1 + b_2 + \ldots b_n + a_1 + a_2 + \ldots a_n$

$= b + a$

$\therefore a + b = b + a \quad \forall a, b \in F^n$

(v) $1. a = 1. (a_1, a_2 \ldots \ldots a_n)$

$= (a_1, a_2 \ldots \ldots a_n)$

$= a$

(vi) $\alpha. (a + b) = \alpha(a_1 + b_1 + a_2 + b_2 + \ldots \ldots a_n + b_n)$

$= \alpha a_1 + \alpha b_1 + \ldots \ldots \alpha a_n + \alpha b_n$

$= \alpha a_1 + \alpha a_2 + \ldots \alpha a_n + \alpha b_1 + \alpha b_2 + \ldots \ldots \alpha b_n$

$= \alpha a + \alpha b$

$\Rightarrow \alpha(a+b) = \alpha a + \alpha b$

(vii) $(\alpha+\beta)a = (\alpha+\beta)(a_1, a_2, a_3 \dots a_n)$

$= (\alpha+\beta)a_1 + (\alpha+\beta)a_2 + \dots (\alpha+\beta)a_n$

$= \alpha a_1 + \alpha a_2 + \dots \alpha a_n + \beta a_1 + \beta a_2 + \dots \beta a_n$

$= \alpha(a) + \beta(b)$

(viii) $(\alpha\beta)a = (\alpha\beta)(a_1, a_2 \dots a_n)$

$= \alpha\beta a_1, \alpha\beta a_2 \dots \alpha\beta a_n$

$= \alpha(\beta a_1, \beta a_2 \dots \beta a_n)$

$= \alpha(\beta a) \quad \forall \alpha, \beta \in F^n$

$\therefore F^n$ is a vector space

~~(The same can be done for element-wise multiplication also)~~

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2) Let $V = \{(a_1, a_2) \mid a_1, a_2 \in R\}$. For $(a_1, a_2), (b_1, b_2) \in V$ is defined as $(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$ and for $\alpha \in R$, $\alpha(a_1, a_2) = (\alpha a_1, \alpha a_2)$. Is V a vector space over R. Justify your answer.

Ans: Let $a = (a_1, a_2 \cdots \cancel{a_n})$

$b = (b_1, b_2 \cdots \cancel{b_n})$

$a+b = (a_1, a_2) + (b_1, b_2)$

$= (a_1 + 2b_1, a_2 + 3b_2)$

$b+a = (b_1, b_2) + (a_1, a_2)$

$= (b_1 + 2a_1, b_2 + 3a_2)$

$\Rightarrow a+b \neq b+a$

commutativity does not hold

$\Rightarrow$ not a vector space

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(3) Determine whether the set of all 2×2 matrices of the form

$\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix}$, $a, b \in R$ with respect to standard matrix addition and scalar multiplication is a vector space over R or not. If not, list all the axioms that fail to hold.

<u>Ans</u> Let $A = \begin{bmatrix} a_1 & a_1 + a_2 \\ a_1 + a_2 & a_2 \end{bmatrix}$ $B = \begin{bmatrix} b_1 & b_1 + b_2 \\ b_1 + b_2 & b_2 \end{bmatrix}$

$C = \begin{bmatrix} c_1 & c_1 + c_2 \\ c_1 + c_2 & c_2 \end{bmatrix}$

(i) $A+(B+C)$

$$= \begin{bmatrix} a_1 & a_1+a_2 \\ a_1+a_2 & a_2 \end{bmatrix} + \begin{bmatrix} b_1+c_1 & b_1+b_2+c_1+c_2 \\ b_1+b_2+c_1+c_2 & c_2+b_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1+b_1+c_1 & a_1+a_2+b_1+b_2+c_1+c_2 \\ a_1+a_2+b_1+b_2+c_1+c_2 & a_2+b_2+c_2 \end{bmatrix}$$

$(A+B)+C$

$$= \begin{bmatrix} a_1+b_1 & a_1+a_2+b_1+b_2 \\ a_1+a_2+b_1+b_2 & a_2+b_2 \end{bmatrix} + \begin{bmatrix} c_1 & c_1+c_2 \\ c_1+c_2 & c_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1+b_1+c_1 & a_1+a_2+b_1+b_2+c_1+c_2 \\ a_1+a_2+b_1+b_2+c_1+c_2 & a_2+b_2+c_2 \end{bmatrix}$$

$\Rightarrow (A+B)+C = A+(B+C)$

$\Rightarrow$ associativity holds true.

(ii) Clearly $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

since $A+O = \begin{bmatrix} a_1 & a_1+a_2 \\ a_1+a_2 & a_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A$

$\Rightarrow \exists$ an identity element

(iii) If $A = \begin{bmatrix} a_1 & a_1+a_2 \\ a_1+a_2 & a_2 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -a_1 & -(a_1+a_2) \\ -(a_1+a_2) & -a_2 \end{bmatrix}$

$\therefore A + A^{-1} = 0$

(iv) $A+B = \begin{bmatrix} a_1+b_1 & a_1+a_2+b_1+b_2 \\ a_1+a_2+b_1+b_2 & a_2+b_2 \end{bmatrix}$

$B+A = \begin{bmatrix} b_1+a_1 & b_1+b_2+a_1+a_2 \\ b_1+b_2+a_1+a_2 & b_2+a_2 \end{bmatrix}$

$A+B = B+A$

commutativity holds true

(v) $1.A = 1.\begin{bmatrix} a_1 & a_1+a_2 \\ a_1+a_2 & a_2 \end{bmatrix}$

$= \begin{bmatrix} a_1 & a_1+a_2 \\ a_1+a_2 & a_2 \end{bmatrix}$

$= A$

(vi) $\alpha(A+B) = \alpha\left\{\begin{bmatrix} a_1 & a_1+a_2 \\ a_1+a_2 & a_2 \end{bmatrix} + \begin{bmatrix} b_1 & b_1+b_2 \\ b_1+b_2 & b_2 \end{bmatrix}\right\}$

$= \begin{bmatrix} \alpha a_1 & \alpha a_1+\alpha a_2 \\ \alpha a_1+\alpha a_2 & \alpha a_2 \end{bmatrix} + \begin{bmatrix} \alpha b_1 & \alpha b_1+\alpha b_2 \\ \alpha b_1+\alpha b_2 & \alpha b_2 \end{bmatrix}$

$= \alpha A + \alpha B$

(vii) $(\alpha+\beta)A = (\alpha+\beta)\begin{bmatrix} a_1 & a_1+a_2 \\ a_1+a_2 & a_2 \end{bmatrix}$

$$= \begin{bmatrix} (\alpha+\beta)a_1 & (\alpha+\beta)(a_2+a_1) \\ (\alpha+\beta)(a_1+a_2) & (\alpha+\beta)a_2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_1 & \alpha(a_1+a_2) \\ \alpha(a_1+a_2) & \alpha a_2 \end{bmatrix} + \begin{bmatrix} \beta a_1 & \beta(a_1+a_2) \\ \beta(a_1+a_2) & \beta a_2 \end{bmatrix}$$

$$= \alpha A + \beta A$$

(viii) $(\alpha\beta)A = \alpha\beta\begin{bmatrix} a_1 & a_1+a_2 \\ a_1+a_2 & a_2 \end{bmatrix}$

$$= \begin{bmatrix} (\alpha\beta)a_1 & (\alpha\beta)(a_1+a_2) \\ (\alpha\beta)(a_1+a_2) & (\alpha\beta)a_2 \end{bmatrix}$$

$$= \alpha\begin{bmatrix} \beta a_1 & \beta(a_1+a_2) \\ \beta(a_1+a_2) & \beta a_2 \end{bmatrix}$$

$$= \alpha(\beta A)$$

$\therefore$ V is a vector space over F

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(4) Prove that the set of all polynomials of degree n over a field F is a vector space.

Ans. Let $V = \{a_0 + a_1x + \cdots a_nx^n \mid a_1, a_2 \cdots a_n \in F\}$

Let $a = a_0 + a_1x + \cdots a_nx^n$ $\quad c = c_0 + c_1x + \cdots c_nx^n$

$b = b_0 + b_1x + \cdots b_nx^n$

(i) $a+(b+c) = (a_0 + a_1x + \cdots a_nx^n) +$

$(b_0 + b_1x + \cdots b_nx^n + c_0 + c_1x + \cdots c_nx^n)$

$= (a_0 + a_1x + \cdots a_nx^n + b_0 + b_1x + \cdots b_nx^n) +$

$(c_0 + c_1x + \cdots c_nx^n)$

$= (a+b)+c$

Associativity holds true.

(ii) Clearly $0 = (0, 0 \cdots 0)$ n zeros

as $a+0 = a = 0+a$

0 is the identity element

(iii) If $a = a_0 + a_1x + \cdots a_nx^n$

$a^{-1} = -a = -a_0 - a_1x + \cdots -a_nx^n$

$a + a^{-1} = (a_0 - a_0) + (a_1x - a_1x) + \cdots (a_nx^n - a_nx^n)$

$= 0$

(iv) $a+b = a_0 + a_1x + \cdots a_nx^n +$

$b_0 + b_1x + \cdots b_nx^n$

$= b_0 + b_1x + \cdots b_nx^n + a_0 + a_1x + \cdots a_nx^n$

$= b+a$

commutativity holds true.

(v) $1.a = 1.(a_0 + a_1x + \cdots a_nx^n)$

$= a_0 + a_1x + \cdots a_nx^n$

$= a$

(vi) $\alpha(a+b) = \alpha(a_0 + a_1x + \cdots a_nx^n + b_0 + b_1x + \cdots b_nx^n)$

$= \alpha a_0 + \alpha a_1x + \cdots \alpha a_nx^n + \alpha b_0 + \alpha b_1x + \cdots \alpha b_nx^n$

$= \alpha a + \alpha b$

(vii) $(\alpha+\beta)a = (\alpha+\beta)(a_0 + a_1x + \cdots a_nx^n)$

$= (\alpha+\beta)a_0 + (\alpha+\beta)a_1x + \cdots (\alpha+\beta)a_nx^n$

$= \alpha a_0 + \alpha a_1x + \cdots \alpha a_nx^n + \beta a_0 + \beta a_1x + \cdots \beta a_nx^n$

$= \alpha a + \beta a$

(viii) $(\alpha\beta)a = (\alpha\beta)(a_0 + a_1x + \cdots a_nx^n)$

$= \alpha\beta a_0 + \alpha\beta a_1x + \cdots \alpha\beta a_nx^n$

$= \alpha(\beta a_0) + \alpha(\beta a_1x) + \cdots \alpha(\beta a_nx^n)$

$= \alpha(\beta a)$

$\therefore$ V is a vector space over the field F.

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(5) Let R^+ denote the set of all positive real numbers where the addition '+' is defined as $a+b = ab$ and where the multiplication is defined as $\alpha\mu = \mu^\alpha$ $\forall \mu \in R^+$, $\alpha \in R$. P.T R^+ is a vector space.

(i) $a + (b + c) = a + bc$

$= a(bc)$

$= (ab)c$

$= (a+b)c$

$= (a+b)+c$

$\Rightarrow$ associativity holds true.

(ii) Clearly $0 \in R^+$ is the identity element

i.e $0 + a = a + 0 = a$

(iii) $1 + a = a^1 = a$

(iv) $a + b = ab$

$= ba$

$= b + a$

$\Rightarrow$ commutativity holds true

(v) For $\forall\, a \in R^+$, $\frac{1}{a}$ is the inverse

$a + (a^{-1}) = a + \left(\frac{1}{a}\right)$

$= a \cdot \frac{1}{a} = 1$

(vi) $\alpha(a+b) = \alpha(ab)$

$= (ab)^\alpha$

$= a^\alpha \cdot b^\alpha$

$= a^\alpha + b^\alpha$

$= \alpha a + \alpha b$

(vii) $(\alpha + \beta)a = a^{\alpha+\beta}$

$= a^{\alpha} \cdot a^{\beta}$

$= \alpha a + \beta a$

(viii) $(\alpha\beta)a = a^{\alpha\beta}$

$= (a^{\beta})^{\alpha}$

$= \alpha \cdot (a^{\beta})$

$= \alpha(\beta a)$

$\therefore R^{+}$ is a vector space.

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(6) Show that a set $W = \{(a_1, a_2, a_3) \in R^3, a_1, a_2, a_3 \in R$ and $2a_1 - 7a_2 + a_3 = 0\}$ is a subspace of $V = R^3$

Ans (i) $0 = (0,0,0) \in R^3$

since $2(0) - 7(0) + 0 = 0$

st $0 \in W$ and $\alpha(x+y) = \alpha x + \alpha y$

(ii) Let $a = 2a_1 - 7a_2 + a_3$

$b = 2b_1 - 7b_2 + b_3$

$\alpha(a+b) = \cancel{\alpha(2a_1 - 7a_2 + a_3 + 2b_1 - 7b_2 + b_3)}$

$= \alpha(a_1 + b_1, a_2 + b_2, a_3 + b_3)$

$= \alpha\{2(a_1 + b_1) - 7(a_2 + b_2) + (a_3 + b_3)\}$

$= \alpha\{(2a_1 - 7a_2 + a_3) + (2b_1 - 7b_2 + b_3)\}$

$= \alpha(0 + 0)$

$= 0$ //

$\therefore$ is a subspace of R^3

(7) Check whether $W = \{ (a_1, a_2, a_3) \in R^3 \mid 5a_1^2 - 3a_2^2 + 6a_3^2 = 0 \}$ is a subspace of $V = R^3$.

<u>Ans</u> $x = (a_1, a_2, a_3)$ $y = (b_1, b_2, b_3)$

$x+y = (a_1+b_1, a_2+b_2, a_3+b_3)$

$= 5(a_1+b_1)^2 - 3(a_2+b_2)^2 + 6(a_3+b_3)^2$

$= 5(a_1^2+b_1^2+2a_1b_1) - 3(a_2^2+b_2^2+2a_2b_2) + 6(a_3^2+b_3^2+2a_3b_3)$

$\neq 0$

$\Rightarrow$ not a subspace

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(8) Let $V = R^3$, then show that $W = \{ (a_1, a_2, a_3) \mid a_1 = a_3 + 2 \}$ is not a subspace

Let $0 = (0, 0, 0) \in R^3$

However : $0 \neq 0 + 2$

$\Rightarrow$ not a subspace

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(9) ~~Let~~ Show that $W = \{ (a_1, a_2, a_3) \mid a_1 = 3a_2, a_3 = -a_2 \}$ is a subspace of R^3

(i) Let $0 = (0, 0, 0)$

$0 = 3(0)$ and $0 = -(0)$

(ii) Let $x = (a_1, a_2, a_3)$
$y = (b_1, b_2, b_3)$

$x+y = (a_1+b_1, a_2+b_2, a_3+b_3)$

$a_1+b_1 = 3(a_2+b_2)$

$\Rightarrow a_1 - 3a_2 = 3b_2 - b_1$

$0 = 0$

$x+y \in W$

111ly $a_3+b_3 = -(a_2+b_2)$

$a_3+a_2 = -b_2-b_3$

$0=0$

$\therefore x+y \in W$

(iii) $\alpha x = \alpha(a_1, a_2, a_3)$

$\alpha a_1 = 3\alpha a_2$ & $\alpha a_3 = -\alpha a_2$

$\alpha x \in W$

$\therefore W$ is a subspace of R^3

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(10) Determine whether the element $(-2,0,3)$ is a linear combination of $(1,3,0)$ and $(2,4,-1)$.

Ans Assume that $(-2,0,3) = a(1,3,0) + b(2,4,-1)$ is true

Then $-2 = a+2b$ — (1)

$0 = 3a+4b$ — (2)

$3 = 0a-b$ — (3)

Solving (1), (2) $\boxed{a=4 \quad b=-3}$

Sub a & b in (3) $\Rightarrow$ The system is consistent

$\Rightarrow (-2,0,3)$ is a linear combination of $(1,3,0)$ & $(2,4,-1)$

(11) Determine whether the vector $(1,1,2)$ is a linear combination of $(1,0,1)$ and $(2,1,3)$ in R^3.

<u>Ans</u> Assume that

$$(1,1,2) = a(1,0,1) + b(2,1,3)$$

Then:

$$1 = a + 2b \quad -(1)$$

$$1 \cancel{0} = \cancel{a} + b \quad -(2)$$

$$2 = a + 3b \quad -(3)$$

$$\boxed{b = 1 \quad \text{and} \quad a = -1}$$

Sub in (3) - the system is consistent - $(1,1,2)$ is a linear combination of $(1,0,1)$ and $(2,1,3)$ in R^3.

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(12) Check whether $x^3 - 3x + 5$ is a linear combination of $x^3 + 2x^2 - x + 1$ and $x^3 + 3x^2 - 1$

<u>Ans</u> Assume that

$$x^3 - 3x + 5 = a(x^3 + 2x^2 - x + 1) + b(x^3 + 3x^2 - 1)$$

compare coefficients

$$1 = a + b \quad -(1)$$

$$0 = 2a + 3b \quad -(2)$$

$$-3 = -a \quad -(3)$$

$$a - b = 5 \quad -(4)$$

$$\boxed{a = 3}$$

$$\boxed{b = -2}$$

Sub in (1) - the system is consistent $\Rightarrow$ $x^3 - 3x + 5$ is a linear combination of $x^3 + 2x^2 - x + 1$ & $x^3 + 3x^2 - 1$.

(13) Test whether the indicated vector is in the span of S.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $S = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

Ans : Let

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + B\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + C\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$1 = A + 0B + C \Rightarrow \boxed{1 = A + C}$

$0 = 0A + 1B + 1C \Rightarrow \boxed{0 = B + C}$

$0 = -A + 0B + 0C \Rightarrow \boxed{A = 0}$

$1 = 0A + B + 0C \Rightarrow \boxed{B = 1}$

$\Rightarrow \boxed{C = 0}$

sub A, C in (1) $\Rightarrow$ not consistent

not a linear combination

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ does not belong in the span S.

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(14) Check whether the indicated value (2,-1,1) is in the span of {(1,0,2), (-1,1,1)} in R3

Ans Let $(2,-1,1) = A(1,0,2) + B(-1,1,1)$

$2 = A - B$ — (1)

$-1 = 0A + B$ $\boxed{B = -1}$

$1 = 2A + B$ $\boxed{A = 1}$

Sub in (1) $\Rightarrow$ is consistent $\Rightarrow$ (2,-1,1) belongs to the span.

(15) Show that vectors $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ generate R^3.

Ans $(x_1, x_2, x_3) = A(1,1,0) + B(1,0,1) + C(0,1,1)$

$x_1 = A + B \quad -(1)$

$x_2 = A + C \quad -(2)$

$x_3 = B + C \quad -(3)$

Solve to find A, B, C

(1) - (2) $\quad x_1 - x_2 = B - C$

$\quad x_3 = B + C \quad (+)$

$2B = x_1 - x_2 + x_3$

$$\boxed{B = \frac{x_1 - x_2 + x_3}{2}}$$

llly $\quad B - C = x_1 - x_2$

$\quad B + C = x_3 \quad (-)$

$-2C = x_1 - x_2 - x_3$

$$\boxed{C = \frac{-x_1 + x_2 + x_3}{2}}$$

(1) + (2) $\Rightarrow \quad x_1 + x_2 = 2A + B + C$

$x_1 + x_2 = 2A + \left(\frac{\cancel{x_1} - \cancel{x_2} + x_3 - \cancel{x_1} + \cancel{x_2} + x_3}{2}\right)$

$x_1 + x_2 = \frac{2A + 2x_3}{2}$

$x_1 + x_2 = 2A + x_3$

$$\boxed{A = \frac{x_1 + x_2 - x_3}{2}}$$

Check the consistency of the equations

Sub in 2

$$A + C = \frac{\cancel{x_1} + x_2 - \cancel{x_3} + -\cancel{x_1} + x_2 + \cancel{x_3}}{2}$$

$$= \frac{2x_2}{2} = x_2$$

The system is consistent.

$$\therefore \left(\frac{x_1 + x_2 - x_3}{2}\right)(1,1,0) + \left(\frac{x_1 - x_2 + x_3}{2}\right)(1,0,1) + \left(\frac{-x_1 + x_2 + x_3}{2}\right)(0,1,1)$$

generate R^3

Span S generates all elements of R^3.

Q16 Show that $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ generate all elements 2×2 matrices $M_{2\times 2}$ over F.

Ans $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + B\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + C\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + D\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$a = A + B + C \quad - (1)$

$b = A + B + D \quad - (2)$

$c = A + C + D \quad - (3)$

$d = B + C + D \quad - (4)$

$a = A + B + C$

$b = A + B + D$

$c = A + C + D$

$d = B + C + D$

① - ② ③ - ④

$a - b = C - D$ $c - d = A - B$

$A - B + C - D = a - b + c - d$ — ✱

$\cancel{(A+C) - (B+D) = a - b + c - d}$

$\Rightarrow A + C = a - b + c - d + B + D$

$\cancel{c - D - (d - C) = a - b + c - d}$

$\cancel{④} - ③$ $\cancel{II}$ ①

Solve ? ②

$\cancel{a - b = B - D}$

$\cancel{2a = C - D}$

$$A = \frac{a + b + c - 2d}{3}$$

$$B = \frac{a + b - 2c + d}{3}$$

$$C = \frac{a + 2b + c + d}{3}$$

$$D = \frac{-2a + b + c + d}{3}$$

The equations are consistent. They generate $M_{2\times 2}$

(17) Show that $S = \{(1,1,1), (0,1,1), (0,1,-1)\}$ generates $R3$

Ans $(x_1, x_2, x_3) = A(1,1,1) + B(0,1,1) + C(0,1,-1)$

$$\boxed{x_1 = A} \quad -(1)$$

$$\boxed{x_2 = A + B + C} \quad -(2)$$

$$\boxed{x_3 = A + B - 11C} \quad -(3)$$

(2) $-$ (3)

$$x_2 + x_3 = 2A + 2B$$

$$2B = x_2 + x_3 - 2A$$

$$2B = -2x_1 + x_2 + x_3$$

$$\boxed{B = \frac{-2x_1 + x_2 + x_3}{2}}$$

$$x_2 - x_3 = 22C$$

$$x_2 - x_3 = 12C$$

$$\boxed{C = \frac{x_2 - x_3}{12}}$$

$$x_2 = A + B + C$$

$$x_2 = x_1 + B + \frac{x_2 - x_3}{12}$$

$$x_2 = \frac{12x_1 + x_2 - x_3 + 12B}{12}$$

$$12x_2 - 12x_1 - x_2 + x_3 = 12B$$

$$\boxed{B = \frac{-12x_1 + 11x_2 + x_3}{3}}$$

The equations are consistent

Any element (x_1, x_2, x_3) can be generated by

$(1,1,1)$, $(0,1,1)$ and $(0,1,-11)$

$$(x_1, x_2, x_3) = x_1(1,1,1) + \left(\frac{11x_2 - 12x_1 + x_3}{12}\right)(0,1,1) + \left(\frac{x_2 - x_3}{12}\right)(0,1,-11)$$

(18) Determine whether the following set of vectors are linearly independent or dependent.

(i) (1,2,1), (2,1,0), (1,-1,2)

$|A|=0 \Rightarrow$ dependent

alternately, convert to row echelon form and solve the equations

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

$|A| = -9 \neq 0 \Rightarrow$ linearly independent.

(ii) (1,0,0), (0,1,0), (1,1,0)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$|A|=0 \Rightarrow$ linearly dependent

(iii) (1,4,-2), (-2,1,3), (-4,11,5)

$$A = \begin{bmatrix} 1 & 4 & -2 \\ -2 & 1 & 3 \\ -4 & 11 & 5 \end{bmatrix}$$

$|A|=0 \Rightarrow$ linearly dependent

(19) Determine if $(3,9,-4,-2)$ belongs to the space spanned by $(1,-2,0,3)$, $(2,3,0,-1)$, $(2,-1,2,1)$

Ans Let $(3,9,-4,-2) = A(1,-2,0,3) + B(2,3,0,-1)$
$+ C(2,-1,2,1)$

$3 = A + 2B + 2C$

$9 = -2A + 3B - C$ $\quad \boxed{C=-2}$

$-4 = 0A + 0B + 2C$ $\quad \boxed{\begin{matrix} a=1 \\ b=3 \end{matrix}}$

$-2 = 3A - B + C \quad -(4)$

Sub in (4) $\quad 3A - B + C$

$= 3 - 3 - 2$

$= -2 = RHS$

The system is consistent.

$\therefore (3,9,-4,-2)$ spans $(1,-2,0,3)$, $(2,3,0,-1)$ and $(2,-1,2,1)$

---

(20) Examine the linear dependency | independency of the following vectors:

$(1,-2,3,4)$, $(-2,4,-1,-3)$, $(-1,2,7,6)$. If they are linearly dependent, find the solution between them

Ans $A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$$

converting to Row Echelon form:

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ -1 & 2 & 7 & 6 \end{bmatrix} \qquad R_2' \leftrightarrow R_2 + 2R_1$$

$-1+6$
$-3+8$

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 10 & 10 \end{bmatrix} \qquad R_3' \leftrightarrow R_3 + R_1$$

$\cancel{a - 2b + 3c + 4d = 0}$

$\cancel{5c + 5d = 0}$

$\cancel{9c + 9d = 0}$.

$$R_3'' \leftrightarrow R_3' - 2R_2'$$

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 = 0 \Rightarrow$ linearly dependent

$\Rightarrow R_3' - 2R_2' = 0$

$R_3 + R_1 - 2(R_2 + 2R_1) = 0$

$R_3 + R_1 - 2R_2 - 4R_1 = 0$

$R_3 - 2R_2 - 3R_1 = 0$

$\therefore (-1, 2, 7, 6)$
$-2(-2, 4, -1, -3) +$
$-3(1, -2, 3, 4)$
$= (0, 0, 0, 0)$

21. Show that the vectors $u = (1,2,3)$, $v = (0,1,2)$, $w = (0,0,1)$ generate R^3.

$\therefore (x_1, x_2, x_3) = A(1,2,3) + B(0,1,2) + C(0,0,1)$

$\boxed{x_1 = A.}$

$x_2 = 2A + B$ $\quad \boxed{B = x_2 - 2x_1}$

$x_3 = 3A + 2B + C$

$x_3 = 3x_1 + 2(x_2 - 2x_1) + C$

$x_3 = 3x_1 + 2x_2 - 4x_1 + C$

$x_3 = 2x_2 - x_1 + C$

$\boxed{C = x_1 - 2x_2 + x_3}$

The system of eqns is consistent $\Rightarrow$ they generate R^3 (u, v, w)

---

22. Find the condition on a, b, c so that $(a,b,c) \in R^3$ belong to the space generated by $u = (2,1,0)$ $v = (1,-1,2)$, $w = (0,3,-4)$

Ans. $(a,b,c) = x_1(2,1,0) + x_2(1,-1,2) + x_3(0,3,-4)$

$a = 2x_1 + x_2 + \cancel{0}$ — (1)

$b = x_1 - x_2 + 3x_3$ — (2) $\times 4$

$c = 2x_2 - 4x_3$ — (3) $\times 3$

$4b = 4x_1 - 4x_2 + 12x_3$

$3c = 6x_2 - 12x_3$

$4b + 3c = 4x_1 + 6x_2 - 4x_2 \Rightarrow 4b + 3c = 4x_1 + 2x_2$

$\boxed{4b + 3c = 2a}$

(23) Show that the vectors $(1, 0, -1)$, $(1, 2, 1)$ and $(0, -3, 2)$ form a basis for R^3. Express each of the standard basis vectors as a linear combination of u, v, w

Ans
$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -3 & 2 \end{bmatrix}$$

$$\left.\begin{aligned} |A| \neq 0 &\Rightarrow \text{linearly independent} \\ &\Rightarrow \text{forms a basis} \end{aligned}\right\}$$ ★★★

Let $(a, b, c) = x_1(1, 0, -1) + x_2(1, 2, 1) + x_3(0, -3, 2)$

$a = x_1 + x_2$ — (1)

$b = 2x_2 - 3x_3$ — (2)

$c = -x_1 + x_2 + 2x_3$ — (3)

(1) + (3)

$a + c = 2x_2 + 2x_3$

$b = 2x_2 - 3x_3$

$\Rightarrow a + c - b = 5x_3$

$$\boxed{x_3 = \frac{a - b + c}{5}}$$

$b = 2x_2 - 3x_3$

$2x_2 = b + 3x_3$

$$2x_2 = b + 3x_3$$

$$2x_2 = b + 3\left(\frac{a-b+c}{5}\right)$$

$$2x_2 = \frac{5b + 3a - 3b + 3c}{5}$$

$$10x_2 = 3a + 2b + 3c$$

check

$$\boxed{x_2 = \frac{3a + 2b + 3c}{10}}$$

$$x_1 = \boxed{\frac{7a + 2b - 3c}{10}}$$

For
The standard basis

$$(1,0,0) = \frac{7}{10}u + \frac{3}{10}v + \frac{1}{5}v$$

chk

$$(0,1,0) = \frac{2}{10}u + \frac{2}{10}v - \frac{1}{5}w$$

$$(0,0,1) = -\frac{3}{10}u + \frac{3}{10}v + \frac{1}{5}w$$

---

(24) Find a basis of the subspace W spanned by the polynomials

(i) $v_1 = t^3 - 2t^2 + 4t + 1$

$v_2 = t^3 + 6t - 5$

$v_3 = 2t^3 - 3t^2 + 9t - 1$

$v_4 = 2t^3 - 3t^2 + 7t + 5$

$$A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 1 & 0 & 6 & -5 \\ 2 & -3 & 9 & -1 \\ 2 & -5 & 7 & 5 \end{bmatrix}$$

convert to row echelon form.

$$A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 2 & 2 & -6 \\ 2 & -3 & 9 & -1 \\ 2 & -5 & 7 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 - R_1$

$$A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 2 & 2 & -6 \\ 0 & 1 & 1 & -3 \\ 2 & -5 & 7 & 5 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - 2R_1$

$-3+4$

$9-8$

$-1-2$

$$A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 2 & 2 & -6 \\ 0 & 1 & 1 & -3 \\ 0 & -1 & -1 & 3 \end{bmatrix}$$

$R_4 \leftrightarrow R_4 - 2R_1$

$-5+4$

$$A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 + R_4$

$$A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 2 & 2 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_4 \leftrightarrow 2R_4 + R_2$

The basis is $\{(1,-2,4,1),(0,1,1,-3)\}$

$\dim(W) = 2 \quad =$ no. of non zero rows

(25) Find the basis and dimension of the subspace W of R^4 spanned by $(1,-2,5,-3)$, $(2,3,1,-4)$ and $(3,8,-3,-5)$

Ans

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 3 & 8 & -3 & -5 \end{bmatrix} \qquad R_2 \leftrightarrow R_2 - 2R_1$$

$3+4$

$1-10$

$-4+6$

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} \qquad R_3 \leftrightarrow R_3 - 3R_1$$

$8+6$

$-3-15$

$-5+9$

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3 \leftrightarrow R_3 - 2R_2$$

Basis $= \{(1,-2,5,-3),(0,7,-9,2)\}$

dim $=$ no. of non zero rows $= 2$

(26) Find the dimension and basis for the solution space W of

↳ $\dim(W) =$ no. of unknowns - no. of non. zero rows

Ex. the system of homogenous eqns.

$$x_1 + 2x_2 + 2x_3 - x_4 + 3x_5 = 0$$

$$x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 0$$

$$3x_1 + 6x_2 + 8x_3 + x_4 + 5x_5 = 0$$

Ans

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix}.$$

Convert to row echelon form

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 2 & 4 & -4 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 - R_1$

$R_3 \leftrightarrow R_3 - 3R_1$

1+3

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

make more rows 0 if possible

$R_3 \leftrightarrow R_3 - 2R_2$

$\dim(W) =$ no. of unknowns - no. of non - zero rows

$= 5 - 2 = 3$

$\boxed{\dim(W) = 3}$

The basis is of the solution space ⇒ cannot directly take coeff from matrix

$\Rightarrow$ must solve the eqn. for the soln. space

$$x_1 + 2x_2 + 2x_3 - x_4 + 3x_5 = 0 \quad -(1)$$

$$x_3 + 2x_4 - 2x_5 = 0 \quad -(2)$$

Let the free variables be x_2, x_4, x_5

If $x_2 = 1 \quad x_4, x_5 = 0$

$x_1 + 2 + 2x_3 = 0$

$\boxed{x_3 = 0 \quad x_1 = -2}$ $\quad \boxed{v_1 = (-2, 1, 0, 0, 0)}$

If $x_2 = 0, \quad x_4 = 1, \quad x_5 = 1$

$\boxed{v_2 = (5, 0, -2, 1, 0)}$

If $x_2 = 0, x_4 = 0, x_5 = 1$

$\boxed{v_3 = (-1, 0, 2, 0, 1)}$

$\{v_1, v_2, v_3\}$ form the basis

Textbooke Questions

(27) Find whether the vector $(2, 4, 6, 7, 8)$ is in the subspace of R^5 spanned by $(1, 2, 0, 3, 0), (0, 0, 1, 4, 0), (0, 0, 0, 0, 1)$

Ans.

$(2, 4, 6, 7, 8) = a(1, 2, 0, 3, 0) + b(0, 0, 1, 4, 0) + c(0, 0, 0, 0, 1)$

$\boxed{2 = a}$ $\quad$ ~~4=2a~~ $\boxed{6 = b}$ $\quad 7 = 3a + 4b$

eqn. not satisfied

$\Rightarrow$ not in subspace.

28 ~~Examine the linear independence/dependence of~~

Find the maximum no. of linearly independent vectors among the following and express each of the remaining as a linear combination of these

$u_1 = (1,2,1)$ $u_2 = (4,1,2)$ $u_3 = (6,5,4)$ $u_4 = (-3,8,1)$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 2 \\ 6 & 5 & 4 \\ -3 & 8 & 1 \end{bmatrix}$$

$R_2' \leftrightarrow R_2 - 4R_1$

$R_3' \leftrightarrow R_3 - 6R_1$

$R_4' \leftrightarrow R_4 + 3R_1$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -2 \\ 0 & -7 & -2 \\ 0 & 14 & 4 \end{bmatrix}$$

$R_3'' \leftrightarrow R_3' - R_2'$

$R_4'' \leftrightarrow R_4' + 2R_2'$

$2-4$

$5 - \cancel{4}$

4

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

max. no. of linearly independent vectors $= 2 = \dim(A)$

$R_3'' = 0$

$R_3'' = R_3' - R_2'$

$0 = R_3' - R_2'$

$0 = R_3 - 6R_1 - (R_2 - 4R_1)$

$0 = R_3 - 6R_1 - R_2 + 4R_1$

$$-R_3 = -2R_1 - R_2$$

$$\boxed{R_3 = 2R_1 - R_2} \rightarrow \text{Eqn } (1)$$

$R_4'' = 0$

$$R_4'' = R_4' + 2R_2'$$

$$0 = R_4' + 2R_2'$$

$$0 = R_4 + 3R_1 + 2(R_2 - 4R_1)$$

$$0 = R_4 + 3R_1 + 2R_2 - 8R_1$$

$$0 = R_4 - 5R_1 + 2R_2$$

$$\boxed{5R_1 - 2R_2 = R_4} \rightarrow \text{Eqn } (2)$$

---

29) Determine whether the set of vectors $(4,1,2,0)$, $(1,2,-1,0)$, $(1,3,1,2)$ and $(6,1,0,1)$ is linearly independent.

Check

Ans

$$A = \begin{bmatrix} 4 & 1 & 2 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & 3 & 1 & 2 \\ 6 & 1 & 0 & 1 \end{bmatrix}$$

converting to Row Echelon form:

$$A = \begin{bmatrix} \cancel{4} & \cancel{7} & \cancel{2} & \cancel{0} \\ \cancel{0} & \cancel{-2} & \cancel{-9} & \cancel{0} \\ & & & \\ & & & \end{bmatrix}$$

~~$R_2 \leftrightarrow 4R_2 - R_1$~~

~~$R_3 \leftrightarrow 4R_3 - 4R_1$~~

2 -4

-1 -8

1 - 8

$$A = \begin{bmatrix} 4 & 1 & 2 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & 3 & 1 & 2 \\ 6 & 1 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow 4R_2 - R_1$

$R_3 \leftrightarrow 4R_3 - R_1$

$R_4 \leftrightarrow 4R_4 - 6R_1$

$-4 -2$

$$A = \begin{bmatrix} 4 & 1 & 2 & 0 \\ 0 & 7 & -6 & 0 \\ 0 & 11 & 2 & 8 \\ 0 & -2 & -12 & 4 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - 11R_1$

~~$R_4 \leftrightarrow R_4 + 2R_1$~~

$R_4 \leftrightarrow R_4 + 2R_1$

$2 - 22$

$$A = \begin{bmatrix} 4 & 1 & 2 & 0 \\ 0 & 7 & -6 & 0 \\ 0 & 0 & -19 & 8 \\ 0 & 2 & \overset{2}{\underset{-8}{\cancel{12}}} & 4 \end{bmatrix}$$

$R_4 \leftrightarrow R_4 + 4R_1$

$-12 + 4$

$$A = \begin{bmatrix} 4 & 1 & 2 & 0 \\ 0 & 7 & -6 & 0 \\ 0 & 0 & -19 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$4a + b + 2c = 0$

~~$d = 4$~~

~~$-19c + 8d =$~~

$7b - 6c + 0d = 0$

$-19d + 8d = 0$

(30) Show that the vectors $u = (1,2,3)$, $v = (0,1,2)$ and $w = (0,0,1)$ generate R^3

Ans $(a,b,c) = A(1,2,3) + B(0,1,2) + C(0,0,1)$

$a = A + 0B + 0C$ $\qquad \boxed{A = a}$

$b = 2A + B + 0C$

$b = 2a + B$ $\qquad \boxed{B = b - 2a}$

$c = 3A + 2B + C$

$c = 3a + 2B + C$

$C = c - 3a + 2B$

$C = c - 3a + 2(b - 2a)$

$= c - 3a + 2b - 4a$

$\boxed{C = c - 2b + a}$

$c = 3A + 2B + C$
$c = 3a + 2(b - 2a) + C$
$C = c - 3a - 2b + 4a$
$C = c + a - 2b$

The system is consistent

$\therefore$ $a(1,2,3) + (b-2a)(0,1,2) + (c-2b+a)(0,0,1)$ generate R^3.

---

(31) Find a homogenous system of equations whose solution set is spanned by $(1,-2,0,3,-1)$, $(2,-3,2,5,-3)$ and $(1,-2,1,2,-2)$

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 & -1 \\ 2 & -3 & 2 & 5 & -3 \\ 1 & -2 & 1 & 2 & -2 \end{bmatrix}$$

$R_2' \leftrightarrow R_2 - 2R_1$

$R_3' \Leftrightarrow R_3 - R_1$

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 & -1 \\ 0 & 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}$$

$-3+4$

$5-6$

$-3+2$

~~make more rows~~ 0

$-2+1.$

$$A = \begin{bmatrix} \cancel{1} & \cancel{-2} & \cancel{0} & \cancel{3} & \cancel{-1} \\ \cancel{0} & \cancel{1} & \cancel{2} & \cancel{-1} & \cancel{-1} \\ & & & & \end{bmatrix}$$

$x_1 - 2x_2 + 3x_4 - x_5 = 0$

$x_2 + 2x_3 - x_4 - x_5 = 0$

$x_3 - x_4 - x_5 = 0$

if $x_3 = 1$