Properties of Matter and Thermal Physics

Elasticity : It is the property by virtue of which a body offers resistance to any deforming force and regains its original condition when the deforming force is removed.

(i) **Perfectly elastic body** : Bodies which can completely regain their original shape and size on removal of the deforming force (eg. quartz fibre)

(ii) **Perfectly plastic body** : Bodies which retain their deformed nature even after the removal of the deforming force. (eg. putty)

(iii) **Perfectly rigid body** : Bodies that do not undergo any deformation / relative displacement on applying a deforming force.

Stress : The restoring | recovering force set up per unit area within the body is called the stress.

- It is equal & opp to the load within the elastic limit.
- If the internal force developed is $\perp^r$ to the surface, then it is called normal stress
- $\text{Stress} = \dfrac{\text{Force}}{\text{Area}}$ dimensions : $[ML^{-1}T^{-2}]$

Strain : The change in dimension produced in a body under a system of forces in equilibrium is called strain.

longitudinal strain $= \dfrac{\Delta l}{l}$

volumetric strain $= \dfrac{\Delta V}{V}$

shear strain $= \dfrac{\Delta\theta}{\theta}$

Elastic Limit: The maximum value of stress within which a body completely regains its original condition of shape and size once the deforming forces are removed.

Hooke's Law: Within the elastic limit, stress $\propto$ strain, i.e the ratio of stress to strain is a constant. This constant is called the modulus of elasticity

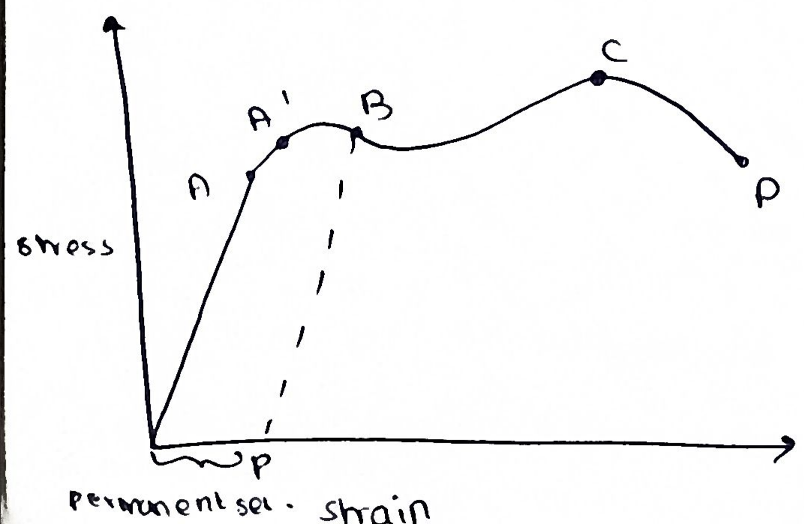
$$\frac{\text{stress}}{\text{strain}} = \text{modulus of elasticity}$$

Elasticity on the basis of intermolecular forces.

- Elasticity is due to high intermolecular forces. When this force is large, the modulus of elasticity is high.
- For even large amounts of stress, the amount of strain developed would be very low.
- When the intermolecular forces are low, like in a gas, the modulus of elasticity is small. For even a small amount of stress, the amount of strain would be very large.

Stress-Strain Diagram

- Let a wire be clamped at one end, and loaded at the other end, gradually from zero value until the wire breaks down.
- The elastic behaviour is studied by plotting a curve of stress vs. strain.



A → proportional limit

A' → elastic limit

B → yield point

C → ultimate tensile strength

D → breaking stress

OA → a straight line, Hooke's law is obeyed.

A → proportional limit: It is measured by the maximum stress that can be developed in the material, without causing a deviation from Hooke's law.

A' → a pt in the vicinity of A, called elastic limit.

Till this point, the wire behaves like a totally elastic body.

Note: not necessary that stress ∝ strain in the AA' portion of the curve.

OP → If the wire is loaded beyond A', it attains a permanent set, i.e there is a permanent deformation.

B → yield point. It is measured by the minimum amount of stress for which the wire begins to deform appreciably without an increase in load.

The corresponding value of stress at that point = yield strength

C → The wire begins to thin down. The value of stress is maximum, it is called the ultimate tensile strength.

$$\text{Tensile strength} = \frac{\text{maximum tensile load}}{\text{original cross-sectional area}}$$

D → wire actually breaks down, called the breaking stress.

Note: The value of breaking stress is lesser than the value of ultimate strength, due to fracture formation. The applied load is much higher at D though.

Other definitions: Creeping: The elongation without an addition of load is called creeping, the behaviour of the metal is called yielding.

* **Subjecting a body to continuous stress**

- It begins to lose its elastic property even within the elastic limit.
- When a wire is continually loaded, it gets tired / ruptures due to the gradual fracture of the material.

Elastic Fatigue : It is defined as the apparent loss of strength of material or the progressive fracture of the material due to repeated stress.

- Some materials like quartz, phosphor bronze & silver fibres regain their original conditions immediately on removal of the deforming forces. This is why they are used as suspension wires in galvanometers, electrometers etc.

- Some materials like glass fibers take hours to recover from the strain.

Elastic After Effect: The delay in regaining the normal condition is called elastic after effect.

$$\text{Safety Factor} = \frac{\text{Ultimate tensile strength}}{\text{working stress}}$$

- The safety factor is kept far below the ultimate tensile strength.
- Lower values are adopted so that the structure has a long life.
- safety factor values

dead load → 4
live load → 6
alternating load → 9
steel → 4 (a good engineering material)
brick → 10

Uses of stress - strain diagram

(i) used to determine yield strength, tensile strength, elastic strength.
(elastic limit is used to determine elastic strength)

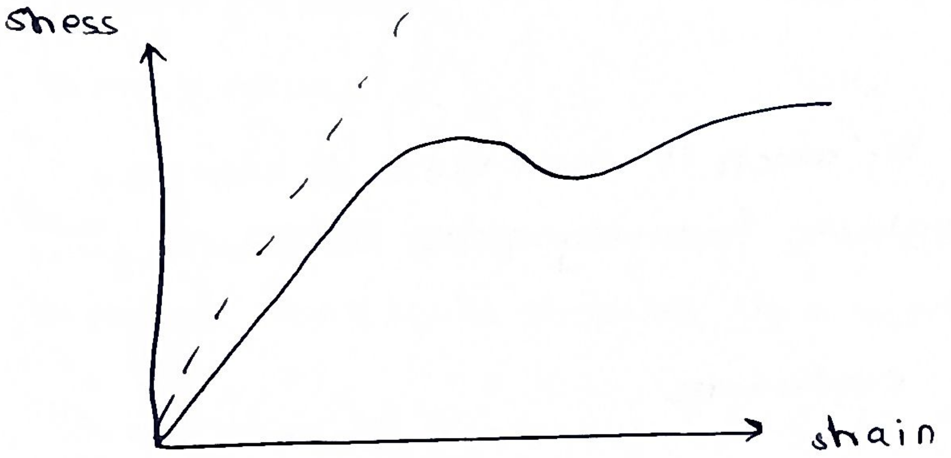
(ii) to determine working stress and safety factor (lower safety factor
⇒ longer life for the structure). It helps to determine

the degree of safety
permanancy
economy
dependability

(iii) Area under the curve = energy required to deform the material elastically.
area under the curve up to UTS gives the energy to deform it plastically.

(iv) used to identify ductile & brittle materials.

Stress - Strain Diagram for ductile & brittle materials



Ductile materials :- Beyond the yield point, the curve continues its upward trend.

- The cross-sectional area goes on decreasing with an increase in tensile load.
- There is an enormous increase in strain for even small increases in stress
- behaviour seen in ductile materials like copper, mild steel
- wire drawing ability w/o fracture.

Brittle material

- a sharp increase in stress is seen for even small amounts of strain
- even in elastic conditions, there is a brittle fracture
- eg. glass, iron.

Factors affecting elasticity?

1. Effect of stress

- Large constant stress / a no. of cycles of stress decreases the elasticity of the body.
- Thus, the working stress of an engineering piece is kept far below ultimate tensile strength.

2. Effect of temperature

- elasticity is higher at lower temperatures.

(i) a carbon filament that is highly elastic becomes plastic at high temp.

(ii) lead, not elastic at normal temperatures, but highly elastic at lower temp.

Creep Resistance

~~creeping~~ :- a property of a material by which it maintains its elastic property without fracture at high temperature & during quick loading.

For eg. invar - an alloy used to make pendulums

- dispersion hardened materials and coarse grained materials have greater creep resistance.

3. Effect of Impurities

(i) Addition of carbon to molten iron → elasticity increases
(if c content > 1% ⇒ elasticity decreases)

(ii) addition of potassium to gold ⇒ elasticity increases

- If impurities atoms distort the lattice structure → elastic property decreases.
- This is because these atoms have different atomic & electronic structures, act as centers of distortion.

Effect of heat treatment & metal processing

(i) Annealing :
- heating and then slowly cooling the metal.
- used to increase softness & ductility
- elastic property decreases, because there is a decrease in the tensile strength and the yield strength
- This is due to the formation of large crystal grains.

(ii) Hammering & rolling :
- used to make thin sheets and plates
- elastic strength increases, because the resulting grains are small

Thus, metals w/ fine grains are stronger than those with large / coarse grain. For high temperature applications, materials with large grains are used as they have high creep resistance.

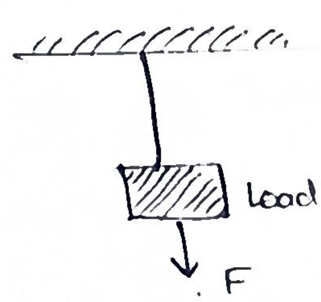
Effect of crystalline nature

crystalline nature	modulus of elasticity
single crystalline	high
polycrystalline	low

Note: For most engineering uses, polycrystalline materials are used due to increased mechanical properties like ductility, malleability, machinability

Three Moduli of Elasticity

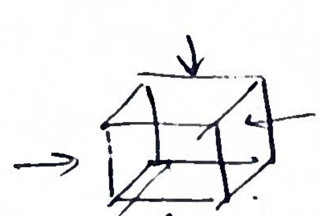
1. Young's modulus (E) : load applied in a particular direction



$$E = \frac{\text{linear stress}}{\text{linear strain}} = \frac{F/A}{\Delta l / l} = \frac{F}{A} \times \frac{l}{\Delta l} \quad Nm^{-2}$$

2. Bulk modulus (K) : Force acting normally on the whole surface producing a change in volume.

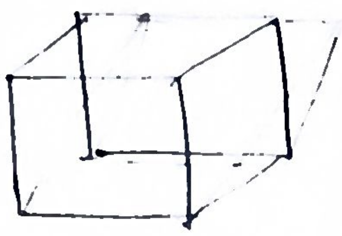
$$K = \frac{\text{bulk stress}}{\text{bulk strain}} = \frac{F/A}{\Delta V / V} = \frac{F}{A} \times \frac{V}{\Delta V} = \frac{PV}{\Delta V}$$



reciprocal of bulk modulus = compressibility

Rigidity modulus (N):

Here, the applied force causes a change in the shape of the body without causing any change in its volume.



The shear strain is defined as the angle through which a line, originally perpendicular to the tangential forces has turned.

$$\text{rigidity modulus} = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F}{A\theta}$$

Note: solids have all 3 moduli of elasticity

gases & liquids have only bulk modulus

Poisson's Ratio

$$\text{Lateral strain} = \frac{\text{change in diameter}}{\text{original diameter}}$$

The ratio of the lateral strain to the longitudinal strain within the elastic limit is called Poisson's ratio.

$$\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{\frac{\Delta D}{D}}{\frac{\Delta l}{l}}$$

Work done per unit volume in deforming a body

① Work done per unit volume in stretching a wire

F = applied force

work done to produce a small change in length $= F \cdot dl$

$$E = \frac{F}{A} \times \frac{l}{\Delta l}$$

$$F = \frac{EA\Delta l}{l}$$

$$F = \frac{EA\Delta l}{l}$$

$$W = \int_0^l F \cdot dl$$

$$W = \int_0^l \frac{EA\Delta l}{l} \cdot dl.$$

$$W = \frac{EA}{l} \cdot \frac{\Delta l^2}{2} = \frac{EA\Delta l}{l} \cdot l$$

$$= F.l \times \frac{1}{2}$$

$W = \frac{1}{2} \times F \times l = \frac{1}{2} \times$ stretching force $\times$ elongation. produced

work done per unit volume

$= \frac{1}{2} \times \frac{\text{stretching force}}{a} \times \frac{\text{elongation}}{l}$

$= \frac{1}{2} \times$ stress $\times$ strain

(2) Work per unit volume in changing the volume of the solid

$$K = \frac{F}{A} \cdot \frac{V}{\Delta V} = \frac{PV}{\Delta V}$$

$$P = \frac{K\Delta V}{V}$$

$$W = \int_0^V \cancel{K}PV \; dV$$

$$W = \int_0^V \frac{K\Delta V}{V} \cdot dv$$

$$W = \frac{K}{V} \cdot \frac{\Delta v^2}{2} = \frac{1}{2} \frac{K\Delta V}{V} \cdot \Delta V$$

$$= \frac{1}{2} \times P \cdot \Delta V$$

$= \frac{1}{2} \times$ bulk stress $\times$ change in volume

w/unit volume $= \frac{1}{2} \times$ stress $\times$ strain

② Work done per unit volume during shearing strain

$$\eta = \frac{F}{A\theta} = \frac{F}{l^2} \times \frac{l}{\Delta l} = \frac{F}{l \Delta l}$$

$$W = \int_0^l F \cdot dl = \int_0^l \eta l \Delta l \, dl = \eta l \cdot \frac{\Delta l^2}{2}$$

$$= \frac{1}{2} \times F \times l$$

Work done per unit volume $= \frac{1}{2} \times$ stress $\times$ strain

Relation between moduli of elasticity

It can be proved that

$$E = 2\eta(1+\sigma) = 3K(1-2\sigma)$$

where σ = Poisson's ratio

$$\frac{E}{\eta} = 2 + 2\sigma \quad \text{and} \quad \frac{E}{3K} = 1 - 2\sigma$$

adding both equations

$$\frac{E}{\eta} + \frac{E}{3K} = 3$$

$$E\left(\frac{1}{\eta} + \frac{1}{3K}\right) = 3$$

$$\frac{3}{E} = \frac{1}{\eta} + \frac{1}{3K}$$

multiplying by 3 throughout

$$\frac{9}{E} = \frac{3}{\eta} + \frac{1}{K}$$

* Poisson's ratio in terms of K and η

$$2\eta(1+\sigma) = 3K(1-2\sigma)$$

$$2\eta + 2\sigma\eta = 3K - 6K\sigma$$

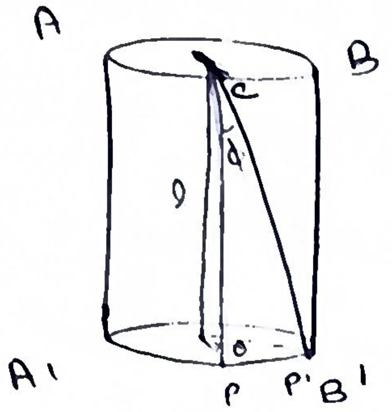
$$2\eta\sigma + 6K\sigma = 3K - 2\eta$$

$$\boxed{\sigma = \frac{3K - 2\eta}{2\eta + 6K}}$$

Twisting couple on a cylinder / wire

Consider a cylinder of length l

radius a

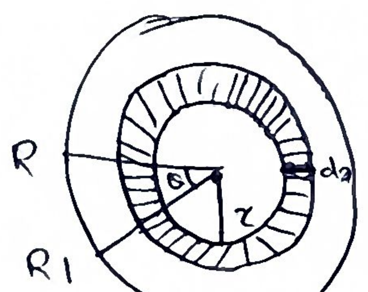


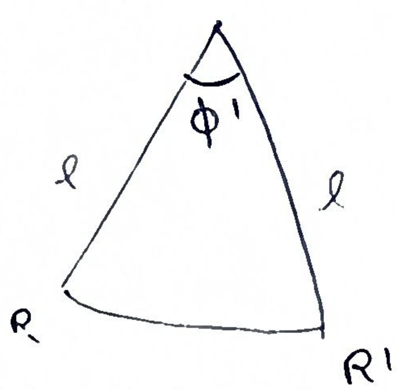
- A twisting couple is applied on the face AB' perpendicular to the length of the cylinder.
- Due to this, each circular cross-section is turned through an angle θ, called the angle of twist. (called pure shear, as there is no change in length or radius, but rather only in shape)

- Due to the elasticity of the material, a restoring torque is set up.
- A line CP on the rim is displaced to CP', through an angle of ϕ called the angle of shear.

To calculate: value of the twisting couple

consider a cylindrical shell of radius x, thickness dx





since θ is small,

$RR' = x\theta$ $\qquad$ $RR' = l\phi'$

$\therefore x\theta = l\phi'$

or $\phi' = \dfrac{x\theta}{l}$

$N = \dfrac{T}{\phi'}$ $\qquad$ T = shearing stress

ϕ' = angle of shear

$$N = \frac{T}{\frac{x\theta}{l}} = \frac{Tl}{x\theta}$$

or $$T = \frac{Nx\theta}{l}$$

base area of the cylinder of thickness 'dx'

$= 2\pi x \, dx$

$$\frac{\text{shearing force}}{\text{area}} = \text{shearing stress}$$

shearing force $= 2\pi x \, dx \times \frac{Nx\theta}{l}$

moment = force × ⊥r distance $= \frac{2\pi x^2 dx \cdot N\theta}{l}$

$= \frac{2\pi x^3 dx \cdot N\theta}{l}$ ⇒ expression for the couple required to twist an infinitesimally thin cylindrical shell of radius x through an angle θ

To calculate the total couple :

$$\int_{x=0}^{x=a} \frac{2\pi N\theta}{l} \cdot x^3 \, dx$$

$$= \frac{2\pi N\theta}{l} \cdot \frac{x^4}{4}$$

$$= \frac{2\pi N\theta}{l} \cdot \frac{a^4}{4}$$

$$= \frac{\pi N\theta a^4}{2l}$$

If $\theta = 1$ radian, then couple per unit twist $= \frac{\pi N a^4}{2l}$

This twisting couple required to produce a twist of unit radian in the cylinder is called torsional rigidity or modulus of torsion.

Shaft : • A shaft is a component of a machine which is used to transmit power from a source to a load.

• It transmits the couple applied at one end to the other end without any appreciable twist in it.

• It can rotate on bearings, where there is an arrangement for applying of a couple at one end with an attachment to a load at the other end.

• Thus any rotating member which transmits torque is called a shaft.

Torsional Pendulum

Torsion Pendulum : A torsional pendulum is a pendulum that performs torsional oscillations and is used to determine the rigidity modulus of the suspension wire and the moment of inertia of the disc.

- The torsional pendulum consists of a metal wire clamped to a metal support at one end and it carries a heavy circular disc at the other end.
- The disc is slightly rotated, and it starts oscillating periodically.
- When a wire of length 'l' and radius 'a' is subjected to an external couple or torque, a restoring couple proportional to the twist is set up due to the elastic reaction.
- The restoring couple sets up an angular acceleration in the wire opp to the twist. During untwisting, it rotates beyond its equilibrium position.
- This again sets up an angular acceleration in the opposite direction.
- This process repeats and thus, torsional oscillations are executed.
- Consider an intermediate stage when the wire is under twist θ & the disc has an angular acceleration

$$\alpha = \frac{d^2\theta}{dt^2}$$

- Potential energy $= \int_0^\theta$ moment of the couple $. d\theta$

$$= \int_0^\theta c\theta \, d\theta = \frac{1}{2} c\theta^2 \quad (c = \text{couple per unit twist})$$

Kinetic energy $= \frac{1}{2} \Sigma m_i v_i^2 = \frac{1}{2}\omega^2 \Sigma m_i r_i^2 = \frac{1}{2} I\omega^2$

$\left.\begin{array}{l} m_i = \text{mass} \\ v_i = \text{velocity} \end{array}\right\}$ of the ith constituent particle of the disc.

ω = angular velocity

· Total energy: $\frac{1}{2}c\theta^2 + \frac{1}{2}I\omega^2$

$$= \frac{1}{2}c\theta^2 + \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2$$

= constant (by the law of conservation of energy)

differentiating w.r.t time

$$\frac{1}{2}c \cdot 2\theta \cdot \frac{d\theta}{dt} + \frac{1}{2}I \cdot 2\frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2}$$

$$0 = c\theta \cdot \frac{d\theta}{dt} + I \cdot \frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2}$$

$$\Rightarrow c\theta + I\frac{d^2\theta}{dt^2} = 0$$

$$I\frac{d^2\theta}{dt^2} = -c\theta$$

$$\frac{d^2\theta}{dt^2} = \frac{-c}{I}\theta$$

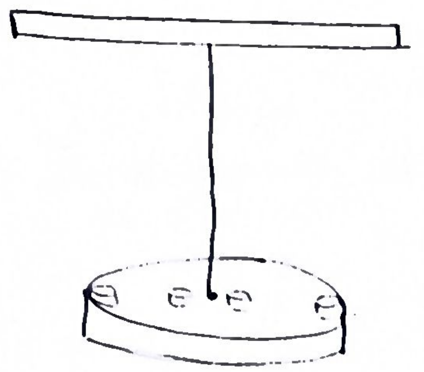
This is similar to the equation $\frac{d^2y}{dx^2} = -\omega^2 y$ (standard SHM eqn)

here $\omega^2 = \frac{c}{I}$, $\omega = \sqrt{\frac{c}{I}}$

∴ Time period $= 2\pi\sqrt{\frac{I}{c}}$

Thus, the oscillations of a ~~simple~~ torsional pendulum are controlled by the moment of inertia of the suspended mass and the couple per unit twist produced in the wire.

Experimental determination of the rigidity modulus and the moment of inertia



- The disc is set into oscillation without any masses on it. The mean period of oscillation T_0 is found out.

$$T_0 = 2\pi\sqrt{\frac{I_0}{c}}$$, where I_0 is the moment of inertia about the axis of suspension w/o any cylindrical masses on it.

- Two equal cylindrical masses of 'm' kg are placed symmetrically on the disc, such that they are very close to the axis of rotation. The distance d_1 from the axis to the center of gravity of the mass is measured. The mean time period T_1 is measured.

$$T_1 = 2\pi\sqrt{\frac{I_1}{c}}$$

$$I_1 = I_0 + 2i_0 + 2md_1^2$$ (from the parallel axis theorem)

i_0 is the moment of inertia of each mass passing through the center & ⊥r to its plane

- The 2 cylindrical masses are then placed farther away, towards the edges of the disc. The mean time for oscillation T_2 is obtained.

$$T_2 = 2\pi\sqrt{\frac{I_2}{c}}$$ where $I_2 = I_0 + 2i_0 + 2md_2^2$

$$I_2 - I_1 = 2m(d_2^2 - d_1^2)$$

$$T_2^2 - T_1^2 = \frac{4\pi^2}{c}\left\{I_2 - I_1\right\}$$

$$\frac{T_0^2}{T_2^2 - T_1^2} = \frac{I_0}{I_2 - I_1}$$

$$\frac{T_0^2}{T_2^2 - T_1^2} = \frac{I_0}{I_2 - I_1}$$

$$= \frac{I_0}{2m(d_2^2 - d_1^2)}$$

$$\therefore \boxed{I_0 = \frac{2m(d_2^2 - d_1^2)\,T_0^2}{T_2^2 - T_1^2}}$$

* Calculation of rigidity modulus

$$C = \frac{\pi n a^4}{2l} \quad - (1)$$

$$T_0^2 = 4\pi^2 \frac{I_0}{C} \qquad = \frac{4\pi^2 I_0}{T_0^2}$$

$$C = \frac{4\pi^2 \cancel{I_0}}{\cancel{T_0^2}} \cdot \frac{2m(d_2^2 - d_1^2)\cancel{T_0^2}}{T_2^2 - T_1^2} \quad - (2)$$

from (1), (2)

$$\frac{\cancel{\pi} n a^4}{2l} = \frac{8\pi \cancel{^2} m(d_2^2 - d_1^2)}{T_2^2 - T_1^2}$$

$$n = \frac{8\pi m(d_2^2 - d_1^2) \times 2l}{(T_2^2 - T_1^2)\,a^4}$$

$$= \frac{16\pi m(d_2^2 - d_1^2)\,l}{a^4 (T_2^2 - T_1^2)}$$

$$\boxed{n = \frac{8\pi l I_0}{T_0^2 a^4}} \quad Nm^{-2}$$

Bending of beams

Assumptions made:

(i) cross-sectional area remains unaltered. (so that shearing stress is small)
(ii) Radius of curvature is large compared to thickness
(iii) Young's modulus remains unchanged.
(iv) minimum deflection is small compared to its length.

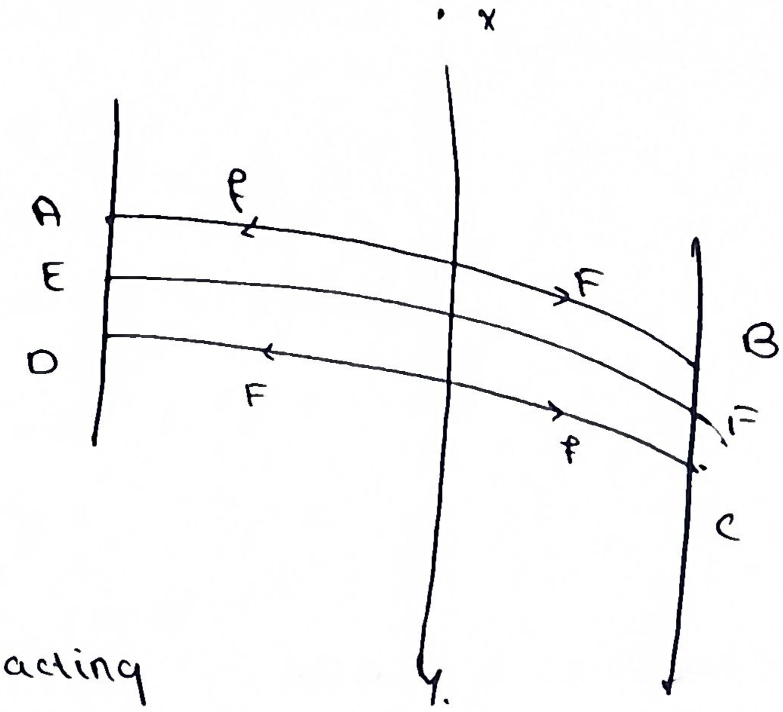
Plane of bending and neutral axis of a bent beam

- A ~~plane~~ beam is made up of a no. of horizontal planes called surfaces.
- Each surface has longitudinal fibres placed parallel ⇒ called longitudinal filaments.

(i) Longitudinal filaments on the convex side = elongated.
(ii) on the concave side = compressed

- Some filaments in the middle are unaltered in length. The plane containing these is called the neutral surface. Neutral surface is defined as the plane in which there is no elongation or compression respectively due to ~~the~~ tensile | compressive force.
- The plane in which the bending takes place is called the plane of bending
- The line obtained from the intersection of the neutral surface & the plane of bending = neutral axis.
- A line ⊥r to the plane of bending, where all the centres of curvatures of the bent filament lie is called the axis of bending.

Bending Moment



- In the bent beam, let EF be the neutral element.
- Under a force of F:
  → AB elongates
  → CD is compressed.

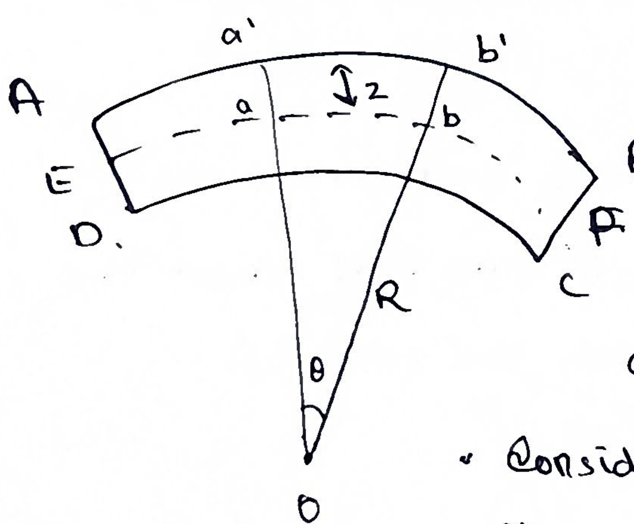
These 2 forces constitute a couple acting clockwise ⇒ Bending couple

- Since the beam is at rest, the moment of this couple must be balanced. ∴
  When AB elongates → an opp. restoring force 'f' acts on them
  and CD is compressed

This constitutes a couple acting anticlockwise ⇒ restoring couple

- In eq. position, moment of bending couple = moment of restoring couple.
- The moment of the internal bending couple is called the bending moment

* Calculating the bending moment of a beam

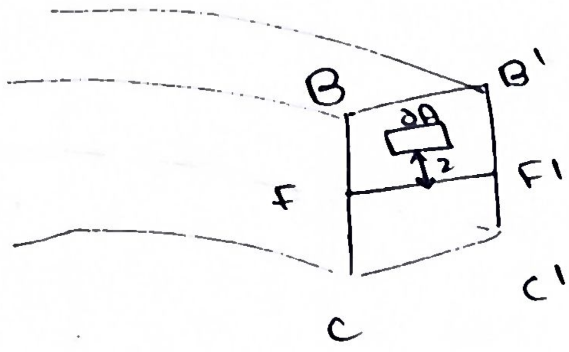


- Consider a bent beam ABCD.
- EF is the neutral axis.
- Let the radius of curvature of the beam be R and let a small element ab subtend an angle θ at the centre.
- Consider another element a'b', z distance above ab.

- When θ is small: $R\theta = ab$ and $(R+z)\theta = a'b'$

The increase in length, $a'b' - ab = (R+z)\theta - R\theta$
$= z\theta$

$$\text{strain} = \frac{\text{change in length}}{\text{original length}} = \frac{z\theta}{R\theta} = \frac{z}{R}$$



- $BB'C'C$ is the cross-sectional area
- ∂A is a small areal element, z distance above the neutral axis FF'
- The Young's modulus of the material:

$$E = \frac{\text{stress}}{\text{strain}}$$

$$\text{stress} = E \times \text{strain}$$
$$= E \times \frac{z}{R}$$

i.e. stress acting on the element $\partial A = E \times \frac{z}{R}$

Force on the area $\partial A = E \times \frac{z}{R} \times \partial A$

Moment of the force along the neutral line $= E \times \frac{z}{R} \times \partial A \times z$

$$= \frac{E}{R} \times \partial A \times z^2$$

Total moment of force $= \frac{E}{R} \sum \partial A z^2$

The quantity $\sum \partial A z^2 = I_g \Rightarrow$ geometric moment of inertia of the cross-sectional area of the beam about a horizontal axis passing through the centroid.

It is also equal to AK^2, where $A =$ cross-sectional area

K = radius of gyration about the horizontal axis through the centroid

$\therefore$ $\boxed{\text{Bending moment} = \frac{E}{R} \times I_g}$

Note: The quantity $EI_g = EAK^2 \Rightarrow$ flexural rigidity.

Note: If the cross-sectional area is rectangular, then the

moment of inertia $= MK^2 = M\frac{d^2}{12}$

i.e $K^2 = \frac{d^2}{12}$

Area $(A) = b.d$

$\therefore I_g = AK^2 = bd.\frac{d^2}{12} = \frac{bd^3}{12}$ //

- If the cross-sectional area is circular:

moment of inertia $= MK^2 = \frac{Mr^2}{4}$

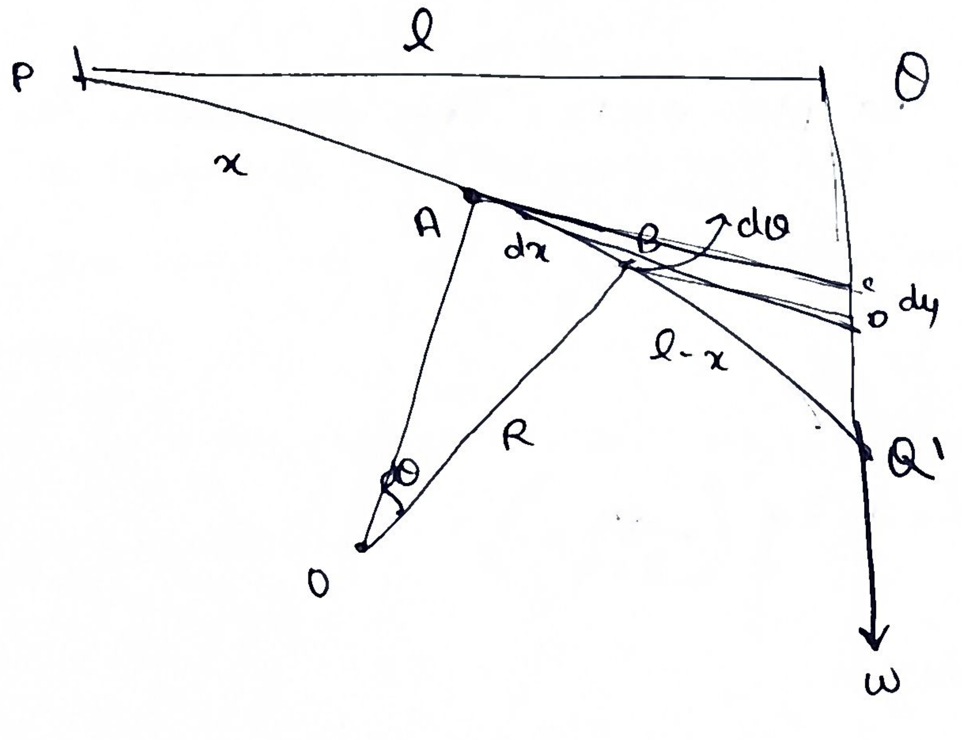
i.e $K^2 = \frac{r^2}{4}$

Area $(A) = \pi r^2$

$\therefore I_g = AK^2 = \pi r^2 . \frac{r^2}{4} = \frac{\pi r^4}{4}$ //

* Depression of a cantilever loaded at the free end

- A cantilever is a beam, fixed horizontally at one end and loaded at the other end.
- The Young's modulus of the material can be calculated, by finding out the depression



- Consider a cantilever, with the neutral axis = PQ
- Let the weight of the cantilever be negligible
- Let a load of w be suspended from Q, such that there is a depression till point Q'.
- Consider a point A, x distance away from P.

$$\text{Bending couple} = w \cdot AQ'$$
$$= w \cdot (l-x)$$

At equilibrium, this is equal to the restoring couple.

$$w \cdot (l-x) = \frac{E}{R} Ig \qquad (1)$$

where R is the radius of curvature of the neutral axis at A. Let AB subtend an angle θ.

For small angles: $dx = R\,d\theta$

or $R = \dfrac{dx}{d\theta}$

(1) becomes: $w(l-x) = EIg \cdot \dfrac{d\theta}{dx}$

or $w(l-x)\,dx = EIg\,d\theta \qquad (2)$

Draw tangents from A & B to the pts C & D. This depression $CD = dy$

$$dy = d\theta\,(l-x)$$

or $d\theta = \dfrac{dy}{l-x}$

(2) becomes: $w(l-x)\,dx = EIg\left(\dfrac{dy}{l-x}\right)$

$$w(l-x)^2\,dx = EIg \cdot dy$$

$$\cdot\ W(l-x)^2\,dx = EI_g \cdot dy$$

depression $\quad dy = \dfrac{W(l-x)^2\,dx}{EI_g}$

Total depression $\quad y = \dfrac{W}{EI_g}\displaystyle\int_0^l (l-x)^2\,dx$

$$y = \frac{W}{EI_g}\int_0^l l^2 + x^2 - 2lx\,dx$$

$$y = \frac{W}{EI_g}\left\{ l^2x + \frac{x^3}{3} - \frac{2lx^2}{2} \right\}_0^l$$

$$= \frac{W}{EI_g}\left\{ \cancel{l^3} + \frac{l^3}{3} - \cancel{l^3} \right\}$$

$$Y = \frac{Wl^3}{3EI_g}$$

$\therefore$ The Young's modulus of the material is:

$$\boxed{E = \frac{Wl^3}{3YI_g}}$$

(ii) When the weight of the cantilever is also taken into account

- In addition to the weight W at Q, there is also the weight of the cantilever at $(l-x)$ also acts along the mid pt. of this section.

- If w is the wt per unit length of the cantilever, then a wt. of $w(l-x)$ acts at $\dfrac{(l-x)}{2}$.

Under equilibrium conditions;

$$W(l-x) + w\frac{(l-x)}{2}.(l-x) = \frac{E\,I_g}{R}$$

i.e $$W(l-x) + w\frac{(l-x)^2}{2} = E\,I_g \cdot \frac{d\theta}{dx}$$

$$W(l-x)dx + \frac{w(l-x)^2}{2}dx = E\,I_g \cdot d\theta$$

$$dy = d\theta(l-x)$$

or $$d\theta = \frac{dy}{l-x}$$

$$W(l-x)\,dx + \frac{w(l-x)^2}{2}dx = E\,I_g\left(\frac{dy}{l-x}\right)$$

$$W(l-x)^2\,dx + \frac{w(l-x)^3}{2} = E\,I_g.dy$$

$$dy = \frac{W}{E\,I_g}(l-x)^2 dx + \frac{w}{2E\,I_g}(l-x)^3$$

Total depression: $$Y = \frac{W}{E\,I_g}\int_0^l (l-x)^2\,dx + \frac{w}{2E\,I_g}\int_0^l (l-x)^3 dx$$

Let $l-x = u$ $\qquad -dx = du$

$l-x$	0	l
u	l	0

$$Y = \frac{W}{E\,I_g}\int_0^l u^2\,du + \frac{W}{2E\,I_g}\int_0^l u^3\,du$$

$$= \frac{W}{E\,I_g}\cdot\left(\frac{u^3}{3}\right)_0^l + \frac{w}{2E\,I_g}\left(\frac{u^4}{4}\right)_0^l$$

$$= \frac{Wl^3}{3EIg} + \frac{wl^4}{8EIg}$$

if $W_1 = wl$

then, $$Y = \frac{Wl^3}{3EIg} + \frac{W_1 l^3}{8EIg}$$

$$Y = \frac{l^3}{3EIg}\left\{ W + \frac{3}{8}W_1 \right\}$$

if $W_1 << W$

then $$\boxed{Y = \frac{Wl^3}{3IgE}}$$

* **Work done in bending a cantilever**

Bending Force: $Mg = W$

$$Mg = \frac{3IgEY}{l^3}$$

Work done = bending Force x displacement

$$= \frac{3IgEY}{l_3} \times Y$$

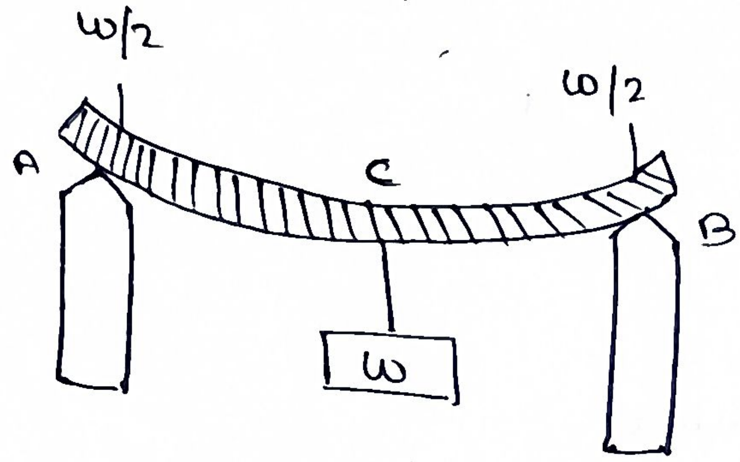
$$= \frac{3IgEy^2}{l^3}$$

This is stored as elastic potential energy in the cantilever.

Uniform and Non-uniform bending

- When a uniform load acts on the beam, the envelope of the bent beam forms an arc of a circle, and thus it is called uniform bending.
- When the beam is loaded only at one point, the bending is called non-uniform bending (eg. in cantilever).

Non-uniforming Bending



- A light beam is supported symmetrically on 2 knife ends.
- A load w is suspended at the center 'C'.
- The reaction at each knife edge is $w/2$.
- Since the middle part is practically horizontal, it can be considered to be 2 cantilevers with a load of $w/2$ acting upwards.

$$y = \frac{wl^3}{3 I_g E}$$

here $w = w/2$ and $l = l/2$

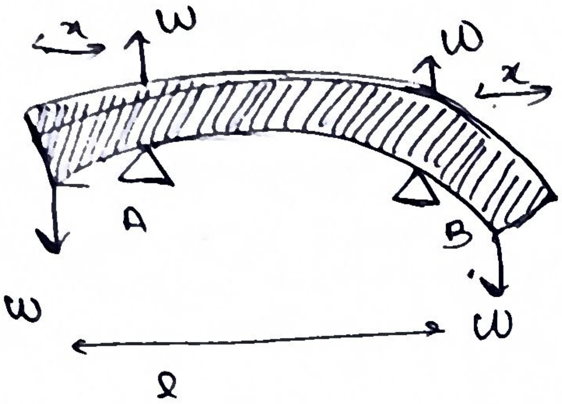
$$\therefore y = \frac{\left(\frac{w}{2}\right)\left(\frac{l}{2}\right)^3}{3 I_g E} = \frac{w l^3}{2 \times 8 \times 3 I_g E}$$

$$= \frac{wl^3}{48 I_g E}$$

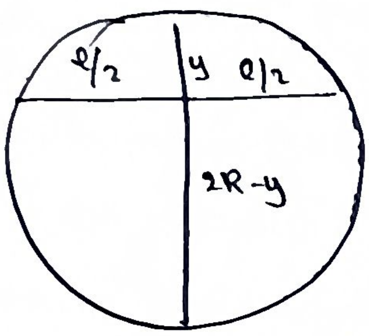
Uniform Bending

- Consider a light beam CD supported symmetrically on 2 knife ends at ends A & B.
- There is a load W suspended at each end.
- The reaction at each knife edge is W.

The bending moment : $W \cdot x = \frac{E}{R} \cdot I_g$ (1)



- If the centre of the beam is elevated through a distance of y, then: by the property of circles (Sagitha's rule)



$$\frac{l}{2} \times \frac{l}{2} = y(2R-y)$$

$$\frac{l^2}{4} = 2Ry - y^2$$

y^2 is very small, so it is neglected.

$$\frac{l^2}{4} = 2Ry$$

or $$\frac{1}{R} = \frac{8y}{l^2} \quad (2)$$

Sub (2) in (1)

$$W \cdot x = EI_g \cdot \frac{8y}{l^2}$$

$$\boxed{y = \frac{Wxl^2}{8I_gE}}$$

Experimental set up

- Support the beam on knife edge(s)
- For non-uniform bending : suspend a load at the centre

For uniform bending: suspend loads on both ends.

- Fix a pin at the centre with wax.
- Align the tip of the pin with the horizontal crosswire of a travelling microscope.
- Add loads to the hanger in ~~[illegible]~~ steps of 50g.
- Realign the tip of the pin with the horizontal crosswire each time, and note the microscopic readings.
- Tabulate the readings

Load (kg)	Microscopic readings			depression 'y' for a mass of M kg
	Loading (m)	unloading (m)	mean (m)	
				mean 'y' ______ m

- The depression is calculated by taking the diff. between 2 successive microscopic readings and then finding their mean.
- use the formula:.

non-uniform bending : $y = \frac{w l^3}{48 I_g E}$

uniform bending : $y = \frac{w l^3}{8 I_g E}$

to find E.

<u>I- shaped girders</u>

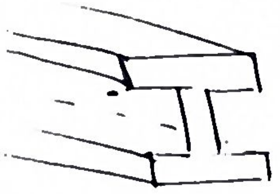
- When a heavy girder is supported at its ends, it bends non-uniformly due to its own weight, in the form of an inverted cantilever.
- $y = \frac{w l^3}{48 I_g E}$

If the cross-section is rectangular, then

$I_g = \frac{bd^3}{12}$

Then $y = \dfrac{wl^3}{48E\left(\dfrac{bd^3}{12}\right)}$

i.e $y = \dfrac{Mgl^3}{4Ebd^3}$



- When a beam is used as a girder, it should have min. depression under its own weight. The depression should be less even when it is loaded.
- This can be done by:
  - (i) decreasing l
  - (ii) increasing E, b or d.
- l can be decreased, but it is not economical.
- Since d ~~ba~~ is in the 3rd degree in the dr, increasing d would drastically reduce y and the volume.
- Thus, it is economical to have large depth and small breadth
- ~~The~~ When a girder is supported at both ends, the middle part is depressed, and the surfaces above and below the neutral surfaces are compressed & extended respectively.
- Stress decreases as we move towards the middle, i.e the middle portion of the girder may be made w/ a smaller amount of material, but the top and bottom parts require additional material, as they are under greater amounts of stress.
- That is why girders are I-shaped.

Advantages

(i) smaller depression even for a large dynamic load.

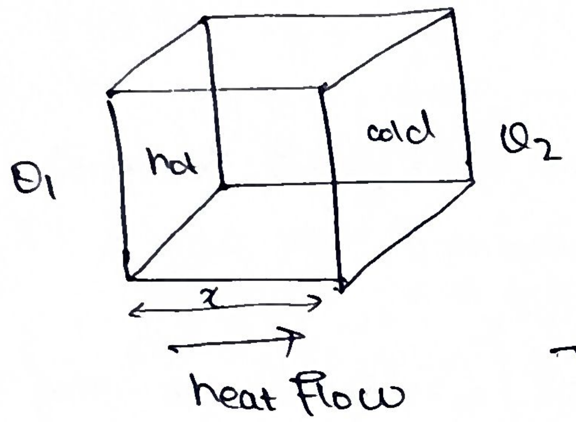
(ii) good amount of material saving w/ no loss of strength.

Thermal Physics

Methods of transfer of heat energy

1. Conduction: transfer of heat energy w/o the actual movement of particles. (eg. heating a metal bar)
2. Convection: transfer of heat w/ the actual movement of particles (eg. sea breeze and land breeze)
3. Radiation: heat transmitted from one place to another w/o the necessity of the intervening medium (how we receive heat from the sun).

Coefficient of Thermal Conductivity



opp. faces have temps θ_1 & θ_2

$\theta_1 > \theta_2$

Area $= A$

length $= x$

The quantity of heat conducted through the body normally (Q)

$\propto$ change in temp $(\theta_1 - \theta_2)$

$\propto$ area (A)

$\propto$ time (t)

$\propto \dfrac{1}{\text{length}}$ (x)

i.e $Q \propto \dfrac{(\theta_1 - \theta_2) \cdot At}{x}$

$$Q = \frac{K(\theta_1 - \theta_2)At}{x}$$

K = coeff. of thermal conductivity

$\dfrac{\theta_1 - \theta_2}{x}$ = temp. gradient

$$Q = K\left(\frac{d\theta}{dx}\right) \cdot At$$

if $A = 1$, $t = 1$ $d\theta/dt = 1$ $\Rightarrow K = Q$

$$\boxed{K = \frac{Q}{A\left(\frac{d\theta}{dx}\right) \cdot t}}$$

K is defined as the quantity of heat flowing per unit area per unit time when the temp. gradient is unity. $\boxed{\text{unit } Wm^{-1}K^{-1}}$

Steady state in Heat conduction

- When an amount of heat is flowing from one end of the body to another, the temperature is diff. at diff. points.
- After a particular amount of time, the temp. is a constant throughout the body. No energy is required to raise its temp.

This is called the steady state.

Thermal Diffusivity

- If K = coeff. of thermal conductivity
  s = specific heat capacity
  ρ = density,

then the ratio $K/\rho s$ = thermal diffusivity.

Thermal diffusivity is defined as the ratio of the coeff. of thermal conductivity to the thermal capacity per unit volume.

Newton's Law of Cooling: The rate of cooling of a body is directly proportional to the temperature diff. between the body & the surroundings.

$\frac{d\theta}{dt}$ = rate of cooling

$$\frac{dQ}{dt} \propto \theta_n - \theta_s$$

$$\boxed{-\frac{dQ}{dt} = k(\theta_n - \theta_s)} \quad (1)$$

k depends on nature & area of exposed surface.

If the body has a mass m, and specific heat s, and if the temp. of the body falls by $d\theta$, then the amount of heat energy lost is given by:

$$dQ = ms\,d\theta$$

rate of heat lost: $\boxed{\frac{dQ}{dt} = \frac{ms\,d\theta}{dt}} \quad (2)$

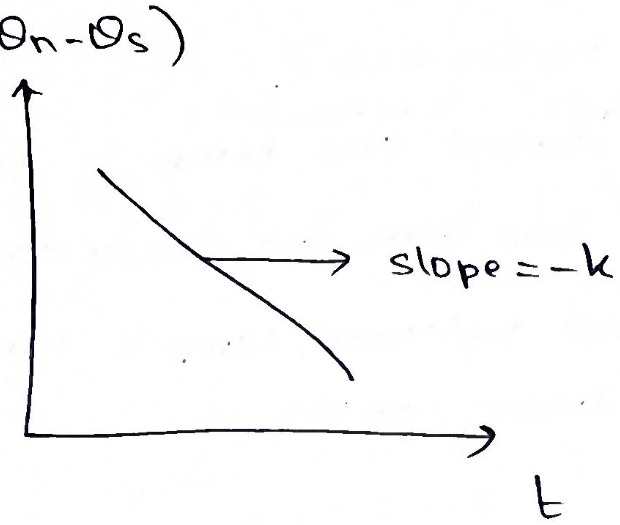
equating 1 , 2

$$-k(\theta_n - \theta_s) = \frac{ms\,d\theta}{dt}$$

$$\frac{d\theta}{(\theta_n - \theta_s)} = \frac{-k\ \cancel{ms}}{ms} . dt$$

$$\frac{d\theta}{(\theta_n - \theta_s)} = -k\,dt \qquad \text{where } -k = \frac{-k}{ms}$$

$$\int \frac{d\theta}{(\theta_n - \theta_s)} = \int -k\,dt$$



$$\boxed{\ln(\theta_n - \theta_s) = -kt + C}$$

If a body cools from θ_1 to θ_2

$$\int_{\theta_2}^{\theta_1} \frac{d\theta}{(\theta_n - \theta_s)} = +\int k\,dt$$

$$\ln\left(\frac{\theta_1 - \theta_s}{\theta_2 - \theta_s}\right) = +kt$$

$$\boxed{t = \frac{1}{k}\ln\left(\frac{\theta_1 - \theta_s}{\theta_2 - \theta_s}\right)}$$

Thermal Conductivity of a Bad Conductor - Lee's Disc method

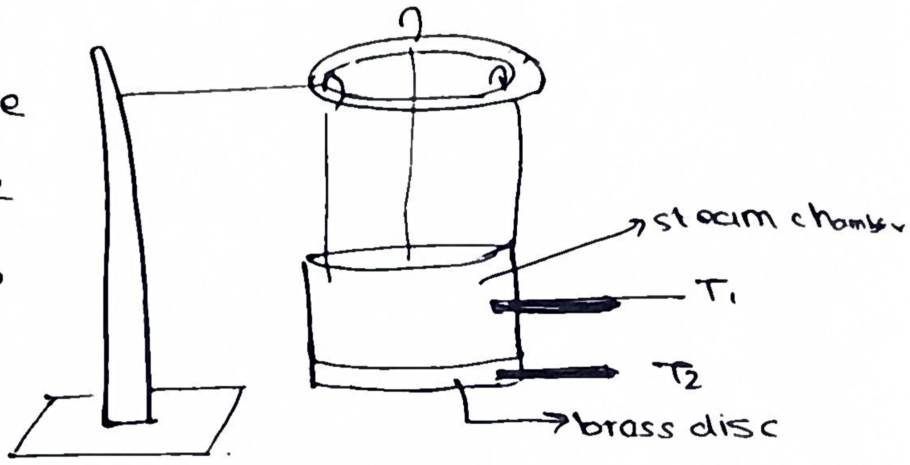
The quantity of heat conducted is directly proportional to A and inversely proportional to x. The thermal conductivity of a bad conductor can be measured when it is a disc with a large surface area.

Apparatus: The setup includes a brass disc suspended by strings. A steam chamber is placed above it. The bad conductor is between the two, in the form of a disc. Two thermometers T_1 & T_2 are inserted into holes to measure the temp. of the brass disc & the chamber.

Experimental setup

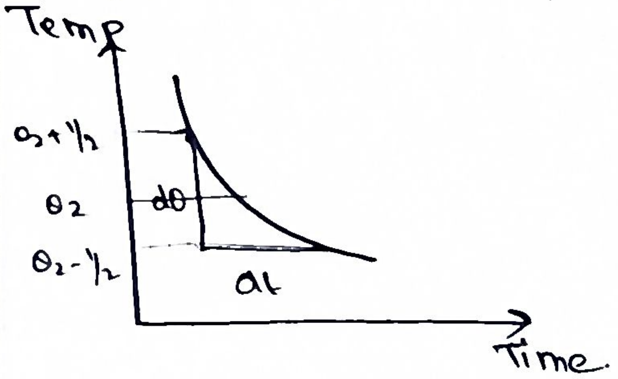
Part 1 : Steam is pumped into the steam chamber. The temp of the steam chamber & the metal disc begin to rise. After the steady state is reached, the temp. of the steam chamber (θ_1) & temp. of the brass disc (θ_2) is measured.

Part 2 : The conductor is removed, and the steam ~~conductor~~ chamber is in direct contact w/ the brass disc. The temperature is allowed to rise till the temp. is 10°C above the steady state (i.e $\theta_2 + 10$).



The steam chamber is removed.
A stopwatch is started and temp is noted every 30 seconds till the temp is 10°c less than the steady state ($\theta_2 - 10$)

A graph is plotted between temp & time. The rate of cooling is determined by the slope drawn at θ_2



Theory

Part 1 : For the 1st part,

$$Q = \frac{K(\theta_1 - \theta_2) A}{x}$$

$$Q = \frac{K(\theta_1 - \theta_2) \cdot \pi r^2}{x} \quad -(1)$$

The heat lost due to radiation is given by : area of exposed surfaces × emissivity

$$\text{i.e } Q = (\pi r^2 + 2\pi r l) \cdot E \quad -(2)$$

E is defined as the quantity of heat lost per unit area per unit time.

At a steady state, (1) = (2)

$$\frac{K(\theta_1 - \theta_2) \cdot \pi r^2}{x} = (\pi r^2 + 2\pi r l) \cdot E$$

$$E = \frac{K(\theta_1 - \theta_2) . \pi r^2}{x(\pi r^2 + 2\pi r l)} \quad (3)$$

Part 2: In the second part of the expt., the top, bottom and side surfaces radiate heat energy.

$$E = \frac{\text{Rate of loss of heat by Radiation}}{\text{Area of exposed surface}}$$

Let M be the mass of the disc

s = heat capacity

$d\theta / dt$ = rate of cooling.

Then,

$$E = \frac{M s \frac{d\theta}{dt}}{2\pi r^2 + 2\pi r l} \quad (4)$$

(3) = (4)

$$\frac{K(\theta_1 - \theta_2) . \pi r^2}{x(\pi r^2 + 2\pi r l)} = \frac{M s \, d\theta/dt}{2\pi r^2 + 2\pi r l}.$$

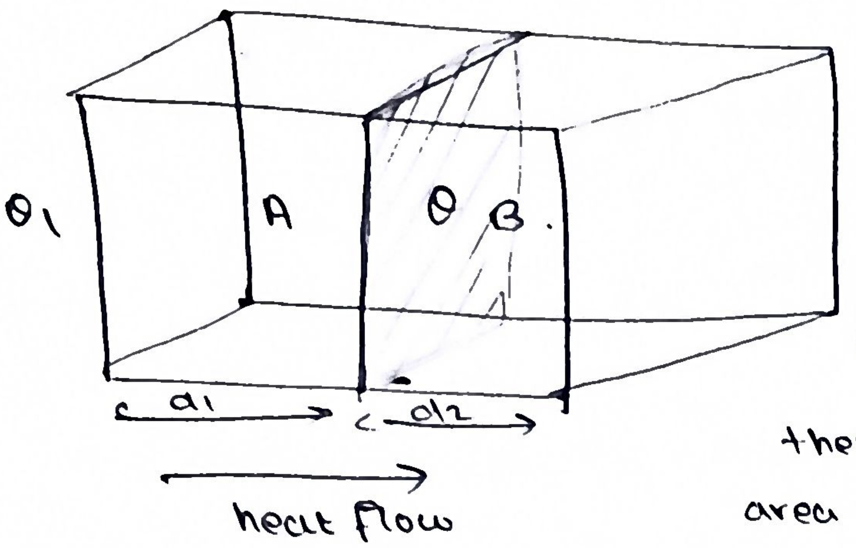
$$K = \frac{M s \frac{d\theta}{dt} . x(\pi r^2 + 2\pi r l)}{(\theta_1 - \theta_2) . \pi r^2 . (2\pi r^2 + 2\pi r l)}$$

$$K = \frac{\cancel{\pi r} . M s \frac{d\theta}{dt} . x \, (r + 2l)}{2\pi r . \cancel{\pi r} \, (\theta_1 - \theta_2) . (r + l)}$$

$$\boxed{K = \frac{M s \frac{d\theta}{dt} . (r + 2l) . x}{2\pi r^2 (\theta_1 - \theta_2)(r + l)} \quad W m^{-1} K^{-1}}$$

Conduction through compound media

A. In series



For the material A:

thermal conductivity : K_1

area : A

length : d_1

For the material B:

thermal conductivity : K_2

area : A

length : d_2

For the material A

Then: $Q = \dfrac{K_1(\theta_1-\theta)A}{d_1}$ $\Rightarrow$ $\theta_1-\theta = \dfrac{Q\,d_1}{A K_1}$ — (1)

For the material B

$Q = \dfrac{K_2(\theta-\theta_2)A}{d_2}$ $\Rightarrow$ $\theta-\theta_2 = \dfrac{Q\,d_2}{A K_2}$ — (2)

Adding (1), (2) : $\theta_1-\theta_2 = \dfrac{Q}{A}\left\{\dfrac{d_1}{K_1}+\dfrac{d_2}{K_2}\right\}$

or ~~in general~~ $Q = \dfrac{A(\theta_1-\theta_2)}{\left\{\dfrac{d_1}{K_1}+\dfrac{d_2}{K_2}\right\}}$

In general : $\boxed{Q = \dfrac{A(\theta_1-\theta_2)}{\Sigma \dfrac{d_i}{K_i}}}$

Also: $\dfrac{K_1(\theta_1-\theta)A}{d_1} = \dfrac{K_2(\theta-\theta_2)A}{d_2}$

$\dfrac{K_1\theta_1 - K_1\theta}{d_1} = \dfrac{K_2\theta - K_2\theta_2}{d_2}$

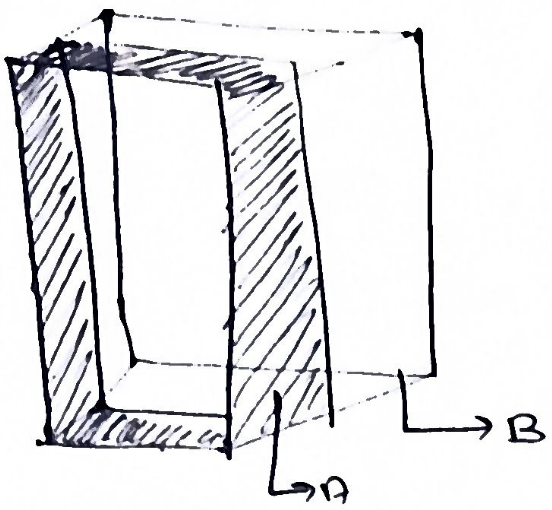
$\Rightarrow K_1\theta_1 d_2 - K_1\theta d_2 = K_2\theta d_1 - K_2\theta_2 d_1$

or $\boxed{\dfrac{K_1\theta_1 d_2 + K_2\theta_2 d_1}{K_1 d_2 + K_2 d_1} = \theta}$ ← interface temp

For numericals use

$\theta = \left(\dfrac{\dfrac{K_1\theta_1}{d_1}+\dfrac{K_2\theta_2}{d_2}}{\dfrac{K_1}{d_1}+\dfrac{K_2}{d_2}}\right)$

B. In parallel



For material A:

$$Q_1 = \frac{K_1 A_1 (\theta_1 - \theta_2)}{d_1}$$

For material B

$$Q_2 = \frac{K_2 A_2 (\theta_1 - \theta_2)}{d_2}$$

Total heat flowing through them:

$$Q = Q_1 + Q_2 = (\theta_1 - \theta_2) \left\{ \frac{K_1 A_1}{d_1} + \frac{K_2 A_2}{d_2} \right\}$$

In general : $\boxed{Q = (\theta_1 - \theta_2) . \sum \frac{K_i A_i}{d_i}}$

Radial Flow of heat

Learn pg 2.72

consider a cylindrical glass tube.

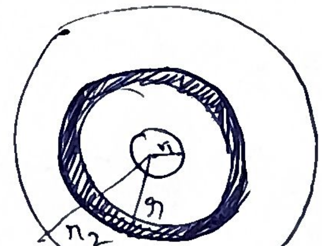
internal radius = r_1

external radius = r_2

length = l

internal temp = θ_1 ($\theta_1 > \theta_2$)

external temp = θ_2



consider a thin cylindrical shell at a distance r from the axis.
Its thickness is dr.

Area of the surface normal to the direction of heat flow = $2\pi r l$

Quantity of heat passing through the cylindrical shell:

$$Q = -KA \frac{d\theta}{dr}$$

$$Q = -K . 2\pi r l . \frac{d\theta}{dr}$$

$$Q \frac{dr}{r} = -K . 2\pi l . d\theta$$

$$\Rightarrow Q \int_{r_1}^{r_2} \frac{dr}{r} = -2\pi K l \int_{\theta_1}^{\theta_2} d\theta$$

$$= Q . \ln\left(\frac{r_2}{r_1}\right) = -2\pi K l (\theta_2 - \theta_1)$$

$$Q \ln\left(\frac{r_2}{r_1}\right) = 2\pi K l \, (\theta_1 - \theta_2)$$

$$K = \frac{Q \ln\left(\frac{r_2}{r_1}\right)}{2\pi l \, (\theta_1 - \theta_2)}$$

To calculate the intermediate temp.

$$Q \int_{r_1}^{r} \frac{dr}{r} = -2\pi K l \int_{\theta_1}^{\theta} d\theta$$

$$Q \ln\left(\frac{r}{r_1}\right) = 2\pi K l \, (\theta_1 - \theta)$$

$$Q \ln\left(\frac{r}{r_1}\right) = \cancel{2\pi l} (\theta_1 - \theta) \cdot \frac{Q \ln\left(\frac{r_2}{r_1}\right)}{\cancel{2\pi l} \cdot (\theta_1 - \theta_2)}$$

$$\ln\left(\frac{r}{r_1}\right) = \frac{\ln\left(\frac{r_2}{r_1}\right) \cdot (\theta_1 - \theta)}{(\theta_1 - \theta_2)}$$

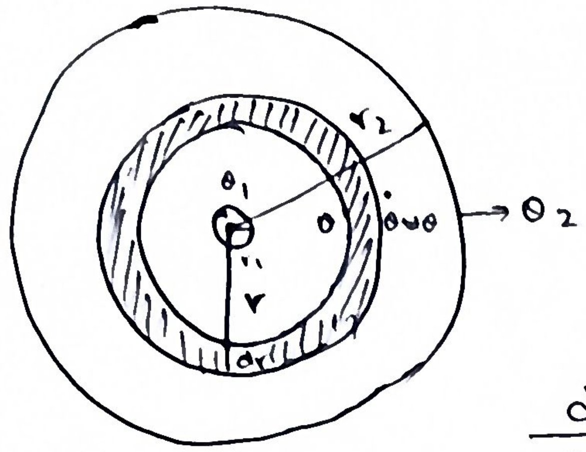
$$\ln\left(\frac{r}{r_1}\right) = \ln\left(\frac{r_2}{r_1}\right)\left(\frac{\theta - \theta_1}{\theta_1 - \theta_2}\right)$$

$$\frac{\ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)} = \frac{\theta_1 - \theta}{\theta_1 - \theta_2}$$

$$\theta_1 - \theta = \frac{(\theta_1 - \theta_2) \ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)} \Rightarrow \theta = \frac{-(\theta_1 - \theta_2) \ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)} + \theta_1$$

$$\boxed{\therefore \theta = \frac{\theta_1 \ln\left(\frac{r_2}{r_1}\right) - (\theta_1 - \theta_2) \ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)}}$$

For a spherical shell



quantity of heat passing through the thin shell

$$Q = -KA\,\frac{d\theta}{dr}$$

$$Q = -K \cdot 4\pi r^2 \cdot \frac{d\theta}{dr}$$

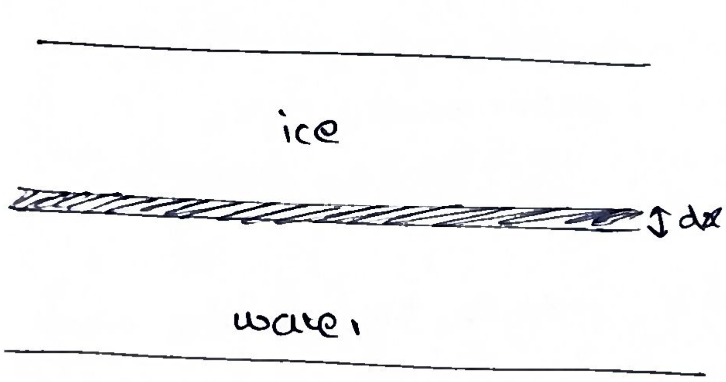
$$\frac{dr}{r^2} = -\frac{K}{Q} \cdot 4\pi \cdot d\theta$$

$$\int_{r_1}^{r_2} \frac{dr}{r^2} = -\frac{K \cdot 4\pi}{Q} \int_{\theta_1}^{\theta_2} d\theta$$

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{+4\pi K}{Q} \cdot (-\theta_2 + \theta_1)$$

or $$K = \frac{Q\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}{4\pi(\theta_1 - \theta_2)} = \boxed{K = \frac{Q(r_2 - r_1)}{r_1 r_2 \cdot 4\pi(\theta_1 - \theta_2)}}$$

Formation of ice on ponds.



↑ heat flow.

A thin layer of ice is formed on ponds when cold air above the pond removes heat from water at 0°C. The thickness of the ice keeps on increasing.

Let dx thickness of ice be formed in time dt.

Temp. of water = 0°C

Temp. below water = $-\theta$

mass of ice formed = $A.\rho.dx$

heat lost = $A\rho dx \cdot L$ (1) (L = latent heat)

Also $$Q = \frac{KA(0-(-\theta))}{x}.dt = \frac{KA\theta.dt}{x} \quad (2)$$

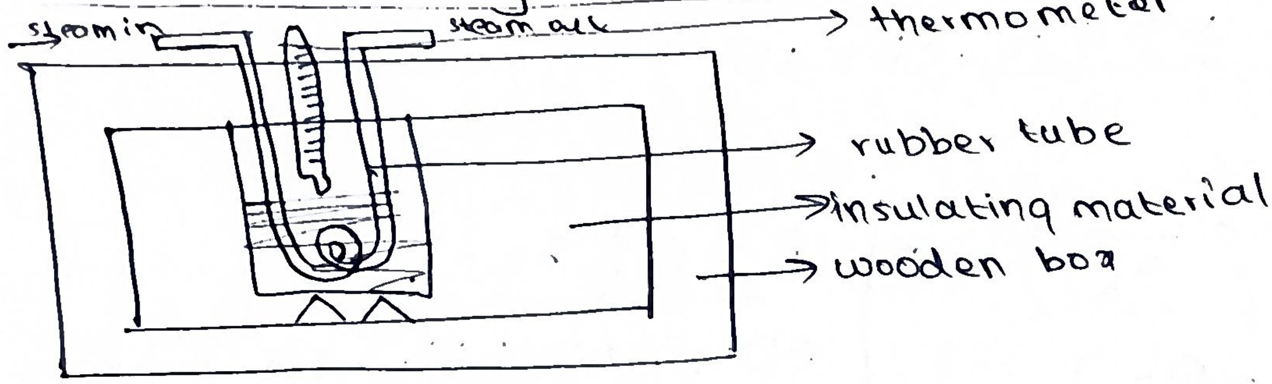
$\therefore A \rho dx . L = \frac{KA.\theta.dt}{x}$

$\frac{dx}{dt} = \frac{K\theta}{\rho L x}$

$dt = \frac{\rho L x \, dx}{K\theta}$

$\therefore \boxed{t = \frac{\rho L x^2}{2K\theta}}$

Thermal conductivity of rubber



Let the mass of the empty calorimeter $= w_1$

Fill it 2/3'ds the way up, mass of calorimeter $= w_2$

mass of water $= w_2 - w_1$

length of tube $= l$, inner radius $= r_1$, outer radius $= r_2$

$\theta_1 =$ initial temp. of water

Steam is passed through the tube for t seconds, until the temp. rises by 10°C

$\theta_2 =$ final temp. of water

$\theta_3 =$ temp. of steam

ava. temp of cold side of tube (θ_s) $= \frac{\theta_1 + \theta_2}{2}$

specific capacity of calorimeter $= s_1$

of water $= s_2$

~~By the radial flow of heat in a cylinder:~~

$$\cancel{K = \frac{Q \ln\left(\frac{r_2}{r_1}\right)}{2\pi l(\theta_1 - \theta_2)}}$$

$$\cancel{\text{or } Q = \frac{2\pi K l(\theta_S - \theta_3)}{\ln\left(\frac{r_2}{r_1}\right)}} \quad \text{— (1)}$$

~~heat gained by calorimeter = $w_1 s_1 (\theta_2 - \theta_1)$~~

By the radial flow of heat in a cylinder:

$$K = \frac{Q \ln\left(\frac{r_2}{r_1}\right)}{2\pi l\left(\theta_3 - \left(\frac{\theta_1 + \theta_2}{2}\right)\right)}$$

i.e

$$Q = \frac{2\pi K l\left(\theta_3 - \left(\frac{\theta_1 + \theta_2}{2}\right)\right)}{\ln\left(\frac{r_2}{r_1}\right)} \quad \text{— (1)}$$

heat gained by calorimeter per second $= \dfrac{w_1 s_1 (\theta_2 - \theta_1)}{t}$

by water per second $= \dfrac{(w_2 - w_1) s_2 (\theta_2 - \theta_1)}{t}$

Total heat gained: $Q = \dfrac{w_1 s_1 (\theta_2 - \theta_1) + (w_2 - w_1) s_2 (\theta_2 - \theta_1)}{t}$

$$Q = \frac{(\theta_2 - \theta_1)\{w_1 s_1 + (w_2 - w_1) s_2\}}{t} \quad (2)$$

equating (1) (2)

$$\frac{(\theta_2 - \theta_1)\{w_1 s_1 + (w_2 - w_1) s_2\}}{t} = \frac{2\pi K l\left(\theta_3 - \left(\frac{\theta_1 + \theta_2}{2}\right)\right)}{\ln\left(\frac{r_2}{r_1}\right)}$$

i.e

$$\boxed{K = \frac{(\theta_2 - \theta_1)\{w_1 s_1 + (w_2 - w_1) s_2\} \cdot \ln\left(\frac{r_2}{r_1}\right)}{2\pi l t\left(\theta_3 - \left(\frac{\theta_1 + \theta_2}{2}\right)\right)}}$$