Discrete Mathematics

Unit-1

Logics and Proofs

1. Proposition : A proposition is a declarative sentence that is either true or false, but not both. Sentences which are exclamatory, interrogative or imperative are not propositions.

* Lower case letters are used to denote propositions.

For eg. 1 New Delhi is the capital of India (Proposition)

2 How beautiful is Rose? (Not a proposition)

3 $2+2=3$ (Proposition)

4 What time is it? (Not a proposition)

5 $x+y=z$ (Not a proposition, statement is neither true nor false)

Look at examples on pg 1.2 GB

2. Atomic Statements : Atomic statements / Primary statements / Simple statements are those declarative sentences which cannot be further split into simple statements

for eg. 1 Ram is a boy

2 It is raining

Do Exercise on pg 1.3 GB

3. Molecular / Compound Statements : Molecular / compound / composite statements are those which contain at least one or more primary statements , and at least one connective.

for eg. Jack and Jill went up the hill.

Connectives

1. Negation : The negation of a statement is formed by adding not at the proper place in the statement, if p is a statement, then its negation is $\neg p$.

Truth Table :

p	$\neg p$
T	F
F	T

Write negations of the following statements.

(a) p : Today is Monday

$\neg p$: Today is not Monday

(b) $p : x < y$

$\neg p = x \geq y$

(c) p: Michael's PC runs on Linux.

$\neg p$: It is not the case that Michael's PC runs on Linux.

* **Conjunction (and):** If p & q are 2 propositions, the proposition 'p and q', is defined as the compound proposition that is true when p and q are true, and is false otherwise.

Truth Table:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Ex (a) p: It is snowing
q : I am cold.
$p \wedge q$: It is snowing and I am cold.

* **Disjunction (or)** : If p and q are 2 propositions, then 'p or q', is called the disjunction of p & q, which is false when both p and q are false, and is true otherwise.

Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Eg. (a) p: ~~x~~ 2 is a +ve integer.
q : $\sqrt{2}$ is a rational number.
$p \vee q$: 2 is a +ve integer or $\sqrt{2}$ is a rational number.

* **Conditional Statements** : If p and q are 2 statements, then $p \to q$ is a conditional statement.

F when Q is False and P is True.

Truth Table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Ex. p : I am hungry

q : I will eat

If I am hungry, I will eat.

* **Biconditional Statements**

If p and q are 2 statements, then $p \leftrightarrow q$ is a biconditional statement. (same as $(p \to q) \wedge (q \to p)$)

T when both p & q have identical values

Truth Table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Ex.1 p : You can take the flight

q : You buy a ticket

$p \leftrightarrow q$: You can take the flight if and only if you buy a ticket.

* Ex-OR of p & q

If p and q are 2 statements, the exclusive or of p and q, denoted by $p \oplus q$, is true when exactly one of p and q is true.

Truth Table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

* Converse, Contrapositive, Inverse

If $p \to q$ is an expression, then

converse: $q \to p$

contrapositive: $\neg q \to \neg p$

inverse: $\neg p \to \neg q$

Ex: Write the converse & contrapositive of the given implication.

$p \to q$: If it is raining, then I get wet.

p: It is raining

q: I get wet

converse: $q \to p$ = If I get wet, then it is raining

contrapositive $\neg q \to \neg p$ = If I do not get wet, then it is not raining.

Do Ex 2 pg. 1.11 RGB
Ex 3 Ex 4

* **Well - Formed Formulas:** If we connect 2/3 or more primary statements at a time, an order should be maintained in finding the truth values of the compound statement.

For eg. $\neg p \wedge q$ is not well - formed, since it could be either $(\neg p) \wedge q$ or $\neg(p \wedge q)$

* Precedence of Operators

$\neg$
$\wedge$
$\vee$
$\rightarrow$
$\leftrightarrow$

* **Tautology**: A statement that is true for all possible values of its propositional variables is called a tautology / universally valid formula / logical truth

* **Contradiction**: A statement that is always false is called a contradiction or an absurdity

* A propositional function that is neither a tautology nor a contradiction is called a contingency.

Do Examples & Exercise pg 1.21 - 1.32 RGB

Do Examples 3, 7, 9 on pg 1.33 - 1.36 to prove that a given statement is a tautology - RGB

Tautological Equivalence / Equivalence of Propositions

2 Formulae ϕ & ψ are said to be equivalent to each other iff. $\phi \Rightarrow \psi$ and $\psi \Rightarrow \phi$. that is $\phi \Leftrightarrow \psi$ should be a tautology.

To prove for tautological equivalence, it is enough to prove that $\psi \leftrightarrow \phi$ is a tautology.

Ex1 P.T $p \to q$ and $\neg p \vee q$ are tautologically equivalent.

Ans: To prove that $p \to q$ and $\neg p \vee q$ are tautologically equivalent, it is enough to prove that $(p \to q) \leftrightarrow (\neg p \vee q)$ is a tautology.

Truth Table

p	q	$\neg p$	$p \to q$ (A)	$\neg p \vee q$ (B)	$(p \to q) \leftrightarrow (\neg p \vee q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

$(p \to q) \leftrightarrow (\neg p \vee q)$ is a tautology

$\Rightarrow \quad (p \to q) \Leftrightarrow (\neg p \vee q)$.

Alternatively, columns A & B match $\Rightarrow$ Tautological equivalence

Ex2 P.T $p \leftrightarrow q \Leftrightarrow (\neg p \vee q) \wedge (p \vee \neg q)$

A = $p \leftrightarrow q$, B = $(\neg p \vee q) \wedge (p \vee \neg q)$

p	q	$\neg p$	$\neg q$	$p \leftrightarrow q$	$\neg p \vee q$	$p \vee \neg q$	$(\neg p \vee q) \wedge (p \vee \neg q)$	$A \leftrightarrow B$
T	T	F	F	T	T	T	T	T
T	F	F	T	F	F	T	F	T
F	T	T	F	F	T	F	F	T
F	F	T	T	T	T	T	T	T

Columns A & B match, and $(p \leftrightarrow q) \leftrightarrow (\neg p \vee q) \wedge (p \vee \neg q)$ is a tautology.

Hence, they are tautologically equivalent.

Ex3 P.T $p \wedge q \Leftrightarrow \neg(\neg p \vee \neg q)$ is a tautological equivalence

A = $p \wedge q$, B = $\neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \vee \neg q$	$\neg B$	$(p \wedge q) \leftrightarrow \neg(\neg p \vee \neg q)$
T	T	F	F	T	F	T	T
T	F	F	T	F	T	F	T
F	T	T	F	F	T	F	T
F	F	T	T	F	T	F	T

$(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$ is a tautology.

$\therefore\ p \wedge q \Leftrightarrow \neg(\neg p \vee \neg q)$.

Do Ex 4(d) pg 1.39 RGB

* Tautological Implication

A formula ϕ is said to tautologically imply a formula ψ iff ~~and only~~ $\phi \to \psi$. In this case, $\phi \Rightarrow \psi$

Ex1 P.T $(p \to q) \Rightarrow (\neg q \to \neg p)$

It is enough to prove that $(p \to q) \to (\neg q \to \neg p)$ is a tautology.

p	q	$p \to q$ (A)	$\neg q$	$\neg p$	$\neg q \to \neg p$ (B)	$A \to B$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Column A = Column B & $(p \to q) \to (\neg q \to \neg p)$ is a tautology.

$\Rightarrow (p \to q) \Rightarrow (\neg q \to \neg p)$ is a tautological implication.

** Logical Equivalence Formulae

(i) $p \to q \Leftrightarrow \neg p \vee q$

(ii) $p \to q \Leftrightarrow \neg q \to \neg p$

(iii) $p \vee q \Leftrightarrow \neg p \to q$

(iv) $p \wedge q \Leftrightarrow \neg (p \to \neg q)$

(v) $\neg (p \to q) \Leftrightarrow p \wedge \neg q$

(vi) $p \leftrightarrow q \Leftrightarrow (p \to q)$

(vii) $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$

Laws

① Idempotent Laws: $p \vee p = p$

$p \wedge p = p$

② Commutative laws: $p \vee q = q \vee p$

$p \wedge q = q \wedge p$

③ Associative Law: $p \vee (q \vee r) = (p \vee q) \vee r$

$p \wedge (q \wedge r) = (p \wedge q) \wedge r$

④ Distributive Law: $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

⑤ Absorption: $p \vee (p \wedge q) = p$

$p \wedge (p \vee q) = p$

⑥ DeMorgan's Law: $\neg(p \vee q) = \neg p \wedge \neg q$

$\neg(p \wedge q) = \neg p \vee \neg q$

Tautological Implication: Ex 2

(i) P.T $p \rightarrow ((p \rightarrow r)) \Rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$

p	q	r	$p \rightarrow r$	$p \rightarrow (p \rightarrow r)$ (A)	$(p \rightarrow q)$	$(p \rightarrow q) \rightarrow (p \rightarrow r)$ (B)	$A \rightarrow B$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
F	T	T	T	T	T	T	T
T	F	F	F	F	F	T	T
F	F	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	F	T	T	T	T	T

The statement is tautologically implied.

∴ (ii) P.T $\underbrace{(p\to(q\to s))}_{A} \wedge \underbrace{(\neg r \vee p)}_{B} \wedge \underbrace{q}_{C} \Rightarrow \underbrace{r\to s}_{D}$

p	q (C)	r	s	$q\to s$	$p\to(q\to s)$ (A)	$\neg r$	$\neg r \vee p$ (B)	$A\wedge B\wedge C$	$r\to s$	$(A\wedge B\wedge C)\to D$
T	T	T	T	T	T	F	T	T	T	T
T	T	T	F	F	F	F	T	F	F	T
T	T	F	T	T	T	T	T	T	T	T
T	F	T	T	T	T	F	T	F	T	T
F	T	T	T	T	T	F	F	F	T	T
T	T	F	F	F	F	T	T	F	F	T
T	F	F	T	T	T	T	T	F	T	T
F	F	T	T	T	T	F	F	F	T	T
F	T	T	F	F	T	F	F	F	F	T
F	T	F	T	T	T	T	T	T	T	T
T	F	T	F	T	T	F	T	F	F	T
T	F	F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	T	T	F	T	T
F	T	F	F	F	T	T	T	T	T	T
F	F	T	F	T	T	F	F	F	F	T
F	F	F	F	T	T	T	T	F	T	T

∴ is a tautological implication

* Duality

The dual of a compound proposition that contains only the logical operators ∨, ∧ and ¬, is the proposition obtained by replacing ∨ by ∧, ∧ by ∨, T by F & F by T.

If A is a proposition, then A* is its dual

* Replacement Process: Any proposition can be replaced by a logically equivalent proposition.

For eg. $P \to (Q\to R)$

can be written as $P\to(\neg Q \vee R)$

* Without using truth tables, prove the following

(i) $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \equiv p \wedge q$

Ans. $= (\neg p \vee q) \wedge (p \wedge (p \wedge q))$

$= (\neg p \vee q) \wedge ((p \wedge p) \wedge q)$ // Associative law

$= (\neg p \vee q) \wedge (p \wedge q)$

$= (p \wedge q) \wedge (\neg p \vee q)$ // commutative law

$= (p \wedge q \wedge \neg p) \vee (p \wedge q \wedge q)$ // distributive law

$= (q \wedge F) \vee (p \wedge q)$ // Idempotent law

$= F \vee (p \wedge q)$ // Domination law

$= p \wedge q$ // Domination law

*(ii) $p \to (\cancel{p}q \to p) \equiv \neg p \to (p \to q)$

$p \to (q \to p) \equiv \cancel{\neg p \to (\neg p \vee o}$

$p \to (\neg q \vee p)$ // conditional as disjunction

$= \neg p \vee (\neg q \vee p)$ // conditional as disjunction

$= (p \vee \neg p) \vee (\neg q)$ // associative

$= T \vee \neg q$ // domination law

$= T$

R.H.S : $p \vee (\neg p \vee q)$

$(p \vee \neg p) \vee q$

$T \vee q = T$

(iii) $\neg(p \leftrightarrow q) \equiv (p \vee q) \wedge \neg(p \vee q) \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

$\boxed{\neg(p \leftrightarrow q) = \neg p \vee (q \rightarrow p)}$

$= \neg p \vee (\neg q \vee p)$

$= (\neg p \vee p) \vee \neg q \quad = T \vee \neg q = T$

R.H.S $(p \wedge \neg q) \vee (\neg p \wedge q)$

$(p \wedge \neg q \vee \neg p) \wedge (p \wedge \neg q \vee q)$

$(T \wedge \neg q) \wedge (p \wedge T)$

$= T$

(iv) $\neg p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \vee r)$

LHS: $\neg p \rightarrow (q \rightarrow r)$

$\neg(\neg p) \vee (q \rightarrow r)$ // conditional as disjunction

$p \vee (\neg q \vee r)$ // conditional as disjunction

$\neg q \vee (p \vee r)$ // associative law

$q \rightarrow (p \vee r)$ // disjunction as a conditional

(v) $p \rightarrow (q \rightarrow r) \equiv p \rightarrow (\neg q \vee r) \equiv (p \wedge q) \rightarrow r$

LHS

$p \rightarrow (q \rightarrow r)$

$\neg p \vee (q \rightarrow r)$ // conditional as disjunction

$\neg p \vee (\neg q \vee r)$ // conditional as disjunction

$= p \rightarrow (\neg q \vee r)$ // disjunction as a condition

$\rightarrow$ 1

$= (\neg p \vee \neg q) \vee r$ // associative law

$\neg(p \wedge q) \vee r$ // De Morgan's Law

$= (p \wedge q) \to r$ // disjunction as conditional

(vi) $((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is a tautology

$((p \vee q) \wedge \neg(\neg p \wedge \neg(q \wedge r))) \vee \neg(p \vee q) \vee \neg(p \vee r)$

// De Morgan's law

$((p \vee q) \wedge (p \vee (q \wedge r))) \vee \neg(p \vee q) \vee \neg(p \vee r)$

// De Morgan's law

$\cancel{((p \vee q) \vee \neg(p \vee q)}$)

$((p \vee q) \wedge (p \vee q) \wedge (p \vee r)) \vee \neg(p \vee q) \vee \neg(p \vee r)$

// Distributive law

$((p \vee q) \wedge (p \vee r)) \vee \neg((p \vee q) \wedge (p \vee r))$ // De Morgan's law

$((p \vee q) \wedge (p \vee r)) \vee \neg(p \vee (q \wedge r))$

$(p \vee (q \wedge r)) \vee \neg(p \vee (q \wedge r))$

$= T$ // ~~Taa~~ of the form $P \vee \neg P = T$

* Prove the following equivalence by proving the equivalence of their duals:

(i) $\neg((\neg p \wedge q) \vee (\neg p \wedge \neg q)) \vee (p \wedge q) \equiv p$

To find duals, change

$\wedge \to \vee$

$\vee \to \wedge$ do not change

$T \to F$ variables or

$F \to T$ negations

dual

$\neg((\neg p \vee q) \wedge (\neg p \vee \neg q)) \wedge (p \vee q)$

$\neg((\neg p \vee q \wedge \neg p) \vee (\neg p \vee q \wedge \neg q)) \wedge (p \vee q)$ distributive law

$\neg((\neg p \wedge \neg p) \vee q) \vee (\neg p \vee (q \wedge \neg q)) \wedge (p \vee q)$ associative law

$\neg((\neg p \vee q) \vee (\neg p \vee F)) \wedge (p \vee q)$ $0 \cdot q$

$x + x = 1$

$\cancel{\neg((q) \vee (\neg p)) \wedge (p \vee q)}$ $\neg p$ $p' \cdot p' = p'$

$\cancel{\neg q \wedge p \wedge (p \vee q)}$ $\neg((\neg p \vee q \vee \neg p) \vee (\neg p \vee q \vee F))$

$\cancel{\neg((\neg p \vee \neg p \vee F) \vee (\neg p \vee q}$ $\neg(\neg p \vee q$

(ii) $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$

~~dual: $(p \wedge q) \to r \equiv (p \to r) \vee (q \vee r)$~~

~~R.H.S $\neg(p \wedge q) \vee r$~~ dual $= \neg(p \vee q) \vee r \equiv (\neg p \vee r) \wedge (\neg q \vee r)$

~~$= \neg p \vee \neg q$~~

dual: $\neg(p \wedge q) \wedge r \equiv (\neg p \wedge r) \vee (\neg q \wedge r)$

R.H.S: $(\neg p \vee \neg q) \wedge r$

$= (\neg p \wedge r) \vee (\neg q \wedge r)$

$=$ R.H.S

(iii) $(p \wedge (p \leftrightarrow q)) \to q \equiv T$

$(p \wedge ((p \to q) \wedge (q \to p))) \to q \equiv T$

$(p \wedge ((\neg p \vee q) \wedge (\neg q \vee p))) \to q \equiv T$

$\neg(p \wedge (\neg p \vee q) \wedge (\neg q \vee p))) \vee q \equiv T$

dual: $\neg(p \vee (\neg p \wedge q) \vee (\neg q \wedge p))) \wedge q \equiv F$

$\neg((p \vee \neg p) \wedge q \vee (\neg q \wedge p))) \wedge q \equiv F$ associative

$xy + z'$

$\neg(T \wedge q \vee (\neg q \wedge p))) \wedge q$

complement

$p \vee \neg p = T$

$\neg(q \vee (\neg q \wedge p)) \wedge q$

$T \wedge q = q$

identity

$\neg(T \wedge p) \wedge q$

$q \vee \neg q$

* Prove the following w/o truth tables.

(i) $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \Rightarrow r$

To prove:

$[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r = T$

$[(p \vee q) \wedge (\neg p \vee r) \wedge (\neg q \vee r)] \rightarrow r$

$[(p \vee q \wedge \neg p) \vee (p \vee q \wedge r) \wedge (\neg q \vee r)] \rightarrow r$

$[(T \vee q) \vee (p \vee q \wedge r) \wedge (\neg q \vee r)] \rightarrow r$

(ii) $((p \vee \neg p) \rightarrow q) \rightarrow ((p \vee \neg p) \rightarrow r) \Rightarrow q \rightarrow r$

L.H.S

$[((p \vee \neg p) \rightarrow q) \rightarrow ((p \vee \neg p) \rightarrow r)] \rightarrow (q \rightarrow r)$

$[(T \rightarrow q) \rightarrow (T \rightarrow r)] \rightarrow (q \rightarrow r)$

$[(F \vee q) \rightarrow (F \vee r)] \rightarrow (q \rightarrow r)$

$[\neg(F \vee q) \vee (F \vee r)] \rightarrow (q \rightarrow r)$

$[T \wedge \neg q \vee F \vee r] \rightarrow (q \rightarrow r)$

$(q \rightarrow r) \rightarrow (q \rightarrow r)$ RGB pg 1.44-1.56

$= T$

(iii) $(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$

LH.S $(\neg p \wedge r \wedge \neg q) \vee (q \wedge r) \vee (p \wedge r)$

$((\neg p \wedge \neg q) \wedge r) \vee (q \wedge r) \vee (p \wedge r)$

ab + ac

a(b+c)

$\neg(p \vee q) \wedge r \vee (r \wedge (p \vee q))$

$(\neg(p \vee q) \wedge r \vee r) \wedge (\neg(p \vee q) \wedge r \vee (p \vee q))$

$(\neg(p \vee q) \wedge r) \wedge (T \wedge r)$

CW Questions

(i) $(p \vee q) \wedge \neg p \Leftrightarrow \neg p \wedge q$

$(p \wedge \neg p) \vee (q \wedge \neg p)$

$F \vee (q \wedge \neg p)$

$= \neg p \wedge q$

$= R.H.S$

$\neg p \wedge q \Leftrightarrow \neg p \wedge q = T$ = Tautologically Equivalent

(ii) $(p \to q) \Leftrightarrow p \to (p \wedge q)$

L.H.S $p \to q$

$= \neg p \vee q$

R.H.S : $p \to (p \wedge q)$

$= \neg p \vee (p \wedge q)$

$= (\neg p \vee p) \wedge (\neg p \vee q)$

$= T \wedge (\neg p \vee q)$

$\cancel{= (p \wedge \neg p) \vee p} = \neg p \vee q$

L.H.S = R.H.S

(iii) $(p \to q) \to q \Leftrightarrow (p \vee q)$

L.H.S $(p \to q) \to q$

$(\neg p \vee q) \to q$

$\neg(\neg p \vee q) \vee q$

$(p \wedge \neg q) \vee q$

$q \vee (p \wedge \neg q) = (p \vee q) \wedge (q \vee \neg q) = (p \vee q) \wedge T = \underline{p \vee q}$

(iv) $(p \to r) \wedge (q \to r) \Leftrightarrow (p \vee q) \to r$

$(\neg p \vee r) \wedge (\neg q \vee r)$

$\cancel{r \vee (\neg p \vee \neg q)}$ $r \vee (\neg p \wedge \neg q)$

$= r \vee \neg(p \vee q)$

$= \neg(p \vee q) \vee r$

$= (p \vee q) \to r$

(v) $q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$

$q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$

$(q \vee p) \wedge (q \vee \neg q) \vee (\neg p \wedge \neg q)$

$(p \vee q) \wedge T \vee (\neg p \wedge \neg q)$

$T \wedge (p \vee q) \vee \neg(p \vee q)$

$T \wedge T = T$

(vi) $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) = p \wedge q$

$(\neg p \vee q) \wedge ((p \wedge p) \wedge q)$

$(\neg p \vee q) \wedge (p \wedge q)$

$\cancel{(\neg p \vee q \wedge p) \wedge (\neg p \vee q \wedge q)}$ $(\neg p \wedge p \wedge q) \vee (q \wedge p \wedge q)$

$\cancel{(p \wedge q) \wedge (\neg p \vee q)}$

$\cancel{(p \wedge q \wedge \neg p) \vee (p \wedge q \wedge q)}$ $(F \wedge q) \vee (p \wedge q)$

$\cancel{(F \wedge q) \vee}$ $= (p \wedge q)$

$(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r)$

* P.T the following are tautologies

(i) $[(p \to q) \to q] \Rightarrow (p \vee q)$

L.H.S $(\neg p \vee q) \to q$

$\neg(\neg p \vee q) \vee q$

$p \wedge \neg q \vee q$

$= p \vee q \quad = R.H.S$

(ii) $p \Rightarrow (p \vee q)$

$\neg p \vee (p \vee q)$

$T \vee q = T$

(iii) $(p \wedge q) \Rightarrow (p \vee q)$

$\neg(p \wedge q) \vee (p \vee q)$

$\neg p \vee \neg q \vee p \vee q$

$(\neg p \vee p) \vee (\neg q \vee q)$

$= T$

(iv) $(p \to q) \wedge \neg q \Rightarrow \neg p$

LH.S

$(\neg p \vee q) \wedge \neg q$

$(\neg p \wedge \neg q) \vee (q \wedge \neg q)$

$\neg p \wedge \neg q \vee F$

$(\neg p \vee F) \wedge (\neg q \vee F)$

$(F \wedge F) \to \neg p$

$F \to \neg p \quad = \quad \neg F \vee \neg p \quad - T \vee \neg p = \neg p$ //

(v) $(p \to q) \wedge p \Rightarrow q$

$(\neg p \vee q) \wedge p \Rightarrow q$

$(\neg p \wedge p) \vee (q \wedge p) \Rightarrow q$

$F \vee (q \wedge \neg p) \Rightarrow q$

$= \neg(F \vee (q \wedge \neg p)) \vee q$

$= \neg F \wedge \neg(q \wedge \neg p)) \vee q$

$= T \wedge \neg q \vee p \vee q$

$= T \wedge T \vee p$

$= T \wedge T = \underline{\underline{T}}$

(vi) $((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

$((p \vee q) \wedge (p \vee \neg(\neg q \vee \neg r)) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

$(p \vee q) \wedge (p \vee (q \wedge r)) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

$(p \vee q) \wedge (p \vee q) \wedge (p \wedge r) \vee \neg(p \vee q) \vee \neg(p \vee r)$

$((p \vee q) \wedge (p \wedge r)) \vee \neg((p \vee q) \wedge (p \wedge r))$

$= T$

(vii) $\neg(p \wedge q) \to (\neg p \vee (\neg p \vee q)) \Leftrightarrow \neg p \vee q$

$(\neg p \vee q) \vee (\neg p \vee \neg p \vee q)$

$(\neg p \vee q) \vee (\neg p \vee q)$

$= (\neg p \vee q) //$

Functionally Complete Set of Connectives

(i) NAND : a combination of NOT & AND $= \uparrow$

(ii) NOR : a combination of NOT & OR $= \downarrow$

For any p & q

$p \uparrow q = \neg(p \wedge q)$

$p \vee q = \neg(p \vee q)$

The connectives $\downarrow$ & $\uparrow$ have been defined in terms of $\wedge, \vee, \neg$.

* A set of connectives is said to be functionally complete if every formula can be expressed in terms of an equivalent formula containing only formulas from this set.

For eg. $\{\wedge, \neg\}$ & $\{\vee, \neg\}$ are functionally complete

Ex1 Write an equivalent formula for $P \wedge (Q \Leftrightarrow R) \vee (R \Leftrightarrow P)$ that does not contain the biconditional

$p \wedge (q \Leftrightarrow r) \vee (r \Leftrightarrow p)$

$p \wedge ((q \rightarrow r) \wedge (r \rightarrow q)) \vee ((p \rightarrow r) \wedge (r \rightarrow p))$

Ex2 write an equivalent formula for $P \wedge (Q \Leftrightarrow r)$ which has neither the biconditional or the conditional.

Ans $p \wedge (q \Leftrightarrow r)$

$= p \wedge ((q \rightarrow r) \wedge (r \rightarrow q))$

$= p \wedge ((\neg q \vee r) \wedge (\neg r \vee q))$

Ex3 Prove that $\{\neg, \vee\}$ is functionally complete.

consider $p \leftrightarrow q$

$= (p \to q) \wedge (q \to p)$

$= (\neg p \vee q) \wedge (\neg q \vee p)$

$= \neg(\neg(\neg p \vee q) \vee \neg(\neg q \vee p))$

$p \wedge q$

$= \neg(\neg p \vee \neg q)$

(Ex 3, 4, 5 pg 1.63 - 1.64 RGB)

P.T $\uparrow$ is functionally complete:

(i) $\neg p = \neg p \vee \neg p$

$= \neg(p \wedge p)$

$= p \uparrow p$

(ii) $p \wedge q = \neg(p \uparrow q)$

$= (p \uparrow q) \uparrow (p \uparrow q)$

(iii) $p \vee q$ $\neg(\neg p \wedge \neg q)$

$= \neg p \uparrow \neg q$

$= (\neg p \vee \neg p) \uparrow (\neg q \vee \neg q)$

$= \neg(p \wedge p) \uparrow \neg(q \wedge q)$

$= (p \uparrow p) \uparrow (q \uparrow q)$

P.T $\downarrow$ is functionally complete

(i) $\neg p = \neg p \wedge \neg p$

$= \neg(p \vee p)$

$= p \downarrow p$

(ii) $p \vee q = \neg(p \downarrow q)$

$= (p \downarrow q) \downarrow (p \downarrow q)$

(iii) $p \wedge q = \neg p \downarrow \neg q$

$= (p \downarrow p) \downarrow (q \downarrow q)$

Minterms : Given a no. of variables, the products in which each variable or its negation, but not both, occurs only once are called the minterms.

For eg, for 2 variables p & q, the minterms are:

$p \wedge q$
$\neg p \wedge q$
$p \wedge \neg q$
$\neg p \wedge \neg q$

Maxterms Given a no. of variables, the disjunctions in which each variable and its negation, but not both, occurs only once are called the maxterms.

The maxterms are the duals of the minterms.

PCNF & PDNF

A formula consisting of disjunction of minterms in the variables only and equivalent to a given formula is called the principal disjunctive normal form (PDNF) or the sum of products canonical form. |||ly, a formula consisting of conjunctions of maxterms in the variables only and equivalent to the given form is called the principal normal conjunctive form (PCNF) or its product of sums canonical form.

Ex 1 Find the PDNF of $\neg p \vee q$:

p	$\neg p$	q	$\neg p \vee q$	minterms
T	F	T	T	$p \wedge q$
T	F	F	F	$p \wedge \neg q$
F	T	T	T	$\neg p \wedge q$
F	T	F	T	$\neg p \wedge \neg q$

PDNF : $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

Ex 2 Find the PDNF of $(p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r)$

p	q	r	$\neg p$	$p \wedge q$	$\neg p \wedge r$	$q \wedge r$	$(p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r)$	minterms
T	T	T	F	T	F	T	T	$p \wedge q \wedge r$
T	T	F	F	T	F	F	T	$p \wedge q \wedge \neg r$
T	F	T	F	F	F	F	F	$p \wedge \neg q \wedge r$
F	T	T	T	F	T	T	T	$\neg p \wedge q \wedge r$
T	F	F	F	F	F	F	F	$p \wedge \neg q \wedge \neg r$
F	F	T	T	F	T	F	T	$\neg p \wedge \neg q \wedge r$
F	T	F	T	F	F	F	F	$\neg p \wedge q \wedge \neg r$
F	F	F	T	F	F	F	F	$\neg p \wedge \neg q \wedge \neg r$

PDNF

$= (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$

Ex 3 Find the PDNF of $p \to ((p \to q) \vee (\neg q \vee \neg p)) = q$

p	q	$\neg p$	$\neg q$	$(p \to q)$	$(\neg q \vee \neg p)$	① $(p \to q) \vee (\neg q \vee \neg p)$	$p \to$ ①	q
T	T	F	F	T	F	T	T	
T	F	F	T	F	T	T	T	
F	T	T	F	T	T	T	T	
F	F	T	T	T	T	T	T	

minterms are: $(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

Ex 4 Find the PCNF of

$(\neg p \to r) \wedge (q \leftrightarrow p)$

p	q	r	$\neg p$	$\neg p \to r$	① $q \leftrightarrow p$	② $\neg p \wedge r$	①∧② minterms	maxterms
T	T	T	F	T	T	T	T	
T	T	F	F	F	T	F	F	
T	F	T	F	T	F	T	F	$p \vee \neg q \vee r$
T	F	F	F	F	F	F	F	$p \vee \neg q \vee \neg r$
F	T	T	T	T	F	F	F	$\neg p \vee q \vee r$
F	T	F	T	T	F	F	F	$\neg p \vee q \vee \neg r$
F	F	T	T	T	F	F	T	
F	F	F	T	T	F	F	F	$\neg p \vee \neg q \vee \neg r$

PDNF : ~~$(p \vee q \vee r) \wedge (p \vee \neg q \vee r)$~~ ,
$(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$

PCNF: ~~$(p \vee \neg q \vee r) \wedge$~~ $(p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r)$
~~$\wedge (\neg p \vee \neg q \vee \neg r)$~~ $(\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee r)$
$\wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r)$

Ex5 PDNF and PCNF of

(i) $(\neg p \vee \neg q) \rightarrow (p \leftrightarrow \neg q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \leftrightarrow \neg q$	$\neg p \vee \neg q$	$(\neg p \vee \neg q) \rightarrow (p \leftrightarrow \neg q)$	min	max
T	T	F	F	F	F	F	T	$p \wedge q$	
T	F	F	T	T	T	T	T	$p \wedge \neg q$	
F	T	T	F	T	T	T	T	$\neg p \wedge q$	
F	F	T	T	T	F	T	F		$p \vee q$

PDNF: $(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q)$

PDNF of $\neg((\neg p \vee \neg q) \rightarrow (p \leftrightarrow \neg q))$

$= p \vee q$

Ex6 $p \vee (\neg p \rightarrow (q \vee (\neg q \rightarrow r)))$

p ①	q	r	$\neg p$	$\neg q$	$\neg q \rightarrow r$	$q \vee (\neg q \rightarrow r)$	$\neg p \rightarrow (q \vee (\neg q \rightarrow r))$ ②	① $\vee$ ②	minterm
T	T	T	F	F	T	T	T	T	
T	T	F	F	F	T	T	T	T	
T	F	T	F	T	T	T	T	T	
T	F	F	F	T	F	F	T	T	
F	T	T	T	F	T	T	T	T	
F	T	F	T	F	T	T	T	T	
F	F	T	T	T	T	T	T	T	
F	F	F	T	T	F	F	F	F	

PDNF: $(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee$

$(\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$

(iii) PDNF and PCNF of

$(p \to (q \wedge r)) \wedge (\neg p \to (\neg q \wedge \neg r))$

p	q	r	$\neg p$	$\neg q$	$\neg r$	$q \wedge r$	① $p \to (q \wedge r)$	$(\neg q \wedge \neg r)$	② $(\neg p \to (\neg q \wedge \neg r))$	① $\wedge$ ②
T	T	T	F	F	F	T	T	F	T	T
T	T	F	F	F	T	F	F	F	T	F
T	F	T	F	T	F	F	F	F	T	F
T	F	F	F	T	T	F	F	T	F	F
F	T	T	T	F	F	T	T	F	F	F
F	T	F	T	F	T	F	T	F	F	F
F	F	T	T	T	F	F	T	F	F	F
F	F	F	T	T	T	F	T	T	T	T

PDNF : $(\cancel{p \vee q \vee r})$: $(p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$

PCNF : $(\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r)$

$\wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee \neg r)$

Predicate Calculus

Property that a subject can have = predicate

The logic based on the analysis of predicates in a statement is called predicate logic or predicate calculus.

Quantifiers : used to measure the nature of the variable

for all : $\longrightarrow \forall$

for some $\longrightarrow \exists$

For eg. (i) Ram is a student

Ram → subject → x

student → predicate → M

$\Rightarrow M(x)$

(ii) Sam is intelligent and Ram is ~~intelligent~~ rich

Sam → x, is intelligent → I, Ram → y, is rich → R

$I(x) \cap R(y)$

(iii) $E(x)$: x is educated

$G(x)$: x is good

$E(x) \rightarrow G(x)$ = If x is educated, then x is good.

1 place predicate: has only a single subject : Ram is rich

2 place predicate: has 2 objects, x is greater than y.

Example : Consider the following statements:

(i) $P(x)$: The no. x is greater than 100

(ii) $Q(x,y)$: $x+y=10$, where x & y are integers

(iii) r: 12 is div by 3:

What is $(P(x) \cap q(x,y)) \rightarrow R$

If the no x is > than 100 and $x+y=10$, where x & y are integers, then 12 is div by 3

Types of quantifiers

universal & existential quantifiers

universal	existential
for all x	For some x
for each / every x	Some x, such that

For example:

(i) Something is green

$G(x)$: x is green

$(\exists x) G(x)$

(ii) Everything is green

$(\forall x) G(x)$

(iii) Something is not green

There exists at least one x, such that x is not green.

$(\exists x) (\overline{G(x)})$

(iv) Nothing is green

$(\forall x) (\overline{G(x)})$

Ex: Write each of the following statements into variables.

(i) All men are good.

$M(x)$: x is a man

$G(x)$: x is good

$(\forall x)(M(x) \rightarrow G(x))$

(ii) No men are good

$(\forall x)(M(x) \rightarrow \overline{G(x)})$

(iii) Some men are good

$(\exists x)(m(x) \cap G(x))$

There exists an x such that x is good and x is a man.

(iv) Some men are not good.

$(\exists x)(m(x) \cap \overline{G(x)})$

Ex Write the following statements in closed form.

(i) Some people who trust others are rewarded

$P(x)$: x is a person

$T(x)$: x trusts others

$R(x)$: x is rewarded

$(\exists x)(P(x) \cap (T(x) \rightarrow R(x)))$

(ii) If anyone is good, then John is good

If there exists an x such that x is a person & x is good, then John is good.

$G(x)$: x is good

$(\exists x)(P(x) \cap G(x) \rightarrow G(\text{John}))$

(iii) Someone is talking

There exists an x such that x is a person & x is talking

$Q(x)$: x is talking

$(\exists x)(P(x) \cap Q(x))$

(iv) It is not true that all roads lead to Roame

There exists some x, such that x is a road,

and x does not lead to Rome

$S(x)$: x is a road

$R(x)$: x leads to Rome

$(\exists x)(S(x) \cap \overline{R(x)})$

Ex $L(x,y)$: x loves y

Write the following in symbolic form.

(i) Everybody loves z.

$\forall x\, L(x,z)$

(ii) Everybody loves somebody

~~$\exists x\, L(x$~~ y $L(x,y)$ is true for some y and all of x

$\Rightarrow$ ~~$\forall y$~~ $\forall \exists y\, L(x,y)$

(iii) There is somebody whom everybody loves

There exists some y who is loved by all x.

$\exists y \forall x\, L(x,y)$

(iv) Nobody loves everybody?

There is not one who loves everybody

$\exists x\, \forall y \neg L(x,y)$

(e) There is someone who nobody loves.

There exists an y, that no x loves

$\neg \forall x \exists y \, L(x, y)$

$\forall x \exists y \neg L(x, y)$

Ex write the following using mathematical and logical operations.

The universe of discourse is all cs, mathematics students.

(a) Every computer science students needs a mathematics course.

$M(x)$: x needs a mathematics course.

$\forall x \, M(x)$

(b) There is a student in the class who owns a PC

$C(x)$: x owns a PC

$(\exists x) C(x)$

(c) Every student in this class has taken at least one mathematics course

$Q(x, y)$: x has taken y

$\forall x \exists y \, Q(x, y)$

(d) There exists a student who has taken at least one mathematics course.

$\exists x \exists y \, Q(x, y)$

Ex Write an expression for the statement : All the world love a lover.

$P(x)$: x is a person

$L(x)$: x is a lover

$R(x,y)$: x loves y

All x love a y, who is a lover

$$(\forall x)\left(P(x) \rightarrow y\left(\left(P(y) \wedge L(y)\right) \rightarrow R(x,y)\right)\right)$$

Ex Symbolize :

For every x, there exists a y such that $x^2 + y^2 >= 100$

$$(\forall x)(\exists y)(x^2 + y^2 >= 100)$$

Ex Symbolize

Every book with a blue cover is a mathematics book.

$M(x)$: x is a mathematics book.

$B(x)$: x has a blue cover

$$(\forall x)\left(B(x) \rightarrow P(x)\right)$$

Ex Every student in this class has studied calculus

$S(x)$: x is a student

$C(x)$: x has studied calculus

$$(\forall x)\left(S(x) \rightarrow C(x)\right)$$

Ex(1) For any value of x, x^2 is non - negative

$$(\forall x)(x^2 \geq 0)$$

(ii) For every value of x, there is some value of y such that $xy = 1$

$(\forall x)(\exists y)(xy = 1)$

(iii) There are positive values of x and y such that $xy > 0$.

$(\exists x)(\exists y)((x > 0) \wedge (y > 0) \wedge (xy > 0))$

(iv) There is a value of x such that if y is positive then $(x+y)$ is negative

$(\exists x)((y > 0) \rightarrow (x+y < 0))$

Ex Write in symbolic form.

All lions are fierce

Some lions do not drink coffee.

Some fierce creatures do not drink coffee.

$P(x)$: x is a lion

$Q(x)$: x is fierce

$R(x)$: x drinks coffee.

(i) $(\forall x)(P(x) \rightarrow Q(x))$

(ii) $(\exists x)(P(x) \wedge \neg R(x))$

(iii) $(\forall x)(Q(x) \wedge \neg R(x))$

Write in symbolic form.

(i) All hummingbirds are richly coloured

(ii) No large birds live on honey

(iii) Birds that do not live on honey are dull in color

(iv) Hummingbirds are small.

$P(x)$: x is a hummingbird

$Q(x)$: x is large

$R(x)$: x lives on honey

$S(x)$: x is richly coloured

(i) $(\forall x)(P(x) \rightarrow S(x))$

(ii) $\neg(\exists x)(Q(x) \wedge R(x))$

(iii) $(\forall x)(\neg R(x) \rightarrow \neg(S(x)))$

(iv) $(\forall x)(P(x) \rightarrow \neg(Q(x)))$

Formulae

I. Addition

(i) $p \vee q = p$

(ii) $p \vee q = q$

II Simplification

(i) $p \wedge q = p$

(ii) $p \wedge q = p$

III Modus Ponens

$p, p \rightarrow q = q$

IV Modus Tollens

$\neg q, p \rightarrow q = \neg p$

V Disjunctive Syllogism

$(p \vee q) \wedge (\neg p) \Rightarrow q$

VI Hypothetical Syllogism

$\cancel{p \wedge r}\ (p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow (p \rightarrow r)$

VII Dilemma

$(p \vee q) \wedge (p \rightarrow r)\ (q \rightarrow r) = r$

VIII Resolution

$(p \vee q) \wedge (\neg p \vee r) \Rightarrow (q \vee r)$

Theory of Inference

When a conclusion is derived from a set of hypothosis by using the accepted ~~valid~~ rules of reasoning, then such a process of derivation is called a formal proof or deduction

The 3 rules of inference :

(i) Rule P

(ii) Rule T

(iii) Rule CP

Rule P : We may introduce a hypothesis at any point in the derivation

Rule T: We may introduce a formula S in a derivation if S is tautologically implied by one or more of the preceding formula in a derivation.

Ex1 Show tha $r \wedge (p \vee q)$ is a valid conclusion from the premises $(p \vee q)$, $(q \to r)$ $(p \to m)$, $\neg m$

$p \to m$	Rule P
$\neg m$	Rule P
$\neg m, \; p \to m = \neg p$	Rule T : Modus Tollens
$p \vee q$	Rule P $\quad \neg q, \; p \to q = \neg p$
$\neg p, \; p \vee q = q$	Rule T Disjunctive Syllogism
$q \to r$	Rule P
r	Rule T, Modus Ponens : $r, \; q \to r = q$

$r \wedge (p \vee q)$ Rule T (conjunction)

Ex2 P.T $\neg q, p \to q \Rightarrow \neg p$ is valid

$\neg q$ Rule P

$p \to q$ Rule P

$\neg q \to \neg p$ Rule T contrapositive

$= \neg p$ Rule T $\neg q, \neg q \to \neg p = \neg p$

(Modus ponens: $q, q \to p = p$)

Ex3 P.T $r \vee s$ is valid from $(c \vee d)$, $(c \vee d) \to \neg H$, $\neg H \to (a \wedge \neg b)$ and $(a \wedge \neg b) \to R \vee S$

~~Cor~~ $(c \vee d) \to \neg h$ Rule P

$\neg h \to (a \wedge \neg b)$ Rule P

$(c \vee d) \to (a \wedge \neg b)$ Rule T (hypothetical syllogism)

$c \vee d$ Rule P $a \to b, b \to c \Rightarrow a \to c$

$a \wedge \neg b$ Rule T Modus Ponens

$(a \wedge \neg b) \to (r \vee s)$ Rule P

$= r \vee s$ Rule T $(a \wedge \neg b)$, $(a \wedge \neg b) \to (r \vee s)$

$= (r \vee s)$

= Modus Ponens

Rule CP

Rule CP means rule of conditional proof. It is also called the deduction theorem.

In general, whenever the conclusion is of the form $r \to s$, that is, in terms of conditional, we should apply Rule CP. In such cases, r is taken as an additional premise.

Ex: Verify the validity of the given statement.

$r \to s$ can be derived from the hypothesis $p \to (q \to s)$, $\neg r \vee p$ and q

$\neg r \vee p$	Rule P	modus ponens: $p,\ p \to q = p$
r	Rule P, additional premise	tollens: $\neg q,\ p \to q = \neg p$
p	Rule T, disjunctive syllogism	$(p \vee q) \wedge (\neg p) = q$
$p \to (q \to s)$	Rule P	
$q \to s$	Rule T	modus ponens
q	Rule P	
s	Rule T	modus ponens
$r \to s$	Rule CP	

Ex Show that s is a valid inference from the premises

$p \to \neg q$, $q \vee r$, $\neg s \to p$, $\neg r$

$q \vee r$ Rule P

$\neg r$ Rule P

q Rule T (disjunctive syllogism) $(q \vee r) \wedge (\neg r) = q$

$p \to \neg q$ Rule P

$\neg p$ Rule T (Modus tollens)

ponens

$p, p \to q = q$

$\neg s \to p$ Rule P

tollens

$\neg q, p \to q = \neg p$

$= s$ Rule T (disjunctive syllogism)

disjunctive

$(p \vee q) \wedge (\neg p) = q$

$\neg p, (\neg s \to p)$

$= s$

$\neg p$, $\neg s \to p$.

Direct and Indirect Method of Proof

Direct method : a conclusion is derived from a set of premises using accepted rules of reasoning.

Indirect method : also called the method of contradiction assume that the conclusion is false, consider $\neg c$ as an additional premise.

Ex Using indirect method of proof, p.t

$\neg q,\ p \to q,\ p \vee t \Rightarrow T$

Ans The conclusion is T, include F as an additional hypothesis

$\neg q$, Rule P

$p \to q$ Rule P

$\neg p$ Rule T: ~~ded~~ $\neg q,\ p \to q = \neg p$

modus tollens

$p \vee t$ Rule P

$\neg p,\ p \vee t = T$ Rule T, disjunctive syllogism ds $\neg p,\ (p \vee q)$

$\neg T$ Rule P additional premise

F RuleT conjunction

Ex Using an indirect method of proof, p.t $p \to \neg s$ follows from $p \to q \vee r,\ q \to \neg p,\ s \to \neg r,\ p$

additional premise: $\neg(p \to \neg s)$

$= \neg(\neg p \vee \neg s)$

$= p \wedge s$

1. $p \to q \vee r$ RuleP
2. p Rulep
3. $q \vee r$ RuleT modus ponens $(p,\ p \to q \vee r)$
4. $s \to \neg r$ Rue P
5. $p \wedge s$ Rule P additional premise

⑥ s Rule T - Simplification

⑦ $\neg r$ Rule T Modus ponens $s, s \rightarrow \neg r = \neg r$

~~⑧ $\neg q$ Rule T Modus tollens~~ $\neg r, q \vee r = \neg q$

modus tollens

⑧ $\neg r, q \vee r \Rightarrow q$ Modus ponens

⑨ $q, q \rightarrow \neg p = \neg p$ = Modus ponens

⑩ $\neg p \wedge p = F$

Theory of Inference for Predicate Calculus

Rule US: Universal Specification

$(\forall x)(P(x)) = P(c)$

Rule ES: Existential Specification

$(\exists x)(P(x)) = P(c)$

Rule UG: Universal Generalization

$P(c) \Rightarrow (\forall x)(P(x))$

Rule EG: Existential Generalization

$P(c) \Rightarrow (\exists x)(P(x))$

Ex1: Verify the validity of the following args.

(i) Lions are dangerous animals

(ii) There are lions

(iii) ∴ There are dangerous animals.

Hypotheses : (i) ~~L~~ $L(x)$: x is a lion

$D(x)$: x is dangerous

① Lions are dangerous animals.

⇒ For all x, if x is a lion, then it is ~~a~~ dangerous animal.

$$(\forall x)(L(x) \rightarrow D(x))$$

② There are lions : $(\exists x)(L(x))$

Conclusion : $(\exists x)(D(x))$

To prove : $(\forall x)(L(x) \rightarrow D(x)), (\exists x)(L(x))$

$$\Rightarrow (\exists x)(D(x)).$$

① $(\exists x)(L(x))$ Rule P

② $L(y)$ Rule ES

③ $(\forall x)(L(x) \rightarrow D(x))$ Rule P

④ $L(y) \rightarrow D(y)$ Rule US

⑤ $D(y)$ Rule T, Modus Ponens

⑥ $(\exists x)(D(x))$ Rule EG.

Ex The famous Socrates argument:

(i) All men are mortal

(ii) Socrates is a mortal

(iii) $\therefore$ Socrates is mortal

$H(x)$: x is a man

$M(x)$: x is mortal

s : Socrates

Hypotheses : (i) $(\forall x)(H(x) \rightarrow M(x))$

(ii) $H(s)$

(iii) $M(s)$

To Prove: $(\forall x)(M(x)), H(s) \Rightarrow M(s)$

① $(\forall x)(H(x) \rightarrow M(x))$ Rule P

② $H(s) \rightarrow M(s)$ Rule US

③ $M(s)$ Rule P

④ $H(s)$ Rule T Modus Ponens

Ex (i) If an integer is divisible by 10, then it is div by 2

(ii) If an integer is divisible by 2, then it is divisible by 3

(iii) $\therefore$ An integer divisibly by 10 is div by 3.

$D_{10}(x)$: x is div by 10

$D_2(x)$: x is div by 2

$D_3(x)$: x is div by 3.

(i) $(\forall x)(D_{10}(x) \rightarrow D_2(x))$

(ii) $(\forall x)(D_2(x) \rightarrow D_3(x))$

Conclusion

(iii) $(\forall x)(D_{10}(x) \rightarrow D_3(x))$

<u>Proof</u> : $(\forall x)(D_{10}(x) \rightarrow D_2(x))$ Rule P

$D_{10}(x) \rightarrow D_2(x)$ Rule US

$(\forall x)(D_2(x) \rightarrow D_3(x))$ Rule P

$D_2(x) \rightarrow D_3(x)$ Rule US

$D_{10}(x) \rightarrow D_3(x)$ Rule T, Hypothetical Syllogism

$(\forall x)(D_{10}(x) \rightarrow D_3(x))$ Rule UG

Ex (i) All integers are rational numbers

(ii) Some integers are powers of 2.

(iii) ∴ Some rational nos. are powers of 2.

<u>Hypothesis</u>:

$R(x)$: x is a rational no.

$P(x)$: x is an integer

$X(x)$: x is a power of 2

(i) $(\forall x)(P(x) \rightarrow R(x))$

(ii) $(\exists x)(P(x) \rightarrow X(x))$

Conclusion

$(\exists x)(R(x) \rightarrow X(x))$

Proof : ① $(\forall x)(P(x) \rightarrow R(x))$ Rule P

② $(P(x) \rightarrow R(x))$ Rule US

③ $P(x)$ Simplification

(Use ∧) ③ $(\exists x)(P(x) \rightarrow X(x))$ Rule P

④ $P(x) \rightarrow X(x)$ Rule ES

⑤ $X(x)$ simplification

⑥ $P(x) \rightarrow X(x)$ Rule T

⑦ $(\exists x)(P(x) \rightarrow X(x))$ Rule EG

Ex3 Prove that

(i) $(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x)(P(x) \wedge (\exists x) Q(x))$

~~using an indirect method of~~ proof

① $(\exists x)(P(x) \wedge Q(x))$ Rule P

② $P(y) \wedge Q(y)$ Rule ES

③ $P(y)$ Simplification, Rule T

④ $Q(y)$ Simplification, Rule T

⑤ $(\exists x) P(x)$ Rule EG

⑥ $(\exists x) Q(x)$ Rule EG

⑦ $(\exists x) P(x) \wedge (\exists x) Q(x)$ Rule T conjunction

(ii) Prove using an indirect method

$(\forall x)(P(x) \vee Q(x)) \Rightarrow ((\forall x)(P(x)) \vee (\exists x)\, Q(x))$

Assume $\neg((\forall x)(P(x) \vee (\exists x)\, Q(x)))$ as an additional premise.

1. $\neg((\forall x)(P(x) \vee (\exists x)\, Q(x)))$ Rule P - addn premise
2. $\neg(\forall x)P(x) \wedge \neg(\exists x)\, Q(x)$ DeMorgan's Law Rule T
3. $(\forall x)\neg P(x) \wedge (\exists x)\neg Q(x)$ Rule T
4. $(\forall x)\neg P(x)$ Rule T Simplification
5. $(\exists x)\neg Q(x)$ Rule T Simplification
6. $\neg P(y)$ Rule US
7. $\neg Q(y)$ Rule ES
8. $\neg P(y) \wedge \neg Q(y)$ Rule T - conjunction
9. $\neg(P(y) \vee Q(y))$ Rule T - DeMorgan's Law
10. $(\forall x)(P(x) \vee Q(x))$ Rule P
11. $P(y) \vee Q(y)$ Rule US
12. F from (9) (11) Rule T, conjunction