Mathomatics for Machine Learning

Unit 5

Probability and Random Variables

Probability - sum and product rules, Bayes theorem, random variables - discrete and continuous random variables - moments - moment generating functions - binomial, poisson, exponential and normal distributions

* **Random Experiment** - experiments that are performed under essentially the same conditions and whose results cannot be predicted

  eg. tossing a coin

* **Sample Space** - the set of all possible outcomes of a random experiment is called the sample space - denoted by S.

* **Event** - outcome of a random event

* **Axioms of probability**

(i) $0 \leq P(x) \leq 1$

(ii) $\sum P(x) = 1$

(iii) probability of disjoint events = sum of probabilities

* **Independent Events** - if the happening or failure of one event does not affect the happening or failure of another event.

* **Bayes Theorem** :

$E_1, E_2 \cdots E_n$ = a set of events associated with a sample space

A is any event associated with S. According to Baye's theorem,

$$P(E_i|A) = \cancel{\frac{P(E_i \cap A)}{P(A)}} \quad \frac{P(E_i)P(A|E_i)}{\sum_{k=1}^{n} P(E_k)P(A|E_k)}$$

* **Random Variables** - If a real variable X is associated with the outcome of a random experiment, and since the values X takes depends on chance, it is called a random / stochastic variable.

* **Discrete Probability Distribution** : Suppose a discrete variate X is the outcome of some experiment, and if the probability that X takes the values x_i is p_i, then

$$P(X = x_i) = p_i$$

The set of values x_i with their probabilities p_i constitute a discrete probability distribution

* **Distribution Function** - The distribution function F(x) of the discrete variate X is defined by:

$$F(x) = P(X \leq x) = \sum_{i=1}^{x} p(x_i)$$ (equivalent to cumulative distribution fn.)

* **Continuous Probability Distribution** : When a variate X takes every value in an interval, it gives rise to a continuous dist. of X.

* **CDF of continuous distributions** - $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$

* **Expectation = mean**

Discrete $= \sum x_i\, p(x)$

Continuous $=$ $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

* **Variance**

Discrete : $\sigma^2 = (x_i - \mu)^2 f(x)$

Cont : $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

* **r^{th} moment**

discrete : $\mu_r = \sum (x_i - \mu)^r f(x)$

cont : $\mu_r = \int_{-\infty}^{\infty} (x-\mu)^r f(x)\,dx$

* **mean deviation** D: $\sum |x_i - \mu|\, f(x)$

C: $\int_{-\infty}^{\infty} |x-\mu|\, f(x)\, dx$

* Binomial Distribution

$$P(x=r) = {}^{n}C_r \, p^r \, q^{n-r}$$

$\boxed{\text{Mean}} = np$ $\boxed{\text{standard deviation}} = \sqrt{npq}$

Derivation: moment generating function about the origin

$$M_0(t) = E(e^{tx})$$

$$= \sum {}^{n}C_x p^x q^{n-x} . e^{tx}$$

$$= \sum {}^{n}C_x \, p^x e^{tx} \, q^{n-x}$$

$$= \sum {}^{n}C_x \, (pe^t)^x \, q^{n-x}$$

$$M_0(t) = (q + pe^t)^n$$

differentiating w.r.t t and sub $t = 0$

$$M_0'(t) = n(q+pe^t)^{n-1} . pe^t$$

$$= n(1)(p)$$

$$\boxed{\text{mean} = np}$$

$$E(X^2) = \sum_{x=0}^{n} x^2 \, {}^{n}C_x \, p^x q^{n-x}$$

write x^2 as $x(x-1) + x$

$$= \sum_{x=0}^{n} \left[x(x-1) + x\right] {}^{n}C_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)\, {}^{n}C_{x}\, p^{x} q^{n-x} + \sum_{x=0}^{n} x\, {}^{n}C_{x}\, p^{x} q^{n-x}$$ (5)

$$= \sum_{x=0}^{n} x(x-1)\, {}^{n}C_{x}\, p^{x} q^{n-x} + np$$

$$= \sum_{x=0}^{n} \frac{x(x-1) \cdot n!}{(n-x)! \cdot x!} \cdot p^{x} q^{n-x} + np$$

$$= \sum_{x=0}^{n} \frac{\cancel{x}\cancel{(x-1)} \cdot n! \cdot p^{x} q^{n-x}}{\cancel{x} \cdot \cancel{(x-1)} \cdot (x-2)!\,(n-x)!} + np$$

$$= \sum_{x=0}^{n} \frac{n! \cdot p^{x} q^{n-x}}{(x-2)!\,(n-x)!} \; p^{x} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,[(n-2)-(x-2)]!} \; p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2} \cdot p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \left[(p+q)^{n-2} \right] + np$$

$$= n(n-1)p^2 + np$$

$$\mathrm{var}(X) = E(X^2) - (E(X))^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= npq$$

$$\boxed{\mathrm{var}(X) = npq}$$

* **Poisson Distribution** - a limiting form of the binomial distribution

$$P(x) = \frac{e^{-\lambda}\lambda^r}{r!}$$

Proof $P(X=r) = {}^nC_r \, p^r q^{n-r}$

$$= \frac{n(n-1)(n-2)\dots(n-r+1)}{r!\ \cancel{\{(n-r)!\}}\ ??} \; p^r (1-p)^{n-r}$$

put $p = \frac{\lambda}{n}$

$$= \frac{n(n-1)\dots(n-r+1)}{r!}\left[\frac{\lambda}{n}\right]^r \left[1-\frac{\lambda}{n}\right]^{n-r}$$

$$\frac{\lambda^r}{r!}\left\{\frac{n(n-1)\dots(n-r+1)}{n^r}\right\}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-r}$$

$$\frac{\lambda^r}{r!}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{r-1}{n}\right)\right]\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-r}$$

$$\lim_{n\to\infty} P(X=r) = \frac{\lambda^r}{r!}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{e^{-\lambda}\lambda^r}{r!}$$

$$\boxed{\text{mean \& variance} = \lambda}$$

$$E(x) = \sum x_r p_r$$

$$= \frac{\sum r \, e^{-\lambda} \lambda^r}{r!}$$

$$= \lambda e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^{r-1} \cdot r}{r!}$$

$$= \lambda e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^{r-1}}{(r-1)!}$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

$$\boxed{\text{mean} = \lambda}$$

$$E(X^2) = \sum_{r=0}^{\infty} r^2 \frac{e^{-\lambda} \lambda^r}{r!}$$

$$= \sum_{r=0}^{\infty} \frac{\{r(r-1)+r\} e^{-\lambda} \lambda^r}{r!}$$

$\vdots$

$$= \cancel{\lambda^2 +} \boxed{\lambda - \lambda^2} \qquad \overbrace{\boxed{\lambda^2+\lambda}}^{E(X^2)} \overbrace{- \lambda^2}^{-(E(X))^2}$$

$$= \lambda \qquad \boxed{\text{variance} = \lambda}$$

* **Exponential Distributions**

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{MGF} = E(e^{tx}) = \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty e^{tx} \cdot e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x}\, dx =$$

$$= \left. \frac{\lambda \cdot \left(e^{-(\lambda-t)}\right)}{-(\lambda - t)} \right]_0^\infty$$

$$= \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1} = 1 + \frac{1}{\lambda} + \frac{1}{\lambda^2} + \cdots$$

$$\text{mean} = \mu_0'(t) = \frac{1}{\lambda}$$

$$\text{variance} = \mu_2' = \frac{2}{\lambda^2}$$

$$\text{var} = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \;//$$

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* (X) Memory-less property of exponential distribution

$$P(X > s+t \mid X > s) = P(X > t)$$

Proof: $$P(X > s+t \mid X > s) = \frac{P\{X > s+t \text{ and } X > s\}}{P\{X > s\}} \quad (1)$$

$$P(X > s) = \int_s^\infty \lambda e^{-\lambda x}\, dx = \frac{-\cancel{\lambda}}{\cancel{\lambda}} (e^{-\lambda x})_s^\infty = e^{-\lambda s} \;//$$

$\Rightarrow$ Eqn 1 becomes

$$\frac{P(X > s+t) \cap P(X > s)}{P(X > s)}$$

$$= \frac{P(X > s+t)}{P(X > s)} = \frac{P(X > s+t)}{e^{-\lambda s}}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t}$$

$$= P(X > t) \;//$$

* Proofs

① If the probability of an event A happening as a result of a trial is P(A) and the probability of a mutually exclusive event B happening is B, then the probability of either events happening is $P(A \cup B) = P(A) + P(B)$.

Proof = n = total no. of equally likely cases

m_1 = favorable to event A

m_2 = favorable to event B

∴ number of cases favorable to A or B is

$m_1 + m_2$

∴ Probability of A or B happening is:

$$\frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B)$$

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② If A, B are any two events (not mutually exclusive), then $P(A+B) = P(A) + P(B) - P(A \cap B)$.

Proof - There are some outcomes which favor both A and B. $= m_3$.

This would be included in both m_1 and m_2.

Then the no. of outcomes favoring A or B or both is $m_1 + m_2 - m_3$

∴ Probability $= \frac{m_1 + m_2 - m_3}{n} = \frac{m_1}{n} + \frac{m_2}{n} - \frac{m_3}{n} = P(A) + P(B) - P(A \cap B)$

(3) Multiplication Law | Theorem of Compound Probability - If the probability of an event A happening is P(A), and the chances of B happening due to A is P(B|A), then the probability of the events A and B happening as a result of 2 trials is ~~P(AB)~~:

~~P(A).P(B|A)~~ $\boxed{P(AB) = P(A) . P(B|A)}$

Proof - n = total no. of outcomes in first trial

m = outcomes favorable to A

$P(A) = m/n$

n_1 = total no. of outcomes in the second trial

m_1 = outcomes in 2nd trial favorable to B

$$P(B|A) = \frac{m_1}{n_1}$$

Each of the n outcomes can be associated with each of the n_1 outcomes.

Total outcomes $= nn_1$

mm_1 outcomes are favorable to A & B

$\therefore \boxed{P(A \cap B) = P(A)\, P(B|A)}$

1) A bag contains 8 white and 6 red balls. Find the probability of drawing two balls of the same color. 13

Ans P(2 balls of the same color) = P(2 white balls) + P(2 red balls)

$$P(2 \text{ white balls}) = \frac{8C_2}{14C_2} = \frac{8!}{6!\,\cancel{2!}} \times \frac{12!\,\cancel{2!}}{14!}$$

$$= \frac{7 \times \cancel{8}^{4}}{\cancel{2}} \times \frac{1}{13 \times \cancel{14}\,7}$$

$$= \frac{28}{91}$$

$$P(2 \text{ red balls}) = \frac{6C_2}{14C_2} = \frac{6!}{4!\,\cancel{2!}} \times \frac{12!\,\cancel{2!}}{14!}$$

$$= \frac{5 \times \cancel{6}\,3}{13 \times \cancel{14}\,7} = \frac{15}{91}$$

$$\therefore P(2 \text{ balls of the same color}) = \frac{28}{91} + \frac{15}{91}$$

$$= \frac{43}{91}$$

2) Find the probability of drawing an ace or a spade or both from a deck of cards.

$$P(\text{ace}) = \frac{4}{52} = 1/13$$

$$P(\text{spade}) = 13/52 = 1/4$$

$$P(\text{ace \& spade}) = 1/52$$

$\therefore P(\text{ace or spade}) = \frac{1}{4} + \frac{1}{13} - \frac{1}{52}$

$= \frac{13 + 4 - 1}{52}$

$= \frac{16}{52} = \frac{4}{13}$ //

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(3) Two cards are drawn in succession from a pack of 52 cards. Find the chance that the first is a king, and the second is a queen if the first card is (i) replaced
(ii) not replaced.

Ans (i) with replacement

$= P(\text{king}) \times P(\text{queen})$

$= \frac{1}{13} \times \frac{1}{13} = 1/169$

(ii) without replacement

$= P(\text{king}) \times P(\text{queen, with total reduced})$

$= \frac{1}{13} \times \frac{4}{51} = \frac{4}{663}$

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(4) A pair of dice is tossed twice. Find the probability ~~that~~ of scoring 7 points

(i) once

(ii) at least once

(iii) twice

(iv) in the first toss, but not the second toss

Ans (i) once in one toss

different cases: (1,6), (6,1), (2,5), (5,2),

(3,4), (4,3) = 6 cases

$\Rightarrow P(\text{one } 7) = \frac{6}{36} = \frac{1}{6}$ //

(ii) at least once

$P(\text{at least once}) = \cancel{1 - P(\text{once})}$

$\cancel{= 1 - \frac{1}{6} = \frac{5}{6}}$

$= 1 - P(x < 1)$

$= 1 - P(0)$

$P(0) = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

$P(\text{at least once}) = 1 - \frac{25}{36}$

$= \frac{11}{36}$ //

(iii) $P(\text{getting } 7 \text{ twice}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

(iv) P (7 in the first toss, but not the second toss) = P(one 7 in two tosses)

$= \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$ //

(5) Given $P(A) = 1/4$, $P(B) = 1/3$ $P(A \cup B) = 1/2$

Find (i) $P(A|B)$

(ii) $P(B|A)$

(iii) $P(A \cap B')$

(iv) $P(A|B')$

Ans (i) $P(A|B) = \dfrac{P(A \cap B)}{\cancel{P(A \cap B)}\ P(B)}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = \frac{1}{4} + \frac{1}{3} - \frac{1}{2}$

$\cancel{= 6+4-3}$ $\dfrac{3+4-6}{12}$

$$\boxed{P(A \cap B) = \frac{1}{12}}$$

$\therefore P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$= \dfrac{1/12}{1/3} = \dfrac{1}{4}$ //

(ii) $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/12}{1/4} = \dfrac{1}{3}$

(iii) $P(A \cap B') = P(A) - P(A \cap B)$

$= \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$

(iv) $P(A|B') = \dfrac{P(A \cap B')}{P(B')} = \dfrac{1/6}{2/3} = \frac{1}{6} \times \frac{3}{2} = \frac{1}{4}$ //

(6) Three machines M_1, M_2, M_3 produce identical item. Of their respective outputs, 5%, 4% and 3% of items are faulty. On a certain day, M_1 has produced 25% of the total output, M_2 has produced 30% and M_3 the remaining. An item selected at random is found to be faulty. What are the chances that it was produced by the machine with the highest output? (17)

Ans

machine with the highest output $= M_3$

$$P(M_3|A) = \frac{P(M_3)\,P(A|M_3)}{P(M_1)P(A|M_1) + P(M_2)P(A|M_2) + P(M_3)P(A|M_3)}$$

chances it was M_3 given that an accident occurred

$P(M_3) = 0.45$ $\quad$ $P(A|M_1) = 0.05$

$P(A|M_3) = 0.03$ $\quad$ $P(M_2) = 0.3$

$P(M_1) = 0.25$ $\quad$ $P(A|M_2) = 0.04$

$$P(M_3|A) = \frac{0.45 \times 0.03}{0.03 \times 0.45 + 0.25 \times 0.05 + 0.3 \times 0.04}$$

$P(M_3|A) = \underline{0.355}$

7. There are 3 bags: 1st containing 1 white, 2 red & 3 green balls, second: 2 white, 3 red, 1 green balls and the third bag: 1 red, 2 green, 3 white. Two balls are chosen from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls so drawn came from the second bag.

Ans $P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$

B_1	B_2	B_3
W : 1	W : 2	W : 3
R : 2	R : 3	R : 1
G : 3	G : 1	G : 2

Let C be the condition of drawing one white & one red ball.

$P(C|B_1) = \frac{{}^1C_1 \, {}^2C_1}{{}^6C_2} = \frac{2}{15}$

$\frac{6!}{4!\,2!}$

$\frac{5 \times 6}{2}$

$P(C|B_2) = \frac{{}^2C_1 \times {}^3C_1}{{}^6C_2} = \frac{6}{15}$

$P(C|B_3) = \frac{{}^3C_1 \times {}^1C_1}{{}^6C_2} = \frac{3}{15}$

To Find: $P(B_2|C) = \frac{P(B_2)\,P(C|B_2)}{P(B_1)P(C|B_1) + P(B_2)P(C|B_2) + P(B_3)P(C|B_3)}$

For the given condition, what are the chances it came from B2 =

$$= \frac{\frac{1}{3} \times \frac{6}{15}}{\frac{1}{3} \times \frac{2}{15} + \frac{1}{3} \times \frac{6}{15} + \frac{1}{3} \times \frac{3}{15}}$$

$$= \frac{\frac{6}{15}}{\frac{2+6+3}{15}} = \frac{6}{11}$$

8\) ★★★ In a certain college, ~~the~~ 4% of the boys and 1% of the girls

Ex.) are taller than 1.8m. Furthermore 60% of the students are girls. If a student is selected at random & is found to be taller than 1.8m, what is the probability that the student is a girl?

Ans. $P(B) = 0.4$ $\quad P(G) = 0.6$

$T \rightarrow$ taller than 1.8 m

$P(T|B) = 0.04$

$P(T|G) = 0.01$

$P(G|T) = ?$ $\rightarrow$ given that the student is taller than 1.8, what are the chances it is a girl?

$$P(G|T) = \frac{P(G)\,P(T|G)}{P(G)P(T|G) + P(B)P(T|B)}$$

$$= \frac{0.6 \times 0.01}{0.6 \times 0.01 + 0.04 \times 0.4} = 0.272$$

⑨ A die is tossed thrice. A success is getting 1 or 6 on a toss. Find the mean and variance of the number of successes.

Ans For a single toss - success $= \frac{2}{6} = \frac{1}{3}$

- failure $= \frac{2}{3}$

For 3 tosses : ~~success~~

x_i = no. of success	0	1	2	3
p_i	$8/27$	$4/9$	$2/9$	$1/27$

$\frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} \times {}^3C_1$

${}^3C_2 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3}$

mean $= \Sigma p_i x_i = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$

variance $= \Sigma p_i x_i^2 - \mu^2$

$\Sigma p_i x_i^2 = 0 + \frac{4}{9} + \frac{8}{9} + \frac{\cancel{9}^{3}}{\cancel{27}_{9}}$

$= \frac{15}{9} = \frac{5}{3}$

$\sigma^2 = \frac{5}{3} - 1 = \frac{2}{3}$ //

(10) A random variable X has the following probability fn. (21)

x	0	1	2	3	4	5	6	7
$p(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find (i) value of k

(ii) $P(X<6)$

(iii) $P(X\geq 6)$

(iv) $P(0<X<5)$

Ans $\sum p(x) = 1$

$9k + 10k^2 - 1 = 0$

$10k^2 + 9k - 1 = 0$

$10k^2 + 10k - k - 1 = 0$

$10k(k+1) - 1(k+1) = 0$

$(10k-1)(k+1)$ $\boxed{k = 1/10}$

(ii) $P(X<6)$

$= 8k + k^2$

$= \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$

(iii) $P(X\geq 6) = \frac{19}{100}$

(iv) $P(0<X<5) = 8k = \frac{8}{10} = \frac{4}{5}$

(11) Is the function defined as follows a density function?

$$f(x) = e^{-x} \quad , \quad x > 0$$
$$= 0 \quad , \quad x < 0$$

If so, determine the probability that the variate having this density will fall in the interval $(1, 2)$?

Also find the CDF : $F(2)$?

<u>Ans</u> $\forall x,\ x \geq 0 \quad f(x) \geq 0 \Rightarrow \therefore$ is a density function

$$P(1 \leq X \leq 2) = \int_1^2 f(x)\,dx$$

$$= \int_1^2 e^{-x}\,dx = -(e^{-x})_1^2$$

$$= e^{-1} - e^{-2} = \underline{\underline{0.233}}$$

$$\text{CDF} : F(2) = \int_0^2 e^{-x}\,dx = -(e^{-x})_0^2$$

$$= 1 - e^{-2} = \underline{\underline{0.865}}$$

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(12) The probability that a pen manufactured by a company will be defective is $1/10$. If 12 such pens are manufactured, find the probability that:

(a) exactly 2 will be defective

(b) at least 2 will be defective

(c) none will be defective.

Ans P(defective) = 0.1

P(not defective) = 0.9

(i) P(exactly 2 defective) $= 12C_2 (0.1)^2 (0.9)^{10} = 0.2301$

(ii) P(at least 2 will be defective) $= 1 - P(X<2)$

$$= 1 - \{12C_0 (0.1)^0 . (0.9)^{12} + 12C_1 (0.1)(0.9)^{11}\}$$

$$= 0.3412$$

(iii) P(none will be defective) $= 12C_{12} (0.1)^0 (0.9)^{12} = 0.2833$

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(13) In 256 sets of 12 tosses of a coin, in how many cases can one expect 8 heads & 4 tails

P(head) = P(tail) = $1/2$

$$P(X=8 \text{ heads}) = 12C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^4 = \frac{495}{4096}$$

$$\text{In 256 trials} = \frac{256 \times 495}{4096}$$

$$= 30.9 \approx 31 \text{ cases}$$

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(14) In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2.

Out of 1000 such samples, how many would be expected to contain at least 3 defective parts

Ans $np = 2$

$p = 2/20 = 0.1$

P(defective) $= 0.1$

P(non-defective) $= 0.9$

P(at least 3 defective parts) $= 1 - P(X<3)$

$$= 1 - \{ {}^{20}C_0 (0.1)^0 (0.9)^{20} + {}^{20}C_1 (0.1)^1 (0.9)^{19} + {}^{20}C_2 (0.1)^2 (0.9)^{18} \}$$

$= 0.323$

For 1000 samples $=$ 323 defectives

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(15) If the probability of a bad reaction from an injection is 0.001, determine the chance that out of 2,000 individuals, more than 2 will get a bad reaction.

Ans probability that more than 2 will get a bad reaction

$= 1 - \{P(\text{no one}) + P(\text{one}) + P(\text{two})\}$

here $\lambda = np = 2000 \times 0.0001 = 2$

$$1 - \left\{ \frac{e^{-2}\, 2^0}{0!} + \frac{e^{-2} \cdot 2^1}{1!} + \frac{e^{-2} \cdot 2^2}{2!} \right\}$$

$$= 1 - \left\{ \frac{1}{e^2} + \frac{2}{e^2} + \frac{2}{e^2} \right\}$$

$$= 1 - \frac{5}{e^2} = \underline{\underline{0.32}}$$

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(16) In a certain factory making razor blades, there is a small chance of 0.002 for any blade to be defective. The blades were supplied in packets of 10, use the Poisson distribution to calculate the approximate no. of packets containing no defective, one defective and two defective blades respectively in a consignment of 10,000 packets.

Ans $\lambda = np = 10 \times 0.002 = 0.02$

$$P(\text{no. defective blade}) = \frac{e^{-\lambda}\lambda^r}{r!} = \frac{e^{-0.02} \cdot (0.02)^0}{0!}$$

$$= e^{-0.02} = 0.9802$$

no. of packets $= 10,000 \times 0.9802 = \underline{\underline{9802}}$

$P(\text{one defective blade}) = e^{-0.02} \times 0.02$

For 10,000 packets $= e^{-0.02} \times 0.02 \times 10,000$

$= 196$

$P(2 \text{ defective blades}) =$

$$10{,}000 \times \frac{(0.02)^2}{2!} = 0.9082 = 2 \text{ approximately}$$

---

(17) The time in hours required to repair a machine is the exponential distribution with $\lambda = 1/2$.

(i) What is the probability that the repair time exceeds 2 hrs?

(ii) What is the conditional probability that a repair takes at least 10 hrs, given that its duration exceeds 9 hrs?

Ans (i) $P(X > 2) = e^{-2\lambda}$

$$= e^{-2 \cdot 1/2} = e^{-1} = 0.3679$$

(ii) $P(X \geq 10 \mid X > 9)$

$$= P(X > 9+1 \mid X > 9) = P(X > 1)$$

$$= e^{-1 \cdot \frac{1}{2}} = e^{-1/2} = 0.6065$$

---

(18) If X follows an exponential distribution, with parameter α, find the probability function of $Y = \log x$

Ans $f(x) = \alpha e^{-\alpha x}$

$y = \log x$

$x = e^y$

$\cancel{f(x) = f(e^y) = \alpha e^{-\alpha e^y}}$

$G(y) = P(Y \le y) = P(\log x \le y) = P(X \le e^y)$

$= 1 - e^{-\alpha e^y}$

$$\boxed{f(y) = G'(y) = \alpha e^y e^{-\alpha e^y}}$$

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(19) X is a normal variable with mean 30 & SD 5, Find the probabilities that

(i) $26 \le X \le 40$

(ii) $X \ge 45$

(iii) $|X - 30| > 5$

Ans $\mu = 30$

$\sigma = 5$

$$z = \frac{X - \mu}{\sigma}$$

when $x = 26$

$$z = \frac{26-30}{5} = \frac{-4}{5} = -0.8$$

when $X = 40$

$$z = \frac{10}{5} = 2$$

$P(26 \le x \le 40) = P(-0.8 \le z \le 2)$

$= P(-0.8 \le z \le 0) + P(0 \le z \le 2)$

$= P(0 \le z \le 0.8) + P(0 \le z \le 2)$

$= 0.2881 + 0.4772 = \underline{0.7653}$

(ii) $P(X>45)$

when $X=45$

$$z = \frac{45-30}{5} = 3$$

$P(X>45) = P(z>3)$ = ~~leo Refxo~~

$= 0.5 - P(z<3)$

$= 0.5 - 0.4987$

$= \underline{\underline{0.0014}}$

(iii) $P(|X-30|>5)$

$= P(25 \le X \le 35)$

$X-30=5$

$X=35$ (or)

~~$X-30=5$~~

~~$30-5$~~

$X=25$

when $X=25$

$$z = \frac{-5}{5} = -1$$

when $z=35$

$z=1$

$P(-1 \le z \le 1) = 2P(0<z<1)$

$= 2 \times 0.3413 = 0.6826$

$P(|X-30|>5) = 1 - P(|X-30| \le 5)$

$= 1 - 0.6826$

$= \underline{\underline{0.3174}}$