Mathematics for Machine Learning

Unit 3

Inner Product Spaces

Inner product and norms - Gram Schmidt orthogonalizationprocess - orthogonal complement - least square approximation

* **Inner Product** - If V is a vector space over F, and there exists a function which assigns a scalar $(u, v) \in F$ corresponding to each ordered pair of vectors $u, v \in V$, then (u, v) is called the inner product in V, provided that it satisfies the following axioms:

If it's $(u, av_1 + bv_2) = \bar{a}(u, v_1) + \bar{b}(u, v_2)$

(i) $(au_1 + bu_2, v) = a(u_1, v) + b(u_2, v)$

(ii) $(u, v) = \overline{(v, u)}$ (If F is a field of real numbers then $\overline{(v, u)} = (v, u)$)

(iii) $(u, u) \geq 0$ and equality holds iff $u = 0$

→ The vector space with an inner product is called an inner product space.

A finite dimensional real inner product space is called a Euclidean space, and a complex inner product space is called a unitary space.

→ If $u = (a_1, a_2 \dots a_n)$ and $v = (b_1, b_2, \dots b_n)$ over R^n

then $(u, v) = (a_1 b_1 + a_2 b_2 \dots a_n b_n)$ is called the standard

inner product in R^n

* Norm : $\sqrt{(u, u)}$, also called length of u.

denoted by $\|u\|$

→ If $\|u\| = 1$ ⇒ called a unit vector

→ To find the angle between 2 vectors

$$\cos\theta = \frac{(u, v)}{\|u\|\|v\|}$$

distance $= \|v - u\|$

* Orthogonality - The vectors $(u, v) \in$ an inner product space V

are said to be orthogonal, if $(u, v) = 0$

$\langle u, v \rangle = 0$

* Orthonormality - The set $\{u_i\}$ of vectors in V is said to be

an orthonormal set, if it is orthogonal and if $\|u_i\| = 1$ for

each u_i.

*** Note that a set $\{u_i\}$ in v is said to be an orthogonal

set, if all pairs of distinct vectors are orthogonal,

i.e $(u_i, u_j) = 0$ when $i \neq j$

* Least Square Approximation

Let $(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$ be a set of observed values of y corresponding to the given time t and let $y = ct + d$ be the line of best fit for this data.

Let $A = \begin{bmatrix} t_1 & 1 \\ t_2 & 1 \\ \vdots & 1 \\ t_n & 1 \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

then $x_0 = \begin{bmatrix} c \\ d \end{bmatrix} = (A^*A)^{-1} A^* y$

Error $E = \|Ax_0 - y\|^2$

A^* is called the conjugate transpose of A

* Adjoint of Linear Operations

Definition : A linear operator T on an inner product space V is said to have an adjoint operator T^* on V, if $(T(u), v) = (u, T^*(v))$ for all $(u, v) \in V$.

Q. (1) If V is an inner product space, then for any vectors (u, v) in V and the scalar c, prove the following: (5)

(i) $\|cu\| = |c| \cdot \|v\|$

(ii) $\|u\| > 0$, for $u \neq 0$

(iii) Cauchy Schwartz Inequality: $|(u,v)| \leq \|u\| \cdot \|v\|$

(iv) Triangle Inequality: $\|u+v\| \leq \|u\| + \|v\|$

Ans $\|cu\| = |c| \cdot \|u\|$

Note that $\|u\| = \sqrt{(u,u)}$

$$\Rightarrow \|u\|^2 = (u,u)$$

IIIly $\|cu\|^2 = (cu, cu)$

$$= c\bar{c}(u,u) \qquad \boxed{(au, bv) = a\bar{b}(u,v)}$$

$$= |c|^2 (u,u)$$

$$= |c|^2 \|u\|^2$$

take square root on both sides

$$\boxed{\|cu\| = |c|^2 \|u\|}$$

(ii) $\|u\| \geq 0$

By the third axiom $(u,u) \geq 0$

$= \|u\|^2 \geq 0 \Rightarrow \|u\| \geq 0$

(iii) ☆☆☆ Cauchy - Schwartz Inequality: $|\langle u,v\rangle| \le \|u\|\cdot\|v\|$

consider $\langle u - cv, u - cv \rangle$ $\qquad$ $\langle au_1 + bu_2, v\rangle$

$a\langle u_1, v\rangle + b\langle u_2, v\rangle$

$\Rightarrow \langle u-cv, u-cv\rangle =$

$$\langle u, u\rangle + \langle u, -cv\rangle + \langle -cv, u\rangle + \langle -cv, -cv\rangle$$

$\boxed{\|u\| = \sqrt{\langle u,u\rangle}}$

$\Rightarrow \|u\|^2 + -\bar{c}\langle u,v\rangle + -c\langle v,u\rangle + +c\bar{c}\langle v,v\rangle$

Set $c = \dfrac{\langle u,v\rangle}{\langle v,v\rangle}$

$$\Rightarrow \|u\|^2 - \frac{\langle v,u\rangle}{\langle v,v\rangle}\cdot\langle u,v\rangle - \frac{\langle u,v\rangle}{\langle v,v\rangle}\langle v,u\rangle + \frac{\langle u,v\rangle}{\langle v,v\rangle}\cdot\frac{\langle v,u\rangle\cancel{\langle v,v\rangle}}{\cancel{\langle v,v\rangle}}$$

$$\Rightarrow \|u\|^2 + \frac{\langle v,u\rangle\langle u,v\rangle}{\langle v,v\rangle} - \frac{2\langle u,v\rangle\langle v,u\rangle}{\langle v,v\rangle}$$

$\boxed{c\bar{c} = |c|^2}$

$$\Rightarrow \|u\|^2 - \frac{2\langle v,u\rangle\langle u,v\rangle}{\langle v,v\rangle} + \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

$\|v\| = \sqrt{\langle v,v\rangle}$.

$$0 \le \|u-cv\|^2 = \|u\|^2 + \frac{|\langle u,v\rangle|^2}{\|v\|^2} - \frac{2\langle u,v\rangle\langle v,u\rangle}{\langle v,v\rangle}$$

$$\|u\|^2 \geq \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

$$\Rightarrow \boxed{\langle u,v\rangle \leq \|u\|.\|v\|}$$

(iv) ☆☆☆ Triangle Inequality: $\|u+v\| \leq \|u\| + \|v\|$

consider $\|u+v\|^2$

$\|v\| = \sqrt{\langle v,v\rangle}$

$= \langle u+v, u+v\rangle$

$= \langle u, u+v\rangle + \langle v, u+v\rangle$

$= \langle u,u\rangle + \langle v,v\rangle + \langle v,u\rangle + \langle u,v\rangle$

$= \|u\|^2 + \|v\|^2 + \langle u,v\rangle + \langle v,u\rangle$

$(x+iy)+(x-iy)$

$z + \bar{z}$

$= 2\,\mathrm{Re}(z)$

$= \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\,|\langle u,v\rangle|$

$= \|u\|^2 + \|v\|^2 + 2\|u\|\|v\|$

$\langle u,v\rangle \leq \|u\|\|v\|$

Cauchy-Schwartz Inequality

$\Rightarrow \|u+v\|^2 \leq (\|u\| + \|v\|)^2$

$$\boxed{\|u+v\| \leq \|u\| + \|v\|}$$

(2) Let V be an inner product space and let $S = \{x_1, x_2 \ldots x_k\}$

(X.) be an orthogonal set of non-zero vectors.

If $y = \sum_{i=1}^{k} a_i x_i$,

then $a_j = \dfrac{\langle y, x_j \rangle}{\|x_j\|^2}$, $\forall j$

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Ans: Given: $y = \sum_{i=1}^{k} a_i x_i$

$\langle y, x_j \rangle = \langle a_1 x_1 + a_2 x_2 + \ldots a_k x_k, x_j \rangle$ —(1)

since for $\langle a u_1 + b u_2, v \rangle = a \langle u_1, v \rangle + b \langle u_2, v \rangle$

(1) becomes:

$\langle y, x_j \rangle = a_1 \langle x_i, x_j \rangle + a_2 \langle x_2, x_j \rangle + \ldots$

$a_k \langle x_k, x_j \rangle$ —(2)

It is given that the set of vectors is orthogonal

$\Rightarrow \langle u, v \rangle = 0$ and $\langle u_i, u_j \rangle = 0$ so long as $i \neq j$

(2) becomes

$\langle y, x_j \rangle = 0 + 0 \ldots a_j \langle x_j, x_j \rangle$

$\langle y, x_j \rangle = a_j \|x_j\|^2$ $\Rightarrow$ $\boxed{a_j = \dfrac{\langle y, x_j \rangle}{\|x_j\|^2}}$

(3) An orthogonal set of non-zero vectors (an orthogonal set of vectors) is linearly independent

(X) ★★★

Ans. Let $u_1, u_2 \ldots u_n$ be an orthogonal set of vectors

$$v = u_1c_1 + u_2c_2 + \ldots u_nc_n$$

$$\langle v, u_j\rangle = \langle u_1c_1 + u_2c_2 + \ldots u_nc_n, u_j\rangle$$

$$= c_1\langle u_1, u_j\rangle + c_2\langle u_2, u_j\rangle + \ldots c_n\langle u_n, u_j\rangle$$

$$= c_j\langle u_j, u_j\rangle$$ (by property of orthonormal set of vectors)

$$= c_j \|u_j\|^2$$

$$c_j = \frac{\langle v, u_j\rangle}{\|u_j\|^2}$$

when $v = 0$

$c_j = 0$

$\Rightarrow u_1, u_2 \ldots u_n$ is linearly independent

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(4) <u>Gram - Schmidt Orthogonalization Process</u>

★★★★

(X) <u>Statement</u>

Let V be an inner product space and let

$S = \{u_1, u_2 \ldots u_n\}$ be a linearly independent subset of V.

Define $S' = \{x_1, x_2 \ldots x_n\}$ is non-zer

where $x_1 = u_1$ and $x_k = u_k - \sum_{j=1}^{k-1} \frac{\langle u_k, x_j\rangle}{\|x_j\|^2} x_j$

then S' is an orthogonal set of non-zero vectors such that

$\boxed{\text{span}(S) = \text{span}(S')}$

Proof by Induction

Step 1 : if $n=1$

$$S = \{y_1\}$$
$$S' = \{x_1\}$$

$$x_1 = y_1$$

$$S = S'$$

hence proved

Induction Step : Assume that the result is true for $n=k$

i.e S_k is orthogonal and $\text{span}(S_k) = \text{span}(S_k')$.

To prove the result for S'_{k+1}

$$S_{k+1}' = \{x_1, x_2 \ \ldots \ x_k, x_{k+1}\}$$

$$x_1 = y_1$$

$$x_k = y_k - \sum_{j=1}^{k-1} \frac{\langle y_k, x_j \rangle}{\|x_j\|^2} \cdot x_j \quad \text{—} \ (*)$$

There are 3 cases:

(i) If $x_{k+1} = 0$

$\Rightarrow y_{k+1} \in \text{span}(S_k) = \text{span}(S_k')$

which contradicts the fact that S_{k+1} is linearly independent.

(ii) For $1 \le i \le k$

$$\langle x_{k+1}, x_i \rangle = \langle y_{k+1} - \sum_{j=1}^{k} \frac{\langle y_{k+1}, x_j \rangle}{\|x_j\|^2} \cdot x_j , x_i \rangle$$

$$= \langle y_{k+1}, x_i \rangle - \sum_{j=1}^{k} \frac{\langle y_{k+1}, x_j \rangle}{\|x_j\|^2} \langle x_j, x_i \rangle$$

$$= \langle y_{k+1}, x_i \rangle - \langle y_{k+1}, x_j \rangle$$ } orthogonal - all like terms become 0

$= 0$

$\langle x_{k+1}, x_i \rangle = 0$

$\Rightarrow S_{k+1}'$ is orthogonal

From equation (*) $\text{Span}(S_{k+1}') \subseteq \text{Span}(S_{k+1})$

By the known corollary, if S is orthogonal, then S is a linear combination

S_{k+1}' is orthogonal $\Rightarrow$ linearly independent

i.e $\text{span}(S_{k+1}') = \text{span}(S_{k+1})$

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(5) If $u = (x_1, x_2)$ and $v = (y_1, y_2)$, prove that $\langle u, v \rangle =$

$x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2$ is an inner product in R^2

Ans . Axiom (1) : $\langle au_1 + bu_2, v \rangle = a\langle u_1, v \rangle + b\langle u_2, v \rangle$

Let $u_1 = (x_1, x_2)$

Let $u_2 = (x_1', x_2')$

$u_1 = (x_1, x_2)$ $\quad (u,v) = x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2$

$u_2 = (x_1', x_2')$ $\quad \Rightarrow \boxed{au_1 + bu_2 = (\underbrace{ax_1 + bx_1'}_{x_1}, \underbrace{ax_2 + bx_2'}_{x_2})}$

Then $(au_1 + bu_2, v) =$

substitute in given equation

$$= (ax_1 + bx_1')y_1 - (ax_1 + bx_1')y_2 - (ax_2 + bx_2')y_1$$

$$+ 4(ax_2 + bx_2')y_2$$

$$= a(x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2)$$

$$+ b(x_1'y_1 - x_1'y_2 - x_2'y_1 + x_2'y_2)$$

$$= a(u_1, v) + b(u_2, v)$$

Axiom 1 holds true

Axiom 2: $(v,u) = (u,v)$

$(v,u) =$ swap x_1 & x_2 & y_1 & y_2

~~$= x_2y_1 - x_2y_2 - x_1y_1 + 4x_1$~~

$x_1 \Rightarrow y_1$

$x_2 \Rightarrow y_2$

$$= y_1x_1 - y_1x_2 - y_2x_1 + 4y_2x_2$$

$= (u,v)$ Axiom 2 holds true

Axiom (3) : $(u,u) \geq 0$

i.e replace everything with x_1 & x_2

$$= x_1^2 - x_1x_2 - x_2x_1 + 4x_2^2$$

$$= x_1^2 - 2x_2x_1 + 4x_2^2$$

$$= x_1^2 + x_2^2 - 2x_1x_2 + 3x_2^2$$

$$= (x_1 + x_2)^2 + 3x_2^2 \geq 0$$

The equation holds only when $(x_1, x_2) = (0, 0)$

Axiom 3 holds true.

(u, v) is an inner product in R^2

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(6) For what values of a, b, c, d for which $f(u,v) = ax_1y_1 + bx_1y_2 + cx_2y_1 = dx_2y_2$ is an inner product space of R^2?

Ans

Given that $f(u,v)$ is an inner product space.

$$f(u,u) = 0$$

$$\Rightarrow ax_1^2 + bx_2x_1 + cx_2x_1 + dx_2^2 \geq 0$$

$$ax_1^2 + bx_1x_2 + cx_1x_2 + dx_2^2 \geq 0$$

$$a\left\{x_1^2 + \frac{bx_1x_2}{a} + \frac{cx_1x_2}{a} + \frac{dx_2^2}{a}\right\} \geq 0$$

$$a\left\{x_1^2 + \left(\frac{b+c}{a}\right)(x_1 + x_2) + \frac{dx_2^2}{a}\right\} \geq 0$$

$$a\left[\underbrace{x_1^2 + \frac{(b+c)}{a} x_1 x_2}_{} + \frac{d}{a} x_2^2\right\} \geq 0$$

try to make into a
whole square term

$$a\left[x_1^2 + \frac{(b+c)}{2.a} \cdot x_1 x_2 \cdot 2 + \frac{x_2^2 (b+c)^2}{a^2} - \frac{x_2^2 (b+c)^2}{4a^2}\right.$$

$$\left. + \frac{d}{a} x_2^2 \right] \geq 0$$

$$a\left[\left\{x_1 + \left(\frac{b+c}{2a}\right) x_2\right\}^2 + \left\{\frac{d}{a} - \frac{(b+c)^2}{4a^2}\right\} x_2^2\right] \geq 0$$

$$a\left[\left\{x_1 + \left(\frac{b+c}{2a}\right) x_2\right\}^2 + \left\{\frac{4ad - (b+c)^2}{4a^2}\right\} x_2^2\right]$$

$a \geq 0$

$$ad \geq \frac{(b+c)^2}{4}$$

$$ad \geq \frac{b^2 + c^2 + 2bc}{4}$$

$b^2 + c^2 > 2bc$

$\Rightarrow ad > bc$

$\boxed{\text{Condns: } a \geq 0 \quad \& \quad ad - bc > 0}$

(7) If V is a vector space of $m \times n$ matrices over $\mathbb{R}$, p.t (15)

$(A, B) = Tr(B^T \cdot A)$ is an inner product on V.

Ans Axiom (1) $(au_1 + bu_2, v) = a(u_1, v) + b(u_2, v)$

$\cancel{u_1 = (A_1, A_2)}$

$\cancel{u_2 = (A_1', A_2')}$

$\cancel{\Rightarrow au_1 + bu_2 = (aA_1 + bA_1', aA_2 + bA_2')}$

$(c_1A_1 + c_2A_2, B) = Tr(B^T(c_1A_1 + c_2A_2))$

$= Tr(c_1 B^TA_1 + c_2 B^TA_2)$

$= c_1 Tr(B^TA_1) + c_2 B^TA_2$

Axiom (1) holds true

Axiom (2) $(u, u) = 0$

here $(u, u) = (A, A)$

$= Tr(A^TA) = Tr(I) = 0$ iff $A = 0$

Axiom (2) $(u, v) = (v, u)$

$(u, v) = Tr(B^TA)$

$= Tr((B^TA)^T)$ (trace does not change if you transpose)

$= Tr(A^T B)$

$= \langle B, A \rangle$ $\therefore (u, v) = (v, u)$

$\therefore (A, B)$ is an inner product

⑧ If V is the vector space of real continuous functions on the real interval $a \le t \le b$, prove that $(f, g) = \int_{a}^{b} f(t)\, g(t)\, dt$ is an inner product on V

Axiom ①

Ans: $(c_1 f_1 + c_2 f_2, g) = \int_a^b (c_1 f_1 + c_2 f_2) g \, dt$

$= c_1 \int_a^b f_1 g \, dt + c_2 \int_a^b f_2 g \, dt$

$= c_1 (f_1, g) + c_2 (f_2, g)$

Axiom 1 holds true

Axiom ② : $(u, v) = (v, u)$

$\int_a^b f(t)\, g(t)\, dt = \int_a^b g(t) f(t)\, dt \Rightarrow (f, g) = (g, f)$

Axiom 2 holds true

Axiom ③ : $(u, u) = 0$

$\int_a^b f(t) \cdot f(t)\, dt = \int_a^b (f(t))^2 \ge 0$ as

$a \le f(t) \le b$

Axiom 3 holds true

(f, g) is an inner product space,

9) If V is the vector space of $m \times n$ matrices over R, find the norm of $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ w.r.t the inner product $(A, B) = Tr(B^T A)$

Ans norm $= \|A\| = \sqrt{\langle A, A \rangle}$

$\langle A, A \rangle = Tr(A^T A)$

$= Tr\left[\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix}\right]$

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$= Tr\begin{bmatrix} 10 & -10 \\ -10 & 20 \end{bmatrix}$

Trace $= 30$

norm $= \sqrt{30}$

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10) Find the vectors which form the orthogonal basis with the vectors $(1, -2, 2, -3)$ and $(2, -3, 2, 4)$ in R^4

Ans orthogonal vector : $\langle u, v \rangle = 0$

$u = (1, -2, 2, -3)$

$v = (2, -3, 2, 4)$

Let the required vector $= w = (a, b, c, d)$

w is orthogonal $w | u \Rightarrow \langle u, w \rangle = 0$

$a - 2b + 2c - 3d = 0$ — (1)

w is orthogonal $w | v \Rightarrow \langle v, w \rangle = 0 \Rightarrow 2a - 3b + 2c + 4d = 0$ — (2)

$a - 2b + 2c - 3d = 0 \quad -(1) \quad \times 2$

$2a - 3b + 2c + 4d = 0 \quad -(2)$

$2a - 4b + 4c - 6d = 0$

$2a - 3b + 2c + 4d = 0 \quad (-)$

$-b + 2c - 10d = 0$

$b - 2c + 10d = 0$

Let c & d be free variables

when $c = 1$ $\quad b - 2 = 0$

$d = 0$ $\quad b = 2$

$2a - 8 + 4 = 0$

$2a - 4 = 0$

$a = 2$

$\Rightarrow \boxed{(a, b, c, d) = (2, 2, 1, 0)}$

when $c = 0$

$d = 1$

$b + 10 = 0 \quad b = -10$

$2a - 3b + 2c + 4d = 0$

$2a + 30 + 4 = 0$

$a = -17$

$\Rightarrow \boxed{(a, b, c, d) = (-17, -10, 0, 1)}$

(11) Find the vectors that form the orthonormal basis with

$u = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$ and $v = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right)$ in R^4

__Ans__ Let the required vector be (x, y, z, t)

$$\frac{x}{2} + \frac{y}{2} + \frac{z}{2} + \frac{t}{2} = 0 \quad -(1)$$

$$\frac{x}{2} + \frac{y}{2} - \frac{z}{2} - \frac{t}{2} = 0 \quad -(2)$$

$x + y = 0$ and $z + t = 0$

$\boxed{x = -y}$ $\boxed{z = -t}$

if $x = 1$, $z = 1$

$y = -1$ $t = -1$

$(1, -1, 1, -1) \quad -(A)$

if $x = -1$ $z = -1$

$y = 1$ $t = 1$

$(-1, 1, -1, 1) \longrightarrow (B)$

To find the orthonormal bases :

$$\left(\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2} \right), \left(\frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2} \right) //$$

(12) Find an orthonormal basis of R^3 given that an arbitrary basis of R^3 is $\{ v_1 = (1,1,1),\ v_2 = (0,1,1),\ v_3 = (0,0,1) \}$ using the Gram-Schmidt Orthogonalization process.

<u>Ans</u> $u_1 = v_1$

$$u_2 = v_2 - \frac{(v_2, u_1)}{\|u_1\|^2} \cdot u_1$$

$$u_3 = v_3 - \frac{(v_3, u_1)}{\|u_1\|^2} \cdot u_1 - \frac{(v_3, u_2)}{\|u_2\|^2} \cdot u_2$$

$$\boxed{u_1 = v_1 = (1,1,1)}$$

$$u_2 = v_2 - \frac{(v_2, u_1)}{\|u_1\|^2} \cdot u_1$$

$$= (0,1,1) - \frac{\langle (0,1,1), (1,1,1) \rangle}{3} \cdot \langle 1,0,1 \rangle$$

$$= (0,1,1) - \frac{0^2 + 1^2 + 1^2}{3} \cdot \langle 1,0,1 \rangle$$

$$= (0,1,1) - \frac{2}{3} (1,0,1)$$

$$\Rightarrow \boxed{u_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right)}$$

$$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} \cdot u_1 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} \cdot u_2$$

$$= \langle 0,0,1 \rangle - \frac{\langle (0,0,1),(1,1,1) \rangle}{3} \cdot \langle 1,1,1 \rangle$$

$$- \frac{\langle (0,0,1), (-2/3, 1/3, 1/3) \rangle}{\frac{4+1+1}{9}} \cdot \langle \frac{-2}{3}, \frac{1}{3}, \frac{1}{3} \rangle$$

$$= \langle 0,0,1 \rangle - 1/3 \cdot \langle 1,1,1 \rangle - \frac{1/3}{2/3} \cdot \langle -2/3, 1/3, 1/3 \rangle$$

$\frac{1}{2} \times \frac{2}{3}$

$$= \langle 0,0,1 \rangle - \langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \rangle - \frac{1}{2} \langle \frac{-2}{3}, 1/3, 1/3 \rangle$$

$\frac{1}{2} \times \frac{3}{2}$

$\frac{3}{4}$

$$= \langle 0,0,1 \rangle - \langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \rangle - \langle \frac{-1}{3}, \frac{1}{6}, \frac{1}{6} \rangle$$

$$\boxed{u_3 = \langle 0, \frac{-1}{2}, \frac{1}{2} \rangle}$$

The orthonormal bases are:

$9+9+9$

$$u_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \quad u_2 = \frac{-2}{\sqrt{27}} (-2, 1, 1)$$

$$= \frac{-1}{\sqrt{6}}$$

$$u_3 = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

(13) Find an orthogonal basis of the subspace of R4 given that an arbitrary basis is:

$\{v_1 = (2,1,3,-1),\ v_2 = (7,4,3,-3),\ v_3 = (5,7,7,8)\}$

Ans By the Gram- Schmidt Orthogonalization process

$$\boxed{u_1 = v_1 = (2,1,3,-1)}$$

$$u_2 = v_2 - \frac{(v_2, u_1)}{\|u_1\|^2} \cdot u_1$$

$$\cancel{= (2,1,3,1)}$$

$$= (7,4,3,-3) - \frac{(\langle 7,4,3,-3\rangle \langle 2,1,3,-1\rangle)}{15} \langle 2,1,3,-1\rangle$$

$4+1+9+1$

$14+4+9+3$

$$= (7,4,3,-3) - \frac{(30)}{15} \langle 2,1,3,-1\rangle$$

$$= (7,4,3,-3) - (4,2,6,-2)$$

$$= (3,2,-3,-1) //$$

$$\boxed{u_2 = (3,2,-3,-1)}$$

$$u_3 = v_3 - \frac{(v_3, u_1)}{\|u_1\|^2} \cdot u_1 - \frac{(v_3, u_2)}{\|u_2\|^2} \cdot u_2$$

$$u_3 = (5,7,7,8) - \frac{(\langle 5,7,7,8\rangle \langle 2,1,3,-1\rangle)}{15} \cdot \langle 2,1,3,-1\rangle - \frac{(\langle 5,7,7,8\rangle, \langle 3,2,-3,-1\rangle)}{23} \cdot u_2$$

$$= \left\{(5,7,7,8) - \frac{30}{15}(2,1,3,-1)\right\}$$

$10+7+21-8$

$17+3$

$9+4+9+1$

$15+14-21-8$

$$= \cancel{0} - \left\{\frac{0}{23}\right\}$$

$$= (5,7,7,8) - (4,2,6,-2)$$

$$\boxed{u_3 = (1,5,1,10)}$$

$\therefore$ The orthogonal basis is :

$$\{u_1 = (2,1,3,-1),\ u_2 = (3,2,-3,-1),\ u_3 = (1,5,1,10)\}$$

---

(14) Apply Gram-Schmidt orthogonalization ~~with~~ to construct the orthonormal basis for R^3 with standard inner product for the basis

$\{v_1 = (1,0,1),\ v_2 = (1,3,1),\ v_3 = (3,2,1)\}$

Ans $\{v_1 = (1,0,1),\ v_2 = (1,3,1),\ v_3 = (3,2,1)\}$

$$\boxed{u_1 = v_1 = (1,0,1)}$$

$$u_2 = v_2 - \frac{(v_2, u_1)}{\|u_1\|^2} \cdot u_1$$

$$u_2 = (1,3,1) - \frac{(\langle 1,3,1\rangle, \langle 1,0,1\rangle)}{2} \cdot (1,0,1)$$

$$= (1,3,1) - \frac{(2)}{2} \cdot (1,0,1)$$

$$\boxed{u_2 = (0,3,0)}$$

$$u_3 = v_3 - \frac{(v_3, u_1)}{\|u_1\|^2} \cdot u_1 - \frac{(v_3, u_2)}{\|u_2\|^2} \cdot u_2$$

$$u_3 = (3,2,1) - \frac{(\langle 3,2,1\rangle, \langle 1,0,1\rangle)}{2} \cdot (1,0,1)$$

$$- \frac{(\langle 3,2,1\rangle \langle 0,3,0\rangle)}{9} \cdot \langle 0,3,0\rangle$$

$$u_3 = (3,2,1) - \left(\frac{4}{2}\right)(1,0,1) - \frac{6}{9}\langle 0,3,0\rangle$$

$$u_3 = (3,2,1) - (2,0,2) - (0,2,0)$$

$$\boxed{u_3 = (1,0,-1)}$$

The orthogonal vector are:

$$\{u_1 = (1,0,1),\ u_2 = (0,3,0),\ u_3 = (1,0,-1)\}$$

The orthonormal vectors are:

$$\left\{ \frac{1}{\sqrt{2}}(1,0,1),\ \frac{1}{3}(0,3,0),\ \frac{1}{\sqrt{2}}(1,0,-1) \right\}$$

---

(15) Let V be the set of polynomials with degree ≤ 2, V is a real inner product space with inner product space defined by:

$$\langle f,g \rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$, starting with the basis $\{1, x, x^2\}$

Obtain an orthogonal basis for V

Ans $v_1 = 1 \quad v_2 = x, \quad v_3 = x^2$

$$\boxed{u_1 = v_1 = 1}$$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} \cdot u_1 \qquad \int_{-a}^{a} \text{odd fn} = 0$$

$$u_2 = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1$$

$$u_2 = x - \frac{\int_{-1}^{1} x \cdot 1\,dx}{\int_{-1}^{1} 1\,dx} \qquad \boxed{u_2 = x}$$

$$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} \cdot u_1 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} \cdot u_2$$

$$u_3 = x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1 - \frac{\langle x^2, x \rangle}{\langle x, x \rangle} \cdot x$$

$$u_3 = x^2 - \frac{\int_{-1}^{1} x^2 \cdot 1\,dx}{\int_{-1}^{1} 1\,dx} - \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx} \cdot x$$

$$u_3 = x^2 - \frac{\left(\frac{x^3}{3}\right)_{-1}^{1}}{2} - \frac{\left(\frac{x^4}{4}\right)_{-1}^{1}}{\left(\frac{x^3}{3}\right)_{-1}^{1}} \cdot x$$

$$u_3 = x^2 - \frac{\frac{1}{3} + \frac{1}{3}}{2} - \left(\frac{\frac{1}{4} - \frac{1}{4}}{dr}\right) \cdot x$$

$$\boxed{u_3 = x^2 - \frac{1}{3}}$$

The orthogonal vectors are:

$$\left\{ u_1 = 1, \ u_2 = x, \ u_3 = x^2 - \frac{1}{3} \right\}$$

★★★

The orthonormal vectors are:

$$\left\{ 1, \frac{x}{\sqrt{\int_{-1}^{1} x \cdot x\,dx}}, \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1} x^2 - \frac{1}{3}}} \right\} = \left\{ 1, \sqrt{\frac{3x}{2}}, \frac{\sqrt{10}}{4}\left(x^2 - \frac{1}{3}\right) \right\}$$

(16) Find the orthogonal basis containing $(1,1,-1), (1,0,1)$ in R^3 with a standard inner product

$v_1 = (1,1,-1)$ $\quad v_2 = (1,0,1)$ $\quad v_3 = (x,y,z)$

v_3 is orthogonal to v_1 & v_2

$\langle v_1, v_3 \rangle = 0$ $\quad x + y - z = 0$ — (1)

v_3 is orthogonal to v_2

$\langle v_2, v_3 \rangle = 0$ $\quad x + z = 0$ — (2)

$x + y - z = 0$

$x + 0y + z = 0$

$$\begin{array}{cccc} & x & y & z \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}$$

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{-1}$$

$$\boxed{v_3 = (1,-2,-1)}$$

---

(17) Find an orthogonal basis containing $(1,3,4)$ over R^3 with standard inner product

$v_1 = \langle 1,3,4 \rangle$; $v_2 = \langle x,y,z \rangle$

$\langle v_1, v_2 \rangle = 0$

$x + 3y + 4z = 0$

Let $z = 1$, $y = 0$, $x = -4$ $\quad \boxed{v_2 = (-4,0,1)}$

v_3 is orthogonal to v_1 & v_2

$$v_3 = \langle a, b, c \rangle$$

$$\langle v_3, v_1 \rangle = 0$$

$$a + 3b + 4c = 0$$

$$\langle v_3, v_2 \rangle = 0$$

$$\Rightarrow -4a + c = 0$$

$$\begin{array}{ccccccc} & a & & b & & c & \\ 3 & & 4 & & 1 & & 3 \\ 0 & & 1 & & -4 & & 0 \end{array}$$

$$\frac{a}{3} = \frac{b}{-17} = \frac{c}{12}$$

$$\boxed{v_3 = \langle 3, -17, 12 \rangle}$$

---

(18) Apply the Gram Schmidt orthogonalization to find the orthonormal basis for $M_{2\times 2}$ over R $(M_{2\times 2}(R))$ with the inner product $\langle A, B\rangle = \text{Trace}(B^T A)$ $\forall A, B \in M_{2\times 2}$

where the given basis is $\left\{ v_1 = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, v_3 = \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right\}$

Ans

$$\boxed{u_1 = v_1 = \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}}$$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} \cdot u_1$$

$$u_2 = \begin{bmatrix} -1 & 9 \\ 5 & -1 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} -1 & 9 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix} \right\rangle} \cdot \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} -1 & 9 \\ 5 & -1 \end{bmatrix} - \frac{Tr\left\{ \begin{bmatrix} 3 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} -1 & 9 \\ 5 & -1 \end{bmatrix} \right\}}{Tr\left\{ \begin{bmatrix} 3 & \cancel{5}\,{-1} \\ \cancel{5} & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix} \right\}} \cdot \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} -1 & 9 \\ 5 & -1 \end{bmatrix} - \frac{Tr \begin{bmatrix} -8 & 28 \\ 0 & 44 \end{bmatrix}}{Tr \begin{bmatrix} 10 & 14 \\ -4 & 26 \end{bmatrix}} \cdot \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} -1 & 9 \\ 5 & -1 \end{bmatrix} - \frac{36}{36} \cdot \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{u_2 = \begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix}}$$

$$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} \cdot u_1 \qquad u_2 = \begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix}$$

$$- \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} \cdot u_2$$

$$u_3 = \begin{bmatrix} 7 & -17 \\ 2 & -6 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 7 & -17 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix} \right\rangle} \cdot \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}$$

$$- \frac{\left\langle \begin{bmatrix} 7 & -17 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix} \right\rangle} \cdot \begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 7 & -17 \\ 2 & -6 \end{bmatrix} - \frac{Tr\left\{ \begin{bmatrix} 3 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 7 & -17 \\ 2 & -6 \end{bmatrix} \right\}}{Tr\left\{ \begin{bmatrix} 3 & -1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix} \right\}} \cdot \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}$$

$$- \frac{Tr\left\{ \begin{bmatrix} -4 & 6 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 7 & -17 \\ 2 & -6 \end{bmatrix} \right\}}{Tr\left\{ \begin{bmatrix} -4 & 6 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix} \right\}} \cdot \begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 7 & -17 \\ 2 & -6 \end{bmatrix} - \frac{Tr\begin{bmatrix} 19 & -45 \\ 37 & -91 \end{bmatrix}}{Tr\begin{bmatrix} 10 & 14 \\ -4 & 26 \end{bmatrix}} \cdot \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix} - T2$$

$$\begin{array}{r} 91 \\ 19 \\ \hline 72 \end{array}$$

$$u_3 = \begin{bmatrix} 7 & -17 \\ 2 & -6 \end{bmatrix} - \frac{36}{72}\begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix} - T2$$

$$u_3 = \begin{bmatrix} 7 & -17 \\ 2 & -6 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix} - \frac{Tr\begin{bmatrix} -16 & 32 \\ 24 & -56 \end{bmatrix}}{Tr\begin{bmatrix} 52 & -28 \\ -28 & 20 \end{bmatrix}} \cdot \begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix}$$

$$\begin{array}{r} 56 \\ 16 \\ \hline 72 \\ 82 \end{array}$$

$$u_3 = \begin{bmatrix} 7 & -17 \\ 2 & -6 \end{bmatrix} - \begin{bmatrix} 1.5 & 2.5 \\ -0.5 & 0.5 \end{bmatrix} - \frac{36}{41}\begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix}$$

?

$$\frac{36}{41}$$

(14) Find an orthonormal basis of R^3, given that an arbitrary is $\{v_1=(3,0,4),\ v_2=(-1,0,7),\ v_3=(2,9,11)\}$. Express (x,y,z) as a linear combination of the orthogonal basis vectors

Ans $\boxed{u_1 = v_1 = (3,0,4)}$

$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\|u_1\|^2} \cdot u_1$$

$$= \langle -1,0,7\rangle - \frac{(\langle -1,0,7\rangle, \langle 3,0,4\rangle)}{25} \cdot \langle 3,0,4\rangle$$

$$= \langle -1,0,7\rangle - \frac{25}{25} \langle 3,0,4\rangle$$

-3+3

$$\boxed{u_2 = \langle -4,0,3\rangle}$$

$$u_3 = v_3 - \frac{\langle v_3, u_1\rangle}{\|u_1\|^2} \cdot u_1 - \frac{\langle v_3, u_2\rangle}{\|u_2\|^2} \cdot u_2$$

$$= \langle 2,9,11\rangle - \frac{(\langle 2,9,11\rangle, \langle 3,0,4\rangle)}{25} \langle 3,0,4\rangle$$

$$- \frac{(\langle 2,9,11\rangle, \langle -4,0,3\rangle)}{25} \langle -4,0,3\rangle$$

-8 + 33

$= \langle 2, 9, 11 \rangle - \frac{50}{25} \langle 3, 0, 4 \rangle - 1 \langle -4, 0, 3 \rangle$

$= \langle 2, 9, 11 \rangle - \langle 6, 0, 8 \rangle - \langle -4, 0, 3 \rangle$

$= \langle 0, 9, 0 \rangle$ //

The orthogonal basis is $\{u_1 = (3, 0, 4), u_2 = (-4, 0, 3), u_3 = (0, 9, 0)\}$

The orthonormal basis is $\{u_1' = \frac{1}{5}(3, 0, 4), u_2' = \frac{1}{5}(-4, 0, 3),$

$u_3' = (0, 1, 0)\}$

Linear combination of the orthogonal basis vectors

Let $u = k_1 u_1 + k_2 u_2 + k_3 u_3$ — (1)

$\therefore (u, u_1) = k_1 \|u_1\|^2$

$k_1 = \frac{(u, u_1)}{\|u_1\|^2} \qquad k_2 = \frac{(u, u_1)}{\|u_2\|^2} \qquad k_3 = \frac{(u, u_1)}{\|u_3\|^2}$

Equation (1) becomes $\qquad u = (x, y, z)$

$$u = \left(\frac{3x + 4z}{25}\right)\left(3, 0, 4\right) + \left(\frac{-4x + 3z}{25}\right)(-4, 0, 3) + \left(\frac{9y}{81}\right)(0, 9, 0)$$

20

Define orthogonal projection of the vector $v \in V$ on W, which is a subspace of V. Find the orthogonal projection $v = (-10, 2, 8)$ on the subspace W spanned by $\{w_1 = (-1, 1, 1), w_2 = (1, -2, 2)\}$

Ans If V is an inner product space and W is a subspace of V and if $v \in V$ and $(w_1, w_2 \ldots w_r)$ span W, then $\sum_{i=1}^{r} \frac{(v, w_i)}{\|w_i\|} \cdot w_i$ is called the orthogonal projection of v on W.

Here the orthogonal projection of $v = (-10, 2, 8)$ would be:

$$w = \frac{(v, w_1)}{\|w_1\|^2} \cdot w_1 + \frac{(v, w_2)}{\|w_2\|^2} \cdot w_2$$

$$w = \frac{(-10+2+8)}{3}(-1, 1, 1) + \frac{(-10-4+16)}{9}(1, -2, 2)$$

$$w = \frac{20}{3}(-1, 1, 1) + \frac{2}{9}(1, -2, 2)$$

$$w = \left(\frac{-58}{9}, \frac{56}{9}, \frac{64}{9}\right)$$

21. Find the angle between $u = f(t) = 2t - 1$ and $v = g(t) = t^2$ in which the inner product is defined by $(f, g) = \int_0^1 f(t)\, g(t)\, dt$

$$\cos\theta = \frac{(u, v)}{\|u\|\,\|v\|}$$

$$(u, v) = \int_0^1 (2t-1)\, t^2 \, dt$$

$$= \int_0^1 2t^3 - t^2 \, dt = \left(\frac{\cancel{2}t^4}{\cancel{4}_2} - \frac{t^3}{3} \right)_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} //$$

$$\|u\|^2 = \int_0^1 (2t-1)^2 dt = \int_0^1 4t^2 + 1 - 4t \, dt$$

$$= \left(\frac{4t^3}{3} + t - 2t^2 \right)_0^1 = \frac{4}{3} + 1 - 2$$

$\frac{4}{3} - 1$

$$= \frac{1}{3} //$$

$$\|v\|^2 = \int_0^1 t^4 \, dt = \frac{1}{5}$$

$$\cos\theta = \frac{1/6}{\sqrt{1/5}\sqrt{1/3}} = \frac{1}{6}\sqrt{15} //$$

Q2 Use the least square approximation to find the best line of

X. fit for :

(i) linear function : $y = ct + d$

(ii) quadratic function : $y = ct^2 + dt + e$

Compute the error in both cases for the following data:

$\{(-3,9), (-2,6), (0,2), (1,1)\}$

<u>Ans</u> Linear Function

$y = ct \cancel{d} + d$

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad y = \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix} \quad x_0 = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$A^*A = \begin{bmatrix} -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^*A = \begin{bmatrix} 14 & -4 \\ -4 & 4 \end{bmatrix}$$

~~$(A*A)^{-1} = \begin{bmatrix} 1/10 & 1/10 \\ 3 & \end{bmatrix}$~~

$$(A*A)^{-1} = \frac{1}{40} \begin{bmatrix} 4 & 4 \\ 4 & 14 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} c \\ d \end{bmatrix} = (A^*A)^{-1}A^*y$$

$$= \frac{1}{40}\begin{bmatrix} 4 & 4 \\ 4 & 14 \end{bmatrix}\begin{bmatrix} -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 4 & 4 \\ 4 & 14 \end{bmatrix}\begin{bmatrix} -38 \\ 8 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -2 \\ 5/2 \end{bmatrix}$$

$$\boxed{\begin{array}{l} c = -2 \\ d = 5/2 \end{array}}$$

The line of best fit is $y = -2t + \frac{5}{2}$

Error $E = \| Ax_0 - y \|^2$

$$= \left\| \begin{bmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 5/2 \end{bmatrix} - \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix} \right\|^2$$

$$= \left\| \begin{bmatrix} 17/2 \\ 13/2 \\ 6/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix} \right\|^2$$

$$E = \left\| \begin{matrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{matrix} \right\|^2 = 1 \qquad \boxed{\text{Error} = 1}$$

Quadratic Function

$$x_0 = \begin{bmatrix} c \\ d \\ e \end{bmatrix} \qquad A = \begin{array}{c} \begin{array}{ccc} t^2 & t & 1 \end{array} \\ \begin{bmatrix} 9 & \cancel{8}-3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \qquad y = \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix}$$

$$(A*A) = \begin{bmatrix} 9 & 4 & 0 & 1 \\ -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$(A*A) = \begin{bmatrix} 98 & -34 & 14 \\ -34 & 14 & -4 \\ 14 & -4 & 4 \end{bmatrix}$$

$$(A*A)^{-1} = \begin{bmatrix} 1/9 & 2/9 & -1/6 \\ 2/9 & 49/90 & -7/30 \\ -1/6 & -7/20 & 3/5 \end{bmatrix}$$

$$A*y = \begin{bmatrix} 9 & 4 & 0 & 1 \\ -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 106 \\ -38 \\ 18 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} c \\ d \\ e \end{bmatrix} = (A^*A)^{-1}(A^*y)$$

$$= \begin{bmatrix} 1/3 \\ -4/3 \\ 2 \end{bmatrix}$$

$c = 1/3 \quad d = -4/3 \quad e = 2$

The line of best fit is $y = \frac{1}{3}t^2 - \frac{4}{3}t + 2$

Error $= E = \|Ax_0 - y\|^2$

$$= \left\| \begin{bmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ -4/3 \\ 2 \end{bmatrix} - \begin{bmatrix} 9 \\ \cancel{6} \\ 2 \\ 1 \end{bmatrix} \right\|^2$$

$\boxed{E=0}$ $\longrightarrow$ error for quadratic function

---

(23) Suppose that the data collected is $\{(1,2), (2,3), (3,5), (4,7)\}$ then determine the line of best fit and also find the error for the linear function

$1+4+9+16$

Ans

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \quad y = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix} \quad x_0 = \begin{bmatrix} c \\ d \end{bmatrix}$$

$\cancel{40-4\ 4x}$

$2 \times 4 \times 4 \times 0$

$$A^*A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}$$

$$A^*A = \begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}$$

$$(A^*A)^{-1} = \frac{1}{20}\begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} c \\ d \end{bmatrix} = (A^*A)^{-1}A^*y$$

$$\cancel{A^*y =} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \qquad A^*y = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

2×4

$2+6+15+28$

$$\begin{array}{r} 23 \\ 28 \\ \hline 51 \end{array}$$

$$A^*y = \begin{bmatrix} 51 \\ 17 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix}\begin{bmatrix} 51 \\ 17 \end{bmatrix}$$

$2 \times 2 \quad 2 \times 1$

$$= \frac{1}{20}\begin{bmatrix} 34 \\ 0 \end{bmatrix}$$

$c = 17/10$

$d = 0$

$$\boxed{y = \frac{17}{10}t + 0}$$

Error : $\|Ax_0 - y\|^2$

(24) Let V be an inner product space and let T, U be linear operators on V, then prove the following

(X)

(i) $(T+U)^* = T^* + U^*$

(ii) $(cT)^* = \bar{c}T^*$

(iii) $(TU)^* = U^*T^*$

Ans (i) $(T+U)^* = T^* + U^*$

$$\langle x, (T+U)^* y \rangle = \langle (T+U)x, y \rangle$$

$$= \langle T(x) + U(x), y \rangle$$

$$= \langle T(x), y \rangle + \langle U(x), y \rangle$$

$$= \langle x, T^*(y) \rangle + \langle x, U^*(y) \rangle$$

$$= \langle x, T^*(y) + U^*(y) \rangle$$

$$= \langle x, (T^* + U^*) y \rangle$$

$$= T^* + U^*$$

(ii) $(cT)^* = \bar{c}T^*$

$$\langle x, (cT)^* y \rangle = \langle (cT)(x), y \rangle$$

$$= c \langle T(x), y \rangle$$

$$= c \langle x, T^*(y) \rangle$$

$$= \langle x, \bar{c}\, T^*(y) \rangle$$

$$= \bar{c}\, T^*(y)$$

(iii) $(TU)^* = U^*T^*$

$$\langle x, (TU)^* y\rangle = \langle (TU)x, y\rangle$$

$$= \langle T(U(x)), y\rangle$$

$$= \langle U(x), T^*(y)\rangle$$

$$= \langle x, U^*T^*(y)\rangle$$

$$= \langle x, U^*T^*(y)\rangle$$

$$= U^*T^*$$

$$\therefore (TU)^* = U^*T^*$$