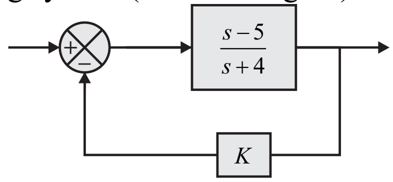
CONTROL SYSTEMS

Chapter 4 : Routh-Hurwitz Stability

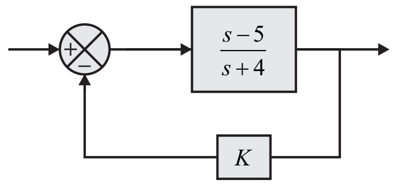
GATE Objective & Numerical Type Solutions

Question 4 [Practice Book] **[GATE EE 1992 IIT-Delhi : 1 Mark]**

For what range of K the following system (shown in figure) is asymptotically stable?



Sol. The given figure is shown below.



Characteristic equation can be written as,

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K(s-5)}{s+4} = 0$$

$$(K+1)s - 5K + 4 = 0$$

Routh Tabulation :

s^1	$(K+1)$	0
s^0	$4-5K$	

For marginal stability,

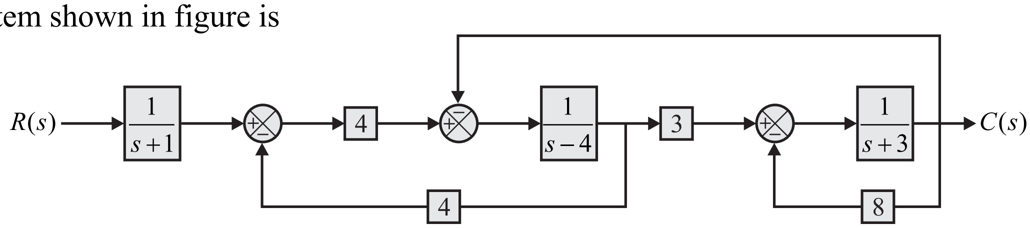
$$K + 1 > 0$$

$$4 - 5K > 0$$

Hence, range of K is $-1 < K < 4/5$.

Question 13 [Practice Book] **[GATE EE 1997 IIT-Madras : 2 Marks]**

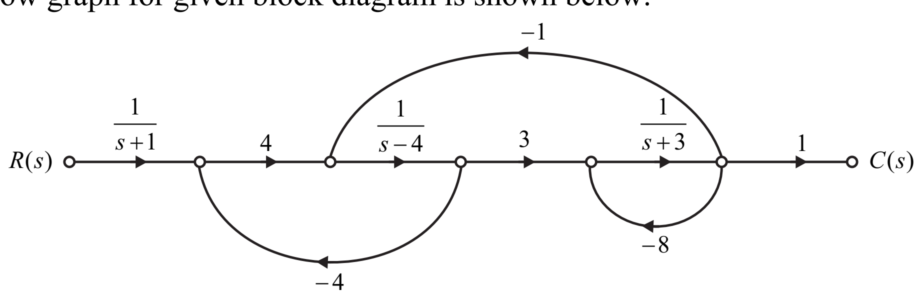
The system shown in figure is



(A) Stable (B) Unstable (C) Marginal Stable (D) Critically stable

Ans. **(A)**

Sol. The signal flow graph for given block diagram is shown below.



Forward path : $P_1 = \frac{1}{s+1}\times 4\times\frac{1}{s-4}\times 3\times\frac{1}{s+3}\times 1 = \frac{12}{(s+1)(s-4)(s+3)}$

Individual loop : $L_1 = \frac{-16}{(s-4)}$, $L_2 = \frac{-3}{(s-4)(s+3)}$, $L_3 = \frac{-8}{(s+3)}$

Two non-touching loop : $L_1L_3 = \frac{-16}{(s-4)}\times\frac{-8}{(s+3)} = \frac{128}{(s-4)(s+3)}$

Determinant : $\Delta = 1-(L_1+L_2+L_3)+(L_1L_3)$

$$\Delta = 1-\left[\frac{-16}{(s-4)}-\frac{3}{(s-4)(s+3)}-\frac{8}{(s+3)}\right]+\frac{128}{(s-4)(s+3)}$$

Path factor : All the loops touch forward path.

$$\Delta_1 = 1-0 = 1$$

Using Mason's gain formula transfer function is given by,

$$\frac{C(s)}{R(s)} = \frac{1}{\Delta}\sum_k P_k\Delta_k$$

$$\frac{C(s)}{R(s)} = \frac{1}{\Delta}\sum_{k=1} P_k\Delta_k = \frac{P_1\Delta_1}{\Delta}$$

$$\frac{C(s)}{R(s)} = \frac{\frac{12}{(s+1)(s-4)(s+3)}}{1-\left[\frac{-16}{s-4}-\frac{8}{s+3}-\frac{3}{(s-4)(s+3)}\right]+\frac{128}{(s-4)(s+3)}}$$

$$TF = \frac{12}{s^3+24s^2+158s+135}$$

Routh Tabulation :

s^3	1	158
s^2	24	135
s^1	$\frac{24(158)-135}{24}$	0
s^0	135	

No of sign changes in the 1^{st} column $=0$

$\therefore$ System is stable.

Hence, the correct option is (A).

Question 20 [Practice Book] **[GATE EC 1999 IIT-Bombay : 5 Marks]**

The loop transfer function of a feedback control system is given by

$$G(s)H(s) = \frac{K(s+1)}{s(1+Ts)(1+2s)},\quad K>0$$

Using Routh-Hurwitz criterion, determine the region of K–T plane in which the closed-loop system is stable.

Sol. The characteristic equation is, $1+G(s)H(s)=0$

$$2Ts^3+(2+T)s^2+(K+1)s+K=0$$

Routh Tabulation :

s^3	$2T$	$K+1$
s^2	$2+T$	K
s^1	$(K+1)-\dfrac{2KT}{2+T}$	0
s^0	K	0

For the system to be stable, all the roots must be in the left-half s-plane, and, thus, all the coefficients in the first column of Routh's tabulation must have the same sign. Therefore, first column of the Routh's array should be positive.

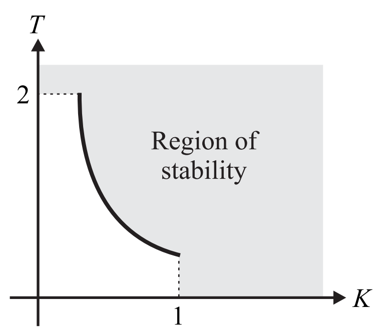
$$2T > 0 \text{ and } 2+T > 0$$
$$T > 0,\ \ K > 0$$

and
$$K+1-\frac{2KT}{2+T} > 0$$
$$2K+2+KT+T-2KT > 0$$
$$K(2-T)+T+2 > 0$$
$$K > \frac{T+2}{T-2}$$

and
$$T > \frac{2(1+K)}{K-1}$$

Hence for stability $K > 1$ and $T > 2$

Region of stability is shown in the figure.

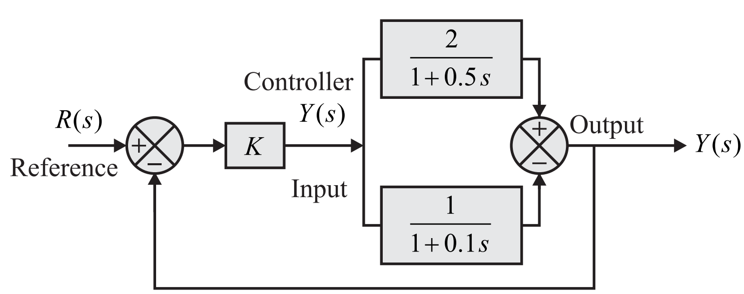


Question 22 [Practice Book] [GATE EE 2000 IIT-Kharagpur : 2 Marks]

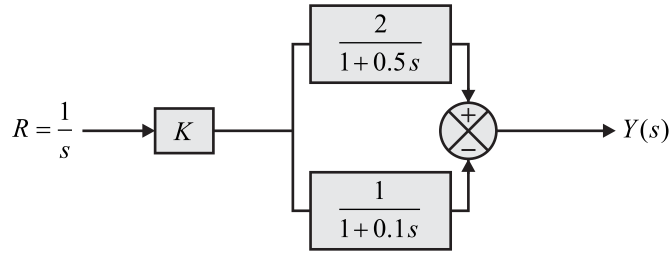
Consider the plant shown in the figure, with a proportional controller with a positive gain

(a) Sketch the open loop unit step response of the plant.

(b) Show that the steady state gain of the above closed loop controlled system, is limited to $\frac{2}{3}$ for stable operation.



Sol. (a) Open loop response of the system is found as,



$$\frac{Y(s)}{R(s)} = K\left[\frac{2}{1+0.5s} - \frac{1}{1+0.1s}\right] = K\left[\frac{4}{s+2} - \frac{10}{s+10}\right]$$

$$\frac{Y(s)}{R(s)} = K\left[\frac{4(S+10)-10(s+2)}{(s+2)(s+10)}\right] = K\left[\frac{-6s+20}{(s+2)(s+10)}\right]$$

$$Y(s) = KR(s)\left[\frac{-6s+20}{(s+2)(s+10)}\right] = K\left[\frac{-6s+20}{s(s+2)(s+10)}\right]$$

$$Y(s) = K\left[\frac{1}{s} - \frac{2}{s+2} + \frac{1}{s+10}\right]$$

Taking Laplace inverse we get,

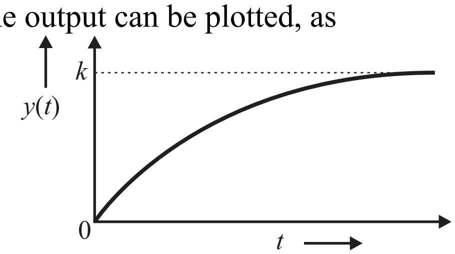
$$y(t) = K[1 - 2e^{-2t} + 1e^{-10t}]$$

At $t = 0$ $y(t) = 0$

$t = 1$ $y(t) = 0.26K$

$t = \infty$ $y(t) = K$

So, the time response of the output can be plotted, as



(b) Characteristics equation is

$$1 + K\left[\frac{2}{1+0.5s} - \frac{1}{1+0.1s}\right] = 0$$

$$1 + K\left[\frac{4}{2+s} - \frac{10}{s+10}\right] = 0$$

$$1 + K\left[\frac{4(s+10)-10(s+2)}{(s+2)(s+10)}\right] = 0$$

$$(s+2)(s+10) + K[-6s+20] = 0$$

$$s^2 + 12s + 20 - 6Ks + 20K = 0$$

$$s^2 + (12-6K)s + 20(1+K) = 0$$

Routh Tabulation :

s^2	1	$20(K+1)$
s^1	$(12-6K)$	
s^0	$20(K+1)$	

For the system to be stable, all the roots must be in the left-half s-plane, and, thus, all the coefficients in the first column of Routh's tabulation must have the same sign. Therefore, first column of the Routh's array should be positive.

$$\therefore \qquad 12-6K>0$$
$$12>6K$$
$$K>2$$
and $$20(K+1)>0$$
$$K>-1$$
$$-1<K<2$$

Taking $K=2$ (critical stability).

Steady state gain of the system, for $K=2$ is

$$\text{Closed loop } T.F. = \frac{\frac{K(20-6s)}{(s+2)(S+10)}}{1+\frac{K(20-6s)}{(s+2)(s+10)}}$$

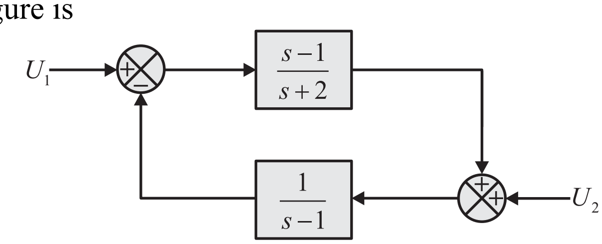
For $K=2,$ and at $s\to 0$

Steady state gain

$$T(0)=\frac{\frac{2(20)}{(2\times 10)}}{1+\frac{2(20)}{(2\times 10)}}=\frac{2}{1+2}=\frac{2}{3}$$

Question 31 [Practice Book] **[GATE EE 2007 - IIT-Kanpur : 1 Mark]**

The system shown in figure is



(A) Stable

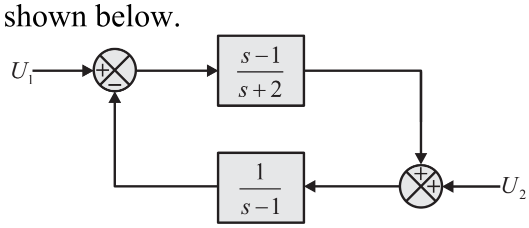
(B) Unstable

(C) Conditionally unstable

(D) Stable for input U_1, but unstable for input U_2

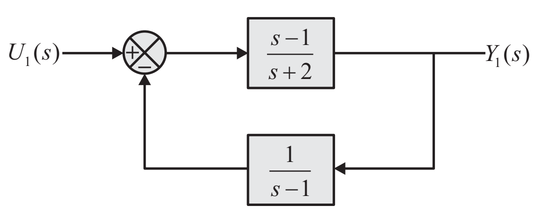
Ans. **(D)**

Sol. The given block diagram is shown below.



Applying superposition, the transfer function

$$\frac{Y_1(S)}{U_1(S)} \quad \text{if } U_2(s)=0$$

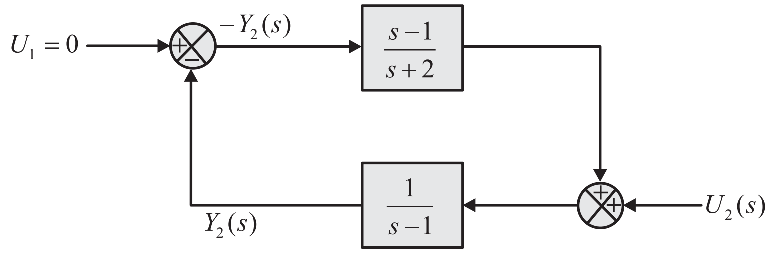


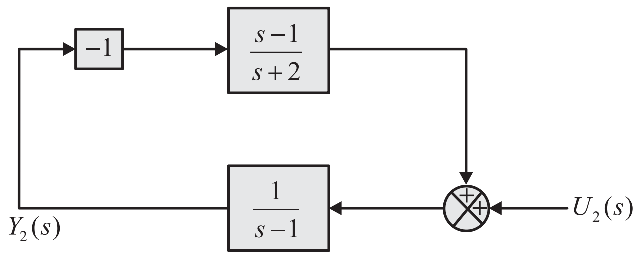
$$\frac{Y_1(s)}{U_1(s)}=\frac{\frac{s-1}{s+2}}{1+\frac{s-1}{s+2}\times\frac{1}{s-1}}=\frac{s-1}{s+2+1}=\frac{s-1}{s+3}$$

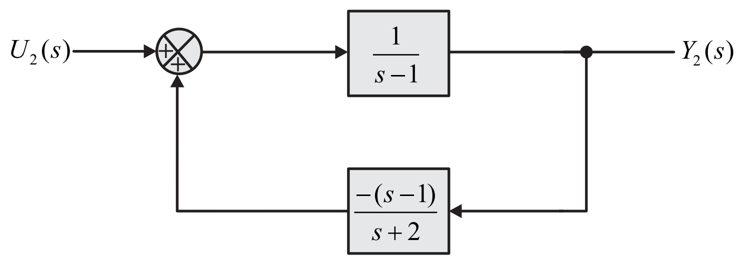
Pole $s=-3$ in left half of s-plane, so it is stable for U_1.

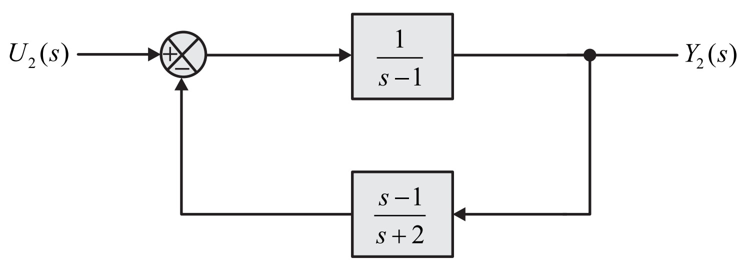
Applying superposition, the transfer function

$$\frac{Y_2(s)}{U_2(s)} \text{ if } U_1(s)=0$$









$$\frac{Y_2(s)}{U_2(s)}=\frac{\frac{1}{s-1}}{1+\frac{1}{s-1}\times\frac{s-1}{s+2}}=\frac{\frac{1}{(s-1)}}{1+\frac{1}{(s+2)}}$$

$$\frac{Y_2(s)}{U_2(s)}=\frac{s+2}{(s+2)(s-1)+(s-1)}=\frac{s+2}{(s-1)(s+3)}$$

Poles $s=1,\,-3$. One pole in right half of s-plane. So it is unstable for U_2.

Hence, the correct option is (D).

Question 32 [Practice Book] **[GATE EE 2007 IIT-Kanpur : 2 Marks]**

If the loop gain K of a negative feedback system having an open loop transfer function $\frac{K(s+3)}{(s+8)^2}$ is to be adjusted to induce a sustained oscillation then

(A) The frequency of this oscillation must be $\frac{4}{\sqrt{3}}$ rad/sec.

(B) The frequency of this oscillation must be 4 rad/sec.

(C) The frequency of this oscillation must be 4 or $\frac{4}{\sqrt{3}}$ rad/sec.

(D) Such a K does not exist.

Ans. **(D)**

Sol. **Given :** $G(s)H(s) = \frac{K(s+3)}{(s+8)^2}$

The characteristics equation is given by, $1+G(s)H(s)=0$

$$1+\frac{K(s+3)}{(s+8)^2}=0$$

$$s^2+16s+64+Ks+3K=0$$

$$s^2+(16+K)s+64+3K=0$$

Routh Tabulation :

s^2	1	$3K+64$
s^1	$16+K$	0
s^0	$3K+64$	

For sustained oscillation, there must be row of zeros. So,

$$16+K=0$$

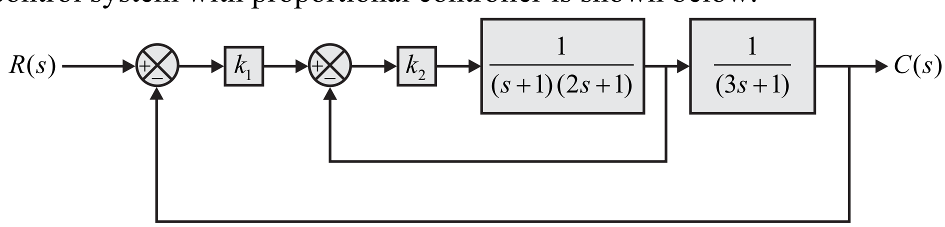
$$K=-16$$

Since K cannot be negative for negative feedback system so oscillation is not possible.

Hence, the correct option is (D).

Question 33 [Practice Book] **[GATE IN 2007 IIT-Kanpur : 2 Marks]**

A cascade control system with proportional controller is shown below.

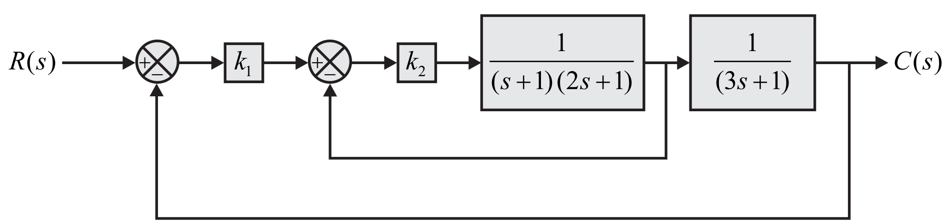


Theoretically, the largest values of the gains k_1 and k_2 that can be set without causing instability of the closed loop system are

(A) 10 and 100 (B) 100 and 10 (C) 10 and 10 (D) ∞ and ∞

Ans.

Sol.



Characteristic equation of the system,

$$1+\frac{k_1}{3s+1}\times\frac{k_2}{(s+1)(2s+1)+k_2}=0$$

$$(3s+1)(s+1)(2s+1)+(3s+1)k_2+k_1k_2=0$$
$$(3s^2+4s+1)(2s+1)+(3s+1)k_2+k_1k_2=0$$
$$6s^3+11s^2+(6+3k_2)s+1+k_2+k_1k_2=0$$

Routh Tabulation :

s^3	6	$6+3k_2$
s^2	11	$1+k_2+k_1k_2$
s^1	$\dfrac{66+33k_2-6(1+k_2+k_1k_2)}{11}$	
s^0	$1+k_2+k_1k_2$	

For the system to be stable all the elements of first column of the array should have same sign. In the above array all the elements of first column must be positive for the system to be stable.

Case 1 : $k_1=10,\ k_2=100$

First element of s^1- row $=\dfrac{66+33k_2-6(1+k_2+k_1k_2)}{11}=\dfrac{66+33\times100-6(1+100+10\times100)}{11}=-\dfrac{3240}{11}$

First element of s^0- row $=1+k_2+k_1k_2=1+100+10\times100=1101$

As first element of s^1- row is negative so system is unstable for $k_1=10$ and $k_2=100$

Case 2 : $k_1=100,\ k_2=10$

First element of s^1- row $=\dfrac{66+33k_2-6(1+k_2+k_1k_2)}{11}=\dfrac{66+33\times10-6(1+10+100\times10)}{11}=-\dfrac{5670}{11}$

First element of s^0- row $=1+k_2+k_1k_2=1+10+100\times10=1011$

As first element of s^1- row is negative so system is unstable for $k_1=100$ and $k_2=10$

Case 3 : $k_1=10,\ k_2=10$

First element of s^1- row $=\dfrac{66+33k_2-6(1+k_2+k_1k_2)}{11}=\dfrac{66+33\times10-6(1+10+10\times10)}{11}=-\dfrac{270}{11}$

First element of s^0- row $=1+k_2+k_1k_2=1+10+10\times10=111$

As first element of s^1- row is negative so system is unstable for $k_1=10$ and $k_2=10$

Case 4 : $k_1=\infty,\ k_2=\infty$

First element of s^1- row $=\dfrac{66+33k_2-6(1+k_2+k_1k_2)}{11}=\dfrac{60+27k_2-6k_1k_2}{11}=\dfrac{\frac{60}{k_2}+27-6k_1}{11}$

$$=\frac{\left(\frac{60}{\infty}\right)+27-6\times\infty}{11}=-\infty$$

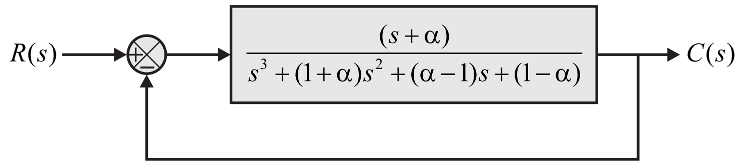
First element of s^0- row $=1+k_2+k_1k_2=1+\infty+\infty\times\infty=\infty$

As first element of s^1- row is negative so system is unstable for $k_1=\infty$ and $k_2=\infty$

Conclusion : None of the option is correct.

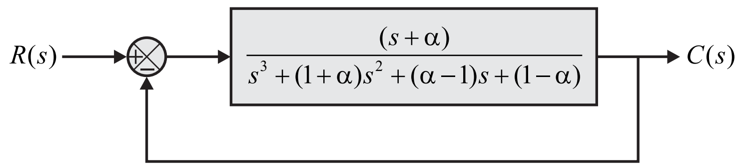
Question 3 [Work Book] **[GATE EE 2014 (Set 1) IIT-Kharagpur : 2 Marks]**

For the given system, it is desired that the system be stable. The minimum value of α for this condition is __________.



Ans. 0.61 to 0.63

Sol. The given figure is shown below.



The characteristics equation is given by, $1+G(s)H(s)=0$

$$1+\frac{(s+\alpha)}{s^3+(1+\alpha)s^2+(\alpha-1)s+(1-\alpha)}=0$$

$$s^3+(1+\alpha)s^2+(\alpha-1)s+(1-\alpha)+(s+\alpha)=0$$

$$s^3+(1+\alpha)s^2+\alpha s+1=0$$

Routh Tabulation :

s^3	1	α
s^2	$1+\alpha$	1
s^1	$\frac{\alpha(1+\alpha)-1}{1+\alpha}$	0
s^0	1	

For the system to be stable, all the roots must be in the left-half s-plane, and, thus, all the coefficients in the first column of Routh's tabulation must have the same sign. Therefore, first column of the Routh's array should be positive.

This leads to the following conditions :

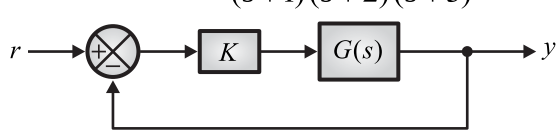
$$1+\alpha>0 \quad \Rightarrow \quad \alpha>-1$$

$$\frac{\alpha(1+\alpha)-1}{1+\alpha}>0 \quad \Rightarrow \quad \alpha=0.618,\ -1.618$$

Hence, the answer is $\alpha=0.618$.

Question 44 [Practice Book] **[GATE EC 2016 (Set 02) IISc-Bangalore : 2 Marks]**

In the feedback system shown below $G(s)=\frac{1}{(s+1)(s+2)(s+3)}$.



The positive value of K for which the gain margin of the loop is exactly 0 dB and the phase margin of the loop is exactly zero degree is_________.

Ans. 60

Sol. Given : $G(s)=\frac{1}{(s+1)(s+2)(s+3)}$

Gain margin = 0 dB and phase margin $=0^0$

Hence system should be marginally stable.

Then find the value of K_{mar} by Routh Hurwitz criteria.

Characteristics equation is given by,

$$1+G(s)+H(s)=0$$

$$1+\frac{K}{(s+1)(s+2)(s+3)}=0$$

$$(s+1)(s+2)(s+3)+K=0$$

$$(s+1)\left[s^2+5s+6\right]+K=0$$

$$s^3+5s^2+6s+s^2+5s+6+K=0$$

$$s^3+6s^2+11s+6+K=0$$

By Routh array

s^3	1	11
s^2	6	$(6+K)$
s^1	$\frac{66-6-K}{6}$	0
s^0	$6+K$	

For marginally stable system

$$\frac{66-6-K}{6}=0$$

$$60-K=0$$

$$K_{mar}=60$$

Note : You will always get questions on marginally stable system, but question will be in different style.

IES Objective Solutions

Question 2 [Practice Book] **[IES EC 1991]**

Which of the following statement is true for the given characteristic equation.

$$2s^4+s^3+3s^2+5s+10=0$$

(A) It has only one root on the imaginary axis

(B) It has two roots in the right half of the *s*-plane

(C) The system is unstable.

(D) The system is stable.

Ans. **(B) and (C)**

Sol. **Given :** Characteristic equation is,

$$2s^4+s^3+3s^2+5s+10=0$$

Routh Tabulation :

s^4	2	3	10
s^3	1	5	0
s^2	-7	10	
s^1	6.42		
s^0	10		

No. of sign changes = 2

No. of roots in RH of *s*-plane = 2

No. of roots in LH of s-plane = 2

For the system to be stable, all the roots must be in the left-half s-plane, and, thus, all the coefficients in the first column of Routh's tabulation must have the same sign. Therefore, first column of the Routh's array should be positive.

Hence, the correct option is (B) and (C).

Question 17 [Practice Book] [IES EC 2003]

The characteristic equation of a control system is given by $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$

The number of the roots of the equation which lie on the imaginary axis of s-plane is

(A) 0 (B) 2 (C) 4 (D) 6

Ans. (C)

Sol. Given : Characteristic equation is,

$$1+G(s)H(s)=0$$

$$s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$$

Routh Tabulation :

s^6	1	8	20	16
s^5	2	12	16	0
s^4	$\frac{16-12}{2}=2$	$\frac{40-16}{2}=12$	16	0
s^3	(0)ε	0	0	0

$\therefore$ Auxiliary equation :

$$A(s)=2s^4+12s^2+16$$

$$\frac{dA(s)}{ds}=8s^3+24s$$

s^3	8	24	0	0
s^2	$\frac{96-48}{8}=6$	16	0	0
s^1	2.67	0	0	0
s^0	16	0	0	0

There is no sign change in the first column of the R-H table, but row s^3 have zero elements. The stability is examined as below.

The roots of the auxiliary equation of the s^4 row are also the roots of the characteristic equation and then determined below :

$$2s^4+12s^2+16=0$$

Put $s^2=x$,

$$2x^2+12x+16=0 \Rightarrow x^2+6x+8=0$$

$$x^2+4x+2x+8=0 \Rightarrow x(x+4)+2(x+4)=0$$

$$(s+4)(s+2)=0$$

The roots are $s_1=\pm j2$ and $s_2=\pm j\sqrt{2}$

Thus, there are four roots which lie on the imaginary axis of s-plane.

Hence, the correct option is (C).

Question 20 [Practice Book] **[IES EE 2004]**

The characteristic equation for a third–order system is : $q(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$.

For the third–order system to be stable, besides that all the coefficients have to be positive, which one of the following has to be satisfied as a necessary and sufficient condition ?

(A) $a_0 a_1 \ge a_2 a_3$ (B) $a_1 a_2 \ge a_0 a_3$ (C) $a_2 a_3 \ge a_1 a_0$ (D) $a_0 a_3 \ge a_1 a_2$

Ans. (B)

Sol. Characteristic equation is $q(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$

For stability of 3rd order system,

Inner product (IP) > Outer product (OP)

$$a_1 a_2 > a_0 a_3$$

For marginal stability of 3rd order system,

Inner product (IP) = Outer product (OP)

$$a_1 a_2 = a_0 a_3$$

Hence, the correct option is (B).

Question 11 [Work Book] **[IES EC 2006]**

The characteristic equation of a control system

$$s^5 + 15s^4 + 85s^3 + 225s^2 + 247s + 120 = 0$$

What are the numbers of roots of the equation which lie to the left of the line $s + 1 = 0$?

(A) 2 (B) 3 (C) 4 (D) 5

Ans. (C)

Sol. Given : Characteristic equation is,

$$s^5 + 15s^4 + 85s^3 + 225s^2 + 247s + 120 = 0$$

Substitute $s = z - 1$

$$(z-1)^5 + 15(z-1)^4 + 85(z-1)^3 + 225(z-1)^2 + 247(z-1) + 120 = 0$$

$$z^5 + 10z^4 + 35z^3 + 50z^2 + 24z = 0$$

Routh Tabulation :

z^5	1	35	24
z^4	10	50	
z^4	1	5	
z^3	30	24	
z^2	4.2	0	
z^1	24	0	
z^0	0		

There are 4 roots which lie to the left of the line $s + 1 = 0$ and one root lies on $s + 1 = 0$.

Hence, the correct option is (C).

Question 30 [Practice Book] **[IES EC 2008, EE 2007]**

The open-loop transfer function of a unity feedback control system is given by $G(s) = Ke^{-Ts}$

Where K and T are variables and are greater than zero. The stability of the closed-loop system depends on

(A) K only (B) Both K and T (C) T only (D) Neither K nor T

Ans. (D)

Sol. **Given :** $G(s) = Ke^{-Ts}$

$$G(s) \approx K(1 - Ts)$$

Characteristic equation is given by,

$$1 + G(s)H(s) = 0$$

$$1 + K(1 - Ts) = 0$$

Routh Tabulation :

s^1	$-KT$	0
s^0	$1+K$	

Since K and T are greater than zero, first column of Routh array will always have sign change and the system will be always unstable. It does not depend on K and T.

Hence, the correct option is (D).

Question 43 [Practice Book] **[IES EE 2009]**

The unit step response of a system is $[1 - e^{-t}(1+t)]u(t)$. What is the nature of the system in turn of stability?

(A) Unstable (B) Stable (C) Critically stable (D) Oscillatory

Ans. **(B)**

Sol. **Given :** Unit step response of a system $= [1 - e^{-t}(1+t)]u(t)$

Impulse response $= \dfrac{d}{dt}$ (step response)

$$= \frac{d}{dt}[1 - e^{-t}(1+t)]$$

Impulse response $= te^{-t}$

Transfer function = Laplace transform of impulse response

$$T.F = \frac{1}{(s+1)^2}$$

$$1 + G(s)H(s) = 0$$

$$1 + \frac{1}{(s+1)^2} = 0$$

$$s^2 + 1 + 2s + 1 = 0$$

$$s^2 + 2s + 2 = 0$$

$$s = (-1 + i) \text{ and } (-1 - i)$$

The roots lie in the left half of s-plane. Therefore, the system is stable.

Hence, the correct option is (B).

Question 60 [Practice Book] **[IES EE 2013]**

The open loop transfer function of an amplifier with resistive negative feedback has two poles in the left half s-plane. Then the amplifier :

(A) Will always be unstable at high frequencies

(B) Will be stable at all frequencies

(C) May become unstable depending upon feedback factor

(D) Will oscillate at low frequencies

Ans. **(B)**

Sol. The open loop transfer function of an amplifier with resistive negative feedback has two poles in the left half s-plane. So, the amplifier is stable.

Hence, the correct option is (B).

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